1.

A. Strong – formulations:

Stress – Strain relation (Hook's Law):

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \quad \sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$$

$$\sigma_{xy} = G\gamma_{xy}.$$

among. $G = \frac{E}{2(1+\nu)}$, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$. $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$.

Boundary Conditions: $\sigma_{xx} = 10\ kPa$. $\sigma_{xy} = 0$.

when $y=0$, $v=0$, $\sigma_{xy}=0$.

when $x=0$, $u=0$, $\sigma_{xy}=0$.

radio $= 0.5m$. $\sigma_{rr}=0$, $\sigma_{\theta r}=0$.

B. Weak Form:

$$\int_\Omega \delta u \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right) d\Omega = 0, \quad \int_\Omega \delta v \left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y}\right) d\Omega = 0.$$

$$\Rightarrow \int_\Omega \sigma_{xx}\frac{\partial\delta u}{\partial x} + \sigma_{xy}\frac{\partial\delta u}{\partial y}\, d\Omega - \int_{\Gamma_t} \delta u\, t_x\, d\Gamma = 0.$$

$$\int_\Omega \sigma_{xy}\frac{\partial\delta v}{\partial x} + \sigma_{yy}\frac{\partial\delta v}{\partial y}\, d\Omega - \int_{\Gamma_t} \delta v\, t_y\, d\Gamma = 0.$$

Γ_t is boundary.

C. Galerkin formulation

$$u(x,y) \approx \sum_{i=1}^{N} u_i \psi_i(x,y) \quad v(x,y) \approx \sum_{i=1}^{N} v_i \psi_i(x,y)$$

Weak form $\Rightarrow \int_\Omega C\,\varepsilon(u_h)\cdot\varepsilon(\delta u)\, d\Omega = \int_{\Gamma_t} \delta u\cdot t\, d\Gamma$

among that: $C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

BCs: Dirichlet BC: $u=0$, at $x=0$, $v=0$ at $y=0$.

Neumann Conditions: $\sigma_{xx} = 10kPa$ at $x=0$. and $\sigma_{xy}=0$.

$\sigma_{rr}=0$, $\sigma_{r\theta}=0$. at hole boundary.

2. The implementation of the element Stiffness matrix.

For the FEM for 2D elasticity problem, I use 3-node linear triangular elements.

a. Numerical Integration:

The stiffness matrix is computed using Gaussian quadrature.
The code uses 3 Gauss points. And the Jacobian matrix (J) is computed to map the natural ~~coo~~ coordinates (ξ, η) to global coordinates (x, y).

b. Strain - Displacement Matrix (B).

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}.$$

c. Material Properties Matrix (D).

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \sigma = D\varepsilon.$$

d. Element Stiffness Matrix (K_{loc}).

$$K_{loc} = \int_{Element} B^T D B \det J \, dA.$$

e. Assembly into Global Stiffness Matrix.

The degrees of freedom (DOFs) for each node in ~~to~~ the element are mapped into the global system using the connectivity matrix IEN.

3. **Analytical solution**

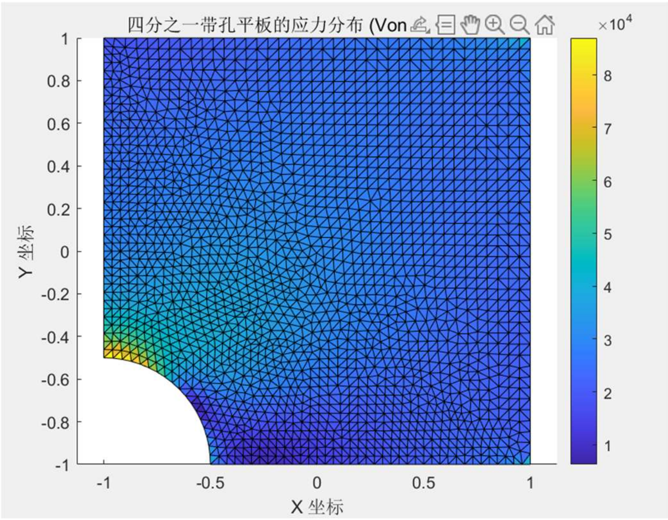
Write the analytical answer using the polar coordinate formula below:

$$\sigma_{rr}(r,\theta)=\frac{T_x}{2}\left(1-\frac{R^2}{r^2}\right)+\frac{T_x}{2}\left(1-4\frac{R^2}{r^2}+3\frac{R^4}{r^4}\right)cos2\theta,$$

$$\sigma_{\theta\theta}(r,\theta)=\frac{T_x}{2}\left(1+\frac{R^2}{r^2}\right)-\frac{T_x}{2}\left(1+3\frac{R^4}{r^4}\right)cos2\theta,$$

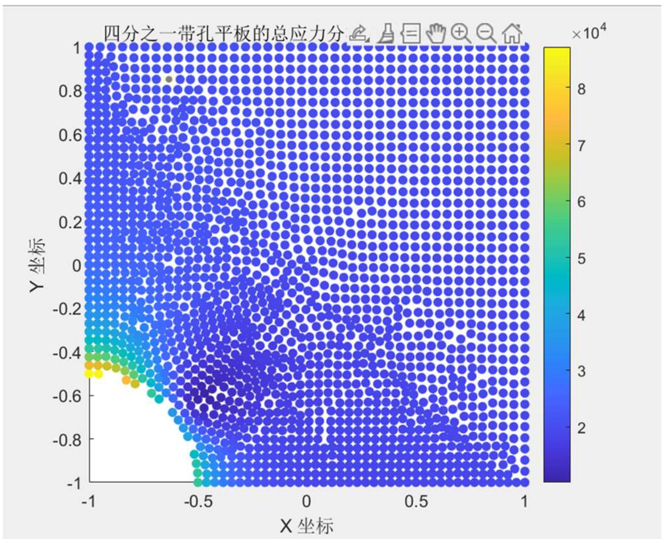
$$\sigma_{r\theta}(r,\theta)=-\frac{T_x}{2}\left(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)sin2\theta.$$

I completed the analysis using MATLAB and visualized the stress distribution by plotting. It is worth noting that the xy coordinate is referenced to the grid coordinate.



4. The solution of finite element analysis

Stress :



Strain: