1.

A. Strong - formulations:

Stress - Strain relation (Hook's Law):

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \quad \sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$$

$$\sigma_{xy} = G\gamma_{xy}.$$

among. $G = \frac{E}{2(1+\nu)}$, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$. $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$.

Boundary Conditions: $\sigma_{xx} = 10\ kPa$. $\sigma_{xy} = 0$.

when $y=0$, $v=0$, $\sigma_{xy}=0$.

when $x=0$, $u=0$, $\sigma_{xy}=0$.

radio $= 0.5m$. $\sigma_{rr}=0$, $\sigma_{\theta r}=0$.

B. Weak Form:

$$\int_\Omega \delta u\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right)d\Omega = 0, \quad \int_\Omega \delta v\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y}\right)d\Omega = 0.$$

$$\Rightarrow \int_\Omega \sigma_{xx}\frac{\partial\delta u}{\partial x} + \sigma_{xy}\frac{\partial\delta u}{\partial y}\,d\Omega - \int_{\Gamma_t}\delta u\, t_x\, d\Gamma = 0.$$

$$\int_\Omega \sigma_{xy}\frac{\partial\delta v}{\partial x} + \sigma_{yy}\frac{\partial\delta v}{\partial y}\,d\Omega - \int_{\Gamma_t}\delta v\, t_y\, d\Gamma = 0.$$

Γ_t is boundary.

C. Galerkin formulation

$$u(x,y) \approx \sum_{i=1}^{N} u_i\psi_i(x,y) \quad v(x,y) \approx \sum_{i=1}^{N} v_i\psi_i(x,y)$$

Weak form $\Rightarrow \int_\Omega C\,\varepsilon(u_h)\cdot\varepsilon(\delta u)\,d\Omega = \int_{\Gamma_t}\delta u\cdot t\,d\Gamma$

among that: $C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

BCs: Dirichlet BC: $u=0$, at $x=0$, $v=0$ at $y=0$.

Neumann Conditions: $\sigma_{xx} = 10kPa$ at $x=0$. and $\sigma_{xy}=0$.

$\sigma_{rr}=0$, $\sigma_{r\theta}=0$ at hole boundary.

2. The implementation of the element Stiffness matrix.

For the FEM for 2D elasticity problem, I use 3-node linear triangular elements.

a. Numerical Integration:

The stiffness matrix is computed using Gaussian quadrature.
The code uses 3 Gauss points. And the Jacobian matrix (J) is computed to map the natural ~~coo~~ coordinates (ξ, η) to global coordinates (x, y).

b. Strain - Displacement Matrix (B).

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}.$$

c. Material Properties Matrix (D).

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad \sigma = D \varepsilon .$$

d. Element Stiffness Matrix (K_{loc}).

$$K_{loc} = \int_{Element} B^T D B \det J \, dA .$$

e. Assembly into Global Stiffness Matrix.

The degrees of freedom (DOFs) for each node in ~~to~~ the element are mapped into the global system using the connectivity matrix IEN.