Dynamics of Thermocapillary Migration with Positive Hysteresis Coefficient

Dong Geng

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1 Physical Model and Coordinate System

We consider a liquid bridge migrating between two parallel plates (gap $2h$) inclined at an angle ϕ. The motion is driven by thermocapillarity from the **hot end** ($x = 0$) towards the **cold end** ($x > 0$) against gravity.

1.1 Field Distributions

A constant linear thermal gradient $G = -\frac{dT}{dx} > 0$ is applied.

- Temperature: $T(x) = T_{\text{hot}} - G \cdot x$
- Surface Tension: $\gamma(x) = \gamma_{\text{hot}} + \gamma_T G \cdot x$
- Viscosity: $\mu(x) = \mu_{\text{hot}} \exp(b \cdot G \cdot x)$

2 Force Analysis

The governing equation is Newton's Second Law:

$$m\ddot{x} = F_{\text{cap}} - F_{\text{visc}} - F_{\text{grav}} \tag{1}$$

where mass $m \approx \frac{\pi}{2}\rho D^2 h$.

2.1 Capillary Driving Force expressed by Advancing Angle θ_a

The capillary force is defined by the pressure difference between the advancing (cold, θ_a) and receding (hot, θ_r) interfaces:

$$F_{\text{cap}} \approx 2hD \left(\frac{\gamma_{\text{cold}} \cos\theta_a}{h} - \frac{\gamma_{\text{hot}} \cos\theta_r}{h} \right) \tag{2}$$

Hysteresis Definition ($\beta > 0$):
We define the positive hysteresis coefficient β as:

$$\beta = \frac{\cos\theta_r}{\cos\theta_a} - 1 \tag{3}$$

From this definition, we can express the receding angle θ_r in terms of the advancing angle θ_a:

$$\cos\theta_r = (1+\beta)\cos\theta_a \tag{4}$$

Since $\theta_a > \theta_r$, $\cos\theta_a < \cos\theta_r$, confirming that $\beta > 0$.

Substituting $\cos\theta_r$ into the force equation:

$$F_{\text{cap}} = 2D\left[\gamma_{\text{cold}}\cos\theta_a - \gamma_{\text{hot}}(1+\beta)\cos\theta_a\right] \tag{5}$$

$$= 2D\cos\theta_a\left[\gamma_{\text{cold}} - \gamma_{\text{hot}} - \beta\gamma_{\text{hot}}\right] \tag{6}$$

Using the approximations $\gamma_{\text{cold}} - \gamma_{\text{hot}} \approx \gamma_T G D$ and $\gamma_{\text{hot}} \approx \gamma(x)$:

$$F_{\text{cap}} \approx \underbrace{2D^2\gamma_T G\cos\theta_a}_{\text{Driving Force (+)}} - \underbrace{2D\gamma(x)\beta\cos\theta_a}_{\text{Hysteresis Drag (-)}} \tag{7}$$

2.2 Viscous and Gravity Forces

$$F_{\text{visc}} = \frac{6\mu(x)D^2}{h}\dot{x}, \quad F_{\text{grav}} = mg\sin\phi \tag{8}$$

3 Governing Differential Equation

Substituting the forces into the equation of motion ($m\ddot{x} = F_{\text{cap}} - F_{\text{visc}} - F_{\text{grav}}$) and dividing by mass $m = \frac{\pi}{2}\rho D^2 h$:

$$\boxed{\frac{d^2x}{dt^2} = \underbrace{\frac{4\gamma_T G\cos\theta_a}{\pi\rho h}}_{\text{Marangoni Accel.}} - \underbrace{\frac{4\gamma(x)\beta\cos\theta_a}{\pi\rho h D}}_{\text{Hysteresis Drag}} - \underbrace{\frac{12\mu(x)}{\pi\rho h^2}\frac{dx}{dt}}_{\text{Viscous Damping}} - \underbrace{g\sin\phi}_{\text{Gravity}}} \tag{9}$$

Interpretation of Terms:

- **Reference Angle**: All capillary terms are now scaled by $\cos\theta_a$ (the advancing angle at the cold end).
- **Marangoni Term (+)**: Constant driving acceleration proportional to $\cos\theta_a$.
- **Hysteresis Term (-)**: Resistive acceleration. Since $\beta > 0$, this term opposes motion. It scales with local surface tension $\gamma(x)$.
- **Viscous Term (-)**: Damping proportional to velocity.
- **Gravity Term (-)**: Constant resistance due to uphill slope.