ON DISCRETE FOURIER ANALYSIS AND APPLICATIONS

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Abstract. In this short note we introduce a few tools in discrete Fourier analysis and prove Meshulam and Roth's theorem. These notes are based on a minicourse given by Victor Souza at IMPA, summer 2024.

1. Lecture 1: Fourier analysis and arithmetic progressions

Let $(G,\cdot)$ be a finite abelian group. A 3-AP in G is simply a set of the form $\{x, x+y, x+2y\} \subseteq G$ with $x \neq y$. Let $S \subseteq G$ be a finite subset. We define

$$r_3(S) = \max\{A \subseteq S : A \text{ is 3-AP free}\}.$$

The main question we are interested in is: what is the size of $r_3([N])$?

In 1953, Roth proved that that dense enough sets $A \subseteq [N]$ must contain 3-AP, for density threshold $\delta \approx \frac{1}{\log \log N}$.

In the late 1980′s, this was improved by Heath–Brown and Szemerédi to $\delta \approx \frac{1}{(\log N)^c}$, for some small $c > 0$. This bound was further refined in the works of Bourgain in 1999 and 2008, and Sanders in 2012 where it is shown that one can take $c = 1/2$, $c = 2/3$, and then $c = 3/4$. In 2011, Sanders then obtained a density-threshold of the form $\delta \approx \frac{(\log \log N)^6}{\log N}$. In 2016, this was further sharpened by a factor $(\log \log N)^2$ by Bloom and then in 2020 again by another factor $\log \log N$ by Schoen. A bit later, Bloom and Sisask showed that a set $A \subseteq [N]$ with no 3-progressions must have density

$$\delta = O\left(\frac{1}{(\log N)^{1+c}}\right)$$

for some small $c > 0$.

1.1. **Behrend's construction.** In the other direction, it was shown by Behrend in 1946 that for infinitely many values N there are indeed subsets $A \subseteq [N]$ of density roughly $\delta \approx 2^{-\sqrt{\log N}}$ which are 3-AP free.

Behrend's major idea is that if we were looking for a progression-free sets in $\mathbb{R}^d$, then we could use spheres. So, consider an d-dimensional cube $[0,n]^d \cap \mathbb{Z}^d$ and family of spheres $x_1^2 + x_2^2 + ... + x_d^2 = t$ for $t = 1, ..., dn^2$. Each point in the cube is contained in one of the spheres, and so at least one of the spheres contains $n^d/(dn^2)$ lattice points. Let us call this set A.

Since spheres do not contain arithmetic progressions, A does not contain any progressions either. Now let f be a Freiman isomorphism from A to a subset of $\mathbb{Z}$ defined as follows. If $x = (x_1, x_2, ..., x_d)$ is a point of A, then $f(x) = x_1 + x_2(2n) + x_3(2n)^2 + ... + x_d(2n)^{d-1}$

That is, we treat x_i as i'th digit of $f(x)$ in base $2n$. Observe that this base was chosen so that f is a Freiman isomorphism of order 2.

Note that $f(A) \subseteq \left\{1, 2, ..., N = (2n)^d\right\}$ and that $f(A)$ is a progression-free set of size at least $n^d/(dn^2)$. Moreover, the density of $f(A)$ in the interval $[1, N]$ is at least

$$\frac{n^d}{dn^2 N} = \frac{1}{d2^d n^2} \approx \frac{1}{2^d n^2} \approx \frac{1}{2^d N^{\frac{2}{d}}}.$$

The last approximation follows from the fact that $N = (2n)^d$. the expression $2^d N^{\frac{2}{d}}$ is minimized when $d = \sqrt{\log N}$, and so we get that the density of $f(A)$ is at least $2^{-\Omega(\sqrt{\log N})}$.

In a major breakthrough, Kelley and Meka showed that Behrend's construction is essentially optimal. More precisely, they proved the following theorem.

Theorem 1.1.1 *Let N be a large integer. Then, there exists a subset $A \subseteq [N]$ of size at least $Ne^{-(\log N)^{\frac{1}{11}}}$ which has no nontrivial 3-progressions.*

1.2. **A bit of discrete Fourier analysis.** Let G be a finite abelian group and let $A \subseteq G$ be an arbitrary subset. Define the indicator function of A as

$$1_A = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ otherwise.} \end{cases}$$

We denote by $L_2(G)$ the space of complex-valued functions on G with the usual inner product, namely

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)} = \mathbb{E}_{x \in G} f(x)\overline{g(x)}.$$

Observe that $L_2(G)$ is a $\mathbb{C}$-vector space of dimension $|G|$.

The **Physical space basis** is the set of functions $\delta_z : G \to \mathbb{C}$ defined by

$$\delta_z(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ otherwise.} \end{cases}$$

Definition 1.2.1 *A character of G is a group homomorphism $\chi : G \to \mathbb{C}^*$. In other words, a character is a function χ such that $\chi(x + y) = \chi(x)\chi(y)$ for all $x, y \in G$.*

The set of characters also defines a group, called the **dual group** of G and denoted by $\hat{G}$. A special character is the identity function $\chi_0(x) = 1$ for all $x \in G$. Now, our next goal is to prove the following theorem.

Theorem 1.2.2 *The set of characters of G forms an orthonormal basis of $L_2(G)$.*

The first step is to show that the characters are indeed orthonormal is to prove the following lemma.

Lemma 1.2.3 *Let $\chi \neq \chi_0$ be a character of G. Then,*

$$\mathbb{E}_{x \in G} \chi(x) = 0.$$

Proof Let χ be a character of G different from χ_0. As $\chi \neq \chi_0$, there exists $y \in G$ such that $\chi(y) \neq 1$. Then, we have

$$\mathbb{E}_{x \in G} \chi(x) = \mathbb{E}_{x \in G} \chi(x+y) = \chi(y) \cdot \mathbb{E}_{x \in G} \chi(x).$$

Since $\chi(y) \neq 1$, we conclude that $\mathbb{E}_{x \in G} \chi(x) = 0$. $\square$

Now we are ready to prove Theorem 1.2.2.

Proof of Theorem 1.2.2 . First, we observe that all characters have norm 1, since

$$\langle \chi, \chi \rangle = \mathbb{E}_{x \in G} \chi(x) \overline{\chi(x)} = \mathbb{E}_{x \in G} \, |\chi(x)|^2 = 1.$$

Now, we show that they form an orthonormal set. Let χ, χ' be two characters of G. If $\chi \neq \chi'$, then we have

References

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