Degree Conditions for Spanning Brooms

Guantao Chen*, Michael Ferrara†, Zhiquan Hu‡,
Michael Jacobson§, and Huiqing Liu¶

December 4, 2013

Abstract

A *broom* is a tree obtained by subdividing one edge of the star an arbitrary number of times. In [E. Flandrin, T. Kaiser, R. Kužel, H. Li and Z. Ryjáček, Neighborhood Unions and Extremal Spanning Trees, *Discrete Math.* **308** (2008), 2343-2350] Flandrin et al. posed the problem of determining degree conditions that ensure a connected graph G contains a spanning tree that is a broom. In this paper, we give one solution to this problem by demonstrating that if G is a connected graph of order $n \geq 56$ with $\delta(G) \geq \frac{n-2}{3}$, then G contains a spanning broom. This result is best possible.

1 Introduction

All graphs considered in this paper are simple and undirected. Given a vertex v in a graph G, we let $N_G(v)$ denote the set of neighbors of v in G and let $d_G(v)$ denote $|N_G(v)|$, the degree of v in G. For simplicity, when the context is clear we will write $N(v)$ and $d(v)$ instead of the more cumbersome $N_G(v)$ and $d_G(v)$. Given a subgraph H of G we let $N_H(v) = N_G(v) \cap V(H)$ and let $d_H(v) = |N_H(v)|$. For an integer $k \geq 1$, let

$$\sigma_k(G) = \min \left\{ d(v_1) + \cdots + d(v_k) \mid \{v_1, \ldots, v_k\} \text{ is an independent set in } G \right\}.$$

*Georgia State University, Atlanta, GA, gchen@gsu.edu. Research supported in part by NSA grant H98230-12-1-0239

†University of Colorado Denver, Denver, CO, michael.ferrara@ucdenver.edu. Research supported in part by Simons Foundation Grant #206692.

‡Central China Normal University, Wuhan, China, hu_zhiq@yahoo.com.cn. Research supported in part by NSFC Grants #11071096, 11271149 and 11371162.

§University of Colorado Denver, Denver, CO, michael.jacobson@ucdenver.edu. Research supported in part by NSF grant DGE-0742434.

¶Hubei University, Wuhan, China, hql_2008@163.com. Research supported in part by NSFC Grant #10971114.

A *block* is a maximal 2-connected subgraph of G, and for a given vertex x in G we refer to a path having x as an endvertex as an *x-path*. If X and Y are subsets of $V(G)$, we let $E(X, Y)$ denote the set of edges uv with $u \in X$ and $v \in Y$.

We consider all cycles to have an implicit clockwise orientation. With this in mind, given a cycle C and a vertex x on C, we let x_C^+ denote the successor of x under this orientation and let x_C^- denote the predecessor. We define x_C^{+i} recursively with $x_C^{+1} = x_C^+$ and $x_C^{+(i+1)} = (x^{+i})_C^+$ for $i > 1$ and define x_C^{-i} analogously. For any any two vertices x and y on C, we let xCy denote the path from x to y on C in the clockwise direction of the orientation and xC^-y denote the path from x to y on C in the counterclockwise direction. Given a set of vertices $X \subseteq V(C)$, we let X_C^+ denote the set of successors of the vertices in X when traversing C in the clockwise direction, and we let X_C^- denote the set of predecessors of the vertices in X when traversing C in the clockwise direction. If $X = N_C(v)$ for some vertex v, we will simply write $N_C(v)^+$ and $N_C(v)^-$ as opposed to the more cumbersome $N_C(v)_C^+$ and $N_C(v)_C^-$.

At times, we will also assign an orientation to a path P. In such a case, given $v \in V(P)$ or $X \subseteq V(P)$, we define $v_P^+, v_P^-, X_P^+, X_P^-$ in a similar manner. For both paths and cycles, we will omit the subscripts "P" and "C" when the context is clear. Also, we let $c(G)$ denote the *circumference* of G, that is the length of a longest cycle in G, and let $p(G)$ denote the order of a longest path in G.

2 Extremal Spanning Trees

In this paper, we are interested in the general problem of finding conditions that ensure a graph G contains a spanning tree with certain extremal properties. There are a number of such results in the literature; see [10] and Chapter 8 of [1] for two excellent and thorough surveys.

As an example, a graph G is *traceable* if it contains a hamiltonian (spanning) path. A *branch vertex* in a tree T is a vertex of degree at least three. A hamiltonian path in a graph G is a spanning tree containing no branch vertices, and as such considering the existence of spanning trees with a bounded number of branch vertices can be viewed as a generalization of traceability. Such trees also have applications to multicasting in optical networks [4, 5, 6]. In [5], Gargano et al. conjectured the following, which would extend a classical result of Ore [8] stating that if G is a graph of order n such that $\sigma_2(G) \geq n - 1$, then G is traceable.

Conjecture 1. *Let $k \geq 0$ be an integer and let G be a connected graph of order n. If $\sigma_{k+2}(G) \geq n - 1$, then G has a spanning tree with at most k branch vertices.*

This conjecture was verified for $k = 1$ in [5]. Ozeki and Yamashita proposed the following conjecture, which they showed would be best possible if true.

Conjecture 2. *Let k be a positive integer, and let G be a connected graph of order n. If $\sigma_{k+3}(G) \geq n-k$, then G has a spanning tree with at most k branch vertices.*

A *spider* is a tree with at most one branch vertex, while a *broom* is a spider that can be obtained from a star by subdividing one edge of the star multiple times. The path P which results from subdividing the edge of the star is referred to as the *handle* of the broom, while the leaves not lying on P are called the *bristles* of the broom. In [3], Flandrin et al. posed the following general problem.

Problem 1. *Find a degree condition on a graph G that guarantees the existence of a spanning broom in G.*

Here, we give a sharp minimum degree condition assuring the existence of a spanning broom in an arbitrary graph G of sufficiently large order.

Theorem 1. *If G is a connected graph of order $n \geq 56$ with $\delta(G) \geq \frac{n-2}{3}$, then G contains a spanning broom. This minimum degree condition is sharp.*

The sharpness of Theorem 1 follows from the example in Figure 1. Interestingly, the largest broom in the sharpness example has order approximately $\frac{2n}{3}$. Therefore for $\frac{2}{3} < c \leq 1$ Theorem 1 implies that the minimum degree threshold for the existence of a broom of order at least cn in a graph of sufficiently large order n is also necessarily $\frac{n-2}{3}$.

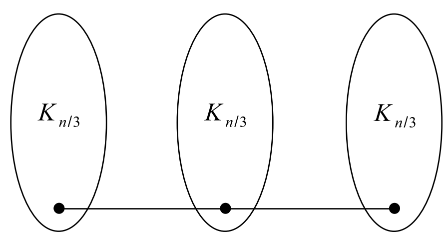

Figure 1: A graph G with $\delta = \frac{n}{3} - 1$ and no spanning broom.

We prove the following two results that together imply Theorem 1.

Theorem 2. *Let G be a connected graph of order $n \geq 56$ such that $\delta(G) \geq \frac{n-2}{3}$.*

If $\kappa(G) = 1$, then G contains a spanning broom. If G is 2-connected, then one of the following two conditions hold:

1. $p(G) - c(G) \leq 1$*, or*

2. G *contains a spanning broom.*

Define a *jellyfish* to be the graph obtained from a broom B by adding an edge between one of the bristles of B and the vertex of degree one on the handle. If a jellyfish J is a spanning subgraph of a graph G, then we call J a *spanning jellyfish* (of G).

Theorem 3. *Let* G *be a 2-connected graph of order* n *such that* $\delta(G) \geq \frac{n-2}{3}$ *and* $p(G) - c(G) \leq 1$*. Then* G *contains a spanning jellyfish.*

3 Proof of Theorem 2

In order to prove Theorem 2, we require several lemmas. We begin by modifying the following from [12].

Lemma 1. *Let* G *be a 3-connected graph of order* $n \geq 3$ *with* $\sigma_4(G) \geq n + 6$ *and let* C *be a longest cycle in* G*. If* H *is component of* $G - V(C)$*, then* $|H| \leq 2$*.*

We utilize Lemma 1 to prove the following.

Lemma 2. *Let* G *be a 2-connected graph of order* $n \geq 2$ *with* $\sigma_4(G) \geq n + 3$ *and let* P *be a longest path in* G*. If* H *is a component of* $G - V(P)$*, then* $|H| \leq 2$*.*

Proof. Let G be as given and let $G' = G \vee K_1$ where $\vee$ denotes the standard graph join and $V(K_1) = \{v\}$. Since G' satisfies the hypotheses of Lemma 1, for any longest cycle C' of G' we have that each component of $G' - V(C')$ has order two. However, C' necessarily contains v and therefore $C' - v$ must be a longest path P in G with $G' - V(C') = G - V(P)$. As each longest path in G similarly corresponds to a longest cycle in G', the result follows. $\square$

Lemma 3. *Let* G *be a graph of order* n *with* $\delta(G) \geq \frac{n-2}{3}$*,* C *be a longest cycle in* G*, and* uv *be a component of* $G - C$*. If* $|C| \leq n - 3$*, then*

$$|N_C(u) \cap N_C(v)| \geq \frac{n-5}{3}.$$

Furthermore, there are at least $\frac{n-11}{3}$ *vertices* $x \in N_C(u) \cap N_C(v)$ *such that* x^{+3} *is also in* $N_C(u) \cap N_C(v)$*.*

Proof. We consider the neighbors of u and v on C so as to bound $|C|$. First, observe that if $x \in N_C(u)$ (respectively $N_C(v)$), then neither x^+ nor x^{+2} is adjacent to v (respectively u), as this would imply the existence of a cycle longer than C in G, contradicting our assumption that C is a longest cycle. Also note that neither u nor v is adjacent to consecutive vertices on C. Thus, if y is a vertex in $N_C(u) - N_C(v)$, then y^+ does not lie in $N_C(u) \cup N_C(v)$ and a similar assertion holds for any vertex y' in $N_C(v) - N_C(u)$.

Taking these two observations into account, we have that

$$|C| \geq 3|N_C(u) \cap N_C(v)| + 2\Big(|N_C(u) \cup N_C(v)| - |N_C(u) \cap N_C(v)|\Big), \tag{1}$$

which together with the above restrictions on the neighbors of u and v implies that $N_C(u) \cap N_C(v)$ is nonempty. However, this alone is not sufficient to obtain the desired bound.

Let $S = N_C(u) \cup N_C(v)$ and let $s_1, \ldots, s_k$ be the vertices of S indexed in the order they appear when traversing C in the clockwise direction. Observe that if either $N_C(u) \subseteq N_C(v)$ or $N_C(v) \subseteq N_C(u)$, then u and v have at least $\frac{n-5}{3}$ common neighbors on C. Thus, we may suppose that there is at least one vertex in each of $N_C(u) - N_C(v)$ and $N_C(v) - N_C(u)$.

As $N_C(u) \cap N_C(v)$ is nonempty, we may therefore assume that (a) s_1 is in $N_C(u) \cap N_C(v)$, but s_k is not, and (b) there is some s_i, $i \neq k$, such that either $s_i \in N_C(u) - N_C(v)$ and $s_{i+1} \in N_C(v)$ or $s_i \in N_C(v) - N_C(u)$ and s_{i+1} is in $N_C(u)$. We have that neither s_k^+ nor s_k^{+2} is in $N_C(u) \cup N_C(v)$ as then we would be able to extend C, and assumption (b) similarly implies that neither s_i^+ nor s_i^{+2} is in $N_C(u) \cap N_C(v)$. Thus we slightly improve (1) and obtain that

$$|C| \geq 3\Big(|N_C(u) \cap N_C(v)| + 2\Big) + 2\Big(|N_C(u) \cup N_C(v)| - |N_C(u) \cap N_C(v)| - 2\Big).$$

If we let $t = |N_C(u) \cap N_C(v)|$, it follows that

$$\begin{aligned} |C| &\geq 3(t+2) + 2\left(|N_C(u) \cup N_C(v)| - t - 2\right) \\ &= 3(t+2) + 2\Big((d_C(u) + d_C(v)) - 2t - 2\Big). \end{aligned}$$

This implies that

$$t \geq 2(d_C(u) + d_C(v)) - |C| + 2 \geq 2\left(\frac{2(n-5)}{3}\right) - (n-3) + 2,$$

so that

$$t = |N_C(u) \cap N_C(v)| \geq \frac{n-5}{3},$$

as desired.

The second part of the claim follows from the fact that for each vertex $x \in N_C(u) \cap N_C(v)$ the sets $S_x = \{x, x^+, x^{+2}\}$ are disjoint and together must account for at least $3|N_C(u) \cap N_C(v)| \geq n-5$ vertices on C. As $|C| \leq n-3$, there are at most two vertices that do not lie in any S_x, and as such, there are at least $\frac{n-5}{3} - 2$ vertices $x \in N_C(u) \cap N_C(v)$ such that $x^{+3} \in N_C(u) \cap N_C(v)$, as desired. $\square$

The following lemma will also be useful as we proceed.

Lemma 4. *Let G be a 2-connected graph of order $n \geq 12$ such that $\delta(G) \geq \frac{n-3}{2}$. Then for every vertex x in G, there is a spanning broom B_x of G such that x is the endpoint of the handle of B_x.*

Proof. We begin by considering P_x, a longest x-path in G, and we claim that P_x has at least $n-3$ vertices. Indeed, consider a longest cycle C in G which, since G is 2-connected and $\delta(G) \geq \frac{n-3}{2}$, necessarily has length at least $n-3$ by a classical result of Dirac [2]. As G is connected, there is a (possibly trivial) path from x to C, so that we may assume that $|P_x| \geq n-3$.

Now assume that amongst all choices of P_x we have selected a path that is contained in a broom B of maximum order amongst all those with x as the endpoint of its handle. Let y be the penultimate vertex of P_x when orienting P_x from x, so that y is adjacent to the bristles of B, and assume that B has b bristles.

Observe that $|G - V(B)| \leq 3$ and that no vertex v in $G - V(B)$ is adjacent to y as this would contradict the maximality of B, and further that v cannot be adjacent to consecutive vertices on P_x or to any bristle of B, as this would contradict the assertion that P_x is a longest x-path in G. Thus

$$d(v) \leq \frac{1}{2}\left(|P_x| - b\right) + d_{G-B}(v). \tag{2}$$

Suppose that v is an isolated vertex in $G - B$ so that (2) and our minimum degree condition together imply that $b \leq 2$. Note first that if $b = 2$, then $|P_x| \leq n-2$. Hence (2) yields that $d(v) \leq \frac{n-4}{2}$, a contradiction.

Thus we may assume that $b = 1$ so that $B = P_x$, and we again let b_1 denote the endpoint of P_x that is distinct from x. For each vertex w in $N_{P_x}(v)$, we have as above that b_1 is not adjacent to w^-. In addition to this, $b_1 w^{-2} \notin E(G)$ as then $xP_x w^{-2} b_1 P_x^- w$ is the handle of a broom with bristles v and w^-, contradicting the maximality of B. As b_1 has no neighbors in $G - P_x$, we have that

$$\begin{aligned} d(b_1) &\leq (|P_x| - 1) - 2(d_{P_x}(v) - 1) \\ &\leq n - 2 - 2\left(\frac{n-5}{2}\right) = 3, \end{aligned}$$

which is a contradiction since $n \geq 12$.

If there is some edge uv in $G - B$ (possibly contained within a P_3 or a K_3), then since $|G - B| \leq 3$ both u and v have at least $\frac{n-7}{2}$ neighbors on $xP_x y^-$. As u and v cannot be adjacent to consecutive vertices on P_x and there is no vertex $q \in N_{P_x}(v)$ (respectively $N_{P_x}(u)$) such that either q^+ or q^{+2} is in $N_{P_x}(u)$ (resp. $N_{P_x}(v)$), it is elementary to demonstrate a contradiction to the assumption that $d(u) + d(v) \geq n - 3$. As $G - V(B)$ either contains an edge or is an independent set, the result follows. $\square$

A graph G is *hamiltonian-connected* if for every u and v in G, there is a hamiltonian path in G with endvertices u and v. We will use the following result from [9], which gives the minimum degree threshold for a graph to be hamiltonian-connected.

Lemma 5. *If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{n+1}{2}$, then G is hamiltonian-connected.*

We are now ready to proceed with the proof of Theorem 2.

Proof. Let G be as given, and assume first that $\kappa(G) = 1$. Note that $\delta(G) \geq \frac{n-2}{3}$ implies that G has at most four blocks. If G has exactly three or four blocks, G necessarily has one of the block configurations given in Figure 2. Given a block B in G, let B^* denote the graph obtained by deleting all cut vertices from B. In the interest of concision, we consider only possibilities (1) and (2) here, as structures (1a) and (2a) are handled in a nearly identical manner to structures (1) and (2), respectively.

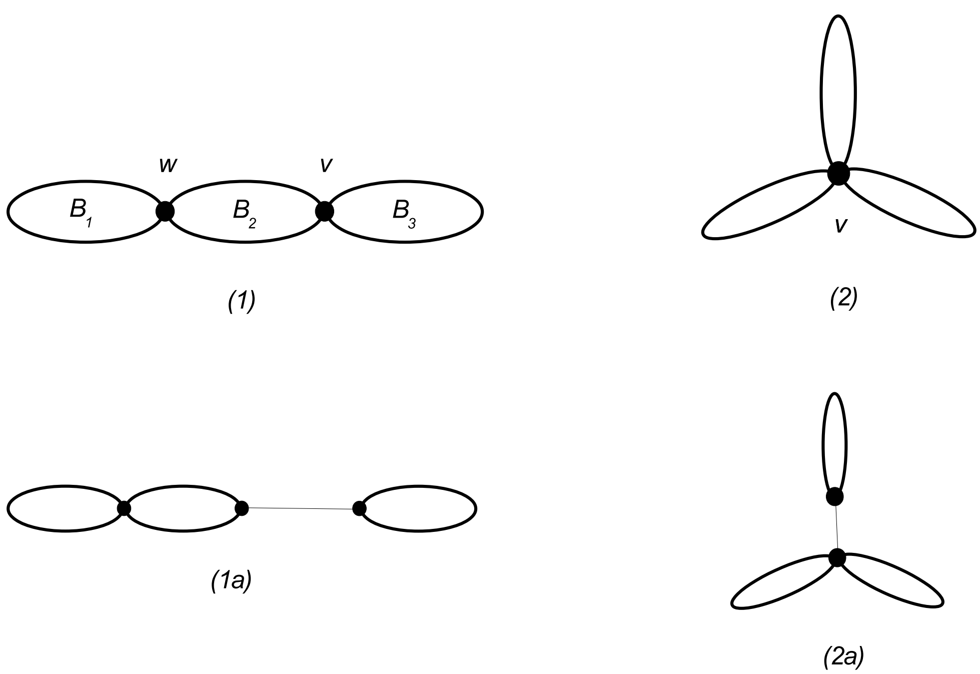

Figure 2: Feasible block structures in Theorem 2

Suppose first that G has block structure (1), and furthermore that B_2 is an edge and $|B_1| \leq |B_3|$. Since $\delta(G) \geq \frac{n-2}{3}$, it follows that $|B_1| \geq \frac{n+1}{3}$ and $|B_3| \leq \frac{2n-1}{3}$. Since v has only one neighbor outside of B_3, $\delta(B_3) \geq \frac{n-5}{3} \geq \frac{|B_3|-3}{2}$, so that by Lemma 4, B_3 contains a spanning broom B with v as the endpoint of its handle. As $|B_1| \leq \frac{n}{2}$, Lemma 5 implies that B_1 is hamiltonian-connected which, together with B, implies that G contains a spanning broom.

If $|B_2| \geq 3$, then as each block of G must have order at least $\frac{n+1}{3}$ and $n \geq 56$, Lemma 5 implies that B_1^*, B_2^* and B_3^* are all hamiltonian-connected. Let $w_i \in N_{B_i}(w)$ for $i \in \{1,2\}$ and $v_i \in N_{B_i}(v)$ for $i \in \{2,3\}$, such that $v_2 \neq w_2$ (which is possible since B_2 is 2-connected). Also choose a hamiltonian path P_1 in B_1^* with endpoint w_1, a hamiltonian path P_2 in B_2^* with endpoints v_2 and w_2 and a hamiltonian path P_3 in B_3^* with endpoint v_3. Then $P_1 w_1 w w_2 P_2 v_2 v v_3 P_3$ is a hamiltonian path in G.

If G has block structure (2), then $\delta(G) \geq \frac{n-2}{3}$ implies that at least two of the blocks of G are complete, so that in particular v is adjacent to every other vertex in these blocks. For the third block, call it B, $n \geq 56$ and Lemma 5 imply that B^* is hamiltonian connected. Thus, for any vertex $v' \in N_B(v)$, there is a hamiltonian path P with v' as an endpoint. As vv' is in $E(G)$ and v is adjacent to every vertex in $G - B$, we see that G has a spanning broom. Note that if G has block structure (2a), then all four blocks of G are necessarily complete.

Thus G has exactly two blocks, so if v is a cut-vertex of G, then $G - v$ must have exactly two components, call them G_1 and G_2. Assume that $|G_1| \leq |G_2|$, so that our assumption that $\delta(G) \geq \frac{n-2}{3}$ implies that $\frac{n-2}{3} \leq |G_1| \leq \frac{n-1}{2}$. Since $n \geq 56$, we have that $\delta(G_1) \geq \frac{n-5}{3} \geq \frac{|G_1|}{2}$, so that G_1 is hamiltonian by Dirac's Theorem [2], and in particular there is a hamiltonian path P in $G - G_2$ that has v as an endpoint.

We wish to apply Lemma 4 to G_2, so suppose first that G_2 is not 2-connected. If v' is a cut vertex in G_2, then each component in $G_2 - v'$ has minimum degree at least $\frac{n-8}{3}$ and hence has order at least $\frac{n-5}{3}$. Now, as $|G_2| \leq \frac{2n-1}{3}$ we therefore have that $G_2 - v'$ has precisely two components, call them F_1 and F_2. From here the remainder of this case continues in a manner similar to our consideration of block structure (1). Both F_1 and F_2 are hamiltonian-connected by Lemma 5, so there is a path P' that spans $G - G_1$ and has v as an endpoint. The union of P and P' is a hamiltonian path in G.

Consequently, we may assume that G_2 is 2-connected. Now, as $|G_2| \leq \frac{2n-1}{3}$ we have that

$$\frac{n-5}{3} \geq \frac{|G_2|-3}{2}.$$

Thus, since G_2 is 2-connected, $|G_2| \geq 12$ (as $n \geq 56$) and $\delta(G_2) \geq \frac{|G_2|-3}{2}$, we conclude that G_2 satisfies the hypotheses of Lemma 4. Thus, for each vertex v_b in $N_{G_2}(v)$ we have that there is a broom B_{v_b} such that v_b is the endpoint of the handle of B_{v_b}. The union of any such B_{v_b} with P and the edge $v_b v$ is a spanning broom in G.

Therefore we may suppose that G is 2-connected and furthermore we assume that $p(G) - c(G) \geq 2$. Let P be a longest path in G, where $(x =)v_1, \ldots, v_k(= y)$ are the vertices of P, in order. For the remainder of this proof we will assume that P has an implicit orientation from x to y, so that for a vertex v on P, v^+, v^-, v^{+i} and v^{-i} are as given in Section 1.

If P is spanning we are done, so let $H = G - P$ and $h = |H|$. We observe that $\sigma_4(G) \geq n+3$. Indeed, if w, x, y and z are pairwise nonadjacent vertices in $V(G)$, then

$$d(w) + d(x) + d(y) + d(z) \geq 4\left(\frac{n-2}{3}\right) \geq n+3$$

since $n \geq 56 \geq 17$. Consequently, by Lemma 2 each component of H has order at most two. The following claim will be useful as we proceed.

Claim 1. *Let z be a vertex in H. Then $N_P(y)^+_P, N_P(x)^-_P$ and $N_P(z)$ are pairwise disjoint and furthermore*

$$(n-2) - d_H(z) \leq d(x) + d(y) + d_P(z) \leq n - h. \tag{3}$$

Proof. Suppose first that there is some vertex $v \in V(P)$ such that v^+_P is adjacent to x and v^-_P is adjacent to y. Then $v^+_P x P v^-_P y P^- v^+_P$ is a cycle in G of length $|P| - 1$, contradicting the assumption that $p(G) - c(G) \geq 2$. Next, assume that there is some vertex w in $N_P(z)$ such that w^+_P is adjacent to x (note that w must be distinct from x by the maximality of P). Then $zwP^-xw^+_P Py$ is a path in G that is longer than P, a contradiction. The case where w^-_P is adjacent to y is handled identically.

Equation (3) follows from the minimum degree condition on G and the observations that $N_P(x) = N(x)$ and $N_P(y) = N(y)$ (as P is a longest path in G) and the fact that $|P| = n - h$. $\blacksquare_{\mathit{Claim\ 1}}$

This allows us to prove the following claim.

Claim 2. $|H| \leq 2$.

Proof. Assume otherwise, and suppose first that there is an isolated vertex z in H. Then $d_P(z) = d(z)$, and we have by Claim 1 that

$$n - 2 \leq d(x) + d(y) + d(z) \leq n - h \leq n - 3,$$

which is a contradiction. Thus, as each component of H has order at most two, it follows that H must be a matching of size at least two. Then for any vertex z in H, $d_H(z) = 1$, so that

$$n - 3 = 2\left(\frac{n-2}{3}\right) + \frac{n-5}{3} \leq d(x) + d(y) + d_P(z) \leq n - h \leq n - 4,$$

again a contradiction. The claim follows. $\blacksquare_{\textit{Claim 2}}$

The next claim will be crucial as we proceed.

Claim 3. *If $v \in N_P(x)$, then neither v_P^{-2} nor v_P^{-3} is in $N_P(y)$.*

Proof. Let v be in $N_P(x)$ and assume first that v_P^{-2} is in $N_P(y)$. Then $vPyv_P^{-2}Pxv$ is a cycle of length $|P|-1$ in G, contradicting the assumption that $p(G)-c(G) \geq 2$.

Next, we assume that v_P^{-3} is in $N_P(y)$ and let a and b denote v_P^- and v_P^{-2}, respectively. Then $C = vPyv_P^{-3}P^-xv$ is a cycle of length $|P|-2$ in G, so that C is necessarily a longest cycle in G. Let

$$S_{ab} = \{x \in N_C(a) \cap N_C(b) \mid x_C^{+3} \in N_C(a) \cap N_C(b)\}$$

and observe that $|S_{ab}| \leq \frac{|C|}{3}$. Note that a and b have no neighbors in H, as otherwise we could extend P, so ab is a component of $V(G)-C$. As $h \geq 1$ we thus have that $|C| \leq n-3$, so by Lemma 3, $|S_{ab}| \geq \frac{n-11}{3}$.

Now, observe that for any vertices $u \in S_{ab}$ and $p \in H$, neither pu_C^+ nor pu_C^{+2} is an edge in G as then $pu_C^+Cu_C^{+3}abuC^-u_C^{+4}$ or $pu_C^{+2}C^-uabu_C^{+3}Cu_C^-$, respectively, are paths in G that are longer than P, a contradiction. Thus

$$\begin{aligned}
d_C(p) &\leq (n-3) - |\{u^+, u^{+2} \ : \ u \in S_{ab}\}| \\
&= (n-3) - 2|S_{ab}| \\
&\leq (n-3) - 2\left(\frac{n-11}{3}\right) \\
&= \frac{n+13}{3}.
\end{aligned}$$

Now, we also have that

$$\begin{aligned}
d_C(p) - |S_{ab}| &\leq (n-3) - |\{u^+, u^{+2} \ : \ u \in S_{ab}\}| - |S_{ab}| \\
&\leq \frac{n+13}{3} - \frac{n-11}{3} \\
&= 8,
\end{aligned}$$

so p has at most 8 neighbors outside of S_{ab}. Since $h \leq 2$, $d_C(p) \geq \frac{n-5}{3}$, which implies that each vertex in H has at least $\frac{n-5}{3} - 8 = \frac{n-29}{3}$ neighbors in S_{ab}. If $H = \{s, t\}$, then $d_{S_{ab}}(s) + d_{S_{ab}}(t) \geq 2\left(\frac{n-29}{3}\right) > \frac{n-3}{3} \geq |S_{ab}|$ since $n \geq 56$. Consequently, there is some vertex q in S_{ab} that is adjacent to all of the vertices in H (this assertion is trivial if $h = 1$). As q is also adjacent to a and b, we conclude that G contains a spanning broom. $\blacksquare_{\textit{Claim 3}}$

We complete the proof of Theorem 2 by considering three cases.

Case 1: $H = K_1$. Let $H = \{z\}$, and note that by Claim 1 (specifically (3)), we have that there is at most one vertex on P that does not lie in $N_P(x)^- \cup N_P(y)^+ \cup N(z)$. Since $h = 1$ we have that for any vertex $v \in N_P(x)$ or $w \in N_P(y)$, neither v_P^{-2} nor w_P^{+2} is a neighbor of z as in either case there is a longest path P' in G such that z is adjacent to the second vertex on P', forming a spanning broom.

We define an *x-gap* to be a vertex $v \in N_P(x)$ such that v^- is not in $N_P(x)$. Similarly, we define a *y-gap* to be a vertex $w \in N_P(y)$ such that w^+ is not in $N_P(y)$. We claim that there is at most one x-gap and at most one y-gap in G. Indeed, consider an x-gap $v \in N_P(x)$. By assumption, v_P^{-2} cannot be in $N_P(x)^-$, by Claim 3, v_P^{-2} cannot be in $N_P(y)^+$ and by the above observation, v_P^{-2} cannot be in $N(z)$. Hence, as there is at most one vertex on P that does not lie in $N_P(x)^- \cup N_P(y)^+ \cup N_P(z)$, there can be at most one x-gap and, by a similar argument, at most one y-gap in G.

Suppose first that there exists no x-gap in G, implying that $N_P(x) = \{v_2, \ldots, v_i\}$ and by Claim 1 that z is not adjacent to any vertex in $\{v_2, \ldots, v_{i-1}\}$.

We claim that there is some integer j such that $0 \le j \le 2$ and v_{i+j} is in $N_P(y)$. Suppose not and note that since $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap (N_P(x)^- \cup N_P(y)^+)$ is empty and $|V(P) - (N_P(x)^- \cup N_P(y)^+ \cup N_P(z))| = 1$, at least three vertices in $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ must be adjacent to z. This implies that two consecutive vertices on P are adjacent to z, contradicting the maximality of P. Thus, as there is at most one y-gap, we have that $|V(v_{i+2}Pv_{n-2}) - N_P(y)|$ is at most one. By Claim 1, z has at most one neighbor in $V(v_{i+3}Pv_{n-1})$. Together with the assertion that z has no neighbor in $\{v_2, \ldots, v_{i-1}\}$, we conclude that $d(z) \le 4$, a contradiction.

By symmetry, all that remains to consider is the possibility that there is exactly one x-gap and exactly one y-gap. This occurs when there is some $v_i \in N_P(x)$ such that $v_{i-4} \in N_P(y)$, which implies as well that v_{i-2} is the unique vertex on P not in $N_P(y)_P^+ \cup N_P(x)_P^- \cup N_P(z)$. Since v_i is the unique x-gap and v_{i-4} is the unique y-gap, there exist indices g_x and g_y such that $\{v_2, \ldots, v_{g_x}\} \subseteq N_P(x)$ and $\{v_{g_y}, \ldots, v_{i-4}\} \subseteq N_P(y)$. Since z cannot be adjacent to consecutive vertices on P and v_{i-2} is the unique vertex not in $N_P(y)_P^+ \cup N_P(x)_P^- \cup N_P(z)$, Claim 1 implies that $g_x = g_y$ and further that $|N_P(z) \cap xPv_{i-4}| \le 1$. By a similar argument, we have that $|N_P(z) \cap v_iPy| \le 1$, so that $d(x) \le 4$, again a contradiction.

Case 2: $H = K_2$. As in Case 1, by (3) we have for either vertex $z \in H$ that there is at most one vertex on P that does not lie in $N_P(x)^- \cup N_P(y)^+ \cup N(z)$. An identical argument to that employed in Case 1 then suffices to reach a contradiction.

Case 3: $H = 2K_1$. In this case, (3) implies that for either vertex $z \in H$,

$$N_P(x)^- \cup N_P^+(y) \cup N(z) = V(P).$$

Using an argument similar to (although somewhat simpler than) that in Case 1, we can again contradict the assertion that $d(z) \geq \frac{n-2}{3}$. $\square$

4 Proof of Theorem 3

Our proof of Theorem 3 utilizes a variant of the Hopping Lemma, which was originally obtained by Woodall [11] as a tool for problems concerning paths and cycles in graphs. Here C is a longest cycle in G and $H_C = G - C$. Note that the assumption that $p(G) - c(G) \leq 1$ implies that each component of H_C is necessarily an isolated vertex.

Let $Y_C^0 = \emptyset$ and for $i \geq 1$, define

$$X_C^i = N_C(H_C \cup Y_C^{i-1}), \quad Y_C^i = (X_C^i)^+ \cap (X_C^i)^-.$$

Further, set $X_C = \cup_{i=1}^{\infty} X_C^i$ and $Y_C = \cup_{i=1}^{\infty} Y_C^i$.

The following is a strengthening of the Hopping Lemma due to van den Heuvel [7] that applies specifically to graphs G that satisfy $c(G) \geq p(G) - 1$. This is stronger than the condition imposed on G in [11], and as a result it enables a stronger conclusion.

Lemma 6 (van den Heuvel [7]). *Given the condition that $p(G) - c(G) \leq 1$, the sets X_C and Y_C satisfy*

(a) $X_C \cap X_C^+ = \emptyset$ *and* $X_C \cap Y_C = \emptyset$*;*

(b) $N(Y_C) \subseteq X_C$*;*

(c) Y_C *is an independent set.*

With Lemma 6 in hand, we are now ready to proceed with our proof of Theorem 3.

Proof. Assume that G is a 2-connected graph of order n such that $\delta(G) \geq \frac{n-2}{3}$ and let C be a longest cycle in G. The assumption that $p(G) - c(G) \leq 1$ implies that $H_C = G - V(C)$ consists of isolated vertices. If $h = |H_C| = 1$, then G necessarily contains a spanning jellyfish, so we may assume that $h \geq 2$. We also have that no pair of consecutive vertices on C both have a neighbor in H_C. If such a pair had a common neighbor in H_C, this would create a cycle of order $|C| + 1$ in G and if such a pair had distinct neighbors in H_C, this would create a path of order $|C| + 2$ in G, contradicting the assumption that $p(G) - c(G) \leq 1$.

If $H_C = \{z_1, z_2\}$ and $N(z_1) \cap N(z_2) \neq \emptyset$, then G contains a spanning jellyfish, so we have that for every vertex $v \in N_C(z_1)$, neither v nor v^+ is in $N_C(z_2)$. Therefore,

$$d(z_2) \leq |C| - 2d(z_1) \leq (n-2) - \frac{2(n-2)}{3} = \frac{n-2}{3}$$

so that every vertex on C is either in $N_C(z_1), N_C(z_1)^+$ or $N_C(z_2)$. Since neither z_1 nor z_2 is adjacent to consecutive vertices on C and we have assumed that $N_C(z_1) \cap N_C(z_2) = \emptyset$, we conclude that z_1 must be adjacent to the successor of some neighbor of z_2 on C, which is impossible. Hence $h \geq 3$.

For $u \in H_C$, set $R_u := N_C(u)^+ \cap N_C(u)^-$, and let $r_u = |R_u|$. If $v \in R_u$, then the cycle C' obtained by replacing the path v^-vv^+ on C with the path v^-uv^+ is also a longest cycle in G. We will refer to C' as *the cycle obtained from C by exchanging v and u.*

Claim 1. *Let $u \in H_C$ and $v \in R_u$. Under the assumption that $p(G) - c(G) \leq 1$, if C' is the cycle obtained from C by exchanging v and u, then $X_{C'} = X_C$, $Y_{C'} = (Y_C - \{v\}) \cup \{u\}$ and $H_{C'} \cup Y_{C'} = H_C \cup Y_C$.*

Proof: As we have assumed that $p(G) - c(G) \leq 1$, by Lemma 6, we have that $X_C \cap Y_C = \emptyset$. This, together with $v \in R_u \subseteq Y_C^1 \subseteq Y_C$, implies that $X_C \subseteq V(C) - \{v\}$, and hence for $i \geq 1$

$$X_C^i = N(H_C \cup Y_C^{i-1}) \cap (V(C) - \{v\}). \tag{4}$$

Also for $i \geq 1$ we have that

$$X_{C'}^i = N(H_{C'} \cup Y_{C'}^{i-1}) \cap (V(C') - \{u\}) = N(H_{C'} \cup Y_{C'}^{i-1}) \cap (V(C) - \{v\}). \tag{5}$$

Recalling that $v \in R_u \subseteq Y_C^1$, we have $H_{C'} = (H_C - \{u\}) \cup \{v\} \subseteq H_C \cup Y_C^1$, and hence

$$X_{C'}^1 = N(H_{C'}) \cap (V(C) - \{v\}) \subseteq N(H_C \cup Y_C^1) \cap (V(C) - \{v\}) = X_C^2. \tag{6}$$

With (6) serving as our base case, it follows by induction from (4) and (5) that $X_{C'}^i \subseteq X_C^{i+1}$ for all $i \geq 1$, which then implies that $Y_{C'}^i - \{u\} \subseteq Y_C^{i+1} - \{v\}$. Therefore, $X_{C'} \subseteq X_C$. Similarly, we have $X_C \subseteq X_{C'}$, and hence $X_{C'} = X_C$. As $Y_{C'}^i - \{u\} \subseteq Y_C^{i+1} - \{v\}$ (and identically $Y_C^i - \{v\} \subseteq Y_{C'}^{i+1} - \{u\}$), it is straightforward to verify that $Y_{C'} = (Y_C - \{v\}) \cup \{u\}$, and hence $H_{C'} \cup Y_{C'} = H_C \cup Y_C$. $\blacksquare_{\textit{Claim 1}}$

The following claim follows immediately from condition (a) of Lemma 6.

Claim 2. *$X_C = N_C(H_C \cup Y_C)$ does not contain consecutive vertices on C. Consequently, $|X_C| \leq \frac{|C|}{2} = \frac{n-h}{2}$.*

Claim 3. *For each $x \in H_C$,*

$$|R_x| \geq \frac{2}{3}(|H_C| - 2) + \frac{|Y_C|}{3}.$$

Proof. Counting vertices in $N_C(x)$ and their successors on C yields that

$$|C| \geq 2r_x + 3(d(x) - r_x) + \frac{1}{2}|Y_C - R_x|. \tag{7}$$

The $2r_x$ and $3(d(x) - r_x)$ terms arise from consideration of the (necessarily disjoint) sets

$$A = \{v, v^+ \mid v^+ \in R_x\} \text{ and } B = \{v, v^+, v^{+2} \mid v^+ \in N_C(x)^+ - R_x\}.$$

In order to explain the $\frac{1}{2}|Y_C - R_x|$ term, we want to show that at least half of the vertices in $Y_C - R_x$ imply the existence of a vertex v on C such that v, v^+ and v^{+2} do not lie in $N_C(x)$. In this case, the vertex v^{+2} would not have yet been counted in either A or B. Choose a vertex $v \in Y_C - R_x$ and note that since $X_C = N(H_C \cup Y_C)$ does not contain consecutive vertices on C, x is not adjacent to v, v^{+2} or v^{-2}. As $v \notin R_x$, either v^- or v^+ is not adjacent to x, forcing either $\{v, v^-, v^{-2}\}$ or $\{v, v^+, v^{+2}\}$ to lie outside of $N_C(x)$. The coefficient of $\frac{1}{2}$ on $|Y_C - R_x|$ arises from the fact that if both v and v^{+2} were in $Y_C - R_x$, then x being nonadjacent to v^+ could generate v, v^+, v^{+2} as a triple that accounts for both v and v^{+2}.

Now, (7) implies that

$$n - h \geq 2r_x + 3(d(x) - r_x) + \frac{1}{2}(|Y_C| - r_x)$$

so that

$$n - h \geq 2r_x + 3(\frac{n-2}{3} - r_x) + \frac{1}{2}(|Y_C| - r_x).$$

Solving for r_x, the desired inequality follows. $\blacksquare_{Claim\ 3}$

We claim that there is a vertex x in X_C such that

$$d_{H_C \cup Y_C}(x) \geq \frac{2}{3}(|H_C| + |Y_C|).$$

Assume otherwise, so that for all $x \in X_C$, we have that

$$d_{H_C \cup Y_C}(x) < \frac{2}{3}(h + |Y_C|).$$

Thus, as $N(H_C \cup Y_C) = X_C$ we have that

$$\left(\frac{n-2}{3}\right)(h + |Y_C|) \leq |E(X_C, H_C \cup Y_C)| < \frac{2}{3}(h + |Y_C|)|X_C| \leq \frac{2}{3}(h + |Y_C|)\left(\frac{n-h}{2}\right).$$

Therefore, $n - 2 < n - h$, which contradicts the assertion that $h \geq 3$.

Consequently, we may choose a cycle C and $x \in X_C$ such that

(i) $d_{H_C \cup Y_C}(x) \geq \frac{2}{3}(|H_C| + |Y_C|)$, and

(ii) subject to (i), $H_C^* := N_{H_C}(x)$ is maximal.
If $H_C^* = H_C$, then G contains a spanning jellyfish and we are done, so choose $y \in H_C - H_C^*$. We have that $R_y \cap N(x)$ is empty, as if not we can exchange y with a vertex z in $R_y \cap N(x)$ to obtain a longest cycle C' with

$$d_{H_{C'} \cup Y_{C'}}(x) = d_{H_C \cup Y_C}(x) \geq \frac{2}{3}(|H_C| + |Y_C|) = \frac{2}{3}(|H_{C'}| + |Y_{C'}|)$$

and $H_{C'}^* = H_C^* \cup \{z\}$.

Thus, as $|H_C| \geq 3$ and $|H_C| - |H_C^*| \geq 1$, we have that

$$\begin{aligned} |Y_C| &\geq |R_y| + |N_{Y_C}(x)| \\ &\geq \left(\frac{2}{3}(|H_C| - 2) + \frac{|Y_C|}{3}\right) + \left(\frac{2}{3}(|H_C| + |Y_C|) - |H_C^*|\right) \\ &= (|H_C| - |H_C^*|) + \frac{|H_C|}{3} - \frac{4}{3} + |Y_C| \\ &> |Y_C|, \end{aligned}$$

a contradiction.

□

5 Conclusion

Problem 1 is quite general, and we have provided one of many possible affirmative answers. We pose the following strengthening of Conjecture 1 in the $k = 1$ case.

Conjecture 3. *If G is a connected graph of order $n \geq 3$ such that $\sigma_3(G) \geq n - 2$, then G contains a spanning broom.*

If true, this conjecture would be best possible via the graph given in Figure 1.

Acknowledgement: The authors would like to thank two anonymous referees, whose detailed reports undoubtedly improved the final version of this paper.

References

[1] J. Akiyama and M. Kano, Factors and Factorizations of Graphs: Proof Techniques in Factor Theory, *Lecture Notes in Mathematics, 2031*, Springer, Heidelberg, 2011.

[2] G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69-81.

[3] E. Flandrin, T. Kaiser, R. Kužel, H. Li and Z. Ryjáček, Neighborhood unions and extremal spanning trees, *Discrete Math.* **308** (2008), 2343-2350.

[4] L. Gargano and M. Hammar, There are spanning spiders in dense graphs (and we know how to find them), *Lect. Notes Comput. Sci.* **2719** (2003), 802-816.

[5] L. Gargano, M. Hammar, P. Hell, L. Stacho and U. Vaccaro, Spanning spiders and light-splitting switches, *Discrete Math.* **285** (2004), 83-95.

[6] L. Gargano, P. Hell, L. Stacho and U. Vaccaro, Spanning trees with bounded number of branch vertices. *Lect. Notes Comput. Sci.* **2380** (2002), 355-365.

[7] J. van den Heuvel, Long cycles in graphs with large degree sums and neighborhood unions, *J. Graph Theory* **21** (1996), 87-102.

[8] O. Ore, A note on Hamilton circuits, *Amer. Math. Monthly* **67** (1960), 55.

[9] O. Ore, Hamiltonian-connected graphs, *J. Math Pures Appl.*, **42** (1963), 21 -27.

[10] K. Ozeki and T. Yamashita, Spanning trees: A survey, *Graphs. Comb.* **27** (2011), 1-26.

[11] D.R. Woodall, The binding number of a graph and its Anderson number. *J. Combin. Theory Ser. B* **15** (1973), 225-255.

[12] Y. Zou, A generalization of a theorem of Jung, *J. Nanjing Normal Univ. Nat. Sci. Ed.* **2** (1987), 8-11.