On the Largest Tree of Given Maximum Degree in a Connected Graph

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ABSTRACT

We prove that every connected graph G contains a tree T of maximum degree at most k that either spans G or has order at least $k\delta(G) + 1$, where $\delta(G)$ is the minimum degree of G. This generalizes and unifies earlier results of Bermond [1] and Win [7]. We also show that the square of a connected graph contains a spanning tree of maximum degree at most three.

1. INTRODUCTION

Graphs in this paper are finite, connected, and have no loops, nor multiple edges. By $\deg_G v$ we denote the degree of a vertex v in G. The maximum and the minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. As usual, $E(G)$ and $V(G)$ denote the edge and the vertex sets of G, respectively. We also put $|V(G)| = n$. The set of all vertices adjacent to a vertex x is denoted by $N(x)$. The induced graph on a subset S of $V(G)$ will be denoted by $\langle S\rangle$. We set

$$h = h(G) := \min\{\Delta(T)\colon T \text{ is a spanning tree of } G\}.$$

A natural generalization of a Hamilton path in a graph is a spanning tree with a maximum degree at most h. Thus in a Hamiltonian graph $h = 2$. Of course, one is interested in establishing conditions on, say, the degrees of a graph that ensure a small value of h. The parameter h plays an important role in many problems, for example, in the Edge Reconstruction Conjecture, since graphs with at least $cn\log h$ edges are edge-reconstructible [5], [2].

Journal of Graph Theory, Vol. 15, 7–13 (1991)

CCC 0364-9024/91/010007-07$04.00

It seems that not much is known about the value of h in general. However, Win [7] demonstrated that the well-known conditions of Ore and Posa on Hamiltonicity can be extended to spanning trees (see Corollaries 1 and 2 below). For some special classes of graphs much more can be established. For instance, if G is a graph with no induced subgraph $K_{1,m}$, then $h \leq m$ [2]. Here we show that $h \leq 3$ for the square of a connected graph (Theorem 3).

It turns out that to obtain upper bounds on h it is convenient to study the more general question of the maximum order of a tree with a given maximum degree k. Notice that for $k = 2$ this is just the problem of finding the length of a longest path in a graph. The purpose of this paper is to show how such an approach permits one to extend the known sufficient conditions from this special case to the general case. In particular, we show that every connected graph contains a tree T of order at least $k\delta(G) + 1$ and $\Delta(T) \leq k$, $k \geq 2$. This generalizes earlier results of Bermond [1] and Win [7].

2. RESULTS

Let k be a positive integer. A set S of independent vertices of G with the property that $G \backslash S'$ is connected for $S' \subseteq S$ will be called a *frame*. Thus, any subset of a frame is also a frame.

If $|S| = k$ we call S a *k-frame*. Let S be a k-frame, put $d(S) := \sum_{v \in S} \deg_G v$

$$\text{and, } D_k = \min\{d(S)\text{: is a } k\text{-frame}\}.$$

Note that if a graph has a k-frame, then D_k is at least the sum of the k smallest degrees.

We start with the following generalization of a result of Win [7]:

Theorem 1. Let G be a connected graph, and let $\Delta(G) \geq k \geq 3$. Then G contains a tree T such that $|T| \geq \min(D_k + 1, n)$ and $\Delta(T) \leq k$.

Proof. Among all trees of G with maximum degree k, let T be a one of maximum order then with the minimum number of endvertices. We may assume that T is not a spanning tree of G, that is, $|T| < n$. If $k = \Delta(G)$, then the claim of the theorem is trivial, so we assume that $\Delta(G) > k \geq 3$.

Denote by $X_i = X_i(T)$ the number of vertices of degree i in T.

Let v be a vertex in T that is adjacent to a vertex not in T; then $\deg_T v = k$, by the maximality of T. Moreover,

(i) $\deg_T v = k$.
(ii) The set of the endvertices of T is a X_1-frame.

To demonstrate (ii), observe that since any subset of endvertices of T is not a cut in G, it is enough to show that they are independent. Assume the contrary. Let x and y be endvertices of T that are adjacent in G, and let z be their ancestor in T. Clearly, $\deg_T z \geq 3$, and we can choose $T' = (T \cup (x, y)) \backslash (t, z)$, where t is the vertex adjacent to z in the unique path connecting x to z (possibly $z = t$). One can see that $X_1(T') \leq X_1(T)$, which contradicts the minimality of $X_1(T)$.

Let $B_1, \ldots, B_k$ be branches of T at v, that is, the subtrees of $T \backslash v$. Consider T as a tree rooted at v. Let v_i be the vertex of B_i adjacent to v in T and let $e_i = (v, v_i)$.

(iii) If $(u_i, u_j) \in E(G)$, $u_i \in B_i$, $u_j \in B_j$, $i \neq j$, then $\max\{\deg_T u_i, \deg_T u_j\} = k$.

To establish (iii), suppose that $\max(\deg_T u_i, \deg_T u_j) < k$. Consider the tree $T' = ((T \cup (u_i, u_j)) \backslash e_i) \cup w$. Then $\Delta(T') \leq k$ and $|T'| = |T| + 1$, a contradiction.

Now in each branch B_i, we fix an endvertex u_i. (Note that the set $\bigcup_{i=1}^k u_i$ forms a k-frame by (ii).)

We define a partition of $N(u_i) \backslash v$ in G as follows:

$$N(u_i) \backslash v = A_i \cup F_i \cup C_i, \text{ where } A_i = \{x \in B_i | \deg_T(x) < k\}.$$
$$F_i = \{x \in B_i | \deg_T(x) = k\}.$$
$$C_i = \{x \notin B_i | \deg_T(x) = k\}.$$

Observe that if $x \in A_i$ then $\deg_T x \geq 2$ by (ii).

Let $a = \sum_{i=1}^k |A_i|, f = \sum_{i=1}^k |F_i|, c = \sum_{i=1}^k |C_i|$.

Denote by $X_k(i)$ the number of vertices of degree k in B_i.

We will establish four inequalities involving $a, f, c,$ which together will imply our theorem.

$$1 + f \leq X_k. \tag{1}$$

This follows from the definitions of f and X_k, and the fact that v is not in any B_i.

$$c \leq (X_k - 1)(k - 1). \tag{2}$$

Indeed, $|C_i| \leq X_k - X_k(i) - 1$; hence

$$c = \sum_{i=1}^{k} |C_i| \leq kX_k - \sum_{i=1}^{k} X_k(i) - k = (k - 1)(X_k - 1),$$

since $\sum_{i=1}^k X_k(i) = X_k - 1$.

$$D_k - k \leq a + f + c. \tag{3}$$

To prove (3), we observe that

$$|A_i| + |F_i| + |C_i| \geq \deg_G u_i - 1$$

by (i). Hence,

$$a + f + c \geq \Sigma_{i=1}^{k} \deg_G u_i - k \geq D_k - k.$$

$$a + \frac{c}{k-1} \leq \sum_{i=2}^{k-1} X_i. \tag{4}$$

To establish (4) consider for each i and each $x \in C_i$ the graph $T \cup (u_i, x)$, and define

$$Y_i = \{y \neq u_i\colon (x,y) \text{ is the edge in the unique cycle of } T \cup (u_i, x), \text{ for some } x \in C_i\}$$

$$\text{Put } Y := \cup Y_i,\ Z_i := A_i \cup Y_i,\ Z := \cup Z_i.$$

Observe that $\deg_T y = 2$ since otherwise for $T' = (T \cup (u_i, x))\backslash(x, y)$, $X_1(T') \leq X_1(T)$. Thus, $|Y_i| = |C_i|$ and $|Z| \leq \Sigma_{i=2}^{k-1} X_i$. Moreover, if $y \in B_j$ then $y \notin A_j$, which means $(y, u_j) \notin E(G)$, and so, $Y \cap (\cup A_i) = \emptyset$. Indeed, $\{T \cup (y, u_j) \cup (u_i, x)\}\backslash(x, y)$ has a unique cycle. By deleting from this cycle any edge having one of its endvertices of degree at least three in T (such an edge does exist; for example, the edge incident to x on the path from x to u_i), we obtain a tree with fewer endvertices.

Since each $x \in C_i$, $x \in B_j$ can belong also to any C_l except C_j, then $|Y| = |\cup C_i| \geq c/(k-1)$. Now we obtain (4) by

$$\sum_{i=2}^{k-1} X_i \geq |Z| = |Y \cup (\cup A_i)| \geq a + |Y| \geq a + \frac{c}{k-1}.$$

Adding inequalities (1)–(4) we get

$$D_k - 1 \leq X_k(k-1) + \sum_{i=2}^{k-1} X_i.$$

Here the right-hand side does not exceed

$$\sum_{i=2}^{k} (i-1)X_i = 2e(T) - |T| = |T| - 2,$$

and so $|T| \geq D_k + 1$. ∎

The parameter D_k arises naturally in this context, but seems hard to compute in general (like many parameters in graph theory). However, for fixed k the complexity of computing D_k, as well as the verification of its existence, is $O(n^{k+1})$. Moreover, in applications D_k can be estimated in terms of other parameters of a graph (e.g., the degrees). In particular we have the following result:

Theorem 2. Let G be a connected graph with degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. Then for each $k \geq 2$, G contains a tree T such that $\Delta(T) \leq k$ and

$$|T| \geq \min\left(1 + \sum_{i=1}^{k} d_i, n\right) \geq \min(\delta k + 1, n).$$

Proof. For $k = 2$, the result is well known and easy to prove. For $k \geq 3$, let T be a maximal tree such that $\Delta(T) = k$. Suppose that $|T| < n$. The endvertices of T form an $X_1(T)$-frame in G and $X_1(T) \geq k$. Hence, $D_k + 1 \geq 1 + \sum_{i=1}^{k} d_i \geq \delta k + 1$, which implies the theorem. ∎

Corollary 1 (Win [7]). For any connected graph G, $h(G) \leq \lceil (n - 1)/\delta \rceil$.

Proof. Choose a maximal tree with $\Delta(T) = \lceil (n - 1)/\delta \rceil$. ∎

Corollary 1 is sharp for a wide class of graphs. For $n \equiv 1(\mathrm{mod}\delta)$, the simplest extremal example is given by the graph obtained by taking k copies of the complete graph $K_{\delta+1}$, with one common vertex to all of them.

We now describe a less trivial example:

Let $H(\delta, n - \delta) = K_\delta + \bar{K}_{n-\delta}$, where + denotes "join." for $2\delta \leq n$ let $G_{\delta,n-\delta}$ be the family of all connected spanning subgraphs of $H(\delta, n - \delta)$ with minimum degree δ. Now we have

Proposition. For any $G \in G_{\delta,n-\delta}$, $h(G) = \lceil (n - 1)/\delta \rceil$.

Proof. Let T be a spanning tree of $G \in G_{\delta,n-\delta}$ with $\Delta(T) = h(G)$. Let C be the set of the endvertices of T that occur in M. Put $|C| = t$. Denote by y_j the number of edges of vertex $v_j \in N$ in T that are incident to the vertices of $M \backslash C$, and by z_j the number of edges left in N. We have

$$\sum_{i=1}^{\delta} (h - y_i - z_i) \geq t.$$

On the other hand, $T \backslash C$ is a tree with $n - t$ vertices. Hence,

$$\sum y_i + \frac{1}{2} \sum z_i + n - t - 1.$$

This yields

$$\delta h \geq t + \sum y_i + \sum z_i \geq t + \sum y_i + \frac{1}{2} \sum z_i = n - 1 .$$

Comparing this with Corollary 1, we get the required result. ∎

Remark. Recently Neumann-Lara and Rivera-Campo [6] established the following result:

If G is a k-connected graph with independence number β_0, then $h(G) \leq \lceil(\beta_0 - 1)/k\rceil + 1$. ∎

Our next and last theorem supplies a "missing link" between the well-known results of Fleischner [3] and Karaganis [4].

Theorem 3. If G is a connected graph, then $h(G^2) \leq 3$.

Proof. The proof gives a linear time algorithm for constructing the required tree. Take a spanning tree T_1 of G, and root it at a cut-vertex v. For each vertex u, let $S(u)$ be the set of its sons in T_1. Now we describe an inductive construction of the required spanning tree T of G^2.

Let T_2 be a spanning path in $v \cup S(v)$ consisting of edges of G^2 ending in v. This is possible since $S(u)$ is always a complete subgraph of G^2. Assume that T_i has been constructed. Now in order to construct T_{i+1} consider a vertex $u \notin T_i$ adjacent to a vertex $w \in T_i$. Then T_{i+1} is obtained from T_i by adding a spanning path in $\langle w \cup S(w)\rangle$ with the endvertex w. Clearly, this process terminates in a spanning tree T. A quick consideration of the construction reveals that the degree of any vertex in T is at most 3, since any vertex is the end of at most one path and is a vertex of degree 2 in at most one path. ∎

ACKNOWLEDGMENT

We would like to thank the referee for his helpful ideas and constructive remarks.

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