On Low Bound of Degree Sequences of Spanning Trees in K-edge-connected Graphs

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Abstract: A graph $G = (V, E)$ is k-edge-connected if for any subset $E' \subseteq E, |E'| < k, G - E'$ is connected. A d_k-tree T of a connected graph $G = (V, E)$ is a spanning tree satisfying that $\forall v \in V, d_T(v) \leq [\frac{d(v)+k-1}{k}] + \alpha$, where $[\cdot]$ is a lower integer form and α depends on k. We show that every k-edge-connected graph with $k \geq 2$, has a d_k-tree, and $\alpha = 1$ for $k = 2, \alpha = 2$ for $k \geq 3$. © 1998 John Wiley & Sons, Inc. J Graph Theory 28: 87–95, 1998

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1. INTRODUCTION

All graphs considered here are finite undirected and without loops or multiple edges. The terminology and notation used here are standard except as indicated. A good reference for any undefined terms is Bondy and Murty's book [1], and the sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively, and the degree of a vertex v of G is denoted by $d_G(v)$ or $d(v)$ for simplicity. For $S \subseteq V(G), G[S]$ denotes the subgraph of G induced by S. An orientation of a

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graph G means that each edge of G is given a direction. After an orientation of G, G becomes a directed graph, denoted by $\vec{G}, G$ is called the underlying graph of $\vec{G}$. If there is a directed (u, v)-path from u to v in $\vec{G}$, then vertex v is said to be reachable from u. A directed graph is diconnected if every pair of vertices can be reachable each other. The indegree (respectively outdegree) of a vertex v in $\vec{G}$ is denoted by $d^-_{\vec{G}}(v)$ (respectively $d^+_{\vec{G}}(v)$). An out-directed tree is a directed graph with its underlying graph being a tree in which the indegree of each vertex being 1 except one, called root. A spanning out-directed tree $\vec{T}$ of a directed graph $\vec{G}$ is an out-directed tree with $V(\vec{T}) = V(\vec{G})$. A d_k-tree (respectively d_k-subtree) T of a connected graph $G = (V, E)$ is a spanning tree (respectively subtree) satisfying that $\forall v \in V, d_T(v) \leq [\frac{d(v)+k-1}{k}] + \alpha$, where $[\cdot]$ is a lower integer form and α depends on k. A vertex v of a d_k-subtree T of G with degree $[\frac{d(v)+k-1}{k}] + \alpha$ in T is called d_k-full.

It is well-known that any connected graph has a spanning tree. Using different parameters researchers have obtained some sufficient conditions for a graph to contain spanning trees with bounded degrees. See, for instance, [2, 3 and 4].

Let $G = (V(G), E(G))$ be a connected graph, $V(G) = \{v_1, v_2, \dots, v_n\}$ and a positive integer sequence $D = (d_1, d_2, \dots, d_n)$. The problem of whether there is a spanning tree T of G satisfying that for all $i, d_T(v_i) \leq d_i$, is very hard, as Hamilton path problem is a special case. In this article we study the connection between the edge-connectivity and the maximum degree of spanning trees of a graph, and obtain a positive integer sequence D, where the i^{th} component d_i is $[\frac{d(v_i)+k-1}{k}]+\alpha$, where $\alpha = 1$ or 2 depends on k. Based on the results obtained, one can easily deduce that: every k-edge-connected k-regular graph has a connected [1, 3] factor. We also note that when $k \geq 3$ the value of α cannot be reduced, if the value of α were less than 2, every k-edge-connected k regular graph would have a Hamilton path, but it is impossible. A counterexample is presented by B. Grunbaum and T. S. Motzkin [5] by constructing a cubic, planar 3-connected graph with 944 vertices in which no elementary chain could contain more than 939 vertices.

2. RESULTS

The main result of this article is the following:

Theorem 1. Let $G = (V(G), E(G))$ be a k-edge-connected graph, then G has a d_k-tree.

In order to prove the Theorem 1, we first give a lemma, which is a generalization of a result of Win [4].

Suppose that there exists some k-edge-connected graph G, which has no d_k-tree. Let H be the subgraph of G induced by the set of vertices of a maximal d_k-subtree of G.

Lemma 1. There is a d_k-tree T of H, and $B, R \subseteq V(H)$, with $B \neq \emptyset$, such that

(i) $B \cup R$ is a subset of d_k-full vertices of T.
(ii) Let $T_i, 1 \leq i \leq m$, be the components of $T \setminus B$, if for some i and j with $i \neq j$ a vertex v_i of T_i is adjacent in H to the vertex v_j of T_j, then v_i or v_j is in R.
(iii) Let $B(T_i)$ be the set of all vertices of B which are adjacent to some vertices of T_i in $T, 1 \leq i \leq m$. If all the vertices of $B(T_i)$ are the endvertices of a d_k-tree of $G[V(T_i) \cup B(T_i)]$. Then in that d_k-tree all vertices of $V(T_i) \cap R$ are d_k-full vertices.
(iv) If v is a vertex of T adjacent in G to a vertex not in $V(H)$, then v is in $B \cup R$.
(v) There is no edge in H between components of $T \setminus (B \cup R)$.

Proof. We first show the existence of T, B and R satisfying (i)–(iv).

Let $\bar{H} = G - V(H)$ and $N(\bar{H})$ be the set of all vertices of $V(H)$ adjacent to some vertex in $V(\bar{H})$. Then $N(\bar{H}) \neq \emptyset$ and for any $t \in N(\bar{H})$, obviously t is d_k-full vertex in any d_k-tree of H, because of the choice of H. Let $T_i, 1 \leq i \leq m$, be the components of $T \setminus \{t\}$, and let t_i be the vertex of T_i adjacent in T to t.

Suppose there is an edge $x_p x_q$ in H between T_p and T_q, then for any d_k-tree T_i^* of $G[V(T_i) \cup \{t\}]$, with t being an end vertex $i = p, q$, at least one of x_p and x_q must be d_k-full. Otherwise we can obtain a d_k-tree T^* by setting

$$T^* = T - (T_p + T_q) + (T_p^* + T_q^*) + x_p x_q - tt_p (\text{or } tt_q).$$

Obviously t is not a d_k-full vertex of T^*, which contradicts our observation about the vertices in $N(\bar{H})$. We conclude that under our supposition, at least one of x_p or x_q, say x_p, has the following property:

(α) Every d_k-tree of $G[V(T_p) \cup \{t\}]$ with t as end vertex, then x_p is d_k-full.

Now let $B = \{t\}$, and R be the union of the set of all vertices x_p of $T \setminus \{t\}$ having property (α) and the set $N(\bar{H}) \setminus \{t\}$. Such vertex x_p is called $d_k - t$-full. Then referring to the statement of Lemma 1, (i) holds, (ii) and (iv) hold because of the choice of R. (iii) follows from (α) by noting that $B(T_i) = \{t\}, 1 \leq i \leq m$. Hence we have established the existence of T, B and R.

Now we show (v) holds by choosing T, B and R such that $|B \cup R|$ is maximal.

By contradiction: suppose that there is an edge e joining two components of $T \setminus (B \cup R)$. Then the path in T joining two end vertices of e contains no vertex of B by (ii) and the choice of R. Thus the two end vertices of e are in the same component T_i of $T \setminus B$ for some $i, 1 \leq i \leq m$, and the path must contain a vertex in R. Now let w be a vertex of R on the path and let $T_{ij}, 1 \leq j \leq l$, be the components of $T_i \setminus \{w\}$, let $B(T_{ij})$ be the set of all vertices of B which are adjacent in T to some vertices of T_{ij} and w_j be the vertex of T_{ij} adjacent in T to w.

Assume that $e = (y_p, y_q)$ connects T_{ip} with T_{iq} and $y_p \in T_{ip}, y_q \in T_{iq}$. If, for $j = p, q$, there exists a d_k-tree T_{ij}^* of $G[V(T_{ij}) \cup B(T_{ij}) \cup \{w\}]$ with the vertices in $B(T_{ij} \cup \{w\}$ being end vertices and y_j is not d_k-full vertex of T_{ij}^*. Then $T_i^* = T[V(T_i) \cup B(T_i) \setminus (V(T_{ip}) \cup V(T_{iq}))] \cup T_{ip}^* \cup T_{iq}^*$ is a d_k-tree of $G[V(T_i) \cup B(T_i)]$, in which all the vertices of $B(T_i)$ are end vertices but both y_p

and y_q are not d_k-full vertices, and $T_i^* + (y_p, y_q) - (w, w_q)(\text{or}(w, w_q))$ is a d_k-tree of $G[V(T_i) \cup B(T_i)]$ in which all vertices of $B(T_i)$ are end vertices, but w is not a d_k-full vertex which contradicts the choice of R. This contradiction shows that at least one of y_p and y_q, say y_p has property:

(β) If w and all the vertices of $B(T_{ip})$ are end vertices in a d_k-tree of $G[V(T_{ip}) \cup B(T_{ip}) \cup \{w\}]$, then in that d_k-tree y_p must be d_k-full.

Now let R' be the set of all vertices of $T_i \setminus \{w\}$ having property (β). Further, let $B^* = B \cup \{w\}$ and $R^* = (R \cup R') \setminus \{w\}$. Then (i), (ii), (iii) and (iv) hold with B^* and R^* replacing B and R, respectively. Moreover $|B^* \cup R^*|$ is greater than $|B \cup R|$, which contradicts our choice of T, B and R. The contradiction shows that there is no edge in H joining components of $T \setminus (B \cup R)$, which completes the proof of Lemma 1. ∎

Lemma 2. Let T be a d_k-tree of H, S be a subset of the set of d_k-full vertices of T. Then the number of components of $T \setminus S$, satisfying that each of them is adjacent to only one vertex of S in T, at least is $\sum_{v \in S}([\frac{d(v)+k-1}{k}] + \alpha - 2) + 2$.

Proof. (By induction on the number of vertices of S). First, we can see that:

$(*)$ For every vertex v of $S, d_T(v) \geq 2$.

If $|S| = 1$, the result is true according to $(^*)$, because the only vertex s of S is not a leaf, and each components of $T \setminus S$ is adjacent to only one vertex of S in T.

Suppose the result holds for $|S| < k$. For $|S| = k$, select a vertex v_0 of S satisfying that one of components of $T \setminus \{v_0\}$ contains all vertices of $S \setminus \{v_0\}$, say ω_0, then ω_0 is a d_k-tree and $|S \setminus \{v_0\}| < k$. By the induction, the number of components of $\omega_0 \setminus (S \setminus \{v_0\})$ satisfying that each is adjacent to only one vertex of $S \setminus \{v_0\}$ in ω_0 at least is $\sum_{v \in (S \setminus \{v_0\})}([\frac{d(v)+k-1}{k}] + \alpha - 2) + 2 - \beta$, where $\beta = 1$ if v_0 is adjacent to some vertex of $S, \beta = 0$, otherwise. Clearly every component of $\omega_0 \setminus (S \setminus \{v_0\})$ is also a component of $T \setminus S$ possibly except one that contains a vertex adjacent to v_0 in T, Recall the property of v_0, one can easily know that all components of $T \setminus \{v_0\}$ except ω_0 are adjacent to no one vertex of $S \setminus \{v_0\}$, then the number of components of $T \setminus S$ each of them is adjacent to only one vertex of S in T at least is:

$$\sum_{v \in (S \setminus \{v_0\})} \left(\left[\frac{d(v) + k - 1}{k} \right] + \alpha - 2 \right) + 2 - \beta + d_T(v_0) - 1$$

$$\geq \sum_{v \in S} \left(\left[\frac{d(v) + k - 1}{k} \right] + \alpha - 2 \right) + 2.$$

where $d_T(v_0) = \frac{d(v_0)+k-1}{k} + \alpha$, and $\beta \leq 1$, which completes the proof. ∎

Proof of Theorem 1. Let H, T, B and R satisfy Lemma 1 and $\omega_1, \omega_2, \ldots, \omega_l$ be all components of $T \setminus (B \cup R)$. Then by the assumption of the Theorem 1 and (v) of Lemma 1, for $1 \le i \le l$, there are at least k edges with one end in $V(\omega_i)$ and one end in $B \cup R$. Since H is a proper subgraph of G, condition (iv) of Lemma 1 implies that there are at least k edges with one end in $B \cup R$ and the other not in $V(H)$. From Lemma 2 the number of components of $T \setminus (B \cup R)$ satisfying that each of them is adjacent in T to exactly one vertex in $B \cup R$ is at least $\sum_{v \in B \cup R}([\frac{d(v)+k-1}{k}] + \alpha - 2) + 2$ components. Thus, the number of edges in $E(G) \setminus E(T)$, with one end in $V(G) \setminus (B \cup R)$ and the other in $B \cup R$ at least is

$$\left(\sum_{v \in B \cup R}\left(\left[\frac{d(v)+k-1}{k}\right] + \alpha - 2\right) + 2\right)(k-1) + k$$
$$= \sum_{v \in B \cup R}\left(\left[\frac{d(v)+k-1}{k}\right] + \alpha - 2\right)(k-1) + 3k - 2 \quad (1)$$

where first item refers to the number of edges of $E(G) \setminus E(T)$ with one end in $B \cup R$ and the other in the components of $T \setminus (B \cup R)$ satisfying that each of them is adjacent in T to exactly one vertex in $B \cup R$. The second item is the minimum number of edges between $V(H)$ and $V(G) \setminus V(H)$.

But on the other hand the number of edges in $E(G) \setminus E(T)$, with one end in $B \cup R$ at most is:

$$\sum_{v \in B \cup R}\left(d(v) - \left[\frac{d(v)+k-1}{k}\right] - \alpha\right) \quad (2)$$

(1)–(2):

$$\sum_{v \in B \cup R}\left(k\left[\frac{d(v)+k-1}{k}\right] - d(v) + k(\alpha - 2) + 2\right) + 3k - 2$$

Because $k[\frac{d(v)+k-1}{k}] - d(v)$ is greater than or equal to 0, one can easily deduce that:

$$(1) - (2)\begin{cases} > 0 & k = 2,\ \ \alpha = 1 \\ > 0 & k \ge 3,\ \ \alpha = 2 \end{cases}$$

which is impossible. The proof of the theorem has been completed. ∎

3. ORIENTATION OF A GRAPH

For a 2-edge-connected graph G, Robbins [7] showed the following theorem:

Theorem 2. If G is 2-edge-connected graph, then G has a diconnected orientation.

Nash-Williams [8] has generalized Robbins' theorem by showing that:

Theorem 3. Every $2k$-edge-connected graph G has a k-arc-connected orientation.

In the following we will give orientation of graphs which develops the result of Theorem 2 and from which one can easily deduce a special case of Theorem 1 when

G is a 2-edge-connected graph. Moreover we also present a polynomial algorithm to find a spanning d_2-tree of a 2-edge-connected graph, based on the following theorem:

Theorem 4. Let $G = (V, E)$ be a 2-edge-connected graph, then there exists a diconnected orientation $\vec{G}$ of G such that for every vertex v of $G, |d^-_{\vec{G}}(v) - d^+_{\vec{G}}(v)| \leq 1$.

In the following we denote the set of all vertices of odd degree in G by $o(G)$. For a directed graph $\vec{H}$, the underlying graph of $\vec{H}$ is denoted by H. First we give some lemmas.

Lemma 3. Let $G = (V, E)$ be a graph and C be a cycle in G then $o(G) = o(G - E(C))$.

Lemma 4. Let $G = (V, E)$ be a graph, u and v be two distinct vertices of odd degree in G, P is a (u, v) path in G, then $o(G - E(P)) = o(G) - \{u, v\}$.

Lemma 5. Let graph G be 2-edge-connected, H a connected subgraph of G, ω a component of $G - E(H)$, if $|V(H) \cap o(\omega)| \leq 1$ then there exists a cycle C in ω such that $|V(H) \cap V(C)| \geq 1$.

Proof. By induction on the number of vertices in $S = V(H) \cap V(\omega)$. One can obviously observe that $|S| \geq 1$ by the connectivity of G.

If $|S| = 1$, then ω is a 2-edge-connected subgraph of G and thus there exists a cycle C in ω containing one edge with one end in S.

Assume the conclusion holds for $|S| = k$ and let $|S| = k + 1$, where $k \geq 1$. Suppose that u and v are two vertices in S, v has even degree and u has odd degree in ω or both have even degree. We denote a (u, v)-path in ω by P, and $e = (v', v)$ is an edge in P. If $\omega - e$ is connected, then there is a cycle in ω containing edge e and so containing at least one vertex in S. If $\omega - e$ is disconnected, then e is a cut edge in ω and $\omega - e$ consists of two connected components, denoted by ω_1 and ω_2. Let $v \in \omega_2$ and $H' = H \cup \omega_1 \cup \{e\}$, then $|V(H') \cap V(\omega_2)| \leq k$, and $|V(H') \cap o(\omega_2)| \leq 1$. Hence by induction hypothesis there is a cycle in ω_2 containing at least one vertex in $V(H')$. As $V(\omega_1) \cap V(\omega_2) = \emptyset$, one can easily see that $V(H) \cap V(\omega) \neq \emptyset$. The result follows by the induction. ∎

Lemma 6. Let $\vec{G}$ be a directed graph, $\vec{H}$ be a diconnected subgraph in $\vec{G}, u$ and v be two vertices in $\vec{H}, \vec{P}$ be a (u, v)-directed-path in $\vec{G} - A(\vec{H})$ then $\vec{H} \cup \vec{P}$ is diconnected. Where $A(\vec{H})$ is the arc set of $\vec{H}$.

Lemma 7. Let $\vec{G}$ be a directed graph, $\vec{H}$ be a diconnected subgraph in $\vec{G}, \vec{C}$ be a directed-cycle in $\vec{G} - A(\vec{H})$ and $V(\vec{H}) \cap V(\vec{C}) \neq \emptyset$, then $\vec{H} \cup \vec{C}$ is diconnected. Where $A(\vec{C})$ is the arc set of $\vec{C}$.

Lemma 8. Let $\vec{G}$ be a diconnected directed graph, then there is an out-directed tree in $\vec{G}$ rooted at u for any $u \in V(G)$.

Let $G = (V, E)$ be the given 2-edge-connected graph, in the following we will present an algorithm for orienting G to be a diconnected digraph $\vec{G}$ with $|d^+_{\vec{G}}(v) - d^-_{\vec{G}}(v)| \leq 1$ for all $v \in V(\vec{G})$.

Algorithm

step 0: Let $G = (V, E)$ and $u_0 \in V$, using DFS algorithm to find a cycle C in G containing u_0. Set $G_0 = C$, orient G_0 to be a directed cycle $\vec{G}_0$. $\vec{G}^* = \vec{G}_0, i = 1$.

step 1: If $G = G^*$ stop. Otherwise take one of the components of $G - E(G^*)$, denoted by ω.

If $|o(\omega) \cap V(G^*)| \leq 1$, using DFS algorithm to find a cycle C in ω such that $V(G^*) \cup V(C) \neq \emptyset$ (from the Lemma 5 such cycle must be found). Set $G_i = C$, orient G_i to be a directed cycle $\vec{G}_i$, set $\vec{G}^* = \vec{G}^* \cup \vec{G}_i, i = i + 1$, goto step 1.

Otherwise, let u and v be two vertices of odd degree in ω, use DFS algorithm to obtain a (u, v)-path P in ω. Set $G_i = P$, orient G_i to be a directed path $\vec{G}_i$ with origin u, $\vec{G}^* = \vec{G}^* \cup \vec{G}_i, i = i + 1$, goto step 1.

We denote $\cup_{j=0,\ldots,i} G_j$ by H_i.

Claim 1. $o(G) \supseteq o(G - E(H_0)) \supseteq \cdots \supseteq o(G - E(H_i))$.

Proof. Otherwise let $i_0, i_0 \geq 1$, be the minimum integer such that $o(G - E(H_{i_0-1})) \not\supseteq o(G - E(H_{i_0}))$, then there exists a vertex $u_0 \in o(G - E(H_{i_0})) \setminus o(G - E(H_{i_0-1}))$, From Lemma 3 and Lemma 4 we know G_{i_0} is a path, if u_0 is not an end of G_{i_0} then $d_{G_{i_0}}(u_0) \equiv 0 (\mathrm{mod}\, 2)$ and $d_{G-E(H_{i_0-1})}(u_0) \equiv d_{G-E(H_{i_0})}(u_0) (\mathrm{mod}\ 2)$, this contradicts $u_0 \in o(G - E(H_{i_0})) \setminus o(G - E(H_{i_0-1}))$. So u_0 is an end of G_{i_0}, but it is impossible since $u_0 \notin o(G - E(H_{i_0-1}))$. ∎

Proof of Theorem 4. In the algorithm for every G_i, the length of G_i is more than 1 or 3, corresponding to a path or cycle, hence at the end of step 1, a graph $G^* = G$ must be obtained. From Lemma 6 and Lemma 7 we know that $\vec{G}^*$ is disconnected at any step.

In the following we will prove $|d^+_{\vec{G}}(v) - d^-_{\vec{G}}(v)| \leq 1$ for all $v \in V(\vec{G})$.

First we show that $|d^+_{\vec{G}}(v) - d^-_{\vec{G}}(v)| = 0$ for all $v \in V(G) \setminus o(G)$.

One can easily see that $d^+_{\vec{G}_i}(u_0) = d^-_{\vec{G}_i}(u_0)$ if u_0 is not an end of a path G_i. If there is a vertex $u_0, u_0 \in V(G) \setminus o(G), |d^+_{\vec{G}_i}(u_0) - d^-_{\vec{G}_i}(u_0)| \geq 1$ then there exists a smallest integer i_0 such that for all $i < i_0, d^+_{\vec{G}_i}(u_0) = d^-_{\vec{G}_i}(u_0)$ and $|d^+_{\vec{G}_{i_0}}(u_0) - d^-_{\vec{G}_{i_0}}(u_0)| = 1$ then u_0 is an end of G_{i_0}, hence $u_0 \in o(G - E(H_{i_0-1}))$, but this contradicts Claim 1 since $u_0 \notin o(G)$.

Second we show that $|d^+_{\vec{G}}(v) - d^-_{\vec{G}}(v)| \leq 1$ for all $v \in o(G)$.

Assume $v_0 \in o(G)$, if, for all i, v_0 is not an end point of G_i, then $|d^+_{\vec{G}}(v) - d^-_{\vec{G}}(v)| = 0$. Otherwise let j_0 be the smallest integer such that v_0 is an end point of a path G_{j_0}, from Lemma 4 we know $v_0 \notin o(G - E(H_{j_0}))$, by Claim 1 for all $j \geq j_0, v_0 \notin o(G - E(H_j))$, hence there is at most one path, say G_{j_0}, with an end v_0. So

$$\begin{aligned}
|d^+_{\vec{G}}(v_0) - d^-_{\vec{G}}(v_0)| &= \left| \sum_{G_i, v_0 \in V(G_i)} (d^+_{\vec{G}_i}(v_0) - d^-_{\vec{G}_i}(v_0)) \right| \\
&= \left| \sum_{G_i \neq G_{j_0}, v_0 \in V(G_i)} (d^+_{\vec{G}_i}(v_0) - d^-_{\vec{G}_i}(v_0)) \right| \\
&\quad + |(d^+_{\vec{G}_{j_0}}(v_0) - d^-_{\vec{G}_{j_0}}(v_0))| \\
&= 1
\end{aligned}$$

which complete the proof of Theorem 4. ∎

By Lemma 8 there exists an out-directed tree in $\vec{G}$ rooted at v for any $v \in V(G)$. Let $\vec{T}$ be an out-directed tree of $\vec{G}$, for every vertex v of $\vec{T}$ it has the property that $d^+_{\vec{T}}(v) \leq d^+_{\vec{G}}(v) \leq [\frac{d(v)+1}{2}]$, which also implies $d_T(v) \leq [\frac{d(v)+1}{2}] + 1$ for every vertex v of T.

Using the *Algorithm* for orienting G to be a diconnected digraph $\vec{G}$ and the algorithm of [6] for finding an out-directed tree $\vec{T}$ of $\vec{G}$, then one can easily see that T satisfying the conclusion of Theorem 1.

The complexity of DFS Algorithm is $O(|E|)$. In step 1 each time we get a path of length at least 1 or get a cycle of length at least 3, thus the total execution time for the *Algorithm* is at most $O(|E|^2)$. Moveover using DFS Algorithm for finding an out-directed tree of a diconnected graph the complexity is at most $O(|E|)$, then we have an algorithm of finding a spanning tree of a 2-edge connected graph satisfying the result of Theorem 1 with the complexity is $O(|E|^2)$.

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