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Spanning trees in graphs of minimum degree 4 or 5

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Abstract

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For a connected simple graph G let $L(G)$ denote the maximum number of leaves in any spanning tree of G. Linial conjectured that if G has N vertices and minimum degree k, then $L(G) \geqslant ((k-2)/(k+1))N + c_k$, where c_k depends on k. We prove that if $k = 4$, $L(G) \geqslant \frac{2}{5}N + \frac{8}{5}$; if $k = 5$, $L(G) \geqslant \frac{1}{2}N + 2$. We give examples showing that these bounds are sharp.

1. Introduction

Is there a spanning tree of a connected simple graph G with many leaves? To find a spanning tree with the maximum number of leaves is an NP-complete problem, even when restricted to cubic (3-regular) graphs [4]. So people want to know for a given graph G with N vertices and minimum degree k, how many leaves at least are there for some spanning tree of G?

Throughout this paper G always denotes a connected simple graph. Let $L(G)$ denote the maximum number of leaves in any spanning tree of G. In 1981, Storer [6] announced that $L(G) \geqslant \frac{1}{4}N + 2$ for any 3-regular graph G with N vertices. The most interesting problem in this area is a conjecture due to Linial [5, cf. [1]], which generalizes Storer's result.

Conjecture. Let the minimum degree of G be k. Then

$$L(G) \geqslant \frac{k-2}{k+1}N + c_k,$$

where c_k depends on k.

This bound is attained with $c_k = 2$ by the following family of k-regular graphs: Construct a 'necklace' with any number of beads, where each bead is $K_{k+1} - e$ (Fig. 1).

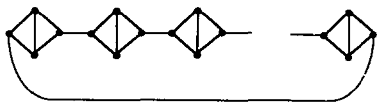


Fig. 1. $K_4 - e$ necklace.

Kleitman and West [3] introduced a new method, the 'dead leaves' approach, with which they gave a proof of Linial's Conjecture for $k = 3$ with a best possible $c_k = 2$. The special case where G is cubic, i.e., Storer's Theorem, had not been proven rigorously before.

Through a complicated proof using dead leaves, Griggs, Kleitman, and Shastri [1] proved that $L(G) \geq \frac{1}{3}(N+4)$ if a cubic graph G with N vertices has no subgraph isomorphic to $K_4 - e$. This bound is also tight, being attained by many graphs.

In Section 2 we prove Linial's Conjecture for $k = 4$ with the best possible value of $c_4 = \frac{8}{5}$. We use the dead leaves approach. Kleitman and West [2] have independently developed a somewhat different proof for this case $k = 4$. While they originally obtained a proof that $L(G) \geq \frac{2}{5}N + c$, we discovered the sharp result presented here.

Building on our work to settle $k = 4$, we prove our main result, which is Linial's Conjecture for $k = 5$, in Section 3. The best possible value for c_5 is 2.

A weaker general result than Linial's Conjecture would be to show that for every $\varepsilon > 0$, $L(G) \geq (1-\varepsilon)N$ for all graphs with sufficiently large minimum degree. This has just been proved by Kleitman and West [2].

It is worth pointing out that the proofs given in Sections 2 and 3, in fact, provide a polynomial algorithm to find a spanning tree which attains the lower bounds on $L(G)$.

We conclude the paper by presenting in Section 4 a new family of graphs attaining Linial's bound.

2. The lower bound for $k = 4$

Suppose T is a partial tree of G. If v is a vertex of G, let $N_T(v)$ denote the set of neighbors of v inside T and $N_{\bar{T}}(v)$ the set of neighbors of v outside T. Let $N(T)$ denote the set of neighbors of T, i.e., $N(T) = \bigcup_{v \in T} N_{\bar{T}}(v)$.

A leaf r of T is *dead* if $|N_{\bar{T}}(r)| = 0$, otherwise it is *alive*. We call r *k-split* if $|N_{\bar{T}}(r)| = k$. We shall form a cost function involving the number of leaves, dead leaves, and vertices of T, and we shall always seek to enlarge T while not decreasing the cost function. To consider dead leaves is a crucial idea, because we cannot gain enough new leaves in many cases, but we do gain some dead leaves to improve the value of the cost function.

Theorem 1. *If G is a connected simple graph with N vertices and minimum degree 4, then $L(G) \geq \frac{2}{5}N + \frac{8}{5}$.*

Proof. First notice that

$$L(G) \geqslant \tfrac{2}{5}N + \tfrac{8}{5} \text{ if and only if } 5L(G) \geqslant 2N + 8$$
$$\text{if and only if } 5L(G) > 2N + 7. \tag{1}$$

Define a cost function

$$\Delta(L, D, N) = \tfrac{13}{3}L + \tfrac{2}{3}D - 2N.$$

Then (1) holds if and only if there exists some spanning tree T for G such that

$$\Delta(L, D, N) > 7,$$

where D is the number of dead leaves of T, since every leaf in T is dead.

Our proof follows such procedures: First we find a partial tree with N_0 vertices, L_0 leaves and D_0 dead leaves such that

$$\Delta(L_0, D_0, N_0) > 7.$$

Then we expand it to a spanning tree of G by a series of steps, where for each step we add some number of vertices n, such that there is a net gain of l leaves and d dead leaves, satisfying the cost function $\Delta(l, d, n) \geqslant 0$. Finally the initial tree becomes a spanning tree T with all leaves dead, and clearly if L is the total number of leaves in T, then $\frac{13}{3}L + \frac{2}{3}L = 5L > 2N + 7$, and we are done.

Initial procedure: Pick one vertex v, and add all edges incident on v along with their endpoints. Such a star is required since $L_0 = \deg(v) \geqslant 4$ implies we have L_0 leaves and $L_0 + 1$ vertices so that $\Delta(L_0, 0, L_0 + 1) > 7$.

Expansion procedure: Let T be the current tree. Before doing the next step, we repeatedly add the vertices, each of which is adjacent to some internal vertex of T, to T. Then only leaves of T may have neighbors outside T. We do this without mentioning it again.

Next we list a collection of acceptable operations, at least one of which is available for the next step, until T becomes a spanning tree of G.

(O1) *There is a leaf r of T with $|N_{\bar{T}}(r)| = k \geqslant 2$.*

Expanding T at r to all $N_{\bar{T}}(r)$ gives $\Delta(k-1, 0, k) > 0$. If we assume (O1) fails, then each live leaf of T has exactly one neighbor outside T. Now we look at the neighbors of T.

(O2) *There is a vertex $x \in N(T)$ with $|N_T(x)| > 4$.*

Adding x to T kills at least $k \geqslant 3$ leaves and $\Delta(0, k, 1) \geqslant 0$.

Assuming (O1) and (O2) both fail, we have $|N_T(v)| \leqslant 3$ for each $v \in N(T)$. Now we consider a neighbor v of T with $|N_T(v)| = 1, 2, 3$ separately.

(O3) *There exists $v \in N(T)$ and $|N_T(v)| = 1$.*

Since $\deg(v) \geqslant 4$, v splits into (at least) 3 vertices outside T. Expanding these 4 vertices gives $\Delta(2, 0, 4) > 0$ (Fig. 2).

If we assume (O1)–(O3) all fail, then $2 \leqslant |N_T(v)| \leqslant 3$ for each $v \in N(T)$.

(O4) *There is $y_1 \in N(T)$ and $|N_T(y_1)| = 2$.*

Assume $\deg(y_1) = 4$ and y_1 splits into x_1 and y_2 outside T (if $|N_{\bar{T}}(y_1)| > 2$,

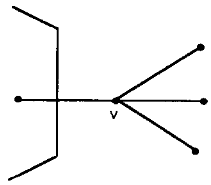


Fig. 2.

expanding at y_1 as in Fig. 2 again, we are done by $\Delta(2, 1, 4) > 0$). We may also assume that $x_1 \sim y_2$, $\deg(x_1) = \deg(y_2) = 4$ and none of them is adjacent to T, since otherwise we are done by $\Delta = \frac{1}{3}$ (Fig. 3) or $\Delta = \frac{5}{3}$ (Fig. 4). Let x_2, y_3 be the neighbors of y_2 besides x_1, y_1. For the same reason, we may assume none of them is adjacent to T. If $\{x_1, x_2, y_2, y_3\}$ form a K_4, we expand T as in Fig. 5, so y_2 is dead, and $\Delta(2, 2, 5) = 0$. If $\{x_1, x_2, y_2, y_3\}$ do not form a K_4 (recall $x_1 \sim y_2$), then one of x_2, y_3 must split, say y_3 splits into x_3, y_4 (as before assume $x_3, y_4 \nsim T$). Notice that y_3 should be adjacent to x_1 or x_2 (or both), otherwise we are done easily.

Set $B = T \cup \{x_i, y_j \colon 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 4\}$. Referring to Fig. 6, so far the cost function $\Delta = -\frac{1}{3}$, so we need just one dead leaf or a 2-split to balance the deficit (each 2-split increases Δ by $\frac{1}{3}$). Clearly if one of $\{x_1, x_2, x_3, y_4\}$ is dead, we are done by $\Delta(3, 2, 7) > 0$; if one of $\{x_1, x_2, x_3, y_4\}$ splits into two vertices outside B, we are done by $\Delta(4, 1, 9) = 0$. In fact, once we get a '4-2-split' structure, i.e., expand T from y_1 by a full binary tree with four internal vertices (Fig. 7), and we win. So each one of the $\{x_1, x_2, x_3, y_4\}$ has exactly one neighbor outside B. Let $a \sim x_1$, $b \sim x_2$, $c \sim x_3$, $d \sim y_4$, where $a, b, c, d \notin B$. If $a = b = c = d$ and $\deg(a) = 4$, expand T to $B \cup \{a\}$, then $\Delta(3, 5, 8) = \frac{1}{3} > 0$ (Fig. 8). If $\deg(a) \geqslant 5$, we are done by $\Delta(4, 1, 8) > 0$. Fig. 9 shows the case $x_1 \sim x_3$.

Now we go back and look at y_3.

(1) $y_3 \sim x_1$.

Assume $x_2 \nsim x_1$, otherwise refer to Fig. 4. Then x_2 must be adjacent to two of $\{x_3, y_3, y_4\}$. If $x_2 \sim y_3$, we are done by Fig. 10, and $\Delta(4, 2, 8) > 0$; otherwise we are done by Fig. 11, killing t and y_3.

(2) $y_3 \nsim x_1$, $y_3 \sim x_2$.

(a) $x_1 \sim x_2$: Since x_3 must be adjacent to two of $\{x_1, x_2, y_4\}$, x_3 should be adjacent to at least one of x_1, x_2. If $x_3 \sim x_1$, expanding T, gives $\Delta(3, 1, 6) > 0$

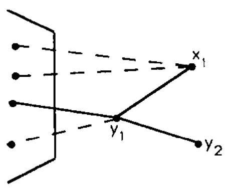


Fig. 3.

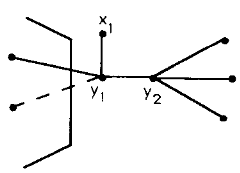
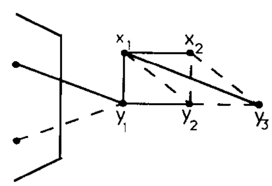


Fig. 4.



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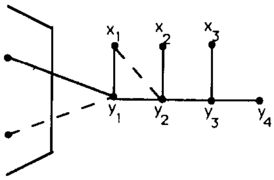

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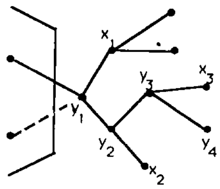

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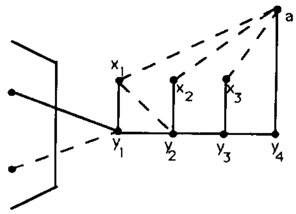

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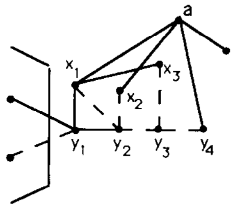

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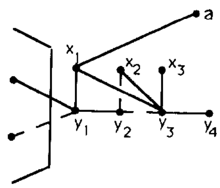

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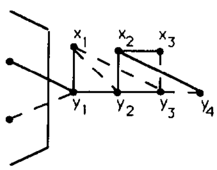


Fig. 11.

(Fig. 12). If $x_3 \sim x_2$, expand T as in Fig. 13, so t and y_2 are killed and $\Delta(3, 2, 7) > 0$.

(b) $x_1 \not\sim x_2$, $x_1 \sim x_3$: Then x_3 must be adjacent to one of x_2, y_4. If $x_3 \sim x_2$, we expand T as in Fig. 14, so that t, y_2 are dead, and $\Delta(3, 2, 7) > 0$. If $x_3 \not\sim x_2$, and $x_3 \sim y_4$, then $x_2 \sim y_4$. Now if $d \neq a$, expand T by Fig. 15 while if $a = d$ and $a \neq b$, expand T as in Fig. 16; if $a = d = b$ but $a \neq c$, expand T as in Fig. 17. We have a 4-2-split for each case.

(c) $x_1 \not\sim \{x_2, x_3\}$, $x_1 \sim y_4$: Then x_3 must be adjacent to x_2 and y_4. Now if $d \neq a$, expand T by Fig. 18; if $a = d$, $a \neq b$ expand T by Fig. 19; if $a = b = d$ but $a \neq c$, expand T by Fig. 20. Again we have a split for each case.

It remains to consider the case that (O1)–(O4) all fail. Then each $v \in N(T)$ has $|N_T(v)| = 3$.

(O5) *There exists* $y_1 \in N(T)$ *with* $|N_T(y_1)| = 3$.

If $|N_{\bar{T}}(y_1)| > 2$, refer to Fig. 2. Hence we may assume $N_{\bar{T}}(y_1) = \{x_1, y_2\}$. Assume $x_1 \not\sim T \not\sim y_2$ (otherwise done by killing many leaves). One of x_1, y_2 must split, say y_2 splits into x_2, y_3, and expanding gives $\Delta(2, 2, 5) = 0$ (Fig. 21). Finally assume $N_{\bar{T}}(y_1) = \{y_2\}$. If $y_2 \sim T$, we get $\Delta(0, 6, 2) = 0$ (Fig. 22). If $y_2 \not\sim T$, y_2 should split into at least 3 vertices outside T, and expanding T gives $\Delta(2, 2, 5) = 0$ (Fig. 23).

Clearly (O1)–(O5) cover all cases, and we are done. □

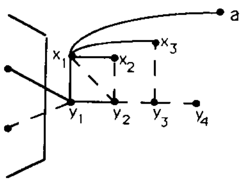


Fig. 12.

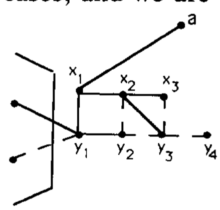


Fig. 13.

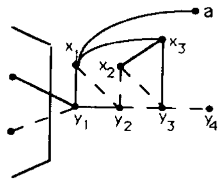


Fig. 14.

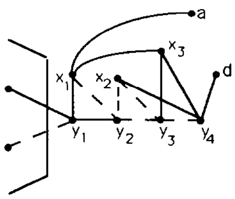


Fig. 15.

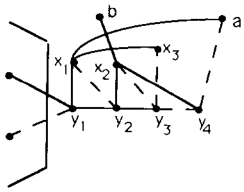


Fig. 16.

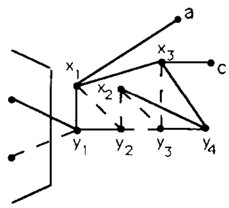
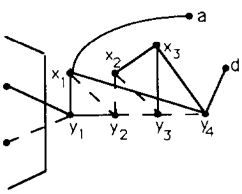


Fig. 17.



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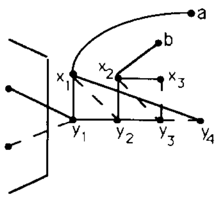

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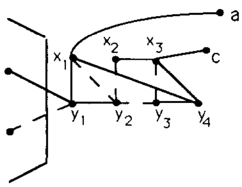

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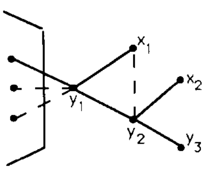

Fig. 21.

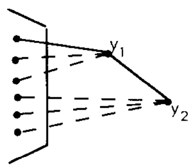

Fig. 22.

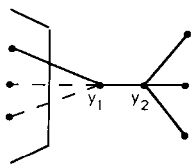

Fig. 23.

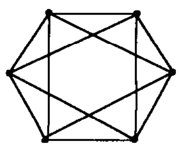

Fig. 24.

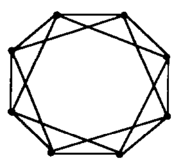

Fig. 25.

Notice that the lower bound of $L(G)$ given in Theorem 1 is sharp. For example, the graph $G(4, 6)$ in Fig. 24, which is 4-regular with 6 vertices, is such a example. Another graph $G(4, 8)$ almost matches this lower bound (Fig. 25). It is not clear whether there are some other graphs matching this lower bound, but we know that such graphs should be 4-regular, and each edge is involved in a triangle.

3. The lower bound for $k = 5$

Now let us consider graphs G with minimum degree 5.

Theorem 2. *If G is a connected simple graph with N vertices and minimum degree at least 5, then $L(G) \geqslant \frac{1}{2}N + 2$.*

Proof. First notice that if N is even, then $L(G) \geq \frac{1}{2}N + 2$ if and only if $L(G) > \frac{1}{2}N + 1$, i.e., $2L(G) > N + 2$. Define the cost function $\Delta(L, D, N) = \frac{4}{7}L + \frac{1}{4}D - N$. It is enough to show that

$$\Delta(L, D, N) > 2. \tag{2}$$

If N is an odd number, then $L(G) \geq \frac{1}{2}N + 2$ if and only if $L(G) > \frac{1}{2}N + \frac{3}{2}$, i.e., $2L(G) > N + 3$, so it is enough to show that

$$\Delta(L, D, N) > 3. \tag{3}$$

As before, we find a partial tree which satisfies (2) or (3) according to whether N is even or odd, and then expand it by a finite sequence of steps, such that each step preserves $\Delta \geq 0$. The proof depends on a series of lemmas.

Initial procedure: Pick $v \in V(G)$ with maximum degree in G, adding all edges incident on v with the end points. If N is even, $d(v) \geq 5$, so this star has $n_0 \geq 6$ vertices and $n_0 - 1$ leaves and $7(n_0 - 1)/4 - n_0 > 2$, while if N is odd, then $d(v) \geq 6$, so this star has $n_0 \geq 7$ vertices and $7(n_0 - 1)/4 - n_0 > 3$.

Expansion procedure: We list a collection of acceptable operations, such that if T is not yet a spanning tree, then certainly at least one of the operations is available for the next step.

We define a *saw path* SP to be a path (no repeated vertices, as usual) of G such that the vertices of SP are alternatively inside T and outside T. A *saw cycle* SC is a saw path such that the first vertex is outside T, and adjacent to the last vertex inside T. The length of the SC is defined as the number of vertices outside T in SC (Fig. 26).

(O1) *If one leaf r is k-split with $k \geq 3$*, we expand r to all of its neighbors, and $\Delta(k - 1, 0, k) > 0$.

Now if we assume (O1) fails, then $|N_{\bar{T}}(r)| \leq 2$ for every leaf r of T.

(O2) *There is a leaf r_1 with $N_{\bar{T}}(r_1) = \{a_1, a_2\}$.*

If one of $\{a_1, a_2\}$, say a_2, is not adjacent to T by at least one other edge, then a_2 has at least 4 neighbors outside T, and expanding at r_1 to all neighbors of a_2, we have $\Delta(3, 0, 5) > 0$. Assume $a_2 \sim r_2 \in T$. If $N_{\bar{T}}(r_2) = \{a_2\}$, we are done by expanding r_1 to a_1, a_2, killing r_2, and $\Delta(1, 1, 2) = 0$. So we may assume $N_{\bar{T}}(r_2) = \{a_2, a_3\}$, where $a_3 \neq a_1$, or otherwise we have a saw cycle with length 2, and we are done by Lemma 1 below. Repeatedly searching, if some $r_i \sim a_j$ for some $j < i$, there is a saw cycle, and we are done by Lemma 1. Otherwise at the very end of the finite saw path, either it ends inside T, so we are done by expanding the last 2-split killing one old leaf, or it ends outside T, so we are done by the argument for a_2.

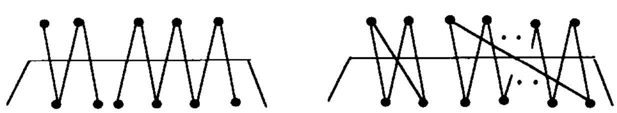

Fig. 26.

If we assume (O1), (O2) both fail, then $|N_{\bar{T}}(r)| \leq 1$ for every leaf r of T. Now we look at the neighbors of T.

(O3) *There is* $v \in N(T)$ *with* $|N_T(v)| = 1$.

Since $|N_{\bar{T}}(v)| = k \geq 4$, expand T to v, then to all $N_{\bar{T}}(v)$, and $\Delta(k-1, 0, k+1) > 0$.

Suppose (O3) also fails, then $|N_T(v)| \geq 2$ for each $v \in N(T)$.

(O4) *There exists* $v \in N(T)$ *with* $|N_T(v)| = 2$.

We are done by Lemma 2.

If (O1)–(O4) all fail, clearly $|N_T(v)| \geq 3$ for each $v \in N(T)$.

(O5) *There exists* $v \in N(T)$ *with* $|N_T(v)| = 3$.

We are done by Lemma 3.

If we assume (O1)–(O5) all fail, then $|N_T(v)| \geq 4$ for every $v \in N(T)$.

(O6) *There exists* $v \in N(T)$ *with* $|N_T(v)| = 4$.

Then $v \sim x \notin T$. If $x \sim T$ (at least 4 times) expand T to v and x, killing 8 leaves, so $\Delta(0, 8, 2) = 0$. If $x \nsim T$, x should split into 4 vertices outside T, expanding these vertices gives $\Delta(3, 3, 6) = 0$. (It is trivial if $\deg(v) > 5$ or $\deg(x) > 5$.)

Finally we assume (O1)–(O6) all fail.

(O7) *There exists a* $v \in N(T)$ *with* $N_T(v) \geq 5$.

Expand T by v, killing at least 4 old leaves, and we are done by $\Delta(0, 4, 1) = 0$.

(O1)–(O7) cover all cases which may appear when expanding T to a spanning tree of G. The summation of the costs of all steps including the initial procedure gives $L(G) \geq \frac{1}{2}N + 2$. We have completed the proof, subject to proving the lemmas that follow.

Lemma 1. *Suppose G is a simple connected graph with minimum degree at least 5. Let T be a tree in G that does not span it. Assume $|N_{\bar{T}}(r)| \leq 2$ for every leaf r of T. If there is a saw cycle SC, then we may expand T preserving $\Delta \geq 0$.*

Proof. Let SC be a shortest saw cycle with length k. Clearly $k \geq 2$. Label all vertices of SC outside T in order by $a_1, a_2, \ldots, a_k$, and set $A = \{a_1, a_2, \ldots, a_k\}$. Correspondingly label vertices of SC inside T by $r_1, r_2, \ldots, r_k$, i.e., $a_1 \sim r_1 \sim a_2 \sim r_2 \sim \cdots$, and so on.

Case 1: $k = 2m$, *integer* $m \geq 1$.

Expand every other 2-split, say expand r_1 to $\{a_1, a_2\}$, r_3 to $\{a_3, a_4\}, \ldots, r_{2m-1}$ to $\{a_{2m-1}, a_{2m}\}$, killing $r_2, r_4, \ldots, r_{2m}$, so $\Delta(m, m, 2m) = 0$ (Fig. 27).

Case 2: $k = 2m+1$, *where* $m \geq 1$.

(1) Assume there is an edge from A to T besides SC. We assume $a_{2m+1} \sim$

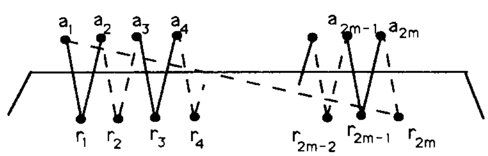

Fig. 27.

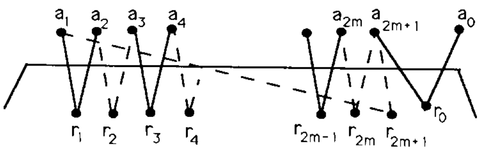

Fig. 28.

$r_0 \in T$. If $|N_{\bar{T}}(r_0)| = 1$, we simply expand r_{2m+1} to a_1 and a_{2m+1}, killing r_0, so $\Delta(1, 1, 2) = 0$. Assume $|N_{\bar{T}}(r_0)| = 2$, say $r_0 \sim a_0 \notin T$, $a_0 \neq a_{2m+1}$. If $a_0 \in A$ we have a shorter saw cycle, a contradiction. Therefore, $a_0 \notin A$, and we expand T as in Case 1, and r_0 to $\{a_{2m+1}, a_0\}$, killing $r_2, \ldots, r_{2m+1}$, giving us $\Delta(m+1, m+1, 2m+2) = 0$ (Fig. 28).

(2) Assume there is no edge from A to the complement of $A \cup T$. While expanding every other 2-split as in Case 1, we expand r_{2m+1} to a_{2m+1}, killing $3m+1$ leaves (all a_i's and $r_2, r_4, \ldots, r_{2m}$), so that $\Delta(m, 3m+1, 2m+1) \geq 0$ for $m > 1$.

Here we must point out that under the assumption of (2), if $m = 1$, then each of $\{a_1, a_2, a_3\}$ must have one more edge incident on T besides SC, so refer to Case (1).

(3) Neither (1) nor (2) happens, say $a_1 \sim x \notin A \cup T$.

Claim. $\{a_1, a_2, \ldots, a_{2m+1}, a_1\}$ *form a cycle, otherwise we may expand* T *preserving* $\Delta \geq 0$.

Proof. To prove the claim, observe that if for some i, $a_i \nsim a_{i+1} \pmod{2m+1}$, then since there are no edges from a_i to T besides SC, a_i should split into 3 vertices outside T other than a_{i+1}. Then we expand r_i to $\{a_i, a_{i+1}\}$, and a_i to the 3 vertices, giving $\Delta(3, 0, 5) > 0$. □

We have $a_1 \sim a_{2m+1}$ by the claim. Expand T by m 2-splits as in Case 1, and expand a_1 to $\{x, a_{2m+1}\}$, killing $r_2, r_4, \ldots, r_{2m}$, and r_{2m+1}, so that $\Delta(m+1, m+1, 2m+2) = 0$ (Fig. 29).

This completes Lemma 1. □

Lemma 2. *Suppose* G *is a simple connected graph with minimum degree at least* 5. *Let* T *be a tree in* G *that does not span it. Assume* $|N_{\bar{T}}(r)| \leq 1$ *for any leaf* r *of* T

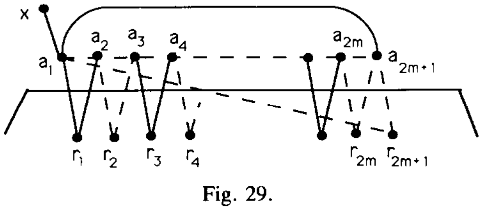

Fig. 29.

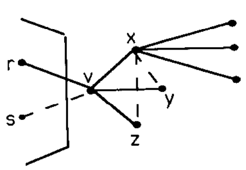

Fig. 30.

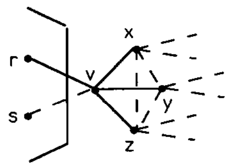

Fig. 31.

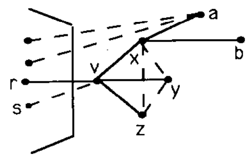

Fig. 32.

and $|N_T(v)| \geqslant 2$ *for each* $v \in N(T)$. *If there exists* $v \in N(T)$ *with* $|N_T(v)| = 2$, *then we may expand* T *preserving* $\Delta \geqslant 0$.

Proof. If $\deg(v) > 5$, then $|N_{\bar{T}}(v)| \geqslant 4$, and expanding T to all $N_{\bar{T}}(v)$, we have $\Delta(3, 1, 5) > 0$. Hence assume $\deg(v) = 5$. Let v split into $\{x, y, z\}$. Assume none of $\{x, y, z\}$ is adjacent to T, since otherwise we are done by $\Delta(2, 3, 4) > 0$. None of $\{x, y, z\}$ has degree >5, since otherwise it should split into 3 new vertices other than its brothers ('brothers' means that they grow from the same vertex in T), and we have $\Delta(4, 1, 7) > 0$ (Fig. 30).

Furthermore, $\{x, y, z\}$ should form a triangle (Fig. 31), since otherwise one of $\{x, y, z\}$ must split into 3 new vertices outside T, and we are done as above.

Let x split into $\{a, b\}$, y into $\{c, d\}$, z into $\{e, f\}$.

Case 1: *One of* $\{a, b, c, d, e, f\}$ *is adjacent to* T.

This gives $\Delta(3, 3, 6) = 0$. Fig. 32 shows the case $a \sim T$.

Case 2: *One of* $\{a, \ldots, f\}$ *is adjacent to only one of* $\{x, y, z\}$.

Then it must split into 3 new vertices other than $\{x, y, z\}$ and its brother, so $\Delta(5, 1, 9) = 0$ (Fig. 33).

If neither Case 1 nor Case 2 happens, then we need to consider the following two more cases.

Case 3: $\{a, b\} = \{c, d\} = \{e, f\}$.

Expand x to a, b, killing y, z, $\Delta(3, 3, 6) = 0$ (Fig. 34).

Case 4: *Assume* $a = f$, $b = d$, $c = e$ (Fig. 35).

We may assume $a \sim b \sim c$, because if there is only one edge among a, b and c, then one of $\{a, b, c\}$ should split into 3 new vertices, and we can refer to Case 2. For the same reason we may assume $\deg(a) = \deg(b) = \deg(c) = 5$.

Assume $a \sim c$. Then each of $\{a, b, c\}$ has exactly one edge to a vertex besides $\{a, b, c, x, y, z\}$, say $a \sim g$. If $g \sim b$ or c, say $g \sim b$, then expand x to $\{a, b\}$, a to $\{g, c\}$, which kills y, z, s, and b, so we have $\Delta(4, 4, 8) = 0$ (Fig. 35). Otherwise, suppose $b \nsim g \nsim c$. If $g \sim T$, we are done by expanding x to $\{a, b\}$, a to $\{c, g\}$,

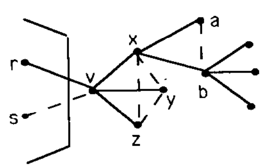

Fig. 33.

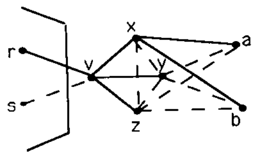

Fig. 34.

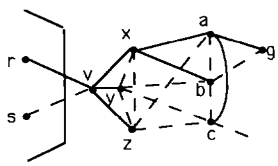

Fig. 35.

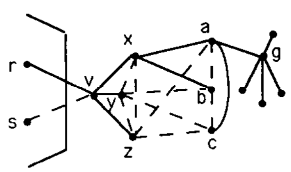

Fig. 36.

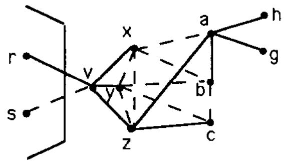

Fig. 37.

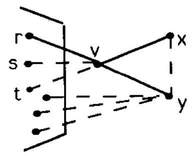

Fig. 38.

which kills many leaves and $\Delta(4, 5, 8) > 0$. If $g \not\sim T$, g splits into 4 new vertices, so expanding all such neighbors gives $\Delta(7, 3, 12) > 0$ (Fig. 36).

It only remains to suppose $a \not\sim c$. Then a has two new neighbors, say g and h. Expand z to $\{a, c\}$, a to $\{b, g, h\}$, killing $\{s, y, x\}$, and we are done by $\Delta(5, 3, 9) > 0$ (Fig. 37). $\square$

Lemma 3. *Suppose G is a simple connected graph with minimum degree at least 5. Let T be a tree in G, that does not span it. Assume $|N_{\bar T}(r)| \leqslant 1$ for every leaf r of T, and $|N_T(v)| \geqslant 3$ for every $v \in N(T)$. If there exists $v \in N(T)$ with $|N_T(v)| = 3$ then we may expand T preserving $\Delta \geqslant 0$.*

Proof. Notice the following points first:

(1) $\deg(v) = 5$. Otherwise $|N_{\bar T}(v)| \geqslant 3$, and expanding v to all $N_{\bar T}(v)$ gives $\Delta(2, 2, 4) = 0$.

(2) Let $N_{\bar T}(v) = \{x, y\}$, and $x \not\sim T \not\sim y$. Otherwise, we are done by killing many leaves (Fig. 38).

(3) $x \sim y$. Otherwise, x has 4 neighbors other than y, and expanding T as in Fig. 39 gives $\Delta(4, 2, 7) > 0$. For the same reason we may assume $\deg(x) = \deg(y) = 5$.

(4) Let x split into $\{a, b, c\}$, y split into $\{d, e, f\}$, where none of $\{a, b, c, d, e, f\}$ is adjacent to T, since otherwise we have $\Delta(3, 5, 6) > 0$ (Fig. 40).

Next we need to take care of the following cases.

Case 1: $|\{a, b, c\} \cap \{d, e, f\}| = 0$.

Expand x to $\{a, b, c\}$, y to $\{d, e, f\}$, and we have $\Delta(5, 2, 9) > 0$.

Case 2: $|\{a, b, c\} \cap \{d, e, f\}| = 1$, *where, say*, $c = f$.

Notice that $\{a, b, c\}$ should form a triangle, for otherwise one of $\{a, b\}$, say a, has 3 neighbors other than its brothers, so we are done by $\Delta(5, 2, 9) > 0$ (Fig. 41). Similarly $\{c, d, e\}$ form a triangle also. But this is impossible if $\deg(c) = 5$. If

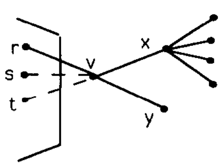

Fig. 39.

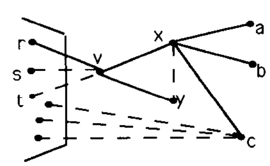

Fig. 40.

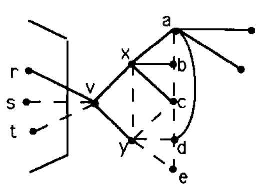

Fig. 41.

$\deg(c)>5$, $\{a, b, c\}$ and $\{c, d, e\}$ do form two triangles, and we are done by Lemma 4 below.

Case 3: $|\{a, b, c\} \cap \{d, e, f\}| = 2$, *and we assume* $b = e$, $c = f$.

It is clear by the above argument that $a, d \sim b, c$.

Subcase 3.1: $a \sim d$, $b \not\sim c$.

We may assume $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 5$, or otherwise one of them must split to 3 neighbors other than its brothers, and we can expand T as in Fig. 41 again. So each one of $\{a, b, c, d\}$ has exactly one more new neighbor. Assume $a \sim g$. If $g \sim T$, we are done by Fig. 42 and $\Delta(4, 6, 8) > 0$. If a is expanded to $\{d, g\}$, killing one of $\{b, c, d\}$, we have $\Delta(4, 4, 8) = 0$ (Fig. 43 but with $b \not\sim c$, and $g \not\sim b$, c or d). Otherwise g splits into 4 vertices other than $\{b, c, d\}$, so $\Delta(7, 3, 12) > 0$ (Fig. 44).

Subcase 3.2: $a \sim d$, $b \sim c$.

As before we are done unless $\deg(a) = \deg(d) = 5$, and $\deg(b)$, $\deg(c) \leq 6$. Let g be another neighbor of a. If one of $\{b, c\}$ is of degree 5, we expand T as in Fig. 43, killing s, t, y, and b (or c), so $\Delta(4, 4, 8) = 0$. Assume $\deg(b) = \deg(c) = 6$. Each one of $\{a, b, c, d\}$ has exactly one new neighbor. Then we may refer to the last part of Subcase 3.1 (Fig. 43 or 44).

Subcase 3.3: $d \not\sim a$ *and* $b \sim c$.

As above we assume $\deg(a) = \deg(d) = 5$, $\deg(b) \leq 6$, $\deg(c) \leq 6$. Then a has two other neighbors g and h, d has i and j. Assume $T \not\sim g, h, i, j$ (else we are done by killing many leaves).

If $\{g, h, i, j\}$ are all distinct, there are at most two of them which may be adjacent to b or c, so one of $\{g, h, i, j\}$ must split into 3 vertices other than its brothers, say h, so $\Delta(6, 2, 11) = 0$ (Fig. 45).

If $|\{g, h\} \cap \{i, j\}| = 1$, assume $h = j$. We should have $g \sim h$ and $h \sim i$, or otherwise one of $\{g, i\}$ must split into at least 3 new vertices (one such instance is

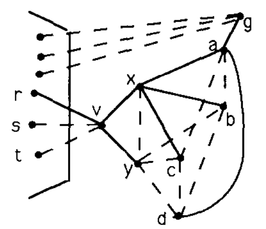

Fig. 42.

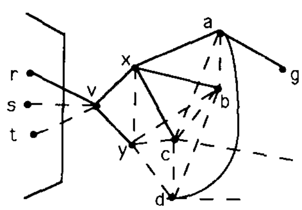

Fig. 43.

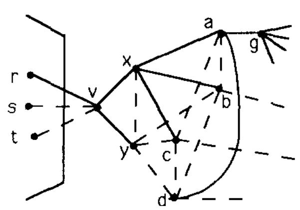

Fig. 44.

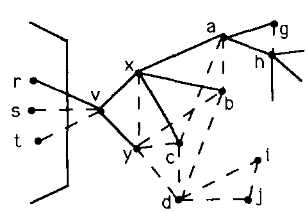

Fig. 45.

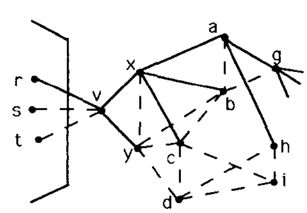

Fig. 46.

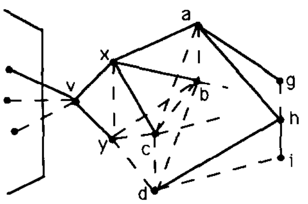

Fig. 47.

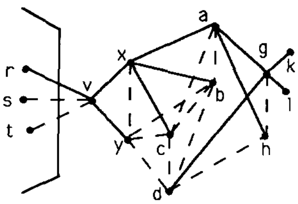

Fig. 48.

shown in Fig. 46), and we get $\Delta(6, 2, 11) = 0$. Also notice that g (as well as i) is adjacent to one of $\{b, c\}$ (otherwise we are done by g (or i) splitting into 3 new neighbors). Then expand a to $\{g, h\}$, h to $\{d, i\}$, killing $\{b, c, d, y, s, t\}$, so $\Delta(5, 6, 10)\} > 0$ (Fig. 47).

Now we assume $\{g, h\} = \{i, j\}$.

If one of $\{g, h\}$ has two neighbors other than $\{a, b, c, d, g, h\}$, say g has neighbors k and l, we expand a to $\{g, h\}$, g to $\{d, k, l\}$, and $\Delta(6, 4, 11) > 0$ (Fig. 48). Otherwise $g \sim b$ (or c), $h \sim c$ (or b), g has one new neighbor k. Expanding T gives $\Delta(5, 6, 10) > 0$ (Fig. 49).

Subcase 3.4: $d \not\sim a$, $b \not\sim c$.

We may follow the proof of Subcase 3.3, while assuming $\deg(b) = \deg(c) = 5$, to expand T preserving $\Delta \geq 0$.

Case 4: $|\{a, b, c\} \cap \{d, e, f\}| = 3$.

Expand x to $\{a, b, c\}$, killing s, t, y, so that $\Delta(3, 3, 6) = 0$ (Fig. 50).

This completes the proof of Lemma 3. $\square$

Lemma 4. *Suppose G is a simple connected graph with minimum degree at least 5. Let T be a tree in G, that does not span it. Suppose $|N_{\bar{T}}(r)| \leq 1$ for every leaf r of T. Assume $|N_T(v)| \geq 3$ for every $v \in N(T)$, and we have the structure as in Fig. 51, where* $\deg(v) = \deg(x) = \deg(y) = 5$. *Then we may suitably expand T from it preserving $\Delta \geq 0$.*

Proof. We notice that expanding out to a, b, c, d, e gives

$$\Delta(4, 3, 8) = -\tfrac{1}{4} < 0,$$

so we need just one dead leaf to finish.

We may assume that $\deg(a) = \deg(b) = \deg(d) = \deg(e) = 5$, and $\deg(c) = 6$,

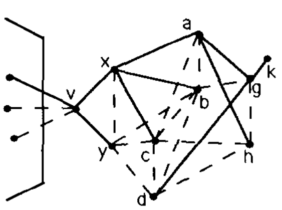

Fig. 49.

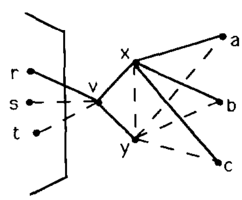

Fig. 50.

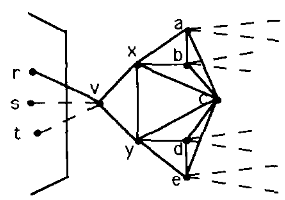

Fig. 51.

and none of them is adjacent to T. In fact, if one of $\{a, b, d, e\}$ is of degree >5 or if $\deg(c)>6$, it should split into at least 3 vertices other than its brothers, so that $\Delta(5, 2, 9)>0$; if one of $\{a, b, c, d, e\} \sim T$, we are done by killing many leaves. Now we discuss the following cases.

Case 1: $a \sim d, e$.

Expand a to $\{e, d\}$, killing 4 leaves, so $\Delta(4, 4, 8)=0$ (Fig. 52).

Because a, b, d, and e are symmetric in Fig. 51, we assume Case 1 does not hold for any one of a, b, d, or e, i.e., a is adjacent to at most one of $\{d, e\}$ (so is b), d is adjacent to at most one of $\{a, b\}$ (so is e).

Case 2: *There are two edges between a or b and d or e, say* $a \sim e$, *and* $b \sim d$.

Each one of $\{a, b, d, e\}$ has a new neighbor. Let $a \sim g$. Assume $g \not\sim T$, or otherwise expand a to $\{g, e\}$ (omitting d), then $\Delta(4, 5, 8)>0$. If $g \not\sim e$ (Fig. 53 shows the case $g \sim d$), or $g \sim e$ and $d \not\sim g \not\sim b$ (Fig. 54), then g has 3 new neighbors other than a, b or c, so if we expand g to those 3 new neighbors, we have $\Delta(6, 2, 11)=0$. If $g \sim e$ and g is adjacent to one of $\{b, d\}$ (notice Fig. 51 is symmetric), expand T as in Fig. 55. If $g \sim \{a, b, d, e\}$, g has another neighbor h, and expanding g to $\{d, h\}$ we have $\Delta(5, 7, 10)>0$.

Case 3: *There is just one edge between a or b and d or e.*

Assume $a \sim e$, $d \not\sim b$. Then b has two new neighbors g and h, and $g, h \not\sim T$ for otherwise we are done by killing many leaves. But one of $\{g, h\}$ must have 3 neighbors other than a, b, c, and its brother, so expanding T gives $\Delta(6, 2, 11)=0$ (Fig. 56).

Case 4: $d, e \not\sim a, b$.

Each one of $\{a, b, d, e\}$ has 2 new neighbors. Assume a splits to $\{g, h\}$, b to $\{k, l\}$ (none of $\{g, h, k, l\}$ is adjacent to T, otherwise we are done by killing many leaves). In fact, if $\{g, h\} \neq \{k, l\}$, one of them should split into 3 new neighbors other than a, b, c, and its brothers, so we are done by $\Delta(6, 2, 11)=0$

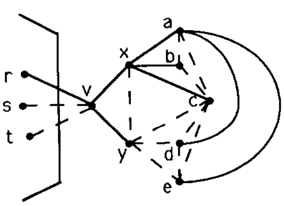

Fig. 52.

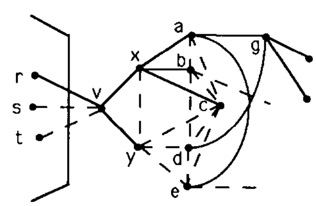

Fig. 53.

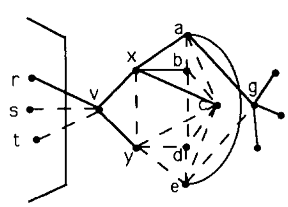

Fig. 54.

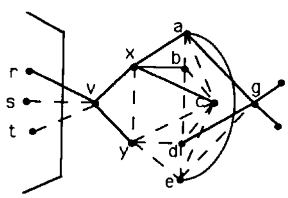

Fig. 55.

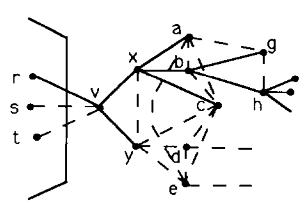

Fig. 56.

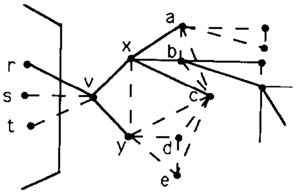

Fig. 57.

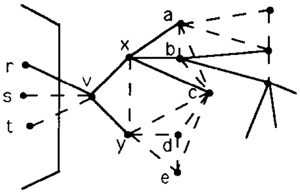

Fig. 58.

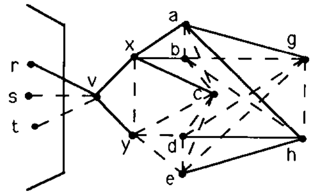

Fig. 59.

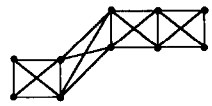
Fig. 60.

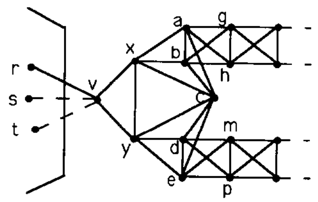

Fig. 61.

(Fig. 57 or Fig. 58). So now we assume $\{g, h\} = \{k, l\}$. By Fig. 56 we are done unless $g \sim h$ and $\deg(g) = \deg(h) = 5$. Similarly if m and p are the neighbors of d, they should be the neighbors of e also, and $m \sim p$, $\deg(m) = \deg(p) = 5$.

We assume $\{g, h\} \cap \{m, p\} = \emptyset$ because if $\{g, h\} = \{m, p\}$, expanding a to $\{g, h\}$, h to $\{d, e\}$ kills many leaves (Fig. 59), while if $|\{g, h\} \cap \{m, p\}| = 1$, say $h = p$, then $\deg(h) = 6$, a contradiction.

According to the above analysis, $\{a, b, g, h\}$ form a K_4. If we expand a to g and h, killing b, we still need one more dead leaf to keep $\Delta \geqslant 0$. We define a *K_2-chain* to be structure formed by using K_2's as beads to form a chain, where each pair of adjacent K_2's forms a complete bipartite graph $K_{2,2}$ besides the edges in the K_2's (Fig. 60).

Repeatedly applying the argument above we find that the problem occurs when there is a K_2-chain starting at a and b, and another K_2-chain at d and e (Fig. 61). But the length of each K_2-chain must be finite, so certainly one of the following cases should happen.

Subcase 4.1: *A K_2-chain stops outside T.*

The structure has to be changed at the very end of it. But one of $\{\alpha, \beta, \rho, \tau\}$ splits into 3 new vertices besides its brother and parent (or is adjacent to T). So we win by gaining at least $\frac{1}{4}$ in Δ (Fig. 62).

Subcase 4.2: *A K_2-chain comes back to T.*

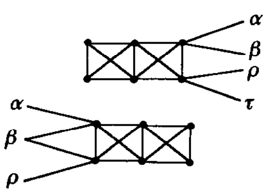

Fig. 62.

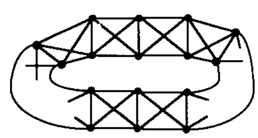
Fig. 63.

Clearly expand until the last 2-split of the K_2-chain, where it kills at least one extra old leaf, and we are done.

Subcase 4.3: *The two K_2-chains meet.*

We may expand the 2-splits along the upper K_2-chain around to d and e, killing d, e, and y, and we win (Fig. 63).

This completes Lemma 4. □

These four lemmas complete the proof of Theorem 2. □

4. A new family of graphs attaining Linial's bound

We have already seen that by taking a necklace of any number A of beads, where each bead is $K_{k+1} - e$, a k-regular graph G is obtained with

$$N = (k+1)A \quad \text{and} \quad L(G) = \frac{k-2}{k+1}N + 2.$$

These graphs are extremal for Storer's result ($k = 3$) and Theorem 2 ($k = 5$).

For $k = 4$ and $N = 5A$, the bound of Theorem 1 is not an integer, but in this case the implied bound is $\lceil \frac{2}{5}N + \frac{8}{5} \rceil = 2A + 2 = \frac{2}{5}N + 2$, and the family of necklaces attains this bound. We also saw that for $k = 4$ and general N, $\frac{8}{5}$ is best possible for c_k.

As a by-product of the proof of Theorem 2, we noticed an interesting new family of examples attaining Linial's bound. For the case $k = 5$ of Theorem 2, let G be a K_2-chain of $B \geqslant 3$ K_2's that closes on itself. Then G is a 5-regular graph with $N = 2B$ and $L(G) = B + 2$. It is extremal in Theorem 2 for all even N (Fig. 64).

This construction extends for arbitrary $m \geqslant 1$ to provide a family of graphs that are regular of degree $k = 3m - 1$: For $B \geqslant 3$ form a K_m-cycle consisting of B K_m's in a circle such that vertices in consecutive K_m's are adjacent. Such a graph G has

$$N = Bm = \tfrac{1}{3}B(k+1) \quad \text{and} \quad L(G) = B(m-1) + 2 = \frac{k-2}{k+1}N + 2.$$

This is the same value for $L(G)$ attained by the necklaces and used to support Linial's Conjecture.

Examples in the new family exist for three times as many values of N as the family of necklaces, although in the new family k is restricted to 2 mod 3. What is

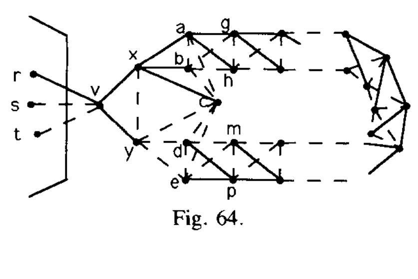

Fig. 64.

significant is that the K_m-cycles are highly connected compared to the necklaces, which are not 3-connected. The connectivity is important to consider since for $k=3$ the bound $L(G) \geqslant \frac{1}{4}N+2$ rises to $L(G) \geqslant \frac{1}{3}(N+4)$ when G is 3-connected, by the result of Griggs, Kleitman, and Shastri [1]. Evidently 3-connectivity does not improve the bound on $L(G)$ for larger k.

The construction can be adapted to provide k-regular graphs G for arbitrary $k \geqslant 2$. Given $A \geqslant 1$ and a, b, $c \geqslant 1$ such that $a+b+c=k+1$, we arrange A copies of the sequence of complete graphs K_a, K_b, K_c in cyclic order. We form a graph G by putting edges between vertices in consecutive complete graphs. Then

$$N=(k+1)A \quad \text{and} \quad L(G)=\frac{k-2}{k+1}N+2$$

In particular, if $a=b=1$ and $c=k-1$, then we have the familiar example of necklaces. We produce examples with high connectivity by taking each of a, b, and c equal to $\lfloor \frac{1}{3}(k+1) \rfloor$ or $\lceil \frac{1}{3}(k+1) \rceil$.

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