Spanning Trees with Many Leaves in Cubic Graphs*

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ABSTRACT

For a connected graph G let $L(G)$ denote the maximum number of leaves in any spanning tree of G. We give a simple construction and a complete proof of a result of Storer that if G is a connected cubic graph on n vertices, then $L(G) \geq \lceil (n/4) + 2 \rceil$, and this is best possible for all (even) n. The main idea is to count the number of "dead leaves" as the tree is being constructed. This method of amortized analysis is used to prove the new result that if G is also 3-connected, then $L(G) \geq \lceil (n/3) + (4/3) \rceil$, which is best possible for many n. This bound holds more generally for any connected cubic graph that contains no subgraph $K_4 - e$. The proof is rather elaborate since several reducible configurations need to be eliminated before proceeding with the many tricky cases in the construction.

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider the problem of finding spanning trees in given graphs that contain many leaves (degree one vertices). All graphs are assumed to be simple (undi-

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rected, no loops or multiple edges). If G is a connected graph, let $L(G)$ denote the maximum number of leaves in any spanning tree of G. We are interested here in $L(G)$ for *cubic* (3-regular) graphs G.

Suppose T is a spanning tree for a connected cubic graph G on n vertices. Necessarily, n is even. Let d_i denote the number of vertices of degree i in T, $i = 1, 2, 3$. Then the number of vertices $n = d_1 + d_2 + d_3$, while the sum of the degrees $2n - 2 = d_1 + 2d_2 + 3d_3$. It follows that $L(T) = d_1 = d_3 + 2$. Consequently, $L(G)$ is *maximized* over such graphs G when it contains T with d_2 as small as possible, that is, $d_2 = 0$. Hence, $L(G) \le (n/2) + 1$. This bound is attained for all (even) n by taking the caterpillar in which $(n/2) - 1$ vertices form a path, and a leg (leaf vertex) is joined to each interior vertex of the path, while two legs are joined to each end of the path. This is the desired tree T, which can be embedded in a suitable graph G by adding a cycle through the leaves of T (see Figure 1).

The more interesting question then is to *minimize* $L(G)$, i.e., to obtain a lower bound on $L(G)$ over all such graphs G in terms of n. This problem was proposed and solved in 1981 by Storer.

Theorem 1 [5]. If G is a connected cubic graph on n vertices, then $L(G) \ge \lceil (n/4) + 2 \rceil$.

This bound is best possible for all (even) n. For example, if $n \equiv 0 \pmod 4$, then take G to be a circular "necklace" of $n/4$ "beads," where each bead is $K_4 - e$ (meaning K_4 with one edge deleted), as shown in Figure 2. If $n \equiv 2$ (mod 4), we may take G to be a necklace in which there are $(n - 6)/4$ beads

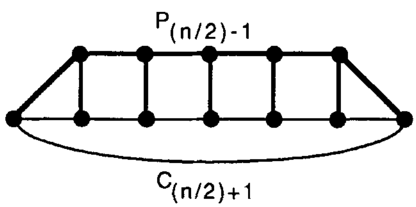


FIGURE 1. A Cubic Graph G with $L(G) = (n/2) + 1$

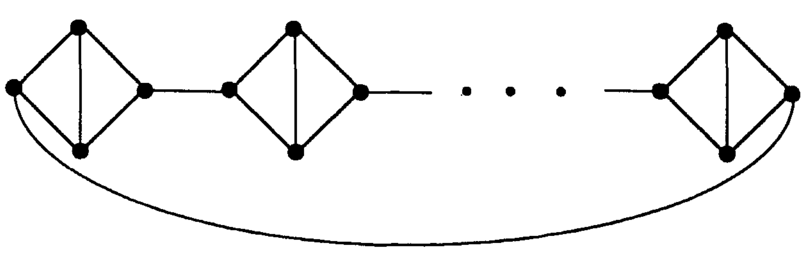


FIGURE 2. Extremal Graph for Theorem 1

that are $K_4 - e$ and one bead that is $K_{3,3} - e$. Then $L(G) = (n/4) + (5/2) = \lceil (n/4) + 2 \rceil$, as claimed.

If one permits vertices of lower degree than 3 in G, then $L(G)$ can drop dramatically, e.g., $L(P_n) = 2$, where P_n is a path on n vertices. Storer works with graphs of maximum degree 3, rather than our more restricted setting of cubic graphs. We consider the effect of introducing the stronger connectivity condition on G that it be *3-connected*, i.e., the removal of any two vertices does not disconnect it. Such graphs cannot contain $K_4 - e$, and our main result applies to this more general situation.

Theorem 2. If G is a connected cubic graph on n vertices that contains no subgraph isomorphic to $K_4 - e$, then $L(G) \geq \lceil (n + 4)/3 \rceil$.

Corollary 1. If G is a 3-connected cubic graph on n vertices, then $L(G) \geq \lceil (n + 4)/3 \rceil$.

Corollary 2. If G is a triangle-free, connected, cubic graph on n *vertices*, then $L(G) \geq \lceil (n + 4)/3 \rceil$.

So the lower bound on $L(G)$ rises to over $n/3$ when $K_4 - e$ is excluded. Recall that a necklace of beads, each $K_4 - e$, was used to attain the lower bound around $n/4$ in Theorem 1.

The bound in Theorem 2 (and both corollaries) is sharp. For $n \equiv 0 \pmod 6$, say $n = 6k$, one can obtain a family of graphs G with $L(G) = 2k + 2 = (n/3) + 2 = \lceil (n + 4)/3 \rceil$ in the following way: Take $2k$ triangles (K_3) and add edges on these $6k$ vertices until a connected cubic graph G is formed. To obtain a tree T in G with $L(T) = 2k + 2$, take any spanning tree T^R in the reduced graph G^R on $2k$ vertices that is obtained by contracting each of the $2k$ triangles to a point and eliminating duplicated edges. Then build T^R up to a tree T in G in the natural way, so that each vertex of degree 1 (respectively, 2, 3) in T^R gives rise to 2 (respectively, 1, 0) leaves in T. These graphs G on $6k$ vertices can be designed so that G is 3-connected.

Another extremal example for $n = 6k$ is to form a necklace of k beads, where each bead looks like the six-vertex graph on the left side of (6) in Figure 4. The graphs in this family are triangle-free, but not 3-connected.

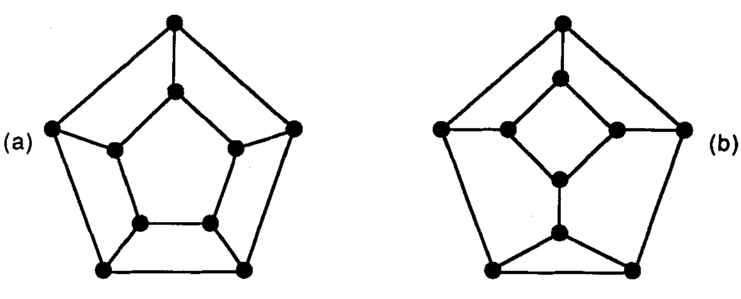


FIGURE 3. Graphs with $n = 10$ and $L = 5$

For $n \equiv 2 \pmod 6$, say $n = 6k + 2$, the graph Q_3 of the usual three-dimensional cube is extremal for $k = 1$. In fact, by carrying out the initial part of the proof of Theorem 2 more carefully, it can be shown that Q_3 is the *unique* extremal graph with $n \equiv 2 \pmod 6$. It follows that for G covered by the theorem, $L(G) \geq (n + 5)/3$ unless $G = Q_3$.

For $n \equiv 4 \pmod 6$, say $n = 6k + 4$, we have that $L(G) \geq 2k + 3$. An extremal graph is obtained for $k = 1$ by taking two 5-cycles and pairing up their vertices, as shown in Figure 3(a). It is tempting to think that this is the only connected cubic graph G not containing $K_4 - 3$, besides Q_3, that has $L(G) <$

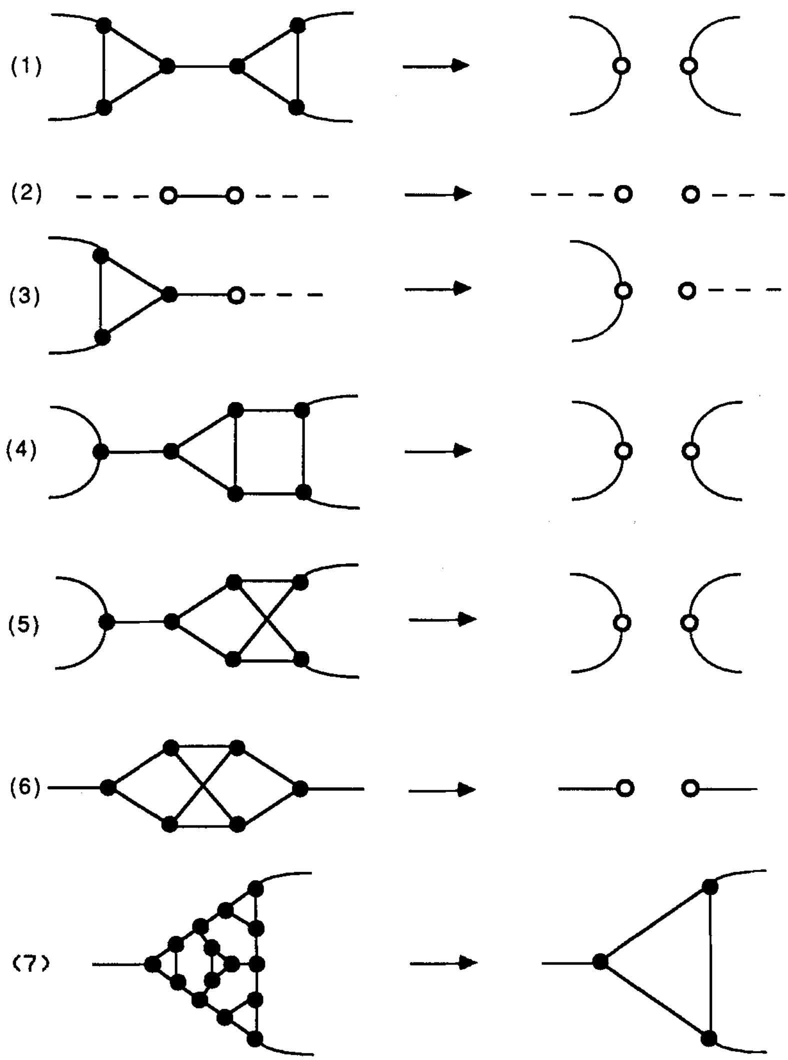


FIGURE 4. List of Reductions

$(n/3) + 2$. (We saw above that many graphs exist with $L(G) = (n/3) + 2$). However, Figure 3(b) shows another example with $n = 10$ and $L = 5$.

Our method of proof is to begin by finding a small tree in G with many leaves and then to grow the tree by adding several vertices in such a way that the number of leaves always grows enough to keep satisfying the theorem. A central idea is to keep track of the number of "dead leaves" as well as the number of leaves. A *dead leaf* is a leaf in the tree under construction, all of whose neighbors are already in the tree. Once a leaf is dead, it remains a leaf during the rest of the construction. To illustrate the power of this approach, we offer a new proof of Theorem 1 in the next section.

In Section 3 we begin the proof of Theorem 2. Another fundamental idea in the proof is that certain configurations are *reducible,* which means they can be replaced by smaller configurations (which may include a special type of vertex we call a *goober*). Once the construction is completed on the reduced graph, the forest obtained can be blown up and reconnected to form the desired tree on G. Section 4 contains the several cases that make up the proof for the reduced graph. In some instances the construction is by necessity quite elaborate.

The paper concludes in Section 5 with some suggestions for further study including, most notably, a conjecture of N. Linial that generalizes Theorem 1.

2. DEAD LEAVES AND THEOREM 1

A natural approach to proving a result such as Theorem 1 is as follows. Starting with a small tree in G that is nice, one tries to add on some number of vertices N in such a way that the tree gains at least $N/4$ leaves. If this can always be done, then the tree eventually constructed must have at least roughly $n/4$ leaves. There is one case that is especially difficult with this approach. Consider a vertex v outside the tree that has all three of its neighbors being leaves in the tree. Then adding v causes no gain in the number of leaves. What can we do if there are many such vertices v? We do gain something by adding v, which is that v itself would have all of its edges inside the tree and that several edges involving neighbors of v are likewise accounted for. So v itself and perhaps some of its neighbors become dead leaves. At any given stage in the construction, it is obvious that the number of dead leaves D is at most the number of leaves L. So if we aim to show that at the end, $L \geq (n/4) + c$, for some c, or equivalently, $4L \geq n + 4c$, then it suffices to show that $aL + bD \geq n + 4c$ for some choice of $a, b \geq 0$ such that $a + b = 4$. We start off by constructing a tree on N vertices with L leaves and D dead leaves such that $\Delta(N, L, D) \geq 4c$, where $\Delta(N, L, D) = aL + bD - N$. It then suffices to show that for any constructed tree that does not yet span G there exists some set of vertices, say N of them, that can be added in such a way as to increase the number of leaves by L and the number of dead leaves by D, where $\Delta(N, L, D) \geq 0$.

This dead leaf method is simply a type of amortized analysis in which some of the benefit of adding new leaves is postponed to the later time that the leaves die. The net gain from adding a leaf is the same. We sacrifice some advantage by using aL, $a < 4$, but not enough to cause failure, while we benefit later from the term $bD, b > 0$, when we create a dead leaf. This allows us to handle a case such as a vertex v as above that is adjacent only to leaves already created. Suitable choices for a and b are determined by carrying out the cases in the proof and solving for a and b, which make the inequalities work out, with some trial and error being necessary.

We are ready to prove Theorem 1. Besides illustrating the value of the dead leaves approach, it may be useful to have a complete proof written down. Storer's approach is to start with a breadth-first spanning tree and then modify it to gain leaves. His approach is natural, but the proof is merely sketched, and we were unable to work out all of the details.

Proof of Theorem 1. Let G be a connected cubic graph on n vertices. Since n is even and $L(G)$ is integral, $L(G) \geq \lceil (n/4) + 2 \rceil$ if and only if $L(G) > (n/4) + (3/2)$. For our dead leaves approach, we seek a and b, $a + b = 4$, so that $aL + bD - N > 6$. It turns out that $a = 3.5$ and $b = .5$ are suitable choices, so we assume these values hereafter. Concerning the notation, vertices shall be denoted by lower case letters, and $v \sim w$ means v and w are adjacent, while $v \not\sim w$ means they are not adjacent. If a vertex v is outside a tree T but adjacent to some vertex in T, we write $v \sim T$. We denote the edge between two vertices v and w by vw.

To start off, select any vertex $v \in G$, and let $w, x, y \sim v$. Begin the tree T by taking the three edges involving v. So we have $\Delta \geq \Delta(4,3,0) = 6.5 > 6$.

For subsequent stages we show that a tree T that only partially spans G can be extended by some amount such that $\Delta \geq 0$. First suppose there exists $v \in T$ such that $v \sim w$, $x \notin T$. Then add vw and vx to T, and $\Delta \geq \Delta(2,1,0) = 1.5$. If no such v exists, suppose there exists $v \in T$ that is not a leaf and $w \notin T$, $w \sim v$. Then add vw to T, giving $\Delta \geq \Delta(1,1,0) = 2.5$. Assume this is not possible either. So vertices $v \notin T$ with $v \sim T$ are adjacent to *no* internal vertices of T, while leaves in T that are not dead lie on only one edge outside T.

Suppose there exists $v \notin T$ such that $v \sim w, x, y \in T$. Adding vw to T creates no new leaves, but v, x, y are all dead leaves, so that $\Delta = \Delta(1,0,3) = .5$. Next suppose there exists $v \notin T$, $v \sim T$, such that v splits, i.e., $v \sim w$, $x \notin T$. Let $t \in T$ such that $v \sim t$. Then add tv, vw, vx to T, and we have $\Delta \geq \Delta(3,1,0) = .5$. Then suppose none of the above operations is possible, so that every $v \notin T$, $v \sim T$, is adjacent to exactly two vertices in T. Since G is connected, there exists such a vertex v, say $v \sim s, t \in T$, and $v \sim w \notin T$. Then if also $w \sim T$, say $w \sim p, q \in T$, add edges tv, vw to T, creating dead leaves at s, w, p, q, so that $\Delta = \Delta(2,0,4) = 0$. Otherwise, we have $w \not\sim T$, so w splits and $w \sim x, y \notin T$. Then add tv, vw, wx, wy to T, so that $\Delta \geq \Delta(4,1,1) = 0$, since s becomes a dead leaf. Thus in every case we can add to T so that $\Delta \geq 0$. By induction, we can eventually extend T to a spanning tree of G with at least $\lceil (n/4) + 2 \rceil$ leaves. ∎

The proof above is essentially a polynomial-time algorithm for constructing a tree with at least $\lceil (n/4) + 2 \rceil$ leaves. This was a concern in [5], where a very different polynomial-time algorithm is provided for the more general class of connected graphs of maximum degree 3. The algorithm there begins with a breadth-first spanning tree for G and then modifies it to obtain many leaves.

3. REDUCIBLE CONFIGURATIONS AND THEOREM 2

We shall describe several "reductions" on G in which suitable configurations are replaced by simpler ones, although the simpler ones may contain a special type of vertex, called a *goober*. All other vertices are called *ordinary*. We carry out a sequence of reductions on G until no more are possible, so that the remaining graph is *irreducible*. We describe below what the possible reductions are. We then state Theorem 3, which gives a lower bound on the number of leaves in a connected irreducible graph. The section concludes with a proof that Theorem 3 implies our main result, Theorem 2. To do this, we must carefully check that a forest with many leaves that spans the reduced graph for G can be lifted to a spanning tree for G with enough leaves to satisfy the claim of Theorem 2. Then Theorem 3 will be proven in Section 4 by examining many cases.

The reductions are shown in Figure 4. In the figures, goobers are shown as open circles. Goobers usually arise by contracting triangles but other possibilities occur with Reductions (4), (5), and (6).

Some conventions must be explained in conjunction with Figure 3. Dashed lines mean that an edge may or may not be incident, and if it is, it is retained after the reduction. In Reductions (1)–(5), two outgoing edges from vertices in the configuration may not meet to form a single edge, or meet at another ordinary vertex when both originate at the left end or both at the right end of the figure. This would never happen anyway, given our hypotheses on G, with the single exception that if the edges on the right side of Reduction (5) meet at an ordinary vertex, we cannot use (5). But in this case, (6) is applicable. On the other hand, edges from opposite ends may meet, e.g., two triangles with two edges joining them (or even three) are reducible using (1). Reduction (7) is unique. We assume that no two of its three outgoing edges join or meet at another ordinary vertex. This prevents us from forming a multiple edge between ordinary vertices or a $K_4 - e$ on ordinary points, both of which are forbidden, by carrying out Reduction (7). We emphasize that outgoing edges from *opposite* sides in (1)–(6) may meet, while those from the *same* side in (1)–(7) may meet only at a goober.

Here is what some reductions do, in words. Reduction (1), for example, destroys any edges between two adjacent triangles and contracts each such triangle to a goober. Reduction (2) destroys edges between goobers. Reduction (6) gets rid of the "bead" on six vertices shown in the figure. In effect, this bead is no worse than a simple edge since we can always gain two leaves from these six vertices.

Goobers are counted in an unusual way which is motivated by the fact that they typically represent the loss of three vertices and one leaf from the graph: Notice the ratio of $\frac{1}{3}$ for leaves to vertices, which is what we seek to obtain for the entire graph. So goobers do *not* count toward the number N of vertices of a tree under construction, but they do count toward L and D, and toward the degrees of their neighbors. Goobers always have degree at most 2, while ordinary vertices have degree 3.

Suppose H is a graph all of whose vertices are ordinary or goobers. We assume ordinary vertices have degree 3 and goobers have degree at most 2. We also assume that H contains no $K_4 - e$ (all 4 vertices are ordinary). Finally, suppose no reduction in Fig. 3 can be applied to H. If H satisfies all of these conditions, we say it is *irreducible*. We are ready to state our result for irreducible graphs, and to show how to use it to derive our main theorem.

Theorem 3. Suppose H is a connected irreducible graph with $a > 0$ ordinary vertices and $b \geq 0$ goobers. Then $L(H) \geq (a/3) + 2$, if $b > 0$, and $L(H) \geq (a/3) + (4/3)$, if $b = 0$.

Proof of Theorem 2 from Theorem 3. Let G^R be the graph obtained after some succession of reductions from G when no further reductions are possible. One can check that G^R contains no multiple edges, nor any of the forbidden graphs $K_4 - e$ (using only ordinary vertices). All ordinary vertices have degree 3 and goobers have degree at most 2. Thus G^R is irreducible.

First suppose that G^R contains no goobers, i.e., only Reduction (7) was performed, if any were. Then G^R is connected, and Theorem 3 can be applied with $b = 0$ to produce a spanning tree for G^R with at least $(a/3) + (4/3)$ leaves, where a is the number of vertices in G^R. It remains to successively restore vertices deleted by Reduction (7), 12 vertices for each reduction, while enlarging the spanning tree each time. We shall see in Fig. 7 that the tree can be enlarged to gain 4 leaves for each reduction, which guarantees that a tree with at least $(n/3) + (4/3)$ leaves is constructed for all of G.

It remains to consider G^R that contains some goober. It can be easily checked that every component in G^R contains a goober since any reduction besides (7) gives a goober to each component if it disconnects any vertices. We say that a component that consists of an isolated goober is *trivial*. Components that are nontrivial will contain at least one ordinary point.

Now we can apply Theorem 3 to each nontrivial component H in G^R, if any such H exists. If a denotes the number of ordinary vertices in H, then our proof will construct a spanning tree for H with at least $(a/3) + 2$ leaves. Take such a tree for every H to obtain a spanning forest for G^R.

For each isolated goober g in G^R, we seek to reattach g to some other component H in such a way that if H is nontrivial, we put back some number of original ordinary vertices $3c$, where the integer $c \geq 0$, while the tree on H is expanded to gain at least c leaves. Alternately, if H is trivial, i.e., H is another goober h, we join g and h, and replace either g or h, or both, by $3c$ ordinary vertices, while producing a tree on this component with at least $c + 2$ leaves. How this is done depends on what reduction separated g and h in the first place. By doing this, we partially restore G^R back to G, call this intermediate graph G^S, in such a way that for every component H of G^S, H is not trivial, and if the number of ordinary vertices in H is a, then the number of leaves in the constructed tree spanning H is at least $(a/3) + 2$.

In Figure 5 we show exactly how this merging of trivial components with nontrivial ones or with each other is carried out. From Figure 3 we see that a goober was isolated when its last incident edge was destroyed, and only Reduction (2) or (3) could have done this. In each row, we give the type of reduction, (2) or (3), that originally destroyed the connection we are to reinstate, followed by a picture showing edges in the forest constructed so far that involve these vertices, followed by the edges used in the forest after we reattach the isolated component. Finally, we list (c, d), where $3c$ vertices have been added and d is the number of leaves gained. One can check in each case that either c or $c + 2$ leaves are gained, as necessary, corresponding to the description above. It can be seen that the vertices in the configuration are spanned by the forest and that no cycle is formed. Goobers will still have degree at most 2. Cases that are identical by symmetry to ones shown are not listed.

The graph G^S typically will still contain some goobers. Our next task is to replace all goobers by the original ordinary vertices while extending the spanning forest by an appropriate number of leaves.

First suppose a goober g was created by contracting a triangle, i.e., by Reduction (1) or (3). Depending on whether the degree of g is 1 or 2 in G^S, we have 2 or 1 leaves in the forest after restoring the triangle. Either way, we gain one leaf while adding 3 ordinary vertices. Specifically, if the triangle corresponding

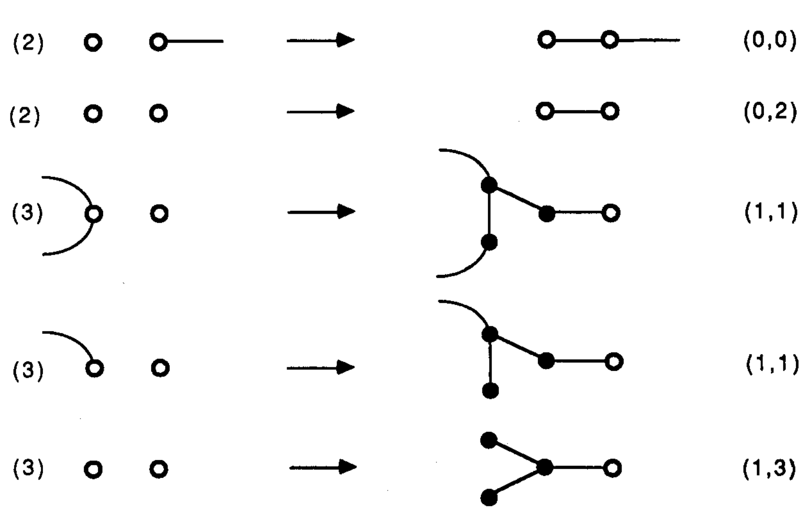


FIGURE 5. Merging Trivial Components

to g is vwx, and if there is just *one* edge in G^S to g from a vertex a that resulted by reducing the edge av, then use the edges av, vw, vx in the forest. If instead g is adjacent to *two* vertices, a and b, that resulted by reduction from edges av and bw, then use the edges av, vw, bw, vx in the forest after restoration.

Next consider goobers created by Reductions (4), (5), or (6). Such goobers occur in pairs. We shall replace each such pair of goobers by the original 6 ordinary vertices. In Figure 6, we see how to expand the forest to span the 6 new vertices, without forming cycles or connecting different components, while gaining 2 leaves in every case. In the event that the two goobers in the pair are from different components in G^S, it is not the case for (4) or (5) that the 6 ordinary vertices and 2 leaves gained are divided equally between the two components. But it is true that such additions are made if one views the forest globally over all its components rather than locally over each separate component. The design of Figure 6 is similar to Figure 5.

To restore all vertices of G, it remains to successively restore vertices deleted by applications of Reduction (7). We describe in Figure 7 the procedure to expand the spanning forest to reach all 12 vertices restored by reversing Reduction (7) so that the number of leaves is increased by 4.

After all of the ordinary vertices have been restored, we have a spanning forest F for G in which every component has at least 2 more leaves than one-third the number of vertices (unless G is itself irreducible, so that Theorem 3 implied directly that $L(G) \geq (n/3) + (4/3)$). If the forest F is not connected, say it has k components. Then we can successively add $k - 1$ edges, $e_1, \ldots, e_{k-1}$ such that for all i, edge e_i joins vertices in separate components of $F \cup \{e_1, \ldots, e_{i-1}\}$. The edge e_i can at worst destroy 2 leaves, but we have at least 2 leaves to spare for every component of F. So after adding all $k - 1$ edges, there remains a spanning tree for G with at least $(n/3) + 2$ leaves.

This completes the proof of Theorem 2 from Theorem 3. ∎

We now proceed to prove Theorem 3.

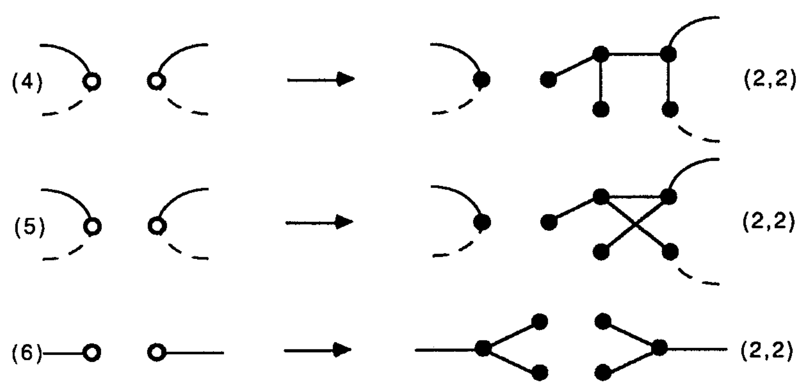


FIGURE 6. Restoring Vertices from Goobers

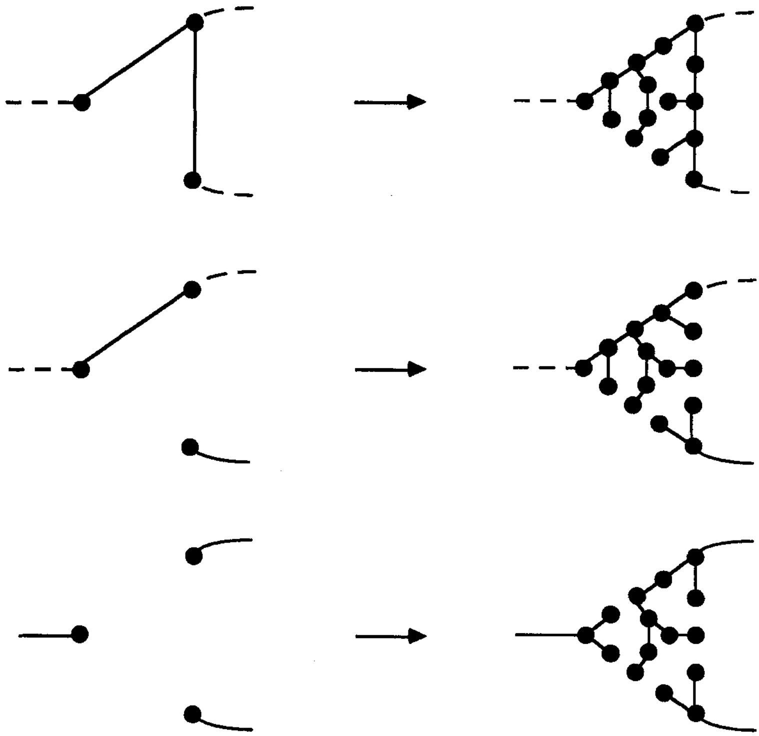


FIGURE 7. Restoration of Reduction (7)

4. PROOF OF THEOREM 3

We use the dead leaf approach as in the proof of Theorem 1. This time, if the triple $S = (N, L, D)$ represents the numbers N of ordinary vertices added, L of leaves gained, and D of dead leaves gained, then we set $\Delta(S) = \Delta(N, L, D) = 2.5L + .5D - N$. Over the entire graph H we seek $\Delta \geq 4$ if $b = 0$ and $\Delta \geq 6$ if $b > 0$. At the end, $2.5L + .5D = 3L$ is integral, so Δ is integral. Thus when we prove $\Delta \geq 5.5$ if $b > 0$, it will be sufficient to prove the theorem. We first show how to start by finding a tree in H for which $\Delta = 4$ if $b = 0$ while $\Delta \geq 5.5$ if $b > 0$. Then we show that by starting with any tree in H we can always find a way to extend the tree such that $\Delta \geq 0$ for the extension. The theorem then follows.

4.1. The Initial Stage

First suppose that $b = 0$, i.e., H has no goobers. Thus, H is connected, cubic, and contains no $K_4 - e$ nor any reducible configuration. Let v a vertex in H. There exists some $w \sim v$ such that v, w belong to no triangle. Let T be the tree of all 5 edges that involved either v or w. Then we have $\Delta = \Delta(6,4,0) = 4$, as required.

Suppose instead that $b > 0$ in H. By hypothesis, the number of ordinary points $a > 0$. Since H is connected, there exist an ordinary vertex v and a goober g in H with $v \sim g$. If there exists another goober $h \sim v$, let T consist of all three edges incident at v, which gives $\Delta \geq \Delta(2,3,0) = 5.5$, as required. Otherwise, $v \sim w, x$, which are both ordinary. By Reduction (3), it must be that $w \not\sim x$. The goober g has degree at most 2, so at least one of w and x, say w, is not adjacent to g. Therefore w splits, i.e., $w \sim y, z$, which are two new vertices. Include all 5 edges that contain v or w in T. If either of y and z is a goober, we are finished, since then $\Delta \geq \Delta(4,4,0) = 6$.

Suppose instead that y and z are ordinary points. If any of x, y, z is adjacent to g, then g is a dead leaf in T, so $\Delta = \Delta(5,4,1) = 5.5$, which is good enough. Similarly, if any one of x, y, z is adjacent to the other two, it is a dead leaf and $\Delta = 5.5$. There remains the case that none of x, y, z is adjacent to g or to both of the other two. Hence one of them, e.g., y, splits into two new vertices p and q. By adding the edges yp and yq to T we obtain that $\Delta \geq \Delta(7,5,0) = 5.5$, which is good enough. This proves that we can always carry out the initial stage of the proof.

4.2. Simple Stages

Given any partial tree T in H, we shall describe how to carry out an additional stage that enlarges T while $\Delta \geq 0$ for the addition. We shall not need to distinguish the cases $b > 0$ and $b = 0$. In this subsection, the simplest extensions are described.

First, suppose that there exists $v \in T$ such that $v \sim w, x$, where $w, x \notin T$. Then it suffices to add vw and vx to T, gaining one leaf at the cost of adding at most two ordinary vertices. Thus, $\Delta \geq \Delta(2,1,0) = .5$. Whenever possible, we carry out such an extension, so assume henceforth it is impossible.

Next suppose that some internal vertex (not a leaf) v in T is adjacent to some $w \notin T$. Then if we add vw to T, it gives $\Delta \geq (1,1,0) = 1.5$. We may assume henceforth that internal vertices in T are adjacent only to vertices in T, while leaves in T have at most one neighbor outside T. Further, no goobers outside T are adjacent to T, or else they could be added to extend T, so that $\Delta \geq \Delta(0,0,0) = 0$.

Now consider any ordinary vertex $v \notin T$ that has all three of its neighbors, $w, x, y \in T$. Then adding vw to T creates dead leaves at v, x, and y, so that $\Delta = \Delta(1,0,3) = .5$. Therefore, we may assume ordinary vertices outside T have at most two neighbors in T.

4.3. Some Vertex v Is Once Adjacent to T

In this case we suppose there exists an ordinary vertex $v \notin T$ that is adjacent to precisely one vertex in T, say $v \sim t, w, x$ where $t \in T$, and $w, x \notin T$. We shall add edges tv, vw, and vx to T. If either w or x is a goober, then we have $\Delta \geq \Delta(2,1,0) = .5$, while if either w or x is adjacent to T, a dead leaf is created in T, so this gives $\Delta \geq \Delta(3,1,1) = 0$. If either w or x splits into two new vertices, say $w \sim y, z$, then also add wy and wz to T, giving $\Delta \geq \Delta(5,2,0) = 0$.

It remains to treat the case that neither w nor x is a goober, $w, x \not\sim T$ (i.e., neither is adjacent to T), and $w \sim x$. There exists $y \sim w, y \notin T$. Since vwx is a triangle, $y \not\sim x$ (H contains no $K_4 - e$) and y is not a goober. (Reduction (3) would apply, but H is irreducible.) If y is twice adjacent to T, adding wy to T would produce 3 dead leaves, so that $\Delta = \Delta(4,1,3) = 0$. Next consider y, that is adjacent only once to T, say $y \sim s \in T$ and $y \sim z \notin T$. Then we do not add vt, vx to T, but instead we add all 5 edges involving w or y, as shown in Figure 8, giving $\Delta \geq \Delta(5,2,1) = .5$. In the figure, the thin line indicates an edge in H that is not used in extending T.

Next we consider the case that $y \not\sim T$, say $y \sim z, a \notin T$. If either of z or a is a goober, then adding the 6 edges that contain either v or y gives $\Delta \geq \Delta(5,2,0) = 0$. Therefore we may assume z and a are ordinary. By Reduction (1), it must be that $z \not\sim a$. If each of z and a fails to expand, i.e., if each is adjacent either to T or to x, then adding the same 6 edges containing v or y creates at least 3 dead leaves. In this case, $\Delta \geq \Delta(6,2,2) = 0$. So we may next assume that at least one of a and z splits, say $z \sim b, c \notin T$. Then expand T as shown in Figure 9. If b or c is a goober, we have $\Delta \geq \Delta(7,3,0) = .5$, so assume both are ordinary. Consider the three points a, b, and c. If any one of them is adjacent to T or x, the extra dead leaf created gives $\Delta \geq \Delta(8,3,1) = 0$. Similarly, if any one of a, b, c is adjacent to the other two, this gives a dead leaf. There remains the case that there is at most one edge between vertices a, b, c, and the other edges for a, b, c go to new vertices. In particular, one of a, b, c splits into two new vertices, and the addition of the corresponding two edges gives altogether $\Delta \geq \Delta(10,4,0) = 0$. This completes the construction for v adjacent just once to T.

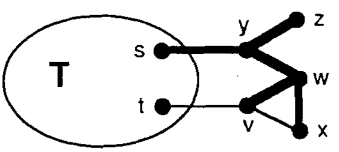


FIGURE 8. The Case $y \sim s \in T, y \sim z \notin T$

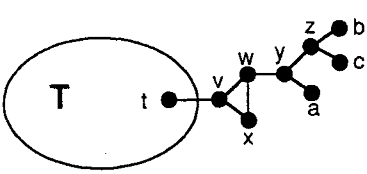


FIGURE 9. $z \sim b, c \notin T$

4.4. v Is Twice Adjacent to T but not Adjacent to a Triangle

From the subsections above, it remains to consider the case that every vertex outside of and adjacent to T is an ordinary vertex and is adjacent to T precisely twice. Unless T spans all of H, which would complete the construction, there exists such a vertex $v \sim T$, say $v \sim t, s \in T$. There exists $w \sim v$, $w \notin T$. We shall add tv, vw to T.

Assume for now that w is an ordinary point. First suppose $w \sim T$. Then w must be adjacent to T twice, say $w \sim r, q$. Then we have gained dead leaves at w, s, r, q and $\Delta = \Delta(2, 0, 4) = 0$. Hence, we may assume instead $w \not\sim T$, so that w splits and $w \sim x, y \notin T$. We add wx, wy to T as well. If either of x or y is adjacent to T, we again get 4 dead leaves, so that $\Delta \geq \Delta(4, 1, 4) = .5$. Alternately, if either x or y is a goober, we have $\Delta \geq \Delta(3, 1, 1) = 0$. We can therefore assume x and y are ordinary points not adjacent to T. If $x \sim y$, then wxy forms a triangle of ordinary points, and there requires a difficult argument which we postpone to Sections 4.5–4.7.

We then may assume that $x \not\sim y$. So x splits, and $x \sim z$, $a \notin T$. We add xz and xa to T as well, as shown in Figure 10. We now argue similarly to the last part of Section 4.3, Figure 9, where now the three points are y, z, a. This takes care of the case that w is ordinary and not part of a triangle outside T.

It remains to consider the case that w is a goober. If w has degree 1 in H, then adding tv, vw to T creates dead leaves at s, w, so that $\Delta = \Delta(1, 0, 2) = 0$. Suppose instead that w is a goober of degree 2 in H. Then there exists $x \notin T$, $x \sim w$. By Reduction (2), x must be ordinary. By Reduction (3), x cannot belong to a triangle of ordinary points. Now add wx to T as well. In passing through the goober w, no contribution is made to Δ at all. Indeed, the entire argument above, where w was ordinary but not part of a triangle of ordinary points, carries over here except the role of w above is played by x here. So the tree T can always be extended in this case.

4.5. v Is Adjacent to a Triangle

The proof has been reduced to the case that, along with the assumptions at the start of Section 4.4, wxy is a triangle of ordinary points. No point is adjacent just once to T, and H contains no $K_4 - e$, so there exist $z, a \notin T$ with $z \sim x$, $a \sim y$. By Reduction (3), both z and a are ordinary. If both z and a are adjacent to T, then extending T by tv, vw, wx, xz, wy, ya gives $\Delta = \Delta(6, 1, 7) = 0$. So assume for the remainder that not both $z, a \sim T$, say $z \not\sim T$. Since $z \not\sim T$, z splits so that $z \sim b, c \notin T$, as shown in Figure 11. If b or c is ordinary and

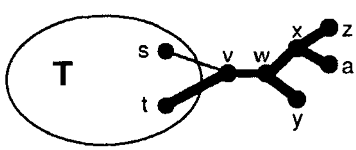


FIGURE 10. $x \sim z, a$

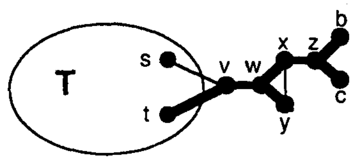


FIGURE 11. z Splits

adjacent to T, say $b \sim T$, then the expansion shown in Figure 11 has $\Delta \geq \Delta(7, 2, 4) = 0$. So we may assume that $b, c \not\sim T$. Next consider that b or c, say b, is a goober. Then $b \not\sim y$ due to Reduction (3). If b has degree 1 in H, or if $b \sim c$, then b is a dead leaf in Figure 11, giving $\Delta \geq \Delta(6, 2, 2) = 0$. If c is also a goober, then $\Delta \geq \Delta(5, 2, 1) = .5$. Consider instead what happens when c is ordinary and $c \not\sim b, T$. If $c \sim y$, then vertices v, w, x, y, z, c create the configuration of Reduction (4), which is impossible since H is reducible. Therefore it must be that c splits, say $c \sim d, e$. Adding the edges cd and ce to Figure 11, we would gain a new leaf, and then $\Delta \geq \Delta(8, 3, 1) = 0$. Therefore, we can handle the case that b or c is a goober.

We may assume then for the rest of the proof that b and c are ordinary points not adjacent to T. By Reduction (1) $b \not\sim c$ while by Reduction (4), $b, c \not\sim y$. So each of b and c splits, but not necessarily disjointly. Let d, e be the other neighbors of b, and add bd, be to Figure 11. If either d or e is a goober, this gives $\Delta \geq \Delta(8, 3, 1) = 0$, so it remains to suppose that d and e are ordinary. If d or e is adjacent to T, then $\Delta \geq \Delta(9, 3, 4) = .5$, so it remains to suppose that $d, e \not\sim T$.

At this time it is useful to reconsider the vertex $a \sim y$. Recall that a is ordinary. Nothing prevents a from being d or e at this stage. But suppose instead for now that $a \sim T$. Then $a \neq d, e$, so the addition of ya to T gives us $S = (10, 3, 4)$, with $\Delta(S) = -.5$, not quite enough. Refer to Figure 12. In this case, we operate on the three points c, d, e as before in Section 4.3, that is, either one of the three is adjacent to the other two, so is a dead leaf, or else one of the three splits and creates a new leaf. Either way, we gain the necessary .5 to achieve $\Delta \geq 0$.

It remains to consider the case that $a \not\sim T$, which we assume hereafter. But then all of the arguments we have been using in this section to extend out from z can be applied similarly to extend out from a. This observation will prove useful to us later on in the proof.

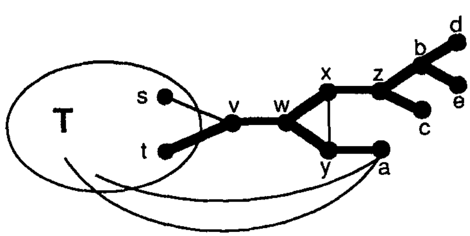


FIGURE 12. $a \sim T$

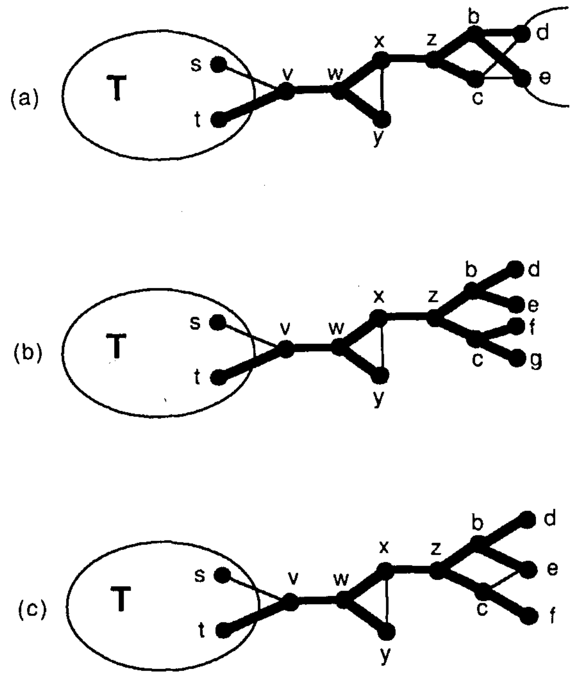


FIGURE 13. The Remaining Cases

We now return to the expansion out from vertex z. We saw that it could be assumed that both b and c split. Arguing as with b, it may be assumed that c also splits into two ordinary vertices not adjacent to T. We divide the remaining possibilities into three cases. The first is that c also splits into d and e, i.e., $b, c \sim d, e$. This is shown in Figure 13(a). The second case is that $c \not\sim d, e$, so that $c \sim f, g$, as shown in Figure 13(b). The third case is that c is adjacent to one of d, e say $c \sim e$ and $c \sim f$, as shown in Figure 13(c). Of course, any of d, e, f, or g could be adjacent to y, i.e., coincide with vertex a defined earlier.

We can immediately dispense with the first of these cases, Figure 13(a). Suppose d and e have no third neighbor in common besides b and c. Then we could apply Reduction (5) to the vertices x, z, b, c, d, e, which is not possible since H is irreducible. Therefore, there exists $g \sim d, e$ with $g \neq b, c$. By Reduction (6) on z, b, c, d, e, g, the new vertex g cannot be ordinary, so it must be a goober. Then extend T by the solid lines in Figure 13(a) together with the edge dg. One then computes that $\Delta = \Delta(9, 3, 4) = .5$.

The two remaining cases are rather more involved. Section 4.6 treats Figure 13(b) while Section 4.7 is devoted to Figure 13(c). We shall see that these two cases are not independent, but instead 4.6 is required for 4.7.

4.6. The Case in Figure 13(b)

It would appear in Figure 13(b) that we are almost through with it, since $S = (11, 4, 1)$ and $\Delta(S) = -.5$, just a hair away from working. This is deceptive.

Indeed, it is so tricky to fully resolve this case that it is surprising it can even be done.

Many possibilities are easy since they increase Δ by at least .5, which is all that is required. We already assume that d, e, f, g are ordinary points not adjacent to T. We are finished if any of them is adjacent to y (giving a dead leaf at y), if any of them splits to two new vertices (giving a new leaf), or if any one of them is adjacent to some two of the other three (giving a new dead leaf). It remains to consider the situation that each of d, e, f, g is adjacent to one of the others. By relabeling, it can be assumed that the matching on d, e, f, g, is one of the two shown in Figure 14. We treat these two major cases separately, beginning with the configuration in Figure 14(a).

4.6.1. The Case in Figure 14(a)

By arguments presented in Section 4.5, it follows that a is adjacent to two ordinary points outside T besides y. Suppose a is adjacent to two points among d, e, f, g. Because $K_4 - e$ is not allowed in H, we cannot have either $a \sim d, e$ (both) or $a \sim f, g$ (both). So we may assume, say, $a \sim e$ and $a \sim g$. Then extending T in the obvious way with leaves at d, e, f, g, a gives $\Delta = \Delta(12, 4, 4) = 0$.

Next consider the case that a is adjacent to just one of d, e, f, g, say $a \sim g$, so that we also have $a \sim h$, where h is a new ordinary vertex. Suppose that h is adjacent to d, e, or f. If $h \sim f$, then z, c, f, g, a, h could have been reduced by Reduction (4), so this is not possible. Since $K_4 - e$ is forbidden, it cannot be that $h \sim d, e$ (both). Thus, h is adjacent to exactly one of d, e, say $h \sim e$, and also $h \sim i$, a new vertex that must be ordinary (by what we assume when extending outward from y). In Figure 15 it is shown how to deal with this situation to achieve $\Delta = \Delta(14, 5, 4) = .5$.

Still assuming that $a \sim g$, $a \sim h$, we must next discuss the case that $h \not\sim d, e, f, g$, say h splits with $h \sim i, j$, two new vertices not shown in Figure 14(a). Extending T out from y instead of x, we are finished by arguments in the last section, unless i and j are ordinary and not adjacent to T, i.e., Figure 13(b)

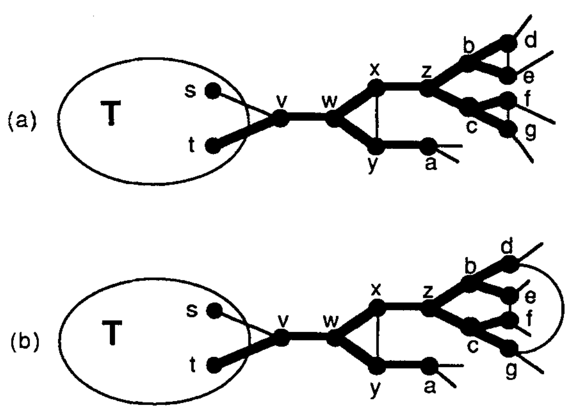


FIGURE 14. There is a Matching on d, e, f, g

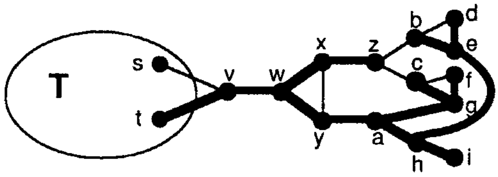


FIGURE 15. $a \sim g$, $a \sim h$, $h \sim e$

applies working out from y. Then by the discussion at the start of this section, we are finished unless there is a matching on the vertices c, f, i, j. If so, since $c \sim f$, it must be that $i \sim j$ and $i, j \not\sim c, f$. This configuration is shown in Figure 16. One can compute that the best we can do now is $S = (15, 5, 2)$ and $\Delta(S) = -1.5$, even worse than Figure 13(b).

In Figure 16 we have that f is not adjacent to any other vertices besides c and g, so there is a new vertex $k \sim f$. By Reduction (3), k cannot be a goober. Suppose the other two neighbors of k are outside T but are among vertices shown in Figure 16, i.e., among d, e, i, j. Since $K_4 - e$ is forbidden, we may assume $k \sim e, i$. In Figure 17 the current situation is displayed. The graph to the right of w, including w itself, is isomorphic to configuration (7) in Figure 3. So the edge leaving d and the edge leaving j either join or meet at an ordinary vertex since otherwise we could have used Reduction (7). Suppose first these two edges join, i.e., $d \sim j$. An extension is shown in Figure 18(a) for this case, which has $\Delta = \Delta(16, 5, 7) = 0$. On the other hand, if there exists an ordinary vertex $l \sim d, j$, let m be the third neighbor of l. An extension of T all the way out to m is shown in Figure 18(b) in which $\Delta \geq \Delta(18, 6, 7) = .5$.

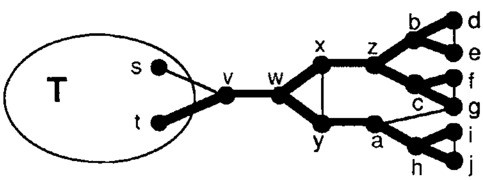


FIGURE 16. h Splits

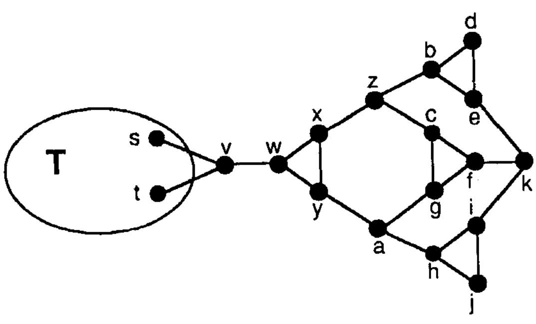


FIGURE 17. $f \sim k$, $k \sim e, i$

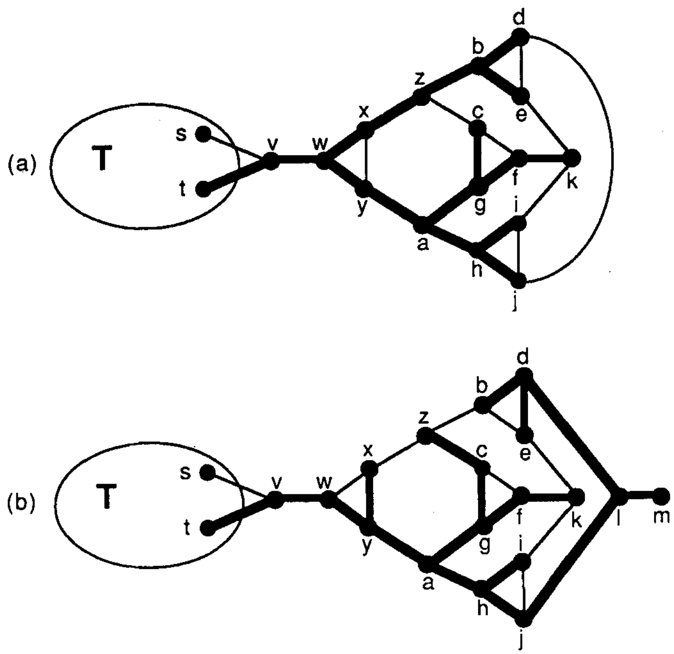


FIGURE 18. Extensions for Fig. 17

Next we may consider the case that vertex k is adjacent to just one of d, e, i, j. By relabelling we may assume $k \sim i$ and that there exists a new vertex $l \sim k$. In this case an appropriate extension of T exists, shown in Figure 19. It has $\Delta \geq \Delta(17, 6, 4) = 0$.

Next suppose that vertex k is not adjacent to any of d, e, i, j. It could be that $k \sim T$. In this case, there seems to be no appropriate extension of T that uses all labeled vertices, but there is one that uses fewer vertices. It is shown in Figure 20, and it has $\Delta = \Delta(12, 4, 4) = 0$.

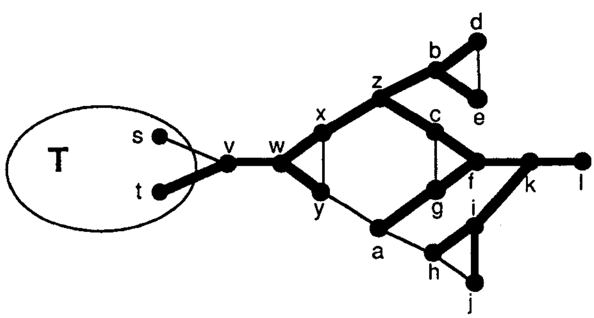


FIGURE 19. k is Once Adjacent to d, e, i, j

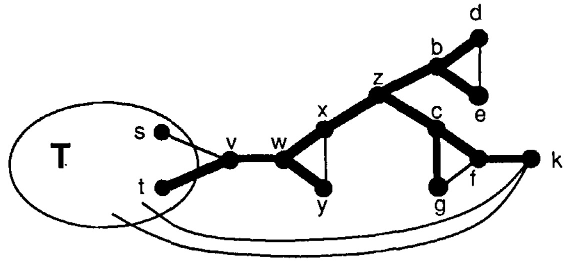


FIGURE 20. $k \sim T$

The remaining possibility for k is that it be ordinary and not adjacent to T or to any labeled vertex besides f. Then it splits, so there exist new vertices $l, m \sim k$. The obvious tree for Figure 16 extended by the edges involving k has $S = (18, 6, 2)$, which is not very good yet. If l and m are each adjacent twice to d, e, i, j, we pick up 6 more dead leaves, so $\Delta = \Delta(18, 6, 8) = 1$, which is suitable. If this is not the case, then at least one of l and m fails to be adjacent to a vertex in at least one of the pairs d, e and i, j. We may assume $m \not\sim d, e$. Consider the tree shown in Figure 21, in which several labelled vertices have been omitted. If either l or m is a goober, the tree has $\Delta \geq \Delta(13, 5, 1) = 0$, while if either l or $m \sim T$, it gives $\Delta \geq \Delta(14, 5, 4) = .5$. We already know that $m \not\sim y, g, d, e$. Further, $m \not\sim l$ by Reduction (1). So it remains here to consider the case that m splits in Figure 21 into two vertices, call them n and o. If we add mn and mo to the tree shown, it gives $\Delta \geq \Delta(16, 6, 1) = -.5$. Then we may argue as usual on the three leaves l, n, o to produce an extension of T with $\Delta \geq 0$.

Finally consider the case that $a \not\sim d, e, f, g$, so that a splits into new ordinary vertices h and i. If h and i are each adjacent to d, e, f, or g, then the tree in Figure 14(a) together with edges ah and ai gives $\Delta \geq \Delta(14, 5, 3) = 0$. If h or i is a goober, we have $\Delta \geq \Delta(13, 5, 1) = 0$. On the other hand, suppose h and i are ordinary and at least one of them, say h, is not adjacent to any of d, e, f, g. Then h splits into new vertices j and k. Adding ah, ai, hj, hk to the tree in Figure 14(a) gives $\Delta \geq \Delta(16, 6, 1) = -.5$. Then we can argue on the three leaves i, j, k as usual to extend T with $\Delta \geq 0$.

4.6.2 The Case in Figure 14(b)

As with the beginning of Section 4.6.1, we may assume in Figure 14(b) that a is adjacent to two ordinary points outside T. If a is adjacent to any two

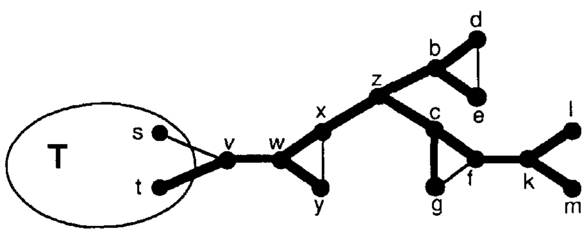


FIGURE 21. $m \not\sim d, e$

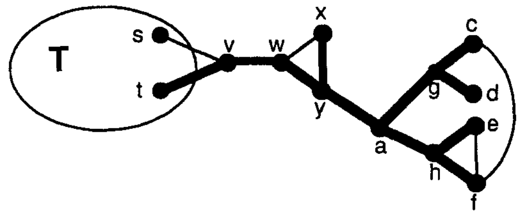


FIGURE 22. $a \sim g$, $h \sim e, f$

of d, e, f, g, then the obvious extension of T for Fig. 14(b) with leaves at d, e, f, g, a has $\Delta = \Delta(12, 4, 4) = 0$.

We next consider the case that a is adjacent to just one of d, e, f, g. By symmetry, we may assume $a \sim g$. Then there exists a new vertex adjacent to a, call it h, which be ordinary. We treat several cases depending on how h relates to d, e, f. First suppose $h \sim e, f$. Extending outward from y rather than x produces Figure 22. The tree in Figure 22 has $\Delta = \Delta(11, 4, 2) = 0$. Suppose next that we have $h \sim d, f$. Then Figure 23 shows a suitable extension with $\Delta = \Delta(13, 4, 6) = 0$. The case that $h \sim d, e$ is similar to the last one. So now suppose h is adjacent to just one of d, e, f, so that also $h \sim i$, which is an ordinary point by the argument in Section 4.5 concerning points near a. Treating each case separately, first suppose $h \sim d$. Figure 24(a) presents an extension with $\Delta \geq \Delta(14, 5, 4) = .5$. The case $h \sim e$ is treated by Figure 24(b), in which $\Delta \geq \Delta(14, 5, 5) = 1$. The case $h \sim f$ is similar to $h \sim e$. Finally, if h is adjacent to none of d, e, f, then it splits, say $h \sim i, j$, and there is an extension with $\Delta \geq \Delta(13, 5, 2) = .5$ shown in Figure 24(c). This completes the cases in which a is adjacent to any of d, e, f, g.

Next suppose $a \nsim d, e, f, g$, so that a splits into new ordinary vertices h and i in Figure 13(b). If any two of d, e, f, g are adjacent either to h or i, they become dead leaves in the obvious tree extension, which yields $\Delta \geq \Delta(14, 5, 3) = 0$. Otherwise, at least one of h, i, say h, is not adjacent to any of d, e, f, g, which means it splits into vertices j and k. If any of i, j, k is a goober, or adjacent to T, or adjacent to any of d, e, f, g, then we have $\Delta \geq 0$. Otherwise, can pick up at least .5 by treating the triple i, j, k as in Section 4.3, Figure 9. This completes the treatment of Figure 13(b).

4.7. The Case in Figure 13(c)

Referring to Figure 13(c), Section 4.5, we claim that $e \nsim d, f$. For suppose, say, that $e \sim d$. There is a third neighbor of d, call it g, which may or may not

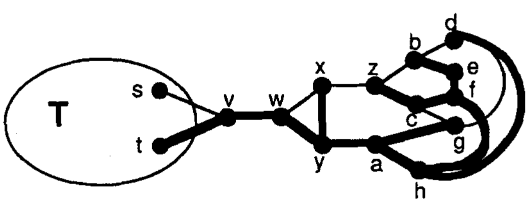
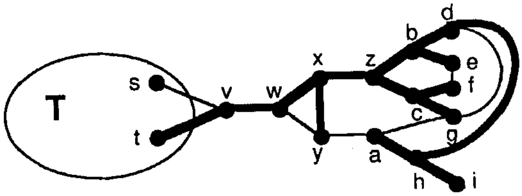
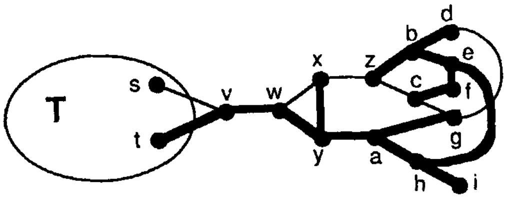


FIGURE 23. $a \sim g$, $h \sim d, f$



(b) h ~ e



(c) h ≁ d,e,f

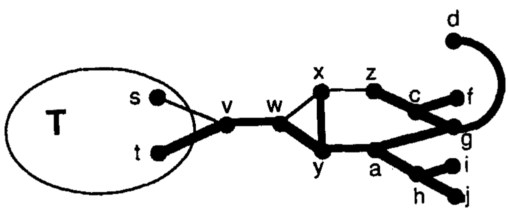


FIGURE 24. $a \sim g$, $h \sim$ at most one of d, e, f

be new. It cannot be that g is goober, due to Reduction (3). But g cannot be ordinary either, or else Reduction (4) could be applied to g, d, b, e, z, c. So our claim holds.

If $e \sim y$, then we can extend T as shown in Figure 25 to achieve $\Delta = \Delta(9,3,3) = 0$. It should be pointed out that vertex f is not involved. We may assume for the remainder that $e \not\sim y$.

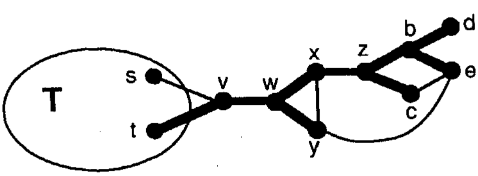


FIGURE 25. $e \sim y$

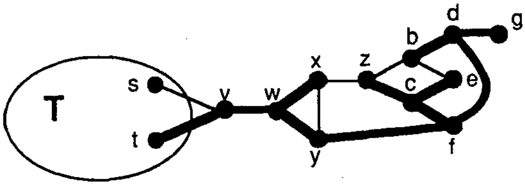


FIGURE 26. $f \sim y, d$

If $f \sim y$, it is also possible to extend T. First, consider the case that we also have $f \sim d$. Then d splits to a third vertex, call it g, which cannot be any of the others in Figure 13(b). So T can be carefully extended as shown in Figure 26 to attain $\Delta \geq \Delta(11, 4, 4) = 1$. Second, consider the case that $f \sim y$ and $f \nsim d$. Then f must split to a third vertex, again call it g, which does not appear in Figure 13(b). Since $f \sim y$, f is the vertex called a, which we learned about in Section 4.5. It follows from this information about $f = a$ that g is ordinary and $g \nsim T$. Further, all neighbors of g are ordinary points. Suppose that $g \nsim e$. Then g splits into vertices, all them h and i, which do not appear elsewhere in Figure 13(c) except possibly at d, which we disregard for now. So growing outward from y instead of x gives us Figure 27. There may be other edges on these vertices in Figure 27, which has been redrawn so that its isomorphism to Figure 13(b) is evident. Therefore, we can continue outward from y (instead of x in Figure 13(b)) and extend T as described in Section 4.6. (For this reason, we needed to treat Figure 13(b) before Figure 13(c).)

We are still assuming $f \sim y$ and $f \nsim d$. Above we dealt with the case that $g \nsim e$. Now assume that $g \sim e$. If it also happens that $g \sim d$, then only d among the labeled vertices outside T does not have all of its neighbors described. So there exists another vertex $h \sim d$. In Figure 28, we present an extension for this case with $\Delta \geq \Delta(12, 4, 5) = .5$. We last assumed $g \sim d$. Instead, consider the case $g \nsim d$, so there exists a new vertex $h \nsim g$. The extension shown in Figure 29 has $\Delta \geq \Delta(12, 4, 4) = 0$.

We have treated now all cases with $e \sim y$ and $f \sim y$, and the cases $d \sim y$ are the same as $f \sim y$ up to relabeling (refer to Figure 13(c)). Assume for the remain-

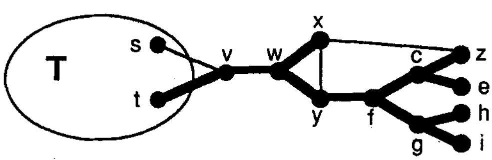


FIGURE 27. $f \sim y$, $f \nsim d$, $g \nsim e$

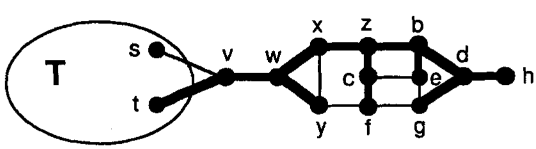


FIGURE 28. $f \sim y$, $f \nsim d$, $g \sim e$

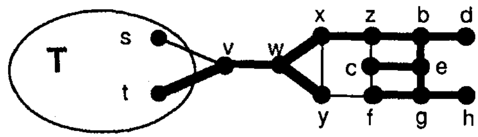


FIGURE 29. $f \sim y, f \not\sim d, g \sim e, g \not\sim d$

der that $d, e, f \not\sim y$. Suppose next that $d \sim f$. There must be a third neighbor of d, call it g. Then the extension shown in Figure 30 gives $\Delta \geq \Delta(11, 4, 2) = 0$.

It remains to proceed from Figure 13(c), where no two of y, d, e, f are adjacent. It must be that d splits, say $d \sim g, h$. Ignoring f for the moment, the tree with leaves at y, c, e, g, h has $S = (11, 4, 1)$ and $\Delta(S) = -.5$, just short of working. Indeed, if either g or h is a goober, is adjacent to T, is adjacent to y or e, or splits, then we pick up the extra .5 we need in $\Delta(S)$. The remaining case is that g and h are ordinary and $g \sim h$. Similarly, working out from f and disregarding d, the remaining case has that f splits into two adjacent ordinary points. Since H contains no $K_4 - e$ it cannot be that $f \sim g, h$. Therefore, f splits into new points, say $f \sim i, j$ where $i \sim j$. As with g, h, we may assume that $i, j \not\sim y$. Therefore, a new vertex, call it a, splits out from y, as shown in Figure 31.

Unless H expands from a in a way that is isomorphic to the way it expands from z, we are finished by one of the earlier cases in this proof. Therefore, we may suppose the expansions from z and a are isomorphic, although not necessarily disjoint. In particular, the two neighbors of a besides y belong to a 4-cycle along with a.

We claim that $a \not\sim g, h, i, j$. For if this were not so, we could assume by symmetry that $a \sim j$. A 4-cycle through a, j could only be a, j, f, i (which creates $K_4 - e$ in H) or a, j, i, k for some new point $k \sim a$ (which creates Reduction (4) on c, f, i, j, k, a). Either way, it is not allowed.

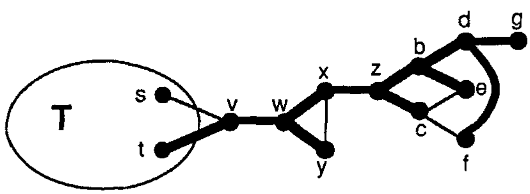


FIGURE 30. $d \sim f$

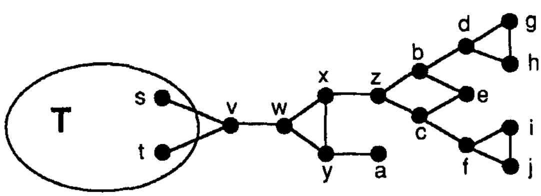


FIGURE 31. No Two of y, d, e, f are Adjacent

We can also exclude $a \sim e$ because no 4-cycle could pass through a, e. Therefore, a splits into new vertices k, l. Then a new vertex m must be adjacent to k, l to create the 4-cycle through a, so that $m \sim k, l$. Next recall that for k, l (working out from a) to be like b, c (working out from z), it must be that k and l are each adjacent to some triangle. Consider the triangle next to k. If it contained any points from the z-side, they would have to be among d, g, h and f, i, j. By symmetry we could suppose $k \sim j$. Then consider the amazing tree extension shown in Figure 32 in which $\Delta \geq \Delta(18, 7, 3) = 1$. There remains the case that $k \nsim g, h, i, j$. Then k splits to a new triangle, say nop where $k \sim n$. Then use Figure 32 except replace triangle fij by nop, using edges kn, no, np in the tree. In Figure 32, the dead leaves at c, f become live leaves at c, o, but we still achieve $\Delta \geq \Delta(18, 7, 1) = 0$, which is just good enough to complete the entire proof of Theorem 3. ∎

5. DIRECTIONS FOR FURTHER STUDY

It clearly would be most enlightening if one could find a proof of Theorem 2 that is not so lengthy, elaborate, and delicate as the one presented here. We are not particularly optimistic that this is possible. Perhaps restricting it to 3-connected cubic G would make it easier.

As we noted earlier, we can show that the cubic graph, Q_3, is the only G in Theorem 2 such that $L(G) = (n/3) + (4/3)$. In view of Theorem 3, any graph G in Theorem 2 that has a reduction using any of the reductions (1)–(6) in Figure 3 satisfies $L(G) \geq (n/3) + 2$. It remains to consider G that are irreducible or can be fully reduced using Reduction (7). One can check the initial cases as in Section 4.1, only being more careful, and show that unless $G = Q_3$, then $\Delta \geq 4.5$ at the start, which forces in the end $L(G) \geq (n/3) + (5/3)$. No graph can be reduced using Reduction (7) to Q_3, as it contains no triangles, so that only Q_3 has $L(G) = (n/3) + (4/3)$.

The next question then is what other graphs besides Q_3 have $L(G) < (n/3) + 2$, i.e., $L(G) = (n/3) + (5/3)$? It suffices to consider irreducible graphs

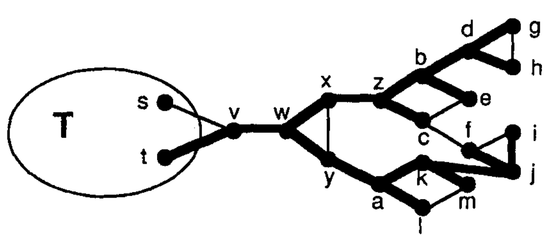


FIGURE 32. $k \sim j$

or graphs that are irreducible after using only Reduction (7). Previously in Figure 3 we saw two examples of such graphs.

The extremal graphs we mentioned for Theorem 2 and its corollaries all contain C_4's. It is reasonable to then restrict our attention to connected cubic graphs that contain no triangle nor C_4. In this case we conjecture that $L(G)$ is at least $\frac{2}{5}n + d$ for some constant d. The idea is that one can pick up two leaves whenever we pass through a C_5. However, we have no family of examples yet to show that an even better bound is impossible. The few examples we studied so far have at least $(n/2)$ leaves.

The most interesting and largest open problem is a conjecture of N. Linial [4] that generalizes Theorem 1: Suppose G is a connected graph that is regular of degree $r \geq 2$. Then there exists some constant d, depending only on r, such that

$$L(G) \geq \frac{r-2}{r+1}n + d.$$

If true, an extremal graph would be the necklace where each bead is $K_{r+1} - e$. For $r = 3$ this inequality is implied by Theorem 1. Kleitman and West [3] extended Theorem 1 to cover all graphs of minimum degree 3. Kleitman and West [3] and Wu [7] independently verified the inequality for $r = 4$. The best-possible value of d is $\frac{8}{5}$. Griggs and Wu [2] have gone on to prove the inequality for $r = 5$ with the best-possible value $d = 2$. Although these proofs get increasingly difficult as r increases, they are not quite as technical as the proof here since no reductions are needed. As with $r = 3$, the results for $r = 4$ and 5 are shown to hold for all graphs of minimum degree r.

For graphs that are not regular, Linial suspects an even stronger bound holds. If a graph G has degree sequence $(d_1 \geq d_2 \geq \cdots \geq d_n \geq 2)$ and is connected, then Linial conjectures that

$$L(G) \geq \sum_i \frac{d_i - 2}{d_i + 1}.$$

Such a lower bound on $L(G)$ in terms of the degree sequence would be analogous to a known bound on the independence number of G due to Wei [6, cf. 1].

ACKNOWLEDGMENT

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References

[1] J. R. Griggs, Lower bounds on the independence number in terms of the degrees. *J. Combinat. Theory Ser. B*, **34** (1983) 22–39.

[2] J. R. Griggs and M. Wu, Spanning trees with many leaves in graphs of minimum degree 4 or 5, submitted (1989).
[3] D. J. Kleitman and D. B. West, Spanning trees with many leaves, submitted (1988).
[4] N. Linial, personal communication (1988).
[5] J. A. Storer, Constructing full spanning trees for cubic graphs. *Info. Proc. Lett.* **13** (1981) 8–11.
[6] V. K. Wei, A lower bound on the stability number of a simple graph. Bell Laboratories Technical Memorandum, 81-1217-9 (1981).
[7] M. Wu, personal communication (1988).