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Operations Research Letters 34 (2006) 269–274

Operations Research Letters

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Exact algorithms for the Hamiltonian cycle problem in planar graphs

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Received 14 January 2005; accepted 11 April 2005
Available online 10 August 2005

Abstract

We construct an exact algorithm for the Hamiltonian cycle problem in planar graphs with worst case time complexity $O(c^{\sqrt{n}})$, where c is some fixed constant that does not depend on the instance. Furthermore, we show that under the exponential time hypothesis, the time complexity cannot be improved to $O(c^{o(\sqrt{n})})$.

Keywords: Hamiltonian cycle; Exact algorithm; Planar separator theorem; Combinatorial optimization

1. Introduction

A *Hamiltonian cycle* in an undirected graph $G = (V, E)$ is a simple cycle that traverses every vertex in V exactly once. The problem of deciding whether a given graph G possesses a Hamiltonian cycle is one of the standard NP-complete graph problems; see for instance Garey and Johnson [6]. Hence, it is highly unlikely that this problem can be solved in polynomial time. Karp [14] (and independently Bax [2]) gave algorithms with a time complexity of $O(n^3 2^n)$ and a polynomial space complexity for this problem; here and throughout the rest of this paper, $n = |V|$ denotes the number of vertices. It is an outstanding open problem whether this time complexity can be improved to $O(1.999^n)$; see, for instance, Woeginger [22]. The Hamiltonian cycle problem becomes substantially easier if we restrict ourselves to the class of planar graphs; this planar variant will be denoted by PLANAR-HC. Although PLANAR-HC is still NP-complete [8], it allows substantially faster solution algorithms. Recursive approaches based on the planar separator theorem of Lipton and Tarjan [15,16] yield algorithms with a time complexity of $O(c^{\sqrt{n}\log n})$, where c is some fixed constant that does not depend on the instance.

In this short technical note, we will remove the $\log n$ factor from the exponent and get a time complexity of $O(c^{\sqrt{n}})$. The main idea is to use Miller's [17] cycle

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doi:10.1016/j.orl.2005.04.013

separator theorem, which helps to clean up the recursion; see Section 2. Furthermore, we will show that it would be quite unlikely to get a time complexity of $O(c^{o(\sqrt{n})})$ where the exponent grows sub-linearly in the square-root of n. This result is derived from the so-called "exponential time hypothesis" (ETH) of Impagliazzo et al. [11]; see Section 3. All our arguments are quite simple, and mainly based on putting together the right pieces from various places in the literature. Our main contribution is to pinpoint the best possible type of time complexity for problem PLANAR-HC (under the ETH): This best possible time complexity is of the form $O(c^{\sqrt{n}})$. The best possible value of the constant c, however, remains unclear.

2. A fast exact algorithm

We discuss the following precedence-constrained generalization GEN-HP of the planar Hamiltonian path problem: An input of GEN-HP consists of a planar graph $G=(V,E)$ with vertex set $V=\{1,\dots,n\}$, a relation $\prec$ on V, and a relation $\to$ on V. The problem is to decide whether G possesses a Hamiltonian path π that starts in vertex 1, ends in vertex n, and satisfies the following two properties: First, $i \prec j$ implies that π visits i before j. Secondly, $i \to j$ implies that π visits i *immediately* before j. Our algorithm is based on Miller's cycle separator theorem.

Proposition 1 (*Miller [17]*). *If $G'=(V,E')$ is an embedded, triangulated* (*every face is a triangle*), *planar graph on n vertices*, *then G' contains a simple cycle C with the following properties*: *C consists of at most $\sqrt{8n}$ vertices. C partitions $G'-C$ into a vertex set A that lies in the region inside of C*, *and into a vertex set B that lies in the region outside of C with $|A|\leqslant 2n/3$ and $|B|\leqslant 2n/3$.* (*Note that there are no edges between vertices in A and vertices in B.*) *Furthermore*, *such a cycle C can be computed in $O(n)$ time.*

We embed the input graph $G=(V,E)$ in the Euclidean plane. We triangulate all faces by adding an appropriate edge set E^* to E. The resulting graph $(V, E\cup E^*)$ satisfies the conditions of Proposition 1, and therefore has a separating cycle C with $|C|=k\leqslant\sqrt{8n}$ as described in the proposition. We stress that the edge set E^* is only needed for finding the cycle C, and that we do *not* add E^* to the input graph G.

We distinguish $2^{k-1}\,k!$ cases that can be solved recursively. Every case is specified by a string $\sigma=\sigma_1\sigma_2\dots\sigma_{k-1}$ of length $k-1$ over the alphabet $\{A,B\}$, and by a permutation $\varphi=\langle\varphi_1,\dots,\varphi_k\rangle$ of the vertices in C. Every case handles a set of potential Hamiltonian paths π that satisfy the following conditions: the Hamiltonian path π visits the vertices in C in the ordering φ. The behavior of the Hamiltonian path π between vertex φ_ℓ and vertex $\varphi_{\ell+1}$ is determined by the letter σ_ℓ in the string σ:

- If $\sigma_\ell=A$, then π visits only vertices from set A between visiting φ_ℓ and $\varphi_{\ell+1}$.
- If $\sigma_\ell=B$, then π visits only vertices from set B between visiting φ_ℓ and $\varphi_{\ell+1}$.

(Note that the case where π visits $\varphi_{\ell+1}$ directly after φ_ℓ is covered by both cases.) If a Hamiltonian path π satisfies these conditions, then π is called *compatible* with σ and φ.

We create two new instances I_A and I_B: The first instance I_A has $A\cup C$ as its vertex set. The edge set consists of the restriction of E to $A\cup C$, together with a number of so-called *extra edges*; these extra edges are all the edges $[\varphi_\ell,\varphi_{\ell+1}]$ for which $\sigma_\ell=B$ holds. The relation $\prec_A$ consists of the restriction of relation $\prec$ to $A\cup C$, together with the total order $\varphi_1\prec\varphi_2\prec\cdots\prec\varphi_k$. The relation $\to_A$ consists of the restriction of $\to$ to $A\cup C$, together with all $\varphi_\ell\to\varphi_{\ell+1}$ for which $\sigma_\ell=B$; note that these additions precisely correspond to the extra edges. The second instance I_B is defined symmetrically around the set B.

Assume that there exists a Hamiltonian path π for G that is compatible with σ and φ. Then this Hamiltonian path π can be divided into a number of maximal subpaths consisting of vertices in $A\cup C$ that alternate with maximal subpaths consisting of vertices in $B\cup C$; these maximal subpaths overlap in the vertices of C. The Hamiltonian path π yields a Hamiltonian path π_A for instance I_A that consists of the maximal subpaths from $A\cup C$ together with the extra edges, and it also yields a Hamiltonian path π_B for instance I_B. Vice versa, if there are Hamiltonian paths π_A and π_B for I_A and I_B, then they can be rearranged into a Hamiltonian path π for G. This naturally leads to a recursive algorithm for GEN-HP.

Here is a crucial observation that cuts down the time complexity of the recursive approach: We do not need to consider all $2^{k-1}\,k!$ possible cases, but only those cases for which the following auxiliary graph is planarly embedded. The auxiliary graph consists of the cycle C together with all the extra edges in I_A embedded in the region inside C and all the extra edges in I_B embedded in the region outside C. (Since the original graph G is planarly embedded, any Hamiltonian cycle π compatible with σ and φ yields a crossing-free traversal of all the extra edges. Therefore, this auxiliary graph must be planarly embedded.)

Lemma 2. *At most $\widetilde{c}^k$ of the cases lead to a planarly embedded auxiliary graph, where $\widetilde{c}$ is an appropriate fixed constant. Furthermore, these cases can be enumerated in $\mathrm{O}(\widetilde{c}^k)$ time.*

Proof. Let $x_1, x_2, \ldots, x_k$ be a clockwise ordering of the vertices along C. Then the extra edges in I_A form a triangulation or part of a triangulation of the region outside C, and the extra edges in I_B form a triangulation or part of a triangulation of the region inside C. The number of combinatorially distinct triangulations of a k-gon is given by the kth Catalan number $\mathrm{CAT}_k = 1/(k+1)\binom{2k}{k}$. Every fixed triangulations has $k-3$ chords, and each of these chords may or may not show up in the auxiliary graph. Therefore, the number of possibilities for the extra edges in I_A is bounded by $2^{k-3}\,\mathrm{CAT}_k < 8^k$, and the number of possibilities for the extra edges in I_B is also bounded by 8^k. Hence the statement in the lemma definitely holds true for $\widetilde{c}=64$.

It is straightforward to generate all possible triangulations and partial triangulations of a k-gon in $\mathrm{O}(\widetilde{c}^k)$ time. □

To summarize, the recursive algorithm for GEN-HP works as follows. If G has at most $n \leqslant 1152$ vertices, then the algorithm simply checks all possible paths by total enumeration. Otherwise, the algorithm computes the cycle C, determines the instances I_A and I_B for each of the $\widetilde{c}^k$ cases in Lemma 2, and solves them recursively. If for one of these cases I_A and I_B both have a compatible Hamiltonian cycle, these two cycles are concatenated into a solution for the original instance. If none of these cases yields a solution, then no solution exists. For $n \geqslant 1152$, the time complexity $T(n)$ for handling an n-vertex instance satisfies

$$T(n) \leqslant \widetilde{c}^k \left(2T\left(\frac{2}{3}n + \sqrt{8n}\right) + \mathrm{O}(n)\right)$$
$$\leqslant \widetilde{c}^{\sqrt{8n}} \left(2\,T\left(\frac{3}{4}n\right) + \mathrm{O}(n)\right).$$

An inductive argument based on this equation yields $T(n) \leqslant c^{\sqrt{n}}$ for an appropriate constant $c < \widetilde{c}^{22}$. Note that the value of c depends on $\widetilde{c}$, but does not depend on n.

If the relations $\prec$ and $\rightarrow$ are empty, problem GEN-HP boils down to the classical Hamiltonian path problem. By checking for every edge $[i, j] \in E$ whether there is a Hamiltonian path from vertex i to vertex j, any instance of the (classical) Hamiltonian cycle problem can be translated into $\mathrm{O}(n)$ instances of the (classical) Hamiltonian path problem. We arrive at the following theorem.

Theorem 3. *PLANAR-HC on n-vertex graphs can be solved in $\mathrm{O}(c^{\sqrt{n}})$ time.*

The running time in Theorem 3 is mainly based on Lemma 2, and the proof of this lemma is mainly based on using Catalan numbers. This use of Catalan numbers to bound the running time of dynamic programming algorithms on planar graphs is quite common in the literature; see for instance the paper [1] by Arora et al. on the weighted planar graph TSP. An input of the weighted planar graph TSP consists of an edge-weighted planar graph G on n vertices/cities. The distance between two vertices is equal to the length of the shortest path in G between these two vertices. The goal is to find the shortest traveling salesman tour under this distance function. We remark that it is straightforward to extend Theorem 3 and to get an $\mathrm{O}(c^{\sqrt{n}})$ time algorithm for the weighted planar graph TSP with n cities.

3. A negative result

An instance of the THREE-SATISFIABILITY problem consists of a set $X = \{x_1, \ldots, x_k\}$ of Boolean variables, and a set $C = \{c_1, \ldots, c_m\}$ of clauses, where each clause consists of three literals over X. The problem is to decide whether there is some truth setting of X that simultaneously satisfies all clauses in C. Since THREE-SATISFIABILITY is an NP-hard problem, it probably cannot be solved in polynomial time. Impagliazzo et al. [11] strengthened this negative statement to the so-called ETH:

(ETH): THREE-SATISFIABILITY cannot be solved in sub-exponential time.

An equivalent, more technical formulation of ETH is the following:

Let d denote the infimum of all real numbers $\delta > 0$ for which THREE-SATISFIABILITY can be solved in time $O(2^{\delta k}\mathrm{poly}(m))$. Then $d > 0$.

Note that ETH implies P≠NP. Impagliazzo et al. [11] and Impagliazzo and Paturi [10] provide evidence that ETH is a reasonable hypothesis. Downey et al. [5] provide more evidence by showing that ETH holds if and only if the parameterized complexity classes $M[1]$ and FPT do not coincide (which is generally believed to be true). Cai and Juedes [3] provide further connections between ETH and parameterized complexity theory. To summarize, ETH seems to be a well-supported and well-accepted hypothesis.

We will now deduce a strong negative result for the planar Hamiltonian cycle problem from ETH. We only need two ingredients from the literature. The first ingredient is the *Sparsification Lemma* from Impagliazzo et al. [11] which states that ETH even holds for sparse instances of THREE-SATISFIABILITY, where the number of clauses is linearly bounded in the number of variables, that is, where $m = O(k)$ holds.

The second ingredient is the NP-hardness reduction from Garey et al. [8] that reduces THREE-SATISFIABILITY to PLANAR-HC. This reduction starts from an arbitrary instance $I = (X, C)$ of THREE-SATISFIABILITY and translates it into a corresponding instance G_I of PLANAR-HC that is embedded in the Euclidean plane: For every variable $x \in X$, the reduction creates a corresponding gadget $\mathrm{GAD}(x)$, a subgraph with $\alpha = 4$ vertices. For every clause $c \in C$, the reduction creates a corresponding gadget $\mathrm{GAD}(c)$, a subgraph with $\beta = 6$ vertices. Furthermore, for every occurrence of variable x in some clause c, there is a gadget $\mathrm{GAD}(x, c)$ that connects $\mathrm{GAD}(x)$ to $\mathrm{GAD}(c)$; this connecting gadget $\mathrm{GAD}(x, c)$ is another graph with a fixed number γ of vertices. Since every clause consists of three literals, there are $3m$ connecting gadgets. The structure described so far is planar, except for crossings of the connecting gadgets $\mathrm{GAD}(x, c)$. Such crossings are resolved by putting an appropriate crossing gadget at the intersection; this crossing gadget is a fixed graph with a fixed number δ of vertices. Since every pair of connecting gadgets crosses at most once, there are at most $9m^2$ crossing gadgets in the construction. To summarize, the reduction in [8] yields a planar graph G_I with at most $\alpha k + \beta m + 3\gamma m + 9\delta m^2$ vertices. The graph G_I has a Hamiltonian cycle if and only if the instance I is satisfiable.

Here is our contribution: We take a sparse instance I of THREE-SATISFIABILITY with $m = O(k)$ clauses, and we perform the reduction of Garey et al. [8] as described in the preceding paragraph for this instance I. This yields a planar graph G_I with $\alpha k + \beta m + 3\gamma m + 9\delta m^2 = O(k^2)$ vertices. Now suppose that there is a sub-exponential algorithm that solved PLANAR-HC on n-vertex graphs in $O(c^{o(\sqrt{n})})$ time, where c is some fixed constant. This algorithm solves PLANAR-HC for the graph G_I in $O(c^{o(k)})$ time, and thereby decides THREE-SATISFIABILITY for instance I in $O(c^{o(k)})$ time. This contradicts ETH for sparse instances of THREE-SATISFIABILITY, and by the results of Impagliazzo et al. [11], it also contradicts ETH.

Theorem 4. *Under the exponential time hypothesis, PLANAR-HC on n-vertex graphs cannot be solved in* $O(c^{o(\sqrt{n})})$ *time.*

4. Discussion

We have shown that the Hamiltonian cycle problem in planar n-vertex graphs can be solved in $O(c^{\sqrt{n}})$ time, and that this type of time complexity is best possible under the ETH. Generally speaking, it seems that $O(c^{\sqrt{n}})$ is the *typical* time complexity for NP-hard graph problems when restricted to the class of planar graphs:

- On the positive side, Lipton and Tarjan [15,16] have shown that all graph optimization problems that can be formulated as non-serial dynamic programming problems (see Rosenthal [20]) can be solved in $O(c^{\sqrt{n}})$ time on planar graphs. As a consequence, problems like 3-colorability of planar graphs and

finding a minimum cardinality vertex cover for planar graphs can be solved in $O(c^{\sqrt{n}})$ time.
- On the negative side, the ETH combined with the classical NP-hardness proofs from the literature usually yields that a time complexity of $O(c^{o(\sqrt{n})})$ is impossible for planar graphs. The classical reductions usually translate a k-variable instance of THREE-SATISFIABILITY into a planar graph with roughly k^2 vertices. Clearly, such a quadratic blow-up yields a lower bound of $c^{\Omega(\sqrt{n})}$. Two examples for this are the reductions of Garey et al. [7] for 3-colorability of planar graphs and for finding a minimum cardinality vertex cover for planar graphs.

The literature contains a number of recent results about designing subexponential-time algorithms for NP-hard problems on planar graphs and their generalizations. Two good surveys for this research branch are Niedermeier [18] and Demaine and Hajiaghayi [4].

Besides the reduction of Garey et al. [8] discussed in Section 3, the literature contains a second NP-hardness reduction for PLANAR-HC: Itai et al. [12] prove that the Hamiltonian cycle problem for grid graphs is NP-hard. The reduction in [12] translates a k-variable instance of THREE-SATISFIABILITY into a grid graph with roughly k^3 vertices. Hence, by combining this reduction with the ETH we conclude that the Hamiltonian cycle problem in n-vertex grid graphs cannot be solved in $O(c^{o(\sqrt[3]{n})})$ time. Open question: Is it possible to get an $O(c^{\sqrt[3]{n}})$ algorithm for this problem?

Papadimitriou [19] proved that the Euclidean traveling salesman problem for n cities is NP-complete. The reduction can be translated into a $c^{\Omega(\sqrt{n})}$ lower bound under the ETH. Hwang et al. [9] design an $O(c^{\sqrt{n}\log n})$ algorithm for the Euclidean TSP. The Ph.D. thesis of Smith [21] contains another $O(c^{\sqrt{n}\log n})$ algorithm for the Euclidean TSP, and the Ph.D. thesis of Kann [13] contains a third algorithm with this time complexity. These three algorithms are based on quite different approaches. Open question: Can we get rid of the logarithmic factor in the exponent?

Acknowledgements

Bettina Klinz acknowledges partial financial support by the Spezialforschungsbereich F 003 “Optimierung und Kontrolle”, Projektbereich Diskrete Optimierung. The research of Gerhard J. Woeginger has been supported by the Netherlands Organisation for Scientific Research (NWO), Grant 639.033.403.

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