POLITECNICO DI TORINO

Electronic and Communications Engineering

Assignment Report 4 - Digital Filtering

Applied Signal Processing Laboratory

01TUMLP

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Exercise 1

In this first exercise I was given the following set of IIR filters:

$$\begin{aligned}
&1.\ B_1 = [1 \quad 1.3 \quad 0.49 \quad -0.013 \quad -0.029] \\
&\quad\ A_1 = [1 \quad -0.4326 \quad -1.6656 \quad 0.1253 \quad 0.2877] \\
&2.\ B_2 = [0.0725 \quad 0.22 \quad 0.4085 \quad 0.4883 \quad 0.4085 \quad 0.22 \quad 0.0725] \\
&\quad\ A_2 = [1 \quad -0.5835 \quad 1.7021 \quad -0.8477 \quad 0.8401 \quad -0.2823 \quad 0.0924] \\
&3.\ B_3 = [1 \quad -1.4 \quad 0.24 \quad 0.3340 \quad -0.1305] \\
&\quad\ A_3 = [1 \quad 0.5913 \quad -0.6436 \quad 0.3803 \quad -1.0091]
\end{aligned} \tag{1}$$

where only one of them is BIBO (Bounded Input Bounded Output) stable. I assumed that all the filters characterize a casual system.
To check which one of them is stable I looked for the one that has all the poles with absolute value less than 1, and consequently has the unitary cycle $|z| = 1$ contained in the ROC (Region of Convergence). To this purpose, I used Matlab roots function to obtain the roots of the denominator, i.e. the poles. As it can be seen in the pole-zero plot in Figure 1, the second filter characterized by the coefficients in eq.(1) is the only to be stable, while both the first and the third filter have at least one pole outside the unitary cycle.

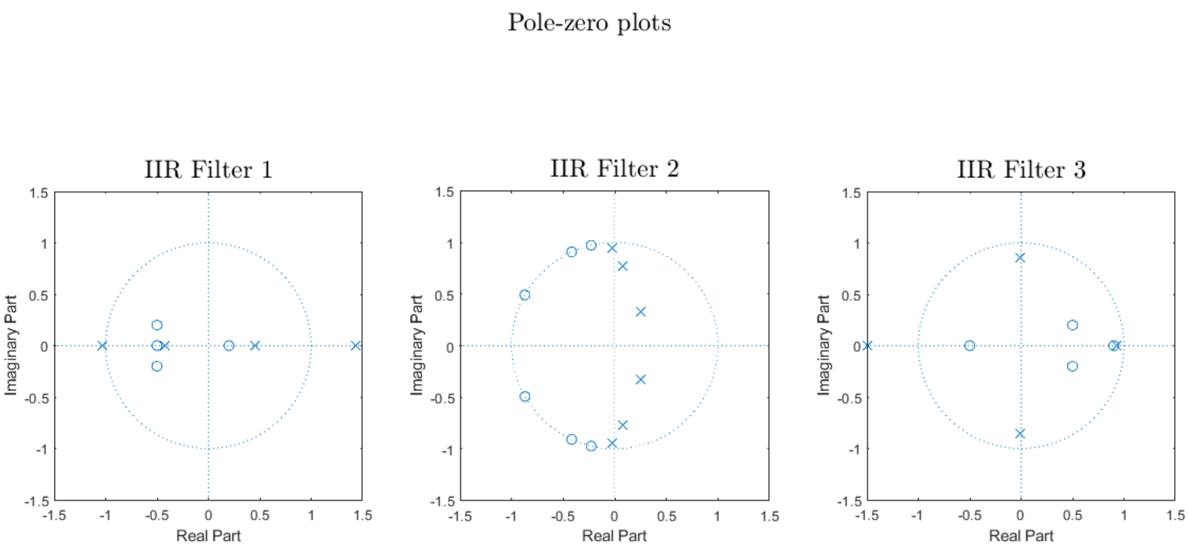


Figure 1: Exercise 1 - *Pole-zero plots of the set of filters*

To obtain the impulse response of eq.(1), I filtered a Kronecker delta function of 101 samples, ranging from -50 to 50, using Matlab filter function and I compared it with the plot generated by impz function. The latter returns the impulse response of the filter directly from its coefficients. The two plots are shown in Figure 2, where it is clear that the two methods end up in the same result. The system converges to zero with increasing n, as it is expected from a stable system. Moreover, it can be verified that the initial assumption about its causality is valid, being the impulse response zero for negative values of n.
Figure 3 compares the stable filter with the third one. However, being the response of the unstable filter diverging towards $\pm\infty$, the plot takes very high values even in the first 50 samples, and as a consequence the values of the impulse response of the stable filter do not show up.

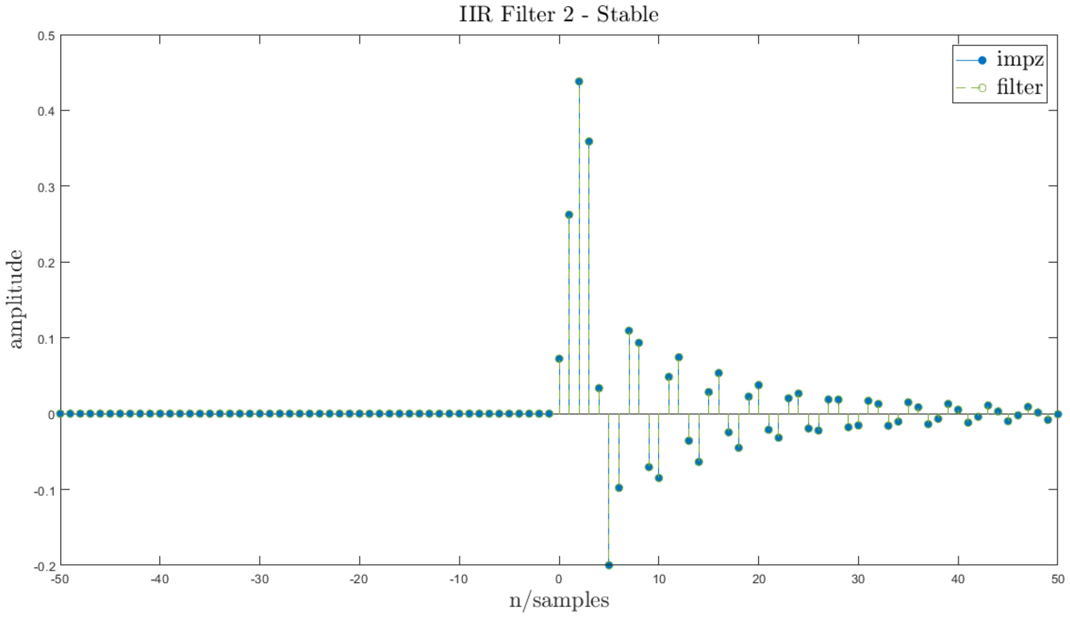


Figure 2: Exercise 1 - *Impulse response of the stable filter*

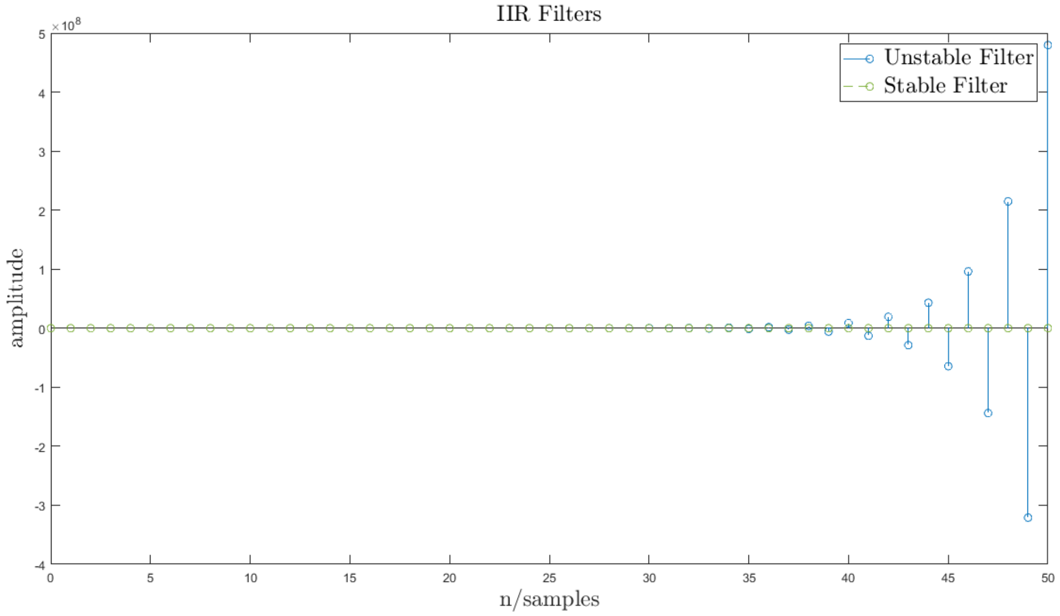


Figure 3: Exercise 1 - *Impulse responses of filter 2 and 3*

Exercise 2

The signal under analysis in this exercise is the following random process:

$$X(t) = cos(2\pi f_1 t) + cos(2\pi f_2 t) + W(t) \tag{2}$$

where $f_1 = 5Hz$, $f_2 = 50Hz$ and $W(t)$ is a zero-mean White Gaussian Noise (WGN) with variance $\sigma^2 = 10$. I sampled the signal at sampling frequency $f_s = 500Hz$ and represented it for a time window $T_0 = 20s$.

To estimate the Power Spectral Density (PSD) of $X(t)$ I computed the Welch periodogram using pwelch function with parameters $M = 25$ segments, 50% of overlap and using the Hamming window. Figure 4 shows the obtained PSD where the harmonic components of $X(t)$ at $\pm f_1$ and $\pm f_2$ can be seen on the top of the spectrum of $W(t)$, which is present along the whole frequency axis.

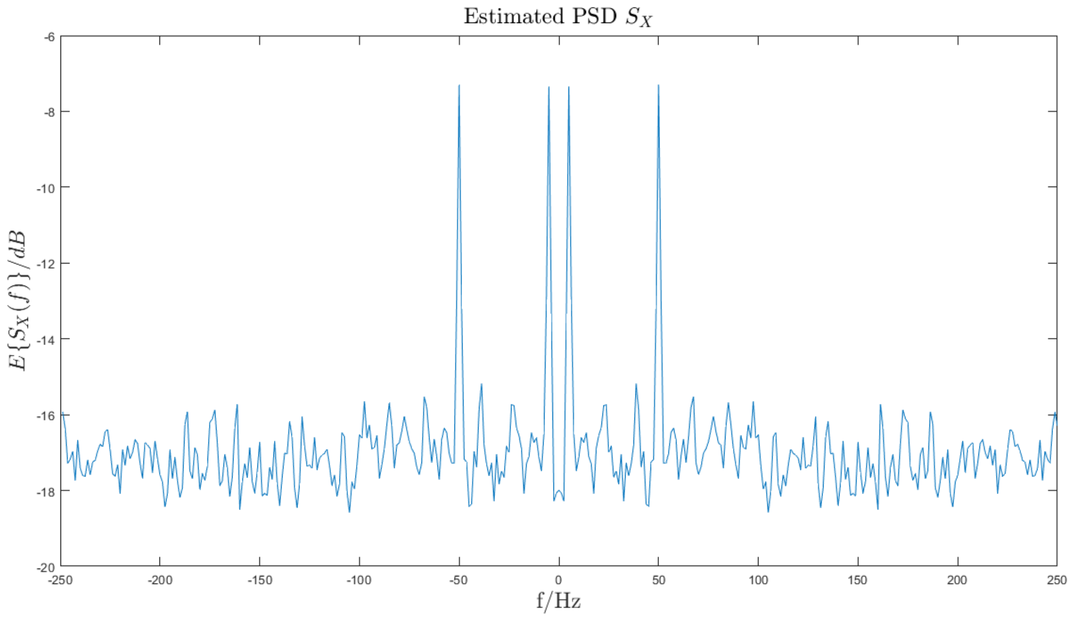
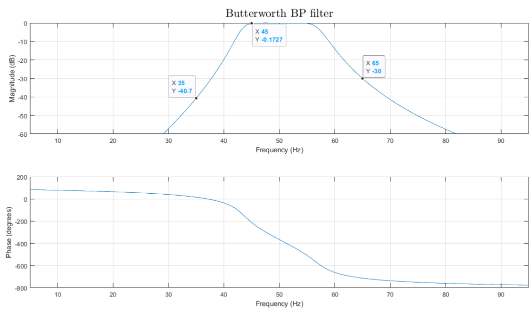
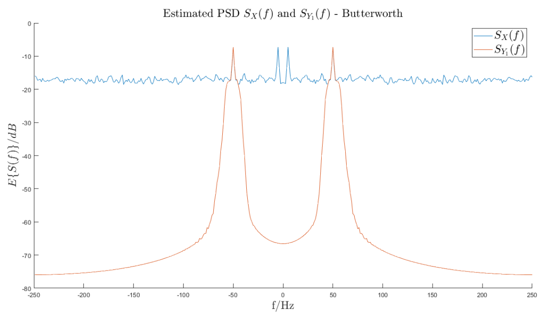


Figure 4: Exercise 2 - *Estimated PSD $S_X(f)$ of signal $X(f)$*

The first filter to be applied is a Butterworth band-pass filter centered at f_2, with band-pass ripple $R_p = 1dB$, transition bands of $10Hz$ and a stop-band attenuation of $30dB$. To design the filter $H_1(z)$ I used the dedicated Matlab functions buttord and butter from which I obtained the tenth-order filter illustrated in Figure 5(a). At the pass-band edge frequencies of $45Hz$ and $55Hz$, the attenuation reaches its pass-band minimum value of $-0.1727dB$, while at the stop-band frequencies of $35Hz$ and $65Hz$ it takes values $-40.7dB$ and $-30dB$ respectively, satisfying the initial requirements. Then I generated signal $Y_1(t)$ by filtering $X(t)$ with $H_1(z)$ and I estimated its PSD using the same Welch periodogram used for $S_X(f)$. In Figure 5(b) it can be noticed that after filtering, only the frequency components around $\pm f_2$ survive, while the frequencies outside the pass-band are strongly attenuated.



(a) *Frequency response*

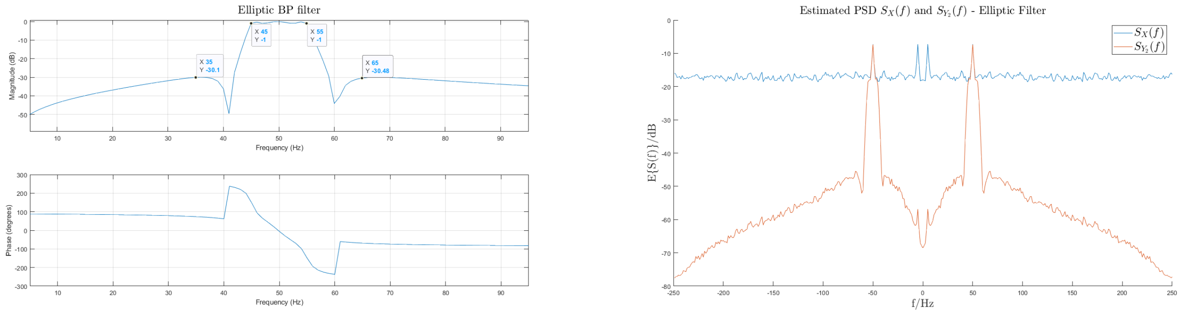


(b) *Estimated PSDs $S_{Y_1}(f)$, $S_X(f)$*

Figure 5: Exercise 2 - *tenth-order Butterworth band-pass filter $H_1(z)$*

Applying the same procedure and referring to the same specifications, I constructed the Elliptic band-pass filter $H_2(z)$ using functions ellipord and ellip. The frequency response of $H_2(z)$ is shown in

Figure 6(a). Then I generated signal $Y_2(t)$ by filtering signal $X(t)$ with $H_2(z)$ and I estimated its PSD using the same Welch periodogram. The estimated PSD $S_{Y_2}(f)$ is shown in Figure 6(b). At the pass-band edge frequencies of $45Hz$ and $55Hz$, the attenuation reaches its pass-band minimum value of $-1dB$, while at the stop-band frequencies of $35Hz$ and $65Hz$ it takes values $-30.1dB$ and $-30.49dB$ respectively, satisfying the initial requirements.

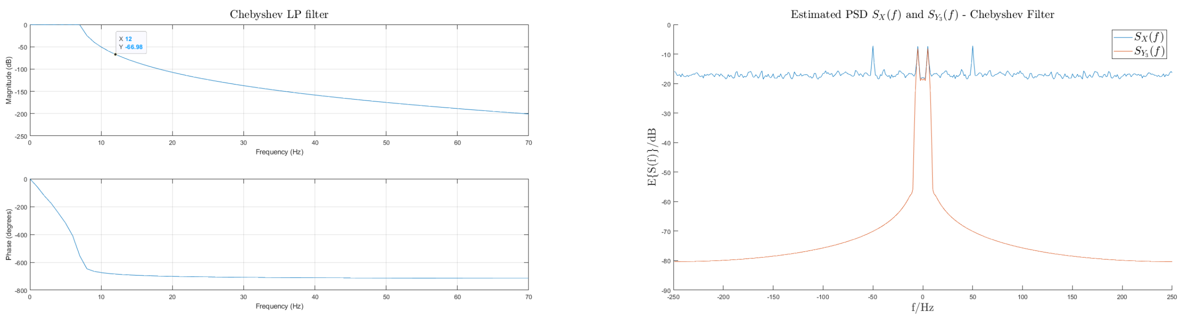


(a) *Frequency response* (b) *Estimated PSDs $S_{Y_2}(f)$, $S_X(f)$*

Figure 6: Exercise 2 - *Sixth-order Elliptic band-pass filter $H_2(f)$*

The Elliptic filter in Figure 6(a) has a sharpest initial transition between pass-band and stop-band, compared to the Butterworth filter. However, only the latter is monotonically decreasing along the whole frequency axis (with reference frequency f_2) and reaches the stop-band edge frequencies with an overall higher attenuation. It can be seen that in the $10Hz$ band centered around f_2, $H_2(z)$ presents some ripple that at the pass-band edge frequencies takes its maximum value of $-1dB$. On the other hand the maximum attenuation in the pass-band of $H_1(z)$ is $-0.1727dB$, more than 80% less and it does not present a ripple. The flatness of the Butterworth filter comes to the price of a higher order needed to meet the requirements. Indeed, $H_1(z)$ is a tenth-order filter, while $H_2(f)$ is only of the sixth-order, but the first one has the sharpest and higher attenuation in the stop-band.

Finally, I constructed a Chebyshev type I low-pass filter $H_3(z)$ with cut off frequency $f_p = 7Hz$, pass-band ripple $R_p = 1dB$, a transition band of $5Hz$ and a stop-band attenuation of $60dB$, using functions $\mathrm{cheb1ord}$ and $\mathrm{cheby1}$. Figure 7 illustrates the impulse response of the obtained eighth order LP filter $H_3(z)$ (Figure 7(a)) and the estimated PSD of signal $Y_3(t)$ (Figure 7(b)) obtained by filtering $X(t)$ with $H_3(z)$ using the same procedure employed in the previous examples.



(a) *Frequency response* (b) *Estimated PSDs $S_{Y_3}(f)$, $S_X(f)$*

Figure 7: Exercise 2 - *eighth-order Chebychev low-pass filter $H_3(z)$*

Figure 8 shows signals $X(t)$ and $Y_3(t)$ for different values of noise variance. The effect of a different variance is evident looking at $X(t)$, however all the filtered signals show a sinusoidal behaviour at 5Hz with some distortions both in amplitude and phase that becomes less evident in the case of $\sigma^2 = 0.1$.

It can be noticed also that $Y_3(z)$ has an initial phase where its value is almost zero, due to the time delay introduced by the filter.

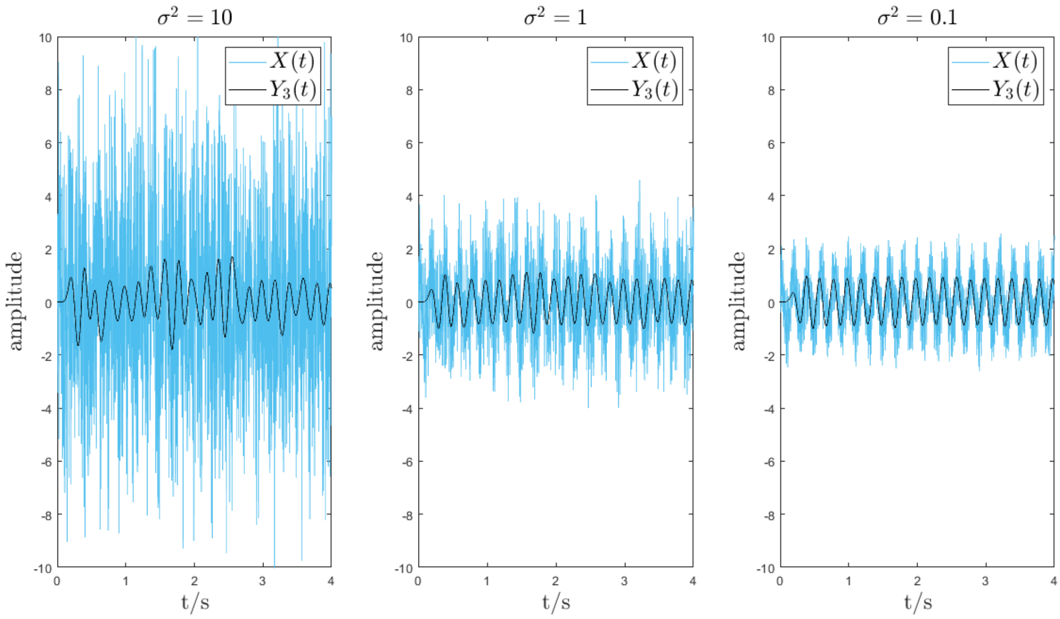


Figure 8: Exercise 2 - *Effect of the noise on $X(t)$ and $Y_3(t)$ at different variances*

Exercise 3

This exercise was focused on the Gibbs phenomenon analyzed in the specific case of an ideal low-pass filter $X_T(f)$ with bandwidth $B = 4Hz$, truncated in time with a flat top window of duration $T = 10s$. With these parameters, the continuous Fourier transform of the impulse response $x_T(t)$ of $X_T(f)$ is the combination of the following sine integral functions:

$$X_T(f) = \frac{1}{\pi} Si(\pi T(f + B/2)) - \frac{1}{\pi} Si(\pi T(f - B/2))$$

$X_T(f)$, together with the individual sine integral functions are depicted in Figure 9.
The ripple of $X_T(f)$ assumes peak value of approximately 1.09, even with different duration T of the window. For example, for $T = 10,\ 100,\ 400s$ the peak takes values $1.092, 1.09, 1.089$ respectively. However, increasing T the frequency of the ripple increases.
The impulse response of the filter $X_T(f)$ is the following:

$$x_T(t) = B \cdot sinc(Bt) \cdot rect_T(t)$$

In Matlab I generated $x_T(t)$ simply defining the function $B \cdot sinc(Bt)$ over a finite time interval $[-T/2, T/2]$ with a sampling frequency of $f_s = 20Hz$. The number of required samples is $N = T \cdot f_s = 200$. Figure 10 shows the obtained function $x_T(t)$.
To compute the discrete frequency response of the truncated filter, I first zero-padded the signal at both sides to reach $10N$ samples so that the spectral resolution is $\Delta f = f_s/N = 0.01Hz$. Then, I computed the DFT $X(k)$ of $x_T(t)$, with the fft function, and I obtained the samples of the CTFT applying the following formula:

$$X\left(\frac{k}{T_0}\right) = \frac{T}{N} \cdot DFT[x(n)]$$

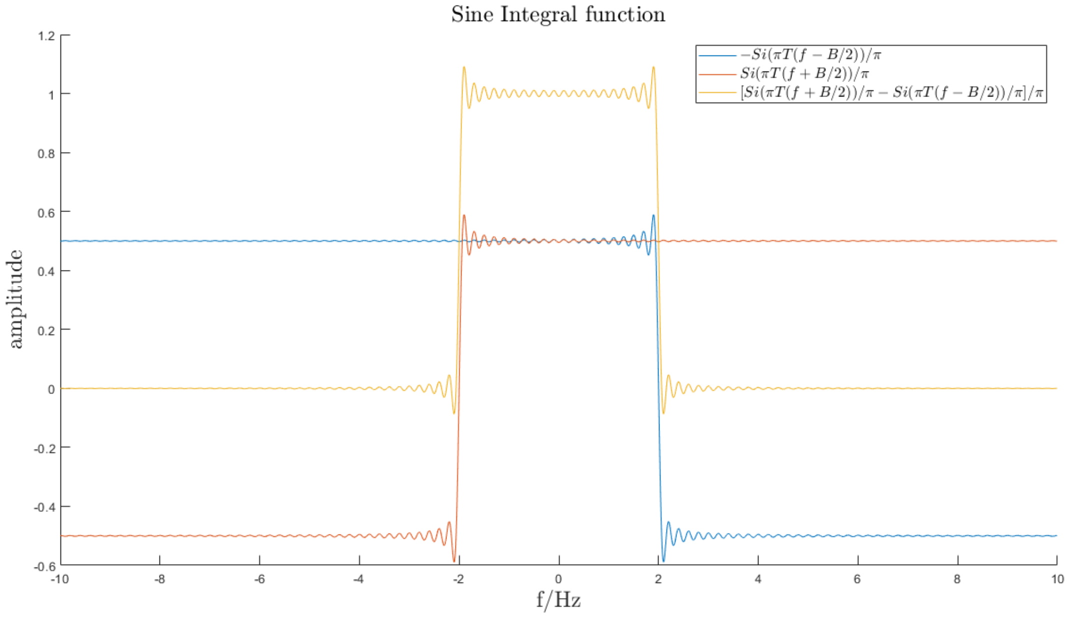


Figure 9: Exercise 3 - *Sine Integral functions related to the Gibbs phenomenon*

Figure 11 compares $|X(\frac{k}{T_0})|$ with $|X_T(f)|$, generated at the beginning of the exercise and it can be seen that they coincides.

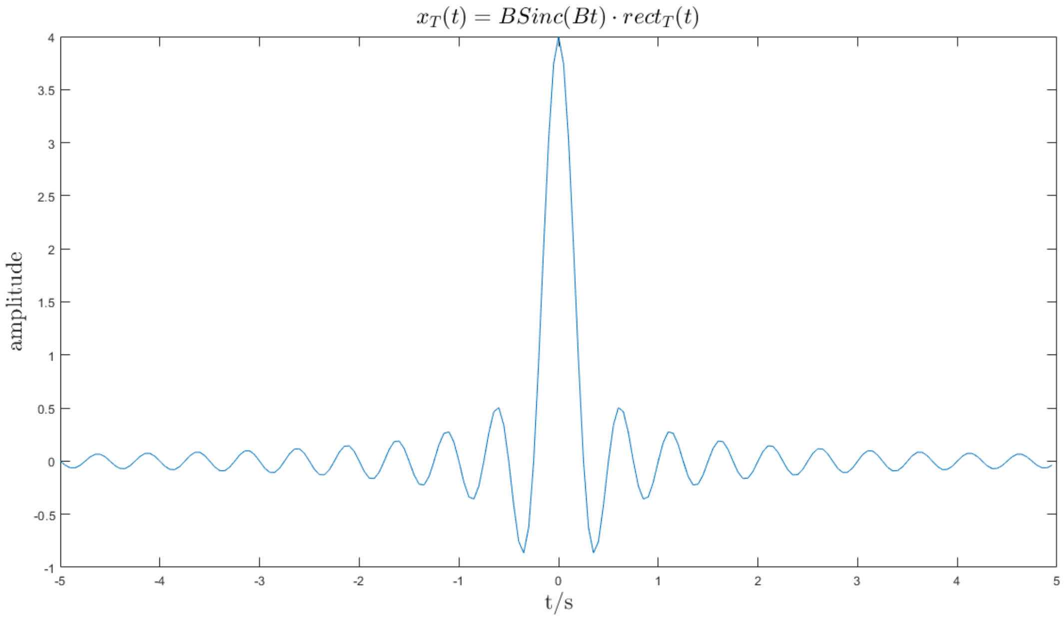


Figure 10: Exercise 3 - *Impulse response of the filter $X_T(f)$*

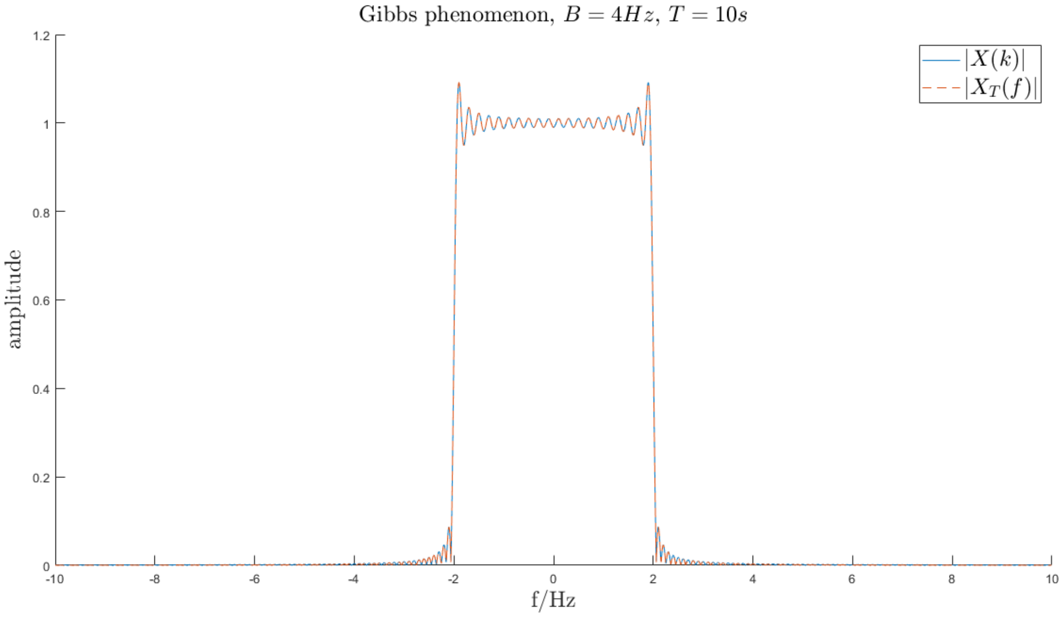


Figure 11: Exercise 3 - *Frequency response of* $x_T(t)$

Exercise 4

In this exercise it was asked to design a low-pass FIR filter with pass-band edge angular frequency $\omega_p = 0.2\pi\ rad/s$, stop-band edge angular frequency $\omega_s = 0.3\pi\ rad/s$, maximum pass-band ripple $R_p = 0.2dB$ and minimum stop-band attenuation $A_s = 40dB$. With these specifications the transition band $B_T = \omega_s - \omega_p$ and the $-6dB$ cut-off frequency are the following:

$$B_T = \omega_s - \omega_p = (0.3 - 0.2)\pi = 0.1\pi$$
$$f_c = \frac{\omega_s + \omega_p}{2 \cdot 2\pi} = \frac{1}{8} = 0.125Hz$$

The windows that ensure the required minimum attenuation A_s are Hann, Hamming and Blackman windows, nevertheless the one that requires the lower number of coefficients is the Hann, which requires $6.2\pi/B_T = 62$ coefficients.
To generate the impulse response of the digital filter I created a vector $x[n] = 2f_c Sinc(2f_c n)$, representing the impulse response of the ideal low-pass filter with bandwidth f_c, and then I computed the element wise multiplication of $x[n]$ with the Hann window $w[n]$ obtained with the hann function. Figure 12 illustrates the samples of the obtained impulse response $h[n] = x[n] \cdot w[n]$. Figure 13 shows the frequency response of $h[n]$ where it can be verified that the generated filter satisfies the required specifications: in pass-band it has a ripple always less than $0.1dB < R_p$, at the stop-band frequency $f_s = w_s/2\pi = 0.15Hz$ it has an attenuation above $40dB$ and at the cutoff frequency it attenuates $-6.021dB$, everything consistent with the design specifications

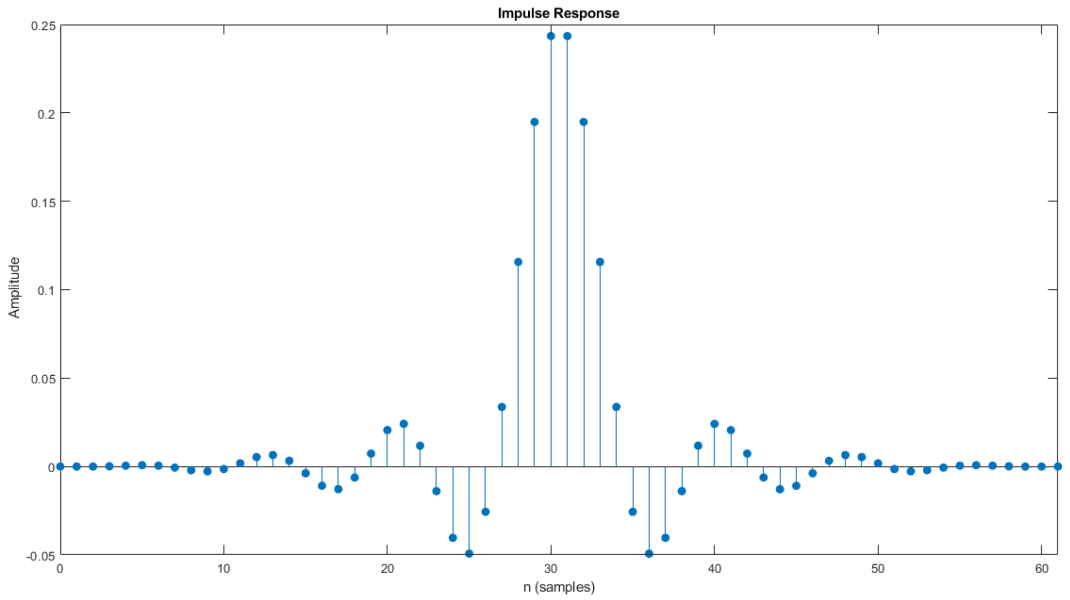


Figure 12: Exercise 4 - *FIR LP filter impulse response* $h[n]$

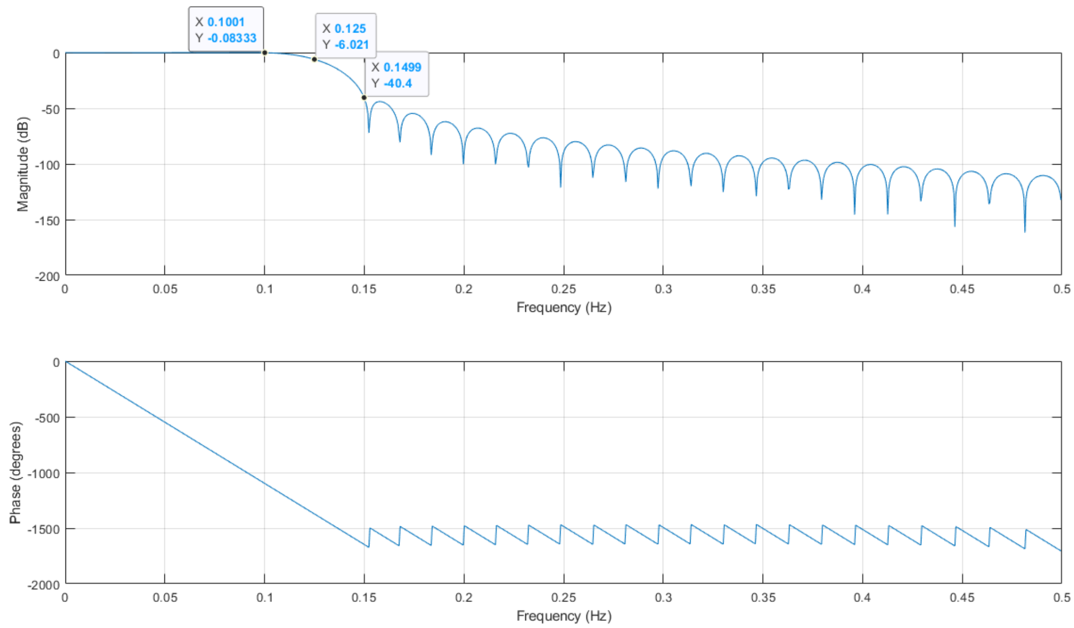


Figure 13: Exercise 4 - *FIR LP filter* $h[n]$ *frequency response*

Exercise 5

In this last exercise it was asked to create a function that generates the impulse response of a FIR filter using the Kaiser window. The input parameters of the function that I created are the required stop-band attenuation A_s, the transition band B_T, the sampling frequency f_s, the cut of frequency f_c and the type of the filter. When the filter that hat to be designed is a band-pass or band-stop, the input parameter f_s must be a vector whose elements are the two edge frequencies put in order. As outputs, the function returns the filter coefficients, the number of the coefficients N and the stop-band attenuation parameter β. To test the function, I generated a low-pass filter and I compared it with the filter generated by Matlab fir1 function, with the parameters β and N obtained through the custom function. The requirements of the filter are the following:

$$A_s = 40dB \quad \text{stop-band attenuation}$$
$$B_T = 400Hz \quad \text{transition band}$$
$$f_s = 8kHz \quad \text{sampling frequency}$$
$$f_c = 800Hz \quad \text{-6dB cut-off frequency}$$

From Figure 14 it can be seen that the two obtained filter are almost the same (zooming it can be noticed that the two function differ for $0.0251dB$, probably associated to a different rounding). At frequency $f_c' = 800.8Hz$, which is not exactly f_c due to the limiting resolution, the function takes value $-6.018dB$ and the stop-band attenuation at $f = 1004Hz$ is $-41.34dB$, as it was requested. The impulse response has $N = 45$ samples and the value of the parameter β is 3.3953. .

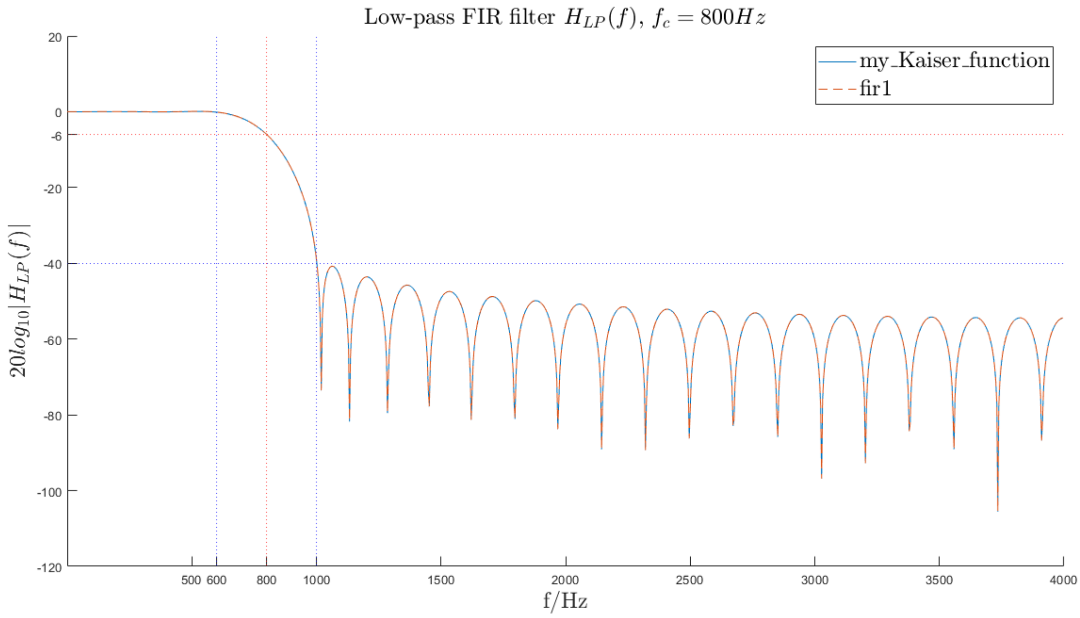


Figure 14: Exercise 5 - *FIR LP filter Frequency response with Kaiser window*