Hidden Markov Models

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If the event A can be broken down into disjoint events A_1, A_2, etc, then **Bayes' Theorem** states that

$$P(A|B) = \frac{P(B|A)P(A)}{\sum P(B|A_i)P(A_i)}$$

Example

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For some calculations, we do not need to worry about the denominator from Bayes' Theorem, and instead can work with the formula

$$P(A|B) \propto P(B|A)P(A)$$

where $\propto$ means "is proportional to."

Example

A sophomore student is selected. Is the student more likely to be a male or a female? Using the above version of Bayes Theorem:

$$
\begin{aligned}
P(M|Soph) &\propto P(Soph|M)P(M) \\
&\propto \left(\tfrac{4}{10}\right)\left(\tfrac{10}{15}\right) \\
&\propto \approx 0.267
\end{aligned}
$$

and

$$
\begin{aligned}
P(F|Soph) &\propto P(Soph|F)P(F) \\
&\propto \left(\tfrac{2}{5}\right)\left(\tfrac{5}{15}\right) \\
&\propto \approx 0.133
\end{aligned}
$$

Since $P(M|Soph) \propto 0.267$ and $P(F|Soph) \propto 0.133$, the selected individual is $\frac{0.267}{0.133} = 2$ times as likely to be a male than a female.

A **Markov chain** is a sequence of random variables (or states) X_1, X_2, X_3, with the property that the next state X_{n+1} depends on the m previous states (including the current one). Usually, m is taken to be 1 in which case the next state depends only on the current one, and the Markov chain is said to have the Markov property and is a first order Markov model. Formally, first order Markov models have the property

$$P(X_{n+1}|X_1, X_2, \ldots X_n) = P(X_{n+1}|X_n)$$

Example

An individual has two coins, a fair (F) coin and a biased (B) coin. Before each coin toss, there is a 10% chance that the individual will switch coins. Initially, there is a 50% chance the individual selects the fair (or biased) coin. Find the probability that the selected coins are FFB. Note that this is a 1st order Markov Chain. The subscript i will be used for the i^{th} selected coin.

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$$P(F_1 F_2 B_3) = P(F_1)P(F_2|F_1)P(B_3|F_2, F_1)$$

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A **Hidden Markov model** (HMM) is a Markov chain where the states are hidden (unobserved), but hidden states emit observed values with certain probabilities. The hidden states can then be deduced based on the observed values.
For a first order HMM with observed states $O_1, O_2, \ldots, O_n$ and hidden states $H_1, H_2, \ldots, H_n$, the HMM is characterized by the following probabilities:

Initial state probabilities:	$P(H_1)$
Transition state probabilities:	$P(H_{i+1}\|H_i)$ for all i
Emission probabilities:	$P(O_i\|H_i)$ for all i

The goal of a HMM is to predict the set of hidden states (such as the unknown gene structure). The most likely sequence of hidden states is the state sequence $H_1, H_2, \ldots, H_n$ that maximizes

$$P(H_1, \ldots, H_n | O_1, \ldots, O_n) \propto \\ P(O_1|H_1)P(H_1) \times P(O_2|H_2)P(H_2|H_1) \times \cdots \times P(O_n|H_n)P(H_n|H_{n-1})$$

Because these probabilities are often extremely small, it is often useful to work on the log scale and find the state sequence that maximizes

$$\log P(H_1, \ldots, H_n | O_1, \ldots, O_n) = \\ \log P(O_1|H_1)P(H_1) + \log P(O_2|H_2)P(H_2|H_1) + \cdots + \log P(O_n|H_n)P(H_n|H_{n-1}) + k,$$

where k is a constant which is not needed for our purposes.

Coin flipping example

We will use the following probabilities:

$$
\begin{aligned}
&P(F_1) = P(B_1) = 0.50\\
&P(F_i|F_{i-1}) = P(B_i|B_{i-1}) = 0.90, i > 1\\
&P(F_i|B_{i-1}) = P(B_i|F_{i-1}) = 0.10, i > 1\\
&P(h|F) = P(\text{tail}|F) = 0.50\\
&P(h|B) = 0.80\\
&P(t|B) = 0.20
\end{aligned}
$$

Suppose we flip 3 coins and observe $(h,\ t,\ h)$. Which is more likely, BFB or BBB?

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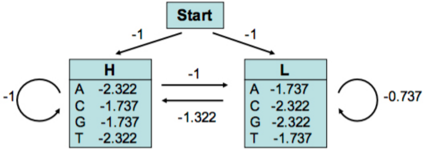
This lends itself to a dynamic programming solution (known as the **Viterbi algorithm**).

Example

Using the HMM described previously, what sequence of coins most likely generated h,t,h?

High (H) vs. Low (L) GC content example

HMM : Viterbi algorithm - a toy example



GGCACTGAA

back-tracking
(= finding the path which corresponds to the highest probability, -24.49)

	G	G	C	A	C	T	G	A	A
H	**-2.73**	**-5.47**	**-8.21**	-11.53	-14.01	...			-25.65
L	-3.32	-6.06	-8.79	**-10.94**	**-14.01**	...			**-24.49**

The most probable path is: **HHHLLLLLL**

Its probability is $2^{-24.49} = 4.25E\text{-}8$
(remember that we used $\log_2(p)$)

Note: probabilities are on the $\log_2$ scale
Source: `http://homepages.ulb.ac.be/~dgonze/TEACHING/viterbi.pdf`