Hidden Markov Models

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Some definitions

- The **probability** of an event *A*, denoted by $P(A)$, represents the proportion of times that *A* occurs over the long run
- For example, if we flip a coin once, $P(H) = 0.50$, indicating that we expect to get heads 50% of the time
- If all outcomes are equally likely, $P(A)$ is the proportion of outcomes where A occurs.
- The **conditional probability** of *A*, given that *B* has occurred, is denoted by $P(A|B)$and has the formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Conditional probability

Class status	M	F	Total
Soph	4	2	**6**
Junior	3	2	**5**
Senior	3	1	**4**
Total	**10**	**5**	**15**

From the conditional probability formula, it is true that

$$P(A \text{ and } B) = P(A|B)P(B)$$

Example:

$$P(Female \text{ and } Soph) = P(Female|Soph)P(Soph)$$
$$= \quad 2/6 \quad \times \quad 6/15$$
$$= \quad 2/15$$

This should make sense, as there are 2 female sophomores and 15 total students

Bayes' Theorem

- $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Class status	M	F	Total
Soph	4	2	**6**
Junior	3	2	**5**
Senior	3	1	**4**
Total	**10**	**5**	**15**

$$P(Soph|Female) = \frac{P(Female|Soph)P(Soph)}{P(Female)}$$

$$= \frac{\frac{2}{6} \times \frac{6}{15}}{\frac{5}{15}}$$

$$= \frac{2}{5}$$

This should make sense, as there are 2 female sophomores and 5 total females

(given the person is female – and there are 5 females – there are 2 sophomores)

Bayes' Theorem

- For some calculations, we do not need to calculate the denominator, $P(B)$.
- Instead, we can use the fact that

$$P(A|B) \propto P(B|A)P(A),$$

where $\propto$ means "is proportional to".

Bayes' Theorem

- A female student is selected. Is the student more likely to be a sophomore or a senior?

Class status	M	F	Total
Soph	4	2	**6**
Junior	3	2	**5**
Senior	3	1	**4**
Total	**10**	**5**	**15**

$$P(Soph|Female) \propto P(Female|Soph)P(Soph)$$
$$= \frac{2}{6} \times \frac{6}{15} = \frac{2}{15}$$

$$P(Senior|Female) \propto P(Female|Senior)P(Senior)$$
$$= \frac{1}{4} \times \frac{4}{15} = \frac{1}{15}$$

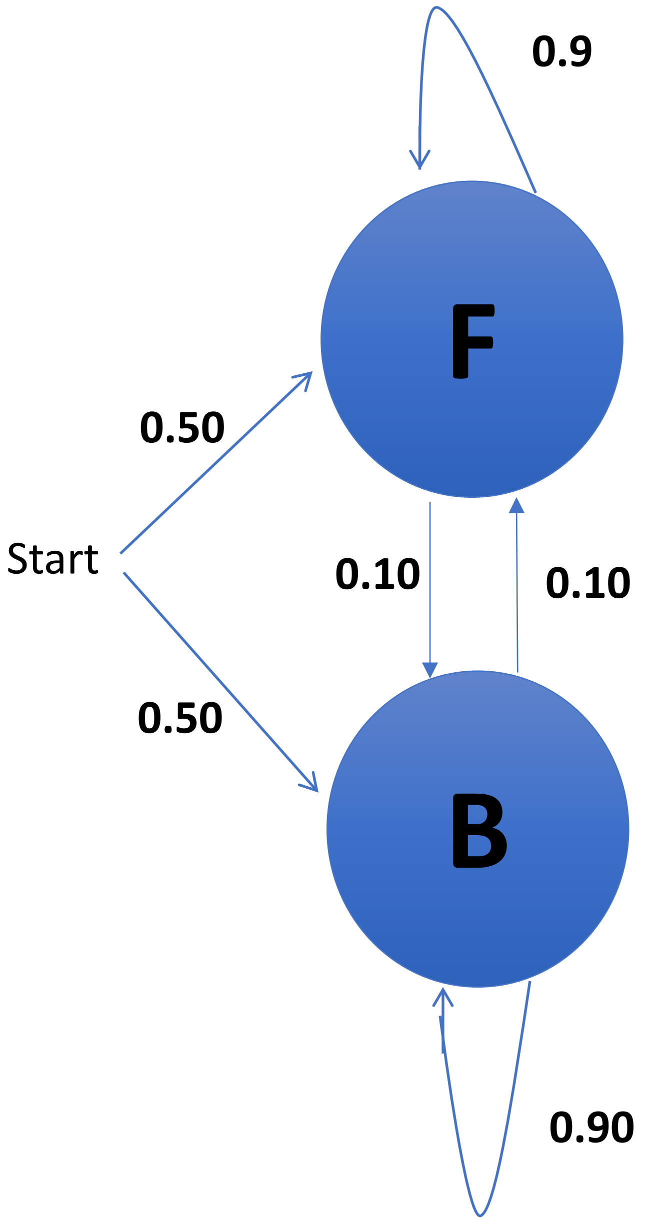
Since $P(Soph|Female) \propto \frac{2}{15}$ and $P(Senior|Female) \propto \frac{1}{15}$ the selected individual is

$\frac{P(Soph|Female)}{P(Senior|Female)} = \frac{2/15}{1/15} =$ 2 times as likely to be a sophomore than a senior

Markov chains

- A **Markov chain** is a sequence of random variables (or states) X_1, X_2, X_3, … with the property that the next state X_{n+1} depends on the m previous states (including the current one).
- Usually, m is taken to be 1 in which case the next state depends only on the current one, and the Markov chain is said to have the Markov property and is a first order Markov model.
- Formally, first order Markov models have the property

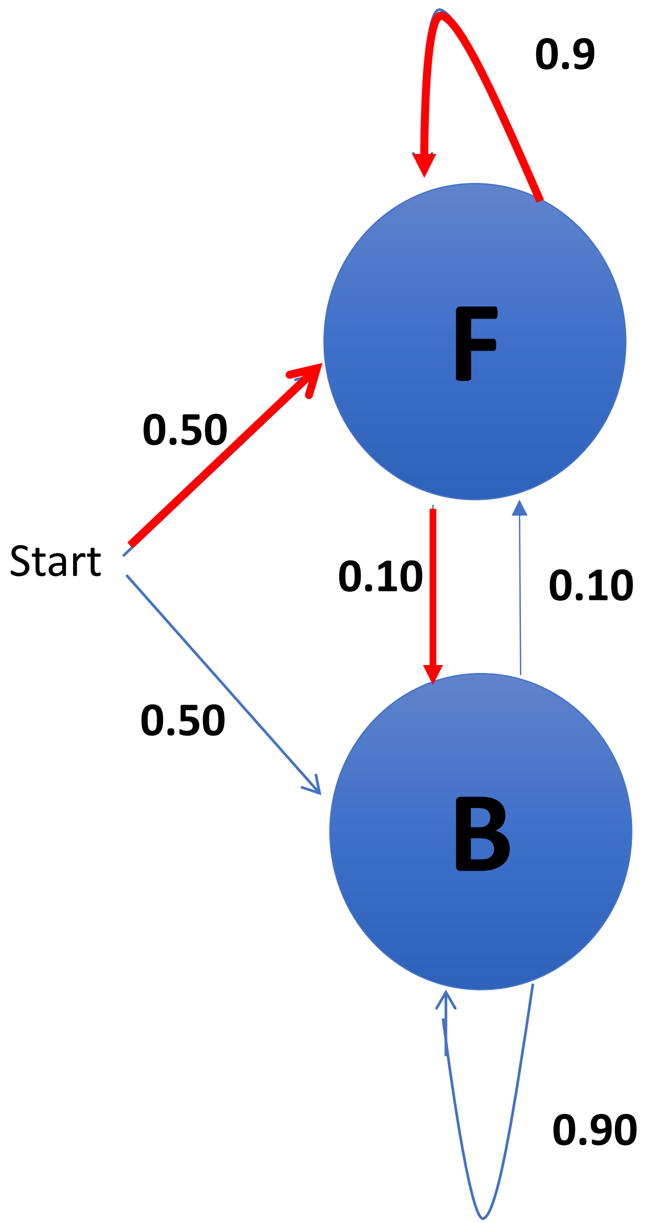
$$P(X_{n+1}|X_1, X_2, \dots, X_n) = P(X_{n+1}|\, X_n)$$



Example:

- An individual has two coins, a fair (F) coin and a biased (B)coin.
- Before each coin toss, there is a 10% chance that the individual will switch coins.
- Initially, there is a 50% chance the individual selects the fair (or biased) coin.
- Find the probability that the selected coins are FFB.
- Note that this is a 1st order Markov Chain. The subscript *i* will be used for the *i*th selected coin

$$P(F_1F_2B_3) = P(F_1)P(F_2|F_1)P(B_3|F_2,F_1) =$$
$$= P(F_1)\,P(F_2|F_1)\,P(B_3|F_2) \quad \text{(Markov assumption)}$$
$$= 0.50 \times 0.90 \quad \times\ 0.10$$
$$= 0.045$$



- Find the probability that the selected coins are FFB.
- We don't need the fancy notation. Just follow the arrows through the Markov Model and write the corresponding probabilities

State	Start	→ F	→ F	→B
Prob	-	**0.50**	**0.90**	**0.10**

- Now multiply the probabilities together to get the probability of the state (e.g., FFB)

$$\mathrm{P(FFB)} = 0.50 \times 0.90 \quad \times \ 0.10$$
$$= 0.045$$