Basic Algebra

Functions, polynomials, coordinate systems, complex numbers

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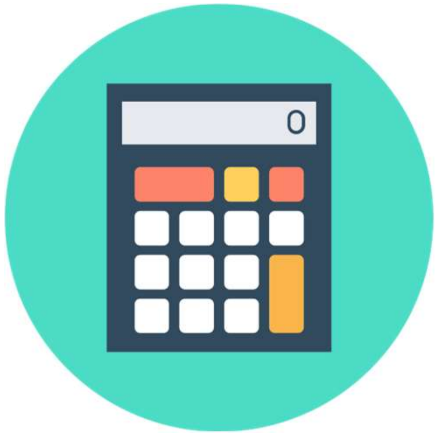

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#MathForDevs

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Polynomials

Definition, storing, basic operations

Polynomials

- We already looked at linear and quadratic polynomials
- Term (monomial): $2x^2$
  - Coefficient (number), variable, power (number ≥ 0)
- Polynomial: sum of monomials
  - $2x^4 + 3x^2 - 0{,}5x + 2{,}72$
  - Degree: the highest degree of the variable (with coefficient $\neq 0$)
- Operations
  - Defined the same way as with numbers
  - Addition and subtraction
    - $(2x^2 + 5x - 8) + (3x^4 - 2) = 3x^4 + 2x^2 + 5x - 10$
  - Multiplication and division
    - $(2x^2 + 5x - 8)(3x^4 - 2) = 6x^6 + 15x^5 - 24x^4 - 4x^2 - 10x + 16$

Polynomials in Python

- numpy has a module for working with polynomials
  - Includes the "general" polynomials,
    as well as a few special cases
    - Chebyshev, Legandre, Hermit
- Storing polynomials
  - As arrays (index ⇒ power, value ⇒ coefficient)
  - Keep in mind this will look "reversed" relative to the way we write

```
import numpy.polynomial.polynomial as p
p.polyadd([-8, 5, 2], [-2, 0, 0, 0, 3])
p.polymul([-8, 5, 2], [-2, 0, 0, 0, 3])
# array([-10., 5., 2., 0., 3.])
# array([ 16., -10., -4., 0., -24., 15., 6.])
```

Polynomials in Python (2)

- Pretty printing
  - Use `sympy` to print the polynomial
    - If it's a list, use it directly
    - If it's a `Polynomial` object, call the `coef` property
  - Reverse the order of coefficients (`sympy` expects them from highest to lowest)

```
import sympy
from sympy.abc import x
polynomial = p.Polynomial([-2, 0, 0, 0, 3])
sympy.init_printing()
print(sympy.Poly(reversed(polynomial.coef), x).as_expr())
# Output: 3.0*x**4 - 2.0
```

Sets

Set notation and basic operations

Set

- An **unordered collection** of things
  - Usually numbers
  - No repetitions
- Set notation: $\{x \in \mathbb{R} \mid x \geq 0\}$
  - "The set of numbers x, which are a subset of the real numbers, which are greater than or equal to zero"
  - Left: example element
  - Right: conditions to satisfy
- Python set comprehensions
  - Very similar to what we already wrote
  - Also very similar to list comprehensions (but with curly braces)

```python
positive_x = {x for x in range(-5, 5) if x >= 0}
# {0, 1, 2, 3, 4}
```

Set Operations

- Cardinality: number of elements
- Checking whether an element is in the set: $x \in S$
- Checking whether a set is subset of another set: $S_1 \subseteq S_2$
- Union $S_1 \cup S_2$, intersection $S_1 \cap S_2$, difference $S_1 \setminus S_2$

```
set1 = { 1, 2, 3, 4 }
set2 = {3, 4, 5, 10, 3, 5, 10, 3, 3}
print(len(set2)) # 4
print(1 in set1) # True
print(10 not in set1) # True
print({1, 2}.issubset(set1)) # True
print(set1.union(set2)) # {1, 2, 3, 4, 5, 10}
print(set1.difference(set2)) # {1, 2}
print(set2.difference(set1)) # {10, 5}
print(set1.symmetric_difference(set2)) # {1, 2, 5, 10}
```

Functions

Mappings from one thing to another

Function

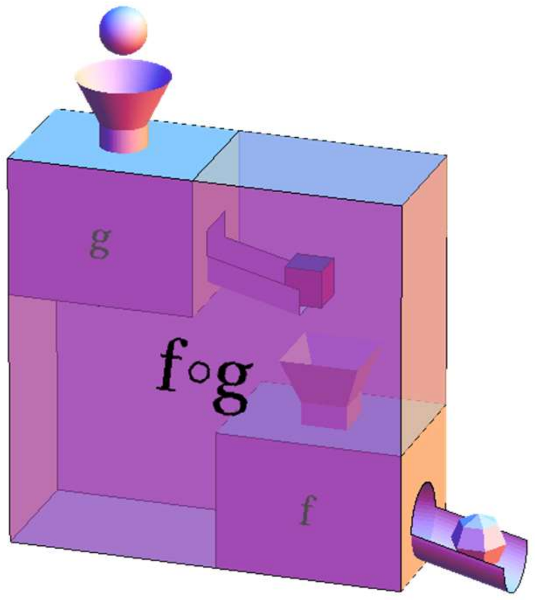
- A relation between
  - A set of inputs X (**domain**)
  - ... and a set of outputs Y (**codomain**)
  - **One input produces exactly one output**
  - The inputs don't need to be numbers
  - Functions don't know how to compute the output, they're just mappings
    - In programming, we write **procedures**
- Math notation: $f : X \to Y$
  - Commonly abbreviated as $y = f(x)$
- Some more definitions
  - **Injective** (one-to-one): unique inputs => unique outputs
  - **Surjective** (onto): every element in the codomain is mapped
  - **Bijective** (one-to-one correspondence): injective and surjective
  - Here is a graphical view

Function Composition

- Also called pipelining in most languages
- Takes two functions and applies them in order
  - **Innermost to outermost**
  - Math notation: $f \circ g = f(g(x))$
  - Can be generalized to more functions
- Note that the order matters

$$f(x) = 2x + 3, \;\; g(x) = x^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$
$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2$$

- This kind of notation can be confusing sometimes
  - x is only a placeholder for the input
  - We've used the same letter x for different inputs
  - Tip: When working with complicated functions, be very careful what the inputs and outputs are, and how variables depend on other variables
- Functions and composition are the basis of functional programming



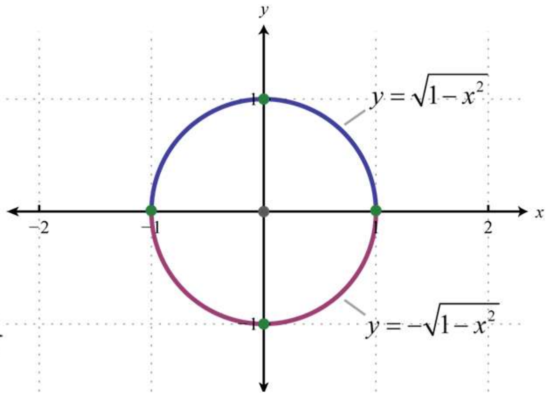
Function Graphs

- One very intuitive way to get to know functions is to plot them
  - We already did that in the last exercise
  - Generate values in the domain (independent variable)
  - For each value compute the output (dependent variable)
  - Create a graph; plot all computed points and connect them with tiny straight lines
- **`lambda`** in Python is a short syntax for a function
  - We can define it outside as well (it's just shorter and simpler to use it inline)

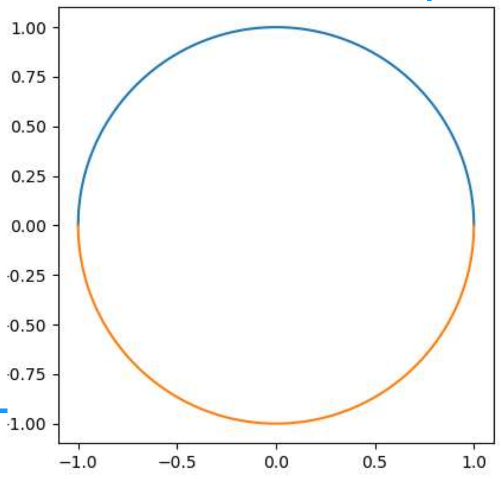
```python
import numpy as np
import matplotlib.pyplot as plt
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
  x = np.linspace(x_min, x_max, n_values)
  y = f(x)
  plt.plot(x, y)
  plt.show()

plot_function(lambda x: np.sin(x))
```

Graphing a Circle

- Let's try to graph the unit circle
  - Equation: $x^2 + y^2 = 1$
- This cannot be represented as one function
  - We have multiple values of y, e. g. $x = 0 \to y = \{-1,\ 1\}$
- We can try two functions (see graph)
  - But we want to represent the circle as one object



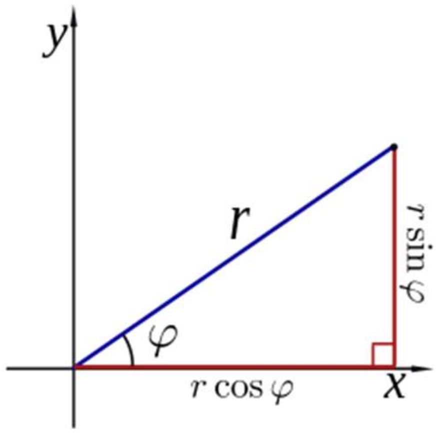
```
def plot_function(f, x_min = -10, x_max = 10, n_values = 2000):
  plt.gca().set_aspect("equal")
  x = np.linspace(x_min, x_max, n_values)
  y = f(x)
  plt.plot(x, y)

plot_function(lambda x: np.sqrt(1 - x**2), -1, 1)
plot_function(lambda x: -np.sqrt(1 - x**2), -1, 1)
plt.show()
```



Graphing a Circle (2)

- In math and science, many problems can be solved by changing our viewpoint
- We can use another type of reference system
  - One which incorporates angles naturally
  - Polar coordinate system (r, φ):
    - (r: distance from origin ($r \geq 0$); φ: angle to x-axis)
  - We can easily convert Cartesian to polar coordinates

$$x^2 + y^2 = 1$$
$$(r\cos\varphi)^2 + (r\sin\varphi)^2 = 1$$
$$r^2\cos^2\varphi + r^2\sin^2\varphi = 1$$
$$r^2(\cos^2\varphi + \sin^2\varphi) = 1$$
$$r^2 = 1,\ r \geq 0 \Rightarrow r = 1$$

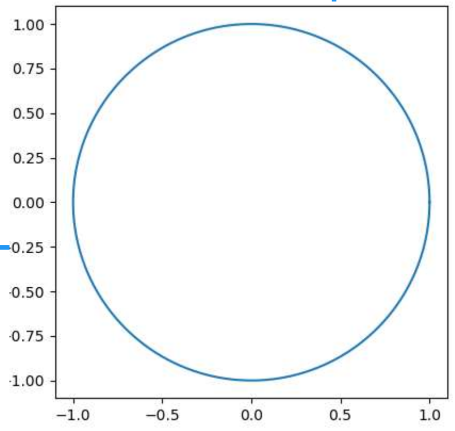


- Now we can see the equation is very, very simple
  - Doesn't even depend on φ
  - This is why we needed the change of viewpoint (coordinates)

Graphing a Circle (3)

- Graphing a function in polar coordinates
  - This applies to any function, circles in particular
  - Generate initial values of r and φ
  - Convert them to rectangular coordinates
  - Plot the rectangular coordinates

```python
import numpy as np
import matplotlib.pyplot as plt
r = 1 # Radius
phi = np.linspace(0, 2 * np.pi, 1000) # Angle (full circle)
x = r * np.cos(phi)
y = r * np.sin(phi)
plt.plot(x, y)
plt.gca().set_aspect("equal")
plt.show()
```



- For most other applications we can do this directly

```python
plt.polar(phi, r)
```

Complex Numbers

Not as complex as they seem

Number Fields

- Field
  - A collection of values with operations "plus" and "times"
  - Algebra is so abstract we can redefine these operations (stay tuned)
- History of number fields
  - Natural (counting) numbers $\mathbb{N} = \{0, 1, 2, \dots\}$
  - Integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
    - Subtraction
  - Rational numbers $\mathbb{Q}$: ratio of two integers
    - Division
    - This is **the smallest field**
  - Real numbers $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
    - Most roots (e.g. $\sqrt{2}$)
  - **Complex numbers** $\mathbb{C}$
    - All roots (including square roots of negative numbers)
    - "Imaginary unit": i is the positive solution of $x^2 = -1$

Complex Numbers

- Pairs of real numbers: $(a; b) : a, b \in \mathbb{R}$
  - Commonly written as $a + bi$
  - Real part: $\mathrm{Re}(a + bi) = a$, imaginary part: $\mathrm{Im}(a + bi) = b$
- In Python, we use `j` instead of `i`

`3j` `1j` `3 + 2j`

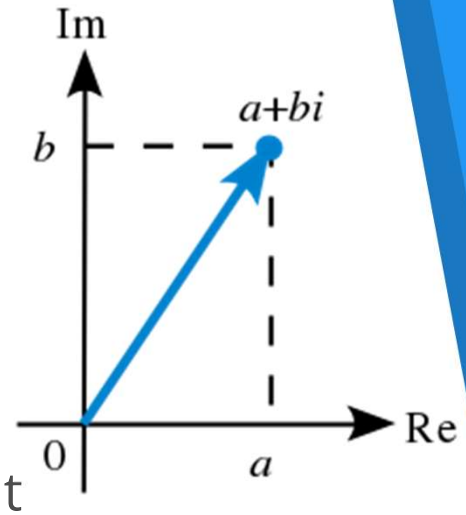
  - Note that we write `1j` to prevent confusion with the variable `j`
  - For the same reason, we don't write `2 * j` if `j` is the imaginary unit
- We can get the real and imaginary parts

```
z = 3 + 2j
print(z.real) # 3
print(z.imag) # 2
```

- Adding and multiplying complex numbers

```
print((3 + 2j) + (8 - 3j)) # (11-1j)
print((3 + 2j) * (8 - 3j)) # (30+7j)
```

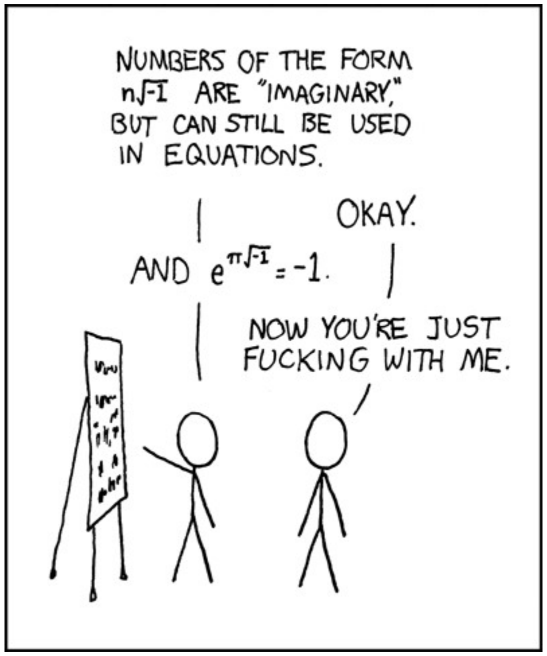
Geometric Interpretation



- Intuition
  - We can plot the coordinate pairs on the plane
  - Each point in the 2D space represents one complex number
- But…
  - We saw that we can change our perspective a little bit
  - Polar coordinates: we can use the same transformation
    - $\rho = |z|$ – **module** of the complex number
    - $\varphi = \arg(z)$ – **argument** of the complex number
    - $a = \rho\cos(\varphi)\,, b = \rho\sin(\varphi)$
  - Why do we do this?
    - Some operations (e.g., multiplication and division) are easier in polar coordinates
    - Powers of complex numbers become extremely easy
  - Polar form
    - $z = a + bi = \rho(\cos(\varphi) + i\sin(\varphi))$

Euler's Formula

- Leonhard Euler proved that $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$
  - Here's a summary of the proof if you're interested
  - It involves series which we haven't covered yet
  - A very beautiful consequence: $e^{i\pi} + 1 = 0$
- Now we can write our complex number as $z = |z|e^{i\varphi}$
- Why and how does multiplication work?
  - Multiplication by a real number
    - Scales the original vector
  - Multiplication by an imaginary number
    - Rotates the original vector
  - You can see a thorough explanation here
- **Main point:** Multiplication of complex numbers is the same as scaling and rotating 2D vectors
  - Algebra is abstract and we love it :)


NUMBERS OF THE FORM $n\sqrt{-1}$ ARE "IMAGINARY," BUT CAN STILL BE USED IN EQUATIONS.
OKAY.
AND $e^{\pi\sqrt{-1}} = -1$.
NOW YOU'RE JUST FUCKING WITH ME.

Fundamental Theorem of Algebra

Roots, roots, and more roots

Fundamental Theorem of Algebra

- "Every non-zero, single-variable, degree-n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots"
  - More simply said, every algebraic equation has as many roots as its power
- Back to quadratic equations
  - How do we get all roots?
  - Simply use the complex math Python module: **`cmath`**

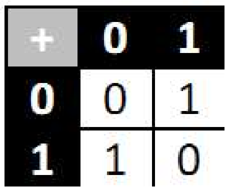
```python
import cmath
def solve_quadratic_equation(a, b, c):
  discriminant = cmath.sqrt(b * b - 4 * a * c)
  return [
    (-b + discriminant) / (2 * a),
    (-b - discriminant) / (2 * a)]

print(solve_quadratic_equation(1, -3, -4)) # [(4+0j), (-1+0j)]
print(solve_quadratic_equation(1, 0, -4)) # [(2+0j), (-2+0j)]
print(solve_quadratic_equation(1, 2, 1)) # [(-1+0j), (-1+0j)]
print(solve_quadratic_equation(1, 4, 5)) # [(-2+1j), (-2-1j)]
```

Some More Notes

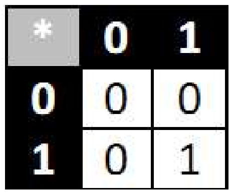
Taking abstraction to the max

Galois Field

- In everyday algebra, we usually think about fields as those we already know
  - E.g., the field of real numbers
- But since algebra is abstract, we can define our own fields
- Galois field: $GF(2)$
  - Elements $\{0, 1\}$
  - Addition: equivalent to XOR
  - Multiplication: as usual
- Usage: in cryptography
  - If you're interested, you can have a look at this paper



+	0	1
0	0	1
1	1	0



*	0	1
0	0	0
1	0	1

A Note about Vectors

- One more application of abstractions
- Vector
  - A line segment with a direction
- We saw that 2D vectors and 2D points have a one-to-one correspondence
  - A point can be represented as its **radius-vector**
- A vector is also an ordered tuple of coordinates
  - That's why we were able to take out thinking of points and apply it to complex numbers
- We usually represent vectors as Python lists: `[2, 3, -5]`
- **Idea**
  - Can we think of the list as a mapping: $0 \Rightarrow 2, 1 \Rightarrow 3,\ 2 \Rightarrow -5$?
  - What does this mean?
  - … we'll find out more next time
- What does this imply about fields?

Summary

- Polynomials
- Sets
- Functions
- Coordinates
- Complex numbers
- Abstraction

Questions?