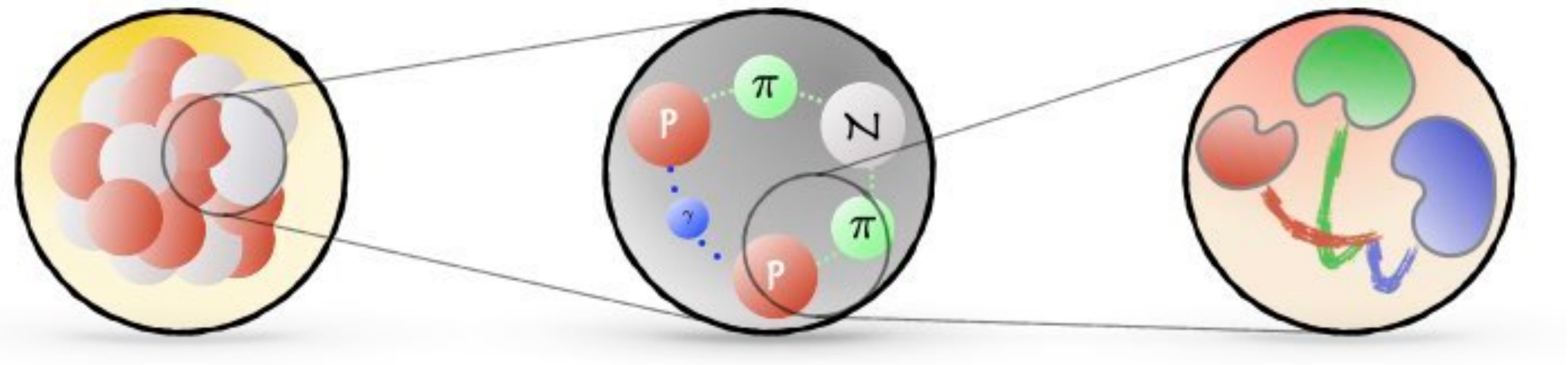

RUHR-UNIVERSITÄT BOCHUM

Theoretical Hadron Physics—160413-WS 20/21

Lecture 12—09.12.2020: Group Theory—The quark model, spin & color

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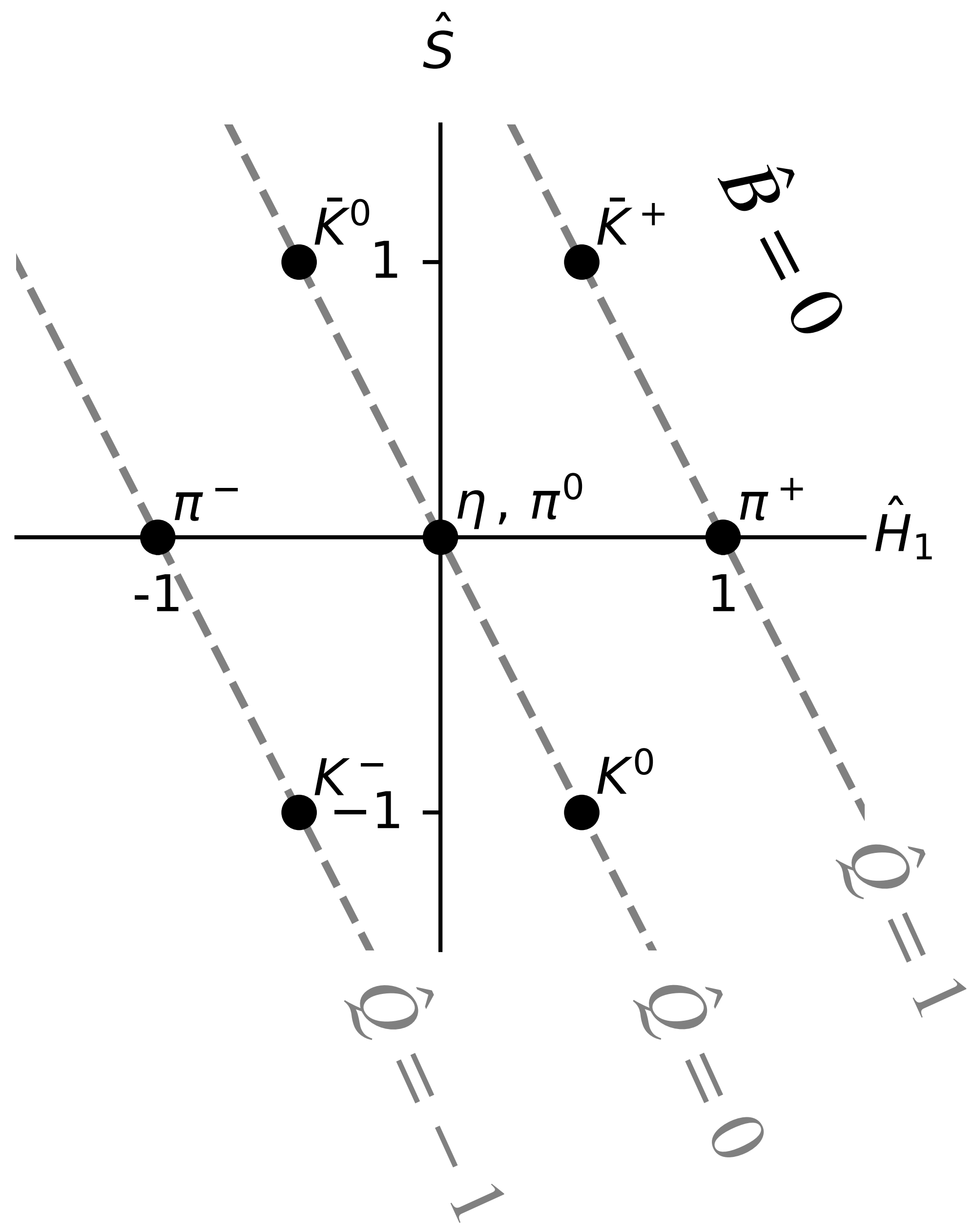
Previous lecture

$\hat{S}(\hat{H}_1;\hat{Q}) + \hat{B} = 2\hat{Q} - 2\hat{H}_1$

Quarks and $SU(3)$ multiplets



- The operators for strangeness ($\hat{S}$), charge ($\hat{Q}$), and Baryon number ($\hat{B}$) can be associated with the $su(3)$ Cartan generators by

$$\hat{S} + \hat{B} \equiv \frac{2}{\sqrt{3}}\hat{H}_2 \text{ and } \hat{Q} \equiv \hat{H}_1 + \frac{1}{\sqrt{3}}\hat{H}_2$$

  → Mesons have $\hat{B} = 0$ and baryons have $\hat{B} = 1$ (homework)

- The $su(3)$ rep for the $(1,1) \equiv 8$-multiplet (octet) aligns with the meson octet
  → Does this also work for other multiplets?
  → What about the fundamental multiplets?

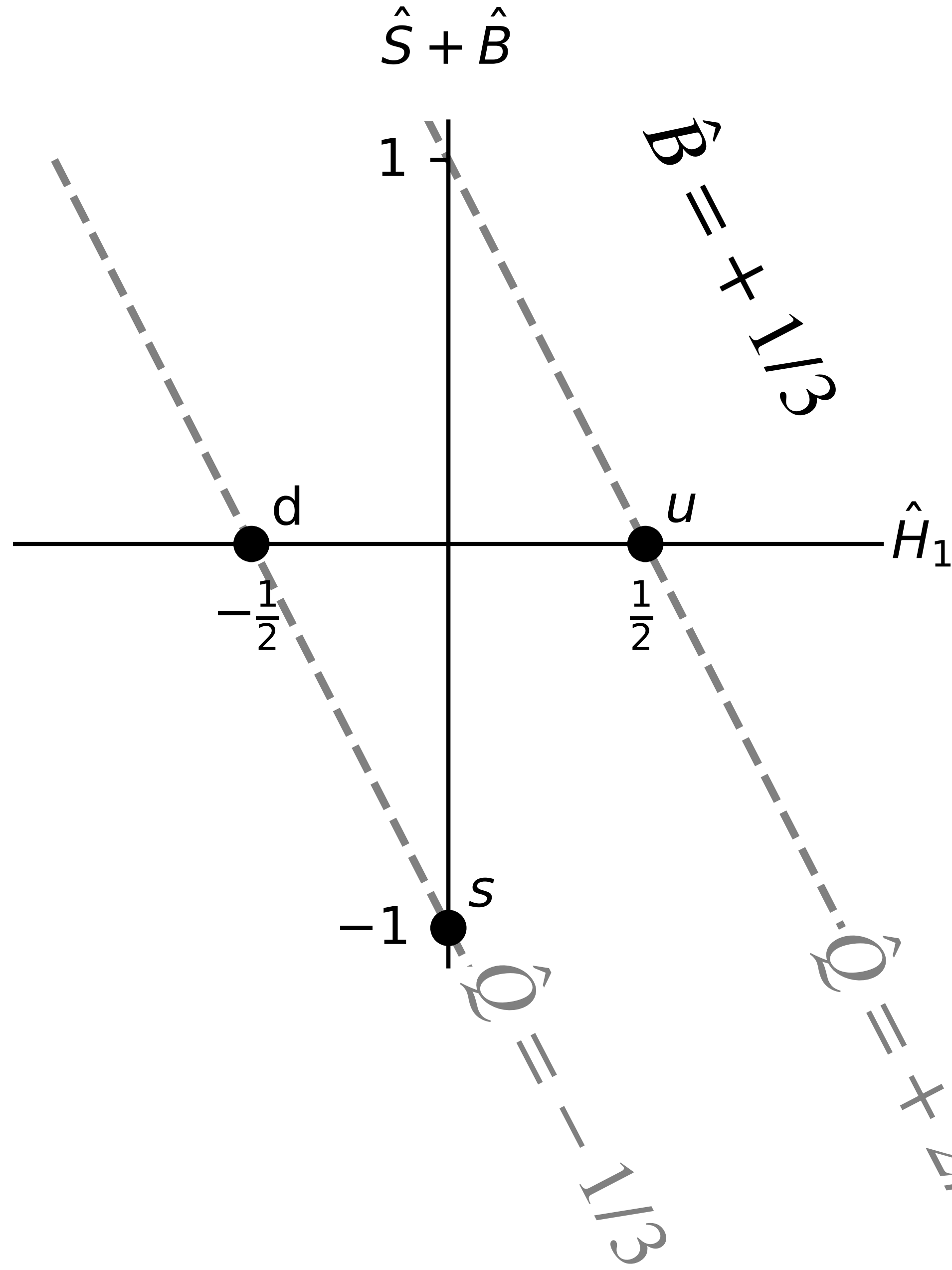
- Particles corresponding to the fundamental $(1,0) \equiv 3$-rep of $su(3)$, named quarks, have fractional charge and baryon number
  → These particles were labeled up- (u), down- (d), and strange-quark (s)
  → The complex conjugate $\bar{3}$-rep introduces their anti particles
  → Quarks have baryon number $\hat{B} = 1/3$

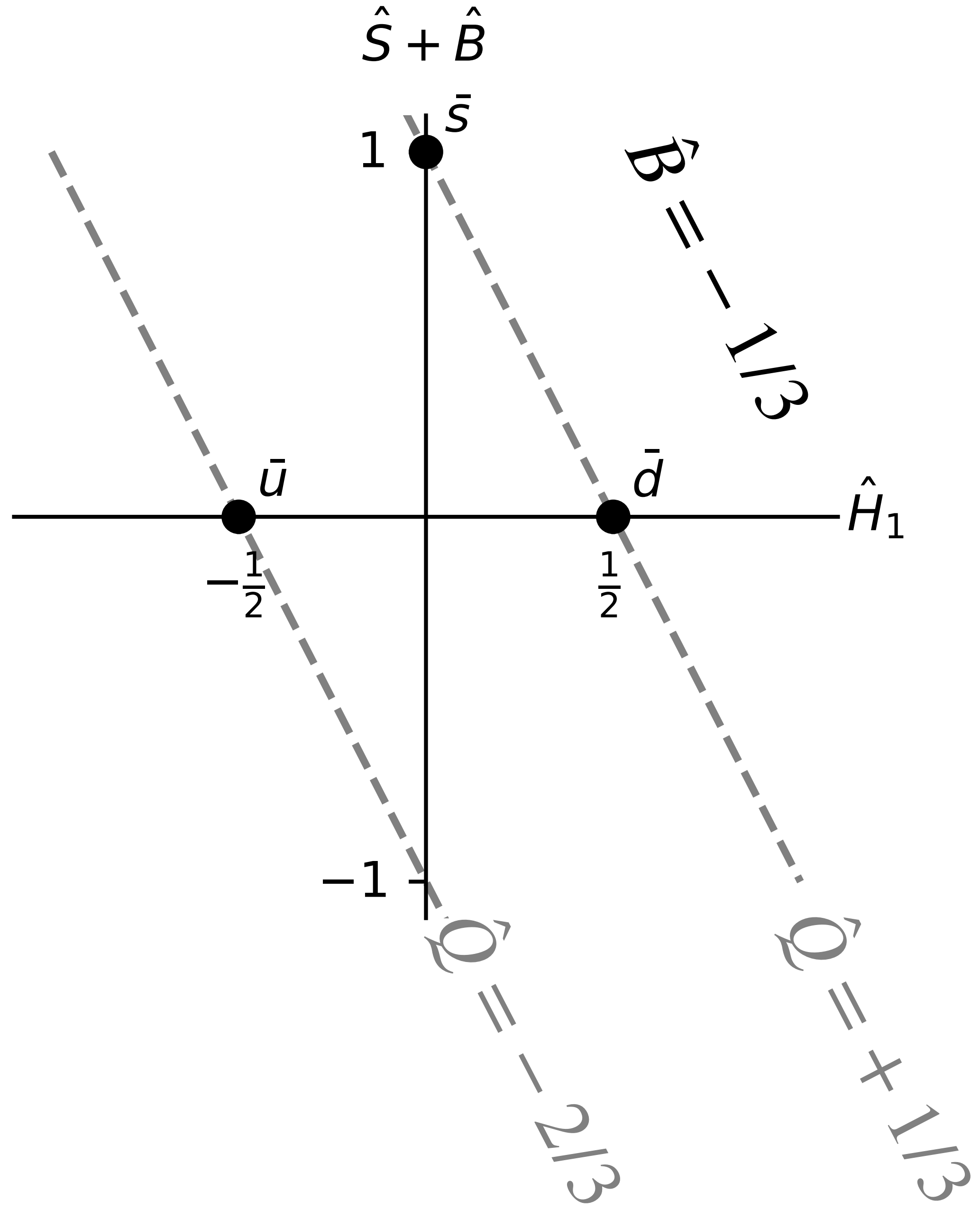
- The two fundamental representations can be associated with quark tensor states

$$|_1\rangle \equiv \left|\frac{1}{2},\frac{\sqrt{3}}{6};(1,0)\right\rangle = |u\rangle \qquad |^1\rangle \equiv \left|-\frac{1}{2},-\frac{\sqrt{3}}{6};(0,1)\right\rangle = |\bar{u}\rangle$$

$$|_2\rangle \equiv \left|-\frac{1}{2},\frac{\sqrt{3}}{6};(1,0)\right\rangle = |d\rangle \text{ and } |^2\rangle \equiv \left|\frac{1}{2},-\frac{\sqrt{3}}{6};(0,1)\right\rangle = |\bar{d}\rangle$$

$$|_3\rangle \equiv \left|0,-\frac{1}{\sqrt{3}};(1,0)\right\rangle = |s\rangle \qquad |^3\rangle \equiv \left|0,\frac{1}{\sqrt{3}};(0,1)\right\rangle = |\bar{s}\rangle$$





6. The quark model

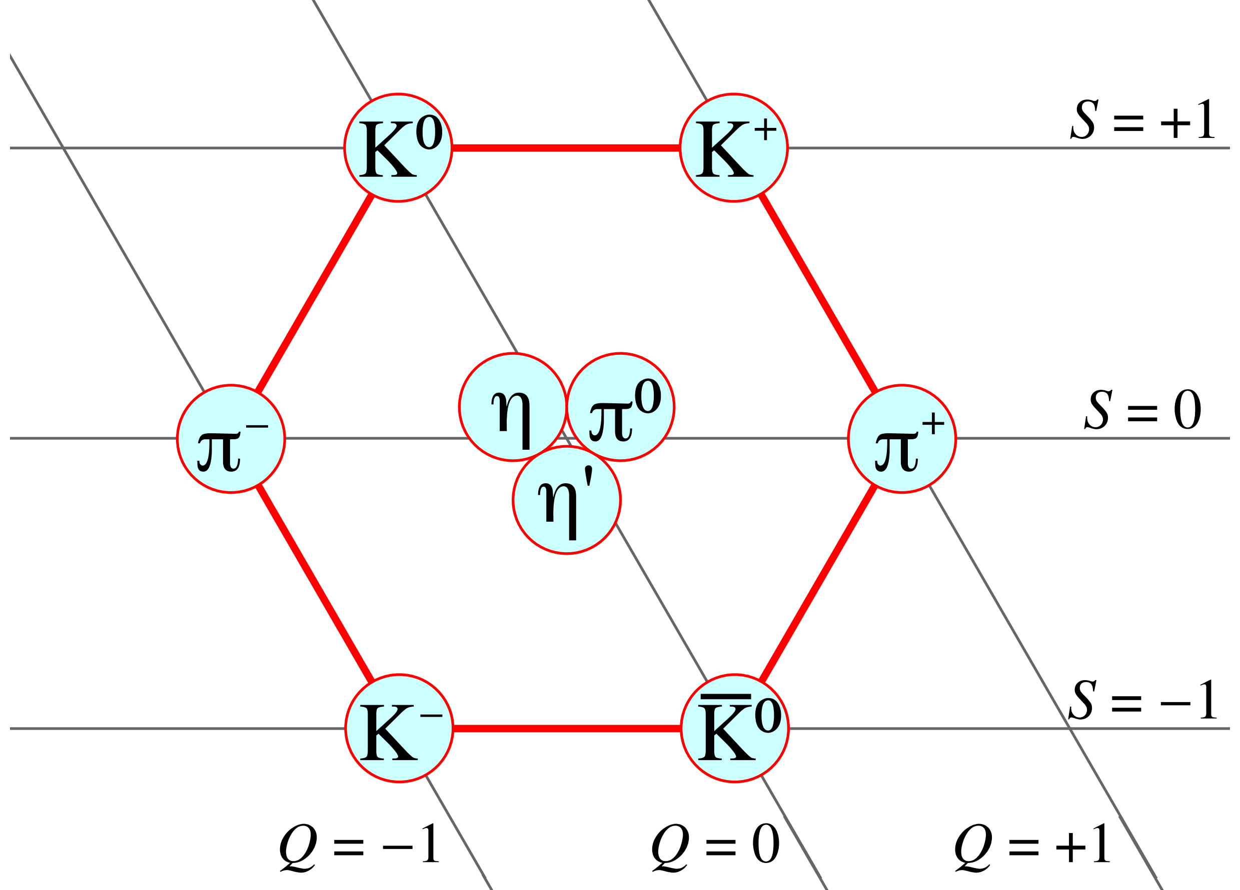
6.1 Overview

Hadrons & Multiplets

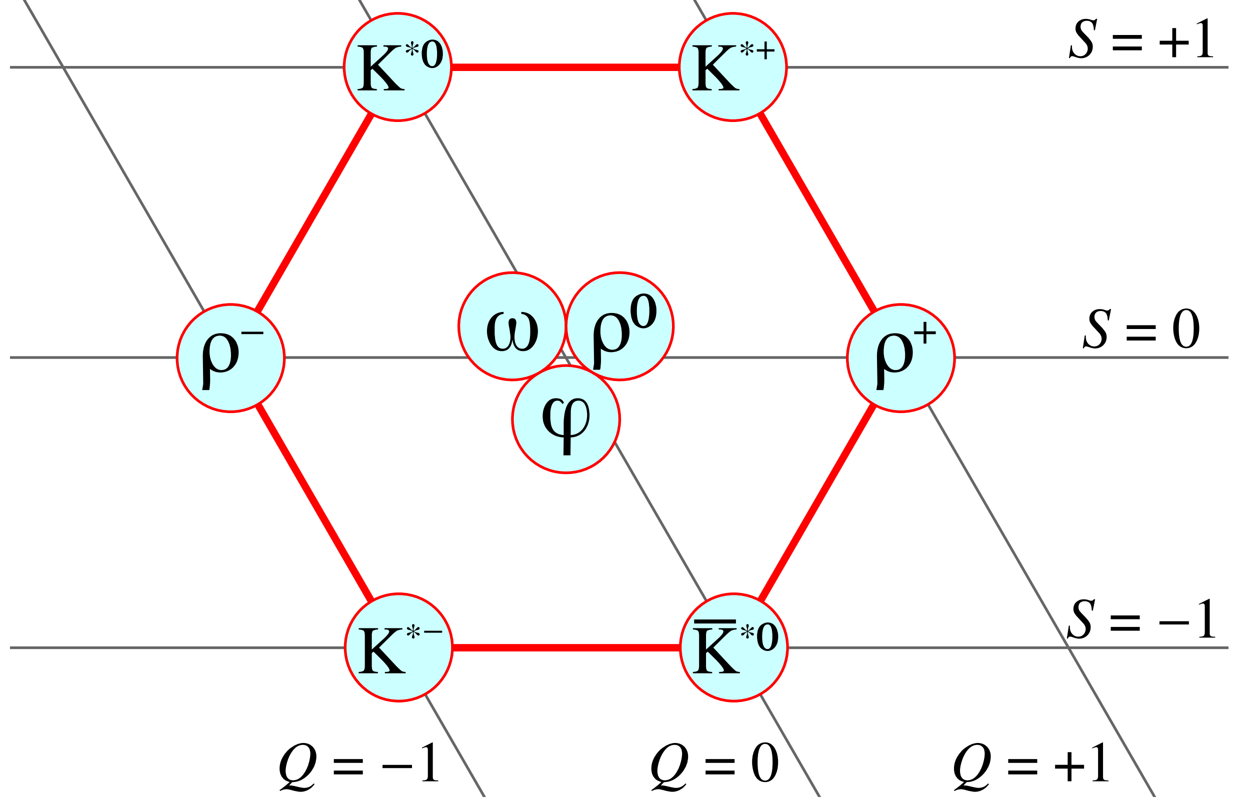
Aligning particles according to spin, charge and strangeness

- Spin-0^- mesons
  - $\rightarrow$ Octet: $\left\{(\pi^0, \pi^\pm), (K^0, K^+, \bar{K}^0, K^-), \eta\right\}$
  - $\rightarrow$ Singlet η'
- Spin-1^- (vector) mesons
  - $\rightarrow$ Octet: $\left\{(\rho^0, \rho^\pm), (K^{*0}, K^{*+}, \bar{K}^{*0}, K^{*-}), \omega\right\}$
  - $\rightarrow$ Singlet: ϕ

Spin-0^-



Spin-1^-



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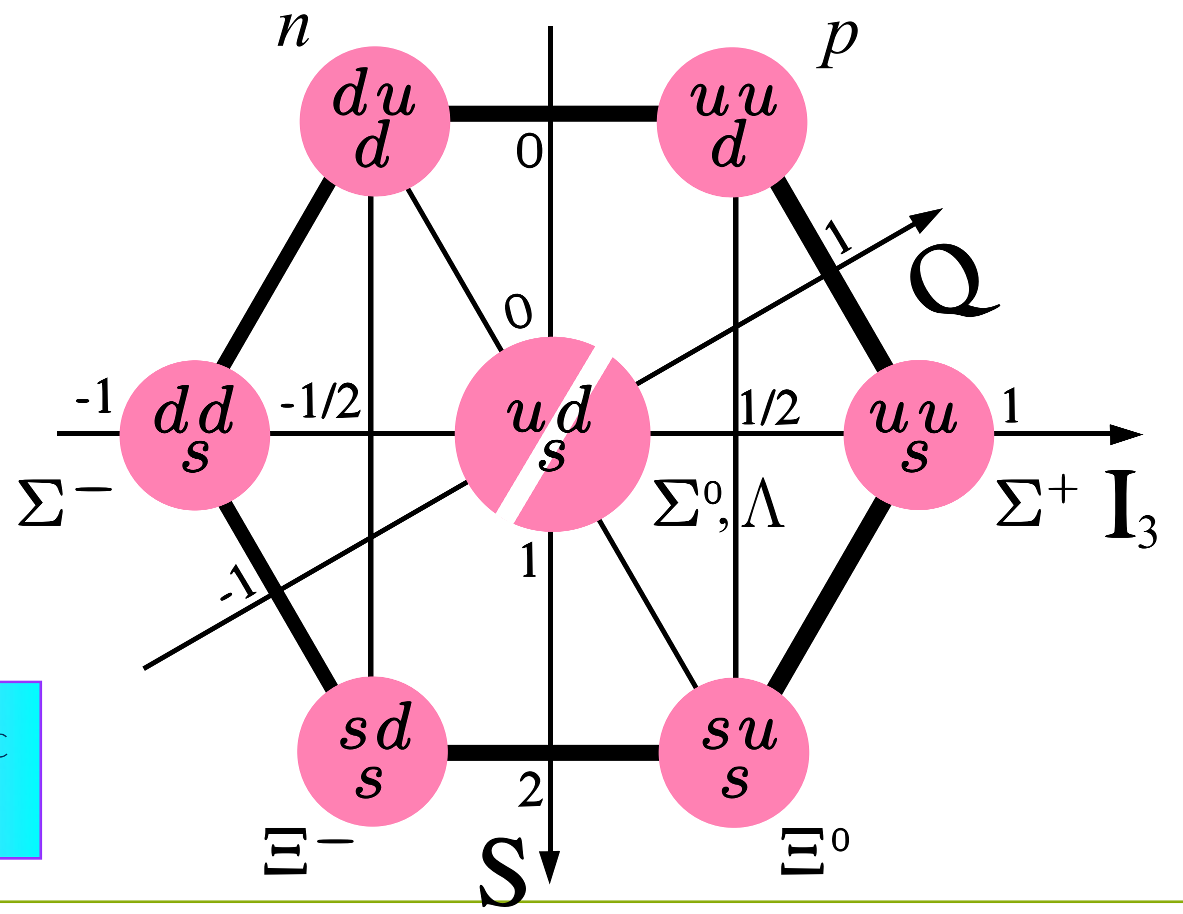

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- Spin-$\frac{1}{2}^+$ baryons
  - $\rightarrow$ Octet: $\left\{(n, p), (\Sigma^0, \Sigma^\pm), (\Xi^0, \Xi^-), \Lambda^0\right\}$
- Spin-$\frac{3}{2}^+$ baryons
  - $\rightarrow$ Decuplet: $\left\{(\Delta^-, \Delta^0, \Delta^+, \Delta^{++}), (\Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+}), (\Xi^{*-}, \Xi^{*0}), \Omega^-\right\}$
- Resonances and non-$SU(3)$ states (e.g., charm quark states)...

Quark-antiquark tensors

- Meson quantum numbers line up with tensor states of quarks and antiquarks

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0)$$

- While components of $Q \neq 0$ or $S \neq 0$ are unique, the $Q = 0 = S$ allows different linear combinations
- The corresponding tensor products can be expressed by

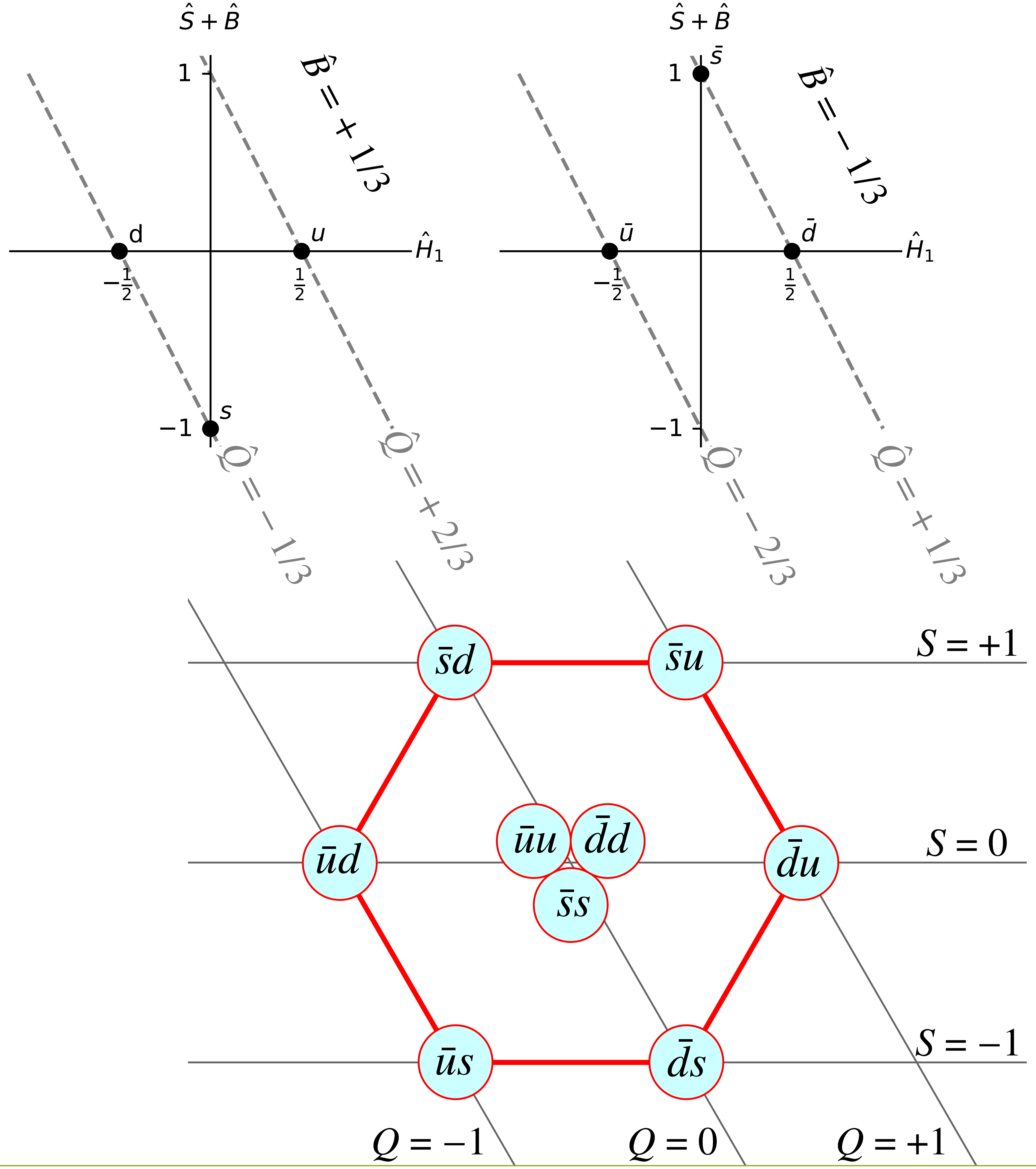
$$\psi^a_b = q^a \bar{q}_b = \tilde{\psi}^a_b + \delta^b_a S$$

→ Octet tensor

$$\tilde{\psi}^b_a = q^a \bar{q}_b - \frac{1}{3}\delta^a_b \bar{q}_c q^c = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}_{ab} - \frac{\bar{u}u + \bar{d}d + \bar{s}s}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ab}$$

→ Singlet tensor

$$S = \frac{1}{3} q_c q^c = \frac{\bar{u}u + \bar{d}d + \bar{s}s}{3}$$



Spin-0^- Mesons

- Comparing meson and quark singlet
  (after normalization, for now using singlet $\eta_1(\eta, \eta')$)

$$\eta_1 \stackrel{!}{=} \frac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}}$$

- Comparing meson and quark octet

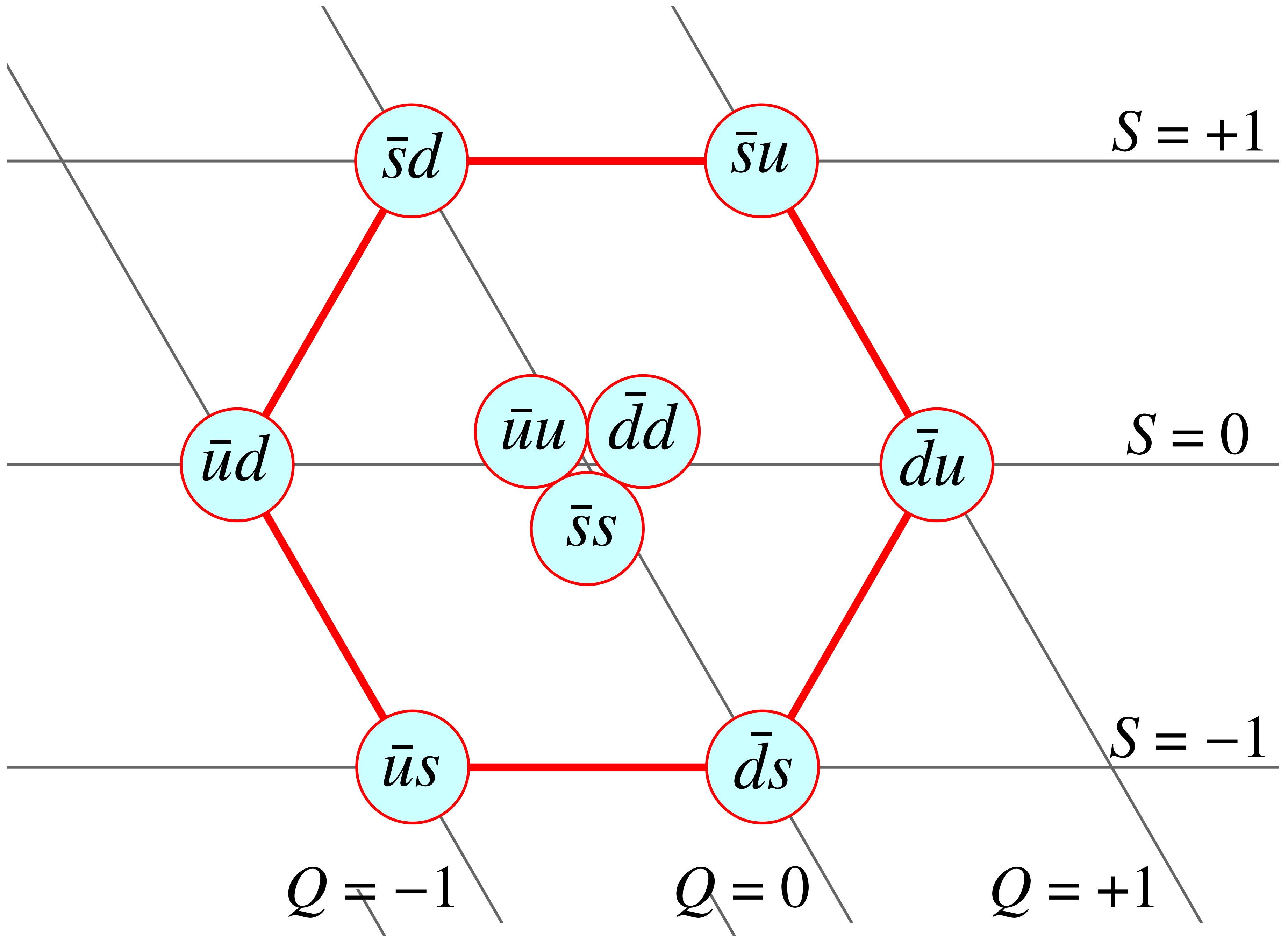
$$\begin{pmatrix} c_1\pi^0 + c_2\eta_8 & \pi^+ & K^+ \\ \pi^- & c_3\pi^0 + c_4\eta_8 & K^0 \\ K^- & \bar{K}^0 & c_5\pi^0 + c_6\eta_8 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \frac{2\bar{u}u - \bar{d}d - \bar{s}s}{3} & \bar{d}u & \bar{s}u \\ \bar{u}d & \frac{-\bar{u}u + 2\bar{d}d - \bar{s}s}{3} & \bar{s}d \\ \bar{u}s & \bar{d}s & \frac{-\bar{u}u - \bar{d}d + 2\bar{s}s}{3} \end{pmatrix}$$

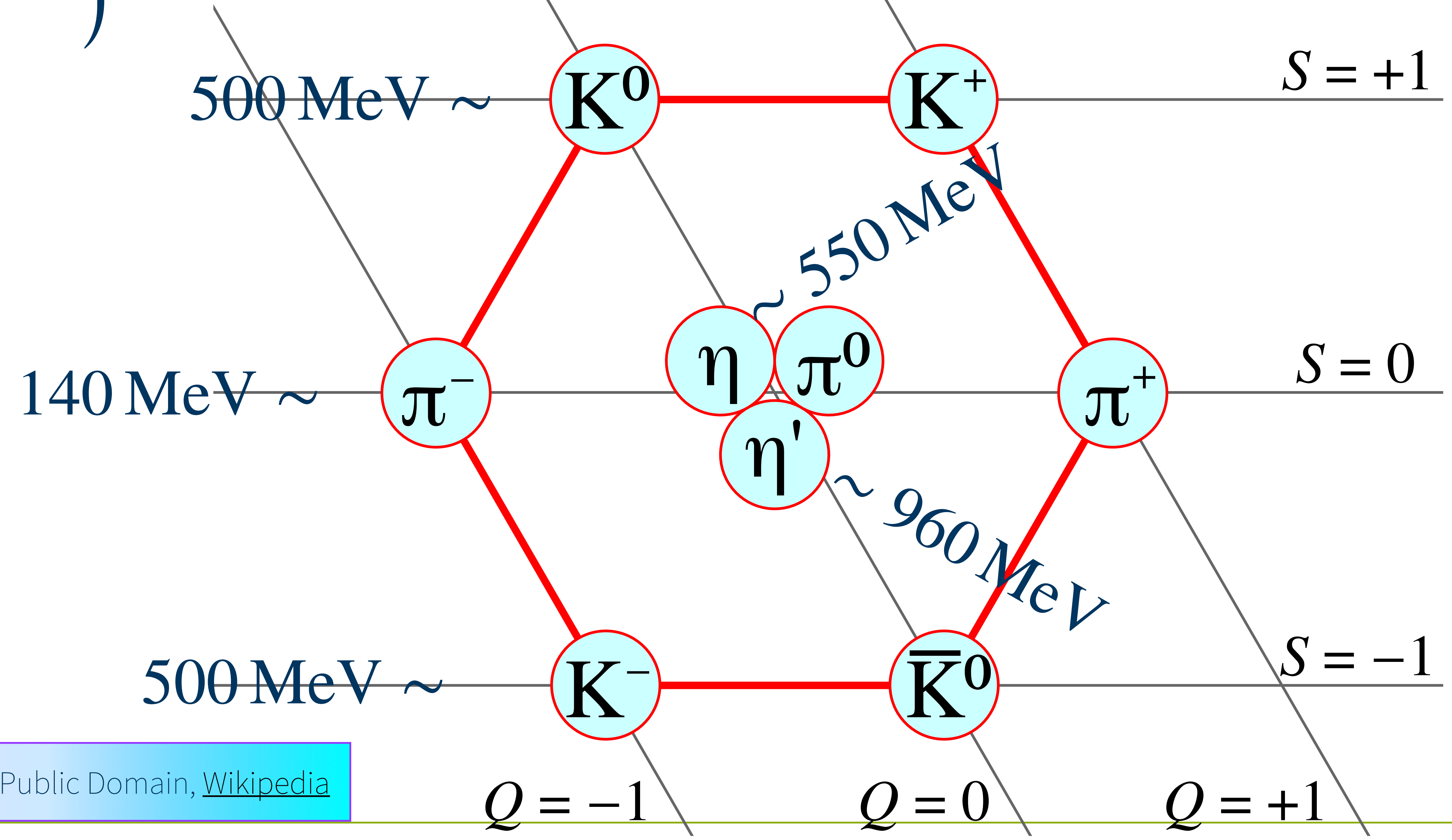
→ Defining octet η_8 to absorb strange bilinears (masses suggest strange content):

$$\eta_8 = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}} \text{ and thus } \pi^0 = \frac{\bar{u}u - \bar{d}d}{\sqrt{2}} \text{ results in the octet tensor}$$

$$M(0)^a_b \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}_{ab}$$

- What are η_1 and η_8 in terms of physical η and η'?
  → Masses suggest some mixing (e.g., $m_K < m_\eta \sim a m_{\eta_8} + b m_{\eta_1}$)





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Spin-1^- Mesons

- One can apply the same logic to find the spin-1^- meson multiplet tensors
- Comparing meson and quark singlet
  (after normalization, for now using singlet $\omega_1(\omega, \phi)$)

$$\omega_1 \overset{!}{=} \frac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}}$$

  → What is ω_1 in terms of physical ω and ϕ?

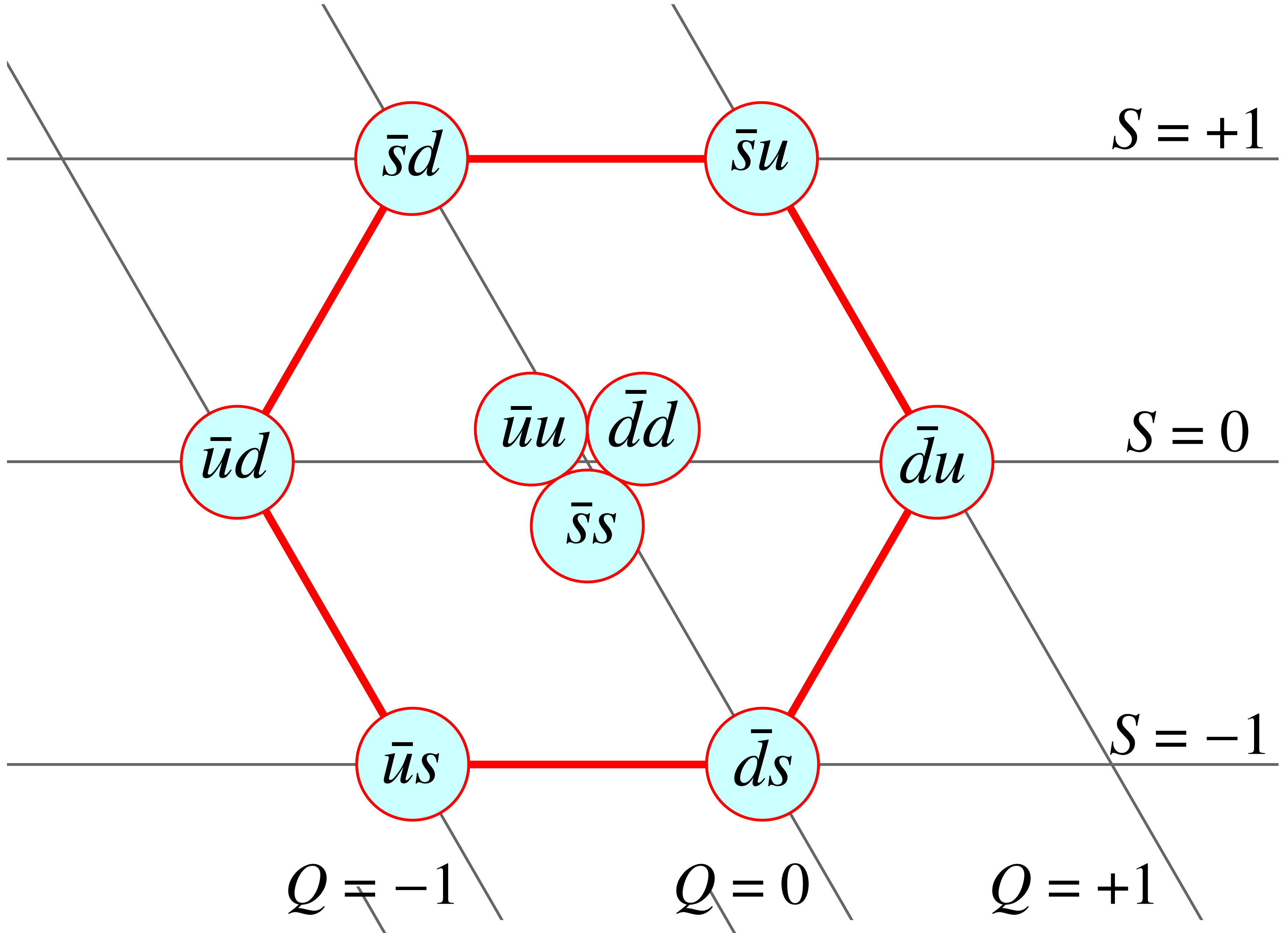
- Comparing meson and quark octet

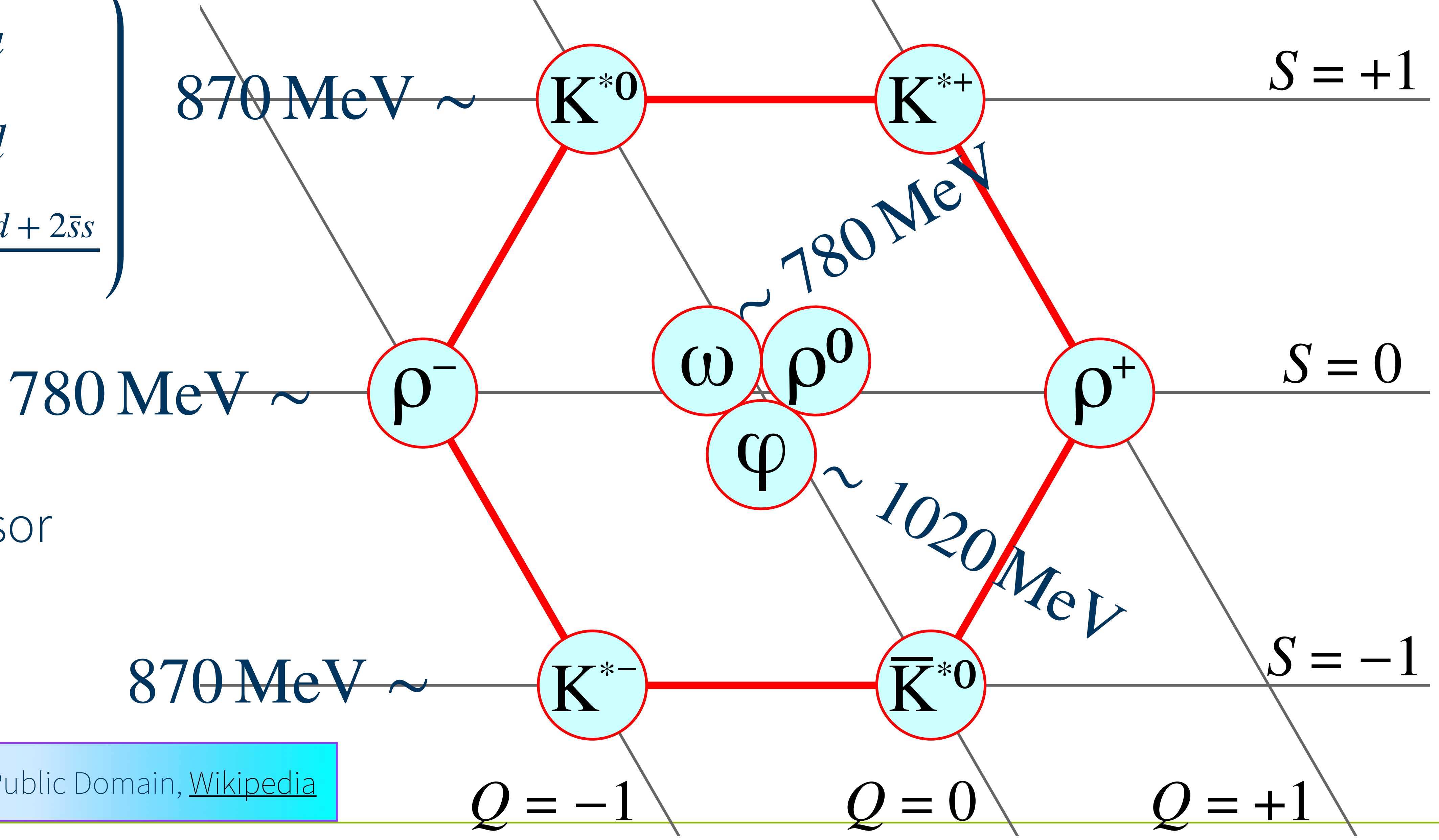
$$M(1)^a_b \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2}{\sqrt{6}}\omega_8 \end{pmatrix}_{ab} \overset{!}{=} \begin{pmatrix} \frac{2\bar{u}u - \bar{d}d - \bar{s}s}{3} & \bar{d}u & \bar{s}u \\ \bar{u}d & \frac{-\bar{u}u + 2\bar{d}d - \bar{s}s}{3} & \bar{s}d \\ \bar{u}s & \bar{d}s & \frac{-\bar{u}u - \bar{d}d + 2\bar{s}s}{3} \end{pmatrix}$$

  → Defining ω_8 to absorb strange bilinears

$$\omega_8 = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}} \quad \text{and thus } \rho^0 = \frac{\bar{u}u - \bar{d}d}{\sqrt{2}} \quad \text{results in the octet tensor}$$

- What are ω_1 and ω_8 in terms of physical ω and ϕ?
  → Masses suggest some mixing (e.g., $m_K < m_\omega \sim am_{\omega_1} + bm_{\omega_2}$)





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Quark-quark-quark tensors

- Baryon quantum numbers line up with tensor states of quarks and antiquarks

$$(1{,}0) \otimes (1{,}0) \otimes (1{,}0) = (0{,}0) \oplus (1{,}1) \oplus (1{,}1) \oplus (3{,}0)$$

- The corresponding tensor products can be expressed by

$$\psi^{abc} = q(1)^a q(2)^b q(3)^c = \epsilon^{abc} A + M_A^{abc} + M_S^{abc} + S^{abc}$$

→ The decuplet tensor S^{abc} is symmetric in the pairwise exchange of all components

→ The singlet tensor A is completely antisymmetric in the pairwise exchange of all components

→ The tensor M_A^{abc} is antisymmetric in $(a \leftrightarrow b)$

$$M_A^{abc} = \epsilon^{abd} B(1)_d^c$$

→ The tensor M_S^{abc} is symmetric in $(a \leftrightarrow b)$

$$M_S^{abc} = \epsilon^{acd} B(2)_d^b + \epsilon^{bcd} B(2)_d^a$$

→ Both $B(1)$ and $B(2)$ are traceless (independent) tensors and thus belong to the octet (homework)

- The completely symmetric and antisymmetric parts are

$$S^{abc} = \frac{1}{6}\left(q(1)^a q(2)^b q(3)^c + \text{Permutations}\right) \text{ and } A = \frac{1}{6}\epsilon_{abc}\psi^{abc}$$

→ The rest is homework :)

- In principle, one has to show that this decomposition reproduces ψ^{abc}, however, since components are linearly independent, dimensional arguments suffice

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$$

$$\varepsilon_{jmn}\varepsilon^{imn} = 2\delta_j^i$$

$$\varepsilon_{ijk}\varepsilon^{ijk} = 6$$

$\psi^{abc} = S^{abc}$ + Asymmetric parts

$\Rightarrow \psi^{111} = S^{111} + 0$

$S^{abc} = \frac{1}{6} \sum_{\text{perm }(a,b,c)} \psi^{abc} = \frac{1}{6}\left(\psi^{abc} + \psi^{acb} + \psi^{bac} + \dots\right.$

$\varepsilon_{abc}\psi^{abc} = ?$

$\varepsilon_{abc}\varepsilon^{abc} A = 6A$

is traceless

$\varepsilon_{abc} M_A^{abc} = \varepsilon_{abc}\varepsilon^{abd} B(1)_d^c = 2\delta_c^d B(1)_d^c = 0$

$\varepsilon_{abc} M_S^{abc} = \varepsilon_{abc}\varepsilon^{acd} B(2)_d^b + \varepsilon^{bcd}\varepsilon_{abc} B(2)_d^a$

ε^{dbc}

$= -2\delta_b^d B(2)_d^b + 2\delta_a^d B(2)_d^a$

$= 0$

$\varepsilon_{abc} S^{abc} = 0$

symmetric

$\Rightarrow \varepsilon_{abc}\psi^{abc} = 6A$

$\Rightarrow A = \frac{1}{6}\varepsilon_{abc}\psi^{abc}$

Quark-quark-quark tensors

The baryon octets

- Both octet components do not allow terms proportional to a product of three times the same quark flavor

$$q(1)^a q(2)^a q(3)^a \notin B(1), B(2)$$

- The octet components differ in their respective symmetries

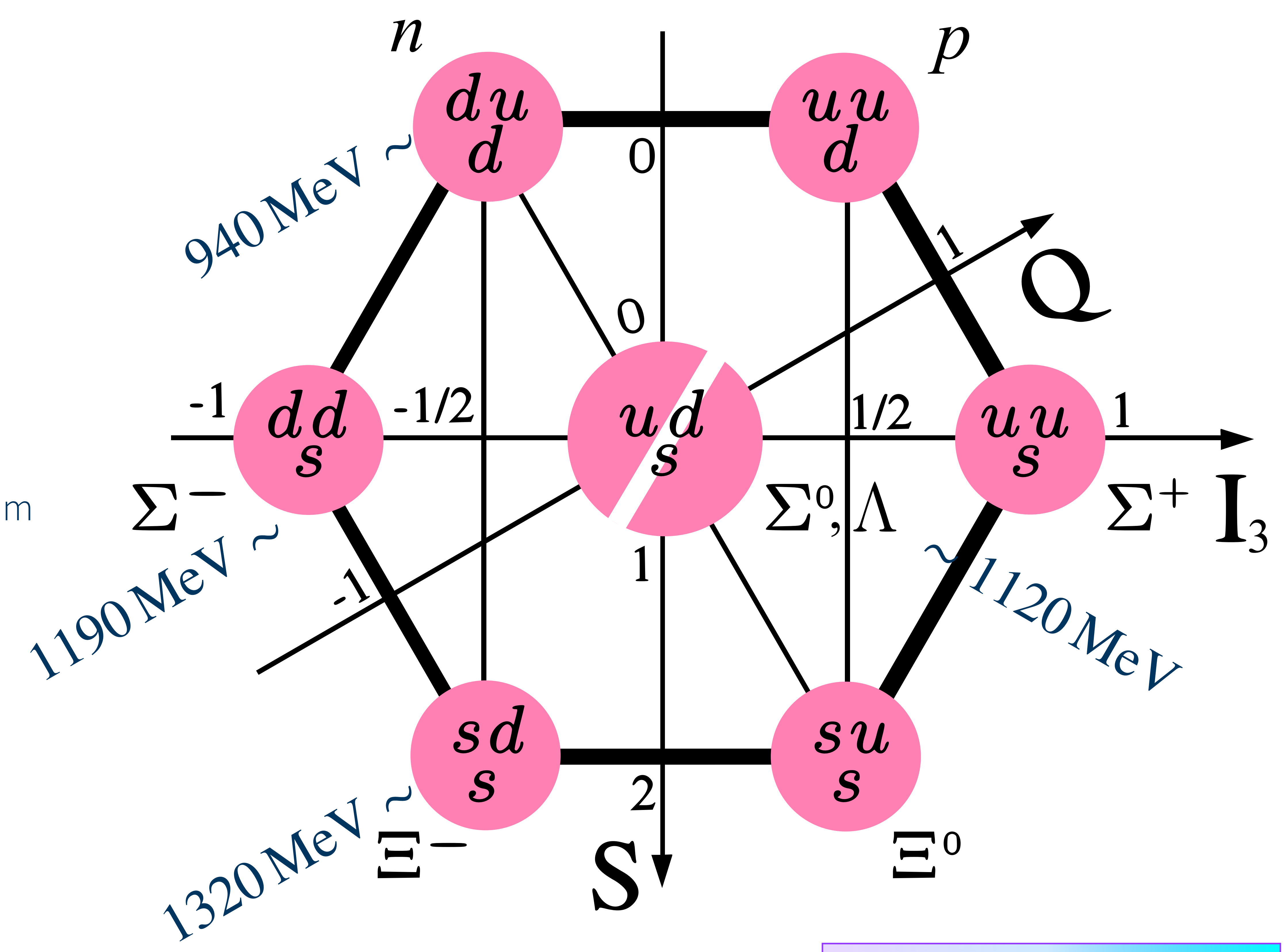
$$B(1)^1_2 = \frac{1}{2}\left(q(1)^3 q(2)^1 - q(1)^1 q(2)^3\right) q(3)^1 = \frac{1}{2}\left(s(1)u(2)u(3) - u(1)s(2)u(3)\right)$$

$$B(2)^1_2 = \frac{1}{6}\left(s(1)u(2)u(3) + u(1)s(2)u(3) - 2u(1)u(2)s(3)\right)$$

- Which octet describes, e.g., nucleons?
  - → This is a difficult question which also depends on spin, color and angular momentum quantum numbers. In general, "every operator which has overlap with the Baryon quantum numbers"
- One can still parameterize baryons effectively without specifying their quark content

$$B^a_b \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}_{ab}$$

- Baryon decuplet allows $q(1)^a q(2)^a q(3)^a$ combinations (Homework)



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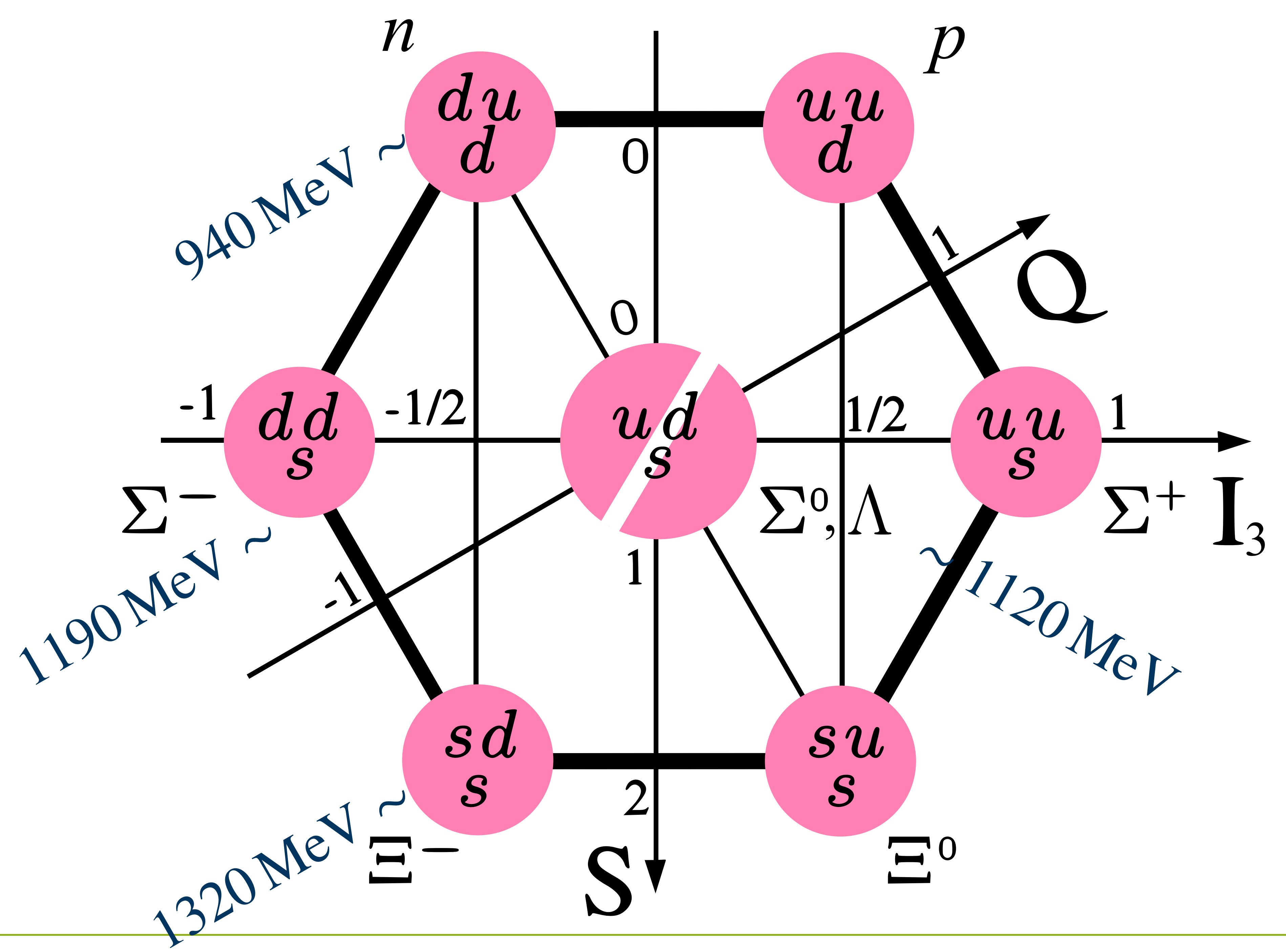
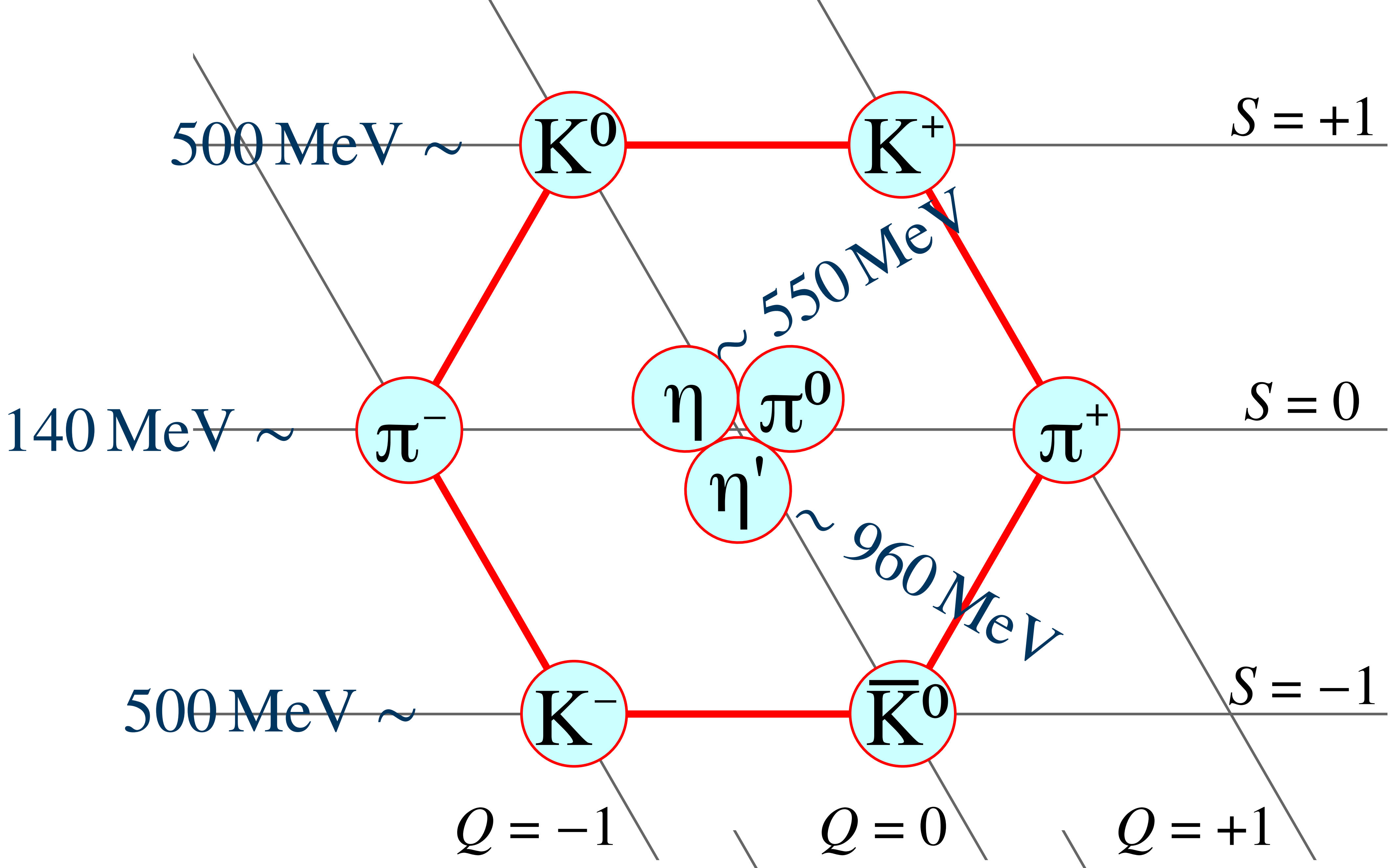
6. The quark model

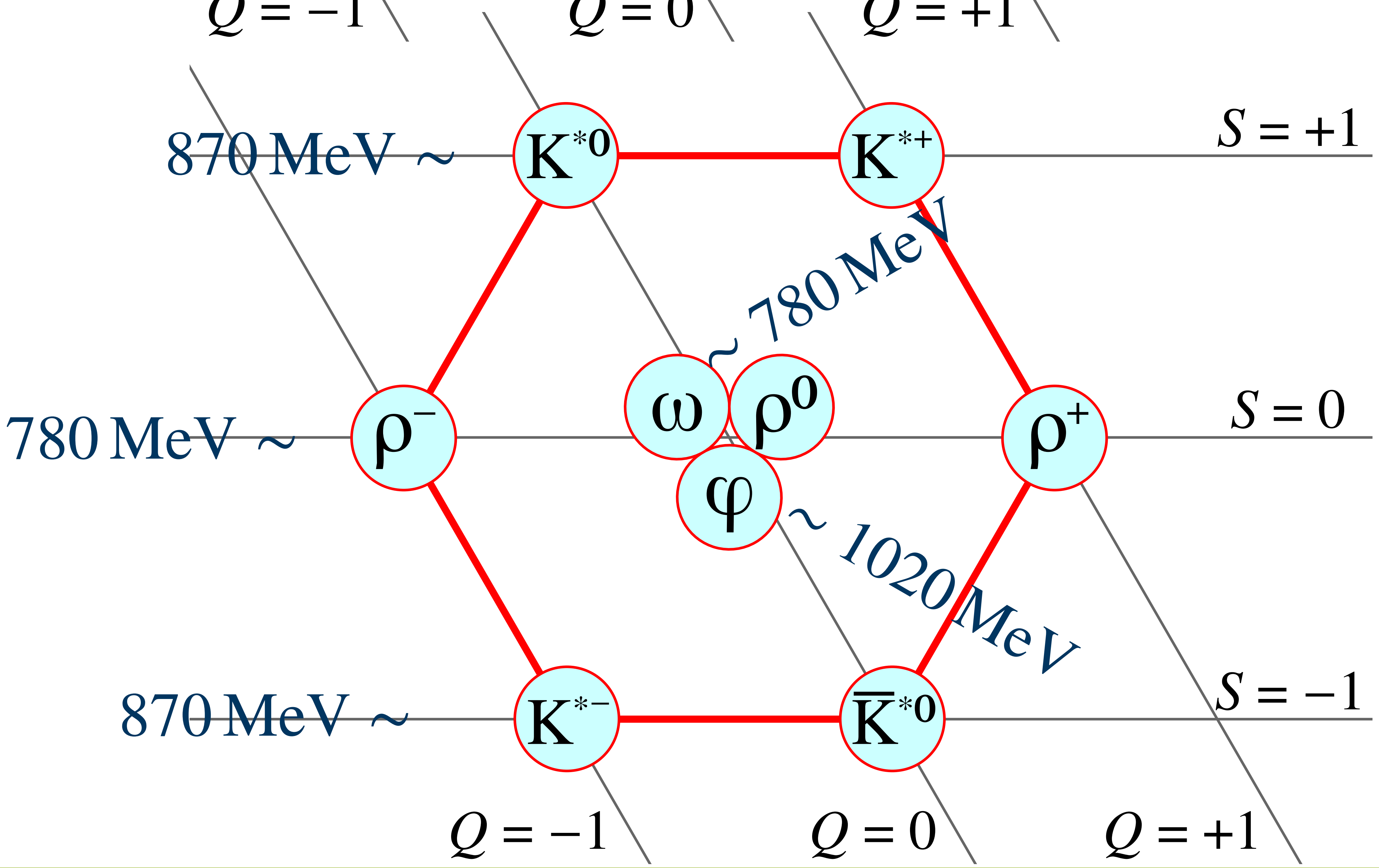
6.2 Gell-Mann Okubo mass relations

Symmetrie breaking

- Even though $SU(3)$ seems to capture the nature (quantum numbers) of the hadrons, masses are slightly different
  - → If $SU(3)$ was an exact symmetry, states in one multiplet should all have the same mass
  - → This suggests that the $SU(3)$ symmetry is broken in a small way





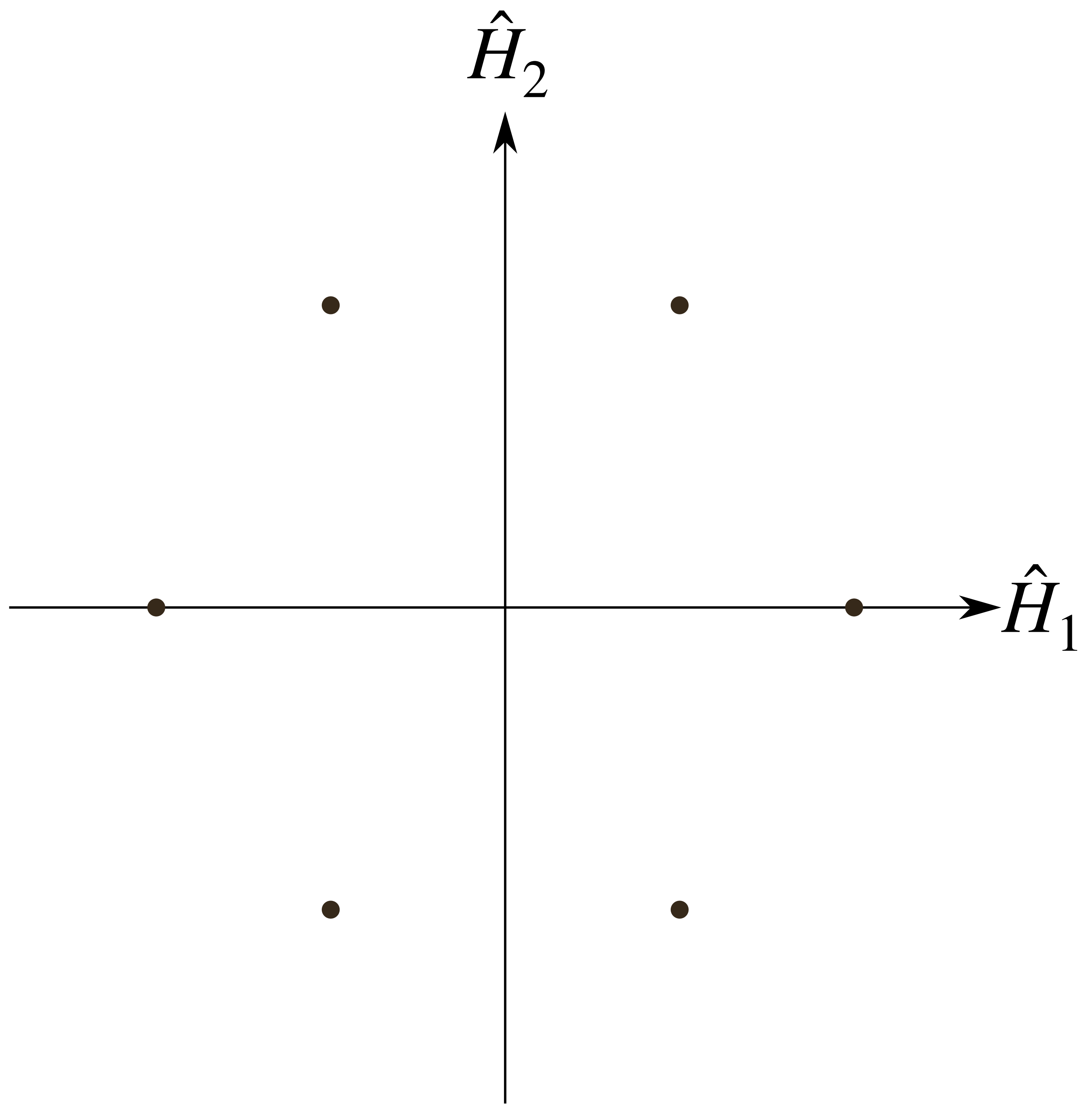


Approximate symmetries

$$\lambda_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \text{ for } a = 1,2,3$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- States orthogonal to strangeness axis approximately have same masses
  → By choice, the strangeness axis is proportional to the Cartan generator $\hat{H}_2$ which can be identified with the Gell-Mann matrix λ_8
  → The orthogonal axis, $\hat{H}_1$, can be identified with λ_3 which is the embedding of the Pauli-Matrix σ_3 in $su(3)$
  ⇒Within $SU(3)$, isospin values for fixed strangeness seem to be conserved
- If we identify the structure which breaks $SU(3)$, we are capable of describing the symmetry breaking effects perturbatively
  → To a good approximation, the symmetry breaking conserves isospin but breaks strangeness



Symmetry breaking

- At the level of individual quarks, different masses break the symmetry

$$H_M \equiv m_u \bar{u}\, u + m_d \bar{d}\, d + m_s \bar{s}\, s = m_a^b\, \bar{q}_b q^a \text{ with } m_a^b \equiv \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}_{ab}$$

→ If all masses where the same, the mass tensor would correspond to $m_a^b = m_0 \delta_a^b$

⇒It would correspond to an invariant tensor and thus vanish by the action of $su(3)$ generators $\hat{T}_c \cdot m_a^b = 0$

- How large is the breaking of $SU(3)$ due to quark masses (how different is m_a^b from $m_0 \delta_a^b$)

→ Depending on the scheme $m_u \approx 2$MeV, $m_d \approx 5$MeV, and $m_s \approx 100$MeV

⇒This can be approximated by $m_u = m_d \neq m_s$ (compared to Hadron masses around 100s or 1000s of MeV)

Decompose mass tensor as invariant tensor plus traceless tensor

$$m_a^b = m_0 \delta_a^b + \Delta m Q_a^b \text{ where } Q_a^b \equiv 2\sqrt{3}\, \left(T_8\right)_a^b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}_{ab}$$

Diagonal and same term for u and d, different for s

$$\text{Thus} \begin{cases} m_0 + \Delta m = m_u \\ m_0 - 2\Delta m = m_s \end{cases} \Rightarrow \begin{cases} m_0 = \dfrac{2m_u + m_s}{3} \\ \Delta m = \dfrac{m_u - m_s}{3} \end{cases}$$

→ This introduces the general form (symmetry) of the symmetry breaking term: Q_a^b is a traceless tensor in the $(1,1)$-rep proportional to $\hat{T}_8$

Symmetry breaking
At the baryon level (octet)

$$m_a^b = m_0\delta_a^b + \Delta m Q_a^b \text{ where } Q_a^b \sim \left(T_8\right)_a^b$$

- $SU(3)$ symmetry breaking has consequences on the level of hadrons

$$\hat{H}_{\text{mass}} = \hat{H}_{m_0} + \underbrace{\hat{H}_{\Delta m}}_{\text{Breaks } SU(3)}$$

→ It is not obvious how quarks translate to baryons,
⇒the only thing we can say is that the symmetry and it's breaking must be present

In total scalar contraction

$8^3 = 512$ combinations
(In the worst case we need 512 parameters to make it scalar)

$$\langle B\,|\,\hat{H}_{\text{mass}}\,|\,B\rangle = \langle B\,|\,\hat{H}_{m_0}\,|\,B\rangle + \langle B\,|\,\hat{H}_{\Delta m}\,|\,B\rangle = m_B\bar{B}_a^b B_e^f\langle {}_a^b\,|{}_f^e\rangle + \bar{B}_a^b\left(Q\right)_c^d B_e^f\langle {}_a^b\,|\,\hat{O}_d^c\,|{}_f^e\rangle$$

- Express Hamiltonian by all allowed combinations of available operators

$$\langle B\,|\,\hat{H}_{\text{mass}}\,|\,B\rangle = m_B\bar{B}_a^b B_b^a + m_1\bar{B}_a^b\left(Q\right)_f^a B_b^f + m_2\bar{B}_a^b B_b^c\left(Q\right)_c^a$$
$$= m_B\text{tr}(B^\dagger B) + m_1\text{tr}(B^\dagger Q B) + m_2\text{tr}(B^\dagger B Q)$$

This is a consequence of Wigner-Eckart

These is a very complicated and unknown functions of the quark masses and other fundamental parameters

→ This is an effective (field theoretical) approach:
⇒Express the theory by degrees relevant for this scale $B^\dagger, B$ (note that $B^\dagger \neq B$ because antibaryons $\neq$ baryons)
⇒Which are constraint by symmetries (scalar contractions involving different powers of Q)
⇒And ordered by significance (e.g., terms at different orders $\mathcal{O}(Q^2) \ll \mathcal{O}(Q)$)

Octet Gell-Mann Okubo mass formula

$$B^a_b \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}_{ab}$$

$$\langle B | \hat{H}_{\text{mass}} | B \rangle = m_B \text{tr}(B^\dagger B) + m_1 \text{tr}(B^\dagger Q B) + m_2 \text{tr}(B^\dagger B Q)$$

- Symmetry breaking adjust the mass terms
  → Without symmetry breaking

$$m_B \text{tr}(B^\dagger B) = m_B \left[|\Lambda|^2 + |\Sigma^0|^2 + |\Sigma^-|^2 + |\Sigma^+|^2 + |\Xi^0|^2 + |\Xi^-|^2 + |n|^2 + |p|^2 \right]$$

  → Symmetry breaking terms

$$m_1 \text{tr}(B^\dagger Q B) = m_1 \left[-|\Lambda|^2 + |\Sigma^0|^2 + |\Sigma^-|^2 + |\Sigma^+|^2 - 2|\Xi^0|^2 - 2|\Xi^-|^2 + |n|^2 + |p|^2 \right]$$
$$m_2 \text{tr}(B^\dagger B Q) = m_2 \left[-|\Lambda|^2 + |\Sigma^0|^2 + |\Sigma^-|^2 + |\Sigma^+|^2 + |\Xi^0|^2 + |\Xi^-|^2 - 2|n|^2 - 2|p|^2 \right]$$

- The isospin families have adjusted masses

$$m_N = m_B + m_1 - 2m_2 \qquad m_\Sigma = m_B + m_1 + m_2$$
$$m_\Lambda = m_B - m_1 - m_2 \qquad m_\Xi = m_B - 2m_1 + m_2$$

- This allows relating different mass terms, for example

$$m_\Lambda = \frac{1}{3}\left(2\left(m_N + m_\Xi\right) - m_\Sigma\right) = \frac{1}{3}\left(2\,(940 + 1320) - 1190\right)\,\text{MeV} = 1110\,\text{MeV}$$

  → Which compares very well against $m_\Lambda^{(\text{exp})} = 1115\,\text{MeV}$
    ⇒This difference is even smaller than isospin breaking effects we have not yet considered

6. The quark model

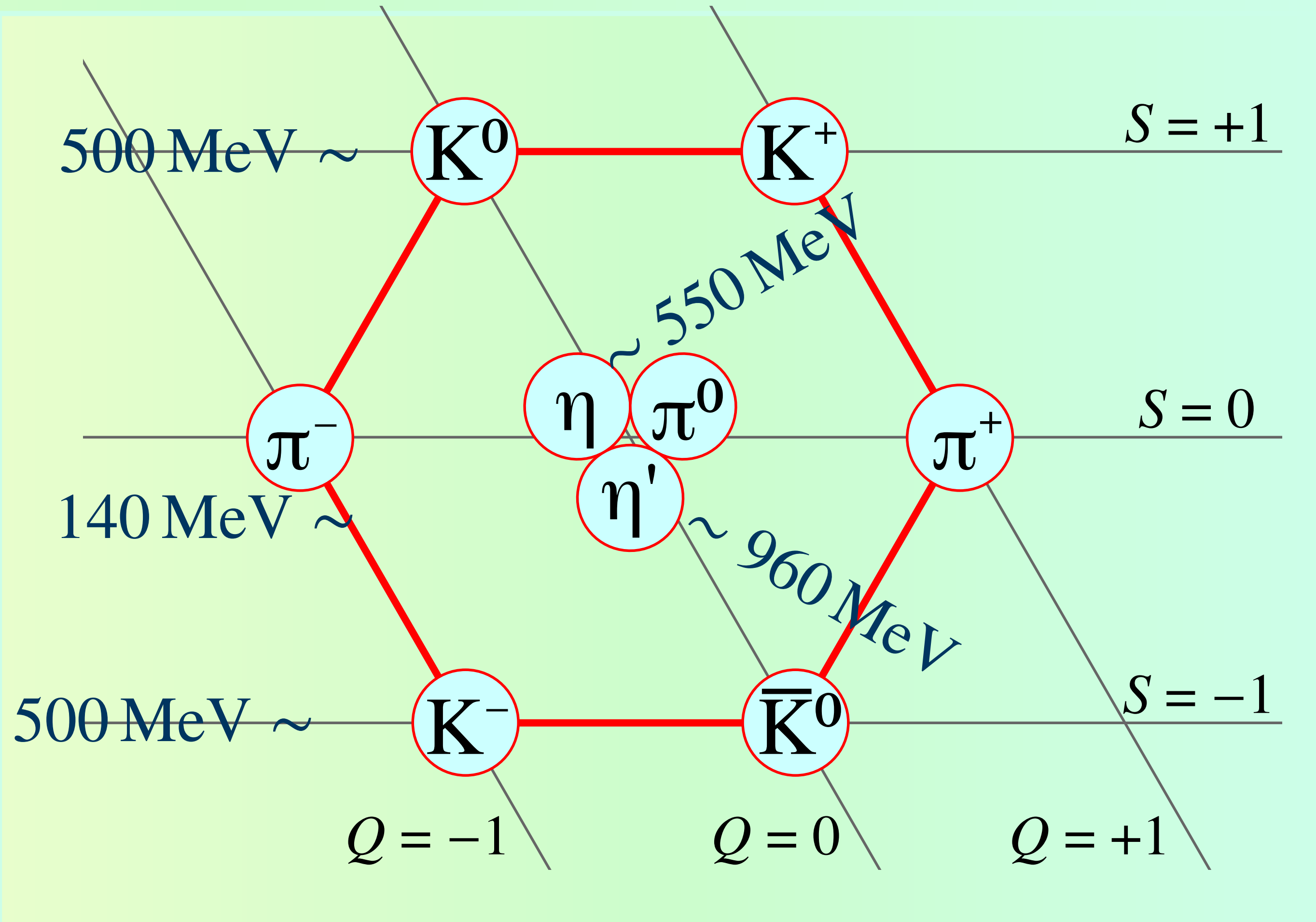
6.3 Mixing angles

$$m_a^b = m_0 \delta_a^b + \Delta m Q_a^b \text{ where } Q_a^b \sim \left(T_8\right)_a^b$$

Symmetry breaking

At the meson level (spin-0^- octet)

The Hamiltonian here is technically a density (depending on fields). Thus we want the integral $\left[d^4 x H(x)\right] = 1$ to be dimensionless.
While fermion fields have mass dimension $[p] \sim \mathrm{MeV}^{3/2}$, and thus need a $m_p \bar{p} p$ term, boson fields have $[\pi] \sim \mathrm{MeV}$ and thus need $m_\pi^2 \pi^2$ terms.



- Similarly to baryons, symmetry breaking at the quark level has consequences for the mesons as well

$$\hat{H}_{\text{mass}} = \hat{H}_{m_0} + \hat{H}_{\Delta m} \qquad \text{Breaks } SU(3)$$

→ Different to baryons, the meson octet is it's own complex conjugated representation

⇒Thus, the tensors at our disposal are only M and Q (no $\bar{M}$) and thus there is only one correction term

$$\langle M | \hat{H}_{\text{mass}} | M \rangle = \langle M | \hat{H}_{m_0} | M \rangle + \langle M | \hat{H}_{\Delta m} | M \rangle = m_M^2 \mathrm{tr}(M^2) + m_1^2 \mathrm{tr}(M^2 Q)$$

- The normalization of Ms components is such that

$$|\vec{\pi}|^2 = |\pi^0|^2 + \pi^+\pi^- + \pi^-\pi^+ = \sum_{i=1}^{3} |\pi_i|^2$$

$$m_M^2 \mathrm{tr}(M^2) = m_M^2 \left[|\vec{K}|^2 + (\eta^8)^2 + |\vec{\pi}|^2 \right]$$

→ Therefore, the symmetry breaking terms induces

$$m_1^2 \mathrm{tr}(M^2 Q) = m_1^2 \left[-\frac{1}{2} |\vec{K}|^2 - (\eta^8)^2 + |\vec{\pi}|^2 \right]$$

→ The physical meson masses are related to the parameters by

$$m_K^2 = m_M^2 - \frac{1}{2} m_1^2,\ m_{\eta_8}^2 = m_M^2 - m_1^2 \text{ and } m_\pi^2 = m_M^2 + m_1^2 \text{ and thus one finds } m_{\eta_8}^2 = \frac{4 m_K^2 - m_\pi^2}{3}$$

$$m_{\eta_8}^2 = \frac{4m_K^2 - m_\pi^2}{3}$$

Octet singlet mixings

- Substituting in physical Kaon and pion masses results in

$$m_{\eta_8} = \sqrt{\frac{4m_K^2 - m_\pi^2}{3}} = 570\,\text{MeV}$$

→ How do we repeat the singlet analysis (there are no other states in the singlet)

- Because there are two pseudo scalar mesons η and η', we can express them as a superposition of the singlet and octet meson

$$\eta_1 \equiv -\eta\sin(\theta) + \eta'\cos(\theta) \text{ and } \eta_8 \equiv +\eta\cos(\theta) + \eta'\sin(\theta)$$

- The actual mass Hamiltonian of all mesons must thus allow a mixing term as well

$$\langle 3\otimes\bar{3}\,|\,H_{\text{mass}}\,|\,3\otimes\bar{3}\rangle = m_{\eta_1}\eta_1^2 + m_M\text{tr}(M^2) + m_1\text{tr}(M^2Q) + m_2\eta_1\text{tr}(MQ)$$

→ Expressing η_1, m_2 with $\eta, \eta', \eta_8, \theta$ results in the mixing angle

$$\tan^2\theta = \frac{m_\eta^2 - m_{\eta_8}^2}{m_{\eta_8}^2 - m_{\eta'}^2} \text{ and } \theta \approx 10^\circ$$

- For spin-0^-, the octet and singlet do not have much overlap

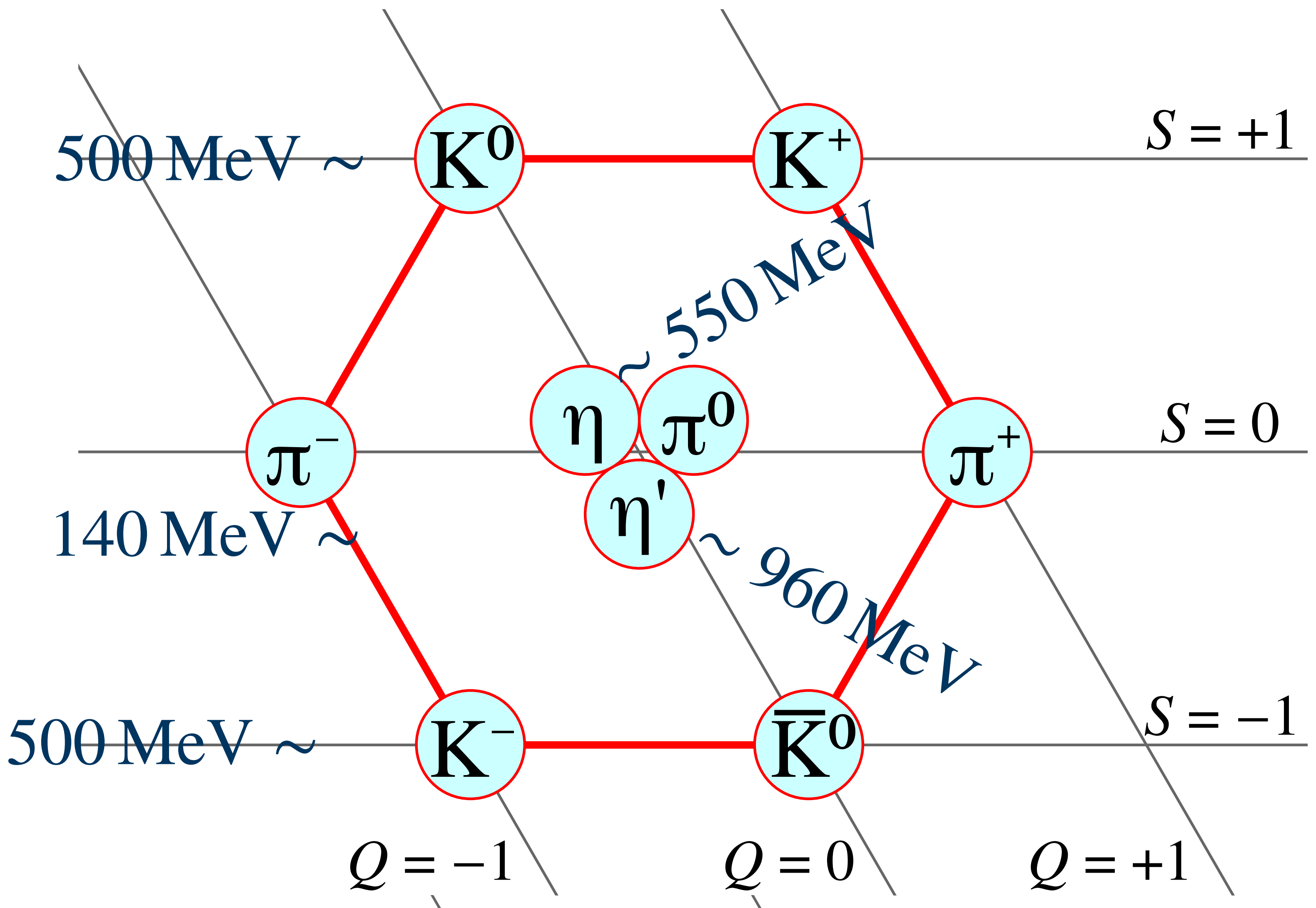
$$\eta \approx \eta_8 = \frac{\bar{u}u + \bar{d}d - 2\bar{s}s}{\sqrt{6}} \text{ and } \eta' \approx \eta_1 = \frac{\bar{u}u + \bar{d}d + \bar{s}s}{\sqrt{3}}$$

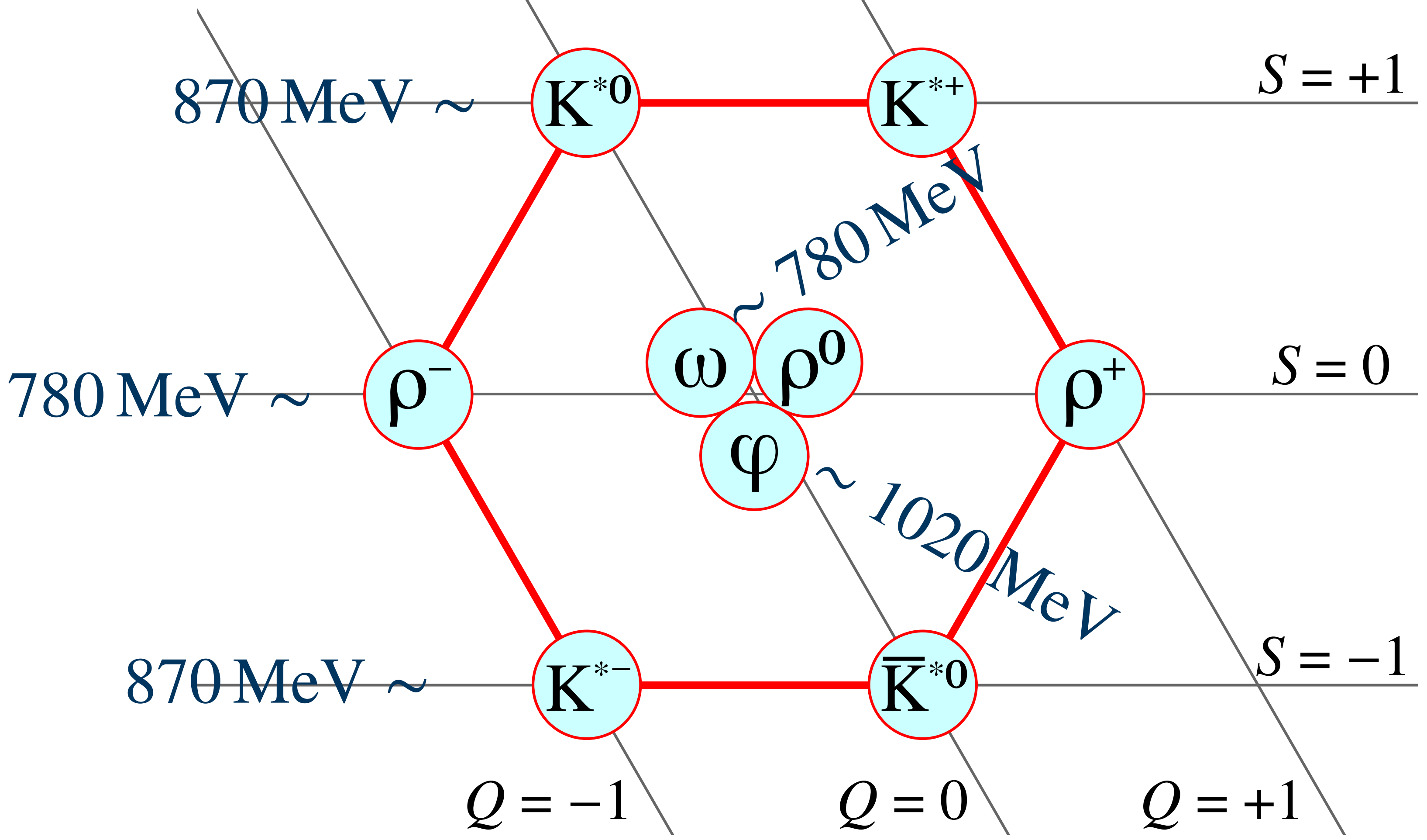
- The same analysis can be repeated for spin-1^- mesons:

$$m_{\omega_8} = \sqrt{\frac{4m_{K^*}^2 - m_\rho^2}{3}} = 930\,\text{MeV and } \theta = \arctan\sqrt{\frac{m_\Phi^2 - m_{\omega_8}^2}{m_{\omega_8}^2 - m_\omega^2}} \approx 40^\circ$$

→ Which is close to the ideal mixing case of $\theta = 1/\sqrt{3}$

$$\Phi \approx -\bar{s}s \text{ and } \omega \approx \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}$$





6. The quark model

6.4 Concluding remarks

Including spin

- So far, we have not explicitly included the quark spin

$$|\psi\rangle = \big(|u\rangle, |d\rangle, |s\rangle\big) \otimes \big(|\uparrow\rangle, |\downarrow\rangle\big)$$

- A straight forward way would be embedding the tensor product in $SU(6)$

$$q^a \stackrel{!}{=} \big(|u,\uparrow\rangle, \quad |u,\downarrow\rangle, \quad |d,\uparrow\rangle, \quad |d,\downarrow\rangle, \quad |s,\uparrow\rangle, \quad |s,\downarrow\rangle\big)_a$$

→ With the generators

$$1_{3x3} \otimes \sigma^i\,, \qquad \lambda^a \otimes 1_{2x2}\,, \qquad \lambda^a \otimes \sigma^i$$

→ Where λ^a are Gell-Mann matrices and σ^i are Pauli matrices ($3 + 8 + 3 \times 8 = 35$ generators)

→ Note that we could classify all generators as $SU(3) \times SU(2)$

- Baryon states now have mixed flavor-spin tensor wave functions

→ Decuplet states are for example (symmetric flavor and symmetric spin)

$$\big|\Delta^{++}, 3/2\big\rangle = |uuu\rangle\,|+++\rangle \qquad \big|\Delta^{+}, 1/2\big\rangle = \frac{|uud\rangle + |udu\rangle + |duu\rangle}{\sqrt{3}} \otimes \frac{|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle}{\sqrt{3}}$$

→ Octet states are complicated because of mixed symmetries

$$|p, 1/2\rangle = \frac{|uud\rangle}{\sqrt{3}}\frac{2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle}{\sqrt{6}} + \frac{|udu\rangle}{\sqrt{3}}\frac{2|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle}{\sqrt{6}} + \frac{|duu\rangle}{\sqrt{3}}\frac{2|\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle}{\sqrt{6}}$$

⇒ Symmetric in $(1 \leftrightarrow 2)$, antisymmetric in others (and as such not in the decuplet)

Including color

- The baryon wave function suggests hidden quantum number because of symmetry

$$|B\rangle = |\text{flavor}\rangle \otimes |\text{spin}\rangle \otimes |\text{space}\rangle$$

→ However, for the decuplet we just saw, spin and flavor were symmetric
→ The ground state in space is also symmetric (angular momentum $l = 0$)
→ This is problematic because, since baryons are fermions, this would violate the Pauli principle

- A new quantum number, color, would safe the Pauli principle for baryons

$$|B\rangle = |\text{flavor}\rangle \otimes |\text{spin}\rangle \otimes |\text{space}\rangle \otimes |\text{color}\rangle$$

→ This means all quarks must carry different color; e.g., they must form a color singlet (confinement)
→ The fundamental color representations should be applicable to quarks and antiquarks: Thus $SU_C(3)$ is a good candidate (we say $N_C = 3$), again
⇒Thus, there are three colors $i = R, B, G$ and three anti colors $\bar{i} = \bar{R}, \bar{B}, \bar{G}$ (not actual colors)

$$\epsilon_{ijk}\xi(1)^i\xi(2)^j\xi(3)^k = \left(\xi^R\xi^B - \xi^B\xi^R\right)\xi^G + \left(\xi^B\xi^G - \xi^G\xi^B\right)\xi^R + \left(\xi^G\xi^R - \xi^R\xi^G\right)\xi^B$$

$$\bar{\xi}_i\xi^i = \bar{\xi}_R\xi^R + \bar{\xi}_B\xi^B + \bar{\xi}_G\xi^G$$

→ Not all wave functions are allowed, e.g., contractions of $\xi^i\xi^j$ don't form a singlet

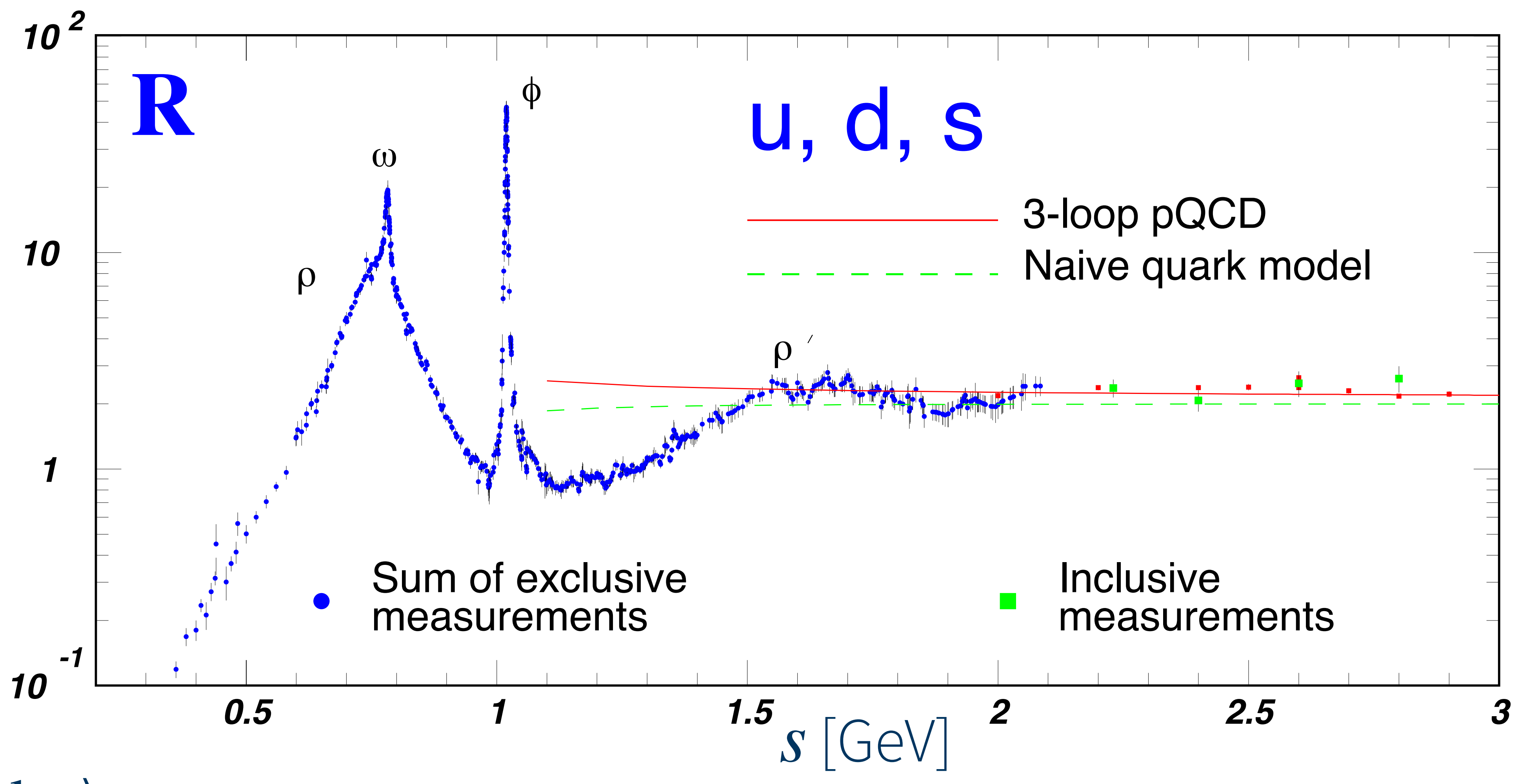
- The color quantum $N_C = 3$ is confirmed by, e.g., electron-positron annihilation experiments

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{CM}^2} \quad \text{and} \quad \sigma(e^+e^- \to \text{hadrons}) \approx \frac{4\pi\alpha^2}{3E_{CM}^2}\sum_f Q_f^2 N_c \quad \text{thus} \quad R(E_{CM}) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx \sum_f Q_f^2 N_c$$

→ For small energies, only the light and strange quark contribute

Quark flavors u, d, s, c, b, t

$$R(E_{CM} < 3\,\text{GeV}) \approx \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2 \text{ which matches the experiment}$$



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Concluding remarks

- The generalization to flavor $SU(4)$, also including the charm quark (c), is straight forward
- We have not yet considered dynamic effects—which are quite relevant
  → The masses of individual up- and down-quarks is significantly smaller than the mass of the proton
- Typically, quark models predict more states than those which are actually observed
- Quarks, even though not asymptotically observed, are real
  → Partons (what should I say here)
- The quark model was eventually replaced (by the much more complicated) Quantum Chromodynamics
  → We still cannot write down an analytic form of a strong Hamiltonian for all hadrons with quarks as degrees of freedom (and it is not obvious that this will ever be the case)
  → However, we can make controlled approximations, which allow us to describe the dynamics of Hadrons to very high accuracy and precision
    ⇒ Placing QCD on in a finite discrete world to simulate it using super computers: Lattice Quantum Chromodynamics (LQCD)
    ⇒ Using an effective theory with more practical degrees of freedom and matching their symmetries to the symmetries of QCD (symmetries are important!): (Heavy Baryon) Chiral Perturbation Theory
- Finding the "*true*" quark content of Baryons is a non-trivial task
  → In general, all quark bilinears which have the same quantum numbers as the baryon can contribute
    ⇒ Lattice QCD is capable of partially addressing this question

Thank you!