$\boxed{\text{Rough notes for the } N \to N,\ \Delta \to N \text{ and the } \Delta \to \Delta \text{ form factors}}$

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1 The two-point correlator

The normalization of the baryon states will follow the appendix of Montvay and Münster [which differs, for example, from the choice used by Wilcox, Draper and Liu, PRD46, 1109 eqs (39) and (40)],

$$1 = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{m_n}{E_{n(\vec{p})}} \sum_s |n(\vec{p},s)\rangle \langle n(\vec{p},s)| \to \frac{1}{V} \sum_{\vec{p}} \frac{m_n}{E_{n(\vec{p})}} \sum_s |n(\vec{p},s)\rangle \langle n(\vec{p},s)|$$

where V is the spatial volume of the lattice and s sums over the possible spin states. We will consider the nucleon and Δ^+ interpolating fields (including Dirac index α and Lorentz index σ), such as

$$\begin{aligned}
\chi^N_{\alpha L}(x) &= \epsilon^{abc} \left(d^{Ta}(x) C\gamma_5 u^b(x) \right) u^c_\alpha(x) \\
\chi^{\Delta^+}_{\sigma,\alpha L}(x) &= \epsilon^{abc} \left[2 \left(d^{Ta}(x) C\gamma_\sigma u^b(x) \right) u^c_\alpha(x) + \left(u^{Ta}(x) C\gamma_\sigma u^b(x) \right) d^c_\alpha(x) \right]
\end{aligned}$$

or some smeared version of these, denoted $\chi^N_{\alpha S}(x)$ or $\chi^{\Delta^+}_{\sigma,\alpha S}(x)$. In the following, the octet and decuplet baryons will be labelled by N and Δ, respectively. However, the results may be applied to any of the octet to decuplet transitions.

The dimensionless correlator from Euclidean time t_i to Euclidean time t_f with momentum $\vec{p}$ is

$$\begin{aligned}
&\Gamma^{NN}_{AB}(t_i, t_f, \vec{p};\ T) \\
&\quad = a^9 \sum_{\vec{x}_f} e^{-i(\vec{x}_f - \vec{x}_i)\cdot\vec{p}}\ T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_f) \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \\
&\quad = a^9 \sum_{n,\vec{k},s} \sum_{\vec{x}_f} e^{-i(\vec{x}_f - \vec{x}_i)\cdot\vec{p}}\ T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_f) \right| n(\vec{k},s) \right\rangle \frac{m_n}{V E_{n(\vec{k})}} \left\langle n(\vec{k},s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \\
&\quad = a^9 \sum_{n,\vec{k},s} \sum_{\vec{x}_f} e^{-i(\vec{x}_f - \vec{x}_i)\cdot\vec{p}}\ T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_i) e^{i(x_f - x_i)\cdot k} \right| n(\vec{k},s) \right\rangle \frac{m_n}{V E_{n(\vec{k})}} \left\langle n(\vec{k},s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \\
&\quad = a^9 \sum_{n,\vec{k},s} \sum_{\vec{x}_f} T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_i) \right| n(\vec{k},s) \right\rangle \frac{m_n e^{-(t_f - t_i)E_{n(\vec{k})}}}{V E_{n(\vec{k})}} \left\langle n(\vec{k},s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle e^{i(\vec{x}_f - \vec{x}_i)\cdot(\vec{k}-\vec{p})} \\
&\quad = a^6 \sum_{n,\vec{k},s} T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_i) \right| n(\vec{k},s) \right\rangle \frac{m_n e^{-(t_f - t_i)E_{n(\vec{k})}}}{E_{n(\vec{k})}} \left\langle n(\vec{k},s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \delta^{(3)}_{\vec{k},\vec{p}} e^{-i\vec{x}_i\cdot(\vec{k}-\vec{p})} \\
&\quad = a^6 \sum_{n,s} T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_i) \right| n(\vec{p},s) \right\rangle \left\langle n(\vec{p},s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \frac{m_n}{E_{n(\vec{p})}} e^{-(t_f - t_i)E_{n(\vec{p})}}
\end{aligned}$$

where $T_{\alpha\beta}$ is some generic 4×4 matrix in Dirac spin space, and α, β are Dirac indices. For $t_f \gg t_i$, the nucleon dominates and the result becomes

$$\Gamma^{NN}_{AB}(t_i, t_f, \vec{p};\ T) \rightarrow a^6 \sum_s T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x_i) \right| N(\vec{p}, s) \right\rangle \left\langle N(\vec{p}, s) \left| \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \frac{m_N}{E_{N(\vec{p})}} e^{-(t_f - t_i) E_{N(\vec{p})}}$$

Similarly, for $t_f \gg t_i$ the Δ correlator becomes

$$\Gamma^{\Delta\Delta}_{\sigma\tau, AB}(t_i, t_f, \vec{p};\ T) \rightarrow a^6 \sum_s T_{\alpha\beta} \left\langle 0 \left| \chi^\Delta_{\sigma, \beta B}(x_i) \right| \Delta(\vec{p}, s) \right\rangle \left\langle \Delta(\vec{p}, s) \left| \bar{\chi}^\Delta_{\tau, \alpha A}(x_i) \right| 0 \right\rangle \frac{m_\Delta}{E_{\Delta(\vec{p})}} e^{-(t_f - t_i) E_{\Delta(\vec{p})}}$$

where the subscripts σ, τ are the Lorentz indices of the spin-3/2 interpolating fields.

The dimensionless matrix elements are given by [see, eg, UKQCD Collaboration, PRD57, 6948 (1998), equation (A4)],

$$\begin{aligned} a^3 \left\langle 0 \left| \chi^N_{\beta B}(x) \right| N(\vec{p}, s) \right\rangle &= \left[\left(Z^{(1)}_B(|\vec{p}|) + \gamma_4 Z^{(2)}_B(|\vec{p}|) \right) u(\vec{p}, s) \right]_\beta e^{ix \cdot p} \\ a^3 \left\langle 0 \left| \chi^\Delta_{\sigma, \beta B}(x) \right| \Delta(\vec{p}, s) \right\rangle &= \left[\left(Z^{(1)}_B(|\vec{p}|) + \gamma_4 Z^{(2)}_B(|\vec{p}|) \right) u_\sigma(\vec{p}, s) \right]_\beta e^{ix \cdot p} \end{aligned}$$

and its adjoint. x and p are Euclidean. (These equations implicitly define $u(\vec{p}, s)$ and $u_\sigma(\vec{p}, s)$. Note in particular that they are implicitly chosen to be dimensionless.) Notice that we are assuming that any smearing is spatially-democratic. If there is no smearing at all, then we will use $Z^{(1)}_L(|\vec{p}|) = 1$ and $Z^{(2)}_L(|\vec{p}|) = 0$. In either case, for the nucleon and Δ we have

$$\begin{aligned} &\Gamma^{NN}_{AB}(t_i, t_f, \vec{p};\ T) \rightarrow \\ &\quad \sum_s T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}|) + \gamma_4 Z^{(2)}_B(|\vec{p}|) \right) u(\vec{p}, s) \bar{u}(\vec{p}, s) \left(Z^{(1)*}_A(|\vec{p}|) + \gamma_4 Z^{(2)*}_A(|\vec{p}|) \right) \right]_{\beta\alpha} \\ &\qquad \frac{m_N}{E_{N(\vec{p})}} e^{-(t_f - t_i) E_{N(\vec{p})}} \\ &\Gamma^{\Delta\Delta}_{\sigma\tau, AB}(t_i, t_f, \vec{p};\ T) \rightarrow \\ &\quad \sum_s T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}|) + \gamma_4 Z^{(2)}_B(|\vec{p}|) \right) u_\sigma(\vec{p}, s) \bar{u}_\tau(\vec{p}, s) \left(Z^{(1)*}_A(|\vec{p}|) + \gamma_4 Z^{(2)*}_A(|\vec{p}|) \right) \right]_{\beta\alpha} \\ &\qquad \frac{m_\Delta}{E_{\Delta(\vec{p})}} e^{-(t_f - t_i) E_{\Delta(\vec{p})}} \end{aligned}$$

In the following, we will consider the spin projection matrices defined as

$$T_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The gamma matrix basis used is

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that the Pauli spin matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the nucleon, the Dirac spin sum is

$$\sum_s u(\vec{p}, s)\bar{u}(\vec{p}, s) = \frac{i\not{p} + m_N}{2m_N}$$

so we arrive at

$$\Gamma^{NN}_{AB}(t_i, t_f, \vec{p};\ T) \quad \to \quad T_{\alpha\beta} \left[\left(Z_B^{(1)}(|\vec{p}|) + \gamma_4 Z_B^{(2)}(|\vec{p}|) \right) \left(\frac{i\not{p} + m_N}{2E_{N(\vec{p})}} \right) \left(Z_A^{(1)*}(|\vec{p}|) + \gamma_4 Z_A^{(2)*}(|\vec{p}|) \right) \right]_{\beta\alpha} e^{-(t_f - t_i)E_{N(\vec{p})}}$$

For $\beta = \alpha$ our result becomes

$$\Gamma^{NN}_{\alpha\alpha,AB}(t_i, t_f, \vec{p}) \quad \to \quad \left(Z_B^{(1)}(|\vec{p}|) + c_\alpha Z_B^{(2)}(|\vec{p}|) \right) \left(Z_A^{(1)*}(|\vec{p}|) + c_\alpha Z_A^{(2)*}(|\vec{p}|) \right) \left(\frac{E_{N(\vec{p})} + c_\alpha m_N}{2E_{N(\vec{p})}} \right) e^{-(t_f - t_i)E_{N(\vec{p})}} \tag{1}$$

where the repeated Dirac index α is *not* summed. Also, we have defined $c_1 = c_2 = 1$ and $c_3 = c_4 = -1$. Thus, for the standard projector T_4, we have

$$\Gamma^{NN}_{AB}(t_i, t_f, \vec{p};\ T_4) \quad \to \quad 2 \left(Z_B^{(1)}(|\vec{p}|) + Z_B^{(2)}(|\vec{p}|) \right) \left(Z_A^{(1)*}(|\vec{p}|) + Z_A^{(2)*}(|\vec{p}|) \right) \left(\frac{E_{N(\vec{p})} + m_N}{2E_{N(\vec{p})}} \right) e^{-(t_f - t_i)E_{N(\vec{p})}} \tag{2}$$

The Rarita-Schwinger spin sum for the Δ in Euclidean space is

$$\sum_s u_\sigma(\vec{p}, s)\bar{u}_\tau(\vec{p}, s) = \frac{i\not{p} + m_\Delta}{2m_\Delta} \left[\delta_{\sigma\tau} + \frac{2p_\sigma p_\tau}{3m_\Delta^2} + i\frac{p_\sigma\gamma_\tau - p_\tau\gamma_\sigma}{3m_\Delta} - \frac{1}{3}\gamma_\sigma\gamma_\tau \right]$$

which results in

$$\Gamma^{\Delta\Delta}_{\sigma\tau,AB}(t_i, t_f, \vec{p};\ T) \to \mathrm{Tr}\Big[T \left(Z_B^{(1)}(|\vec{p}|) + \gamma_4 Z_B^{(2)}(|\vec{p}|) \right) \left(\frac{i\not{p} + m_\Delta}{2E_{\Delta(\vec{p})}} \right) \left(\delta_{\sigma\tau} + \frac{2p_\sigma p_\tau}{3m_\Delta^2} + i\frac{p_\sigma\gamma_\tau - p_\tau\gamma_\sigma}{3m_\Delta} - \frac{1}{3}\gamma_\sigma\gamma_\tau \right) \left(Z_A^{(1)*}(|\vec{p}|) + \gamma_4 Z_A^{(2)*}(|\vec{p}|) \right) \Big] e^{-(t_f - t_i)E_{\Delta(\vec{p})}}$$

where σ and τ are Lorentz indices. In the case $\sigma = \tau$ and the projector T_4, we find (with no implied sum over σ)

$$\Gamma^{\Delta\Delta}_{\sigma\sigma,AB}(t_i, t_f, \vec{p};\ T_4) \quad \to \quad \left(Z_B^{(1)}(|\vec{p}|) + Z_B^{(2)}(|\vec{p}|) \right) \left(Z_A^{(1)*}(|\vec{p}|) + Z_A^{(2)*}(|\vec{p}|) \right) \frac{2}{3} \left(1 + \frac{p_\sigma^2}{m_\Delta^2} \right) \left(\frac{E_{\Delta(\vec{p})} + m_\Delta}{2E_{\Delta(\vec{p})}} \right) e^{-(t_f - t_i)E_{\Delta(\vec{p})}} \tag{3}$$

2 The three-point correlator

The dimensionless correlator from Euclidean time t_i (incoming momentum $\vec{p}_i$) to Euclidean time t_f (outgoing momentum $\vec{p}_f$) with a vector insertion at Euclidean time t is

$$
\begin{aligned}
&\Gamma^{NN}_{\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f\,;\,T)\\
&\quad= a^{12}\sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta}e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f}e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i}\left\langle 0\left|\chi^N_{\beta B}(x_f)V_\mu(x)\bar{\chi}^N_{\alpha A}(x_i)\right|0\right\rangle\\
&\quad= a^{12}\sum_{n,\vec{k},s}\sum_{m,\vec{l},s'}\sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta}e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f}e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i}\left\langle 0\left|\chi^N_{\beta B}(x_f)\right|m(\vec{l},s')\right\rangle\frac{m_m}{VE_{m(\vec{l})}}\\
&\qquad\left\langle m(\vec{l},s')\left|V_\mu(x)\right|n(\vec{k},s)\right\rangle\frac{m_n}{VE_{n(\vec{k})}}\left\langle n(\vec{k},s)\left|\bar{\chi}^N_{\alpha A}(x_i)\right|0\right\rangle\\
&\quad= a^{12}\sum_{n,\vec{k},s}\sum_{m,\vec{l},s'}\sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta}e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f}e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i}\left\langle 0\left|\chi^N_{\beta B}(x)e^{i(x_f-x)\cdot l}\right|m(\vec{l},s')\right\rangle\\
&\qquad\frac{m_m}{VE_{m(\vec{l})}}\left\langle m(\vec{l},s')\left|V_\mu(x)\right|n(\vec{k},s)\right\rangle\frac{m_n}{VE_{n(\vec{k})}}\left\langle n(\vec{k},s)\left|e^{-i(x_i-x)\cdot k}\bar{\chi}^N_{\alpha A}(x)\right|0\right\rangle\\
&\quad= a^{12}\sum_{n,\vec{k},s}\sum_{m,\vec{l},s'}\sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta}\left\langle 0\left|\chi^N_{\beta B}(x)\right|m(\vec{l},s')\right\rangle\frac{m_m}{VE_{m(\vec{l})}}e^{-(t_f-t)E_{m(\vec{l})}}e^{i(\vec{x}_f-\vec{x})\cdot(\vec{l}-\vec{p}_f)}\\
&\qquad\left\langle m(\vec{l},s')\left|V_\mu(x)\right|n(\vec{k},s)\right\rangle\frac{m_n}{VE_{n(\vec{k})}}e^{-(t-t_i)E_{n(\vec{k})}}e^{i(\vec{x}-\vec{x}_i)\cdot(\vec{k}-\vec{p}_i)}\left\langle n(\vec{k},s)\left|\bar{\chi}^N_{\alpha A}(x)\right|0\right\rangle\\
&\quad= a^{6}\sum_{n,\vec{k},s}\sum_{m,\vec{l},s'} T_{\alpha\beta}\left\langle 0\left|\chi^N_{\beta B}(x)\right|m(\vec{l},s')\right\rangle\delta^{(3)}_{\vec{l},\vec{p}_f}\frac{m_m}{E_{m(\vec{l})}}e^{-(t_f-t)E_{m(\vec{l})}}e^{-i\vec{x}\cdot(\vec{l}-\vec{p}_f)}\\
&\qquad\left\langle m(\vec{l},s')\left|V_\mu(x)\right|n(\vec{k},s)\right\rangle\delta^{(3)}_{\vec{k},\vec{p}_i}\frac{m_n}{E_{n(\vec{k})}}e^{-(t-t_i)E_{n(\vec{k})}}e^{i\vec{x}\cdot(\vec{k}-\vec{p}_i)}\left\langle n(\vec{k},s)\left|\bar{\chi}^N_{\alpha A}(x)\right|0\right\rangle\\
&\quad= a^{6}\sum_{n,s}\sum_{m,s'} T_{\alpha\beta}\left\langle 0\left|\chi^N_{\beta B}(x)\right|m(\vec{p}_f,s')\right\rangle\frac{m_m}{E_{m(\vec{p}_f)}}e^{-(t_f-t)E_{m(\vec{p}_f)}}\left\langle m(\vec{p}_f,s')\left|V_\mu(x)\right|n(\vec{p}_i,s)\right\rangle\\
&\qquad\frac{m_n}{E_{n(\vec{p}_i)}}e^{-(t-t_i)E_{n(\vec{p}_i)}}\left\langle n(\vec{p}_i,s)\left|\bar{\chi}^N_{\alpha A}(x)\right|0\right\rangle
\end{aligned}
$$

For $t_f \gg t \gg t_i$, the nucleon dominates and the result becomes

$$
\begin{aligned}
&\Gamma^{NN}_{\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f;\ T) \rightarrow \\
&\quad a^6 \sum_s \sum_{s'} T_{\alpha\beta} \left\langle 0 \left| \chi^N_{\beta B}(x) \right| N(\vec{p}_f,s') \right\rangle \left\langle N(\vec{p}_f,s') \left| V_\mu(x) \right| N(\vec{p}_i,s) \right\rangle \\
&\quad \left\langle N(\vec{p}_i,s) \left| \bar{\chi}^N_{\alpha A}(x) \right| 0 \right\rangle \frac{m_N^2}{E_{N(\vec{p}_f)} E_{N(\vec{p}_i)}} e^{-(t_f-t)E_{N(\vec{p}_f)}} e^{-(t-t_i)E_{N(\vec{p}_i)}} \\
&\quad = e^{ix\cdot(p_f-p_i)} \frac{m_N^2}{E_{N(\vec{p}_f)} E_{N(\vec{p}_i)}} e^{-(t_f-t)E_{N(\vec{p}_f)}} e^{-(t-t_i)E_{N(\vec{p}_i)}} \\
&\quad \sum_{s,s'} T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|) \right) u(\vec{p}_f,s') \right]_\beta \\
&\quad \left\langle N(\vec{p}_f,s') \left| V_\mu(x) \right| N(\vec{p}_i,s) \right\rangle \left[\bar{u}(\vec{p}_i,s) \left(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|) \right) \right]_\alpha \\
&\quad = \frac{m_N^2}{E_{N(\vec{p}_f)} E_{N(\vec{p}_i)}} e^{-(t_f-t)E_{N(\vec{p}_f)}} e^{-(t-t_i)E_{N(\vec{p}_i)}} \\
&\quad \sum_{s,s'} T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|) \right) u(\vec{p}_f,s') \right]_\beta \\
&\quad \left\langle N(\vec{p}_f,s') \left| V_\mu(0) \right| N(\vec{p}_i,s) \right\rangle \left[\bar{u}(\vec{p}_i,s) \left(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|) \right) \right]_\alpha
\end{aligned} \tag{4}
$$

Similarly, for $t_f \gg t \gg t_i$, the result for the Δ becomes

$$
\begin{aligned}
&\Gamma^{\Delta\Delta}_{\sigma\tau,\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f;\ T) \\
&\quad = a^{12} \sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta} e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f} e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i} \left\langle 0 \left| \chi^\Delta_{\sigma,\beta B}(x_f) V_\mu(x) \bar{\chi}^\Delta_{\tau,\alpha A}(x_i) \right| 0 \right\rangle \\
&\quad \rightarrow \frac{m_\Delta^2}{E_{\Delta(\vec{p}_f)} E_{\Delta(\vec{p}_i)}} e^{-(t_f-t)E_{\Delta(\vec{p}_f)}} e^{-(t-t_i)E_{\Delta(\vec{p}_i)}} \\
&\quad \sum_{s,s'} T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|) \right) u_\sigma(\vec{p}_f,s') \right]_\beta \\
&\quad \left\langle \Delta(\vec{p}_f,s') \left| V_\mu(0) \right| \Delta(\vec{p}_i,s) \right\rangle \left[\bar{u}_\tau(\vec{p}_i,s) \left(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|) \right) \right]_\alpha
\end{aligned} \tag{5}
$$

Nothing is conceptually different for the transition form-factors, thus for $t_f \gg t \gg t_i$ the result for the $\Delta \rightarrow N$ (the Δ has incoming momentum $\vec{p}_i$ and the nucleon has outgoing momentum $\vec{p}_f$) becomes

$$\begin{aligned}
&\Gamma^{\Delta N}_{\sigma,\mu,AB}(t_i, t, t_f, \vec{p}_i, \vec{p}_f;\ T) \\
&\qquad = a^{12} \sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta} e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f} e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i} \left\langle 0 \left| \chi^N_{\beta B}(x_f) V_\mu(x) \bar{\chi}^\Delta_{\sigma,\alpha A}(x_i) \right| 0 \right\rangle \\
&\qquad \rightarrow \frac{m_\Delta m_N}{E_{N(\vec{p}_f)} E_{\Delta(\vec{p}_i)}} e^{-(t_f-t)E_{N(\vec{p}_f)}} e^{-(t-t_i)E_{\Delta(\vec{p}_i)}} \\
&\qquad \sum_{s,s'} T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|) \right) u(\vec{p}_f, s') \right]_\beta \\
&\qquad \langle N(\vec{p}_f, s') \left| V_\mu(0) \right| \Delta(\vec{p}_i, s) \rangle \left[\bar{u}_\sigma(\vec{p}_i, s) \left(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|) \right) \right]_\alpha
\end{aligned} \tag{6}$$

and the result for the $N \to \Delta$ (the nucleon has incoming momentum $\vec{p}_i$ and the Δ has outgoing momentum $\vec{p}_f$)

$$\begin{aligned}
&\Gamma^{N\Delta}_{\sigma,\mu,AB}(t_i, t, t_f, \vec{p}_i, \vec{p}_f;\ T) \\
&\qquad = a^{12} \sum_{\vec{x}_i,\vec{x}_f} T_{\alpha\beta} e^{-i(\vec{x}_f-\vec{x})\cdot\vec{p}_f} e^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i} \left\langle 0 \left| \chi^\Delta_{\sigma,\beta B}(x_f) V_\mu(x) \bar{\chi}^N_{\alpha A}(x_i) \right| 0 \right\rangle \\
&\qquad \rightarrow \frac{m_\Delta m_N}{E_{\Delta(\vec{p}_f)} E_{N(\vec{p}_i)}} e^{-(t_f-t)E_{\Delta(\vec{p}_f)}} e^{-(t-t_i)E_{N(\vec{p}_i)}} \\
&\qquad \sum_{s,s'} T_{\alpha\beta} \left[\left(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|) \right) u_\sigma(\vec{p}_f, s') \right]_\beta \\
&\qquad \langle \Delta(\vec{p}_f, s') \left| V_\mu(0) \right| N(\vec{p}_i, s) \rangle \left[\bar{u}(\vec{p}_i, s) \left(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|) \right) \right]_\alpha
\end{aligned} \tag{7}$$

3 The nucleon electromagnetic form factors

In conventional (but Euclidean) notation, the matrix element of interest is

$$\begin{aligned}
\langle N(\vec{p}_f, s') \left| V_\mu(0) \right| N(\vec{p}_i, s) \rangle_{\text{continuum}} &= Z_V \langle N(\vec{p}_f, s') \left| V_\mu(0) \right| N(\vec{p}_i, s) \rangle \\
&= \bar{u}(\vec{p}_f, s') \left[\gamma_\mu F_1(q^2) - \frac{\sigma_{\mu\nu} q_\nu}{2m_N} F_2(q^2) \right] u(\vec{p}_i, s) \qquad (8)
\end{aligned}$$

where Z_V is the renormalization factor ($Z_V = 1$ for a conserved current), $q = p_f - p_i$ and the electric and magnetic form factors are

$$G_E(q^2) = F_1(q^2) - \frac{q^2}{4m_N^2} F_2(q^2) \tag{9}$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2) \tag{10}$$

This allows the three-point correlator with $t_f \gg t \gg t_i$ to be written as

$$\Gamma^{NN}_{\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f\,;\,T) = \frac{e^{-(t_f-t)E_{N(\vec{p}_f)}}e^{-(t-t_i)E_{N(\vec{p}_i)}}}{4Z_V E_{N(\vec{p}_f)}E_{N(\vec{p}_i)}}\left[\mathrm{Tr}\left(T\,M^{(1)}_\mu\right)F_1(q^2)+\mathrm{Tr}\left(T\,M^{(2)}_\mu\right)F_2(q^2)\right]$$

where

$$\begin{aligned}
M^{(1)}_\mu &= \left(Z_B^{(1)}(|\vec{p}_f|)+\gamma_4 Z_B^{(2)}(|\vec{p}_f|)\right)(i\not{p}_f+m_N)\,\gamma_\mu\,(i\not{p}_i+m_N) \\
&\quad \left(Z_A^{(1)*}(|\vec{p}_i|)+\gamma_4 Z_A^{(2)*}(|\vec{p}_i|)\right) \\
M^{(2)}_\mu &= \left(Z_B^{(1)}(|\vec{p}_f|)+\gamma_4 Z_B^{(2)}(|\vec{p}_f|)\right)(i\not{p}_f+m_N)\left(\frac{-\sigma_{\mu\nu}q_\nu}{2m_N}\right)(i\not{p}_i+m_N) \\
&\quad \left(Z_A^{(1)*}(|\vec{p}_i|)+\gamma_4 Z_A^{(2)*}(|\vec{p}_i|)\right)
\end{aligned}$$

and

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu,\gamma_\nu\right]$$

3.1 The result for $\mu = 4$

The $\mu = 4$ matrices are

$$\begin{aligned}
M^{(1)}_4 &= \left(Z_B^{(1)}(|\vec{p}_f|)+\gamma_4 Z_B^{(2)}(|\vec{p}_f|)\right)\left[\gamma_4\left(E_{N(\vec{p}_f)}+\gamma_4 m\right)\left(E_{N(\vec{p}_i)}+\gamma_4 m\right)-2\gamma_4 i\sigma_{jk}p_{fj}p_{ik}\right. \\
&\quad \left.+\gamma_4\vec{p}_i\cdot\vec{p}_f - i\left(E_{N(\vec{p}_i)}-\gamma_4 m\right)\vec{p}_f\cdot\vec{\gamma} - i\left(E_{N(\vec{p}_f)}+\gamma_4 m\right)\vec{p}_i\cdot\vec{\gamma}\right] \\
&\quad \left(Z_A^{(1)*}(|\vec{p}_i|)+\gamma_4 Z_A^{(2)*}(|\vec{p}_i|)\right) \\
M^{(2)}_4 &= \left(Z_B^{(1)}(|\vec{p}_f|)+\gamma_4 Z_B^{(2)}(|\vec{p}_f|)\right)\frac{\gamma_4}{2m}\left[-\left(E_{N(\vec{p}_f)}+\gamma_4 m\right)i\vec{\gamma}\cdot\vec{q}\left(E_{N(\vec{p}_i)}+\gamma_4 m\right)\right. \\
&\quad -\vec{q}^{\,2}\gamma_4\left(E_{N(\vec{p}_i)}+\gamma_4 m\right)-\vec{p}_i\cdot\vec{q}\gamma_4\left(E_{N(\vec{p}_i)}-E_{N(\vec{p}_f)}\right)+(\vec{p}_i+\vec{p}_f)\cdot\vec{q}i\vec{\gamma}\cdot\vec{p}_i \\
&\quad \left.+\vec{p}_i^{\,2}i\vec{\gamma}\cdot\vec{q}+2i\sigma_{jk}p_{ij}q_k\gamma_4\left(E_{N(\vec{p}_i)}+E_{N(\vec{p}_f)}+2\gamma_4 m\right)\right]\left(Z_A^{(1)*}(|\vec{p}_i|)+\gamma_4 Z_A^{(2)*}(|\vec{p}_i|)\right)
\end{aligned}$$

Thus, for T_4 the result is

$$\begin{aligned}
\mathrm{Tr}\left(T_4 M_4^{(1)}\right) &= 2\left(Z_B^{(1)}(|\vec{p}_f|) + Z_B^{(2)}(|\vec{p}_f|)\right) \\
&\quad \left[2E_{N(\vec{p}_i)}E_{N(\vec{p}_f)} + m_N\left(E_{N(\vec{p}_i)} + E_{N(\vec{p}_f)}\right) - \frac{q^2}{2}\right] \\
&\quad \left(Z_A^{(1)*}(|\vec{p}_i|) + Z_A^{(2)*}(|\vec{p}_i|)\right) \\
\mathrm{Tr}\left(T_4 M_4^{(2)}\right) &= 2\left(Z_B^{(1)}(|\vec{p}_f|) + Z_B^{(2)}(|\vec{p}_f|)\right) \\
&\quad \left[\frac{-q^2}{4m_N^2}\left(m_N E_{N(\vec{p}_i)} + m_N E_{N(\vec{p}_f)} + 2m_N^2\right) - \frac{1}{2}\left(E_{N(\vec{p}_f)} - E_{N(\vec{p}_i)}\right)^2\right] \\
&\quad \left(Z_A^{(1)*}(|\vec{p}_i|) + Z_A^{(2)*}(|\vec{p}_i|)\right)
\end{aligned} \tag{11}$$

Consider the ratio

$$R_4 = \frac{Z_V \Gamma^{NN}_{4,AB}(t_i, t, t_f, \vec{p}_i, \vec{p}_f\,;\, T_4)\, \Gamma^{NN}_{CL}(t_i, t, \vec{p}_f\,;\, T_4)}{\Gamma^{NN}_{AL}(t_i, t, \vec{p}_i\,;\, T_4)\, \Gamma^{NN}_{CB}(t_i, t_f, \vec{p}_f\,;\, T_4)} \tag{12}$$

$$\begin{aligned}
&= \frac{1}{2E_{N(\vec{p}_f)}\left(E_{N(\vec{p}_i)} + m_N\right)} \left[\left(2E_{N(\vec{p}_i)}E_{N(\vec{p}_f)} + m_N\left(E_{N(\vec{p}_i)} + E_{N(\vec{p}_f)}\right) - \frac{q^2}{2}\right) F_1(q^2)\right. \\
&\quad \left. + \left(\frac{-q^2}{4m_N}\left(E_{N(\vec{p}_i)} + E_{N(\vec{p}_f)} + 2m_N\right) - \frac{1}{2}\left(E_{N(\vec{p}_f)} - E_{N(\vec{p}_i)}\right)^2\right) F_2(q^2)\right]
\end{aligned} \tag{13}$$

For the special case of $\vec{p}_f = \vec{0}$, we have $q^2 = 2m_N\left(E_{N(\vec{p}_i)} - m_N\right)$ and we arrive at

$$R_4 = G_E(q^2)$$

3.2 The result for $\mu \neq 4$

We now consider other projection matrices T_k

$$\begin{aligned}
\mathrm{Tr}\left[M_j^{(1)} T_k\right] &= 2\left(Z_B^{(1)}(|\vec{p}_f|) + Z_B^{(2)}(|\vec{p}_f|)\right)\epsilon_{jkl}\left[p_{fl}\left(E_{N(\vec{p}_i)} + m_N\right)\right. \\
&\quad \left. - p_{il}\left(E_{N(\vec{p}_f)} + m_N\right)\right]\left(Z_A^{(1)*}(|\vec{p}_i|) + Z_A^{(2)*}(|\vec{p}_i|)\right) \\
\mathrm{Tr}\left[M_j^{(2)} T_k\right] &= \frac{1}{m_N}\left(Z_B^{(1)}(|\vec{p}_f|) + Z_B^{(2)}(|\vec{p}_f|)\right)\left[\epsilon_{jkl}p_{fl}\left(E_{N(\vec{p}_i)} + m_N\right)^2\right. \\
&\quad - \epsilon_{jkl}p_{il}\left(2m_N\left(E_{N(\vec{p}_f)} + m_N\right) + \vec{p}_i \cdot \vec{p}_f\right) - p_{ik}\epsilon_{jlm}p_{il}p_{fm} \\
&\quad \left. - p_{fj}\epsilon_{klm}p_{il}p_{fm}\right]\left(Z_A^{(1)*}(|\vec{p}_i|) + Z_A^{(2)*}(|\vec{p}_i|)\right)
\end{aligned}$$

Consider the ratio

$$
\begin{aligned}
R_{jk} &= \frac{Z_V\left(E_{N(\vec{p}_i)}+m_N\right)}{(-\epsilon_{jkl}p_{il})}\frac{\Gamma^{NN}_{j,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f\,;\ T_k)\Gamma^{NN}_{CL}(t_i,t,\vec{p}_f\,;\ T_4)}{\Gamma^{NN}_{AL}(t_i,t,\vec{p}_i\,;\ T_4)\Gamma^{NN}_{CB}(t_i,t_f,\vec{p}_f\,;\ T_4)} \qquad (14)\\
&= \frac{-1}{4E_{N(\vec{p}_f)}\epsilon_{jkl}p_{il}}\Bigg[2\epsilon_{jkl}\left\{p_{fl}\left(E_{N(\vec{p}_i)}+m_N\right)-p_{il}\left(E_{N(\vec{p}_f)}+m_N\right)\right\}F_1(q^2)\\
&\quad+\Bigg\{\epsilon_{jkl}p_{fl}\left(E_{N(\vec{p}_i)}+m_N\right)^2-\epsilon_{jkl}p_{il}\left(2m_N\left(E_{N(\vec{p}_f)}+m_N\right)+\vec{p}_i\cdot\vec{p}_f\right)\\
&\quad-p_{ik}\epsilon_{jlm}p_{il}p_{fm}-p_{fj}\epsilon_{klm}p_{il}p_{fm}\Bigg\}\frac{F_2(q^2)}{m_N}\Bigg] \qquad (15)
\end{aligned}
$$

For the special case of $\vec{p}_f=\vec{0}$, the expression simplifies to

$$R_{jk}=G_M(q^2)$$

In the general case, Eqs. (13) and (15) can be used to determine $G_E(q^2)$ and $G_M(q^2)$.

4 The $\gamma N \to \Delta$ electromagnetic form factors

In conventional (but Euclidean) notation, the matrix element of interest is

$$\begin{aligned}
\langle \Delta(\vec{p}_f, s') \left| V_\mu(0) \right| N(\vec{p}_i, s) \rangle_{\text{continuum}} &= Z_V \langle \Delta(\vec{p}_f, s') \left| V_\mu(0) \right| N(\vec{p}_i, s) \rangle \\
&= i\sqrt{\frac{2}{3}} \bar{u}_\tau(\vec{p}_f, s') \mathcal{O}^{\tau\mu} u(\vec{p}_i, s) \qquad (16)
\end{aligned}$$

where Z_V is the renormalization factor ($Z_V = 1$ for a conserved current), $q = p_f - p_i$, $u_\tau(\vec{p}, s)$ is a spin-vector in the Rarita-Schwinger formalism, and $u(\vec{p}, s)$ is a Dirac spin vector. The operator $\mathcal{O}^{\tau\mu}$ can be decomposed into

$$\mathcal{O}^{\tau\mu} = G_{M1}(q^2) K_{M1}^{\tau\mu} + G_{E2}(q^2) K_{E2}^{\tau\mu} + G_{C2}(q^2) K_{C2}^{\tau\mu} \,,$$

where the form-factors $G_{M1}(q^2)$, $G_{E2}(q^2)$, and $G_{C2}(q^2)$ are referred to as the magnetic dipole $M1$, the electric quadrupole $E2$ and the electric charge or scalar quadrupole $C2$ transition form factors. Definitions come from Leinweber PRD48, and Alexandrou. The kinematical factors are, in Euclidean notation,

$$\begin{aligned}
K_{M1}^{\tau\mu} &= -\frac{3}{(m_\Delta + m_N)^2 + q^2} \frac{(m_\Delta + m_N)}{2m_N} \, i\epsilon^{\tau\mu\alpha\beta} P_\alpha q_\beta \qquad (17) \\
K_{E2}^{\tau\mu} &= -K_{M1}^{\tau\mu} + 6\Omega^{-1}(q^2) \frac{(m_\Delta + m_N)}{2m_N} \, i\gamma_5 \, \epsilon^{\tau\lambda\alpha\beta} P_\alpha q_\beta \epsilon^{\mu\lambda\gamma\delta} (2P_\gamma + q_\gamma) q_\delta \qquad (18) \\
K_{C2}^{\tau\mu} &= -6\Omega^{-1}(q^2) \frac{(m_\Delta + m_N)}{2m_N} \, i\gamma_5 \, q_\tau \left(q^2 P_\mu - q \cdot P q_\mu \right) \qquad (19)
\end{aligned}$$

with $\Omega(q^2) = \left[(m_\Delta + m_N)^2 + q^2\right]\left[(m_\Delta - m_N)^2 + q^2\right]$. Recall the momenta are Euclidean, so q_4 is imaginary. The $P^\mu = (p_f^\mu + p_i^\mu)/2$. This allows the three-point correlator with $t_f \gg t \gg t_i$ to be written as

$$\begin{aligned}
\Gamma^{N\Delta}_{\sigma,\mu,AB}(t_i, t, t_f, \vec{p}_i, \vec{p}_f \,;\, T) &= i\sqrt{\frac{2}{3}} \, \frac{e^{-(t_f - t)E_{\Delta(\vec{p}_f)}} e^{-(t - t_i)E_{N(\vec{p}_i)}}}{4 Z_V E_{\Delta(\vec{p}_f)} E_{N(\vec{p}_i)}} \\
&\quad \left[M^{(1)}_{\sigma\mu} G_{M1}(q^2) + M^{(2)}_{\sigma\mu} G_{E2}(q^2) + M^{(3)}_{\sigma\mu} G_{C2}(q^2) \right] \qquad (20)
\end{aligned}$$

where

$$
\begin{aligned}
M^{(1)}_{\sigma\mu} &= \mathrm{Tr}\Big(T\ \Big(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|)\Big)\left(i\not{p}_f + m_\Delta\right)\left[\delta_{\sigma\tau} + \frac{2p_{f\sigma}p_{f\tau}}{3m_\Delta^2} + i\frac{p_{f\sigma}\gamma_\tau - p_{f\tau}\gamma_\sigma}{3m_\Delta} - \frac{1}{3}\gamma_\sigma\gamma_\tau\right] \\
&\quad K^{\tau\mu}_{M1}\left(i\not{p}_i + m_N\right)\Big(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|)\Big)\Big) \\
M^{(2)}_{\sigma\mu} &= \mathrm{Tr}\Big(T\ \Big(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|)\Big)\left(i\not{p}_f + m_\Delta\right)\left[\delta_{\sigma\tau} + \frac{2p_{f\sigma}p_{f\tau}}{3m_\Delta^2} + i\frac{p_{f\sigma}\gamma_\tau - p_{f\tau}\gamma_\sigma}{3m_\Delta} - \frac{1}{3}\gamma_\sigma\gamma_\tau\right] \\
&\quad K^{\tau\mu}_{E2}\left(i\not{p}_i + m_N\right)\Big(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|)\Big)\Big) \\
M^{(3)}_{\sigma\mu} &= \mathrm{Tr}\Big(T\ \Big(Z^{(1)}_B(|\vec{p}_f|) + \gamma_4 Z^{(2)}_B(|\vec{p}_f|)\Big)\left(i\not{p}_f + m_\Delta\right)\left[\delta_{\sigma\tau} + \frac{2p_{f\sigma}p_{f\tau}}{3m_\Delta^2} + i\frac{p_{f\sigma}\gamma_\tau - p_{f\tau}\gamma_\sigma}{3m_\Delta} - \frac{1}{3}\gamma_\sigma\gamma_\tau\right] \\
&\quad K^{\tau\mu}_{C2}\left(i\not{p}_i + m_N\right)\Big(Z^{(1)*}_A(|\vec{p}_i|) + \gamma_4 Z^{(2)*}_A(|\vec{p}_i|)\Big)\Big)
\end{aligned}
$$

Consider the ratio

$$
R_{\sigma\mu j} = \frac{Z_V^2\,\Gamma^{N\Delta}_{\sigma,\mu,AB}(t_i,t,t_f,\vec{p}_i,\vec{p}_f\,;\ T)\ \Gamma^{\Delta N}_{\sigma,\mu,DC}(t_i,t,t_f,-\vec{p}_f,-\vec{p}_i\,;\ T)}{\Gamma^{NN}_{AC}(t_i,t_f,-\vec{p}_i\,;\ T_4)\ \sum_{j=1}^{3}\Gamma^{\Delta\Delta}_{BD}(t_i,t_f,\vec{p}_f\,;\ T_4)} \tag{21}
$$

It can be used to obtain the three form factors, $G_{M1}(q^2)$, $G_{E2}(q^2)$ and $G_{C2}(q^2)$. All Z factors and exponentials cancel. Three different choices for the indices σ, μ and j will suffice to determine the three form factors for any given momenta. Technically, Eq. (21) only gives their magnitudes since $R_{\sigma\mu j}$ is quadratic in the form factors. This same type of ratio is used by Alexandrou et al, PRD69,114506 (2004), eq 12.

4.1 The $M^{(n)}_{\sigma\mu}$ for $T = T_4$

In these expressions, Greek indices run from 1 to 4 and Roman indices run from 1 to 3.

$$
\begin{aligned}
M^{(1)}_{\sigma\mu} &= \frac{-(m_N+m_\Delta)p_{i\alpha}p_{f\beta}}{m_N((m_N+m_\Delta)^2+q^2)}\left(Z_B^{(1)}(|\vec{p}_f|)+Z_B^{(2)}(|\vec{p}_f|)\right)\left(Z_A^{(1)*}(|\vec{p}_i|)+Z_A^{(2)*}(|\vec{p}_i|)\right) \\
&\left[(E_\Delta+m_\Delta)(E_N+m_N)\left(2i\epsilon_{\sigma\mu\alpha\beta}-\frac{p_{f\sigma}}{m_\Delta}\epsilon_{4\mu\alpha\beta}\right)\right. \\
&+(E_\Delta+m_\Delta)\left(\frac{-ip_{f\sigma}}{m_\Delta}\epsilon_{j\mu\alpha\beta}p_{ij}+\epsilon_{j\mu\alpha\beta}p_{ij}\delta_{4\sigma}-\epsilon_{4\mu\alpha\beta}p_{i\sigma}(1-\delta_{4\sigma})\right) \\
&+(E_N+m_N)\left(\frac{-ip_{f\sigma}}{m_\Delta}\epsilon_{j\mu\alpha\beta}p_{fj}-\epsilon_{j\mu\alpha\beta}p_{fj}\delta_{4\sigma}+\epsilon_{4\mu\alpha\beta}p_{f\sigma}(1-\delta_{4\sigma})\right) \\
&\left.-2i\epsilon_{\sigma\mu\alpha\beta}\vec{p}_i\cdot\vec{p}_f+\frac{p_{f\sigma}}{m_\Delta}\epsilon_{4\mu\alpha\beta}\vec{p}_i\cdot\vec{p}_f+i\epsilon_{j\mu\alpha\beta}(p_{ij}p_{f\sigma}-p_{i\sigma}p_{fj})(1-\delta_{4\sigma})\right] \qquad (22) \\
M^{(2)}_{\sigma\mu} &= -M^{(1)}_{\sigma\mu}+\left(Z_B^{(1)}(|\vec{p}_f|)+Z_B^{(2)}(|\vec{p}_f|)\right)\left(Z_A^{(1)*}(|\vec{p}_i|)+Z_A^{(2)*}(|\vec{p}_i|)\right)\mathcal{K}_{\tau\mu}\Big[\\
&\Big((m_\Delta+E_\Delta)p_{ik}-(m_N+E_N)p_{fk}\Big)(1-\delta_{4\sigma})(1-\delta_{4\tau})\epsilon_{\sigma\tau k} \\
&\left.+ip_{fk}p_{il}\delta_{4\tau}(1-\delta_{4\sigma})\epsilon_{k\sigma l}-ip_{fk}p_{il}\left(\delta_{4\sigma}+\frac{ip_{f\sigma}}{m_\Delta}\right)(1-\delta_{4\tau})\epsilon_{k\tau l}\right] \qquad (23) \\
M^{(3)}_{\sigma\mu} &= 0 \qquad (24)
\end{aligned}
$$

where

$$
\mathcal{K}_{\alpha\mu} = \frac{4(m_N+m_\Delta)}{m_N\Omega(q^2)}\left(m_N^2m_\Delta^2\delta_{\alpha\mu}+m_N^2p_{f\alpha}p_{f\mu}+m_\Delta^2p_{i\alpha}p_{i\mu}+p_i\cdot p_f(p_{i\alpha}p_{f\mu}+p_{f\alpha}p_{i\mu}-p_i\cdot p_f\delta_{\alpha\mu})\right)
$$

4.2 The $M^{(n)}_{\sigma\mu}$ for $T = T_j$

In these expressions, Greek indices run from 1 to 4 and Roman indices run from 1 to 3.

$$\begin{aligned}
M^{(1)}_{\sigma\mu} &= \frac{-(m_N+m_\Delta)p_{i\alpha}p_{f\beta}}{m_N((m_N+m_\Delta)^2+q^2)}\left(Z^{(1)}_B(|\vec{p}_f|)+Z^{(2)}_B(|\vec{p}_f|)\right)\left(Z^{(1)*}_A(|\vec{p}_i|)+Z^{(2)*}_A(|\vec{p}_i|)\right)\\
&\Bigg[(E_\Delta+m_\Delta)(E_N+m_N)\epsilon_{\tau\mu\alpha\beta}\epsilon_{j\sigma\tau}(1-\delta_{4\tau})\\
&+\Bigg((E_N+m_N)p_{fk}-(E_\Delta+m_\Delta)p_{ik}\Bigg)\frac{p_{f\sigma}}{m_\Delta}(1-\delta_{4\tau})\epsilon_{\tau\mu\alpha\beta}\epsilon_{jk\tau}\\
&+i\Bigg((E_N+m_N)p_{fk}-(E_\Delta+m_\Delta)p_{ik}\Bigg)\epsilon_{\tau\mu\alpha\beta}\left(\delta_{4\tau}(1-\delta_{4\sigma})\epsilon_{jk\sigma}-\delta_{4\sigma}(1-\delta_{4\tau})\epsilon_{jk\tau}\right)\\
&+p_{fk}p_{il}\epsilon_{jkl}\left(3\epsilon_{\sigma\mu\alpha\beta}+i\frac{p_{f\sigma}}{m_\Delta}\epsilon_{4\mu\alpha\beta}-\epsilon_{4\mu\alpha\beta}\delta_{4\sigma}\right)\\
&-p_{fk}p_{il}\epsilon_{\tau\mu\alpha\beta}(1-\delta_{4\sigma})(1-\delta_{4\tau})(\delta_{jk}\epsilon_{\sigma\tau l}+\delta_{\sigma\tau}\epsilon_{jkl}+\delta_{\tau l}\epsilon_{jk\sigma}-\delta_{\sigma l}\epsilon_{jk\tau})\Bigg]
\end{aligned} \tag{25}$$

$$\begin{aligned}
M^{(2)}_{\sigma\mu} &= -M^{(1)}_{\sigma\mu}+\left(Z^{(1)}_B(|\vec{p}_f|)+Z^{(2)}_B(|\vec{p}_f|)\right)\left(Z^{(1)*}_A(|\vec{p}_i|)+Z^{(2)*}_A(|\vec{p}_i|)\right)\Bigg[\\
&-2i\mathcal{K}_{\sigma\mu}\Bigg((E_N+m_N)p_{fj}-(E_\Delta+m_\Delta)p_{ij}\Bigg)\\
&+\mathcal{K}_{4\mu}\Bigg(-\delta_{\sigma j}(E_N+m_N)(E_\Delta+m_\Delta)-\frac{p_{fj}p_{f\sigma}}{m_\Delta}(E_N+m_N)\\
&-\frac{p_{ij}p_{f\sigma}}{m_\Delta}(E_\Delta+m_\Delta)+p_{fj}p_{i\sigma}-\delta_{j\sigma}\vec{p}_i\cdot\vec{p}_f+p_{ij}p_{f\sigma}\Bigg)\\
&+\mathcal{K}_{j\mu}\Bigg(\left(\delta_{4\sigma}-\frac{ip_{f\sigma}}{m_\Delta}\right)(E_N+m_N)(E_\Delta+m_\Delta)+ip_{f\sigma}(E_N+m_N)(1-\delta_{4\sigma})\\
&+ip_{i\sigma}(E_\Delta+m_\Delta)(1-\delta_{4\sigma})+\frac{ip_{f\sigma}}{m_\Delta}\vec{p}_i\cdot\vec{p}_f+\delta_{4\sigma}\vec{p}_i\cdot\vec{p}_f\Bigg)\Bigg]
\end{aligned} \tag{26}$$

$$\begin{aligned}
M^{(3)}_{\sigma\mu} &= \frac{i(m_N+m_\Delta)}{m_N\Omega(q^2)}\left(q^2(p_i+p_f)_\mu-q\cdot(p_i+p_f)q_\mu\right)\\
&\left(Z^{(1)}_B(|\vec{p}_f|)+Z^{(2)}_B(|\vec{p}_f|)\right)\left(Z^{(1)*}_A(|\vec{p}_i|)+Z^{(2)*}_A(|\vec{p}_i|)\right)\Bigg[\\
&(m_N+E_N)p_{fj}\left(-3p_{i\sigma}+\frac{p_{f\sigma}}{m_\Delta}\left(m_N-\frac{2p_i\cdot p_f}{m_\Delta}\right)-i\delta_{4\sigma}\left(m_N+\frac{p_i\cdot p_f}{m_\Delta}\right)\right)\\
&-(m_\Delta+E_\Delta)p_{ij}\left(-3p_{i\sigma}+\frac{p_{f\sigma}}{m_\Delta}\left(m_N-\frac{2p_i\cdot p_f}{m_\Delta}\right)+i\delta_{4\sigma}\left(m_N+\frac{p_i\cdot p_f}{m_\Delta}\right)\right)\\
&+\delta_{j\sigma}(m_N+E_N)(m_\Delta+E_\Delta)\left(m_N+\frac{p_i\cdot p_f}{m_\Delta}\right)\\
&+(1-\delta_{4\sigma})\left(m_N+\frac{p_i\cdot p_f}{m_\Delta}\right)\left(p_{i\sigma}p_{fj}-\vec{p}_i\cdot\vec{p}_f\delta_{j\sigma}+p_{ij}p_{f\sigma}\right)\Bigg]
\end{aligned} \tag{27}$$

where $\mathcal{K}_{\alpha\mu}$ was defined in the previous subsection.

4.3 Special case: $T = T_j$, $\mu = 4$, $\sigma \neq 4$, $\vec{p}_i = \vec{0}$

$$\begin{aligned}
M^{(1)}_{\sigma\mu} &= 0 \\
M^{(2)}_{\sigma\mu} &= 0 \\
M^{(3)}_{\sigma\mu} &= \left(Z^{(1)}_B(|\vec{q}|) + Z^{(2)}_B(|\vec{q}|)\right)\left(Z^{(1)*}_A(0) + Z^{(2)*}_A(0)\right)\frac{m_N + m_\Delta}{m_\Delta}\left[q_j q_\sigma\left(1 + \frac{2E_\Delta}{m_\Delta}\right) - \vec{q}^2\delta_{\sigma j}\right]
\end{aligned}$$

Eq. (21) simplifies to

$$G_{C2}(q^2) = \pm\frac{4\sqrt{6}m_\Delta E_\Delta m_N}{(m_N + m_\Delta)}\sqrt{1 + \frac{m_\Delta}{E_\Delta}}\sqrt{1 + \frac{\vec{q}^2}{3m_\Delta^2}}\left(\frac{\sqrt{R_{\sigma\mu j}}}{q_j q_\sigma(1 + 2E_\Delta/m_\Delta) - \vec{q}^2\delta_{\sigma j}}\right)$$

in agreement with Alexandrou et al, PRD69,114506 (2004), eq 19.

4.4 Special case: $T = T_4$, $\mu \neq 4$, $\sigma \neq 4$, $\vec{p}_i = \vec{0}$

$$\begin{aligned}
M^{(1)}_{\sigma\mu} &= 2\left(Z^{(1)}_B(|\vec{q}|) + Z^{(2)}_B(|\vec{q}|)\right)\left(Z^{(1)*}_A(0) + Z^{(2)*}_A(0)\right)(m_N + m_\Delta)\epsilon_{\sigma\mu k}q_k \\
M^{(2)}_{\sigma\mu} &= 0 \\
M^{(3)}_{\sigma\mu} &= 0
\end{aligned}$$

Eq. (21) simplifies to

$$G_{M1}(q^2) = \pm\frac{2\sqrt{6}E_\Delta m_N}{(m_N + m_\Delta)q_k}\sqrt{1 + \frac{m_\Delta}{E_\Delta}}\sqrt{1 + \frac{\vec{q}^2}{3m_\Delta^2}}\sqrt{R_{\sigma\mu j}}$$

where μ, σ and k are three distinct spatial directions. This equation is in agreement with Alexandrou et al, PRD69,114506 (2004), eq 20(a).

4.5 Special case: $T = T_j$, $\mu \neq 4$, $\sigma \neq 4$, $\vec{p}_i = \vec{0}$

$$
\begin{aligned}
M^{(1)}_{\sigma\mu} &= -i\left(Z^{(1)}_B(|\vec{q}|) + Z^{(2)}_B(|\vec{q}|)\right)\left(Z^{(1)*}_A(0) + Z^{(2)*}_A(0)\right)\left(\frac{m_N + m_\Delta}{m_\Delta + E_\Delta}\right) q_k \epsilon_{l\mu k} \\
&\quad \left((E_\Delta + m_\Delta)\epsilon_{j\sigma l} + \frac{q_\sigma q_m}{m_\Delta}\epsilon_{jml}\right) \\
M^{(2)}_{\sigma\mu} &= -M^{(1)}_{\sigma\mu} + i\left(Z^{(1)}_B(|\vec{q}|) + Z^{(2)}_B(|\vec{q}|)\right)\left(Z^{(1)*}_A(0) + Z^{(2)*}_A(0)\right)(m_N + m_\Delta)\Bigg[4q_j\left(\delta_{\sigma\mu} - \frac{q_\sigma q_\mu}{\vec{q}^2}\right) \\
&\quad + 3\frac{E_\Delta}{m_\Delta} q_\sigma \left(\delta_{j\mu} - \frac{q_j q_\mu}{\vec{q}^2}\right)\Bigg] \\
M^{(3)}_{\sigma\mu} &= -i\left(Z^{(1)}_B(|\vec{q}|) + Z^{(2)}_B(|\vec{q}|)\right)\left(Z^{(1)*}_A(0) + Z^{(2)*}_A(0)\right)(m_N + m_\Delta)\frac{q_\mu}{m_\Delta}(E_\Delta - m_N) \\
&\quad \left[\delta_{\sigma j} - \frac{q_j q_\sigma}{\vec{q}^2}\left(1 + \frac{2E_\Delta}{m_\Delta}\right)\right]
\end{aligned}
$$

Eq. (21) simplifies to

$$
\begin{aligned}
G_{M1}(q^2) &= \pm\frac{2\sqrt{6}E_\Delta m_N}{(m_N + m_\Delta)(q_j^2 - q_k^2)}\sqrt{1 + \frac{m_\Delta}{E_\Delta}}\sqrt{1 + \frac{\vec{q}^2}{3m_\Delta^2}} \\
&\quad \left[\left(q_j\sqrt{R_{kkj}} - q_k\sqrt{R_{jjk}}\right) - \frac{m_\Delta}{E_\Delta}\left(q_j\sqrt{R_{jkk}} - q_k\sqrt{R_{kjj}}\right)\right] \\
G_{E2}(q^2) &= \pm\frac{2\sqrt{6}E_\Delta m_N}{3(m_N + m_\Delta)(q_j^2 - q_k^2)}\sqrt{1 + \frac{m_\Delta}{E_\Delta}}\sqrt{1 + \frac{\vec{q}^2}{3m_\Delta^2}} \\
&\quad \left[\left(q_j\sqrt{R_{kkj}} - q_k\sqrt{R_{jjk}}\right) + \frac{m_\Delta}{E_\Delta}\left(q_j\sqrt{R_{jkk}} - q_k\sqrt{R_{kjj}}\right)\right]
\end{aligned}
$$

where μ, σ and k are three distinct spatial directions. These equations are in agreement with Alexandrou et al, PRD69,114506 (2004), eqs 20(b) and 21.

5 The $\Delta \to \Delta$ electromagnetic form factors

In conventional (but Euclidean) notation, the matrix element of interest is

$$
\begin{aligned}
&\langle \Delta(\vec{p}_f, s') \,| V_\mu(0) |\, \Delta(\vec{p}_i, s) \rangle_{\text{continuum}} \\
&\qquad = Z_V \, \langle \Delta(\vec{p}_f, s') \,| V_\mu(0) |\, \Delta(\vec{p}_i, s) \rangle \\
&\qquad = -\bar{u}_\alpha(\vec{p}_f, s') \mathcal{O}^{\alpha\mu\beta} u_\beta(\vec{p}_i, s)
\end{aligned}
$$

where Z_V is the renormalization factor ($Z_V = 1$ for a conserved current), $q = p_f - p_i$, $u_\alpha(\vec{p}, s)$ is a spin-vector in the Rarita-Schwinger formalism, The operator $\mathcal{O}^{\alpha\mu\beta}$ can be decomposed into

$$
\mathcal{O}^{\alpha\mu\beta} = -\delta^{\alpha\beta} \left(a_1 \gamma_\mu + \frac{a_2}{2m_\Delta} P^\mu \right) + \frac{q^\alpha q^\beta}{(2m_\Delta)^2} \left(c_1 \gamma_\mu + \frac{c_2}{2m_\Delta} P^\mu \right)
$$

where $P^\mu = (p_f + p_i)/2$. The parameters a_1, a_2, c_1, c_2 are independent covariant vertex function coefficients related to the multipole form factors (Leinweber PRD46).