

Let \vec{n} be a four dimensional vector

$$\vec{n} = (n_x, n_y, n_z, n_t)$$

where n_i takes values 0, 1. We often equivalently label these 16 vectors by

$$n = \sum_{\mu=1}^4 n_{\mu} 2^{\mu-1}.$$

Let us define

$$\Gamma(\vec{n}) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4}.$$

There are obviously 16 of these Γ 's and they form a basis for 4×4 matrices.

The multiplication table for the Γ 's is pretty simple to obtain,

$$\Gamma(\vec{n})\Gamma(\vec{m}) = \eta(\vec{n}, \vec{m})\Gamma(\vec{n} \oplus \vec{m})$$

where $\vec{n} \oplus \vec{m}$ is component-wise addition modulo 2 ($= \vec{n} \text{ .XOR. } \vec{m}$), and

$$\eta(\vec{n}, \vec{m}) = \prod_{\mu < \nu} (-)^{n_{\nu} m_{\mu}}.$$

Furthermore, with $\eta(\vec{n}) = \eta(\vec{n}, \vec{n})$,

$$\Gamma^{\dagger}(\vec{n}) = \eta(\vec{n})\Gamma(\vec{n}).$$

Obviously the 16 operators $\bar{\psi}\Gamma(\vec{n})\psi$ exhaust all of the possible mesons. **Ex.**

$$\vec{n} = (0, 0, 0, 0) \rightarrow \bar{\psi}\Gamma(\vec{n})\psi = \bar{\psi}\psi$$

$$\vec{n} = (1, 1, 1, 1) \rightarrow \bar{\psi}\Gamma(\vec{n})\psi = \bar{\psi}\gamma_5\psi$$

$$\vec{n} = (0, 0, 1, 0) \rightarrow \bar{\psi}\Gamma(\vec{n})\psi = \bar{\psi}\gamma_3\psi$$

etc.

A generic 4×4 matrix A can be decomposed as

$$A = \sum_{\vec{n}} a_{\vec{n}} \Gamma^{\dagger}(\vec{n})$$

where

$$a_{\vec{n}} = \frac{1}{4} \text{Tr} \Gamma(\vec{n}) A.$$

Now consider the correlation function of a meson of type \vec{n}

$$\begin{aligned} C_{\vec{n}}(x) &= \langle \bar{\psi}(x) \Gamma(\vec{n}) \psi(x) \bar{\psi}(0) \Gamma^{\dagger}(\vec{n}) \psi(0) \rangle \\ &= \eta(\vec{n}) \text{Tr} [\Gamma(\vec{n}) G(x, 0) \Gamma(\vec{n}) G(0, x)]. \end{aligned}$$

Here $G(x, 0)$ is the quark propagator with $G(0, x) = \gamma_5 G^{\dagger}(x, 0) \gamma_5$.

Using the decomposition

$$G(x, 0) = \sum_{\vec{p}} g_{\vec{p}}(x) \Gamma^{\dagger}(\vec{p})$$

we then find

$$C_{\vec{n}}(x) = \eta(\vec{n}) \sum_{\vec{p}, \vec{q}} \text{tr}_c \left[g_{\vec{p}}(x) g_{\vec{q}}^\dagger(x) \right] \text{tr}_d \left[\Gamma(\vec{n}) \Gamma^\dagger(\vec{p}) \Gamma(\vec{n}) \Gamma(\vec{1}) \Gamma(\vec{q}) \Gamma(\vec{1}) \right],$$

where $\vec{1} = (1, 1, 1, 1)$, *i.e.*, $\Gamma(\vec{1}) = \gamma_5$. Using the multiplication table for the Γ 's we find

$$C_{\vec{n}}(x) = \eta(\vec{n}) \sum_{\vec{p}, \vec{q}} \eta(\vec{p}) c_m(\vec{n}, \vec{p}, \vec{q}) \text{tr}_c \left[g_{\vec{p}}(x) g_{\vec{q}}^\dagger(x) \right] \text{tr}_d [\Gamma(\vec{q} \oplus \vec{p})],$$

where

$$c_m(\vec{n}, \vec{p}, \vec{q}) = \eta(\vec{n}, \vec{p}) \eta(\vec{n}, \vec{1}) \eta(\vec{q}, \vec{1}) \eta(\vec{n} \oplus \vec{1}, \vec{q} \oplus \vec{1}) \eta(\vec{n} \oplus \vec{p}, \vec{n} \oplus \vec{q}).$$

Since all Dirac matrices, except the identity matrix, are traceless we find

$$\text{tr}_d \Gamma(\vec{q} \oplus \vec{p}) = 4 \delta_{\vec{p}, \vec{q}}.$$

Simple algebra, using the definition for the η 's reduces the result to

$$C_{\vec{n}}(x) = 4 \eta(\vec{n}) \sum_{\vec{q}} \eta(\vec{q}) \eta(\vec{n} \oplus \vec{q}) \mathbf{SGN}(\vec{q}) |g_{\vec{q}}(x)|^2,$$

where

$$\mathbf{SGN}(\vec{q}) = (-)^{q_x + q_y + q_z + q_t}.$$

So in order to do the meson spectroscopy we just have to compute once and for all the 16 diagonal η 's and the \mathbf{SGN} 's. A table is attached at the end. Note that both η and \mathbf{SGN} are γ -representation independent.

B. Vector Currents

We can consider three different vector currents on the lattice, all of which have the same (naive) continuum limit. On the lattice only one current is conserved. Note that we only consider Wilson fermions with the Wilson parameter r set to 1. The vector currents are

$$V_\mu^{loc}(x) = \bar{\psi}(x) \gamma_\mu \psi(x),$$

$$V_\mu^{cvc}(x) = \frac{1}{2} \left\{ \bar{\psi}(x) (\gamma_\mu - 1) U_\mu(x) \psi(x + \mu) + \bar{\psi}(x + \mu) (\gamma_\mu + 1) U_\mu^\dagger(x) \psi(x) \right\},$$

which is the conserved current, and

$$V_\mu^{ncvc}(x) = \frac{1}{2} \left\{ \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + \mu) + \bar{\psi}(x + \mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) \right\}.$$

We want to compute the correlators of these currents with a ρ of ‘polarization’ μ , where we take μ to be a space like direction, *i.e.*, 1, 2 or 3,

$$C_\mu^A(x) = \langle V_\mu^A(x) \bar{\psi}(0) \gamma_\mu \psi(0) \rangle,$$

where A is one of *loc*, *cvc* or *ncvc*. The correlator of the local current is just the γ_μ -meson already computed.

We consider the non-conserved current correlator next. We find

$$C_\mu^{ncvc}(x) = \frac{1}{2} \left\{ \text{Tr} [\gamma_\mu U_\mu(x) G(x + \mu, 0) \gamma_\mu \gamma_5 G^\dagger(x, 0) \gamma_5] + \text{Tr} [\gamma_\mu U_\mu^\dagger(x) G(x, 0) \gamma_\mu \gamma_5 G^\dagger(x + \mu, 0) \gamma_5] \right\}.$$

Introducing the vector $\vec{\mu}$ as the unit vector in direction μ and going through the same steps as in the computation of the meson propagators we find

$$C_\mu^{ncvc}(x) = 2 \sum_{\vec{q}} \eta(\vec{q}) \eta(\vec{\mu} \oplus \vec{q}) \mathbf{SGN}(\vec{q}) \left\{ \text{tr}_c [U_\mu(x) g_{\vec{q}}(x + \mu) g_{\vec{q}}^\dagger(x)] + \text{tr}_c [U_\mu^\dagger(x) g_{\vec{q}}(x) g_{\vec{q}}^\dagger(x + \mu)] \right\}.$$

The second term is the hermitian conjugate of the first and hence

$$C_\mu^{ncvc}(x) = 4 \sum_{\vec{q}} \eta(\vec{q}) \eta(\vec{\mu} \oplus \vec{q}) \mathbf{SGN}(\vec{q}) \text{Re} \left(\text{tr}_c [U_\mu(x) g_{\vec{q}}(x + \mu) g_{\vec{q}}^\dagger(x)] \right).$$

We can write the conserved vector current as

$$V_\mu^{cvc}(x) = V_\mu^{ncvc}(x) + V_\mu^{wils}(x),$$

where

$$V_\mu^{wils}(x) = \frac{1}{2} \left\{ -\bar{\psi}(x) U_\mu(x) \psi(x + \mu) + \bar{\psi}(x + \mu) U_\mu^\dagger(x) \psi(x) \right\}.$$

Obviously the correlator has the same decomposition. Thus we need to consider now only

$$C_\mu^{wils}(x) = \frac{1}{2} \left\{ \text{Tr} [-U_\mu(x) G(x + \mu, 0) \gamma_\mu \gamma_5 G^\dagger(x, 0) \gamma_5] + \text{Tr} [U_\mu^\dagger(x) G(x, 0) \gamma_\mu \gamma_5 G^\dagger(x + \mu, 0) \gamma_5] \right\}.$$

Again we insert the decomposition of $G(x, 0)$ into Dirac matrices. The spinor part of the correlator is

$$\text{tr}_d [\Gamma^\dagger(\tilde{p}) \gamma_\mu \Gamma(\tilde{\mathbf{1}}) \Gamma(\tilde{q}) \Gamma(\tilde{\mathbf{1}})] = 4c_{vw}(\tilde{\mu}, \tilde{q}) \delta_{\tilde{p}, \tilde{\mu} \oplus \tilde{q}}.$$

Here we have defined

$$c_{vw}(\vec{\mu}, \vec{q}) = \eta(\vec{\mu} \oplus \vec{q}) \eta(\vec{\mu}, \vec{\mathbf{1}}) \eta(\vec{q}, \vec{\mathbf{1}}) \eta(\vec{\mu} \oplus \vec{\mathbf{1}}, \vec{q} \oplus \vec{\mathbf{1}}) \eta(\vec{\mu} \oplus \vec{q}) = \mathbf{SGN}(\vec{q}) \eta(\vec{\mu}, \vec{q})$$

The second equation follows after some algebra. Then we obtain

$$C_\mu^{wils}(x) = 2 \sum_{\vec{q}} c_{vw}(\vec{\mu}, \vec{q}) \left\{ -\text{tr}_c [U_\mu(x) g_{\vec{\mu} \oplus \vec{q}}(x + \mu) g_{\vec{q}}^\dagger(x)] + \text{tr}_c [U_\mu^\dagger(x) g_{\vec{\mu} \oplus \vec{q}}(x) g_{\vec{q}}^\dagger(x + \mu)] \right\}.$$

Some algebra shows that

$$c_{vw}(\vec{\mu}, \vec{\mu} \oplus \vec{q}) = -c_{vw}(\vec{\mu}, \vec{q}).$$

Thus, changing the summation variable in the second term of $C_\mu^{wils}(x)$ from \vec{q} to $\vec{\mu} \oplus \vec{q}$ shows that it is just the hermitian conjugate of the first term. So we find

$$C_\mu^{wils}(x) = -4 \sum_{\vec{q}} c_{vw}(\vec{\mu}, \vec{q}) \text{Re} \left(\text{tr}_c [U_\mu(x) g_{\vec{\mu} \oplus \vec{q}}(x + \mu) g_{\vec{q}}^\dagger(x)] \right).$$

The $\eta(\vec{n}, \vec{m})$'s necessary for the computation of the 'Wilson part' of the conserved current correlators have been generated by a small Fortran program.

C. Axial Currents

There is no (partially) conserved axial current on the lattice. So we just have to do with lattice axial currents that have the correct naive continuum limit. We consider two such currents

$$A_\mu^{loc}(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x),$$

and

$$A_\mu^{nloc}(x) = \frac{1}{2} \left\{ \bar{\psi}(x)\gamma_\mu\gamma_5 U_\mu(x)\psi(x+\mu) + \bar{\psi}(x+\mu)\gamma_\mu\gamma_5 U_\mu^\dagger(x)\psi(x) \right\}.$$

We want to compute the correlators of the 4th component of these currents with a pion

$$C_4^B(x) = \langle A_4^B(x) \bar{\psi}(0)\gamma_5\psi(0) \rangle,$$

where B stands for *loc* or *nloc*.

We consider the non-local current correlator first. We find

$$C_4^{nloc}(x) = \frac{1}{2} \left\{ \text{Tr} \left[\gamma_4\gamma_5 U_4(x) G(x+\hat{4}, 0) \gamma_5\gamma_5 G^\dagger(x, 0) \gamma_5 \right] + \text{Tr} \left[\gamma_4\gamma_5 U_4^\dagger(x) G(x, 0) \gamma_5\gamma_5 G^\dagger(x+\hat{4}, 0) \gamma_5 \right] \right\}.$$

Once more we insert the decomposition of $G(x, 0)$ into Dirac matrices. This time the spinor part becomes

$$\text{tr}_d \left[\Gamma(\vec{4}) \Gamma(\vec{1}) \Gamma^\dagger(\vec{p}) \Gamma(\vec{q}) \Gamma(\vec{1}) \right] = 4c_a(\vec{4}, \vec{q}) \delta_{\vec{p}, \vec{4} \oplus \vec{q}}.$$

Here we have introduced

$$c_a(\vec{4}, \vec{q}) = \eta(\vec{4} \oplus \vec{q}) \eta(\vec{4}, \vec{1}) \eta(\vec{q}, \vec{1}) \eta(\vec{q} \oplus \vec{1}, \vec{4} \oplus \vec{1}) \eta(\vec{4} \oplus \vec{q}) = \mathbf{SGN}(\vec{4}) \eta(\vec{q}, \vec{4})$$

Then the correlator becomes

$$C_4^{nloc}(x) = 2 \sum_{\vec{q}} c_a(\vec{4}, \vec{q}) \left\{ \text{tr}_c \left[U_4(x) g_{\vec{4} \oplus \vec{q}}(x+\hat{4}) g_{\vec{q}}^\dagger(x) \right] + \text{tr}_c \left[U_4^\dagger(x) g_{\vec{4} \oplus \vec{q}}(x) g_{\vec{q}}^\dagger(x+\hat{4}) \right] \right\}.$$

c_a satisfies

$$c_a(\vec{4}, \vec{4} \oplus \vec{q}) = c_a(\vec{4}, \vec{q})$$

and the second term above becomes, after changing the summation variable from \vec{q} to $\vec{4} \oplus \vec{q}$, the hermitian conjugate of the first term. Thus we get

$$C_4^{nloc}(x) = 4 \sum_{\vec{q}} c_a(\vec{4}, \vec{q}) \text{Re} \left(\text{tr}_c \left[U_4(x) g_{\vec{4} \oplus \vec{q}}(x+\hat{4}) g_{\vec{q}}^\dagger(x) \right] \right).$$

Analogously we find for the local axial current correlator

$$C_4^{loc}(x) = 4 \sum_{\vec{q}} c_a(\vec{4}, \vec{q}) \text{Re} \left(\text{tr}_c \left[g_{\vec{4} \oplus \vec{q}}(x) g_{\vec{q}}^\dagger(x) \right] \right).$$

D. Improved Currents

The local currents considered previously can be $\mathcal{O}(a)$ -improved by adding

$$\delta A_\mu(x) = \tilde{\partial}_\mu \bar{\psi}(x) \gamma_5 \psi(x),$$

and

$$\delta V_\mu(x) = \tilde{\partial}_\nu \bar{\psi}(x) \gamma_\nu \gamma_\mu \psi(x)$$

respectively with appropriately chosen coefficients. Note that in the last equation we used $i\sigma_{\mu\nu} = \gamma_\nu \gamma_\mu$. Further we have defined

$$\tilde{\partial}_\mu f(x) = \frac{1}{2} [f(x + \mu) - f(x - \mu)].$$

We note that the correlator of $\delta A_4(x)$ with $\bar{\psi}(0) \gamma_5 \psi(0)$ is, apart from the $\tilde{\delta}_4$, just the pion correlator, pion_1 .

For the vector current we need the correlator of $\delta V_k(x)$ with the rho, $\bar{\psi}(0) \gamma_k \psi(0)$. Since we will project on zero momentum by summing over all \vec{x} , only the $\nu = 4$ part in $\delta V_k(x)$ will give a non-vanishing contribution

$$C_k^{\delta V}(x) = \langle \delta V_k(x) \bar{\psi}(0) \gamma_k \psi(0) \rangle = \tilde{\partial}_4 \text{Tr} [\gamma_4 \gamma_k G(x, 0) \gamma_k \gamma_5 G^\dagger(x, 0) \gamma_5].$$

The spinor part of the correlator is

$$\text{tr}_d [\gamma_4 \gamma_k \Gamma^\dagger(\tilde{\mathbf{p}}) \gamma_k \Gamma(\tilde{\mathbf{1}}) \Gamma(\tilde{\mathbf{q}}) \Gamma(\tilde{\mathbf{1}})] = 4c_{\delta v}(\tilde{\mathbf{k}}, \tilde{\mathbf{q}}) \delta_{\tilde{\mathbf{p}}, \tilde{\mathbf{4}} \oplus \tilde{\mathbf{q}}}.$$

Here we have defined

$$c_{\delta v}(\vec{k}, \vec{q}) = \eta(\vec{4} \oplus \vec{q}) \eta(\vec{4}, \vec{k}) \eta(\vec{k}, \vec{1}) \eta(\vec{q}, \vec{1}) \eta(\vec{4} \oplus \vec{k}, \vec{4} \oplus \vec{q}) \eta(\vec{k} \oplus \vec{1}, \vec{q} \oplus \vec{1}) \eta(\vec{k} \oplus \vec{q}) = -\mathbf{SGN}(\vec{q}) \eta(\vec{k} \oplus \vec{q})$$

One checks easily that

$$c_{\delta v}(\vec{k}, \vec{4} \oplus \vec{q}) = c_{\delta v}(\vec{k}, \vec{q})$$

and hence we obtain

$$C_k^{\delta V}(x) = 4 \sum_{\vec{q}} c_{\delta v}(\vec{k}, \vec{q}) \tilde{\partial}_4 \text{Re} \left(\text{tr}_c \left[g_{\vec{4} \oplus \vec{q}}(x) g_{\vec{q}}^\dagger(x) \right] \right).$$

Table

n	(n_x, n_y, n_z, n_t)	$\eta(n)$	SGN (n)
0	(0,0,0,0)	1	1
1	(1,0,0,0)	1	-1
2	(0,1,0,0)	1	-1
3	(1,1,0,0)	-1	1
4	(0,0,1,0)	1	-1
5	(1,0,1,0)	-1	1
6	(0,1,1,0)	-1	1
7	(1,1,1,0)	-1	-1
8	(0,0,0,1)	1	-1
9	(1,0,0,1)	-1	1
10	(0,1,0,1)	-1	1
11	(1,1,0,1)	-1	-1
12	(0,0,1,1)	-1	1
13	(1,0,1,1)	-1	-1
14	(0,1,1,1)	-1	-1
15	(1,1,1,1)	1	1