Finite Volume Corrections

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We derive finite volume corrections for a few common integrals.

CONTENTS

Lattice QCD calculations are performed in a finite volume and so loop integrals get replaced with loop-sums. When we consider these corrections, we work in the approximation that $T \to \infty$ and only the spatial volume is held finite.

When nothing in the integrand can go on-shell, the convenient identity to use to determine the finite volume corrections is the Poisson Summation Formula

$$\sum_{\vec{n}} \delta^3(\vec{n} - \vec{y}) = \sum_{\vec{m}} e^{2\pi i \vec{m}\cdot\vec{y}} \,. \tag{1}$$

1. *Tadpole integral*

Consider the tadpole integral in finite volume

$$\begin{aligned} i\mathcal{I}(m) &= \sum\!\!\!\!\!\!\!\int \frac{i}{k^2 - m^2 + i\epsilon} \\ &= \sum\!\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{i}{(k_0 - \omega_k)(k_0 + \omega_k)} \\ &= \sum\!\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \,, \end{aligned} \tag{2}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2 - i\epsilon}$ and we closed the contour on the upper-half plane. Let us start by considering the finite volume integral for which

$$\int \frac{d^3k}{(2\pi)^3} \longrightarrow \frac{1}{L^3} \sum_{\vec{k}} , \tag{3}$$

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giving us

$$\begin{aligned} i\mathcal{I}^{\rm FV}(m) &= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{k}}\frac{1}{\sqrt{\vec{k}^2+m^2}} \\ &= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{n}}\frac{1}{\frac{2\pi}{L}\sqrt{\vec{n}^2+\left(\frac{mL}{2\pi}\right)^2}} \\ &= \frac{1}{4\pi L^2}\sum_{\vec{n}}\frac{1}{\sqrt{\vec{n}^2+\left(\frac{mL}{2\pi}\right)^2}} \\ &= \frac{1}{4\pi L^2}\sum_{\vec{n}}\int d^3y\frac{\delta^3(\vec{y}-\vec{n})}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} \\ &= \frac{1}{4\pi L^2}\int d^3y\frac{1}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\sum_{\vec{n}}\delta^3(\vec{y}-\vec{n}) \qquad \text{use Poisson Summation Formula, Eq. (1)} \\ &= \frac{1}{4\pi L^2}\int d^3y\frac{1}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\sum_{\vec{n}}e^{2\pi i\vec{n}\cdot\vec{y}} \\ &= \frac{1}{4\pi L^2}\sum_{\vec{n}}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\,. \end{aligned} \tag{4}$$

What we are really interested in is the difference between the finite and infinite volume integrals. Notice, the $\vec{n}=\vec{0}$ contribution to $i\mathcal{I}^{\rm FV}(m)$ is the infinite volume expression (which has the UV divergences). Therefore, we can construct

$$\begin{aligned} i\delta\mathcal{I}(m) &\equiv i\mathcal{I}^{\rm FV}(m) - i\mathcal{I}(m) \\ &= \frac{1}{4\pi L^2}\sum_{\vec{n}\neq 0}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\,, \end{aligned} \tag{5}$$

which is the FV correction we are interested in and it is free of UV divergences. Now let us evaluate the integral

$$\begin{aligned} \int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} &= \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\int_0^\infty dy y^2\frac{e^{2\pi i n y\cos\theta}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} \\ &= \frac{2}{n}\int_0^\infty dy\frac{y\sin(2\pi n y)}{\sqrt{y^2+\left(\frac{mL}{2\pi}\right)^2}} \end{aligned} \tag{6}$$

Gradshteyn and Ryzhik, 7^{th} edition, Section 3.771 (pg 442), Eq. (5), has the relation

$$\int_0^\infty dy\frac{y\,\sin(ay)}{(y^2+\beta^2)^{1/2-\nu}} = \sqrt{\pi}\beta\left(\frac{2\beta}{a}\right)^\nu\frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)}K_{\nu+1}(a\beta)\,, \qquad a>0,\ \mathrm{Re}\ \beta>0,\ \mathrm{Re}\ \nu<0\,. \tag{7}$$

For us, this yields

$$\begin{aligned} \frac{1}{4\pi L^2}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} &= \frac{1}{4\pi L^2}\frac{2}{n}\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})}\frac{mL}{2\pi}K_1(nmL) \qquad [n=|\vec{n}|] \\ &= \frac{m^2}{(4\pi)^2}4\frac{K_1(nmL)}{nmL} \end{aligned} \tag{8}$$

and so the finite volume correction to the tadpole integral is given by

$$i\delta\mathcal{I}(m) = \frac{m^2}{(4\pi)^2}\sum_{\vec{n}\neq 0}4\frac{K_1(|\vec{n}|mL)}{|\vec{n}|mL}\,. \tag{9}$$

After using dim-reg, we can then express the finite volume tadpole integral as

$$\begin{aligned} i\mathcal{I}^{\rm FV}_{\overline{\rm MS}}(m,\mu) &= i\mathcal{I}(m) + i\delta\mathcal{I}(m) \\ &= \frac{m^2}{(4\pi)^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) + \sum_{\vec{n}\neq 0}4\frac{K_1(|\vec{n}|mL)}{|\vec{n}|mL}\right]\,. \end{aligned} \tag{10}$$

2. Leading Heavy Baryon Mass correction

A common integral that arises for heavy baryon/meson formula is

$$\mathcal{F}_{\mu\nu}(m,\Delta) = \sumint \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2-m^2+i\epsilon)((p+k)\cdot v - \Delta + i\epsilon)}\,. \tag{11}$$

In the integrand, p_μ is the momentum of the heavy baryon, which in the rest frame is given by $p_\mu = Mv_\mu$ where v_μ is the four-velocity of the baryon, which in the rest frame is given by $v_\mu^T = (1,\vec{0})$. The quantity Δ is the mass splitting between the mass of the virtual heavy-baryon in the loop and the external heavy-baryon.

This integral will be contracted with either the nucleon spin vector $S_\mu S_\nu$ or the decuplet propagator, $P^{\mu\nu}$. In heavy-baryon χPT, for both of these, we have the constraints

$$\begin{aligned} S^\mu v_\mu &= 0\,, \\ v_\mu P^{\mu\nu} = v_\nu P^{\mu\nu} &= 0\,. \end{aligned} \tag{12}$$

The relevance of this is that, when the denominators are combined in the integrand, and the loop-momentum is shifted, the shift is proportional to v_μ, and thus vanishes. Therefore, from these constraints, the $\mu = 0$ components of the integrand variables will not contribute and so we can focus on the spatial components. Since $\vec{p} = \vec{v} = 0$ in the rest frame, the integrand has no dependence upon an external vector, therefore we can set $k_i k_j = \vec{k}^2 g_{ij}$. For the wave-function renormalization, we can treat $p\cdot v$ as a small parameter giving us the two integrals

$$\sumint \frac{d^4k}{(2\pi)^4}\left[\frac{k_i k_i}{(k^2-m^2+i\epsilon)(k\cdot v-\Delta+i\epsilon)} - \frac{p\cdot v\, k_i k_i}{(k^2-m^2+i\epsilon)(k\cdot v-\Delta+i\epsilon)^2}\right]. \tag{13}$$

We see that, we can obtain the second integral from the first through a derivative with respect to Δ, and so we will focus on the first integral

$$\begin{aligned} \mathcal{F}(m,\Delta) &= \sumint \frac{d^4k}{(2\pi)^4}\frac{\vec{k}^2}{(k^2-m^2+i\epsilon)(k\cdot v-\Delta+i\epsilon)}\,, \\ &= \sumint \frac{d^4k}{(2\pi)^4}\frac{\vec{k}^2}{(k_0^2-\vec{k}^2-m^2+i\epsilon)(k_0-\Delta+i\epsilon)}\,, \\ &= \sumint \frac{d^4k}{(2\pi)^4}\frac{\vec{k}^2}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)(k_0-\Delta+i\epsilon)}\,, \end{aligned} \tag{14}$$

To evaluate this integral, we can close the contour in the upper half-plane, picking up the $k_0 = -\omega_k + i\epsilon$ pole

$$\begin{aligned} \mathcal{F}(m,\Delta) &= \frac{i}{2}\frac{1}{L^3}\sum_{\vec{k}}\frac{\vec{k}^2}{\omega_k(\omega_k+\Delta)} \\ &= \frac{i}{2}\frac{1}{L^3}\sum_{\vec{n}}\frac{\left(\frac{2\pi}{L}\right)^2\vec{n}^2}{\left(\frac{2\pi}{L}\right)^2\sqrt{\vec{n}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{n}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\ &= \frac{i}{2}\frac{1}{L^3}\sum_{\vec{n}}\int d^3y \frac{\vec{y}^2\delta^3(\vec{y}-\vec{n})}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\ &= \frac{i}{2}\frac{1}{L^3}\int d^3y \frac{\vec{y}^2}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)}\sum_{\vec{n}} e^{i2\pi\vec{n}\cdot\vec{y}}\,. \end{aligned} \tag{15}$$

We will start with the internal nucleon loop with $\Delta = 0$

$$\begin{aligned} \mathcal{F}(m,\Delta=0) &= \frac{i}{2}\frac{1}{L^3}\sum_{\vec{n}}\int d^3y \frac{y^2 e^{i2\pi ny\cos\theta}}{y^2+(\frac{mL}{2\pi})^2} \\ &= \frac{i}{2}\frac{1}{L^3}\sum_{\vec{n}}\int d^3y e^{i2\pi ny\cos\theta}\left[\frac{y^2+(\frac{mL}{2\pi})^2}{y^2+(\frac{mL}{2\pi})^2} - \frac{(\frac{mL}{2\pi})^2}{y^2+(\frac{mL}{2\pi})^2}\right]. \end{aligned} \tag{16}$$

FIG. 1. Diagrams contributing to nucleon mass at NLO.

The first term is infinite. Since we are interested in the difference between infinite and finite volume, this infinity is the same in both. Focussing on the difference, we then have

$$\begin{aligned}\delta\mathcal{F}(m,\Delta=0) &= -\frac{i}{2}\frac{1}{L^3}\left(\frac{mL}{2\pi}\right)^2\sum_{\vec{n}\neq 0}\int_0^{2\pi}d\phi\int_{-1}^{+1}d\cos\theta\int_0^\infty dy\frac{y^2e^{i2\pi ny\cos\theta}}{y^2+(\frac{mL}{2\pi})^2}\\ &= -\frac{i}{(4\pi)^2}4m^3\sum_{\vec{n}\neq 0}\frac{1}{nmL}\int_0^\infty\frac{y\sin(2\pi ny)}{y^2+(\frac{mL}{2\pi})^2}\\ &= -\frac{i}{(4\pi)^2}4m^3\sum_{\vec{n}\neq 0}\sqrt{\frac{\pi}{2}}\frac{K_{\frac{1}{2}}(nmL)}{\sqrt{nmL}}\end{aligned} \tag{17}$$

Up to some overall normalizations, related to different factorizations of the total self-energy diagram into this integral and the rest of the term, this agrees with Eq. (A8) of Ref. [1].
TODO: Massage this to be closer to the mass correction, with the $1/F^2$ terms and the correct spin-projections etc. so that we can see the relation to the infinite volume formula more easily. Get the formula to work with a delta.

3. Heavy Baryon Integral a different way

Both loop integrals in Fig. 1 will be proportional to the integral

$$\mathcal{F}_{\mu\nu}(m,p,\Delta) = \int_R\frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m^2+i\epsilon][(p+k)\cdot v-\Delta+i\epsilon]} \tag{18}$$

With the nucleon loop evaluated with $\Delta = 0$ and we denoted a regularization/renormalization scheme R which we almost always take to be dim-reg in modified $\overline{\text{MS}}$. In infinite volume, we have

$$\begin{aligned}\mathcal{F}_{\mu\nu}(m,p,\Delta) &= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu}{\left[k^2-m^2-2\lambda\Delta+2\lambda k\cdot v+2\lambda p\cdot v\right]^2}\\ &= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{i(l_\mu-\lambda v_\mu)(l_\nu-\lambda v_\nu)}{\left[l^2-\lambda^2-m^2-2\lambda\Delta+2\lambda p\cdot v\right]^2}\,.\end{aligned} \tag{19}$$

The integrals will be contracted either with the nucleon spin-vectors S_μ or the delta propagator $\mathcal{P}_{\mu\nu}$ which are orthogonal to v_μ, $v\cdot S=0$, $v_\mu\mathcal{P}^{\mu\nu}=v_\nu\mathcal{P}^{\mu\nu}=0$. Further, to get the wave function renormalization, we can take the linear approximation of the integral with respect to $p\cdot v$, leaving us with

$$\mathcal{F}_{\mu\nu}(m,p,\Delta) = \mathcal{F}_{\mu\nu}(m,0,\Delta)+p\cdot v\mathcal{J}_{\mu\nu}(m,0,\Delta)\,, \tag{20}$$

$$\mathcal{F}_{\mu\nu}(m,0,\Delta) = 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{il_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^2} \tag{21}$$

$$\mathcal{J}_{\mu\nu}(m,0,\Delta) = -8\int_0^\infty d\lambda\ \mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{i\lambda l_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^3} \tag{22}$$

In infinte volume, we have

$$\begin{aligned}\mathcal{F}_{\mu\nu}(m,0,\Delta) &= \frac{-1}{(4\pi)^2}g_{\mu\nu}\left\{\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi+1\right]\left(\frac{2}{3}\Delta^3-\Delta m^2\right)+\frac{10}{9}\Delta^3-\frac{4}{3}\Delta m^2-\frac{2}{3}\bar{\mathcal{F}}(m,\Delta,\mu)-\frac{2}{3}\Delta^3\ln\left(\frac{4\Delta^2}{\mu^2}\right)\right\}\\ \bar{\mathcal{F}}(m,\Delta,\mu) &= (\Delta^2-m^2+i\epsilon)^{3/2}\ln\left(\frac{\Delta+\sqrt{\Delta^2-m^2+i\epsilon}}{\Delta-\sqrt{\Delta^2-m^2+i\epsilon}}\right)-\frac{3}{2}\Delta m^2\ln\left(\frac{m^2}{\mu^2}\right)-\Delta^3\ln\left(\frac{4\Delta^2}{m^2}\right)\,.\end{aligned} \tag{23}$$

The $i\epsilon$ informs us how to take the limit of $m > \Delta$. The function $\bar{\mathcal{F}}$ is defined such that

$$\begin{aligned}\bar{\mathcal{F}}(m=0,\Delta,\mu) &= 0\\ \bar{\mathcal{F}}(m,\Delta=0,\mu) &= \pi m^3\,.\end{aligned} \tag{24}$$

Now let us evaluate the finite volume correction to Eq. (21).

$$\begin{aligned}\mathcal{F}_{\mu\nu}(m,0,\Delta) &= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{il_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^2}\\ &= 2i\int_0^\infty d\lambda\ \mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{\left[(l_0-\omega_l+i\epsilon)(l_0+\omega_l-i\epsilon)\right]^2}\,,\qquad w_l^2=\vec{l}^2+\lambda^2+2\lambda\Delta+m^2\\ &= 2i\int_0^\infty d\lambda\ \mu^{2\epsilon}\int_R\frac{d^3l}{(2\pi)^3}\frac{2\pi i}{2\pi}\frac{l_\mu l_\nu}{4\omega_l^3}\bigg|_{l_0=-\omega_l}\,.\end{aligned} \tag{25}$$

This integral is contracted with either $S_\mu S_\nu$ or $\mathcal{P}_{\mu\nu}$, both of which satisfy $v\cdot S=0$ and $v_\mu\mathcal{P}^{\mu\nu}=0$. There are no external vectors to pick a preferred direction for l_μ and therefore, we can focus on the integral

$$\begin{aligned}&\frac{1}{L^3}\sum_{\vec{l}}\frac{\vec{l}^2}{[\vec{l}^2+\lambda^2+2\lambda\Delta+m^2]^{3/2}}\\ &=\frac{1}{L^3}\sum_{\vec{n}}\frac{(2\pi/L)^2\vec{n}^2}{(2\pi/L)^3[\vec{n}^2+\beta_L^2]^{3/2}}\,,\qquad \beta_L=\frac{\beta L}{2\pi}\,,\qquad \beta^2=\lambda^2+2\lambda\Delta+m^2\\ &=\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}}\frac{n^2}{[n^2+\beta_L^2]^{3/2}}\\ &=\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}}\int d^3y\frac{y^2e^{2\pi iny\cos\theta}}{[y^2+\beta_L^2]^{3/2}}\end{aligned} \tag{26}$$

Putting this back into our full expression, and determining the finite volume difference, we have

$$\begin{aligned}\delta F_{\mu\nu}(m,0,\Delta) &\propto -\frac{2}{4}\int_0^\infty d\lambda\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}\neq0}\int d^3y\frac{y^2e^{2\pi iny\cos\theta}}{[y^2+\beta_L^2]^{3/2}}\\ &= -\frac{1}{2}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\int_0^\infty dy\frac{y^4}{[y^2+\beta_L^2]^{3/2}}\frac{e^{2\pi iny}-e^{-2\pi iny}}{2\pi iny}\\ &= -\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\int_0^\infty dy\frac{y^3\sin(2\pi ny)}{[y^2+\beta_L^2]^{3/2}}\end{aligned} \tag{27}$$

Gradshteyn and Ryzhik, 7^{th} edition, Section 3.773 (pg 444), Eq. (3), has the relation

$$\int_0^\infty dx\frac{x^{2m+1}\sin(ax)}{(\beta^2+x^2)^{n+1/2}}=\frac{(-1)^{m+1}\sqrt{\pi}}{2^n\beta^n\Gamma(n+\frac{1}{2})}\frac{d^{2m+1}}{da^{2m+1}}\left[aK_n(a\beta)\right]\,,\qquad a>0,\mathrm{Re}\ \beta\ >0,-1\le m\le n \tag{28}$$

which brings us to

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu}(m,0,\Delta) &\propto -\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{(-1)^2\sqrt{\pi}}{2\beta_L\Gamma(3/2)}\frac{d^3}{d(2\pi n)^3}\left[(2\pi n)K_1(2\pi n\beta_L)\right]\\ &= -\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{(-1)^2\sqrt{\pi}}{2\beta_L\Gamma(3/2)}\beta_L^2\left[K_1(nL\beta)-nL\beta K_0(nL\beta)\right]\\ &= -\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{\beta L}{2\pi}\left[K_1(nL\beta)-nL\beta K_0(nL\beta)\right]\\ &= -\frac{4}{(4\pi)^2}\int_0^\infty d\lambda\sum_{\vec{n}\neq0}\frac{\beta^2}{nL\beta}\left[K_1(nL\beta)-nL\beta K_0(nL\beta)\right]\\ &= -\frac{4}{(4\pi)^2}\sum_{\vec{n}\neq0}\int_0^\infty d\lambda\ \beta^2\left[\frac{K_1(nL\beta)}{nL\beta}-K_0(nL\beta)\right]\,.\end{aligned} \tag{29}$$

Recall that $\beta^2 = \lambda^2 + 2\lambda\Delta + m^2$ and so

$$\begin{aligned} 2\beta d\beta &= 2(\lambda+\Delta)d\lambda \\ d\lambda &= \frac{\beta d\beta}{\lambda+\Delta} \\ &= \frac{\beta d\beta}{\sqrt{\beta^2 - m^2 + \Delta^2}} \end{aligned} \tag{30}$$

and so this leaves us with

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta) \propto -\frac{4}{(4\pi)^2}\sum_{\vec{n}\neq 0}\int_m^\infty d\beta\ \frac{\beta^3}{\sqrt{\beta^2-m^2+\Delta^2}}\left[\frac{K_1(nL\beta)}{nL\beta} - K_0(nL\beta)\right]. \tag{31}$$

Now, let us make the variable transformation

$$\begin{aligned} x &= \frac{\beta^2}{m^2} \\ dx &= \frac{2\beta d\beta}{m^2} \\ dx\frac{m^3 x}{2\sqrt{x-1+\delta^2}} &= \frac{d\beta\beta^3}{m\sqrt{\beta^2/m^2 - 1 + \Delta^2/m^2}} \end{aligned} \tag{32}$$

which brings our correction to

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta=0) \propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\int_1^\infty dx\ \frac{x}{\sqrt{x-1+\delta^2}}\left[\frac{K_1(nmL\sqrt{x})}{nmL\sqrt{x}} - K_0(nmL\sqrt{x})\right]. \tag{33}$$

From here, the trick is to recognize

$$\frac{1}{a^2 b}\frac{\partial}{\partial a}(a^2 K_1(ab)) = \frac{K_1(ab)}{ab} - K_0(ab) \tag{34}$$

which lets us write

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta=0) \propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\frac{1}{(nmL)^2}\frac{\partial}{\partial nmL}\left[(nmL)^2\int_1^\infty dx\ \frac{\sqrt{x}}{\sqrt{x-1+\delta^2}}K_1(nmL\sqrt{x})\right]. \tag{35}$$

For $\delta = 0$, we can use the relation $K_{-\nu}(z) = K_\nu(z)$ and Gradshteyn and Ryzhik, 7^{th} edition, Section 5.592 (pg 691), Eq. (12),

$$\int_1^\infty dx \frac{K_\nu(a\sqrt{x})}{x^{\nu/2}(x-1)^{1-\mu}} = \Gamma(\mu)2^\mu\frac{K_{\nu-\mu}(a)}{a^\mu}\,, \quad \mathrm{Re}\ a>0, \mathrm{Re}\ \mu>0\,, \tag{36}$$

which yields

$$\begin{aligned} \delta\mathcal{F}_{\mu\nu}(m,0,\Delta=0) &\propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\sqrt{2\pi}\frac{1}{(nmL)^2}\frac{\partial}{\partial nmL}\left[(nmL)^2\frac{K_{3/2}(nmL)}{\sqrt{nmL}}\right] \\ &= +\frac{4m^3}{(4\pi)^2}\sqrt{\frac{\pi}{2}}\sum_{\vec{n}\neq 0}\frac{K_{1/2}(nmL)}{\sqrt{nmL}} \end{aligned} \tag{37}$$

which agrees with Eq. (24) up to the overall phase of i we started with differently in the new way.

I. USEFUL INTEGRALS

A. Lorentz Integrals

$$\begin{aligned}\tilde{\mathcal{I}}^{(2n)}(\delta) &\equiv \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^n}{k^2-m^2} \\ &= \frac{i(-)^{n-1}}{(4\pi)^{d/2}}\frac{d(d+2)...(d+2(n-1))}{2^n}\Gamma\left(-2(n-1)-\frac{d}{2}\right)\delta^{2(n-1)-\epsilon}\end{aligned} \tag{1}$$

$$\tilde{\mathcal{I}}^{(0)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{1-\epsilon} \tag{2}$$

$$\tilde{\mathcal{I}}^{(2)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{2-\epsilon} \tag{3}$$

$$\tilde{\mathcal{I}}^{(4)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{3-\epsilon} \tag{4}$$

$$\tilde{\mathcal{I}}^{(6)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{4-\epsilon} \tag{5}$$

$$\tilde{\mathcal{I}}^{(8)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{5-\epsilon} \tag{6}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{l^2-\delta} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{4}\left[\frac{1}{\epsilon}-\gamma+\frac{3}{2}+\ln 4\pi\right]\delta^{2-\epsilon} \tag{7}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\delta]^2} = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi\right]\delta^{-\epsilon} \tag{8}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2-\delta]^2} = \frac{2i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+\frac{1}{2}+\ln 4\pi\right]\delta^{1-\epsilon} \tag{9}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\delta]^2} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{1-\epsilon} \tag{10}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^2} = \frac{3i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + \frac{2}{3} + \ln 4\pi\right] \delta^{2-\epsilon} \tag{11}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\delta]^3} = \frac{-i}{(4\pi)^2} \frac{1}{2} \frac{1}{\delta} \tag{12}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma - \frac{1}{2} + \ln 4\pi\right] \delta^{-\epsilon} \tag{13}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \frac{g_{\mu\nu}}{4} \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right] \delta^{-\epsilon} \tag{14}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu l_a l_b}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \frac{g_{\mu\nu} g_{ab} + g_{\mu a} g_{\nu b} + g_{\mu b} g_{\nu a}}{8} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi\right] \delta^{1-\epsilon} \tag{15}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\delta]^4} = \frac{i}{(4\pi)^2} \frac{1}{3!} \frac{1}{\delta^2} \tag{16}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^4} = \frac{-i}{(4\pi)^2} \frac{1}{3} \frac{1}{\delta} \tag{17}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^4} = \frac{i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi - \frac{11}{6}\right] \delta^{-\epsilon} \tag{18}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\delta]^6} = \frac{i}{(4\pi)^2} \frac{1}{20} \frac{1}{\delta^4} \tag{19}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \frac{1}{30} \frac{1}{\delta^3} \tag{20}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^6} = \frac{i}{(4\pi)^2} \frac{1}{20} \frac{1}{\delta^2} \tag{21}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^6}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \frac{1}{5} \frac{1}{\delta} \tag{22}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^8}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \left[\ln\delta + \frac{137}{60} - \left(\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi \right) \right] \tag{23}$$

B. λ-Parameter Integrals

Here are some integrals which frequently occur in the study of HBχPT. This general parameter integral I found in the talks by Jenkins and Manohar [1].

$$I(\alpha,b,c)=\int_0^\infty d\lambda\left(\lambda^2+2\lambda b+c\right)^\alpha \tag{24}$$

This integral has a recursion realtion,

$$(1+2\alpha)I(\alpha,b,c)=(\lambda^2+2\lambda b+c)^\alpha(\lambda+b)\Big|_0^\infty+2\alpha(c-b^2)I(\alpha-1,b,c) \tag{25}$$

Some functions I will use are

$$\begin{aligned}\mathcal{F}(m,\delta,\mu)&=(m^2-\delta^2)\left[\sqrt{\delta^2-m^2}\,\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)-\delta\ln\left(\frac{m^2}{\mu^2}\right)\right]-\frac{1}{2}\delta m^2\ln\left(\frac{m^2}{\mu^2}\right)\\ \mathcal{J}(m,\delta,\mu)&=(m^2-2\delta^2)\ln\left(\frac{m^2}{\mu^2}\right)+2\delta\sqrt{\delta^2-m^2}\,\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\end{aligned} \tag{26}$$

Specific integrals are;

$$I(-1,\delta,m^2)\;=\;\int_0^\infty d\lambda\frac{1}{\lambda^2+2\lambda\delta+m^2-i\epsilon}=\frac{-1}{2\sqrt{\delta^2-m^2}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right) \tag{27}$$

$$I(-1,0,m^2)\;=\;\frac{\pi}{2m} \tag{28}$$

$$\begin{aligned}\mu^{2\epsilon}I(-\epsilon,\delta,m^2)\;&=\;\mu^{2\epsilon}\int_0^\infty d\lambda\frac{1}{[\lambda^2+2\lambda\delta+m^2-i\epsilon']^\epsilon}\\ &=\;-\delta+\epsilon\left\{-2\delta+\delta\,\ln\frac{m^2}{\mu^2}-\sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\right\}\end{aligned} \tag{29}$$

$$\mu^{2\epsilon}I(-\epsilon,0,m^2)\;=\;-\epsilon\pi m \tag{30}$$

$$\begin{aligned}\mu^{2\epsilon}I(1-\epsilon,\delta,m^2)\;&=\;\mu^{2\epsilon}\int_0^\infty d\lambda\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon}\\ &=\;\frac{2}{3}\delta^3-\delta m^2+\epsilon\left\{\frac{10}{9}\delta^3-\frac{4}{3}\delta m^2-\frac{2}{3}\mathcal{F}(m,\delta,\mu)\right\}\end{aligned} \tag{31}$$

$$\mu^{2\epsilon}I(1-\epsilon,0,m^2)\;=\;-\epsilon\frac{2}{3}\pi m^3 \tag{32}$$

$$\begin{aligned}\mu^{2\epsilon}\int_0^\infty d\lambda\ 2\lambda\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon} &= \mu^{2\epsilon}\int_0^\infty d\lambda\ (2\lambda+2\delta)\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon}-2\delta\mu^{2\epsilon}I(1-\epsilon,\delta,m^2)\\ &= -\frac{1}{2}m^4+2\delta^2m^2-\frac{4}{3}\delta^4\\ &\quad+\epsilon\left[-\frac{1}{4}m^4+\frac{8}{3}\delta^2m^2-\frac{20}{9}\delta^4+\frac{1}{2}m^4\ln\frac{m^2}{\mu^2}+\frac{4}{3}\delta\mathcal{F}(m,\delta,\mu)\right]\end{aligned} \tag{33}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda\ 2\lambda\left[\lambda^2+m^2\right]^{1-\epsilon} = -\frac{1}{2}m^4\left(1+\epsilon\left[\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right]\right) \tag{34}$$

$$\begin{aligned}\mu^{2\epsilon}\int_0^\infty d\lambda\ \frac{2\lambda}{\left[\lambda^2+2\lambda\delta+m^2\right]^{-\epsilon}} &= \mu^{2\epsilon}\int_0^\infty d\lambda\ (2\lambda+2\delta)\left[\lambda^2+2\lambda\delta+m^2\right]^{-\epsilon}-2\delta\mu^{2\epsilon}I(-\epsilon,\delta,m^2)\\ &= 2\delta^2-m^2+\epsilon\left[4\delta^2-m^2+\mathcal{J}(m,\delta,\mu)\right]\end{aligned} \tag{35}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda\ \frac{2\lambda}{\left[\lambda^2+m^2\right]^{-\epsilon}} = -m^2\left(1+\epsilon\left[1-\ln\frac{m^2}{\mu^2}\right]\right) \tag{36}$$

C. Combined Lorentz and λ-parameter Integrals

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^2} = \frac{-i}{(4\pi)^2}\Bigg\{\delta\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]+\delta\left(1-\ln\left(\frac{m^2}{\mu^2}\right)\right) + \sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{37}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-m^2+i\epsilon\right]^2} = \frac{-i\pi}{(4\pi)^2}m \tag{38}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\lambda^2-2\lambda\delta-m^2]^2} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{2}\left\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\left(\frac{2}{3}\delta^3-\delta m^2\right) +\frac{10}{9}\delta^3-\frac{4}{3}\delta m^2-\frac{2}{3}\mathcal{F}(m,\delta,\mu)\right\}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\lambda^2-m^2]^2} = \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{3}\pi m^3 \tag{39}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-m^2]^2} = \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}m^4\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi+\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right] \tag{40}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \frac{i}{4(4\pi)^2}\frac{1}{\sqrt{\delta^2-m^2+i\epsilon}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right) \tag{41}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{2\lambda}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \frac{i}{2(4\pi)^2}\Bigg\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right] -1-\ln\left(\frac{m^2}{\mu^2}\right)+\frac{\delta}{\sqrt{\delta^2-m^2}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{42}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \Bigg\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta \quad \delta-\delta\ln\left(\frac{m^2}{\mu^2}\right)+\sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{43}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-2\lambda\delta-m^2]^3} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}\left\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right](2\delta^2-m^2) +2\delta^2+\mathcal{J}(m,\delta,\mu)\right\} \tag{44}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}m^2\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi-\ln\frac{m^2}{\mu^2}\right] \tag{45}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\ l_a l_b l_c l_d}{[l^2-\lambda^2-2\lambda\delta-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}}{16} \times \left\{ \left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\left(\frac{1}{2}m^4-2\delta^2 m^2+\frac{4}{3}\delta^4\right) + \frac{1}{4}m^4-\frac{8}{3}\delta^2 m^2+\frac{20}{9}\delta^4-\frac{1}{2}m^4\ln\frac{m^2}{\mu^2}-\frac{4}{3}\delta\mathcal{F}(m,\delta,\mu)\right\} \quad (46)$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\ l_a l_b l_c l_d}{[l^2-\lambda^2-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}}{32}\ m^4\times\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi+\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right] \quad (47)$$

[1] E. Jenkins and A. V. Manohar (1991), talk presented at the Workshop on Effective Field Theories of the Standard Model, Dobogoko, Hungary, Aug 1991.

[1] Silas R. Beane, "Nucleon masses and magnetic moments in a finite volume," Phys. Rev. D **70**, 034507 (2004), arXiv:hep-lat/0403015.