$SU(2)$ Heavy Baryon Chiral Perturbation Theory

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Abstract

Notes on $SU(2)$ heavy baryon chiral perturbation theory.

Contents

Chapter 1

Lagrangian

We begin by writing down the Lagrangian and deriving some Feynman Rules. We postpone a discussion of how to derive these operators until a later date. These notes, unless otherwise noted, are in Minkowski space with the mostly minus metric.

We are using what is known as HBχPT (Heavy Baryon Chiral Perturbation Theory) [1] which is an expansion about the infinite mass limit of the heavy baryon (a non-relativistic EFT). Some of our conventions come from the review article [2]. But, our normalization and definitions of the fields and their transformation properties come from Ref. [3].

1.1 Meson Lagrangian

The low-energy chiral Lagrangian for the pions begins with the LO terms

$$\mathcal{L} = \frac{F^2}{4}\langle D_\mu U^\dagger D^\mu U\rangle + \frac{F^2}{4}\langle \chi^\dagger U + \chi U^\dagger\rangle\,, \tag{1.1}$$

with $F \approx 92$ MeV. $\langle A\rangle$ denotes a flavor trace of the operator A. The field U contains the pions

$$U = u^2 = \exp\left(\frac{i\pi\cdot\tau}{F}\right)\,, \tag{1.2}$$

where τ are the Pauli-matrices and π are the pion fields. Explicitly,

$$\pi\cdot\tau = \phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\,. \tag{1.3}$$

Under global chiral transformations, the fields U and χ transform as

$$U \to LUR^\dagger\,, \qquad \chi \to L\chi R^\dagger \tag{1.4}$$

The covariant derivative acting on the U field is

$$\begin{aligned} D_\mu U &= \partial_\mu U - il_\mu U + iUr_\mu \\ &= \partial_\mu U - i(v_\mu - a_\mu)U + iU(v_\mu + a_\mu)\,, \end{aligned} \tag{1.5}$$

where l_μ, r_μ, v_μ and a_μ are sources for the left, right, vector and axial-vector currents, respectively. The spurion field χ contains the sources for the scalar and pseudoscalar density

$$\chi = 2B(s + ip)\,, \tag{1.6}$$

where the quantity B is and LEC proportional to the condensate of quark-antiquark pairs in the vacuum, which drive spontaneous symmetry breaking. In the chiral limit

$$B = -\frac{\langle\Omega|\bar{q}q|\Omega\rangle}{F^2}\,. \tag{1.7}$$

The fields s and p transform under chiral symmetry such that Eq. (1.1) is invariant under the global chiral transformations. When we compute explicit quantities in the QCD vacuum, then $p = 0$ and $s = m_Q$, the quark mass matrix. Making this explicit choice causes an (perturbatively small) explicit breaking of chiral symmetry proportional to the quark masses.

Considering only the *up* and *down* quark, the quark mass operator can be written

$$m_Q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \begin{pmatrix} \hat{m}-\delta & 0 \\ 0 & \hat{m}+\delta \end{pmatrix} = \hat{m}\mathbb{1} - \delta\tau_3\,. \tag{1.8}$$

The quantity B and m_Q are themselves renormalization scheme and scale dependent, but the product does not depend upon these details. This can easily be understood by realizing that the LO (leading order) contribution to the pion mass, an observable quantity, is

$$M_\pi^2 = 2B\hat{m} + \text{higher order corrections}\,. \tag{1.9}$$

1.2 Nucleon Lagrangian

The counting of the Lagrangian order counts in powers of $Q \sim M_\pi \sim \Delta$. We will closely follow the comlete Lagrangian given in Ref. [4], but make a few different choices (and match to that Lagrangian when possible). The pion-nucleon Lagrangian, as it is often referred, is given by (**COMMENT: still completing this**)

$$\mathcal{L}_{N\pi}^{(1)} = \overline{N}\left[iv\cdot D + \mathring{g}_A S\cdot u\right]N - \overline{T}^\mu\left[iv\cdot D - \Delta - \mathring{g}_A^\Delta S\cdot u\right]T_\mu + \frac{\mathring{g}_A^{N\Delta}}{2}\left[\overline{T}_\mu u^\mu N + \overline{N}u^\mu T_\mu\right]\,, \tag{1.10}$$

$$\begin{aligned}
\mathcal{L}_{N\pi}^{(2)} = \overline{N}\Bigg[&\frac{-D_\perp^2}{2M_0} + \frac{i\mathring{g}_A}{2M_0}\left(S\cdot\overleftarrow{D}v\cdot u - v\cdot uS\cdot\overrightarrow{D}\right) + \frac{\bar{c}_1}{4\pi F}\langle\chi_+\rangle + \frac{\bar{c}_5}{4\pi F}\chi_+^\delta \\
&+\left(\frac{\bar{c}_2}{4\pi F} - \frac{\mathring{g}_A^2}{8M_0}\right)\frac{1}{2}\langle(v\cdot u)^2\rangle + \frac{\bar{c}_3}{4\pi F}\frac{1}{2}\langle u\cdot u\rangle + \left(\frac{\bar{c}_4}{4\pi F} + \frac{1}{4M_0}\right)\frac{1}{2}[S_\mu,S_\nu][u_\mu,u_\nu] \\
&- \frac{i(c_6+1)}{4M_0}[S^\mu,S^\nu]F_{\mu\nu}^+ - \frac{ic_7}{4M_0}[S^\mu,S^\nu]\langle F_{\mu\nu}^+\rangle\Bigg]N \\
&- \frac{F}{4\pi}\overline{N}\left[Z_0^N\langle\mathcal{Q}^2\rangle + Z_1^N\mathcal{Q}\langle\mathcal{Q}\rangle + Z_2^N\mathcal{Q}^2\right]N \\
&+ \overline{T}^\mu\left[\frac{D_\perp^2}{2M_0} + \frac{i\mathring{g}_A^{\Delta\Delta}}{2M_0}\left(S\cdot\overleftarrow{D}v\cdot u - v\cdot uS\cdot\overrightarrow{D}\right) + \frac{t_1}{4\pi F}\langle\chi_+\rangle + \frac{t_5}{4\pi F}\chi_+^\delta + \cdots\right]T_\mu \\
&+ \overline{T}[O]N + \overline{N}[O]T\,,
\end{aligned} \tag{1.11}$$

$$\mathcal{L}_{N\pi}^{(3)} \supset \overline{N}\left[\frac{\bar{d}_{16}}{(4\pi F)^2}S\cdot u\langle\chi_+\rangle + \frac{\bar{d}_{17}}{(4\pi F)^2}\langle S\cdot u\chi_+\rangle\right]N\,, \tag{1.12}$$

$$\begin{aligned}
\mathcal{L}_{N\pi}^{(4)} \supset \frac{1}{(4\pi F)^3}\overline{N}\Big[&\bar{e}_{38}\langle\chi_+\rangle\langle\chi_+\rangle + \bar{e}_{39}\chi_+^\delta\langle\chi_+\rangle + \bar{e}_{40}\langle\chi_+^\delta\chi_+^\delta\rangle + \bar{e}_{41}\chi_-^\delta\langle\chi_-\rangle \\
&+ \bar{e}_{115}\langle\chi\chi^\dagger\rangle + \bar{e}_{116}(\det\chi + \det\chi^\dagger)\Big]N\,.
\end{aligned} \tag{1.13}$$

In these expressions, $\langle\mathcal{O}\rangle$ denotes a trace over the flavor indices, $\mathrm{Tr}(\mathcal{O})$ of the field content of $\mathcal{O}$. The leading $1/M$ corrections to the LECs, as defined in the right-most column of Table 3 of Ref. [4] are also included. I believe this is to keep the definition of the LECs consistent with those in the relativistic Lagrangian. The $\chi_\pm$ fields are related to Eq. (1.6) by

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \qquad\qquad \chi^\delta_\pm = \chi_\pm - \frac{1}{2}\mathrm{tr}(\chi_\pm)\,. \tag{1.14}$$

The other fields (and flavor contractions) will be defined in detail in the subsequent sub-sections.

In Eq. (1.13), we have made heavy use of the Cayley-Hamilton identity, which for general 2×2 matrices, provides the relation

$$\{A,B\} = A\langle B\rangle + \langle A\rangle B + \langle AB\rangle - \langle A\rangle\langle B\rangle\,. \tag{1.15}$$

We can match the LECs in these Lagrangians to those in Ref. [4], where it is understood that we are matching to the $\hat{}$ operators of the heavy baryon Lagrangian and not those of the relativistic Lagrangian. The main difference is that with the Lagrangian we have written, we use powers of $4\pi F$ such that all the LECs remain dimensionless, while it is more common in the literature to have LECs of inverse mass dimension. We will describe the flavor contractions in detail in the following sub-sections.

1.2.1 $\mathcal{L}_{N\pi^{(1)}}$

We begin by explicitly describing the implicit flavor contractions in Eq. (1.10). The nucleon is described by an iso-doublet and the delta by an iso-tensor field

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$
$$T_{111} = \Delta^{++}\,,\quad T_{112} = T_{121} = T_{211} = \frac{1}{\sqrt{3}}\Delta^+\,,\quad T_{122} = T_{212} = T_{221} = \frac{1}{\sqrt{3}}\Delta^0\,,\quad T_{222} = \Delta^-\,. \tag{1.16}$$

We assert (but do not prove the final term) the covariant derivative on the nucleon is given by

$$(D_\mu N)_k = \partial_\mu N_k + \Gamma_{\mu,kj}N_j + N_k\langle\Gamma_\mu\rangle \tag{1.17}$$
$$\begin{aligned}\Gamma_\mu &= \frac{1}{2}\left[u^\dagger(\partial_\mu - il_\mu)u + u(\partial_\mu - ir_\mu)u^\dagger\right]\\ &= \frac{1}{2}\left[u^\dagger(\partial_\mu - i(\mathbf{v}_\mu - \mathbf{a}_\mu))u + u(\partial_\mu - i(\mathbf{v}_\mu + \mathbf{a}_\mu))u^\dagger\right]\\ &= \frac{1}{2}\left[u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right] - \frac{i}{2}\left[u^\dagger(\mathbf{v}_\mu - \mathbf{a}_\mu)u + u(\mathbf{v}_\mu + \mathbf{a}_\mu)u^\dagger\right]\end{aligned} \tag{1.18}$$

The pure derivative terms are traceless. The $SU(2)$ vector and axial vector sources are also traceless, and so, the only non-vanishing part of $\langle\Gamma_\mu\rangle$ comes from the electromagnetic interaction. Therefore, we can replace

$$\begin{aligned}\langle\Gamma_\mu\rangle &= -\frac{ieA_\mu}{2}\langle u^\dagger Q u + uQu^\dagger\rangle\\ &= -ieA_\mu\langle Q\rangle\end{aligned} \tag{1.19}$$

where A_μ is the photon field and the charge matrix is

$$Q = \begin{pmatrix} \frac{2}{3} & \\ & -\frac{1}{3}\end{pmatrix}\,. \tag{1.20}$$

The practical effect of the $\langle\Gamma_\mu\rangle$ contribution to the electromagnetic coupling is that the charge operator can be replaced by the hadronic nucleon level charge operator such that the covariant derivative is as in Eq. (1.17) without the $\langle\Gamma_\mu\rangle$ term and $Q \to \mathrm{diag}(1,0)$.

The explicit flavor contractions of the remaining terms are

$$\mathcal{L}^{(1)}_{N\pi} \supset \mathring{g}_A \overline{N}^i S\cdot u_i^j N_j + \frac{\mathring{g}_A^{N\Delta}}{2}\left(\overline{T}^{kji}\cdot u_i^{i'}\epsilon_{ji'}N_k + \overline{N}^k \epsilon^{ij} u_i^{i'}\cdot T_{i'jk}\right) + \mathring{g}_A^{\Delta}\overline{T}^{kji}_\mu S\cdot u_i^{i'} T^\mu_{i'jk}\,, \tag{1.21}$$

where the field u_μ is given by

$$\begin{aligned} u_\mu &= iu(\partial_\mu - ir_\mu)u^\dagger - iu^\dagger(\partial_\mu - il_\mu)u \\ &= iu(\partial_\mu - i(\mathbf{v}_\mu + \mathbf{a}_\mu))u^\dagger - iu^\dagger(\partial_\mu - i(\mathbf{v}_\mu - \mathbf{a}_\mu))u \\ &= i(u\partial_\mu u^\dagger - u^\dagger\partial_\mu u) + u(\mathbf{a}_\mu + \mathbf{v}_\mu)u^\dagger + u^\dagger(\mathbf{a}_\mu - \mathbf{v}_\mu)u \end{aligned} \tag{1.22}$$

The couplings, $\mathring{g}_A$, $\mathring{g}_A^{N\Delta}$ and $\mathring{g}_A^{\Delta}$ are the leading order contributions to the nucleon axial charge, the $N \to \Delta$ transition, and the Δ axial charge respectively.

It is useful for our purposes of computing the nucleon mass and axial form-factors to expand the Lagrangian up to two powers of the pion field, for which we find

$$\begin{aligned} \overline{N} iv\cdot DN = \overline{N} iv\cdot\partial N & \\ + \overline{N}\Bigg[& iv^\mu \frac{\phi\partial_\mu\phi - \partial_\mu\phi\phi}{8F^2} + v\cdot\mathbf{v}' + ev\cdot A(Q + \langle Q\rangle) \\ & + \frac{i}{2F}[\phi, v\cdot\mathbf{a}] + \frac{2\phi v\cdot\mathbf{v}\phi - \phi^2 v\cdot\mathbf{v} - v\cdot\mathbf{v}\phi^2}{8F^2}\Bigg]N \end{aligned} \tag{1.23}$$

where $\mathbf{v}' = \mathbf{v} - eAQ$. Notice, in the final term, the contribution to $\mathbf{v}_\mu$ that is proportional to the flavor identity exactly vanishes, and so we can use either the quark or nucleon charge operator in this term and will obtain the same interactions.

For the nucleon axial coupling operator, we have

$$\mathring{g}_A \overline{N} S\cdot uN = \mathring{g}_A \overline{N} S^\mu \left[\frac{\partial_\mu\phi}{F} + 2\mathbf{a}_\mu + \frac{i}{F}[\mathbf{v}_\mu, \phi] + \frac{2\phi\mathbf{a}_\mu\phi - \phi^2\mathbf{a}_\mu - \mathbf{a}_\mu\phi^2}{4F^2}\right]N\,. \tag{1.24}$$

Performing the flavor contractions of the first term, we see that the Lagrangian includes

$$\mathring{g}_A \overline{N} S^\mu \frac{\partial_\mu\phi}{F} N = \frac{\mathring{g}_A}{F}\left[\overline{p} S^\mu \partial_\mu \pi^0 p - \overline{n} S^\mu \partial_\mu\pi^0 n + \sqrt{2}\left(\overline{p} S^\mu \partial_\mu \pi^+ n + \overline{n} S^\mu \partial_\mu \pi^- p\right)\right]. \tag{1.25}$$

Chapter 2

Nucleon Masses

NOTE: wave-function renormalization described in Ref. [5], to treat $p \cdot v$ =?.

NOTE on Power Counting: we will adopt the description of the power counting from the Bochum group - Nolan will love this:

$$
\begin{aligned}
\text{LO} \quad & M_0 \\
\text{NLO} \quad & \mathcal{O}(M_\pi^2) \\
\text{N}^2\text{LO} \quad & \mathcal{O}(M_\pi^3) \\
\text{N}^3\text{LO} \quad & \mathcal{O}(M_\pi^4) \\
\text{N}^n\text{LO} \quad & \mathcal{O}(M_\pi^{n+1})
\end{aligned}
$$

2.1 Feynman Rules

There are a few key aspects to working out the Feynman Rules. First, we have to understand all the flavor contractions that are implicit in the Lagrangian. Second, we have to carefully treat the interactions involving derivatives. Third, the delta states have extra minus signs associated with the Lorentz contraction of the spinors and the mostly minus metric.

2.2 NLO Mass correction

Let us begin with the leading mass shift which comes from the operators

$$
\begin{aligned}
\mathcal{L}_{N\pi}^{(2)} &\supset \overline{N}\left[\frac{\bar{c}_1}{4\pi F}\langle\chi_+\rangle + \frac{\bar{c}_5}{4\pi F}\chi_+^\delta\right]N \\
&= \overline{N}\left[\frac{\bar{c}_1}{4\pi F}4B\langle m_q\rangle - \frac{\bar{c}_5}{4\pi F}4B\delta\tau_3\right]N + \cdots \\
&= \overline{p}\left[\frac{\bar{c}_1}{4\pi F}4(2B\hat{m}) + \frac{\bar{c}_5}{4\pi F}2(2B\delta)\right]p + \overline{n}\left[\frac{\bar{c}_1}{4\pi F}4(2B\hat{m}) - \frac{\bar{c}_5}{4\pi F}2(2B\delta)\right]n + \cdots
\end{aligned}
\tag{2.1}
$$

where the $\cdots$ are from higher orders in the expansion of u in terms of ϕ fields.

When computing a diagram, or "amplitude", with a single insertion of an interaction Lagrangian, we have

$$
= i\mathcal{A} = \langle \text{out}|i\mathcal{L}|\text{in}\rangle\,. \tag{2.2}
$$

Recalling, from the all-order sum of the 1PI diagrams and how they shift the pole of the propagator, we have

$$-i\delta\Sigma = -i\mathcal{A}\,. \tag{2.3}$$

Therefore, from Eq. (**??**), the leading correction to the nucleon masses are

$$\begin{aligned}
\delta M_p &= -4\bar{c}_1 \frac{2B\hat{m}}{4\pi F} - 2\bar{c}_5 \frac{2B\delta}{4\pi F}\,,\\
\delta M_n &= -4\bar{c}_1 \frac{2B\hat{m}}{4\pi F} + 2\bar{c}_5 \frac{2B\delta}{4\pi F}\,.
\end{aligned} \tag{2.4}$$

At this order, we can replace the LO terms with on-shell quantities (where possible)

$$\begin{aligned}
\delta M_p &= \Lambda_\chi \left[-4\bar{c}_1 \frac{M_\pi^2}{(4\pi F_\pi)^2} - 2\bar{c}_5 \frac{2B\delta}{(4\pi F_\pi)^2}\right] + \mathrm{N^3LO+}\,,\\
\delta M_n &= \Lambda_\chi \left[-4\bar{c}_1 \frac{M_\pi^2}{(4\pi F_\pi)^2} + 2\bar{c}_5 \frac{2B\delta}{(4\pi F_\pi)^2}\right] + \mathrm{N^3LO+}\,,
\end{aligned} \tag{2.5}$$

where the $\mathrm{N^3LO+}$ denotes specific corrections arising from changing $2B\hat{m} = M_\pi^2 + \mathrm{N^2LO}$ and higher and $F = F_\pi + \mathrm{N^2LO}$ and higher terms. There are of course $\mathrm{N^3LO}$ corrections from higher order baryon mass diagrams also. There is not any formula to relate $2B\delta$ to a measured hadronic quantity, so we can not easily make this replacement.

2.3 NNLO Mass correction

The first interesting mass correction arises at $\mathrm{N^2LO}$ in the power counting which is the "sunset" or "sunrise" diagrams, depicted in Fig. 2.1 and arising from the interactions given in Eq. (1.21). Before evaluating this diagram, let us first do a power-counting exercise to understand what we expect the answer to be. The matrix element that will give rise to this radiative correction is

$$\delta M_N^{\mathrm{N^2LO}} = \langle N|\mathring{g}_A \overline{N} S\cdot u N\ \mathring{g}_A \overline{N} S\cdot u N|N\rangle\,. \tag{2.6}$$

There will also be a four-momentum integral over the momentum of the virtual states. We know the leading part of u_μ without sources is $u_\mu \propto \partial_\mu\phi/F$, and so the leading interaction will be a derivative interaction between the pions and the nucleon. So, the self-energy correction will be propotional to

$$\delta M_N^{\mathrm{N^2LO}} \propto \frac{\mathring{g}_A^2}{F^2} S^\mu S^\nu \int_R \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{[k^2 - m_\pi^2][k\cdot v]}\,. \tag{2.7}$$

In the heavy-baryon limit, we see the self-energy correction depends on an integral with only one scale, the pion mass. When we use dimension-regularization to regulate the integral, which respects chiral symmetry, the only scale that this integral can depend upon is m_π. Therefore, we can simply count all the powers of momentum in the numerator and denominator to understand the scaling in m_π: there are four powers of k from the integral measure, two from the derivative interactions and three in the denominator. So, the mass-dimension of the integral is $[\int \mathrm{NLO}] = 4 + 2 - 3 = 3$. Therefore, we know that the self-energy correction will be proportional to (loops always come with powers of $(4\pi)^2$)

$$\delta M_N^{\mathrm{N^2LO}} \propto \mathring{g}_A^2 \frac{m_\pi^3}{(4\pi F)^2}\,. \tag{2.8}$$

Figure 2.1: Diagrams contributing to nucleon mass at NLO. The dashed lines are pions, the solid lines are nucleons and the thick solid line is a delta-resonance.

All the work of getting the factors of 2, π, and (-1) correct amount to determining the precise pre-factor of this result. This correction is interesting for a few reasons:

1. We know that $m_\pi^2 = 2B\hat{m}$ at LO, and therefore, this correction is non-analytic in the quark mass, $\delta M_N^{\mathrm{N^2LO}} \propto \hat{m}^{3/2}$. Such non-analytic corrections are the hall-mark of interesting QFT corrections as they can not *a priori* be parameterized with a small number of local operators that scale with integer powers of $\hat{m}$. In some sense, the interesting predictions of EFT are the coefficients of these leading non-analytic quark mass dependent corrections. If we can tease out such non-analytic corrections in our comparison with results from LQCD calculations, this lends great confidence to our understanding of low-energy QCD.

2. There was another point - but I'm blanking at the moment

2.3.1 Nucleon-pion loop

We will start by evaluating the diagram with virtual nucleon-pion states. The self-energy correction from this diagram is given by

$$
\begin{aligned}
-i\delta\Sigma &= \frac{1}{2!} 2 \left(\frac{i\mathring{g}_A}{F}\right)^2 C_{NN\phi}^2 \int_R \frac{d^4k}{(2\pi)^4} \frac{ik_\mu S^\mu i}{k^2 - m_\phi^2 + i\epsilon} \frac{i(-ik_\nu S^\nu)}{(p+k)\cdot v - i\epsilon}\,, \\
\delta\Sigma &= \left(\frac{\mathring{g}_A}{F}\right)^2 C_{NN\phi}^2 S^\mu S^\nu \int_R \frac{d^4k}{(4\pi)^4} \frac{ik_\mu k_\nu}{[k^2 - m_\phi^2 + i\epsilon][(p+k)\cdot v + i\epsilon]}
\end{aligned}
\tag{2.9}
$$

Where do each of these factors come from?

1. The $1/2!$ arises because we need 2 insertions of the interaction Lagrangian

$$
e^{i\mathcal{L}_I} = 1 + i\mathcal{L} + \frac{1}{2!}(i\mathcal{L})^2 + \cdots
\tag{2.10}
$$

2. The factor of 2 arises from the 2 possible Wick contractions

$$
\langle N|\overline{N}S\cdot uN\ \overline{N}S\cdot uN|N\rangle\,, \qquad\qquad \langle N|\overline{N}S\cdot uN\ \overline{N}S\cdot uN|N\rangle\,,
\tag{2.11}
$$

3. The $C_{NN\phi}^2$ is a Clebsch-Gordon coefficient arising from the various flavor contractions for a given external nucleon. Consider the self-energy graph for the proton. The leading ϕ/F term arising from the $\mathring{g}_A\overline{N}^i S\cdot u_i^j N_j$ operator (with $v_\mu = a_\mu = 0$) is

$$
\mathring{g}_A\overline{N}^i S\cdot u_i^j N_j = \frac{\mathring{g}_A}{F}\overline{N}^i S\cdot\partial\phi_i^j N_j + \mathrm{O}\left(\mathring{g}_A\overline{N}S\cdot\partial\frac{\phi^3}{F^3}N\right)
\tag{2.12}
$$

Performing the flavor contractions of the nucleons with the ϕ field, Eq. (1.3), the set of operators are

$$\frac{\mathring{g}_A}{F}\overline{N}^i S\cdot\partial\phi_i^j N_j = \frac{\mathring{g}_A}{F}\left[\overline{p}S\cdot\partial\pi^0 p - \overline{n}S\cdot\partial\pi^0 n + \sqrt{2}\left(\overline{p}S\cdot\partial\pi^+ n + \overline{n}S\cdot\partial\pi^- p\right)\right] \tag{2.13}$$

If we define the generic self-energy correction from this diagram for the proton, and keep track of the flavor content of the virtual states, we see that we have

$$\delta\Sigma^{\mathrm{N^2LO}}(p;N,\phi) = \delta\Sigma^{\mathrm{N^2LO}}(p;p,\pi^0) + 2\delta\Sigma(p;n,\pi^+) \tag{2.14}$$

and thus, $C^2_{pp\pi^0} = 1$ and $C^2_{pn\pi^+} = 2$.

4. The factor of $(i\mathring{g}_A/F)^2$ is the i from $i\mathcal{L}$ and $\mathring{g}_A/F$ is the prefactor of the operator at LO in the ϕ/F expansion of u_μ.
5. Each interaction vertex is proportional to a spin-vector, S_μ which is Lorentz contracted with a derivative operator. If the momentum at a given vertex of the ϕ-field is flowing into the vertex, the interaction is proportional to $-ik_\mu$. If the momentum is flowing out of the vertex, then the interaction is proportional to $+ik_\mu$. From the relative sign of the external p momentum and the virtual k momentum, we see that I have chosen to run the momentum in the loop such that they add on the virtual nucleon line. This means the initial vertex (on the right) is proportional to $-iS^\nu k_\nu$ and the second vertex (on the left) is proportional to $+iS^\mu k_\mu$.
6. We then have a propagator for each particle in the loop and an integral over all virtual momentum k. The integral must be regulated and renormalized, denoted by the subscript R on the integral.

Let us now proceed to evaluate this integral. First, we can isolate the contribution from the integral that will lead to the wave-function renormalization that we will need for many processes (including higher order corrections to m_N). Recall, the wave-function renormalization is obtained from the derivative of the self-energy with respect to the external momentum, evaluated at the on-shell point, which in the heavy-baryon formalism is $p\cdot v = 0$, thus we can take the linear approximation of the self-energy with respect to $p\cdot v$ to isolate the wave-function correction;

$$\begin{aligned}
\delta\Sigma_N^{\mathrm{N^2LO}} &= \left(\frac{\mathring{g}_A}{F}C_{NN\phi}\right)^2 S^\mu S^\nu\left[\mathcal{F}_{\mu\nu}(m,0) - p\cdot v\,\mathcal{J}_{\mu\nu}(m,0)\right]\\
\mathcal{F}_{\mu\nu}(m,0) &= \int_R \frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][k\cdot v+i\epsilon]}\\
\mathcal{J}_{\mu\nu}(m,0) &= \int_R \frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][k\cdot v+i\epsilon]^2}
\end{aligned} \tag{2.15}$$

For integrals like these that appear with a heavy-matter field, we want to use the λ-parameter method to combine the denominators, see App. A. We will also use dimensional regularization to regulate the integrals (with $d = 4-2\epsilon$) (and the ϵ in d and the shift in the propagator pole are NOT the same - context alone keeps these as separate)

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(m,0) &= 2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\lambda\frac{ik_\mu k_\nu}{[k^2-m^2+2\lambda k\cdot v+i\epsilon]^2}\\
&= 2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\lambda\frac{ik_\mu k_\nu}{[(k+\lambda v)^2-m^2-\lambda^2+i\epsilon]^2}\\
&= 2\mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\int_0^\infty d\lambda\frac{i(l_\mu-\lambda v_\mu)(l_\nu-\lambda v_\nu)}{[l^2-m^2-\lambda^2+i\epsilon]^2}\,.
\end{aligned} \tag{2.16}$$

The spin-vectors satisfy $v \cdot S = 0$, so the pieces of the integral we need are

$$
\begin{aligned}
\mathcal{F}_{\mu\nu}(m,0) &= 2\mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} \int_0^\infty d\lambda \frac{i l_\mu l_\nu}{[l^2 - m^2 - \lambda^2 + i\epsilon]^2} \\
&= \frac{g_{\mu\nu}}{d} 2\mu^{2\epsilon} \int_0^\infty d\lambda \int \frac{d^d l}{(2\pi)^d} \frac{i l^2}{[l^2 - m^2 - \lambda^2 + i\epsilon]^2} \\
&= \frac{g_{\mu\nu}}{d} 2\mu^{2\epsilon} \int_0^\infty d\lambda \, i \frac{2i}{(4\pi)^2} \left[\frac{2}{4-d} - \gamma_E + \frac{1}{2} + \ln 4\pi\right] (\lambda^2 + m^2)^{1-\epsilon} \\
&= -\frac{g_{\mu\nu}}{d} \frac{4}{(4\pi)^2} \mu^{2\epsilon} \left[\frac{2}{4-d} - \gamma_E + \frac{1}{2} + \ln 4\pi\right] \int_0^\infty d\lambda (\lambda^2 + m^2)^{1-\epsilon} \\
&= -\frac{g_{\mu\nu}}{d} \frac{4}{(4\pi)^2} \mu^{2\epsilon} \left[\frac{2}{4-d} - \gamma_E + \frac{1}{2} + \ln 4\pi\right] \left(-\epsilon \frac{2}{3} \pi m^3\right) \\
&= g_{\mu\nu} \frac{4}{d} \frac{2}{3} \frac{\pi m^3}{(4\pi)^2} \epsilon \mu^{2\epsilon} \left[\frac{2}{4-d} - \gamma_E + \frac{1}{2} + \ln 4\pi\right] . \qquad (2.17)
\end{aligned}
$$

Since the wave-function contribution to the self-energy comes at $p \cdot v = 0$, the total self-energy from this graph is given by ($S^2 = \frac{1-d}{4}$ in d-dimensions)

$$
\begin{aligned}
\delta\Sigma^{\mathrm{N^2LO}} &= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^\mu S^\nu F_{\mu\nu}(m,0) \\
&= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 \frac{1-d}{4} \frac{4}{d} \frac{2}{3} \frac{\pi m^3}{(4\pi)^2} \epsilon \mu^{2\epsilon} \left[\frac{2}{4-d} - \gamma_E + \frac{1}{2} + \ln 4\pi\right] . \qquad (2.18)
\end{aligned}
$$

We are only interested in the $\epsilon \to 0$ limit of this contribution. Normally, we have to track finite pieces that arise from $d = 4 - 2\epsilon$. However, since the contribution from the λ-parameter integral iteself is proportional to ϵ, all these finite pieces vanish as $\epsilon \to 0$ with the only surviving term arising from the $1/\epsilon$ pole,

$$
\begin{aligned}
\delta\Sigma_N^{\mathrm{N^2LO}} &= -\frac{(\mathring{g}_A C_{NN\phi})^2}{2} \frac{\pi m_\phi^3}{(4\pi F)^2} \\
&= -\frac{\pi \mathring{g}_A^2}{2} \left[\frac{m_{\pi^0}^3}{(4\pi F)^2} + 2\frac{m_{\pi^\pm}^3}{(4\pi F)^2}\right] . \qquad (2.19)
\end{aligned}
$$

In the isospin limit, these two terms are the same. We can also change $F \to F_\pi$ and $\mathring{g}_A \to g_A$ with the first corrections to these changes appearing two orders higher at $\mathrm{O}(m_\pi^5)$, thus leaving the NLO self-energy correction as ($\Lambda_\chi = 4\pi F_\pi$)

$$
\delta\Sigma_N^{\mathrm{N^2LO}} = -\frac{3\pi g_A^2}{2} \Lambda_\chi \left(\frac{m_\pi}{4\pi F_\pi}\right)^3 . \qquad (2.20)
$$

2.3.1.1 Wave function correction

Now, let us evaluate the wave-function correction arising from the $J_{\mu\nu}(m,0)$ term in Eq. (2.15). After we include the virtual delta corrections in the next section, we'll see an easier way, but for now, we'll do the integral for completeness.

$$
\begin{aligned}
\delta Z_N^{\mathrm{N^2LO}} &= \frac{\partial}{\partial p \cdot v} \delta\Sigma_N^{\mathrm{N^2LO}} \bigg|_{p\cdot v = 0} \\
&= -\left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^\mu S^\nu J_{\mu\nu}(m,0) . \qquad (2.21)
\end{aligned}
$$

We start by evaluating the integral $J_{\mu\nu}$ in dim-reg

$$
\begin{aligned}
J_{\mu\nu}(m,0) &= \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][k\cdot v-i\epsilon]^2}\\
&= 8\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\lambda\frac{i\lambda k_\mu k_\nu}{[k^2-m_\phi^2+2\lambda k\cdot v+i\epsilon]^3}\\
&= 8\mu^{2\epsilon}\int_0^\infty d\lambda\int\frac{d^dl}{(2\pi)^d}\frac{i\lambda(l_\mu-\lambda v_\mu)(l_\nu-\lambda v_\nu)}{[l^2-m_\phi^2-\lambda^2+i\epsilon]^3}\,.
\end{aligned}
\tag{2.22}
$$

We can drop the terms proportional to v_μ, as above, leaving us with

$$
\begin{aligned}
J_{\mu\nu}(m,0) &= \frac{8g_{\mu\nu}}{d}\mu^{2\epsilon}\int_0^\infty d\lambda\lambda\int\frac{d^dl}{(2\pi)^d}\frac{il^2}{[l^2-m_\phi^2-\lambda^2+i\epsilon]^3}\\
&= \frac{8g_{\mu\nu}}{d}\int_0^\infty d\lambda\lambda\frac{-1}{(4\pi)^2}\left[\frac{2}{4-d}-\gamma_E-\frac{1}{2}+\ln 4\pi\right]\left(\frac{\mu^2}{\lambda^2+m^2}\right)^\epsilon\\
&= \frac{8g_{\mu\nu}}{d}\frac{-1}{(4\pi)^2}\left[\frac{2}{4-d}-\gamma_E-\frac{1}{2}+\ln 4\pi\right]\mu^{2\epsilon}\int_0^\infty d\lambda\frac{\lambda}{(\lambda^2+m^2)^\epsilon}\\
&= \frac{8g_{\mu\nu}}{d}\frac{-1}{(4\pi)^2}\left[\frac{2}{4-d}-\gamma_E-\frac{1}{2}+\ln 4\pi\right]\frac{-m^2}{2}\left[1+\epsilon\left(1-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\\
&= \frac{m^2g_{\mu\nu}}{(4\pi)^2}\left(1+\frac{1}{2}\epsilon\right)\left[\frac{1}{\epsilon}-\gamma_E-\frac{1}{2}+\ln 4\pi\right]\left[1+\epsilon\left(1-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\\
&= \frac{m^2g_{\mu\nu}}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi-\ln\left(\frac{m^2}{\mu^2}\right)\right],
\end{aligned}
\tag{2.23}
$$

where we have kept track of all finite terms that will not vanish in the $\epsilon\to 0$ limit. To retain the full finite terms, we also have to recall that $S^2=(1-d)/4$ in d-dimensions.

$$
\begin{aligned}
\delta Z_N^{\mathrm{N^2LO}} &= -\left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^\mu S^\nu J_{\mu\nu}(m,0)\\
&= -\left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2\frac{1-d}{4}\frac{m^2}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi-\ln\left(\frac{m^2}{\mu^2}\right)\right]\\
&= -C_{NN\phi}^2\frac{3\mathring{g}_A^2}{4}\frac{m^2}{(4\pi F)^2}\left[\frac{2}{3}+\ln\left(\frac{m^2}{\mu^2}\right)-\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi\right]\right]
\end{aligned}
\tag{2.24}
$$

If we work in the isospin limit, express the bare quantities in terms of on-shell quantities, and take the sum over all internal states, the full wave-function renormalization from the virtual nucleon-pion loop is

$$
\delta Z_N^{\mathrm{N^2LO}} = -\frac{9g_A^2}{4}\frac{m_\pi^2}{(4\pi F_\pi)^2}\left[\frac{2}{3}+\ln\left(\frac{m_\pi^2}{\mu^2}\right)-\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi\right]\right]. \tag{2.25}
$$

We see that the wave-function correction needs a counterterm and has finite correction after the standard modified MS-bar subtraction [6] (modified MS-bar subtracts the $-\gamma_E+1+\ln 4\pi$ along with the $1/\epsilon$ pole).

2.3.2 Delta-pion loop

Now we turn to the self-energy correction arising from the virtual delta-pion loop, Fig. 2.1. The counting of Wick contractions is similar to that for the nucleon loop, except the multiplicative factor of 2 arises not from having two ways to tie the nucleons, but rather from the cross term of two different interaction Lagrangians at second order:

$$e^{i(\mathcal{L}_1+\mathcal{L}_2)} \supset 2i\mathcal{L}_1 i\mathcal{L}_2 : \qquad \langle N|\overline{N}u\cdot T\;\overline{T}\cdot uN|N\rangle \tag{2.26}$$

The self-energy contribution is

$$\begin{aligned}
-i\delta\Sigma_N^{\mathrm{NLO},\Delta} &= \frac{1}{2!}\times 2\times\left(\frac{i\mathring{g}_A^{N\Delta}C_{TN\phi}}{2F}\right)^2\int_R \frac{d^4k}{(2\pi)^4}\frac{i\;ik_\mu U_T^\mu}{k^2-m_\phi^2+i\epsilon}\frac{\overline{U}_T^\nu(-ik_\nu)\;i}{(p+k)\cdot v-\Delta+i\epsilon}\\
&= (i)^5(-1)\left(\frac{\mathring{g}_A^{N\Delta}C_{TN\phi}}{2F}\right)^2 P^{\mu\nu}\int_R\frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][(p+k)\cdot v-\Delta+i\epsilon]}\,.
\end{aligned}\tag{2.27}$$

In this expression, the sum over spin-polariztions implicit in the self-energy gives first the spin-projector for the Rarita-Schwinger polarization tensor for a spin-3/2 field

$$P^{\mu\nu} = v^\mu v^\nu - g^{\mu\nu} - \frac{4}{d-1}S^\mu S^\nu\,, \tag{2.28}$$

which we have written for $d = 4-2\epsilon$ dimensions. This contribution to the self-energy is therefore given by

$$\begin{aligned}
\delta\Sigma_N^{\mathrm{NLO},\Delta} &= \left(\frac{\mathring{g}_A^{N\Delta}C_{TN\phi}}{2F}\right)^2 P^{\mu\nu}\left[\mathcal{F}_{\mu\nu}(m,\Delta) - p\cdot v\,\mathcal{J}_{\mu\nu}(m,\Delta)\right]\,,\\
\mathcal{F}_{\mu\nu}(m,\Delta) &= \int_R\frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][k\cdot v-\Delta+i\epsilon]}\,,\\
\mathcal{J}_{\mu\nu}(m,\Delta) &= \int_R\frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m_\phi^2+i\epsilon][k\cdot v-\Delta+i\epsilon]^2}\,.
\end{aligned}\tag{2.29}$$

The integrals we have to evaluate are very similar to those with the nucleon-pion loops but with the extra mass parameter Δ. Using the same λ-parameter integral, shifting the momentum variable and recognizing $v_\mu P^{\mu\nu} = v_\nu P^{\mu\nu} = 0$, we have

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(m,\Delta) &= -\frac{g_{\mu\nu}}{(4\pi)^2}\mu^{2\epsilon}\left[\frac{2}{4-d}-\gamma_E+1+\ln 4\pi\right]\int_0^\infty d\lambda(\lambda^2+2\lambda\Delta+m^2)^{1-\epsilon}\\
&= -\frac{g_{\mu\nu}}{(4\pi)^2}\left[\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi\right]\left(\frac{2}{3}\Delta^3-\Delta m^2\right)+\frac{10}{9}\Delta^3-\frac{4}{3}\Delta m^2-\frac{2}{3}\overline{\mathcal{F}}(m,\Delta,\mu)\right]\\
\overline{\mathcal{F}}(m,\Delta,\mu) &= (\Delta^2-m^2+i\epsilon)^{3/2}\ln\left(\frac{\Delta+\sqrt{\Delta^2-m^2+i\epsilon}}{\Delta-\sqrt{\Delta^2-m^2+i\epsilon}}\right)+\Delta\left(\Delta^2-\frac{3}{2}m^2\right)\ln\left(\frac{m^2}{\mu^2}\right)\,.
\end{aligned}\tag{2.30}$$

There is an additional finite contribution coming from

$$P^{\mu\nu}g_{\mu\nu} = 2-d = -2(1-\epsilon)\,, \tag{2.31}$$

such that

$$\begin{aligned}
\delta\Sigma_N^{\mathrm{N^2LO},\Delta} = \frac{1}{2}\left(\frac{\mathring{g}_A^{N\Delta}C_{TN\phi}}{(4\pi F)}\right)^2\Bigg[&\left[\frac{1}{\epsilon}-\gamma_E+1+\ln 4\pi\right]\left(\frac{2}{3}\Delta^3-\Delta m^2\right)\\
&+\frac{4}{9}\Delta^3-\frac{1}{3}\Delta m^2-\frac{2}{3}\overline{\mathcal{F}}(m,\Delta,\mu)\Bigg].
\end{aligned}\tag{2.32}$$

Unlike the virtual nucleon-pion correction, we see that the delta-pion correction requires a renormalization of the M_0 and NLO LEC, $\bar{c}_1$ in order to properly renormalize the theory. Let us consider the wave-function correction from the delta-pion loop. Examining Eq. (2.29), we see that the contribution from $\mathcal{J}_{\mu\nu}(m,\Delta)$ can simply be determined from $\mathcal{F}_{\mu\nu}(m,\Delta)$ from a derivative

$$\mathcal{J}_{\mu\nu}(m,\Delta) = \frac{\partial}{\partial\Delta}\mathcal{F}_{\mu\nu}(m,\Delta) \tag{2.33}$$

from which we can determine

$$\begin{aligned}
\delta Z_N^{\mathrm{N^2LO},\Delta} &= -\left(\frac{\mathring{g}_A^{N\Delta} C_{TN\phi}}{2F}\right)^2 P^{\mu\nu}\partial_\Delta \mathcal{F}_{\mu\nu}(m,\Delta) \\
&= -\frac{1}{2}\left(\frac{\mathring{g}_A^{N\Delta} C_{TN\phi}}{4\pi F}\right)^2 \left[\left[\frac{1}{\epsilon} - \gamma_E + 1 + \ln 4\pi\right](2\Delta^2 - m^2) + m^2 + \overline{\mathcal{J}}(m,\Delta,\mu)\right]
\end{aligned} \tag{2.34}$$

with the new function

$$\overline{\mathcal{J}}(m,\Delta,\mu) = -2\Delta\sqrt{\Delta^2 - m^2 + i\epsilon}\,\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}\right) + (m^2 - 2\Delta^2)\ln\left(\frac{m^2}{\mu^2}\right). \tag{2.35}$$

Neither of these $\overline{\mathcal{F}}$ or $\overline{\mathcal{J}}$ scalar functions vanishes in the chiral limit. It is convenient to define versions which do such that, for example, M_0 is the mass of the nucleon in the chiral limit. Similar choices also impacts the interpretation of $\mathring{g}_A$. Therefore, let us define

$$\begin{aligned}
\mathcal{F}(m,\Delta,\mu) &= \overline{\mathcal{F}}(m,\Delta,\mu) - \overline{\mathcal{F}}(0,\Delta,\mu) \\
&= (\Delta^2 - m^2)^{3/2}\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2}}{\Delta - \sqrt{\Delta^2 - m^2}}\right) + \Delta\left(\Delta^2 - \frac{3}{2}m^2\right)\ln\left(\frac{m^2}{\mu^2}\right) \\
&\quad - \Delta^3 \ln\left(\frac{4\Delta^2}{\mu^2}\right) \\
&= (\Delta^2 - m^2)^{3/2}\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2}}{\Delta - \sqrt{\Delta^2 - m^2}}\right) - \Delta^3\ln\left(\frac{4\Delta^2}{m^2}\right) - \frac{3}{2}\Delta m^2 \ln\left(\frac{m^2}{\mu^2}\right),
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\mathcal{J}(m,\Delta,\mu) &= \overline{\mathcal{J}}(m,\Delta,\mu) - \overline{\mathcal{J}}(0,\Delta,\mu) \\
&= -2\Delta\sqrt{\Delta^2 - m^2}\,\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2}}{\Delta - \sqrt{\Delta^2 - m^2}}\right) + (m^2 - 2\Delta^2)\ln\left(\frac{m^2}{\mu^2}\right) \\
&\quad + 2\Delta^2 \ln\left(\frac{4\Delta^2}{\mu^2}\right) \\
&= -2\Delta\sqrt{\Delta^2 - m^2}\,\ln\left(\frac{\Delta + \sqrt{\Delta^2 - m^2}}{\Delta - \sqrt{\Delta^2 - m^2}}\right) + 2\Delta^2\ln\left(\frac{4\Delta^2}{m^2}\right) + m^2\ln\left(\frac{m^2}{\mu^2}\right),
\end{aligned} \tag{2.37}$$

such that

$$\begin{aligned}
\delta\Sigma_N^{\mathrm{N^2LO},\Delta} &= \frac{1}{2}\left(\frac{\mathring{g}_A^{N\Delta} C_{TN\phi}}{4\pi F}\right)^2 \Bigg[\left[\frac{1}{\epsilon} - \gamma_E + 1 + \ln 4\pi\right]\left(\frac{2}{3}\Delta^3 - \Delta m_\phi^2\right) \\
&\qquad + \frac{4}{9}\Delta^3 - \frac{1}{3}\Delta m_\phi^2 - \frac{2}{3}\mathcal{F}(m_\phi,\Delta,\mu) - \frac{2}{3}\Delta^3\ln\left(\frac{4\Delta^2}{\mu^2}\right)\Bigg],
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
\delta Z_N^{\mathrm{N^2LO},\Delta} &= -\frac{1}{2}\left(\frac{\mathring{g}_A^{N\Delta} C_{TN\phi}}{4\pi F}\right)^2 \Bigg[\left[\frac{1}{\epsilon} - \gamma_E + 1 + \ln 4\pi\right](2\Delta^2 - m^2) \\
&\qquad + m_\phi^2 + \mathcal{J}(m_\phi,\Delta,\mu) - 2\Delta^2\ln\left(\frac{4\Delta^2}{\mu^2}\right)\Bigg].
\end{aligned} \tag{2.39}$$

2.4 $\mathrm{N^3LO}$ Contributions

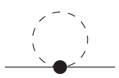
2.4.1 c_1 operator

$$
\begin{aligned}
\mathcal{L}^{(2)}_{N\pi\bar{c}_1} &= \frac{\bar{c}_1}{4\pi F}\overline{N}N\langle\chi_+\rangle \\
&= \frac{\bar{c}_1}{4\pi F}\overline{N}N\,2B\hat{m}\left[4 - \frac{2}{F^2}\left((\pi^0)^2 + 2\pi^+\pi^-\right)\right]
\end{aligned}
\tag{2.40}
$$

This leads to the NLO mass shift

$$
\begin{aligned}
&= -i\delta\Sigma = i\frac{4\bar{c}_1}{4\pi F}2B\hat{m} \\
\delta\Sigma &= -4\bar{c}_1\frac{2B\hat{m}}{4\pi F}
\end{aligned}
\tag{2.41}
$$

The Lagrangian also leads to an $\mathrm{N^3LO}$ mass correction



$$
\begin{aligned}
&= -i\delta\Sigma = -i\frac{\bar{c}_1}{4\pi F}2B\hat{m}\frac{2}{F^2}\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left[\frac{i}{k^2 - m^2_{\pi^0} + i\epsilon} + \frac{2i}{k^2 - m^2_{\pi^\pm} + i\epsilon}\right] \\
\delta\Sigma &= 6\bar{c}_1\frac{2B\hat{m}}{(4\pi F)^3}(4\pi)^2\mathcal{I}(m_\pi)\,.
\end{aligned}
\tag{2.42}
$$

Where we reduced to the isospin limit as isospin breaking corrections are also higher order. We would like to express the answer in terms of m_π and F_π instead of the bare quantities $B\hat{m}$ and F. In the $\mathrm{N^3LO}$ term, we can interchange $F \to F_\pi$ and $2B\hat{m} \to m_\pi^2$ since the corrections will appear at $\mathrm{O}(m_\pi^6)$ order ($\mathrm{N^5LO}$). For the LO correction, we have

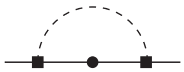
$$
\begin{aligned}
\delta\Sigma^{\mathrm{NLO}} &= -4\bar{c}_1\frac{m_\pi^2}{4\pi F_\pi}\frac{F_\pi}{F}\frac{2B\hat{m}}{m_\pi^2} \\
&= -4\bar{c}_1\Lambda_\chi\epsilon_\pi^2\left[1 + \epsilon_\pi^2\bar{l}_4^r - \frac{\mathcal{I}(m_\pi)}{F_\pi^2}\right]\left[1 - 2\epsilon_\pi^2\bar{l}_3^r - \frac{\mathcal{I}(m_\pi)}{2F_\pi^2}\right] + \mathrm{O}\left(\Lambda_\chi\epsilon_\pi^6\right) \\
&= -4\bar{c}_1\Lambda_\chi\epsilon_\pi^2\left[1 - \epsilon_\pi^2(2\bar{l}_3^r - \bar{l}_4^r) - \frac{3}{2}\frac{\mathcal{I}(m_\pi)}{F_\pi^2}\right] + \mathrm{O}\left(\Lambda_\chi\epsilon_\pi^6\right)\,.
\end{aligned}
\tag{2.43}
$$

2.4.1.1 Mass insertion in loop and wave-function correction

The wave function correction comes from correcting the external legs from the LO mass insertion, giving rise to

$$
\begin{aligned}
\delta\Sigma_{WF} &= \delta\Sigma^{\mathrm{NLO}} \times \delta Z \\
&= -4\bar{c}_1\frac{2B\hat{m}}{4\pi F}\frac{-3\mathring{g}A^2}{F^2}S^\mu S^\nu\mathcal{J}_{\mu\nu}(m_\pi, 0) \\
&= 4\bar{c}_1\frac{2B\hat{m}}{4\pi F}\frac{3\mathring{g}_A^2}{F^2}S^\mu S^\nu\mathcal{J}_{\mu\nu}(m_\pi, 0)\,.
\end{aligned}
\tag{2.44}
$$

We can also insert the c_1 operator in the virtual nucleon-pion loop



$$= -i\delta\Sigma = \left(\frac{i\mathring{g}_A C_{NN\phi}}{F}\right)^2 \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d} iS^\mu k_\mu \frac{i}{k^2-m_\phi^2}\frac{i}{k\cdot v}\frac{4i\bar{c}_1 2B\hat{m}}{4\pi F}\frac{i}{k\cdot v}(-iS^\nu k_\nu)$$

$$\begin{aligned}
\delta\Sigma &= (i)^8(-)\frac{3\mathring{g}_A^2}{F^2}\frac{4\bar{c}_1 2B\hat{m}}{4\pi F}S^\mu S^\nu \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu}{(k^2-m_\pi^2)(k\cdot v)^2} \\
&= -4\bar{c}_1 \frac{2B\hat{m}}{4\pi F}\frac{3\mathring{g}_A^2}{F^2}S^\mu S^\nu \mathcal{J}_{\mu\nu}(m_\pi, 0)\,,
\end{aligned} \tag{2.45}$$

which is equal and opposite to the wave-function renormalization of the LO mass shift.

2.4.2 $\bar{c}_{2,3}$ operators

$$\begin{aligned}
\mathcal{L}^{(2)}_{N\pi\bar{c}_{2,3}} &= \frac{\tilde{\bar{c}}_2 v^\mu v^\nu + \bar{c}_3 g^{\mu\nu}}{4\pi F}\overline{N}N\,\frac{\langle u_\mu u_\nu\rangle}{2} \\
&= \frac{\tilde{\bar{c}}_2 v^\mu v^\nu + \bar{c}_3 g^{\mu\nu}}{4\pi F}\overline{N}N\,\frac{1}{F^2}\left(\partial_\mu\pi^0\partial_\nu\pi^0 + 2\partial_\mu\pi^+\partial_\nu\pi^-\right)
\end{aligned} \tag{2.46}$$

where from Eq. (1.11)

$$\frac{\tilde{\bar{c}}_2}{4\pi F} = \frac{\bar{c}_2}{4\pi F} - \frac{\mathring{g}_A^2}{8M_0}\,. \tag{2.47}$$

The self energy graph from these operators is very similar to that of the $\bar{c}_1$ operator leading to

$$\begin{aligned}
-i\delta\Sigma_{\bar{c}_{2,3}} &= \frac{i(\tilde{\bar{c}}_2 v^\mu v^\nu + \bar{c}_3 g^{\mu\nu})}{4\pi F}\frac{1+2}{F^2}\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu}{k^2-m_\pi^2} \\
\delta\Sigma_{\bar{c}_{2,3}} &= -3\frac{\tilde{\bar{c}}_2 v^\mu v^\nu + \bar{c}_3 g^{\mu\nu}}{4\pi F}\frac{g_{\mu\nu}}{dF^2}\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{ik^2}{k^2-m_\pi^2} \\
&= -3\frac{\frac{1}{d}\tilde{\bar{c}}_2 + \bar{c}_3}{4\pi F}\frac{\tilde{\mathcal{I}}(m_\pi)}{F^2}\,,
\end{aligned} \tag{2.48}$$

where

$$\begin{aligned}
\tilde{\mathcal{I}}(m) &= \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{ik^2}{k^2-m_\pi^2} \\
&= \frac{-m^4}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln\frac{m^2}{\mu^2}\right].
\end{aligned} \tag{2.49}$$

Then, we can write

$$\frac{1}{d}\tilde{\mathcal{I}}(m) = \frac{1}{4}\left(\tilde{\mathcal{I}}(m) - \frac{m^4}{2(4\pi)^2}\right)\,, \tag{2.50}$$

such that the mass correction becomes

$$\delta\Sigma_{\bar{c}_{2,3}} = -3\left(\frac{\tilde{\bar{c}}_2}{4} + \bar{c}_3\right)\frac{(4\pi)^2\tilde{\mathcal{I}}(m_\pi)}{(4\pi F)^3} + \frac{3}{8}\tilde{\bar{c}}_2\frac{m_\pi^4}{(4\pi F)^3}\,. \tag{2.51}$$

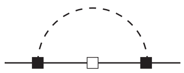
Note, the pre-factor of 3 is 1+2 from the strength of the π^0 and $\pi^\pm$ vertices in Eq. Eq. (2.46).

2.4.3 Relativistic corrections

The leading relativistic corrections arise from the operators

$$\mathcal{L}_{N\pi}^{(2)} \supset \overline{N}\left[-\frac{D_\perp^2}{2M_0} + \frac{i\mathring{g}_A}{2M_0}\left(S\cdot\overleftarrow{D}v\cdot u - v\cdot uS\cdot\overrightarrow{D}\right)\right]N\,. \tag{2.52}$$

The baryon kinetic correction gives the self energy correction



$$\begin{aligned} = -i\delta\Sigma &= \left(\frac{i\mathring{g}_A C_{NN\phi}}{F}\right)^2 \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m_\phi^2}\frac{i(iS^\mu k_\mu)}{k\cdot v}\frac{-i(-k^2+(v\cdot k)^2)}{2M_0}\frac{i(-iS^\nu k_\nu)}{k\cdot v} \\ &= i^7(-)^3\frac{\mathring{g}_A^2 C_{NN\phi}^2}{2M_0F^2}S^\mu S^\nu(g^{\rho\sigma}-v^\rho v^\sigma)\mathcal{J}_{\mu\nu\rho\sigma}(m_\phi,0)\,, \end{aligned} \tag{2.53}$$

where

$$\begin{aligned} \mathcal{J}_{\mu\nu\rho\sigma}(m,0) &= \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu k_\rho k_\sigma}{(k^2-m^2)(k\cdot v)^2} \\ &= 8\int_0^\infty d\lambda\lambda\mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{i(l-\lambda v)_\mu(l-\lambda v)_\nu(l-\lambda v)_\rho(l-\lambda v)_\sigma}{[l^2-(\lambda^2+m^2)]^3} \end{aligned} \tag{2.54}$$

The Lorentz tensor this integral is contracted with vanishes for all terms proportional to v_α. The non-vanishing contribution is

$$\mathcal{J}_{\mu\nu\rho\sigma}(m,0) = \frac{g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}}{(4\pi)^2}\frac{m^4}{4}\left[L_\epsilon+\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right]. \tag{2.55}$$

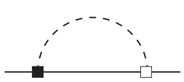
The contraction of these metric tensors with the Lorentz structure yields

$$S^\mu S^\nu(g^{\rho\sigma}-v^\rho v^\sigma)(g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}) = S^2(d-1)+2S^2 = \frac{1-d}{4}(d+1) = -\frac{15}{4}\left[1-\frac{16}{15}\epsilon\right]. \tag{2.56}$$

Thus, the self energy is given by

$$\delta\Sigma = \mathring{g}_A^2C_{NN\phi}^2\frac{15}{32}\frac{m_\phi^4}{M(4\pi F)^2}\left[L_\epsilon-\frac{17}{30}-\ln\frac{m^2}{\mu^2}\right]. \tag{2.57}$$

The self-energy from the relativistic correction of the vertex functions is



$$\begin{aligned} = -i\delta\Sigma &= \left(\frac{i\mathring{g}_A C_{NN\phi}}{F}\right)\left(\frac{i^2\mathring{g}_A C_{NN\phi}}{2M_0}\right)\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i(iS^\mu k_\mu)}{k^2-m_\phi^2}\frac{iS^\nu ik_\nu}{k\cdot v}\frac{v^\alpha(-ik_\alpha)}{F} \\ &= i^7(-)\frac{\mathring{g}_A^2C_{NN\phi}^2}{2M_0F^2}S^\mu S^\nu v^\alpha\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu k_\alpha}{(k^2-m_\phi^2)(k\cdot v)} \\ &= i\frac{\mathring{g}_A^2C_{NN\phi}^2}{2M_0F^2}S^\mu S^\nu v^\alpha\int_0^\infty d\lambda\mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{2i(l-\lambda v)_\mu(l-\lambda v)_\nu(l-\lambda v)_\alpha}{[l^2-(\lambda^2+m_\phi^2)]^2} \\ &= -i\frac{\mathring{g}_A^2C_{NN\phi}^2}{2M_0F^2}S^\mu S^\nu\int_0^\infty d\lambda\lambda\mu^{2\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{2il_\mu l_\nu}{[l^2-(\lambda^2+m_\phi^2)]^2} \\ \delta\Sigma &= \frac{-3}{32}\mathring{g}_A^2C_{NN\phi}^2\frac{m_\phi^4}{M_0(4\pi F)^2}\left[L_\epsilon-\frac{1}{6}-\ln\frac{m^2}{\mu^2}\right]. \end{aligned} \tag{2.58}$$

Combining the self-energy corrections from both vertex corrections and the kinetic recoil correction, we have

$$\begin{aligned}\delta\Sigma &= \frac{9\mathring{g}_A^2 C_{NN\phi}^2}{32}\frac{m_\phi^4}{M_0(4\pi F)^2}\left[L_\epsilon - \frac{15}{18} - \ln\frac{m_\phi^2}{\mu^2}\right] \\ &= \frac{27\mathring{g}_A^2}{32}\frac{m_\pi^4}{M_0(4\pi F)^2}\left[L_\epsilon - \frac{15}{18} - \ln\frac{m_\pi^2}{\mu^2}\right].\end{aligned} \tag{2.59}$$

2.4.4 Counterterms

$$\mathcal{L}^{(4)} \supset \frac{1}{(4\pi F)^3}\overline{N}\Big[\bar{e}_{38}\langle\chi_+\rangle\langle\chi_+\rangle + \bar{e}_{115}\langle\chi\chi^\dagger\rangle + \bar{e}_{116}(\det\chi + \det\chi^\dagger)\Big]N \tag{2.60}$$

Taking the tree-level graph from these operators, the self-energy correction is

$$\delta\Sigma = -\frac{1}{(4\pi F)^3}\Big\{16\bar{e}_{38}(2B\hat{m})^2 + 2\bar{e}_{115}\left[(2B\hat{m})^2 + (2B\delta)^2\right] + \bar{e}_{116}\left[(2B\hat{m})^2 - (2B\delta)^2\right]\Big\}. \tag{2.61}$$

In the isospin limit, we can define this correction as

$$\delta\Sigma = -\frac{(2B\hat{m})^2}{(4\pi F)^3}\bar{e}\,. \tag{2.62}$$

2.4.5 All terms through N^{3}LO

Let us begin with an expression as it naturally appears from the sum of all graphs, defined as

- The quark mass dependence arising in operators will be left as $\hat{m}_l$;
- The pion mass appearing in the pion propagators will be the on-shell pion mass;
- The pion decay constant and nucleon axial coupling appearing in the expressions will be the leading order values;

2.4.5.1 HBχPT($\not\Delta$)

With this prescription, we find the renormalized (subtract all L_ϵ contributions) nucleon mass corrections are (The first N^{3}LO term needs to be written in terms of loop functions to make it easier to add FV corrections)

$$M_N[\mathrm{LO}] = M_0 \tag{2.63a}$$

$$M_N[\mathrm{NLO}] = -4\bar{c}_1\frac{2B\hat{m}_l}{4\pi F} \tag{2.63b}$$

$$M_N[\mathrm{N^2LO}] = -\frac{3\pi\mathring{g}_A^2}{2}\frac{M_\pi^3}{(4\pi F)^2} \tag{2.63c}$$

$$\begin{aligned}M_N[\mathrm{N^3LO}] = &-\frac{27\mathring{g}_A^2}{32}\frac{M_\pi^4}{M_0(4\pi F)^2}\left[\frac{5}{6} + \ln\frac{M_\pi^2}{\mu^2}\right] + 6\bar{c}_1\frac{2B\hat{m}}{(4\pi F)^3}(4\pi)^2\mathcal{I}(M_\pi) \\ &- 3\left(\frac{\tilde{\bar{c}}_2}{4} + \bar{c}_3\right)\frac{(4\pi)^2\tilde{\mathcal{I}}(M_\pi)}{(4\pi F)^3} + \frac{3\tilde{\bar{c}}_2}{8}\frac{M_\pi^4}{(4\pi F)^3} - \bar{e}\frac{(2B\hat{m})^2}{(4\pi F)^3}\end{aligned} \tag{2.63d}$$

We can then replace the integral functions with their infinite volume expressions to obtain

$$
\begin{aligned}
M_N &= M_0 - 4\bar{c}_1 \frac{2B\hat{m}_l}{4\pi F} - \frac{3\pi \mathring{g}_A^2}{2}\frac{M_\pi^3}{(4\pi F)^2} + \frac{M_\pi^4}{(4\pi F)^2}\frac{3}{8}\left[\frac{\tilde{\bar{c}}_2}{4\pi F} - \frac{15\mathring{g}_A^2}{8M_0}\right] - \bar{e}\frac{(2B\hat{m})^2}{(4\pi F)^3} \\
&\quad - \ln\frac{M_\pi^2}{\mu^2}\frac{M_\pi^4}{(4\pi F)^2}\frac{3}{4}\left[\frac{9\mathring{g}_A^2}{8M_0} - \frac{2B\hat{m}}{M_\pi^2}\frac{8\bar{c}_1}{4\pi F} + \frac{\tilde{\bar{c}}_2 + 4\bar{c}_3}{4\pi F}\right] \\
&= M_0 - 4\bar{c}_1 \frac{2B\hat{m}_l}{4\pi F} - \frac{3\pi \mathring{g}_A^2}{2}\frac{M_\pi^3}{(4\pi F)^2} + \frac{M_\pi^4}{(4\pi F)^2}\frac{3}{8}\left[\frac{\bar{c}_2}{4\pi F} - 2\frac{\mathring{g}_A^2}{M_0}\right] - \bar{e}\frac{(2B\hat{m})^2}{(4\pi F)^3} \\
&\quad - \ln\frac{M_\pi^2}{\mu^2}\frac{M_\pi^4}{(4\pi F)^2}\frac{3}{4}\left[\frac{\mathring{g}_A^2}{M_0} - \frac{2B\hat{m}}{M_\pi^2}\frac{8\bar{c}_1}{4\pi F} + \frac{\bar{c}_2 + 4\bar{c}_3}{4\pi F}\right]
\end{aligned} \tag{2.64}
$$

This expression matches that in Ref. [7] with the replacements $\bar{c}_i \to (4\pi F)c_i$, $\bar{e} \to (4\pi F)^3 e$ and $F \to f$ (note their normalization of $f \approx 92$ MeV), and at this order, we can also send all $M_\pi^2 \to 2B\hat{m} = M^2$.

2.4.5.2 HBχPT($\not\Delta$): bare parameters

In order to add a little confusion, let us define leading pion mass as

$$
M^2 \equiv 2B\hat{m}\,. \tag{2.65}
$$

Then, the nucleon mass at N^{3}LO is given by

$$
\begin{aligned}
M_N &= M_0 - 4\bar{c}_1\frac{M^2}{4\pi F} - \frac{3\pi\mathring{g}_A^2}{2}\frac{M^3}{(4\pi F)^2} + \frac{M^4}{(4\pi F)^3}\left[\frac{3}{8}\left(\bar{c}_2 - 2\mathring{g}_A^2\frac{4\pi F}{M_0}\right) - \bar{e}\right] \\
&\quad - \frac{M^4}{(4\pi F)^3}\ln\frac{M^2}{\mu^2}\frac{3}{4}\left[\mathring{g}_A^2\frac{(4\pi F)}{M_0} - 8\bar{c}_1 + \bar{c}_2 + 4\bar{c}_3\right]
\end{aligned} \tag{2.66}
$$

COMMENT: OLD below In infinite volume, after subtracting all L_ϵ contributions, the self-energy is given by

$$
\begin{aligned}
M_N &= M_0 - 4\bar{c}_1\epsilon_\pi^2\Lambda_\chi - \frac{3\pi\mathring{g}_A^2}{2}\epsilon_\pi^3\Lambda_\chi + \frac{27\mathring{g}_A^2}{32}\frac{m_\pi^4}{M_0(4\pi F)^2}\left[\frac{15}{18} + \ln\frac{m_\pi^2}{\mu^2}\right] \\
&\quad + 4\bar{c}_1\Lambda_\chi\epsilon_\pi^2\left[\epsilon_\pi^2(2\bar{l}_3^r - \bar{l}_4^r) + \frac{9}{4}\frac{\mathcal{I}(m_\pi)}{F_\pi^2}\right] - \left(\frac{\tilde{\bar{c}}_2}{4} + \bar{c}_3\right)\frac{(4\pi)^2\tilde{\mathcal{I}}(m_\pi)}{(4\pi F)^3} + \frac{\tilde{\bar{c}}_2}{8}\frac{m_\pi^4}{(4\pi F)^3} - \frac{16\bar{d}_{38}(2B\hat{m})^2}{(4\pi F)^3} \\
&= M_0 - 4\bar{c}_1\epsilon_\pi^2\Lambda_\chi - \frac{3\pi\mathring{g}_A^2}{2}\epsilon_\pi^3\Lambda_\chi + \frac{27\mathring{g}_A^2}{32}\epsilon_\pi^4\frac{\Lambda_\chi}{M_N}\left[\frac{15}{18} + \ln(\epsilon_\pi^2)\right]\Lambda_\chi \\
&\quad + 4\bar{c}_1\epsilon_\pi^4\left[(2\bar{l}_3^r - \bar{l}_4^r) + \frac{9}{4}\ln(\epsilon_\pi^2)\right]\Lambda_\chi + \left(\frac{\tilde{\bar{c}}_2}{4} + \bar{c}_3\right)\epsilon_\pi^4\ln(\epsilon_\pi^2)\Lambda_\chi + \frac{\tilde{\bar{c}}_2}{8}\epsilon_\pi^4\Lambda_\chi - 16\bar{d}_{38}\epsilon_\pi^4\Lambda_\chi \\
\frac{M_N}{\Lambda_\chi} &= \frac{M_0}{\Lambda_\chi} - 4\bar{c}_1\epsilon_\pi^2 - \frac{3\pi g_A^2}{2}\epsilon_\pi^3 + \epsilon_\pi^4\ln\epsilon_\pi^2\left[\frac{27\mathring{g}_A^2}{32}\frac{\Lambda_\chi}{M_N} + 9\bar{c}_1 + \frac{\tilde{\bar{c}}_2}{4} + \bar{c}_3\right] \\
&\quad + \epsilon_\pi^4\left[\frac{27\mathring{g}_A^2}{32}\frac{15\Lambda_\chi}{18M_N} + 4\bar{c}_1(2\bar{l}_3^r - \bar{l}_4^r) + \frac{\tilde{\bar{c}}_2}{8} - 16\bar{d}_{38}\right]
\end{aligned} \tag{2.67}
$$

Chapter 3

Nucleon sigma term

A quantity of phenomenological interest is the nucleon sigma term, sometimes called the pion-nucleon sigma term, defined as

$$\sigma_{\pi N} = \hat{m}\langle N|\bar{u}u + \bar{d}d|N\rangle = \hat{m}\frac{\partial}{\partial\hat{m}}M_N\,. \tag{3.1}$$

This quantity is of interest for interpreting the prospective scattering of dark matter off nuclei if the dark matter couples through the Higgs, which would couple to nucleons and nuclei through this scalar condensate in the nucleon.

It is often convenient to re-express this sigma term in terms of the pion mass dependence since the pion mass can be "measured" from the two-point correlation functions and does not require renormalization. This form is also more convenient when using the Feynman-Hellmann method to extract the sigma-term from an analysis of the nucleon mass as a function of the pion mass.

3.0.1 Useful expressions and relations

Ref. [8] express the relation between the physical m_π and F_π and their leading parameters M^2 and F in three useful expressions to NNLO in Eqs. (80-82). Following the notation of this reference, we have the definitions

$$x = \frac{M^2}{(4\pi F)^2}\,, \qquad M^2 = 2B\hat{m}\,, \qquad \xi = \epsilon_\pi^2 = \frac{m_\pi^2}{(4\pi F_\pi)^2}\,. \tag{3.2}$$

Eq. (80) provides the physical quantities in terms of the bare ones, useful for the $\hat{m}\partial_{\hat{m}}$ derivative

$$\begin{aligned} m_\pi^2 &= M^2\left\{1 + x\left(\frac{1}{2}\ln\frac{M^2}{\mu^2} + l_M^r\right) + x^2\left(\frac{17}{8}\ln^2\frac{M^2}{\mu^2} + c_{1M}^r\ln\frac{M^2}{\mu^2} + c_{2M}^r\right)\right\} \\ F_\pi &= F\left\{1 + x\left(-\ln\frac{M^2}{\mu^2} + l_F^r\right) + x^2\left(-\frac{5}{4}\ln^2\frac{M^2}{\mu^2} + c_{1F}^r\ln\frac{M^2}{\mu^2} + c_{2F}^r\right)\right\}\,. \end{aligned} \tag{3.3}$$

We can relate these to the FLAG expressions Eq.(88) of FLAG 2019 [9]

$$\begin{aligned} M_\pi^2 &= M^2\left\{1 - \frac{1}{2}x\ln\frac{\Lambda_3^2}{M^2} + \frac{17}{8}x^2\left(\ln\frac{\Lambda_M^2}{M^2}\right)^2 + x^2k_M\right\} \\ F_\pi &= F\left\{1 + x\ln\frac{\Lambda_4^2}{M^2} - \frac{5}{4}x^2\left(\ln\frac{\Lambda_F^2}{M^2}\right)^2 + x^2k_F\right\}\,. \end{aligned} \tag{3.4}$$

In these expressions

$$
\begin{aligned}
\ln\frac{\Lambda_M^2}{M^2} &= \frac{1}{51}\left(28\ln\frac{\Lambda_1^2}{M^2} + 32\ln\frac{\Lambda_2^2}{M^2} - 9\ln\frac{\Lambda_3^2}{M^2} + 49\right)\\
\ln\frac{\Lambda_F^2}{M^2} &= \frac{1}{30}\left(14\ln\frac{\Lambda_1^2}{M^2} + 16\ln\frac{\Lambda_2^2}{M^2} + 6\ln\frac{\Lambda_3^2}{M^2} - 6\ln\frac{\Lambda_4^2}{M^2} + 23\right)
\end{aligned}
\tag{3.5}
$$

and [6]

$$
\bar{\ell}_n = \ln\frac{\Lambda_n^2}{M_{\pi,\text{phys}}^2} = \frac{2}{\gamma_i}\bar{l}_n^r(\mu) - \ln\frac{M_{\pi,\text{phys}}^2}{\mu^2}
\tag{3.6}
$$

with

$$
\gamma_1 = \frac{1}{3}, \qquad \gamma_2 = \frac{2}{3}, \qquad \gamma_3 = -\frac{1}{2}, \qquad \gamma_4 = 2.
\tag{3.7}
$$

Using these relations, we can express these ln contributions in terms of the renormalized LECs appearing in the Lagrangian

$$
\begin{aligned}
-\frac{1}{2}\ln\frac{\Lambda_3^2}{M^2} &= \frac{1}{2}\ln\frac{M^2}{\mu^2} + 2\bar{l}_3^r(\mu)\\
\ln\frac{L_4^2}{M^2} &= -\ln\frac{M^2}{\mu^2} + \bar{l}_4^r(\mu)\\
\ln\frac{\Lambda_M^2}{M^2} &= \frac{1}{51}\left(-51\ln\frac{M^2}{\mu^2} + 168\bar{l}_1^r + 96\bar{l}_2^r + 36\bar{l}_3^r + 49\right)\\
&= -\ln\frac{M^2}{\mu^2} + \frac{4}{17}\left(14\bar{l}_1^r + 8\bar{l}_2^r + 3\bar{l}_3^r + \frac{49}{12}\right)\\
\ln\frac{\Lambda_F^2}{M^2} &= \frac{1}{30}\left(-30\ln\frac{M^2}{\mu^2} + 84\bar{l}_1^r + 48\bar{l}_2^r - 24\bar{l}_3^r - 6\bar{l}_4^r + 23\right)\\
&= -\ln\frac{M^2}{\mu^2} + \frac{1}{5}\left(14\bar{l}_1^r + 8\bar{l}_2^r - 4\bar{l}_3^r - \bar{l}_4^r + \frac{23}{6}\right)
\end{aligned}
\tag{3.8}
$$

Then, we have

$$
\begin{aligned}
\frac{17}{8}\left(\ln\frac{\Lambda_M^2}{M^2}\right)^2 &= \frac{17}{8}\ln^2\frac{M^2}{\mu^2} - \ln\frac{M^2}{\mu^2}\left(\frac{49}{12} + 14\bar{l}_1^r + 8\bar{l}_2^r + 3\bar{l}_3^r\right) + \frac{2}{17}\left(\frac{49}{12} + 14\bar{l}_1^r + 8\bar{l}_2^r + 3\bar{l}_3^r\right)^2\\
&\to l_M^r = 2\bar{l}_3^r\,,\\
&\quad c_{1M}^r = -\left(\frac{49}{12} + 14\bar{l}_1^r + 8\bar{l}_2^r + 3\bar{l}_3^r\right)\\
&\quad c_{2M}^r = k_M + \frac{2}{17}\left(\frac{49}{12} + 14\bar{l}_1^r + 8\bar{l}_2^r + 3\bar{l}_3^r\right)^2\\
-\frac{5}{4}\left(\ln\frac{\Lambda_F^2}{M^2}\right)^2 &= -\frac{5}{4}\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}\frac{1}{2}\left(14\bar{l}_1^r + 8\bar{l}_2^r - 4\bar{l}_3^r - \bar{l}_4^r + \frac{23}{6}\right) - \frac{1}{20}\left(14\bar{l}_1^r + 8\bar{l}_2^r - 4\bar{l}_3^r - \bar{l}_4^r + \frac{23}{6}\right)^2\\
&\to l_F^4 = \bar{l}_4^r\\
&\quad c_{1F}^r = \frac{1}{2}\left(14\bar{l}_1^r + 8\bar{l}_2^r - 4\bar{l}_3^r - \bar{l}_4^r + \frac{23}{6}\right)\\
&\quad c_{2F}^r = k_F - \frac{1}{20}\left(14\bar{l}_1^r + 8\bar{l}_2^r - 4\bar{l}_3^r - \bar{l}_4^r + \frac{23}{6}\right)^2 .
\end{aligned}
\tag{3.9}
$$

We see that c^r_{1M} and c^r_{1F} only depend upon $\bar{l}^r_i$ and finite terms, so there is a relation between them

$$c^r_{1M} = -2c^r_{1F} - 7\bar{l}^r_3 - \bar{l}^r_4 - \frac{1}{4}\,. \tag{3.10}$$

Eq. (81) of Ref. [8] expresses the bare parameters as functions of the physical ones

$$\begin{aligned}
M^2 &= m_\pi^2 \left\{1 + \xi\left(-\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + \tilde{l}^r_M\right) + \xi^2\left(-\frac{5}{8}\ln^2\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{1M}\ln\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{2M}\right)\right\} \\
&= m_\pi^2 \left\{1 + \xi\left(-\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} - 2\bar{l}^r_3\right) + \xi^2\left(-\frac{5}{8}\ln^2\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{1M}\ln\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{2M}\right)\right\}\,, \\
F &= F_\pi \left\{1 + \xi\left(\ln\frac{m_\pi^2}{\mu^2} + \tilde{l}^r_F\right) + \xi^2\left(-\frac{1}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{1F}\ln\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{2F}\right)\right\} \\
&= F_\pi \left\{1 + \xi\left(\ln\frac{m_\pi^2}{\mu^2} - \bar{l}^r_4\right) + \xi^2\left(-\frac{1}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{1F}\ln\frac{m_\pi^2}{\mu^2} + \tilde{c}^r_{2F}\right)\right\}\,.
\end{aligned} \tag{3.11}$$

It is useful to note, that at NLO, we can relate $\tilde{l}^r_{M,F} = -l^r_{M,F}$. We can also use these expressions to determine the relation between x and ξ which we will need through NLO

$$\begin{aligned}
M^2 &= m_\pi^2\left(1 - \xi\left(\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}^r_3\right)\right) \\
\frac{M^2}{(4\pi F)^2} &= \frac{m_\pi^2}{(4\pi F_\pi)^2}\frac{1 - \xi\left(\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}^r_3\right)}{\left[1 + \xi\left(\ln\frac{m_\pi^2}{\mu^2} - \bar{l}_4\right)\right]^2} \\
&\simeq \frac{m_\pi^2}{(4\pi F_\pi)^2}\left[1 - \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}^r_3 - 2\bar{l}^r_4\right)\right] \\
x &= \xi\left(1 - \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}^r_3 - 2\bar{l}^r_4\right)\right) + \text{NNLO}
\end{aligned} \tag{3.12}$$

For analyzing our results, the most useful versions are from Eq. (82) (though, to make use of our m_π^2 results, we require our NPR to be complete)

$$\begin{aligned}
m_\pi^2 &= M^2 + m_\pi^2\left\{\xi\left(\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}^r_3\right) + \xi^2\left(\frac{5}{8}\ln^2\frac{m_\pi^2}{\mu^2} + \hat{c}^r_{1M}\ln\frac{m_\pi^2}{\mu^2} + \hat{c}^r_{2M}\right)\right\}\,, \\
F_\pi &= F\left\{1 + \xi\left(-\ln\frac{m_\pi^2}{\mu^2} + \bar{l}^r_4\right) + \xi^2\left(\frac{5}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \hat{c}^r_{1F}\ln\frac{m_\pi^2}{\mu^2} + \hat{c}^r_{2F}\right)\right\}\,.
\end{aligned} \tag{3.13}$$

To relate the LECs in this expression to those in Eq. (3.3), we can start with that expression and use the NLO relation between x and ξ in Eq. (3.12). For the pion mass, we have

$$
\begin{aligned}
m_\pi^2 &= M^2 + M^2 x \left(\frac{1}{2} \ln \frac{M^2}{\mu^2} + l_M^r \right) + M^2 x^2 \left(\frac{17}{8} \ln^2 \frac{M^2}{\mu^2} + c_{1M}^r \ln \frac{M^2}{\mu^2} + c_{2M}^r \right) \\
&= M^2 + m_\pi^2 \left(1 - \xi \left(\frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r \right) \right) \xi \left(1 - \xi \left(\frac{5}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r \right) \right) \left(\frac{1}{2} \ln \frac{M^2}{\mu^2} + 2\bar{l}_3^r \right) \\
&\quad + m_\pi^2 \xi^2 \left(\frac{17}{8} \ln^2 \frac{m_\pi^2}{\mu^2} + c_{1M}^r \ln \frac{m_\pi^2}{\mu^2} + c_{2M}^r \right) \\
&= M^2 + m_\pi^2 \xi \left(\frac{1}{2} \ln \frac{M^2}{\mu^2} + 2\bar{l}_3^r \right) - m_\pi^2 \xi^2 \left(\frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r \right) \left(\frac{6}{2} \ln \frac{m_\pi^2}{\mu^2} + 4\bar{l}_3^r - 2\bar{l}_4^r \right) \\
&\quad + m_\pi^2 \xi^2 \left(\frac{17}{8} \ln^2 \frac{m_\pi^2}{\mu^2} + c_{1M}^r \ln \frac{m_\pi^2}{\mu^2} + c_{2M}^r \right) \\
&= M^2 + m_\pi^2 \xi \left(\frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - \frac{1}{2} \xi \left(\frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r \right) \right) \\
&\quad + m_\pi^2 \xi^2 \left(\frac{5}{8} \ln^2 \frac{m_\pi^2}{\mu^2} + (c_{1M}^r - 8\bar{l}_3^r + \bar{l}_4^r) \ln \frac{m_\pi^2}{\mu^2} + (c_{2M}^r - 4\bar{l}_3^r(2\bar{l}_3^r - \bar{l}_4^r)) \right) \\
&= M^2 + m_\pi^2 \xi \left(\frac{1}{2} \ln \frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r \right) \\
&\quad + m_\pi^2 \xi^2 \left(\frac{5}{8} \ln^2 \frac{m_\pi^2}{\mu^2} + \left(c_{1M}^r - 8\bar{l}_3^r + \bar{l}_4^r - \frac{1}{4} \right) \ln \frac{m_\pi^2}{\mu^2} + c_{2M}^r - 4\bar{l}_3^r \left(2\bar{l}_3^r - \bar{l}_4^r + \frac{1}{4} \right) \right) \\
\to \hat{c}_{1M}^r &= c_{1M}^r - 8\bar{l}_3^r + \bar{l}_4^r - \frac{1}{4} \\
\hat{c}_{2M}^r &= c_{2M}^r - \bar{l}_3(8\bar{l}_3 - 4\bar{l}_4^r + 1)
\end{aligned}
\tag{3.14}
$$

For the pion decay constant, we have

$$
\begin{aligned}
F_\pi &= F\left\{1 + x\left(-\ln\frac{M^2}{\mu^2} + \bar{l}_4^r\right) + x^2\left(-\frac{5}{4}\ln^2\frac{M^2}{\mu^2} + c_{1F}^r\ln\frac{M^2}{\mu^2} + c_{2F}^r\right)\right\} \\
&= F\Bigg\{1 + \xi\left(1 - \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r\right)\right)\left(-\ln\frac{M^2}{\mu^2} + \bar{l}_4^r\right) \\
&\qquad + \xi^2\left(-\frac{5}{4}\ln^2\frac{m_\pi^2}{\mu^2} + c_{1F}^r\ln\frac{m_\pi^2}{\mu^2} + c_{2F}^r\right)\Bigg\} \\
&= F\Bigg\{1 + \xi\left(-\ln\frac{M^2}{\mu^2} + \bar{l}_4^r\right) + \xi^2\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r\right)\left(\ln\frac{m_\pi^2}{\mu^2} - \bar{l}_4^r\right) \\
&\qquad + \xi^2\left(-\frac{5}{4}\ln^2\frac{m_\pi^2}{\mu^2} + c_{1F}^r\ln\frac{m_\pi^2}{\mu^2} + c_{2F}^r\right)\Bigg\} \\
&= F\left\{1 + \xi\left(-\ln\frac{m_\pi^2}{\mu^2} + \bar{l}_4^r + \xi\left(\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r\right)\right)\right\} \\
&\qquad + \xi^2\left(\frac{5}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \left(c_{1F}^r + 2\bar{l}_3^r - \frac{9}{2}\bar{l}_4^r\right)\ln\frac{m_\pi^2}{\mu^2} + c_{2F}^r + 2\bar{l}_4^r(\bar{l}_4^r - \bar{l}_3^r)\right) \\
&= F\Bigg\{1 + \xi\left(-\ln\frac{m_\pi^2}{\mu^2} + \bar{l}_4^r\right) \\
&\qquad + \xi^2\left(\frac{5}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \left(c_{1F}^r + 2\bar{l}_3^r - \frac{9}{2}\bar{l}_4^r + \frac{1}{2}\right)\ln\frac{m_\pi^2}{\mu^2} + c_{2F}^r + 2\bar{l}_4^r(\bar{l}_4^r - \bar{l}_3^r) + 2\bar{l}_3^r\right)\Bigg\} \\
\rightarrow \hat{c}_{1F}^r &= c_{1F}^r + 2\bar{l}_3^r - \frac{9}{2}\bar{l}_4^r + \frac{1}{2} \\
\hat{c}_{2F}^r &= c_{2F}^r + 2\bar{l}_4^r(\bar{l}_4^r - \bar{l}_3^r) + 2\bar{l}_3^r
\end{aligned} \tag{3.15}
$$

A fourth choice for expressing the pion mass is

$$
\begin{aligned}
m_\pi^2 &= M^2\left\{1 + \xi\left(\frac{1}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r\right) + \xi^2\left(\frac{7}{8}\ln^2\frac{m_\pi^2}{\mu^2} + \hat{\hat{c}}_{1M}^r\ln\frac{m_\pi^2}{\mu^2} + \hat{\hat{c}}_{2M}^r\right)\right\}, \\
\hat{\hat{c}}_{1M}^r &= c_{1M}^r - 6\bar{l}_3^r + \bar{l}_4^r - \frac{1}{4}, \\
\hat{\hat{c}}_{2M}^r &= c_{2M}^r - \bar{l}_3^r(1 + 4(\bar{l}_3 - \bar{l}_4^r)).
\end{aligned} \tag{3.16}
$$

3.0.2 $\hat{m}\partial_{\hat{m}} = \hat{m}\frac{\partial\epsilon_\pi}{\partial_{\hat{m}}}\partial_{\epsilon_\pi}$

We want to relate the derivative of interest, $\hat{m}\partial_{\hat{m}}$ to a derivative with respect to ϵ_π

$$
\begin{aligned}
\hat{m}\partial_{\hat{m}} &= \hat{m}\frac{\partial\epsilon_\pi}{\partial_{\hat{m}}}\partial_{\epsilon_\pi} \\
&= \hat{m}\left(\frac{\partial}{\partial_{\hat{m}}}\frac{\sqrt{m_\pi^2}}{4\pi F_\pi}\right)\partial_{\epsilon_\pi} \\
&= \left(\frac{\hat{m}\partial_{\hat{m}}m_\pi^2}{2m_\pi 4\pi F_\pi} - \frac{m_\pi}{4\pi F_\pi}\frac{\hat{m}\partial_{\hat{m}}F_\pi}{F_\pi}\right)\partial_{\epsilon_\pi} \\
&= \left(\frac{\hat{m}\partial_{\hat{m}}m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}}F_\pi}{F_\pi}\right)\frac{\epsilon_\pi}{2}\partial_{\epsilon_\pi}.
\end{aligned} \tag{3.17}
$$

We can use Eq. (3.3) to take the derivatives

$$
\begin{aligned}
\hat{m}\partial_{\hat{m}} m_\pi^2 &= M^2 \left\{ 1 + x\left(\ln\frac{M^2}{\mu^2} + 4\bar{l}_3 + \frac{1}{2}\right) + x^2\left(\frac{51}{8}\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}\left(3c_{1M}^r + \frac{17}{4}\right) + 3c_{2M}^r + c_{1M}^r\right)\right\} \\
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{2m_\pi^2} &= \frac{1}{2}\Bigg\{ 1 + \frac{x}{2}\left(\ln\frac{M^2}{\mu^2} + 4\bar{l}_3^r + 1\right) \\
&\qquad + x^2\left(4\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}(2c_{1M}^r + 4 - 2\bar{l}_3^r) + 2c_{2M}^r + c_{1M}^r - \bar{l}_3^r(1 + 4\bar{l}_3^r)\right)\Bigg\} \\
\hat{m}\partial_{\hat{m}} F_\pi &= F\left\{ x\left(\bar{l}_4^r - 1 - \ln\frac{M^2}{\mu^2}\right) + x^2\left(-\frac{5}{2}\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}\left(2c_{1F}^r - \frac{5}{2}\right) + 2c_{2F}^r + c_{1F}^r\right)\right\} \\
\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= x\left(\bar{l}_4^r - 1 - \ln\frac{M^2}{\mu^2}\right) \\
&\quad + x^2\left(-\frac{7}{2}\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}\left(2c_{1F}^r + 2\bar{l}_4^r - \frac{7}{2}\right) + 2c_{2F}^r + c_{1F}^r - \bar{l}_4^r(\bar{l}_4^r - 1)\right)
\end{aligned} \tag{3.18}
$$

and this brings us to our derivative conversion factor

$$
\begin{aligned}
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= 1 + x\left(\frac{5}{2}\ln\frac{M^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + x^2\Bigg(11\ln^2\frac{M^2}{\mu^2} + \ln\frac{M^2}{\mu^2}\left(2c_{1M}^r - 4c_{1F}^r + 11 - 2\bar{l}_3^r - 4\bar{l}_4^r\right) \\
&\qquad + 2c_{2M}^r + c_{1M}^r - 4c_{2F}^r - 2c_{1F}^r - \bar{l}_3^r(1 + 4\bar{l}_3^r) + 2\bar{l}_4^r(\bar{l}_4^r - 1)\Bigg).
\end{aligned} \tag{3.19}
$$

Now, we want to do the $x \to \xi$ conversion to put this in terms of our lattice quantities

$$
\begin{aligned}
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= 1 + \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\frac{m_\pi^2}{\mu^2} + \ln\frac{m_\pi^2}{\mu^2}\left(2c_{1M}^r - 4c_{1F}^r - 12\bar{l}_3^r + 6\bar{l}_4^r + \frac{7}{2}\right) \\
&\qquad + 2c_{2M}^r + c_{1M}^r - 4c_{2F}^r - 2c_{1F}^r - 11\bar{l}_3^r + 3\bar{l}_4^r - 2(2\bar{l}_3^r - \bar{l}_4^r)^2\Bigg) \\
&= 1 + \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\frac{m_\pi^2}{\mu^2} - \ln\frac{m_\pi^2}{\mu^2}\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 7\right) \\
&\qquad + 2\hat{c}_{2M}^r - 4(\hat{c}_{1F}^r + \hat{c}_{2F}^r) + 8\bar{l}_3^r(\bar{l}_3^r - \bar{l}_4^r) + 2\bar{l}_4^r(3\bar{l}_4^r - 16) + \frac{7}{4}\Bigg). \\
&= 1 + \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\frac{m_\pi^2}{\mu^2} - \ln\frac{m_\pi^2}{\mu^2}\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 7\right) \\
&\qquad + 2\hat{\hat{c}}_{2M}^r - 4(\hat{c}_{1F}^r + \hat{c}_{2F}^r) + \bar{l}_4^r(6\bar{l}_4^r - 8\bar{l}_3^r - 16) + \frac{7}{4}\Bigg).
\end{aligned} \tag{3.20}
$$

As with our F_K/F_π analysis [10], we can change work with $\mu = 4\pi F$ and then replace

$$\begin{aligned}\ln\frac{m_\pi^2}{\mu^2} &= \ln\left(\xi\frac{F_\pi^2}{F^2}\right)\\ &= \ln(\xi) + 2\xi(\bar{l}_4^r - \ln\xi) + \mathrm{O}(\xi^2)\end{aligned} \tag{3.21}$$

in Eq. (3.20) to arrive at

$$\begin{aligned}\frac{\hat{m}\partial_{\hat{m}}m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}}F_\pi}{F_\pi} &= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right)\\ &\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\xi + \ln\xi\left(2c_{1M}^r - 4c_{1F}^r - 12\bar{l}_3^r + 6\bar{l}_4^r - \frac{3}{2}\right)\\ &\qquad + 2c_{2M}^r + c_{1M}^r - 4c_{2F}^r - 2c_{1F}^r - 11\bar{l}_3^r + 8\bar{l}_4^r - 2(2\bar{l}_3^r - \bar{l}_4^r)^2\Bigg)\\ &= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right)\\ &\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\xi - \ln\xi\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 2\right)\\ &\qquad + 2\hat{c}_{2M}^r - 4(\hat{c}_{2F}^r + \hat{c}_{1F}^r) + 8\bar{l}_3^r(\bar{l}_3^r - \bar{l}_4^r) + \bar{l}_4^r(6\bar{l}_4^r - 11) + \frac{7}{4}\Bigg)\\ &= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right)\\ &\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\xi - \ln\xi\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 2\right)\\ &\qquad + 2\hat{\hat{c}}_{2M}^r - 4(\hat{c}_{2F}^r + \hat{c}_{1F}^r) + \bar{l}_4^r(6\bar{l}_4^r - 8\bar{l}_3^r - 11) + \frac{7}{4}\Bigg)\end{aligned} \tag{3.22}$$

In these expressions, we can estimate $\bar{l}_3^r$ by converting it to $\bar{\ell}_3$ and taking the value from FLAG [9]. All other LECs except for $\hat{c}_{2M}^r$ will be determined in the fit to F_π.

As an alternative, in Ref. [11], it was observed that the NNLO corrections to M_π^2 were numerically very small (see Fig. 7). Therefore, we can try an analysis where we truncate the contribution to the derivative conversion factor using only the NLO contributions to M_π^2. With such a truncation, we have

$$
\begin{aligned}
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= 1 + \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{3}{4}\ln^2\frac{m_\pi^2}{\mu^2} - \ln\frac{m_\pi^2}{\mu^2}\left(4c_{1F}^r + 10\bar{l}_3^r - 6\bar{l}_4^r + \frac{1}{2}\right) \\
&\qquad - 2c_{1F}^r - 4c_{2F}^r - 10\bar{l}_3^r + 3\bar{l}_4^r - 4(\bar{l}_3^r)^2 + 8\bar{l}_3^r\bar{l}_4^r - 2(\bar{l}_4^r)^2\Bigg) \\
&= 1 + \xi\left(\frac{5}{2}\ln\frac{m_\pi^2}{\mu^2} + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{3}{4}\ln^2\frac{m_\pi^2}{\mu^2} - \ln\frac{m_\pi^2}{\mu^2}\left(4\hat{c}_{1F}^r + 2\bar{l}_3^r + 12\bar{l}_4^r - \frac{3}{2}\right) \\
&\qquad - 2\hat{c}_{1F}^r - 4\hat{c}_{2F}^r + 2\bar{l}_3^r - 6\bar{l}_4^r - 4(\bar{l}_3^r)^2 + 6(\bar{l}_4^r)^2 + 1\Bigg) \\
&= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{3}{4}\ln^2\xi - \ln\xi\left(4\hat{c}_{1F}^r + 2\bar{l}_3^r + 12\bar{l}_4^r + \frac{7}{2}\right) \\
&\qquad - 2\hat{c}_{1F}^r - 4\hat{c}_{2F}^r + 2\bar{l}_3^r - \bar{l}_4^r - 4(\bar{l}_3^r)^2 + 6(\bar{l}_4^r)^2 + 1\Bigg)
\end{aligned} \tag{3.23}
$$

3.0.2.1 Summary of conversion factor

The infinite volume, continuum limit formula for F_π used in the analysis is

$$
F_\pi = F\left\{1 + \xi(-\ln\xi + \bar{l}_4^r) + \xi^2\left(\frac{5}{4}\ln^2\xi + (\hat{c}_{1F}^r + 2)\ln\xi + \hat{c}_{2F} - 2\bar{l}_4^r\right)\right\}. \tag{3.24}
$$

This begins from Eq. (3.13) and then utilizes $\mu = 4\pi F_\pi$ with suitable modifications at NNLO. We then have 3 estimates for the derivative conversion factor under the three options:

1. Use Eq. (3.13) for M_π^2;

$$
\begin{aligned}
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\xi - \ln\xi\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 2\right) \\
&\qquad + 2\hat{c}_{2M}^r - 4(\hat{c}_{2F}^r + \hat{c}_{1F}^r) + 8\bar{l}_3^r(\bar{l}_3^r - \bar{l}_4^r) + \bar{l}_4^r(6\bar{l}_4^r - 11) + \frac{7}{4}\Bigg);
\end{aligned} \tag{3.25}
$$

2. Use Eq. (3.16) for M_π^2

$$
\begin{aligned}
\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} &= 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) \\
&\quad + \xi^2\Bigg(\frac{19}{4}\ln^2\xi - \ln\xi\left(8\hat{c}_{1F}^r + 10\bar{l}_3^r + 32\bar{l}_4^r - 2\right) \\
&\qquad + 2\hat{\hat{c}}_{2M}^r - 4(\hat{c}_{2F}^r + \hat{c}_{1F}^r) + \bar{l}_4^r(6\bar{l}_4^r - 8\bar{l}_3^r - 11) + \frac{7}{4}\Bigg);
\end{aligned} \tag{3.26}
$$

3. Truncate the M_π^2 expansion at NLO

$$\frac{\hat{m}\partial_{\hat{m}} m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}} F_\pi}{F_\pi} = 1 + \xi\left(\frac{5}{2}\ln\xi + 2\bar{l}_3^r - 2\bar{l}_4^r + \frac{5}{2}\right) + \xi^2\Bigg(\frac{3}{4}\ln^2\xi - \ln\xi\left(4\hat{c}_{1F}^r + 2\bar{l}_3^r + 12\bar{l}_4^r + \frac{7}{2}\right) - 2\hat{c}_{1F}^r - 4\hat{c}_{2F}^r + 2\bar{l}_3^r - \bar{l}_4^r - 4(\bar{l}_3^r)^2 + 6(\bar{l}_4^r)^2 + 1\Bigg). \quad (3.27)$$

3.0.2.2 Cross check of derivative using LECs determined in analysis

$$\begin{aligned} M_N &= M_0 + \Lambda_\chi\epsilon_\pi^2\beta - \Lambda_\chi\frac{3\pi\mathring{g}_A^2}{2}\epsilon_\pi^3 + \Lambda_\chi\epsilon_\pi^4\delta M_N^{\rm NNLO} \\ \frac{M_N}{\Lambda_\chi} &= \frac{M_0}{4\pi F(1+\epsilon_\pi^2(\bar{l}_4^r - \ln\epsilon_\pi^2) + \epsilon_\pi^4\delta_F^{\rm NNLO})} + \beta\epsilon_\pi^2 - \frac{3\pi\mathring{g}_A^2}{2}\epsilon_\pi^3 + \epsilon_\pi^4\delta M_N^{\rm NNLO} \\ &= \frac{M_0}{4\pi F} + \epsilon_\pi^2(\beta - (\bar{l}_4^r - \ln\epsilon_\pi^2)) - \frac{3\pi\mathring{g}_A^2}{2}\epsilon_\pi^3 + \epsilon_\pi^4\left(\delta M_N^{\rm NNLO} + (\bar{l}_4^r - \ln\epsilon_\pi^2)^2 - \delta_F^{\rm NNLO}\right) \end{aligned} \quad (3.28)$$

3.0.3 Relating $\Lambda_\chi\partial_{\epsilon_\pi}(M_N/\Lambda_\chi)$ to $\hat{m}\partial_{\hat{m}}M_N$

A convenient derivative to take with LQCD data is

$$\Lambda_\chi\frac{\partial}{\partial\epsilon_\pi}\frac{M_N}{\Lambda_\chi}\,. \quad (3.29)$$

To relate this to the derivative of interest, let us begin with

$$\Lambda_\chi\hat{m}\partial_{\hat{m}}\left(\frac{M_N}{\Lambda_\chi}\right) = \Lambda_\chi\left(\frac{\hat{m}\partial_{\hat{m}}M_N}{\Lambda_\chi} - \frac{M_N}{\Lambda_\chi^2}\hat{m}\partial_{\hat{m}}\Lambda_\chi\right) \quad (3.30)$$

which allows us to express the quark mass derivative to one in terms of dimensionless hadronic quantities

$$\begin{aligned} \hat{m}\partial_{\hat{m}}M_N &= \Lambda_\chi\hat{m}\partial_{\hat{m}}\left(\frac{M_N}{\Lambda_\chi}\right) + \frac{M_N}{\Lambda_\chi}\hat{m}\partial_{\hat{m}}\Lambda_\chi \\ &= \left(\frac{\hat{m}\partial_{\hat{m}}m_\pi^2}{m_\pi^2} - 2\frac{\hat{m}\partial_{\hat{m}}F_\pi}{F_\pi}\right)\frac{\epsilon_\pi}{2}\left[\Lambda_\chi\partial_{\epsilon_\pi}\left(\frac{M_N}{\Lambda_\chi}\right) + M_N\frac{\partial_{\epsilon_\pi}\Lambda_\chi}{\Lambda_\chi}\right]. \end{aligned} \quad (3.31)$$

The conversion (first) term is given by Eq. (3.20). The first derivative we obtain from our fit to M_N/Λ_χ and the last term can easily be determined from our fit to F_π using Eq. (3.13)

$$\frac{\epsilon_\pi}{2}\frac{\partial_{\epsilon_\pi}\Lambda_\chi}{\Lambda_\chi} = \epsilon_\pi^2\left(\bar{l}_4^r - \ln\frac{m_\pi^2}{\mu^2} - 1\right) + \epsilon_\pi^4\left(\frac{3}{2}\ln^2\frac{m_\pi^2}{\mu^2} + \ln\frac{m_\pi^2}{\mu^2}\left(2\hat{c}_{1F}^r + 2\bar{l}_4^r + \frac{3}{2}\right) + 2\hat{c}_{2F}^r + \hat{c}_{1F}^r - \bar{l}_4^r(\bar{l}_4^r - 1)\right) \quad (3.32)$$

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3.0.4 Relating $\hat{m}\partial_{\hat{m}}$ to $M_\pi^2\partial_{M_\pi^2}$

To use the pion mass dependence, we have to know the higher order corrections to relate the pion mass to the quark mass:

$$\hat{m}\frac{\partial}{\partial\hat{m}} = \hat{m}\left(\frac{\partial M_\pi^2}{\partial\hat{m}}\right)\frac{\partial}{\partial M_\pi^2}\,. \tag{3.33}$$

The pion mass is given by

$$M_\pi^2 = 2B\hat{m}\,(1+\delta_{M^2}) \tag{3.34}$$

where δ_M^2 is the infinite tower of higher order corrections in terms of the quark mass. At NLO, we have

$$\begin{aligned}
\delta_{M^2}^{\text{NLO}} &= \frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}\left[\ln\left(\frac{2B\hat{m}}{\mu^2}\right)+4\bar{l}_3^r(\mu)\right]\\
&= -\frac{1}{2}\bar{l}_3\frac{2B\hat{m}}{(4\pi F)^2}\,,\\
\hat{m}\frac{\partial\delta_{M^2}}{\partial\hat{m}} &= \delta_{M^2}+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}\\
&= \frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}(1-\bar{l}_3)
\end{aligned} \tag{3.35}$$

The renormalized LECs are related to the "bar" LECs through the relation [6]

$$\bar{l}_i^r(\mu) = (4\pi)^2 l_i^r(\mu) = \frac{\gamma_i}{2}\left[\bar{l}_i+\ln\left(\frac{M^2}{\mu^2}\right)\right]\,, \tag{3.36}$$

with

$$\gamma_1 = \frac{1}{3},\qquad \gamma_2 = \frac{2}{3},\qquad \gamma_3 = -\frac{1}{2},\qquad \gamma_4 = 2, \tag{3.37}$$

In general, we have

$$\begin{aligned}
\hat{m}\frac{\partial M_\pi^2}{\partial\hat{m}} &= \hat{m}\left\{2B(1+\delta_{M^2})+2B\hat{m}\frac{\partial\delta_{M^2}}{\partial\hat{m}}\right\}\\
&= M_\pi^2+2B\hat{m}\left(\delta_{M^2}+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}\right)\\
&= M_\pi^2\left[1+\delta_{M^2}+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}+\text{NNLO}\right]\\
&= M_\pi^2\left[1+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}(1-\bar{l}_3)+\text{NNLO}\right]
\end{aligned} \tag{3.38}$$

Therefore, we can express the sigma term as

$$\begin{aligned}
\hat{m}\partial_{\hat{m}}M_N &= \left[1+\delta_{M^2}+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}+\text{NNLO}\right]M_\pi^2\partial_{M_\pi^2}M_N\\
&= \left[1+\delta_{M^2}+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}+\text{NNLO}\right]\frac{M_\pi}{2}\frac{\partial M_N}{\partial M_\pi}\\
&= \left[1+\frac{1}{2}\frac{M_\pi^2}{(4\pi F_\pi)^2}\left(\ln\left(\frac{M_\pi^2}{\mu^2}\right)+4\bar{l}_3^r(\mu)+1\right)+\text{NNLO}\right]\frac{M_\pi}{2}\frac{\partial M_N}{\partial M_\pi}\\
&= \left[1+\frac{1}{2}\frac{M_\pi^2}{(4\pi F_\pi)^2}\left(1-\bar{l}_3\right)+\text{NNLO}\right]\frac{M_\pi}{2}\frac{\partial M_N}{\partial M_\pi}
\end{aligned} \tag{3.39}$$

3.0.5 Relating $\hat{m}\partial_{\hat{m}} M_N$ to $\epsilon_\pi \partial_{\epsilon_\pi} M_N$

An even more convenient expression is to differentiate with respect to $\epsilon_\pi = M_\pi/(4\pi F_\pi)$ for which we also need to track the quark mass dependence of F_π. For F_π we have

$$F_\pi = F\left(1+\delta_F\right)\,, \tag{3.40}$$

and at NLO,

$$\begin{aligned}
\delta_F^{\rm NLO} &= \frac{2B\hat{m}}{(4\pi F)^2}\left[-\ln\left(\frac{2B\hat{m}}{\mu^2}\right)+\bar{l}_4^r(\mu)\right] \\
&= \frac{2B\hat{m}}{(4\pi F)^2}\bar{l}_4\,,
\end{aligned} \tag{3.41}$$

such that

$$\begin{aligned}
\hat{m}\partial_{\hat{m}}\delta_F &= \delta_F - \frac{2B\hat{m}}{(4\pi F)^2} \\
&= \frac{2B\hat{m}}{(4\pi F)^2}(\bar{l}_4 - 1)\,.
\end{aligned} \tag{3.42}$$

The derivative is given by

$$\hat{m}\partial_{\hat{m}} = \hat{m}\frac{\partial \epsilon_\pi}{\partial \hat{m}}\partial_{\epsilon_\pi}\,. \tag{3.43}$$

With

$$\epsilon_\pi = \frac{m_\pi}{4\pi F_\pi} = \frac{\sqrt{2B\hat{m}(1+\delta_{M^2})}}{4\pi F(1+\delta_F)} \tag{3.44}$$

we have

$$\begin{aligned}
\hat{m}\partial_{\hat{m}}\epsilon_\pi &= \frac{1}{2M_\pi}\frac{M_\pi^2 + 2B\hat{m}\,\hat{m}\partial_{\hat{m}}\delta_{M^2}}{4\pi F_\pi} - \frac{M_\pi}{(4\pi F_\pi)^2}4\pi F\,\hat{m}\partial_{\hat{m}}\delta_F \\
&= \frac{\epsilon_\pi}{2}\left[1+\hat{m}\partial_{\hat{m}}\delta_{M^2} - 2\frac{\hat{m}\partial_{\hat{m}}\delta_F}{F_\pi/F}\right] \\
&= \frac{\epsilon_\pi}{2}\left[1+\hat{m}\partial_{\hat{m}}\delta_{M^2} - 2\hat{m}\partial_{\hat{m}}\delta_F + \text{NNLO}\right] \\
&= \frac{\epsilon_\pi}{2}\left\{1+\frac{1}{2}\frac{2B\hat{m}}{(4\pi F)^2}\left[\ln\left(\frac{2B\hat{m}}{\mu^2}\right)+4\bar{l}_3^r(\mu)+1\right] - 2\frac{2B\hat{m}}{(4\pi F)^2}\left[-\ln\left(\frac{2B\hat{m}}{\mu^2}\right)+\bar{l}_4^r(\mu)-1\right]\right\} \\
&= \frac{\epsilon_\pi}{2}\left\{1+\frac{2B\hat{m}}{(4\pi F)^2}\left[\frac{5}{2}\ln\left(\frac{2B\hat{m}}{\mu^2}\right)+\frac{5}{2}+2(\bar{l}_3^r-\bar{l}_4^r)\right]\right\} \\
&= \frac{\epsilon_\pi}{2}\left\{1+\epsilon_\pi^2\left[\frac{5}{2}(\ln\epsilon_\pi^2+1)+2\left(\bar{l}_3^r(\Lambda_\chi)-\bar{l}_4^r(\Lambda_\chi)\right)\right]\right\} = \frac{\epsilon_\pi}{2}\left\{1+\epsilon_\pi^2\left[\frac{5}{2}-\frac{1}{2}\bar{l}_3-2\bar{l}_4\right]\right\}\,,
\end{aligned} \tag{3.45}$$

and this leaves us with the sigma term

$$\begin{aligned}
\hat{m}\partial_{\hat{m}}M_N &= \left\{1+\epsilon_\pi^2\left[\frac{5}{2}\left(\ln\epsilon_\pi^2+1\right)+2\left(\bar{l}_3^r-\bar{l}_4^r\right)\right]\right\}\frac{\epsilon_\pi}{2}\frac{\partial M_N}{\partial \epsilon_\pi} \\
&= \left\{1+\epsilon_\pi^2\left[\frac{5}{2}-\frac{1}{2}\bar{l}_3-2\bar{l}_4\right]\right\}\frac{\epsilon_\pi}{2}\frac{\partial M_N}{\partial \epsilon_\pi}\,,
\end{aligned} \tag{3.46}$$

where in the first line, the LECs are understood to be

$$\bar{l}_i^r = (4\pi)^2 l_i^r(\mu = 4\pi F_\pi)\,. \tag{3.47}$$

Chapter 4

Finite Volume Corrections

Lattice QCD calculations are performed in a finite volume and so loop integrals get replaced with loop-sums. When we consider these corrections, we work in the approximation that $T \to \infty$ and only the spatial volume is held finite.

When nothing in the integrand can go on-shell, the convenient identity to use to determine the finite volume corrections is the Poisson Summation Formula

$$\sum_{\vec{n}} \delta^3(\vec{n} - \vec{y}) = \sum_{\vec{m}} e^{2\pi i \vec{m}\cdot\vec{y}} \,. \tag{4.1}$$

4.0.0.1 Tadpole integral

Consider the tadpole integral in finite volume

$$\begin{aligned} i\mathcal{I}(m) &= \sum\!\!\!\!\!\!\int \frac{i}{k^2 - m^2 + i\epsilon} \\ &= \sum\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{i}{(k_0 - \omega_k)(k_0 + \omega_k)} \\ &= \sum\!\!\!\!\!\!\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \,, \end{aligned} \tag{4.2}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2 - i\epsilon}$ and we closed the contour on the upper-half plane. Let us start by considering the finite volume integral for which

$$\int \frac{d^3k}{(2\pi)^3} \longrightarrow \frac{1}{L^3} \sum_{\vec{k}} \,, \tag{4.3}$$

giving us

$$
\begin{aligned}
i\mathcal{I}^{\text{FV}}(m) &= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{k}}\frac{1}{\sqrt{\vec{k}^2+m^2}} \\
&= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{n}}\frac{1}{\frac{2\pi}{L}\sqrt{\vec{n}^2+\left(\frac{mL}{2\pi}\right)^2}} \\
&= \frac{1}{4\pi L^2}\sum_{\vec{n}}\frac{1}{\sqrt{\vec{n}^2+\left(\frac{mL}{2\pi}\right)^2}} = \frac{1}{4\pi L^2\sqrt{\left(\frac{mL}{2\pi}\right)^2}} + \sum_{\vec{n}\neq 1}\frac{1}{\sqrt{\vec{n}^2+\left(\frac{mL}{2\pi}\right)^2}} \\
&= \frac{1}{4\pi L^2}\sum_{\vec{n}}\int d^3y\frac{\delta^3(\vec{y}-\vec{n})}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} \\
&= \frac{1}{4\pi L^2}\int d^3y\frac{1}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\sum_{\vec{n}}\delta^3(\vec{y}-\vec{n}) \qquad \text{use Poisson Summation Formula, Eq. (4.1)} \\
&= \frac{1}{4\pi L^2}\int d^3y\frac{1}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\sum_{\vec{n}}e^{2\pi i\vec{n}\cdot\vec{y}} \\
&= \frac{1}{4\pi L^2}\sum_{\vec{n}}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\,.
\end{aligned}
\tag{4.4}
$$

What we are really interested in is the difference between the finite and infinite volume integrals. Notice, the $\vec{n}=\vec{0}$ contribution to $i\mathcal{I}^{\text{FV}}(m)$ is the infinite volume expression (which has the UV divergences). Therefore, we can construct

$$
\begin{aligned}
i\delta\mathcal{I}(m) &\equiv i\mathcal{I}^{\text{FV}}(m) - i\mathcal{I}(m) \\
&= \frac{1}{4\pi L^2}\sum_{\vec{n}\neq 0}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}}\,,
\end{aligned}
\tag{4.5}
$$

which is the FV correction we are interested in and it is free of UV divergences. Now let us evaluate the integral

$$
\begin{aligned}
\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} &= \int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta\int_0^\infty dy y^2\frac{e^{2\pi i n y\cos\theta}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} \\
&= \frac{2}{n}\int_0^\infty dy\frac{y\sin(2\pi n y)}{\sqrt{y^2+\left(\frac{mL}{2\pi}\right)^2}}
\end{aligned}
\tag{4.6}
$$

Gradshteyn and Ryzhik, 7^{th} edition, Section 3.771 (pg 442), Eq. (5), has the relation

$$
\int_0^\infty dy\frac{y\,\sin(ay)}{(y^2+\beta^2)^{1/2-\nu}} = \sqrt{\pi}\beta\left(\frac{2\beta}{a}\right)^\nu\frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)}K_{\nu+1}(a\beta)\,, \qquad a>0,\ \text{Re}\ \beta>0,\ \text{Re}\ \nu<0\,. \tag{4.7}
$$

For us, this yields

$$
\begin{aligned}
\frac{1}{4\pi L^2}\int d^3y\frac{e^{2\pi i\vec{n}\cdot\vec{y}}}{\sqrt{\vec{y}^2+\left(\frac{mL}{2\pi}\right)^2}} &= \frac{1}{4\pi L^2}\frac{2}{n}\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)}\frac{mL}{2\pi}K_1(nmL) \qquad [n=|\vec{n}|] \\
&= \frac{m^2}{(4\pi)^2}4\frac{K_1(nmL)}{nmL}
\end{aligned}
\tag{4.8}
$$

and so the finite volume correction to the tadpole integral is given by

$$i\delta\mathcal{I}(m) = \frac{m^2}{(4\pi)^2}\sum_{\vec{n}\neq 0} 4\frac{K_1(|\vec{n}|mL)}{|\vec{n}|mL}\,. \tag{4.9}$$

After using dim-reg, we can then express the finite volume tadpole integral as

$$\begin{aligned} i\mathcal{I}^{\text{FV}}_{\overline{\text{MS}}}(m,\mu) &= i\mathcal{I}(m) + i\delta\mathcal{I}(m) \\ &= \frac{m^2}{(4\pi)^2}\left[\ln\left(\frac{m^2}{\mu^2}\right) + \sum_{\vec{n}\neq 0} 4\frac{K_1(|\vec{n}|mL)}{|\vec{n}|mL}\right]. \end{aligned} \tag{4.10}$$

4.0.0.2 Leading Heavy Baryon Mass correction

Both loop integrals in Fig. 2.1 will be proportional to the integral

$$\mathcal{F}_{\mu\nu}(m,p,\Delta) = \int_R \frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2 - m^2 + i\epsilon][(p+k)\cdot v - \Delta + i\epsilon]}\,, \tag{4.11}$$

with the nucleon loop evaluated with $\Delta = 0$ and we denoted a regularization/renormalization scheme R which we almost always take to be dim-reg in modified $\overline{\text{MS}}$. In the literature, the FV correction to this integral is evaluated after first introducing the λ parameterization to combine the denominators of the integrand. We repeat this derivation in Sec. 4.0.0.3. In this section, we attempt to determine these FV corrections without introducing the λ parameterization.

The infinite volume self-energy with a nucleon-pion virtual loop is given by Eq. (2.18)

$$\begin{aligned} \delta\Sigma &= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^\mu S^\nu F_{\mu\nu}(m_\phi, \Delta = 0) \\ &= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^\mu S^\nu \frac{2}{3}\frac{g_{\mu\nu}\pi m_\phi^3}{(4\pi)^2}\,. \end{aligned}$$

A similar correction will arise with an intermediate $\Delta\pi$ virtual state with $S^\mu S^\nu$ replaced by a Δ spin-projector. In both cases, the 0^{th} components will be zero as $v\cdot S = v_\mu P^{\mu\nu} = v_\nu P^{\mu\nu} = 0$. As with the infinite volume self-energy, at this order, we can expand the integrand for small external momentum around the point $p\cdot v = 0$ and thus, our integral can be replaced by

$$F_{\mu\nu}(m,p,\Delta) = \int_R \frac{d^4k}{(2\pi)^4}\left[\frac{ik_\mu k_\nu}{(k_0^2 - \omega_k^2 + i\epsilon)(k_0 - (\Delta - i\epsilon))} - \frac{ip\cdot v k_\mu k_\nu}{(k_0^2 - \omega_k^2 + i\epsilon)(k_0 - (\Delta - i\epsilon))^2}\right], \tag{4.12}$$

with $\omega_k^2 = \vec{k}^2 + m^2$.

Let us focus on the first integral, which we can replace with

$$F_{ij}(m,p=0,\Delta) = \int \frac{d^4k}{(2\pi)^4}\frac{i\vec{k}^2\delta_{ij}/3}{(k_0 + \omega_k - i\epsilon)(k_0 - \omega_k + i\epsilon)(k_0 - (\Delta - i\epsilon))}\,. \tag{4.13}$$

We can start with the k_0 integral, which can be evaluated with the Cauchy Integral Formula using either the upper or lower half-plane contour integral. Let us start by using the upper half plane, which picks up the single pole $k_0 = -\omega_k + i\epsilon$, and we will convert the three-dimensional integral to the FV sum,

$$\begin{aligned}
F_{ij}(m,\Delta,L) &= \frac{\delta_{ij}}{3}\frac{2\pi i}{2\pi}\frac{1}{L^3}\sum_{\vec{k}}\frac{i\vec{k}^2}{-2\omega_k(-\omega_k-\Delta)} \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{k}}\frac{\vec{k}^2}{\omega_k(\omega_k+\Delta)}\,, \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}}\frac{\left(\frac{2\pi}{L}\right)^2\vec{n}^2}{\left(\frac{2\pi}{L}\right)^2\sqrt{\vec{n}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{n}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}}\int d^3y\frac{\vec{y}^2\delta^3(\vec{y}-\vec{n})}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\int d^3y\frac{\vec{y}^2}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)}\sum_{\vec{n}}e^{i2\pi\vec{n}\cdot\vec{y}}\,, \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}}\int d^3y\frac{y^2e^{i2\pi ny\cos\theta}}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}}2\pi\int_0^\infty dy\frac{y^4(e^{i2\pi ny}-e^{-i2\pi ny})}{i2\pi ny\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)} \\
&= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}}\frac{2}{n}\int_0^\infty dy\frac{y^3\sin(2\pi ny)}{\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}\left(\sqrt{\vec{y}^2+(\frac{mL}{2\pi})^2}+\frac{\Delta L}{2\pi}\right)}\,,
\end{aligned} \tag{4.14}$$

where $n = |\vec{n}|$. Let us start with the $\Delta = 0$ term from a virtual $N\pi$ loop. In this case, the integral is easy to evaluate using Cauchy Contour Integrals. Also, recall, we are interested in the difference between finite and infinite volume, and the $\vec{n} = 0$ term corresponds to the infinite volume integral. Therefore, the FV correction will be proportional to

$$\begin{aligned}
F_{ij}(m,\Delta=0,L) &= \frac{-\delta_{ij}}{6}\frac{1}{L^3}\sum_{\vec{n}\neq 0}\frac{2}{n}\int_0^\infty dy\frac{y^3\sin(2\pi ny)}{y^2+(\frac{mL}{2\pi})^2} \\
&= -\frac{\delta_{ij}}{3L^3}\sum_{\vec{n}\neq 0}\frac{1}{n}\int_0^\infty dy\frac{y^3(e^{i2\pi ny}-e^{-i2\pi ny})}{2i(y+i(\frac{mL}{2\pi}))(y-i(\frac{mL}{2\pi}))} \\
&= -\frac{\delta_{ij}}{3L^3}\sum_{\vec{n}\neq 0}\frac{1}{n}\frac{1}{2}\int_{-\infty}^\infty dy\frac{y^3(e^{i2\pi ny}-e^{-i2\pi ny})}{2i(y+i(\frac{mL}{2\pi}))(y-i(\frac{mL}{2\pi}))}\,.
\end{aligned} \tag{4.15}$$

For the first term, for the infinite arc to vanish, we want to use the upper half plane, and for the second term, the lower-half plane, such that

$$
\begin{aligned}
F_{ij}(m,\Delta=0,L) &= -\frac{\delta_{ij}}{6L^3}\sum_{\vec{n}\neq 0}\frac{1}{n}\frac{2\pi i}{2i}\left[\left.\frac{y^3 e^{i2\pi ny}}{y+i(\frac{mL}{2\pi})}\right|_{y=i\frac{mL}{2\pi}} - \left.\frac{-y^3 e^{-i2\pi ny}}{y-i(\frac{mL}{2\pi})}\right|_{y=-i\frac{mL}{2\pi}}\right] \\
&= -\frac{\delta_{ij}}{6L^3}\sum_{\vec{n}\neq 0}\frac{\pi}{n}(-)\left(\frac{mL}{2\pi}\right)^2 e^{-nmL} \\
&= \frac{\delta_{ij}}{6}\frac{4\pi m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\frac{e^{-nmL}}{nmL} \\
&= \frac{\delta_{ij}}{6}\frac{4\pi m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\sqrt{\frac{2}{\pi}}\frac{K_{1/2}(nmL)}{\sqrt{nmL}}\,. \qquad (4.16)
\end{aligned}
$$

Comparing to Eq. (2.18), we see the FV corrections are given by

$$
\begin{aligned}
\delta_L\delta\Sigma^{\text{nlo}} &= \delta\Sigma^{\text{nlo}}(m,L) - \delta\Sigma^{\text{nlo}}(m,\infty) \\
&= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 S^i S^j \frac{2\delta_{ij}}{3}\frac{\pi m_\phi^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\frac{e^{-nm_\phi L}}{nm_\phi L} \\
&= \left(\frac{\mathring{g}_A C_{NN\phi}}{F}\right)^2 \frac{\pi m_\phi^3}{2(4\pi)^2}\sum_{\vec{n}\neq 0}\frac{e^{-nm_\phi L}}{nm_\phi L}\,. \qquad (4.17)
\end{aligned}
$$

Performing the flavor contractions and working in the isospin limit, we have

$$
\delta_L\delta\Sigma^{\text{nlo}} = \frac{3\pi\mathring{g}_A^2}{2}\frac{m_\pi^3}{(4\pi F)^2}\sum_{\vec{n}\neq 0}\frac{e^{-nm_\pi L}}{nm_\pi L}\,, \qquad (4.18)
$$

which agrees with Eqs. (15-16) of Ref. [12]. Combining with the infinite volume self-energy correction at this order, Eq. (2.18), we find

$$
\begin{aligned}
\delta M_N^{\text{NLO}} &= -\frac{3\pi\mathring{g}_A^2}{2}\frac{m_\pi^3}{(4\pi F)^2}\left[1-\sum_{\vec{n}\neq 0}\frac{e^{-nm_\pi L}}{nm_\pi L}\right] \\
&= -\frac{3\pi g_A^2}{2}\Lambda_\chi\epsilon_\pi^3\left[1-\sum_{\vec{n}\neq 0}\frac{e^{-nm_\pi L}}{nm_\pi L}\right] + \text{O}(\Lambda_\chi\epsilon_\pi^5) \\
&= -\frac{3\pi g_A^2}{2}\Lambda_\chi\epsilon_\pi^3\left[1-\sum_{\vec{n}\neq 0}\sqrt{\frac{2}{\pi}}\frac{K_{1/2}(nm_\pi L)}{\sqrt{nm_\pi L}}\right] + \text{O}(\Lambda_\chi\epsilon_\pi^5)\,, \qquad (4.19)
\end{aligned}
$$

with $\epsilon_\pi = m_\pi/(4\pi F_\pi)$, $\Lambda_\chi = 4\pi F_\pi$ and $n = |\vec{n}|$.

Question: Can we follow this strategy of not introducing the λ parameter integral and evaluate the FV corrections from the virtual $\Delta\pi$ loop, $F_{ij}(m,\Delta,L)$ =?

4.0.0.3 Heavy Baryon Integral as in the literature

Both loop integrals in Fig. 2.1 will be proportional to the integral

$$
\mathcal{F}_{\mu\nu}(m,p,\Delta) = \int_R \frac{d^4k}{(2\pi)^4}\frac{ik_\mu k_\nu}{[k^2-m^2+i\epsilon][(p+k)\cdot v-\Delta+i\epsilon]} \qquad (4.20)
$$

With the nucleon loop evaluated with $\Delta = 0$ and we denoted a regularization/renormalization scheme R which we almost always take to be dim-reg in modified $\overline{\text{MS}}$. In infinite volume, we have

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(m,p,\Delta) &= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{ik_\mu k_\nu}{\left[k^2-m^2-2\lambda\Delta+2\lambda k\cdot v+2\lambda p\cdot v\right]^2} \\
&= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{i(l_\mu-\lambda v_\mu)(l_\nu-\lambda v_\nu)}{\left[l^2-\lambda^2-m^2-2\lambda\Delta+2\lambda p\cdot v\right]^2}\,.
\end{aligned} \tag{4.21}$$

The integrals will be contracted either with the nucleon spin-vectors S_μ or the delta propagator $\mathcal{P}_{\mu\nu}$ which are orthogonal to v_μ, $v\cdot S=0$, $v_\mu\mathcal{P}^{\mu\nu}=v_\nu\mathcal{P}^{\mu\nu}=0$. Further, to get the wave function renormalization, we can take the linear approximation of the integral with respect to $p\cdot v$, leaving us with

$$\mathcal{F}_{\mu\nu}(m,p,\Delta) = \mathcal{F}_{\mu\nu}(m,0,\Delta) + p\cdot v\,\mathcal{J}_{\mu\nu}(m,0,\Delta)\,, \tag{4.22}$$

$$\mathcal{F}_{\mu\nu}(m,0,\Delta) = 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{il_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^2} \tag{4.23}$$

$$\mathcal{J}_{\mu\nu}(m,0,\Delta) = -8\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{i\lambda l_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^3} \tag{4.24}$$

In infinte volume, we have

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(m,0,\Delta) &= \frac{-1}{(4\pi)^2}g_{\mu\nu}\left\{\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi+1\right]\left(\frac{2}{3}\Delta^3-\Delta m^2\right)+\frac{10}{9}\Delta^3-\frac{4}{3}\Delta m^2-\frac{2}{3}\bar{\mathcal{F}}(m,\Delta,\mu)-\frac{2}{3}\Delta^3\ln\left(\frac{4\Delta^2}{\mu^2}\right)\right\} \\
\bar{\mathcal{F}}(m,\Delta,\mu) &= (\Delta^2-m^2+i\epsilon)^{3/2}\ln\left(\frac{\Delta+\sqrt{\Delta^2-m^2+i\epsilon}}{\Delta-\sqrt{\Delta^2-m^2+i\epsilon}}\right)-\frac{3}{2}\Delta m^2\ln\left(\frac{m^2}{\mu^2}\right)-\Delta^3\ln\left(\frac{4\Delta^2}{m^2}\right)\,.
\end{aligned} \tag{4.25}$$

The $i\epsilon$ informs us how to take the limit of $m>\Delta$. The function $\bar{\mathcal{F}}$ is defined such that

$$\begin{aligned}
&\bar{\mathcal{F}}(m=0,\Delta,\mu) = 0 \\
&\bar{\mathcal{F}}(m,\Delta=0,\mu) = \pi m^3\,.
\end{aligned} \tag{4.26}$$

Now let us evaluate the finite volume correction to Eq. (4.23).

$$\begin{aligned}
\mathcal{F}_{\mu\nu}(m,0,\Delta) &= 2\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{il_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\Delta-m^2+i\epsilon\right]^2} \\
&= 2i\int_0^\infty d\lambda\ \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{\left[(l_0-\omega_l+i\epsilon)(l_0+\omega_l-i\epsilon)\right]^2}\,, \quad w_l^2=\vec{l}^2+\lambda^2+2\lambda\Delta+m^2 \\
&= 2i\int_0^\infty d\lambda\ \mu^{2\epsilon}\int_R \frac{d^3l}{(2\pi)^3}\frac{2\pi i}{2\pi}\frac{l_\mu l_\nu}{4\omega_l^3}\bigg|_{l_0=-\omega_l}\,.
\end{aligned} \tag{4.27}$$

This integral is contracted with either $S_\mu S_\nu$ or $\mathcal{P}_{\mu\nu}$, both of which satisfy $v\cdot S=0$ and $v_\mu\mathcal{P}^{\mu\nu}=0$. There are no external vectors to pick a preferred direction for l_μ and therefore, we can focus on the integral

$$
\begin{aligned}
&\frac{1}{L^3}\sum_{\vec{l}}\frac{\vec{l}^2}{[\vec{l}^2+\lambda^2+2\lambda\Delta+m^2]^{3/2}}\\
&=\frac{1}{L^3}\sum_{\vec{n}}\frac{(2\pi/L)^2\vec{n}^2}{(2\pi/L)^3[\vec{n}^2+\beta_L^2]^{3/2}}\,,\quad \beta_L=\frac{\beta L}{2\pi}\,,\quad \beta^2=\lambda^2+2\lambda\Delta+m^2\\
&=\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}}\frac{n^2}{[n^2+\beta_L^2]^{3/2}}\\
&=\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}}\int d^3y\frac{y^2e^{2\pi i n y\cos\theta}}{[y^2+\beta_L^2]^{3/2}}
\end{aligned}
\tag{4.28}
$$

Putting this back into our full expression, and determining the finite volume difference, we have

$$
\begin{aligned}
\delta F_{\mu\nu}(m,0,\Delta)&\propto-\frac{2}{4}\int_0^\infty d\lambda\frac{1}{2\pi}\frac{1}{L^2}\sum_{\vec{n}\neq0}\int d^3y\frac{y^2e^{2\pi i n y\cos\theta}}{[y^2+\beta_L^2]^{3/2}}\\
&=-\frac{1}{2}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\int_0^\infty dy\frac{y^4}{[y^2+\beta_L^2]^{3/2}}\frac{e^{2\pi i n y}-e^{-2\pi i n y}}{2\pi i n y}\\
&=-\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\int_0^\infty dy\frac{y^3\sin(2\pi n y)}{[y^2+\beta_L^2]^{3/2}}
\end{aligned}
\tag{4.29}
$$

Gradshteyn and Ryzhik, 7^{th} edition, Section 3.773 (pg 444), Eq. (3), has the relation

$$
\int_0^\infty dx\frac{x^{2m+1}\sin(ax)}{(\beta^2+x^2)^{n+1/2}}=\frac{(-1)^{m+1}\sqrt{\pi}}{2^n\beta^n\Gamma(n+\frac{1}{2})}\frac{d^{2m+1}}{da^{2m+1}}[aK_n(a\beta)]\,,\quad a>0,\mathrm{Re}\ \beta>0,-1\leq m\leq n
\tag{4.30}
$$

which brings us to

$$
\begin{aligned}
\delta\mathcal{F}_{\mu\nu}(m,0,\Delta)&\propto-\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{(-1)^2\sqrt{\pi}}{2\beta_L\Gamma(3/2)}\frac{d^3}{d(2\pi n)^3}[(2\pi n)K_1(2\pi n\beta_L)]\\
&=-\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{(-1)^2\sqrt{\pi}}{2\beta_L\Gamma(3/2)}\beta_L^2[K_1(nL\beta)-nL\beta K_0(nL\beta)]\\
&=-\frac{1}{2\pi}\int_0^\infty d\lambda\frac{1}{L^2}\sum_{\vec{n}\neq0}\frac{1}{|\vec{n}|}\frac{\beta L}{2\pi}[K_1(nL\beta)-nL\beta K_0(nL\beta)]\\
&=-\frac{4}{(4\pi)^2}\int_0^\infty d\lambda\sum_{\vec{n}\neq0}\frac{\beta^2}{nL\beta}[K_1(nL\beta)-nL\beta K_0(nL\beta)]\\
&=-\frac{4}{(4\pi)^2}\sum_{\vec{n}\neq0}\int_0^\infty d\lambda\ \beta^2\left[\frac{K_1(nL\beta)}{nL\beta}-K_0(nL\beta)\right].
\end{aligned}
\tag{4.31}
$$

Recall that $\beta^2=\lambda^2+2\lambda\Delta+m^2$ and so

$$
\begin{aligned}
2\beta d\beta&=2(\lambda+\Delta)d\lambda\\
d\lambda&=\frac{\beta d\beta}{\lambda+\Delta}\\
&=\frac{\beta d\beta}{\sqrt{\beta^2-m^2+\Delta^2}}
\end{aligned}
\tag{4.32}
$$

and so this leaves us with

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta) \propto -\frac{4}{(4\pi)^2}\sum_{\vec{n}\neq 0}\int_m^\infty d\beta\ \frac{\beta^3}{\sqrt{\beta^2-m^2+\Delta^2}}\left[\frac{K_1(nL\beta)}{nL\beta}-K_0(nL\beta)\right]. \tag{4.33}$$

Now, let us make the variable transformation

$$\begin{aligned} x &= \frac{\beta^2}{m^2} \\ dx &= \frac{2\beta d\beta}{m^2} \\ dx\frac{m^3 x}{2\sqrt{x-1+\delta^2}} &= \frac{d\beta\beta^3}{m\sqrt{\beta^2/m^2-1+\Delta^2/m^2}} \end{aligned} \tag{4.34}$$

which brings our correction to

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta) \propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\int_1^\infty dx\ \frac{x}{\sqrt{x-1+\delta^2}}\left[\frac{K_1(nmL\sqrt{x})}{nmL\sqrt{x}}-K_0(nmL\sqrt{x})\right]. \tag{4.35}$$

From here, the trick is to recognize

$$\frac{1}{a^2b}\frac{\partial}{\partial a}(a^2K_1(ab)) = \frac{K_1(ab)}{ab}-K_0(ab) \tag{4.36}$$

which lets us write

$$\delta\mathcal{F}_{\mu\nu}(m,0,\Delta) \propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\frac{1}{(nmL)^2}\frac{\partial}{\partial nmL}\left[(nmL)^2\int_1^\infty dx\ \frac{\sqrt{x}}{\sqrt{x-1+\delta^2}}K_1(nmL\sqrt{x})\right]. \tag{4.37}$$

For $\delta = 0$, we can use the relation $K_{-\nu}(z) = K_\nu(z)$ and Gradshteyn and Ryzhik, 7^{th} edition, Section 5.592 (pg 691), Eq. (12),

$$\int_1^\infty dx\frac{K_\nu(a\sqrt{x})}{x^{\nu/2}(x-1)^{1-\mu}} = \Gamma(\mu)2^\mu\frac{K_{\nu-\mu}(a)}{a^\mu}\,, \quad \text{Re } a>0, \text{Re } \mu>0\,, \tag{4.38}$$

which yields

$$\begin{aligned} \delta\mathcal{F}_{\mu\nu}(m,0,\Delta=0) &\propto -\frac{2m^3}{(4\pi)^2}\sum_{\vec{n}\neq 0}\sqrt{2\pi}\frac{1}{(nmL)^2}\frac{\partial}{\partial nmL}\left[(nmL)^2\frac{K_{3/2}(nmL)}{\sqrt{nmL}}\right] \\ &= +\frac{4m^3}{(4\pi)^2}\sqrt{\frac{\pi}{2}}\sum_{\vec{n}\neq 0}\frac{K_{1/2}(nmL)}{\sqrt{nmL}} \end{aligned} \tag{4.39}$$

which agrees with Eq. (4.26) up to the overall phase of i we started with differently in the new way.

Appendix A

Integrals

I. USEFUL INTEGRALS

A. Lorentz Integrals

$$\tilde{\mathcal{I}}^{(2n)}(\delta) \equiv \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^n}{k^2-m^2}$$
$$= \frac{i(-)^{n-1}}{(4\pi)^{d/2}}\frac{d(d+2)...(d+2(n-1))}{2^n}\Gamma\left(-2(n-1)-\frac{d}{2}\right)\delta^{2(n-1)-\epsilon} \tag{1}$$

$$\tilde{\mathcal{I}}^{(0)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{1-\epsilon} \tag{2}$$

$$\tilde{\mathcal{I}}^{(2)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{2-\epsilon} \tag{3}$$

$$\tilde{\mathcal{I}}^{(4)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{3-\epsilon} \tag{4}$$

$$\tilde{\mathcal{I}}^{(6)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{4-\epsilon} \tag{5}$$

$$\tilde{\mathcal{I}}^{(8)}(\delta) = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{5-\epsilon} \tag{6}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{l^2-\delta} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{4}\left[\frac{1}{\epsilon}-\gamma+\frac{3}{2}+\ln 4\pi\right]\delta^{2-\epsilon} \tag{7}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\delta]^2} = \frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+\ln 4\pi\right]\delta^{-\epsilon} \tag{8}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2-\delta]^2} = \frac{2i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma+\frac{1}{2}+\ln 4\pi\right]\delta^{1-\epsilon} \tag{9}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\delta]^2} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{2}\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta^{1-\epsilon} \tag{10}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^2} = \frac{3i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + \frac{2}{3} + \ln 4\pi\right] \delta^{2-\epsilon} \tag{11}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{1}{[l^2-\delta]^3} = \frac{-i}{(4\pi)^2} \frac{1}{2} \frac{1}{\delta} \tag{12}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma - \frac{1}{2} + \ln 4\pi\right] \delta^{-\epsilon} \tag{13}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l_\mu l_\nu}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \frac{g_{\mu\nu}}{4} \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right] \delta^{-\epsilon} \tag{14}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l_\mu l_\nu l_a l_b}{[l^2-\delta]^3} = \frac{i}{(4\pi)^2} \frac{g_{\mu\nu}g_{ab} + g_{\mu a}g_{\nu b} + g_{\mu b}g_{\nu a}}{8} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi\right] \delta^{1-\epsilon} \tag{15}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{1}{[l^2-\delta]^4} = \frac{i}{(4\pi)^2} \frac{1}{3!} \frac{1}{\delta^2} \tag{16}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^4} = \frac{-i}{(4\pi)^2} \frac{1}{3} \frac{1}{\delta} \tag{17}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^4} = \frac{i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi - \frac{11}{6}\right] \delta^{-\epsilon} \tag{18}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{1}{[l^2-\delta]^6} = \frac{i}{(4\pi)^2} \frac{1}{20} \frac{1}{\delta^4} \tag{19}$$

$$\int \frac{d^dl}{(2\pi)^d} \frac{l^2}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \frac{1}{30} \frac{1}{\delta^3} \tag{20}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^4}{[l^2-\delta]^6} = \frac{i}{(4\pi)^2} \frac{1}{20} \frac{1}{\delta^2} \tag{21}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^6}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \frac{1}{5} \frac{1}{\delta} \tag{22}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^8}{[l^2-\delta]^6} = \frac{-i}{(4\pi)^2} \left[\ln\delta + \frac{137}{60} - \left(\frac{1}{\epsilon} - \gamma + 1 + \ln 4\pi\right)\right] \tag{23}$$

B. λ-Parameter Integrals

Here are some integrals which frequently occur in the study of HBχPT. This general parameter integral I found in the talks by Jenkins and Manohar [1].

$$I(\alpha,b,c)=\int_0^\infty d\lambda\left(\lambda^2+2\lambda b+c\right)^\alpha \tag{24}$$

This integral has a recursion realtion,

$$(1+2\alpha)I(\alpha,b,c)=(\lambda^2+2\lambda b+c)^\alpha(\lambda+b)\Big|_0^\infty+2\alpha(c-b^2)I(\alpha-1,b,c) \tag{25}$$

Some functions I will use are

$$\begin{aligned}
\mathcal{F}(m,\delta,\mu)&=(m^2-\delta^2)\left[\sqrt{\delta^2-m^2}\,\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)-\delta\ln\left(\frac{m^2}{\mu^2}\right)\right]-\frac{1}{2}\delta m^2\ln\left(\frac{m^2}{\mu^2}\right)\\
\mathcal{J}(m,\delta,\mu)&=(m^2-2\delta^2)\ln\left(\frac{m^2}{\mu^2}\right)+2\delta\sqrt{\delta^2-m^2}\,\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)
\end{aligned} \tag{26}$$

Specific integrals are;

$$I(-1,\delta,m^2) = \int_0^\infty d\lambda\frac{1}{\lambda^2+2\lambda\delta+m^2-i\epsilon}=\frac{-1}{2\sqrt{\delta^2-m^2}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right) \tag{27}$$

$$I(-1,0,m^2) = \frac{\pi}{2m} \tag{28}$$

$$\begin{aligned}
\mu^{2\epsilon}I(-\epsilon,\delta,m^2) &= \mu^{2\epsilon}\int_0^\infty d\lambda\frac{1}{[\lambda^2+2\lambda\delta+m^2-i\epsilon']^\epsilon}\\
&= -\delta+\epsilon\left\{-2\delta+\delta\,\ln\frac{m^2}{\mu^2}-\sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\right\}
\end{aligned} \tag{29}$$

$$\mu^{2\epsilon}I(-\epsilon,0,m^2) = -\epsilon\pi m \tag{30}$$

$$\begin{aligned}
\mu^{2\epsilon}I(1-\epsilon,\delta,m^2) &= \mu^{2\epsilon}\int_0^\infty d\lambda\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon}\\
&= \frac{2}{3}\delta^3-\delta m^2+\epsilon\left\{\frac{10}{9}\delta^3-\frac{4}{3}\delta m^2-\frac{2}{3}\mathcal{F}(m,\delta,\mu)\right\}
\end{aligned} \tag{31}$$

$$\mu^{2\epsilon}I(1-\epsilon,0,m^2) = -\epsilon\frac{2}{3}\pi m^3 \tag{32}$$

$$
\begin{aligned}
\mu^{2\epsilon}\int_0^\infty d\lambda\; 2\lambda\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon} &= \mu^{2\epsilon}\int_0^\infty d\lambda\;(2\lambda+2\delta)\left[\lambda^2+2\lambda\delta+m^2\right]^{1-\epsilon} - 2\delta\mu^{2\epsilon} I(1-\epsilon,\delta,m^2) \\
&= -\frac{1}{2}m^4 + 2\delta^2 m^2 - \frac{4}{3}\delta^4 \\
&\quad + \epsilon\left[-\frac{1}{4}m^4 + \frac{8}{3}\delta^2 m^2 - \frac{20}{9}\delta^4 + \frac{1}{2}m^4\ln\frac{m^2}{\mu^2} + \frac{4}{3}\delta\mathcal{F}(m,\delta,\mu)\right]
\end{aligned}
\tag{33}
$$

$$
\mu^{2\epsilon}\int_0^\infty d\lambda\; 2\lambda\left[\lambda^2+m^2\right]^{1-\epsilon} = -\frac{1}{2}m^4\left(1+\epsilon\left[\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right]\right)
\tag{34}
$$

$$
\begin{aligned}
\mu^{2\epsilon}\int_0^\infty d\lambda\; \frac{2\lambda}{\left[\lambda^2+2\lambda\delta+m^2\right]^{-\epsilon}} &= \mu^{2\epsilon}\int_0^\infty d\lambda\;(2\lambda+2\delta)\left[\lambda^2+2\lambda\delta+m^2\right]^{-\epsilon} - 2\delta\mu^{2\epsilon} I(-\epsilon,\delta,m^2) \\
&= 2\delta^2 - m^2 + \epsilon\left[4\delta^2 - m^2 + \mathcal{J}(m,\delta,\mu)\right]
\end{aligned}
\tag{35}
$$

$$
\mu^{2\epsilon}\int_0^\infty d\lambda\; \frac{2\lambda}{\left[\lambda^2+m^2\right]^{-\epsilon}} = -m^2\left(1+\epsilon\left[1-\ln\frac{m^2}{\mu^2}\right]\right)
\tag{36}
$$

C. Combined Lorentz and λ-parameter Integrals

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^2} = \frac{-i}{(4\pi)^2}\Bigg\{\delta\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]+\delta\left(1-\ln\left(\frac{m^2}{\mu^2}\right)\right) + \sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{37}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-m^2+i\epsilon\right]^2} = \frac{-i\pi}{(4\pi)^2}m \tag{38}$$

$$\begin{aligned}\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\lambda^2-2\lambda\delta-m^2]^2} &= \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{2}\left\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\left(\frac{2}{3}\delta^3-\delta m^2\right)\right. \\ &\left. +\frac{10}{9}\delta^3-\frac{4}{3}\delta m^2-\frac{2}{3}\mathcal{F}(m,\delta,\mu)\right\} \\ \mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{[l^2-\lambda^2-m^2]^2} &= \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{3}\pi m^3\end{aligned} \tag{39}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-m^2]^2} = \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}m^4\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi+\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right] \tag{40}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \frac{i}{4(4\pi)^2}\frac{1}{\sqrt{\delta^2-m^2+i\epsilon}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right) \tag{41}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{2\lambda}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \frac{i}{2(4\pi)^2}\Bigg\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right] - 1-\ln\left(\frac{m^2}{\mu^2}\right)+\frac{\delta}{\sqrt{\delta^2-m^2}}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{42}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{\left[l^2-\lambda^2-2\lambda\delta-m^2+i\epsilon\right]^3} = \Bigg\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\delta \quad \delta-\delta\ln\left(\frac{m^2}{\mu^2}\right)+\sqrt{\delta^2-m^2}\log\left(\frac{\delta-\sqrt{\delta^2-m^2+i\epsilon}}{\delta+\sqrt{\delta^2-m^2+i\epsilon}}\right)\Bigg\} \tag{43}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-2\lambda\delta-m^2]^3} = \frac{i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}\Bigg\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\left(2\delta^2-m^2\right) +2\delta^2+\mathcal{J}(m,\delta,\mu)\Bigg\} \tag{44}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\, l_\mu l_\nu}{[l^2-\lambda^2-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{\mu\nu}}{8}m^2\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi-\ln\frac{m^2}{\mu^2}\right] \tag{45}$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\ l_a l_b l_c l_d}{[l^2-\lambda^2-2\lambda\delta-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}}{16}$$
$$\times\left\{\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi\right]\left(\frac{1}{2}m^4-2\delta^2m^2+\frac{4}{3}\delta^4\right)\right.$$
$$\left.+\frac{1}{4}m^4-\frac{8}{3}\delta^2m^2+\frac{20}{9}\delta^4-\frac{1}{2}m^4\ln\frac{m^2}{\mu^2}-\frac{4}{3}\delta\mathcal{F}(m,\delta,\mu)\right\} \quad (46)$$

$$\mu^{2\epsilon}\int_0^\infty d\lambda \int \frac{d^dl}{(2\pi)^d}\frac{\lambda\ l_a l_b l_c l_d}{[l^2-\lambda^2-m^2]^3} = \frac{-i}{(4\pi)^2}\frac{g_{ab}g_{cd}+g_{ac}g_{bd}+g_{ad}g_{bc}}{32}\ m^4\times\left[\frac{1}{\epsilon}-\gamma+1+\ln 4\pi+\frac{1}{2}-\ln\frac{m^2}{\mu^2}\right] \quad (47)$$

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