October 2, 2013 Name:________________________________

CSCI 2830
Exam 1

Solve all problems. For full credit, you must show all of your work on each problem. Partial credit will be given for correct but incomplete progress toward a solution.

1. (20 points) Let A and B be $n \times n$ matrices, C be an $m \times n$ matrix, y be a column vector of dimension m, and x and w be column vectors of dimension n. What is the result of each of the following calculations? (Specify scalar, column vector, row vector, or matrix. For each vector or matrix, give its dimensions.)

   Solution:

   For this problem, you have to remember the rule of multiplying an $m \times n$ matrix by an $n \times m$ matrix. First check that the center values match–for these two sizes, those numbers are both n. Then the size of the result comes from the outer values–here $m \times m$.

   (a) CB is an $m \times n$ matrix

   (b) $x^T A$ is a row vector of dimension n

   (c) $x^T Bw$ is a scalar

   (d) $A + C$ is undefined unless $m = n$ (dimensions of A and C don't match)

   (e) $B^T w$ is a column vector of dimension n

   (f) CC^T is an $m \times m$ matrix

   (g) $x^T y$ is undefined unless $m = n$ because the vector dimensions do not match

   (h) yx^T is an $m \times n$ matrix

2. (20 points) True or False? Explain your answer.

   (a) If vectors u and v are linearly independent and vector w is in Span$\{u,\ v\}$, then the set $\{u,v,w\}$ is linearly independent.

   Solution: False. w is a linear combination of u and v, so it is dependent on u and v.

   (b) If you are able to find two correct solutions to a system of linear equations then that system has infinitely many solutions.

   Solution: True. The only possibilities are one unique, zero, and infinitely many solutions. If there is more than one, there must be infinitely many.

   (c) Partitioning of matrices is always just an unnecessary complication.

   Solution: False. Partitioning of matrices can lead to better usage of the computer's memory.

   (d) If A is 3×2 matrix with two nonzero pivots then the system $Ax = b$ is always consistent.

   Solution: False. There are choices of b for which there is no solution. (See Theorem 4 on p.37 of the text.)

3. (20 points) Consider a system of 4 equations in 5 unknowns $Ax = b$. Suppose we form the augmented matrix $(\, A \quad | \quad b\,)$ and apply row operations to put it in the following reduced row echelon form:

$$(\, U \quad | \quad c\,) = \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

(a) Find the general solution to this system. Be sure to indicate which are basic and which are free variables.

(b) How does the solution change if the vector b is replaced with the zero vector?

Solution:

(a) The system to be solved is

$$(\, U \quad | \quad c\,) = \begin{pmatrix} 1 & 3 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 4 \\ 0 \end{pmatrix}.$$

The nonzero pivots are in columns 1, 3, and 4, so the basic variables are x_1,x_3, and x_4. The free variables are x_2 and x_5.
We find the solution via backsubstitution. We find the basic variables in terms of the free variables as follows:

$$\begin{aligned} \text{third row}: \;\; x_4 + 0x_5 = 4 \;\; &\rightarrow \;\; x_4 = 4, \\ \text{second row}: \;\; x_3 + 2x_5 = 7 \;\; &\rightarrow \;\; x_3 = 7 - 2x_5, \\ \text{first row}: \;\; x_1 + 3x_2 - 3x_5 = 5 \;\; &\rightarrow \;\; x_1 = 5 - 3x_2 + 3x_5. \end{aligned}$$

We can then write the general solution as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 - x_2 + 3x_5 \\ x_2 \\ 7 - 2x_5 \\ 4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 7 \\ 4 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

(b) The first vector in the general solution becomes the zero vector.

4. (20 points) Two unrelated subproblems.

(a) Rewrite the following as a product of any of the nonsingular square matrices A, B, A^{-1}, B^{-1}, A^T, and B^T (no other inverses or transposes):

$$B(AB)^{-1}(BB^TA^T)^T(BB^T)^{-1}A^{-1}.$$

Solution:

This solution depends on the following identities:

$$\begin{array}{rcl} AA^{-1} & = & A^{-1}A = I \\ (ABC)^{-1} & = & C^{-1}B^{-1}A^{-1} \\ (ABC)^T & = & C^TB^TA^T \\ (A^T)^T & = & A \\ (A^T)^{-1} & = & (A^{-1})^T \end{array}$$

The above expression can be reduced by any number of steps. In what follows, square brackets show one order in which to group and rearrange terms in order to reduce them:

$$\begin{array}{rcl} B(AB)^{-1}(BB^TA^T)^T(BB^T)^{-1}A^{-1} & = & B(B^{-1}A^{-1})[(A^T)^T(B^T)^TB^T][(B^T)^{-1}B^{-1}]A^{-1} \\ & = & (BB^{-1})A^{-1})[ABB^T][(B^T)^{-1}B^{-1}]A^{-1} \\ & = & (I)(A^{-1}A)B[B^T(B^T)^{-1}]B^{-1}A^{-1} \\ & = & (I)(I)B[I]B^{-1}A^{-1} \\ & = & (I)(I)(I)A^{-1} = A^{-1} \end{array}$$

(b) How many floating-point operations are required to compute a linear combination of four column vectors when each vector is of dimension 5?

Solution:

$$\begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix} = \alpha * \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix} + \beta * \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix} + \gamma * \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix} + \delta * \begin{pmatrix} \times \\ \times \\ \times \\ \times \\ \times \end{pmatrix}$$

Every vector elements gets multiplied by a scalar for a total of $4 * 5 = 20$ multiplications. Every scaled vector element gets added to four others for a total of $3 * 5 = 15$ additions. The operation total is then $20 + 15 = 35$.

There are no divisions in a linear combination in general. Divisions are required to compute the multipliers in the special linear combinations done in Gaussian elimination.

5. (20 points) Consider the following system of linear equations:

$$\begin{aligned} 2x + \alpha y &= 4 \\ 4x + 8y &= \beta \end{aligned}$$

Answer the following questions. In each case, demonstrate that your answer is correct by manipulating the equations *and* by drawing a picture of the system in the xy-plane. You may find it helpful to rewrite the system in matrix-vector form.

(a) Find one pair of values α and β for which the system has a unique solution.

(b) Let $\alpha = 4$. For what value(s) of β does the system have infinitely many solutions?

(c) Let $\alpha = 4$. For what value(s) of β does the system have no solutions?

(d) Let $\alpha = -4$ and $\beta = -8$. Add a third linear equation in x and y to the system such that the new three-equation system has exactly one solution.

Solution:

Note that you also needed to draw pictures of all of these cases. Let me know if you've got questions about the pictures–they are too hard to post here.

(a) One example is $\alpha = \beta = 0$ in which case $x = 2$ and $y = -1$.

(b) When $\alpha = 4$, the equations represent two lines of the same slope $m = -\frac{1}{2}$. For the lines to be the same, we require that they have the same y-intercept, i.e., for $\beta = 8$. The equation of the line in this case is $y = -\frac{1}{2}x + 1$.

(c) For the lines to be parallel, we just require that they have different intercepts, i.e., for $\beta \neq 8$. If $\beta = 0$, the two lines are $y = -\frac{1}{2}x + 1$ and $y = -\frac{1}{2}x$.

(d) In this case, the lines intersect at the point $x = 0$ and $y = -1$. A good third equation is thus the equation of any line passing through that point. The line $y = -1$ is one example; the line $x = 0$ is another.