November 3, 2013 Name: ____________________

CSCI 2830
Solutions to Exam 2

Solve all problems. **For full credit, you must show your work on each problem.** Partial credit will be given for correct but incomplete progress toward a solution.

1. (20 points) Find the determinant of the following matrix:

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Solution:

For a 4×4 determinant, you must use full-blown cofactor expansion. It's easiest if you do the expansion using a row with lots of zeros, so I choose the last row of A.

$$\det A = 0 + (-1)^{4+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} + 0 + (-1)^{4+4} 2 \det \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second 3×3 is zero as the matrix is obviously singular (repeated columns). The first determinant can be evaluated via the shorthand method. Its value is 1. Thus,

$$\det A = (4 + (-3) + 0) - (2 + 0 - 2) + 0 = 1.$$

2. (20 points) Let $A = \begin{pmatrix} 1 & -1 & 1 & -1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Describe the space spanned by the columns of A?

Solution: This space is a 2D plane in $\mathcal{R}^3$. It is NOT in $\mathcal{R}^2$.

(b) What is a basis for the column space of A?

Solution: An easy basis is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, but any pair of linearly independent 3-dimensional vectors works.

(c) What is a basis for the nullspace of A?

Solution: If you solve the system $Ax = 0$, you'll first find that the basic variables are x_1 and x_2 and the free variables are x_3, x_4, x_5, x_6. It follows that the solution is

$$x = x_3 \begin{pmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -\frac{11}{3} \\ \frac{4}{3} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The basis is then these four vectors. You have to split the four vectors out to display the basis.

I think the arithmetic is now correct. I did not penalize for arithmetic mistakes as long as the result had four vectors with +1s in the right places (on the diagonal if the vectors are in order). This problem was an arithmetic nightmare.

(d) What is the rank of A?

Solution: The rank is 2. That's the number of linearly independent rows or columns in A.

3. (20 points) Let v_1, v_2, and v_3 be linearly independent n-dimensional vectors.

(a) Are the vectors $w_1 = v_1 + v_2$, $w_2 = v_3 + v_2$, and $w_3 = v_1 + v_3$ also linearly independent? Explain.

Solution:

Yes, there's no way to create one of these vectors as a linear combination of the other two.

(b) What is one formula for the orthogonal projection of w_3 onto v_1?

Solution:

$$proj_{v_1} w_3 = \frac{v_1 v_1^T}{v_1^T v_1} w_3 \frac{v_1^T w_3}{v_1^T v_1} v_1$$

Either one of these is sufficient.

(c) Now suppose that v_1, v_2, and v_3 are orthogonal. What is the result of the projection in the preceding step?

Solution: The result is the coordinate of w_3 in the v_1 direction.

4. (20 points) Let A be a 72×11 matrix of rank 9.

   (a) What is the dimension of the column space of A?

   Solution:
   $9 = \text{rank}(A)$

   (b) What is the dimension of the nullspace of A?

   Solution:
   $2 =$ number of columns - rank(A)

   (c) The set of all 72×11 matrices constitutes a vector space. Does the set of all 72×11 matrices of rank 9 constitute a subspace of that vector space? Why or why not?

   Solution:
   No. It is not closed under vector addition because the rank of the sum of two matrices may not equal the rank of either of those matrices. Also multiplication by zero creates a matrix of rank zero, so it's not closed under scalar multiplication either. It also does not contain the zero matrix because the zero matrix is not of rank 9. Any one of these complaints suffices for an answer.

   [Hint: try an example using smaller matrices with dependent columns.]

5. (20 points) Short answers about n-dimensional vectors u,v,w.

(a) Suppose that w is orthogonal to u and v. Show that w is also orthogonal to all vectors in span$\{u,\ v\}$.

Solution: $w^T u = 0$ and $w^T v = 0$. Any vector z in the span is $z = au + bv$ for some scalars a, b by definition. Then $w^T z = auw^T + bvw^T = 0 + 0 = 0$, so w is orthogonal to every such z.

(b) Compute the angle between the vectors $u = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix}$.

Solution: You need $\cos\theta = \frac{u^v}{\|u\|\|v\|}$. Note that $u^T v =$, so that cosine equals zero no matter the norms of the vectors. Thus, the angle is $\frac{\pi}{2}$.

(c) Normalize the vector $u = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix}$. That is, convert u into a vector that has norm equal to one.

Solution: $\| \ u \ \| = \sqrt{9+1+1} = \sqrt{11}$. The normalized vector is $\frac{u}{\sqrt{11}}$.