October 17, 2013

CSCI 2830
Assignment 5

Due Wednesday, October 17

1. (10 points) Describe the column space and the nullspace of the following matrices:
$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Solution:

   The column space of A is the set of all linear combinations of the two column vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. That is, it is the set of all vectors of the form $\begin{pmatrix} \alpha - \beta \\ 0 \end{pmatrix}$ for some scalars α and β or, in other words, all vectors of dimension 2 with a zero second element. We can describe this space as the x-axis which is a line, a one dimensional space.

   The nullspace of A is the set of all solutions to the homogeneous system $Av = 0$. If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution, then
$$Av = \begin{pmatrix} v_1 - v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   So the nullspace of A is the set of all vectors v with equal elements.

   The column space of B is the zero vector of dimension 2 because all columns of B are the zero vector and all linear combinations of the zero vector are the zero vector itself. The nullspace of B is all of $\mathcal{R}^3$ because any vector multiplied by B gives the zero vector.

2. (15 points) Describe geometrically the following subspaces. In each case, give a basis for the subspace and state the dimension of the subspace.

   (a) $(\,0 \quad 0 \quad 0\,)$, $(\,0 \quad 1 \quad 0\,)$, $(\,0 \quad 2 \quad 0\,)$,

   Solution: If the vectors are expressed as the coordinates $(\,x \quad y \quad z\,)$, the subspace spanned by the vectors is the y-axis. One basis for the y-axis is $(\,0 \quad 1 \quad 0\,)$. (It is fine here to say a line or 1D subspace without specifying that it is the y-axis.)

   (b) $(\,0 \quad 0 \quad 1\,)$, $(\,0 \quad 1 \quad 1\,)$, $(\,0 \quad 2 \quad 1\,)$,

   Solution: If the vectors are expressed as the coordinates $(\,x \quad y \quad z\,)$, the subspace spanned by the vectors is the yz-plane. OneA basis for the plane is $(\,0 \quad 1 \quad 0\,)$ and $(\,0 \quad 0 \quad 1\,)$. (It is fine here to say a plane or 2D subspace without specifying that it is the xy-plane. And any linearly independent pair of vectors in that plane consitutes a basis.)

   (c) all six of these vectors,

   Solution: Since the y-axis is included in the yz-plane, the six vectors still span that plane. And a basis for the plane is still $(\,0 \quad 1 \quad 0\,)$ and $(\,0 \quad 0 \quad 1\,)$ or any other pair of linearly independent pair of vectors in that plane.

   (d) the nullspace of the 4×4 identity matrix,

   Solution: The basis for the nullspace is either the empty set or the zero vector, depending on the convention you want to follow.

   (e) all vectors with all negative components. (Is this a subspace?)

   Solution: This set is not a subpsace. If you multiply such a vector by a negative scalar, you get a vector with positive components, so the set is not closed under scalar multiplication.

   Whether or not the zero vector is included depends on whether you want to include -0 (as on a computer) or if you say 0 is unsigned (as in usual real arithmetic).

   It is also ok to work with the set of vectors with negative components. In this case, the set is the portion of R^n where all components are negative. In 2D that's the negative quadrant of the xy-plane. It is an n-dimensional entity, so one basis would be the n n-dimensional canonical vectors.

3. (15 points) In class, we saw that the set of all 2×2 matrices forms a vector space $\mathcal{V}$.

(a) Show that the sum of two 2×2 nonsingular matrices may be singular.

Solution: Let A be any 2×2 nonsingular matrix. Then $-A$ is also nonsingular. But $A + (-A) = A - A = 0$ is the zero matrix which is singular.

(b) Show that the sum of two 2×2 singular matrices may be nonsingular.

Solution: Consider the two diagonal matrices

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both D_1 and D_2 are singular matrices, but their sum is I which is nonsingular. Any such example is fine here.

(c) Does the set of all 2×2 nonsingular matrices form a subspace of $\mathcal{V}$? Explain.

Solution: This set is not a subspace of $\mathcal{V}$ because it is not closed under vector addition. See above example. Also, the zero matrix is singular so it is not in this set.

(d) Does the set of all 2×2 singular matrices form a subspace of $\mathcal{V}$? Explain.

Solution: This set is not a subspace of $\mathcal{V}$ because it is not closed under vector addition. See above example.

(e) Does the set of all 2×2 matrices with all four elements equal form a subspace of $\mathcal{V}$? Prove your answer.

Solution: Yes.

The zero matrix has all four elements equal to zero.

$\begin{pmatrix} a & a \\ a & a \end{pmatrix} + \begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ a+b & a+b \end{pmatrix}$ which is a 2×2 matrix with all elements equal, so the set is closed under vector addition.

$kA = k \begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} ka & ka \\ ka & ka \end{pmatrix}$ which is a 2×2 matrix with all elements equal, so the set is closed under scalar multiplication.

(f) What is the smallest subspace of $\mathcal{V}$? Explain.

Solution: It is the 2×2 zero matrix. The sum of two 2×2 zero matrices is a 2×2 zero matrix. A scalar times a 2×2 zero matrix is a 2×2 zero matrix. So, the set that is only the 2×2 zero matrix is closed under vector addition and scalar multiplication.

Any other single matrix set is not a subspace because scaling by k changes the values of that matrix meaning that the set is not closed under scalar multiplication.