November 7, 2012

CSCI 2830
Solutions to Assignment 7

1. (15 points) Find the projection of b onto the column space of A when

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

Split b into $p + q$ with p in the column space of A and q perpendicular to that space. Which subspace associated with the matrix A (column space or nullspace) contains q? Why?

Solution:

The projection of b onto the column space of A is

$$Pb = A(A^TA)^{-1}A^Tb = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}.$$

Use $p = Pb$, then $q = b - p = \begin{pmatrix} -2 & 2 & 1 \end{pmatrix}^T$. Check that this q is orthogonal to p: $p^Tq = 0$. Thus, q is in the left nullspace of A.

2. (15 points) Find a third column so that the matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \end{pmatrix}$$

is orthogonal. Is there more than one choice for that third column? Why or why not?

Solution:

We can build the third column out of any vector that is linearly independent of the first two. To make the math a little easier, we choose a starting vector of $v = \begin{pmatrix} 42 & 0 & 0 \end{pmatrix}^T$. But ANY linearly independent starting vector will give the same answer or its negative. (There are two possible solutions along the same line but with opposite directions.)

Applying the Gram-Schmidt process gives

$$\begin{aligned}
\hat{q}_3 &= v - (v^T q_1) q_1 - (v^T q_2) q_2 \\
&= \begin{pmatrix} 42 \\ 0 \\ 0 \end{pmatrix} - (\frac{42}{\sqrt{3}}) \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} - (\frac{42}{\sqrt{14}}) \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \end{pmatrix} \\
&= \begin{pmatrix} 42 \\ 0 \\ 0 \end{pmatrix} - (\frac{42}{3}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - (\frac{42}{14}) \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \\
&= \begin{pmatrix} 42 \\ 0 \\ 0 \end{pmatrix} - 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \\
&= \begin{pmatrix} 25 \\ -20 \\ -5 \end{pmatrix}.
\end{aligned}$$

Since $\parallel \hat{q}_3 \parallel = 5\sqrt{42}$,

$$q_3 = \frac{\hat{q}_3}{\parallel \hat{q}_3 \parallel} = \frac{1}{42} \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}.$$

Check: $Q^T Q = Q Q^T = I$.

3. (15 points) Apply the Gram-Schmidt process to

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \; y = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; z = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and express the result in the form $A = QR$.

Solution:

$$
\begin{aligned}
q_1 &= \frac{x}{\sqrt{x}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\
\hat{q}_2 &= y - (y^T q_1) q_1 \\
&= = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - (\sqrt{2}) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
q_2 &= \frac{\hat{q}_2}{\sqrt{\hat{q}_2}} = \hat{q}_2.
\end{aligned}
$$

$$\begin{aligned}
\hat{q}_3 &= z - (z^T q_1)q_1 - (z^T q_2)q_2 \\
&= = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - (\frac{1}{\sqrt{2}})\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - (1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
&= = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}^T . \\
q_3 &= \frac{\hat{q}_3}{\sqrt{\hat{q}_3}} = \sqrt{\frac{2}{3}}\hat{q}_3.
\end{aligned}$$

We have computing the QR factorization $A = \begin{pmatrix} x & y & z \end{pmatrix} = QR$ with $Q = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$ and

$$R = \begin{pmatrix} -\sqrt{2} & -\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 0 & -1 & -1 \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix} .$$