December 16, 2013 Name:__________________________________

CSCI 2830
Solutions to Exam 3

Solve all problems. **For full credit, you must show your work on each problem.** Partial credit will be given for correct but incomplete progress toward a solution.

1. (20 points) Let $x1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $x2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $x3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Then $\{x_1, x_2, x_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathcal{R}^4$. Construct an orthogonal basis for W. (Orthogonal is good enough. You do not need to construct an orthonormal basis.)

   Solution: This is Example 2 from Section 6.4 of the textbook. See the solution there. The result is $q1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $q2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $q3 = \begin{pmatrix} 0 \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$.

2. (20 points) True or false (give a proof if true and a proof or counterexample if false):

   (a) The eigenvalues of a matrix are always distinct.

   Solution: FALSE. Consider $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This 3×3 matrix A has three copies of the eigenvalue 2.

   (b) If the eigenvalues of the 3×3 matrix A are 0, 2, and 5, then A is certainly invertible.

   Solution: FALSE

   Consider $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ which has those eigenvalues but is singular due to the zero row.

   More generally, if y is the eigenvector of A corresponding to the zero eigenvalue, $Ay = 0y = 0$. Thus, y is a nonzero vector in the nullspace of A, so A is singular.

   (c) If the eigenvectors of A are all multiples of $\begin{pmatrix} 1 & 4 \end{pmatrix}^T$, then A is diagonalizable.

   Solution: FALSE

   A matrix is diagonalizable if and only if its eigenvectors are linearly independent. That is, if the matrix of evecs S is invertible so that $\Lambda = S^{-1}AS$ is diagonal.

   (d) If λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

   Solution: TRUE

$$\begin{aligned} Ax &= \lambda x \\ x &= \lambda A^{-1} x \\ \frac{1}{\lambda} x &= \lambda A^{-1} x. \end{aligned}$$

3. (20 points) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find the eigenvector corresponding to the smallest eigenvalue.

(b) What is the determinant of A?

Solution:

(a) First, get the eigenvalues by finding the roots of the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^3 - (2-\lambda) = 0.$$

Rearranging gives

$$\begin{aligned} (2-\lambda)^3 &= (2-\lambda) \\ (2-\lambda)^2 &= 1 \text{ OR } 2-\lambda = 0 \\ 2-\lambda &= \pm 1 \\ \lambda &= 1, 2, or 3 \end{aligned}$$

For the eigenvector for $\lambda^{(1)} = 1$, solve $(A - 1 \cdot I)x = 0$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x = 0 \to \begin{cases} x_3 = 0 \\ x_1 = -x_2 \end{cases} \to x^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

To make all the eigenvectors have unit norm, the eigenvector needs to be multiplied by $\frac{1}{\sqrt{2}}$. (You didn't have to do this.)

(b) The easy way to compute the determinant is to multiply the eigenvalues. That is, $\det(A) = 1 \cdot 2 \cdot 3 = 6$.

4. (20 points) What 4×4 matrices represent the homogeneous transformations that

(a) project every vector onto the yz-plane?

Solution: $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This matrix just removes the x component.

(b) rotate the yz-plane through 90 degrees, leaving the x-axis alone?

Solution: $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Substituting in $\theta = 90$ gives $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) translate the homogeneous vector $(\, x \quad y \quad z \quad 1\,)^T$ to $(\, x+1 \quad y+3 \quad z+5 \quad 1\,)^T$?

Solution: $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

5. (20 points) Short answers (really!) for $m \times n$ A, with $m >> n$, and m-dimensional vector b.

   (a) Why does the linear system $Ax = b$ typically have no exact solution?

   Solution: For a tall, skinny coefficient matrix, it is generally the case that the righthand side vector b is not in the column space of A.

   (b) Write down the steps for any correct method of solving the least squares problem $Ax = b$.

   Solution: The easiest way is to form the normal equations $A^T Ax = A^T b$. As long as the rank of A is n, you can solve these equations via Gaussian elimination. You can also write down $A = QR$ and solve $Rx = A^T b$.

   (c) Let A be a 3×2 matrix and b be a 3-dimensional vector. Give an example of a matrix A for which the least squares problem $Ax = b$ has a unique solution.

   Solution: Any matrix with linearly independent columns (rank = 2) will lead to normal equations $A^T Ax = A^T b$ with a nonsingular coefficient matrix. Those normal equations have the unique solution $x = (A^T A)^{-1} A^T b$.