September 26, 2012

CSCI 2830
Assignment 4
due September 25

For full credit, you must show your work.

1. (15 points) An elementary matrix is an identify matrix with one element of the lower triangle (left of and below the diagonal) set to a non-zero value. Multiplying a matrix A by an elementary matrix has the effect of replacing one of the rows of A with a linear combination of that row with another row.

   (a) Show how to apply a sequence of elementary matrices to carry out the sequence of row operations that convert the following matrix A to upper triangular form without row exchanges.

$$A = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 2 & -2 & 3 & -3 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 4 & 3 \end{pmatrix}.$$

Solution: Let M be the augmented matrix $\begin{pmatrix} A & b \end{pmatrix}$.
Apply three elementary matrices to introduce zeros in the first column

$$E_3E_2E_1M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M$$

leads to

$$M' = E_3E_2E_1M = \begin{pmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & -1 & 3 & -3 & -28 \\ 0 & -2 & 7 & -6 & 6 \\ 0 & 1 & 0 & 7 & 12 \end{pmatrix}.$$

Apply two elementary matrices to introduce zeros in the second column

$$E_5E_4M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M'$$

leads to

$$M'' = \begin{pmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & -1 & 3 & -3 & -28 \\ 0 & 0 & 1 & 0 & 62 \\ 0 & 0 & 3 & 4 & -16 \end{pmatrix}.$$

Apply one elementary matrix to introduce a zero in the third column

$$E_6M'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} M'\prime$$

leads to

$$\begin{pmatrix} 1 & -1 & 2 & -1 & -8 \\ 0 & -1 & 3 & -3 & -28 \\ 0 & 0 & 1 & 0 & 62 \\ 0 & 0 & 0 & 4 & -202 \end{pmatrix}.$$

Experiment with some sample elementary matrices before embarking on this problem so that you know how they work. Including b in your solution to the first part will save you some effort in the second part.

2. (10 points)

(a) The transformation T is a shear transformation axis that maps $\mathbf{e}_2$ to $\mathbf{e}_2$ and maps $\mathbf{e}_1$ to $\mathbf{e}_1 + 2\mathbf{e}_2$. Give the matrix A such that $T(x) = Ax$. What does this transformation do to the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$?

Solution: $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ which maps $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ onto $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$. You can use the definition of linearity to check your answer.

(b) Suppose the linear transformation $T(x) = Ax$ maps R^3 to R^3, where A is a 3×3 matrix. If T maps (5,0,0) to $(1, 2, 2)$ and maps $(0, 1, 1)$ to $(6, 0, 3)$, what does it do to $(5, 1, 1)$?

Solution: Since the transformation is linear, we know that $T(u) + T(v) = T(u + v)$ for vectors u and $v4$, so T maps $(5, 0, 0) + (0, 1, 1) = (5, 1, 1)$ to $(1, 2, 2) + (6, 0, 3) = (7, 2, 5)$.