September 23, 2013

CSCI 2830
Solutions to Assignment 3

1. Study the backsubstitution handout.
2. Work through Sections 1, 2, 3, 4, 6 and 7 of *Elements of MATLAB.*
3. (10 points) Build some test codes.

   (a) Write a MATLAB function called `mvrow` that performs the operation $b = Ax$ by the inner product algorithm.

   ```
   function b = mvrow(n,A,x)
   for j = 1:n
      b(j) = A(j,:)*x;
   end
   b = b';
   ```

   Note that `b` is a vector and so should have only one index (i.e., not `b(i,j)`). The for loop constructs a row vector, and so I have transposed it at the end to get the correct column vector. Another, harder to write, approach would have initialized `b` as a column vector of all 0's.

   I didn't penalize for using two indices for vectors this time, but I may in future programs. The two problems with extra indices are loss of readability of the code and an ill-defined result (as the second index is never accessed).

   (b) Write a MATLAB function called `mvcol` that performs the operation Ax by taking a linear combination of the columns of the matrix A.

   ```
   function b = mvcol(n,A,x)
   for j = 1:n
      b(j) = 0;
   end
   b = b';
   for j = 1:n
      b = b + A(:,j)*x(j);
   end
   ```

This script will not run without initializing `b` as a column vector. (Matlab will detect disagreement of dimensions.) I suspect that there are cleaner ways to do it.

(c) Determine the number of flops performed by each function when multiplying a 5×5 matrix and a 5×1 vector. Why do we expect the flop counts to be the same? If yours are different, explain why.

Ideally, the flop counts are the same because the two scripts are performing the same operation. The different operations just come in a different order. Any differences between them should be minor.

My `mvrow` above performs 5 multiplications and 4 additions per iteration for 5 iterations. That gives a total of 45 flops. My `mvcol` performs 5 multiplications and 5 additions for 5 iterations for a total of 50 flops. Both scripts do the same operation but one is a bit tighter than the other in terms of flops. It's possible to make them both do 45 if you are smart about how you set up the sums. When n gets large, these minor differences don't matter much.

(d) Find the function of $n = 5$ that gives the number of flops for these matrix-vector multiplications.

You need to show that the function is about $2n^2$. For my scripts, it is $2n^2 - n$ for `mvrow` and $2n^2$ for `mvcol`.

4. Define four scalars $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and four column vectors:

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^T, \\ v_2 &= \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T, \\ v_3 &= \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}^T, \\ v_4 &= \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T. \end{aligned}$$

(a) Determine whether the vectors are linearly independent by solving the system $\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 = 0$.

Solution: The system to be solved is

$$\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 = \begin{pmatrix} \gamma_1 + \gamma_2 \\ \gamma_1 + \gamma_4 \\ \gamma_2 + \gamma_3 \\ \gamma_3 + \gamma_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that this is the same system we would have if we'd created a 4×4 matrix V with the four vectors as its columns and multiplied it times a vector c with the four scalars as its components: $Vc = 0$. If there is a nonzero solution to the homogeneous system $Vc = 0$, the matrix V is singular and the vectors are dependent.

Solving the four equations leads to

$$\begin{aligned}
\gamma_1 + \gamma_2 = 0 &\quad\rightarrow\quad \gamma_2 = -\gamma_1 \\
\gamma_1 + \gamma_4 = 0 &\quad\rightarrow\quad \gamma_4 = -\gamma_1 \\
\gamma_2 + \gamma_3 = 0 &\quad\rightarrow\quad \gamma_3 = -\gamma_2 = \gamma_1 \\
\gamma_3 + \gamma_4 = 0 &\quad\rightarrow\quad \gamma_4 = -\gamma_3 = -\gamma_1.
\end{aligned}$$

Thus, the homogeneous system has a nonzero solution $c = \gamma_1 \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}^T$. Thus, the vectors are dependent.

(b) Determine whether or not the vectors span $\mathcal{R}^4$ by solving the system $\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$.

Solution: Now the system $Vc = b$ to be solved is

$$\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 + \gamma_4 v_4 = \begin{pmatrix} \gamma_1 + \gamma_2 \\ \gamma_1 + \gamma_4 \\ \gamma_2 + \gamma_3 \\ \gamma_3 + \gamma_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and the equations become

$$\begin{aligned}
\gamma_1 + \gamma_2 = 0 &\quad\rightarrow\quad \gamma_2 = -\gamma_1 \\
\gamma_1 + \gamma_4 = 0 &\quad\rightarrow\quad \gamma_4 = -\gamma_1 \\
\gamma_2 + \gamma_3 = 0 &\quad\rightarrow\quad \gamma_3 = -\gamma_2 = \gamma_1 \\
\gamma_3 + \gamma_4 = 1 &\quad\rightarrow\quad \gamma_4 = 1 - \gamma_3 = 1 - \gamma_1.
\end{aligned}$$

These equations are inconsistent because they require that $-\gamma_1 = 1 - \gamma_1$ which can never be true.

Thus, there is no linear combination of the vectors that gives $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$ and the vectors do not span $\mathcal{R}^4$.