Assignment 1

Problem 1 (25 points)

Throughout this problem, we use n to range over natural numbers $\mathbb{N}$.

1. We have $a_n = 1, 2, 4, 8, 16, \ldots$.

- Closed Form: $a_n = 2^n$ for $n \geq 0$.
- Recurrence: $a_n = 2a_{n-1},\ n \geq 1$ and $a_0 = 1$.

2. $b_n = 1, 3, 2, 9, 4, 27, 8, 81, \ldots$ (altenating powers of 2 and 3).

- Closed Form: $b_n = \begin{cases} 2^{\frac{n}{2}} & n \text{ is even} \\ 3^{\frac{n+1}{2}} & n \text{ odd} \end{cases}$.
- Recurrence: $b_n = \begin{cases} 2a_{n-2} & n \text{ is even and } n \geq 2 \\ 3a_{n-2} & n \text{ is odd and } n \geq 3 \end{cases}$. Bases cases $b_0 = 1, b_1 = 3$.

3. $c_n = 0, 1, 3, 6, 10, 15, \ldots$.

- Closed Form: $c_n = \frac{n(n+1)}{2}$ for $n \geq 0$.
- Recurrence: $c_n = c_{n-1} + n,\ n \geq 1$ and base case $c_0 = 0$.

4. $d_n = 1, 0, 1, 0, 1, 0, 1, 0, \ldots$

- Closed Form: $d_n = (n+1)\text{mod}2$ for $n \geq 0$.
- Alternative Closed Form: $d_n = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$.
- Recurrence: $d_n = d_{n-2}, n \geq 2$ with base cases $d_0 = 1, d_1 = 0$.

5. $e_n = 1, 1, 0, 0, 1, 1, 0, 0, \ldots$

- Closed Form: $e_n = \begin{cases} 1 & \text{if } n \mod 4 = 0 \\ 1 & \text{if } n \mod 4 = 1 \\ 0 & \text{if } n \mod 4 = 2 \\ 0 & \text{if } n \mod 4 = 3 \end{cases}$.
- Alternative Closed form $e_n = 1 - \lfloor \frac{n \mod 4}{2} \rfloor$.
- Recurrence: $e_n = e_{n-4}, n \geq 4$ with base cases $e_0 = 1, e_1 = 1, e_2 = 0, e_3 = 0$.

Problem 2 (25 points)

1. $p_n = 2^{n+2}$. The recurrence is $p_n = 2p_{n-1},\ n \geq 1$ with $p_0 = 4$.

2. $q_n = n!$. The recurrence is $q_n = nq_{n-1},\ n \geq 1$ with $q_0 = 1$.

3. $r_n = 2n^2 - 3n + 5$. We have $r_{n-1} = 2(n-1)^2 - 3(n-1) + 5 = r_n - 4n + 5$. Therefore, the recurrence is

$$r_n = r_{n-1} + 4n - 5,\ n \geq 1,\ \text{and } r_0 = 5\,.$$

3. $s_n = s_{n-2} + 1,\ n \geq 2$ with base cases $s_0 = 0, s_1 = 1$.

4. Note $t_n = \frac{1}{n+1}$. We can therefore write $n = \frac{1}{t_n} - 1$. We have $t_{n+1} = \frac{1}{n+2} = \frac{1}{\frac{1}{t_n} - 1 + 2} = \frac{t_n}{1+t_n}$.

Therefore, the answer is

$$t_n = \frac{t_{n-1}}{1 + t_{n-1}},\ n \geq 1,\ \text{with } t_0 = 1\,.$$

5. $u_n = u_{n-1} + (2n+1), n \geq 2$ with $u_1 = 3$. Note that we start this series from $n \geq 1$ rather than our usual $n \geq 0$. This is because the summation starts from 1.

6. $v_n = v_{n-1} \times 2^{2n+1},\ n \geq 2$ with $v_1 = 2$. Note that the series starts from $n \geq 1$ since the lower limit of the product is $j = 1$.

Problem 3 (20 points)

The sequence itself is given by $s_n = 2, 7, 16, 29, 46, 67, \ldots$.

1. The first differences are $d_n = 5, 9, 13, 17, 21, 25, \ldots$.

2. The closed form is therefore $d_n = 4n + 5, n \geq 0$.

3. The second difference sequence is $e_n = 4, 4, 4, 4, 4, \ldots$. Its closed form is $e_n = 4$ for $n \geq 0$.