CSCI 2824 Discrete Structures
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Midterm 2

1. Use proof by contradiction to prove that no integer can be both even and odd.

To set up a proof by contradiction, you need a case where the hypothesis is true, but the conclusion is false. Here, you could say: if n is an integer, then n is both even and odd.

Let an integer n be given.

If n is even, then $n = 2k, k \in \mathbb{Z}$, and if n is odd, then $n = 2m + 1, m \in \mathbb{Z}$.

This means that $2k = 2m + 1$,
Which re-arranges to $2k = 2m + 1$, $k - m = 1/2$
Since k and m are both integers, then $k - m$ is also an integer. However, ½ is not an integer, so we have a contradiction. From this we conclude that an integer cannot be both even and odd.

2. Use proof by contradiction to prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

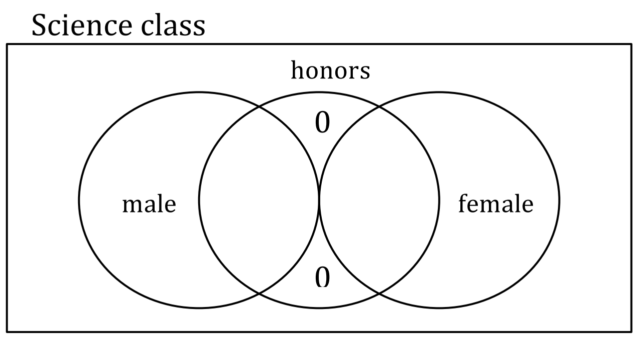
The counterexample to this statement would be if n is an integer and $n^3 + 5$ is odd, then n is odd. Let's assume that such a case has been found. This means that:

$$(2k + 1)^3 + 5 = 2m + 1, where\ k, m \in \mathbb{Z}$$

which is:

$$(2k + 1)(2k + 1)(2k + 1) + 4 = 2m$$

3. In Mrs. Smith's science class, there are a total of 25 students. Of these, 15 are female and 15 are honor students. Draw a Venn diagram for this information, filling in as much information as possible and answer the following question: What is the smallest possible number of female students who are on the honor roll?



The smallest number of females that could be on the honor role is 5.

4. Prove whether the following statements are true or false:

   a. $\{6k + 1 : k \in \mathbb{Z}\} \subseteq \{2n + 1 : n \in \mathbb{Z}\}$

   Let set A be $6k + 1 : k \in \mathbb{Z}$, and set B be $2n + 1 : n \in \mathbb{Z}$. If A is a subset of B, then every element of set A is included in set B. To show this, let

   $$6k + 1 = 2n + 1$$

   and then,

   $$3k = n$$

   which shows that set A consists of every third element of set B. Therefore, it is true that A is a subset of B.

   b. $\{2n + 1 : n \in \mathbb{Z}\} \subseteq \{6m - 1 : m \in \mathbb{Z}\}$

   This statement is false and you can show this by listing out elements in the first set and showing that they are not included in the second set. Let set A be $2n + 1 : n \in \mathbb{Z}$, and set B be $6m - 1 : m \in \mathbb{Z}$.

   A = {1, 3, 5, 7, 9…} and B = {-1, 5, 11, 17…}.
   The numbers 1 and 3 are examples of elements in set A that are not included in set B. Therefore, A cannot be a subset of B.

   The other way to solve this would be using proof by contradiction. Let $2n + 1 = 6m - 1$. Solving for m produces a value that is not an integer, and yet we know from the problem definition that m is an integer. Therefore, we have a contradiction, and can conclude that A is not a subset of B.

c. $\{6k + 1: k \in \mathbb{Z}\} = \{2n + 1: n \in \mathbb{Z}\}$

This is also false. If the sets are equal then it means that they are both subsets of each other. Let set A be $6k + 1: k \in \mathbb{Z}$, and set B be $2n + 1: n \in \mathbb{Z}$.

A = {1, 7, 13, …} and B = {1, 3, 5, 7,…}.

Since set B includes numbers not included in set A, such as 3 and 5, the sets are not equal.

You can also solve this using a proof by contradiction similar to the approach used in part b of this question.

5. How many permutations of *{a,b,c,d,e,f,g}* end with *a*?

   There are seven letters to choose from, however, the a needs to be fixed in the final position, which leaves six letters. In the first position, there are 6 choices. In the second position, there are 5 choices, and 4 choices for the third position. The number of permutations is 6!.

6. A local baseball team has a group of 10 outfield players and a group of 15 infield players. From these groups, how many ways are there to field a team of 9 players, where 3 players are needed from the outfielders and 6 players are needed from the infielders? Each arrangement of players is unique, i.e. the position assigned to each player matters. (Note: we're assuming that pitchers are infielders, and that all players in a group can play all positions for that group.)

   The arrangement of the players matters, making this a permutation and not a combination. There are 10 outfielders, from which to select 3, and there are 15 infield players, from which to select 6. The solution then is:

$$\frac{10!}{7!} * \frac{15!}{9!}$$

7. The Rubiks cube is a popular puzzle from the 1980's that features six sides with unique colors. Each side has nine squares, each of which has a color. The center square is fixed on each side and cannot be changed. However, all other squares can be rearranged to be one of the six colors. How many arrangements are possible for one side of a Rubiks cube?

   We only care about one side here. There are 9 squares on each side, but the middle square is fixed, which means that only 8 squares can change. Each of those squares can be one of 6 colors. All squares can have any of those colors, which gives a solution of 6^8

8. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

   This is a combination with repetition of indistinguishable items. We need to select 20 coins from 5 types. In this case, n = 5, and k = 20. The equation is C(n+k-1,k), which is C(24,20). Solving this, you get:

$$\frac{24!}{20!\,4!}$$

9. In the following pseudo-code for a nested for loop, how many times will the line "select an item" be executed if n = 20:

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For i = 1 to n
     For j = i+1 to n
          For k = j+1 to n
               "select an item"
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   These for loops produce a C(20,3). If you use a small value for n, such as n=4, you can see the pattern that develops for the values of i,j,k. You will get {1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}. There are no repeats for any of the values in each set. This example is a C(4,3), which is 4!/3!=4. When n = 20, then it's 20!/3!17! = 1140.