CSCI 2824 Discrete Structures
Instructor: Hoenigman
Assignment 4
Due Tuesday, Oct 1 at the beginning of class

1. Write a proof for the following statement:

   Let n be a positive integer. If n is odd, then $n^3 - n$ is divisible by 4.

   Proof: Let n be odd, then $n = 2k + 1, k \in \mathbb{Z}$. If $n^3 - n$ is divisible by 4, then $(n^3 - n) = 4q, q \in \mathbb{Z}$. Substitute to get, $(2k+1)^3 - n = 4s, s \in \mathbb{Z}$.

   $$(2k+1)^3 - (2k+1) = 8k^3 + 12k^2 + 6k + 1 - 2k - 1$$

   $$= 8k^3 + 12k^2 + 8k$$
   $$= 4(2k^3 + 3k^2 + 2k)$$
   $$= 4(s).$$

   By closure property of integers, since k is an integer, so is s. Therefore, $n^3 - n$ is divisible by 4.

2. Write a proof for the following statement:

   Let n be a positive integer. If n is divisible by 4, then $n^3 - n$ is divisible by 4.

   Proof: Let n is divisible by 4, then $n = 4q, q \in \mathbb{Z}$. If $n^3 - n$ is divisible by 4, then $(n^3 - n) = 4q, q \in \mathbb{Z}$. Substitute to get $4q^3 - 4q = 4(q^3 - q)$. Since $(q^3 - q)$ is an integer, by closure property of integers, $n^3 - n$ is divisible by 4.

3. Write the contrapositive statement, and then prove the original statement by proving the contrapositive for the following:

   Let n be a positive integer. If m + n is odd, then m or n must be even.

   Contrapositive: If m and n are even, then m + n must be even.

4. Write a proof for the following statement:

   Let n be a positive integer. If n is even, then the product of n and its successor is even.

   Proof: Let n be a positive integer, then $n = 2q, q \in \mathbb{Z}$. The product of n and $n+1$ is $2q * (2q + 1) = 4q^2 + 2q = 2(2q^2 + q)$. Since q is an integer, $2q^2 + q$ is also an integer by closure property of integers. Therefore, the product of n and n+1 is even.

5. Provide a counterexample to the following statements:
    a. If (a%c)=(b%c), then a = b
    b. If (a%b)=c, then ((a+1)%b)=c+1

A counterexample to part a. is a = 8 and b = 4. In both cases, a%c and b%c, the result is 0, but a $\neq$b.

A counterexample to part b. is a = 5 and b = 3. (a%b) = 2. However ((a+1)%b) = 0.

6. Write a proof for the following statement:

Let n be a positive integer. If n is not divisible by 3, then n^2+2 is divisible by 3.

Proof by cases:
If n is not divisible by 3, then $n = 3k + 1$ *or* $n = 3k + 2$, $k \in \mathbb{Z}$.
Case 1: n = 3k + 1
$(3k+1)^2 + 2 = (9k^2 + 6k + 1) + 2 = 9k^2 + 6k + 3 = 3(3k^2 + 2k + 1)$. Since k is an integer, $3k^2 + 2k + 1$ is also an integer. Therefore $n^2 + 2$ is divisible by 3.
Case 2: n = 3k + 2
$(3k+2)^2 + 2 = (9k^2 + 12k + 4) + 2 = 9k^2 + 12k + 6 = 3(3k^2 + 4k + 2)$. Since k is an integer, $3k^2 + 2k + 2$ is also an integer. Therefore $n^2 + 2$ is divisible by 3.