CSCI 5454 Problem Set 4

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No Collaborators

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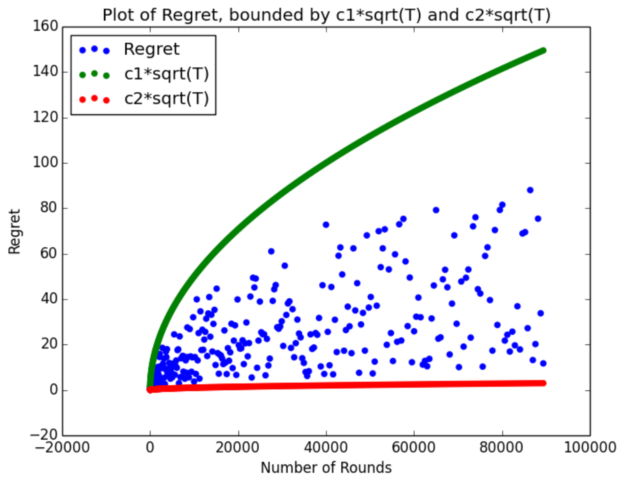
1. *Implement the action setting of Hedge and use it to complete a few tasks.*

   (a) *Use Hedge to write a game AI and display some sample output.*

```
  Payoff Matrix
[-8, 10, 2]
[-4, -2, -1]
[-8, -6, 1]
##########
Round 0
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.3333333333333333,0.3333333333333333,0.3333333333333333]
Action chosen by AI 1
AI loss vector [-8, 10, 2]
AI weight vector [2980.9579870417283, 4.5399929762484854e-05, 0.1353352832366127]
old AI score 0.0 , new AI score -10.0 , difference -10.0
old user score 0.0 , new user score 10.0 , difference 10.0
##########
Round 1
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.9999545869027009,
1.522928810416062e-08, 4.5397868011057175e-05]
Action chosen by AI 0
AI loss vector [-8, 10, 2]
AI weight vector [8886110.520507872, 2.061153622438558e-09, 0.018315638888734182]
old AI score -10.0 , new AI score -2.0 , difference 8.0
old user score 10.0 , new user score 2.0 , difference -8.0
##########
Round 2
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.9999999979388461,2.319522825462676e-16,2.0611536181902033e-09]
Action chosen by AI 0
AI loss vector [-8, 10, 2]
AI weight vector [26489122129.84347, 9.357622968840175e-14, 0.002478752176666359]
old AI score -2.0 , new AI score 6.0 , difference 8.0
old user score 2.0 , new user score -6.0 , difference -8.0
```

   (b) *Give a plot that exhibits $\Theta(\sqrt{T})$ regret.*



I let $c_1 = .5$ and $c_2 = .01$ and $\eta = \sqrt{\frac{8ln\,N}{T}}$. The graph shows that regret is bounded by $\Theta(\sqrt{T})$ where T is the number of rounds. Regret remained $\leq \sqrt{(\frac{T}{2}ln\,N)}$. At times regret would get really close to the bound but other times it would not be close. It never went above the bound. Example output of regret in the first column and $\sqrt{(\frac{T}{2}ln\,N)}$ in the second column is below.

```
0.0854754326956 1.16388787282
0.711207959794 1.55185049709
0.915738476176 1.93981312136
1.86659059913 2.32777574563
1.11868740447 2.7157383699
0.942107984927 3.10370099418
1.20108924348 3.49166361845
2.33209799223 3.87962624272
2.39799236386 4.26758886699
3.71141140061 4.65555149126
3.10138106514 5.04351411554
1.19507429131 5.43147673981
3.95892727292 5.81943936408
0.926603427611 6.20740198835
2.69832933191 6.59536461262
0.594289924894 6.9833272369
6.25916816147 7.37128986117
3.62512254793 7.75925248544
0.855690231446 8.14721510971
```

2. *Show that squared loss guarantees that the best prediction in terms of expected loss is going to be $x = p$. Give a distribution where this is not the case for absolute loss.*

We want to prove that the expected loss $E[L(x, y)]$ is minimized by letting $x = p$. If $y = 1$ with probability p and $y = 0$ with probability $1 - p$ then

$$\begin{aligned} E[L(x, y)] &= p \cdot L(x, 1) + (1 - p) \cdot L(x, 0) \\ &= p \cdot (x - 1)^2 + (1 - p) \cdot x^2 \\ &= p \cdot (x^2 - 2x + 1) + (1 - p) \cdot x^2 \\ &= px^2 - 2px + p + x^2 - px^2 \\ &= x^2 - 2px + p \end{aligned}$$

The minimum of a quadratic function of the form $f(x) = ax^2 + bx + c$ can be found with the equation

$$x = -\frac{b}{2a}$$

Solving for x we get

$$-1 \cdot \frac{(-2px)}{2 \cdot 1} = \frac{2p}{2} = p$$

This shows that the minimum expected loss occurs when is $x = p$.

Now we want to show that there is a probability distribution for $y = 1$ and $y = 0$ where the expected value of the absolute loss function is not minimized when $x = p$. We can write the expected value of the absolute loss function when $y = 1, 0$ as

$$E[L(x, y = 1, 0)] = z|x - 1| + w|x - 0|$$

where z is the probability of $y = 1$ and w is the probability of $y = 0$. Since we can assume $x \geq 0$ we can write the expected value of the absolute loss function above as

$$E[L(x, y = 1, 0)] = z(x - 1) + wx$$

If we let $z = p$ and $w = 1 - p$ then the expected value of the absolute loss function is

$$(p)(x - 1) + (1 - p)x = px - p + x - px = -p + x \tag{1}$$

I have no solution for this part of the problem.

3. *Use hedge as a subroutine and give an algorithm with will always achieve $O(\sqrt{T})$ regret without knowing T ahead of time.*

Pseudocode

```
def MetaHedge():
  set learning rate to the 1/sqrt(time_horizon_guess).
  start Hedge with inputs time_horizon_guess and learning_rate.
  update the learning rate with a new time_horizon_guess everytime
  the rounds (T) of the learning algorithm exceed time_horizon_guess.
```

Correctness

The solution to this question is using the time horizon guess $\hat{T}$ change the learning rate η as the algorithm runs in order to keep the regret $O(\sqrt{T})$. Two cases arise with this algorithm when we don't know how many rounds it will run for.

(a) The algorithm stops before or at the time horizon guess $\hat{T}$. In this case $\hat{T}$ will be $\geq T$, where T is the actual number of rounds the algorithm ran for. Since we overestimate or extimate exactly T with $\hat{T}$, the learning rate η and the weights and the losses will be as expected or smaller then expected. Because of this when we sum the algorithm loss and the minimum loss of the single best predition/action to get regret it will be bounded by $O(\sqrt{T})$.

(b) The number of rounds T the algorithm runs for exceeds the time horizon guess $\hat{T}$. An example of this is if we set $\hat{T} = 3$ but T ends up running for 4 rounds instead. In this case once $T = 3$ we would guess $\hat{T} > 3$ taking us back to the first case above. We would do this over and over again until the algorithm stopped.

In both of these cases the regret remains below $\sqrt{T}$ because we overestimate the number of rounds T the algorithm runs for with time horizon guess $\hat{T}$ which gives us a smaller learning rate *eta* and consistently smaller loss and weight vectors.

4. *Design an online selling algorithm which will maximize the number of customers helped in a hardward store with limited supplies.*

   (a)

   (b)

   (c)

5. *Find a mixed nash equilibrium of the zero-sum game. Give both strategies and the value of the game and show the vector of expected payoffs for player 2 under player 1's strategy and vice versa.*

$$M = \begin{bmatrix} 1 & 0 & 1 & 8 & 0 \\ 5 & 8 & 9 & 2 & 1 \\ 0 & 1 & 8 & 0 & 5 \\ 8 & 9 & 2 & 1 & 0 \\ 1 & 8 & 0 & 5 & 8 \end{bmatrix}$$

Nash equilibrium means that player 1 and player 2 won't want to deviate from their strategies. The row player wants to minimize and the column player wants to maximize.