CSCI 5454 Problem Set 2

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1. *Probability boot camp*

   (a) *Prove Markov's inequality, $Pr[X \geq c] \leq E[X]/c$, with $c > 0$*

   The formula for the probability of a continuous random variable X with probability density function $f(x)$ is

   $$Pr[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} f(x)dx$$

   And the formula for the expected value of a continuous random variable X with probability density function $f(x)$ is

   $$E[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} xf(x)$$

   so for $Pr[X \geq c]$ we have

   $$Pr[X \geq c] = \int_{c}^{\infty} f(x)dx$$

   and since X is a nonnegative random variable we have

   $$E[X] = \int_{0}^{\infty} xf(x)dx$$

   Notice that $0 < c \leq \infty$. This tells us that the bounds of $E[X]$ are greater than $Pr[X \geq c]$. We can break up the integral formed by $E[X]$ to create an inequality that will begin to look similar to the integral of $Pr[X \geq c]$.

   $$\begin{aligned} E[X] &= \int_{0}^{\infty} xf(x)dx \\ &= \int_{0}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx \\ &\geq \int_{c}^{\infty} xf(x)dx \end{aligned}$$

We can assume that $x \geq c$ because c is one of the bounds of the integral. This means we can substitute c for x.

$$\int_c^\infty xf(x)dx \geq \int_c^\infty cf(x)dx \geq c\int_c^\infty f(x)dx$$

We now have an equation for $E[X]$ that has $Pr[X \geq c]$.

$$E[X] \geq c\int_c^\infty f(x)dx = cPr[X \geq c]$$

Dividing both sides by c gives us

$$E[X]/c \geq Pr[X \geq c]$$

which is Markov's inequality. We have just shown that $Pr[X \geq c] \leq E[X]/c$, with $c > 0$ is true based on the probability and expected value of the continuous random variable X.

(b) *Prove Chebyshev's inequality* $Pr[|X - \mu| \geq c \cdot \sigma] \leq 1/c^2$

If we let $k = (c \cdot \sigma)$ we have

$$Pr[|X - \mu| \geq c \cdot \sigma] = Pr[|X - \mu| \geq k]$$

One of the properties of $|a|$ is that it can also be represented as $\sqrt{a^2}$ so we can change

$$Pr[|X - \mu| \geq k] = Pr[\sqrt{(X - \mu)^2} \geq k]$$

If we take the square root of both sides of the inequality we get

$$Pr[(X - \mu)^2 \geq k^2]$$

From here we can use Markov's inequality since we know $(X - \mu)^2$ is nonnegative. If we let $(X - \mu)^2 = X$ and $k^2 = c$ and substitute those into Markov's inequality we have

$$Pr[(X - \mu)^2 \geq k^2] \leq E[(X - \mu)^2]/k$$

The variance σ^2 of a continuous random variable X with mean μ is

$$\sigma^2 = E[(X - \mu)^2]$$

So

$$Pr[(X - \mu)^2 \geq k^2] \leq E[(X - \mu)^2]/k$$

Now becomes

$$Pr[(X - \mu)^2 \geq k^2] \leq \sigma^2/k$$

If we now change k back to $c \cdot \sigma$ we have the equation

$$Pr[(X - \mu)^2 \geq (c \cdot \sigma)^2] \leq \sigma^2/c^2 \cdot \sigma^2$$

Reducing the right side of the equation results in

$$Pr[(X - \mu)^2 \geq (c \cdot \sigma)^2] \leq 1/c^2$$

And taking the square root of both sides of the inequality on the left hand side of the equation results in

$$Pr[\sqrt{(X - \mu)^2} \geq c \cdot \sigma] \leq 1/c^2$$

Which can be further reduced to

$$Pr[|X - \mu| \geq c \cdot \sigma] \leq 1/c^2$$

Which is Chebyshev's inequality. We have just shown that Chebyshev's inequality can be proven using Markov's Inequality.

(c) *Show that for any discrete random variables* $X, X', E[X] = E[E[X|X']]$.

If we let $Y = X'$ we have

$$E[X] = E[E[X|Y]]$$

The expected value of a discrete random variable X is

$$E[X] = \sum_x x \cdot Pr[X = x]$$

The conditional probability for any two discrete random variables X, Y is defined to be

$$Pr[X = x|Y = y] = \frac{Pr[X = x \cap Y = y]}{Pr[Y = y]}$$

The conditional expectation for any two discrete random variable X, Y is defined to be

$$E[X|Y = y] = \sum_x x \cdot Pr[X = x|Y = y]$$

Given the above assertions

$$\begin{aligned}
E[E[X|Y]] &= \sum_y E[X|Y = y] \cdot Pr[Y = y] \\
&= \sum_y \sum_x x \cdot Pr[X = x|Y = y] \cdot Pr[Y = y] \\
&= \sum_x x \cdot \sum_y Pr[X = x|Y = y] \cdot Pr[Y = y] \\
&= \sum_x x \cdot Pr[X = x] \\
&= E[X]
\end{aligned}$$

Sources I used to complete this problem:

- http://www.maths.qmul.ac.uk/ pettit/MTH5122/notes15.pdf
- https://en.wikipedia.org/wiki/Expected_value

(d) *Prove by induction that $E[X_t] = 0$ for a martingale.*

We want to show that in a martingale sequence the expected value for a random variable X_{t+1} is the random variable X_t before it.

Base case:

When $t = 0$ with $X_0 = 0$ we have

$$\begin{aligned} E[X_1|X_0] &= \sum_x x \cdot Pr[X_1 = x|X_0 = 0] \\ &= 0 \end{aligned}$$

Inductive step:

We showed that the base case $E[X_1|X_0] = 0$ is true. Therefore, we can say that $E[X_2|X_1] = E[E[X_1|X0]] = 0$ no matter what X_2 is. This leads to the equation

$$E[X_t|X_0, ..., X_{t-1}] = E[E[...E[X_1|X_0]]] = 0$$

So by induction we have just shown that $E[X_t] = 0$.