CSCI 5454 Problem Set 3

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No Collaborators

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1. (a) *Use the subroutine probefluxcapacitor() to implement the routine randbit() that outputs a uniformly random bit.*

**Pseudocode**

```
def randbit():
  while true:
    (bitA, p) = probefluxcapacitor()
    (bitB, p) = probefluxcapacitor()
    if bitA does not equal bitB:
      return bitA
```

**Correctness**

If the bits were uniformly random then a 1 or a 0 would be equally probable. This would mean that the probability that either one would occur would need to be $\frac{1}{2}$ in order to be uniformly random.

For the problem we are given that $P(1) = p$ and $P(0) = (1 - p)$. The first thing we have to figure out is how do we make an algorithm such that the probability of returning a 1 is equal to the probability of returning a 0. The second thing we have to figure out is how do we make sure the final probability for any run of randbit() is equal $\frac{1}{2}$.

Notice the probability of two independent events, $A$ and $B$, occurring together is $P(A \cap B) = P(A) \cdot P(B)$ [1]. If we let those two events be the probability of randbit() returning a 1, $P(1)$, and the probability of randbit() returning a 0, $P(0)$, then we can say that the probability of them both occurring, $P(1 \cap 0)$, is $p \cdot (1 - p)$. This helps because we know that one call to probefluxcapacitor() will either return $p$, for bit value 1, or $(1 - p)$, for bit value 0. But if we combine two calls to probefluxcapacitor() sequentially and then compare the results we can create a probability like $P(1 \cap 0)$.

Two calls to probefluxcapacitor() could result in 4 four combinations:

- $P(1 \cap 1) = p \cdot p$
- $P(0 \cap 0) = (1 - p) \cdot (1 - p)$
- $P(1 \cap 0) = p \cdot (1 - p)$
- $P(0 \cap 1) = (1 - p) \cdot p$

The pseudocode above only returns a bit if the probability combination was $p \cdot (1 - p)$ or $(1 - p) \cdot p$ which are in fact equal to each other. This tells us that returning a 1 — bitA is 1 and bitB is 0 — and returning a 0 — bitA is 0 and bitB is 1 — have equal probability now. And since the if statement only looks at two possible combinations from the two calls to probefluxcapacitor() and out of only one of those possible combinations will a 1 be returned, the probability of returning a 1, $P(1)$, is $\frac{1}{2}$.

**Runtime**

In order to find the running time of a randomize algorithm, we need to look at the expectation of the running time. The two calls to probefluxcapacitor() each take $O(1)$ because they are just looking at at most two numbers. The checking of the two bits in the if statement is also $O(1)$. This means that one time through the loop takes $O(1)$. The probability of getting out of the while loop and returning is $p \cdot (1 - p)$ for a 1 or $(1 - p) \cdot p$ for a 0.

Let $X$ be the indicator random variable associated with the returning of randbit(). $X_1$ means a 1 is returned and $X_0$ means a 0 is returned. This also means $E[X_1] = P(1)$, the probability of 1 being returned, and $E[X_0] = P(0)$, the probability of 0 being returned. This gives us

$$\begin{aligned} E[X] &= \sum_{i=0}^{1} E[X_i] \\ &= E[X_0] + E[X_1] \\ &= p \cdot (1-p) + (1-p) \cdot p \\ &= 2p(1-p) \end{aligned}$$

This means the running time of randbit() is $O(2p(1-p) \cdot 1) = O(2p(1-p))$.

(b) *Implement the algorithm randbit(p) that outputs an independent random bit with P(randbit(p)=1)=p.*

**Pseudocode**

```
def randbit(p):
  (k, n) = p
  l = list of bits
  for i from 1 to n:
    bit = randbit()
    append bit to l
  if 1 in l:
    return 1
  else:
    return 0
```

**Correctness**

randbit() will return a 1 with a probability $p$ of $\frac{1}{2}$. We consider putting the independent event $A$ of running randbit() a single time into a collection. Then running randbit() $n$ times would give us a collection of $n$ events/bits. Since these events are all independent then the probability of them happening together is

$$
\begin{aligned}
&= P(A_1) \cdot P(A_2) \cdot \ldots \cdot P(A_n) \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2} \\
&= \frac{1}{2^n}
\end{aligned}
$$

$k$ is the number of outcomes we want out of the combined runs of randbit(). For example, if we want $P(randbit(\frac{3}{4}) = 1) = \frac{3}{4}$, then we would need look at the combined output of calling randbit() two times because $n = 2$. This would give us four possible possible outcomes: (0,0), (1,0), (0,1), (1,1). Out of those four possible outcomes there is a 1 contained in three of them. This means that if we return a 1 when a 1 is in a possible outcome we get $p = 3/4$.

In the pseudocode above we generalize this idea by running randbit() $n$ times and keeping a list of the bits that are returned by randbit(). If a 1 exists in the list then we return a 1 otherwise we return a 0.

This technique doesn't appear to work for any other example so it is not correct. I was unable to figure out a solution to this problem.

(c) *Show the correctness and runtime for randbit(p) when p is not of the form* $k/2^n$

**Correctness**

Let's look at the alorithm when $p = 3/4$. This means $p = 0.11$ and $i$ will be in the range 0 to 1. $d$ is first set to $b_1$ which is a 1. Then randbit() is called and its output is compared to $d$. The probability of randbit() returning a 0 which would end the function is 1/2. The next time through $d$ is set to $b_2$ which is a 1. Again randbit() is called and its output is compared to $d$ and the function returns if the output of randbit() is not the same as $d$. Each time, the probability of the output of randbit() not being equal to $d$ is 1/2.

I don't have any answer for proving the correctness of this problem.

**Runtime**

We are given that getdigit(p,i) runs in $O(i^c)$ time for some $c$. The if statement where the output of randbit() is compared to $d$ has a probability of 1/2 of being correct each time through the loop. The loop itself could run a maximum total of $i$ times which is equal to the number of digits in the binary representation of $p$.

We can then say the upper bound on this function is $O(i(i^c + 1/2))$

2. *Give a linear time algorithm that determines if an element appears more than $n/2$ times in an array with probability $1 - p$ where $p$ is small.*

**Pseudocode**

```
def AppearsOften(A):
  n = total elements in A
  counter = 0
  i = 1
  if n > 1:
    for j from 1 to n:
      if arethesame(i,j):
        counter = counter + 1
      elif !arethesame(i,j):
        counter = counter - 1
      if counter < 1:
        if j < n:
          i = j + 1
          counter = 0
    if counter > 0:
      return 'YES'
    else:
      return 'NO'
  else:
    return 'YES' # Single element in the array so it must occur > n/2 times
```

**Correctness**

The algorithm AppearsOften() goes through all of the elements in array $A$ and attempts to keep a count of the element that appears most often in $A$. The base case, when $i = 1$, occures when $A$ has only one element and the algorithm returns "YES" because $n/2 = 1/2$ and the element occures one time which is greater than $1/2$.
For the inductive step we assume $n > 1$ and $i, j \leq n$. Each iteration through the loop, we compare the $i$th element with the $j$th element. If they are the same we increment the counter otherwise we decrement the counter. The counter helps to determine if after having gone through the elements in the array there is an element $x$ that occurs $> n/2$ times. The counter would need to be greater than 1 in order for there to be an element that occurs $> n/2$ times. We can think of this as breaking the array into comparisons

between pairs. $x$ would occur in a lot of pairs because it is the element that occurs the most so the counter would end up being higher after the completion of the for loop.

This algorithm will return “NO” when there isn't an element that occurs $> n/2$ times but it will also return “NO” in some cases when there is an element $> n/2$. This also occurs for “YES” so I am unable to prove it correctly says “NO” if there is no such element and “YES” with probability $1 - p$.

**Runtime**

The algorithm goes all $n$ elements of array $A$ at most one time. Keeping a counter and checking if elements are the same and checked the value of the counter take constant time, O(1), each. This means the runtime for this algorithm is $O(n)$.

3. *Help Gru understand his minion dispensing situation*

   (a) *What is the expected number of minions in the first pod after t seconds?*

   If a minion is added to a pod every second we expect there to be a total of $t$ minions combined from all of the pods after $t$ seconds. Let $X$ be the random variable that represents the number of minions in the first pod. After time $t$, $X$ can be between 0 and $t$ which means the pod can contain between 0 and $t$ minions. The expected value of the number of minions $i$ in the first pod after time $t$ can be represented as the binomial distribution [2]

$$\begin{aligned} E[X] &= \sum_{i=1}^{t} i \binom{t}{i} \frac{1}{k^i} (1 - \frac{1}{k})^{t-i} \\ &= \frac{t}{k} \end{aligned}$$

   (b) *In expectation, when will the first pod get its first minion?*

   Let $X$ be the random variable that represents the number of minions that are dispensed before a minion goes into the first pod. The expected value of the number of minions $i$ that are dispensed before the first pod will get a minion can be represented as the geometric distribution [3]

$$\begin{aligned} E[X] &= \sum_{i=1}^{\infty} i(1 - \frac{1}{k})^{i-1} \cdot \frac{1}{k} \\ &= k \end{aligned}$$

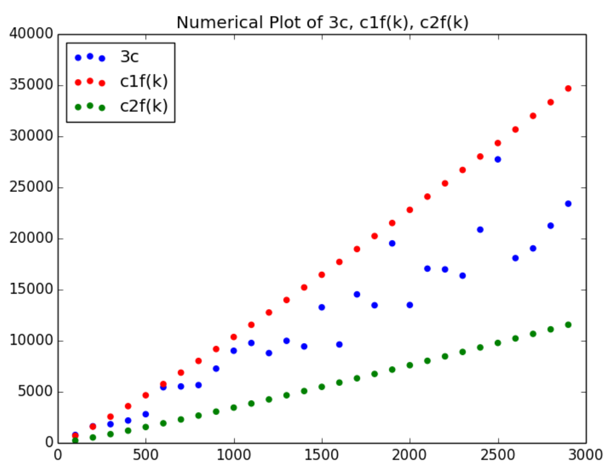
   (c) *In expectation, when will all pods have at least one minion?*

   Let $n$ be the number of minions we need to dispense in order to have at least one minion in all $k$ pods. Let $n_i$ be the number of minions we have to dispense in order to fill the ith pod. The probability that the ith pod will be filled is $(k - i + 1)/k$. The expected time until the ith pod gets a minion is $k/(k - i + 1)$.

The expected value of $n$ is then [4]

$$\begin{aligned} E[n] &= \sum_{i=1}^{k} E[n_i] = \sum_{i=1}^{k} k/(k-i+1) \\ &\leq \sum_{i=1}^{k} k/(k-i) = k \sum_{i=1}^{k} 1/(k-i) \\ &\leq k \sum_{i=1}^{k} 1/i = \theta(k \, log \, k) \end{aligned}$$

(d) *Plot 3c,* $c_1 f(k)$, $c_2 f(k)$.




I let $c_1 = 1.5$ and $c_2 = .5$.

4. *Implementation of Karger's min-cut algorithm using Kruskal's algorithm.*

   (a) *Explain how to view Kargers algorithm as constructing a spanning tree.*

   A spanning tree is a subgraph that is a tree that includes all of the vertices in the graph [6]. It won't necessarily include all of the edges in a graph. Karger's randomly chooses an edge and contracts it making the two vertices at the endpoints of the edge into one vertex. It then reconnects all of the edges from those two vertices to the now single vertex except the one edge it contracted. By the end of the algorithm it will have chosen enough edges to contract that all of the vertices in the graph would have been one of the endpoints of a contraction. Karger's returns a subgraph made up of two vertices of the original graph which can be said to be a spanning tree of the original graph.

   (b) *Show that the probability of spanning tree $T$ being selected is the same in both.*

   When Karger's algorithm chooses a random edge and then contracts it, it almost like running Kruskal's with uniform random weights [5]. Kruskal's algorithm will choose the minimum weight edge which has the same probability as Karger's choosing a random edge.

   We can say that in a uniform distribution the probability of choosing an edge is the same for every edge. In the case of Karger's if we have $k$ edges then in a uniform probability distribution the probability of choosing any edge is $1/k$. In the case of Kruskal's algorithm, if we have $k$ edges and each with a uniformly random weight choosed the probability of choosing the lowest weight will be $1/k$ assuming there are no duplicate weights.

   Let $X$ be the random variable that the spanning tree $T$ will be selected by Karger's and $Y$ be the random variable that the spanning tree $T$ will be selected by Kruskal's. For their distributions to be the same [7]

$$P(X \leq x) = P(Y \leq x) \, \text{for all} \, x$$

   This means the moment generating functions for these variables need to be the same in order for the their distributions to be the same. Since we know both are continous uniform distributions and the probability of selecting an edge is the same, their moment generating funcitons are the same and the probability of selecting $T$ is the same.

(c) *Assuming $T$ was chosen by both algorithms, explain why the max-weight edge of $T$ in Kruskal defines the final cut output by Karger.*

In Kruskal's the max-weight edge will be the last edge chosen since Kruskal's algorithm chooses edges that are the minimum weight left in the graph. In Karger's algorithm the final cut is involves the last edge left after all contractions of the graph have taken place. If the probabilities are uniform for choosing an edge in both algorithm's, the max-weight edge in Kruskal's will be the final cut edge in Karger's since it is the last edge remaining in the graph.

(d) *Put the above three steps together to prove the equivalence of the two algorithms: the distribution over cuts they return is the same.*

Kruskal's algorithm produces a spanning tree and in part (a) we showed that Karger's algorithm also produces a spanning tree. In part (b) we showed that the probability of selecting a spanning tree is the same. In part (c) we showed that the last (max-weight) edge chosen by Kruskal's is the same as the edge chosen by Karger's that the min-cut crosses.

Combining all three parts we can say both produce a spanning tree, both produce the same spanning tree, and if we were to remove the max-weight edge from the spanning tree Kruskal's algorithm produces it would be the same edge that the min-cut crosses in Karger's. All things being equal, randomized Kruskal's with max-weight edge removal is the same a Karger's min-cut algorithm because they both return the same cut of vertices.

5. *Implement Karger's min-cut algorithm.*

6. *Help minion captain A maximize the sum of the player's abilities on his team.*

**Pseudocode**

**Correctness**

**Runtime**

# References


[1] https://www.mathsisfun.com/data/probability-events-independent.html

[2] https://en.wikipedia.org/wiki/Binomial_distribution

[3] https://en.wikipedia.org/wiki/Geometric_distribution

[4] http://tuvalu.santafe.edu/ aaronc/courses/5454/csci5454_spring2013_L4.pdf

[5] https://en.wikipedia.org/wiki/Karger%27s_algorithm

[6] https://en.wikipedia.org/wiki/Spanning_tree

[7] https://en.wikipedia.org/wiki/Random_variable#Equality_in_distribution