CSCI 5454 Problem Set 4

Robert Werthman

No Collaborators

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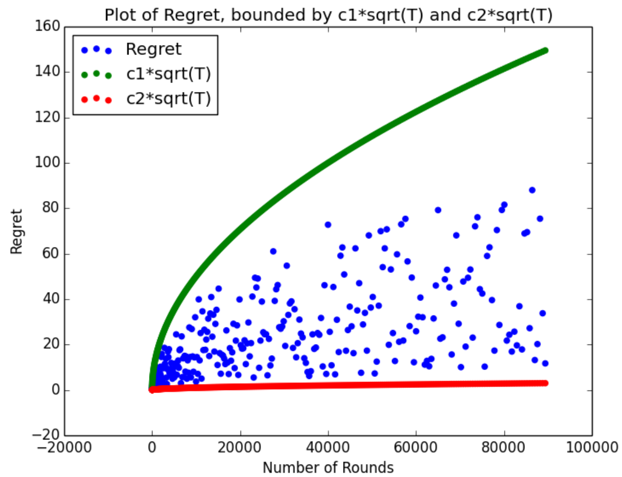
1. *Implement the action setting of Hedge and use it to complete a few tasks.*

   (a) *Use Hedge to write a game AI and display some sample output.*

```
  Payoff Matrix
[-8, 10, 2]
[-4, -2, -1]
[-8, -6, 1]
##########
Round 0
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.3333333333333333,0.3333333333333333,0.3333333333333333]
Action chosen by AI 1
AI loss vector [-8, 10, 2]
AI weight vector [2980.9579870417283, 4.5399929762484854e-05, 0.1353352832366127]
old AI score 0.0 , new AI score -10.0 , difference -10.0
old user score 0.0 , new user score 10.0 , difference 10.0
##########
Round 1
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.9999545869027009,
1.522928810416062e-08, 4.5397868011057175e-05]
Action chosen by AI 0
AI loss vector [-8, 10, 2]
AI weight vector [8886110.520507872, 2.061153622438558e-09, 0.018315638888734182]
old AI score -10.0 , new AI score -2.0 , difference 8.0
old user score 10.0 , new user score 2.0 , difference -8.0
##########
Round 2
##########
Enter the row you wish to choose: 0
Action chosen by user 0
Probability distribution of AI actions [0.9999999979388461,2.319522825462676e-16,2.0611536181902033e-09]
Action chosen by AI 0
AI loss vector [-8, 10, 2]
AI weight vector [26489122129.84347, 9.357622968840175e-14, 0.002478752176666359]
old AI score -2.0 , new AI score 6.0 , difference 8.0
old user score 2.0 , new user score -6.0 , difference -8.0
```

   (b) *Give a plot that exhibits $\Theta(\sqrt{T})$ regret.*



I let $c_1 = .5$ and $c_2 = .01$ and $\eta = \sqrt{\frac{8 ln\, N}{T}}$. The graph shows that regret is bounded by $\Theta(\sqrt{T})$ where T is the number of rounds. Regret remained $\leq \sqrt{(\frac{T}{2} ln\, N)}$. At times regret would get really close to the bound but other times it would not be close. It never went above the bound. Example output of regret in the first column and $\sqrt{(\frac{T}{2} ln\, N)}$ in the second column is below.

```
0.0854754326956 1.16388787282
0.711207959794 1.55185049709
0.915738476176 1.93981312136
1.86659059913 2.32777574563
1.11868740447 2.7157383699
0.942107984927 3.10370099418
1.20108924348 3.49166361845
2.33209799223 3.87962624272
2.39799236386 4.26758886699
3.71141140061 4.65555149126
3.10138106514 5.04351411554
1.19507429131 5.43147673981
3.95892727292 5.81943936408
0.926603427611 6.20740198835
2.69832933191 6.59536461262
0.594289924894 6.9833272369
6.25916816147 7.37128986117
3.62512254793 7.75925248544
0.855690231446 8.14721510971
```

2. *Show that squared loss guarantees that the best prediction in terms of expected loss is going to be $x = p$. Give a distribution where this is not the case for absolute loss.*

We want to prove that the expected loss $E[L(x, y)]$ is minimized by letting $x = p$. If $y = 1$ with probability p and $y = 0$ with probability $1 - p$ then

$$\begin{aligned} E[L(x, y)] &= p \cdot L(x, 1) + (1 - p) \cdot L(x, 0) \\ &= p \cdot (x - 1)^2 + (1 - p) \cdot x^2 \\ &= p \cdot (x^2 - 2x + 1) + (1 - p) \cdot x^2 \\ &= px^2 - 2px + p + x^2 - px^2 \\ &= x^2 - 2px + p \end{aligned}$$

The minimum of a quadratic function of the form $f(x) = ax^2 + bx + c$ can be found with the equation

$$x = -\frac{b}{2a}$$

Solving for x we get

$$-1 \cdot \frac{(-2px)}{2 \cdot 1} = \frac{2p}{2} = p$$

This shows that the minimum expected loss occurs when is $x = p$.

Now we want to show that there is a probability distribution for $y = 1$ and $y = 0$ where the expected value of the absolute loss function is not minimized when $x = p$. We can write the expected value of the absolute loss function when $y = 1, 0$ as

$$E[L(x, y = 1, 0)] = z|x - 1| + w|x - 0|$$

where z is the probability of $y = 1$ and w is the probability of $y = 0$.

I do not have a solution for this part of the problem.

3. *Use hedge as a subroutine and give an algorithm with will always achieve $O(\sqrt{T})$ regret without knowing T ahead of time.*

Pseudocode

```
def MetaHedge():
  set learning rate to the 1/sqrt(time_horizon_guess).
  start Hedge with inputs time_horizon_guess and learning_rate.
  update the learning rate with a new time_horizon_guess everytime
  the rounds (T) of the learning algorithm exceed time_horizon_guess.
```

Correctness

The solution to this question is using the time horizon guess $\hat{T}$ to change the learning rate η as the algorithm runs in order to keep the regret $O(\sqrt{T})$. Two cases arise with this algorithm when we don't know how many rounds it will run for.

(a) The algorithm stops before or at the time horizon guess $\hat{T}$. In this case $\hat{T}$ will be $\geq T$, where T is the actual number of rounds the algorithm ran for. Since we overestimate or extimate exactly T with $\hat{T}$, the learning rate η and the weights and the losses will be as expected or smaller then expected. Because of this when we sum the algorithm loss and the minimum loss of the single best predition/action to get regret it will be bounded by $O(\sqrt{T})$.

(b) The number of rounds T the algorithm runs for exceeds the time horizon guess $\hat{T}$. An example of this is if we set $\hat{T} = 3$ but T ends up running for 4 rounds instead. In this case once $T = 3$ we would guess $\hat{T} > 3$ taking us back to the first case above. We would do this over and over again until the algorithm stopped.

In both of these cases the regret remains below $\sqrt{T}$ because we overestimate the number of rounds T the algorithm runs for with time horizon guess $\hat{T}$ which gives us a smaller learning rate *eta* and consistently smaller loss and weight vectors.

4. *Design an online selling algorithm which will maximize the number of customers helped in a hardward store with limited supplies.*

   (a) *Give a deterministic algorithm with a competitive ratio ≤ 2 between the cost of itself and the cost of the optimal offline solution like vertex cover.*

Pseudocode

```
def OnlineHelpCustomer(customers, store_inventory):
  for each customer:
    for item in customer.set:
      if item in store_inventory:
        return item to sell to customer
    return nothing to sell to customer
```

Correctness

OnlineCustomerHelp acts like maximal matching. It chooses an item in a customers set and and if that item has not already been chosen that item is then sold to that customer. The customer and item are never seen again. In maximal matching the item and customer represent vertices and once they have been matched the vertices cannot be used again in another match. For each customer we want to choose an item in their set that maximizes the ability of other customers to choose items in their sets that have not already been chosen by other customers.

The difference between the online solution and the optimal offline solution is that the offline solution knows all of the items in the sets of all the customers. It can then choose which items are sold to which customer to maximize the number of customers who walk away with an item sold to them.

The offline optimal solution is as follows.

```
def OfflineHelpCustomer(customers):
  create a bipartite graph G that contains customers and items
  find the maximal matching in the graph
  sort the customers in descending order by their set size
  start with the customer with the largest set and sell them
    the item found in the maximal matching
```

```
do this for every customer possible
```

The worst case for either of these aglorithms is when one customer has all of the items in their set and the other customers have just one item and that item is the same for everyone else. This leads to the smallest amount of customers helped. For instance if we have two items x and y and customers A = {y,x,z} , B = {y}, C = {y}. In this case OfflineCustomerHelp would at worst sell two items – item x or z to customer A and then item y to customer B or C. The OnlineCustomerHelp would at worst sell item y because it saw customer A first and chose the first item that had not been sold which was y. In this case, OnlineHelpCustomer ends up selling one item which is at least half of the items sold in OfflineCustomerHelp's solution.

The cost of the solution to these algorithms is the number of customers who don't walk away with an item sold to them. If you sold k items you saw at least k customers which means that if there are n customers then $n - k$ customers were not sold an item. The pattern for the cost of the solution of OfflineCustomerHelp depends on the ceiling of the mean size of the all the customers' sets which indicates how many items could be sold. The cost of the solution for the online algorithm OnlineCustomerHelp depends on the median of the size of all the customer's sets. The median gives the number of items sold to customers. These patterns for the offline and online algorithms give a competitive cost ratio of

$$OPT/ALG = \frac{n - mean(|S_n|)}{n - median(|S_n|)} \leq 2.$$

(b) *Show that every deterministic algorithm has competitive ratio ≥ 2.*

ALG is the cost found by a deterministic algorithm for a solution and OPT is the cost of the optimal solution. In order for every deterministic algorithm to have a competitive ratio of at least 2 we have to assume that

$$OPT/ALG \leq 2$$

This means that

$$OPT/2 \leq ALG$$

This says that the cost of the solution returned by the deterministic algorithm is at least half of the cost of the optimal solution. We showed this was the case for the single deterministic algorithm found in 4a. How do we show that this is the case for every deterministic algorithm?

We know that the cost of a solution given by a deterministic algorithm will either be worse or the same as the cost of the optimal solution. This gives us the inequality

$$OPT \leq ALG$$

In fact the cost of the solution given by a deterministic algorithm could be any constant c, $c > 1$, times the optimal solution. This produces the inequality

$$OPT \leq c \cdot ALG$$

I do not have a solution for why $c = 2$.

(c) *Prove that the expected value of the competitive ratio of randomized version of the online selling algorithm is < 2.*

If customer i arrives at the shop and wishes to purchase an item the probability of them receiving any item from their set is

$$p_i = \frac{1}{S_i \cup U_i}$$

If X is the random variable that represents the number of customers that receive an item from the store, the expected value of all n customers receiving an item is

$$E[X] = \sum_{i=1}^{n} i \cdot p_i = \sum_{i=1}^{n} i \cdot \frac{1}{S_i \cup U_i}$$

The cost of the randomized version of the online selling algorithm is then equal to

$$n - E[X]$$

which is the number of customers that do not receive an item.

We want to show that the competitive ratio of the expectation of this randomized algorithm is less than 2 show in the equation below

$$\frac{OPT}{ALG} < 2 \Rightarrow \frac{OPT}{n - E[X]} < 2$$

5. *Find a mixed nash equilibrium of the zero-sum game. Give both strategies and the value of the game and show the vector of expected payoffs for player 2 under player 1's strategy and vice versa.*

$$M = \begin{bmatrix} 1 & 0 & 1 & 8 & 0 \\ 5 & 8 & 9 & 2 & 1 \\ 0 & 1 & 8 & 0 & 5 \\ 8 & 9 & 2 & 1 & 0 \\ 1 & 8 & 0 & 5 & 8 \end{bmatrix}$$

Nash equilibrium means that player 1 and player 2 won't want to deviate from their strategies. The row player wants to minimize and the column player wants to maximize.