CSCI 5454 Problem Set 2

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1. *Probability boot camp*

   (a) *Prove Markov's inequality, $Pr[X \geq c] \leq E[X]/c$, with $c > 0$*

   The formula for the probability of a continuous random variable X with probability density function $f(x)$ is

   $$Pr[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} f(x)dx$$

   And the formula for the expected value of a continuous random variable X with probability density function $f(x)$ is

   $$E[x_1 \leq X \leq x_2] = \int_{x_1}^{x_2} xf(x)$$

   so for $Pr[X \geq c]$ we have

   $$Pr[X \geq c] = \int_{c}^{\infty} f(x)dx$$

   and since X is a nonnegative random variable we have

   $$E[X] = \int_{0}^{\infty} xf(x)dx$$

   Notice that $0 < c \leq \infty$. This tells us that the bounds of $E[X]$ are greater than $Pr[X \geq c]$. We can break up the integral formed by $E[X]$ to create an inequality that will begin to look similar to the integral of $Pr[X \geq c]$.

   $$\begin{aligned} E[X] &= \int_{0}^{\infty} xf(x)dx \\ &= \int_{0}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx \\ &\geq \int_{c}^{\infty} xf(x)dx \end{aligned}$$

We can assume that $x \geq c$ because c is one of the bounds of the integral. This means we can substitute c for x.

$$\int_c^\infty xf(x)dx \geq \int_c^\infty cf(x)dx \geq c\int_c^\infty f(x)dx$$

We now have an equation for $E[X]$ that has $Pr[X \geq c]$.

$$E[X] \geq c\int_c^\infty f(x)dx = cPr[X \geq c]$$

Dividing both sides by c gives us

$$E[X]/c \geq Pr[X \geq c]$$

which is Markov's inequality. We have just shown that $Pr[X \geq c] \leq E[X]/c$, with $c > 0$ is true based on the probability and expected value of the continuous random variable X.

(b) *Prove Chebyshev's inequality* $Pr[|X - \mu| \geq c \cdot \sigma] \leq 1/c^2$

One of the properties of $|a|$ is that it can also be represented as $\sqrt{a^2}$.
We can then change $|X - \mu|$ to $\sqrt{(X-\mu)^2}$.
If we take the square root of both sides of $\sqrt{(X-\mu)^2} \geq c \cdot \sigma$ we get

$$(X-\mu)^2 \geq (c \cdot \sigma)^2$$

The variance σ^2 of a continuous random variable X with mean μ is

$$\sigma^2 = E[(X-\mu)^2]$$

(c) *Show that for any discrete random variables* $X, X', E[X] = E[E[X|X']]$.

(d) *Prove by induction that* $E[X_t] = 0$ *for a martingale.*