CSCI 5654-Fall16: Assignment #2
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P1.
(A)

Entering variables	Leaving variables
x_2	w_1 or w_5
x_5	w_5
x_6	w_2 or x_3
w_3	w_2 or x_3

(B)
To find the corresponding solutions, we need to solve for x_2 in the w_5 row of the dictionary:

$$x_2 = 2 - \frac{x_1}{2} - 2x_5 + \frac{w_4}{2} - \frac{w_5}{2}$$

This equation is then plugged into all the places x_2 is a nonbasic variable.

Basic variables	Solutions
w_1	0
w_2	2
x_3	4
x_4	1
x_2	2
z	15

Nonbasic variables
x_1
w_5
x_5
x_6
w_3
w_4

Yes, this dictionary is degenerate. The solution b of the basic variable w_1 is 0.

(C)
The next dictionary looks like:

$$\begin{array}{r|cccccccc}
w_1 & 0 & -7x_1 & -7x_5 & +w_4 & -2w_5 & -2x_6 & +2w_3 \\
w_2 & 2 & -x_1 & -x_5 & & & -x_6 & -w_3 \\
x_3 & 4 & -\frac{x_1}{2} & -\frac{w_5}{2} & -\frac{3w_4}{2} & -\frac{w_5}{2} & -x_6 & -w_3 \\
x_4 & 1 & -\frac{x_1}{2} & +x_5 & +\frac{w_4}{2} & +\frac{w_5}{2} & -x_6 & +2w_3 \\
x_2 & 2 & -\frac{x_1}{2} & -2x_5 & +\frac{w_4}{2} & -\frac{w_5}{2} & & \\
\hline
z & 15 & -2x_1 & -3x_5 & -w_4 & -w_5 & +x_6 & +3w_3
\end{array}$$

We can choose betwen x_6 and w_3 to be the entering variable. We should choose w_3 to be the entering variable, because it increases the value of the objective function z the most. Since the leaving variable is automatically chosen based on the which basic variable constrains the entering variable the most, the value of the objective function does not depend on the choice of the leaving variable.

P2. (15 points) Consider the following feasible dictionary:

$$
\begin{array}{c|cccccc}
x_{B,1} & b_1 & +a_{11}x_{N,1} & +\cdots & +{\color{red}a_{1j}}x_{N,j} & \cdots & +a_{1n}x_{N,n} \\
\vdots & \vdots & & \ddots & \vdots & \ddots & \\
x_{B,i} & b_i & +a_{i1}x_{N,1} & +\cdots & +{\color{red}a_{ij}}x_{N,j} & \cdots & +a_{in}x_{N,n} \\
\vdots & & \ddots & & & & \\
x_{B,m} & b_m & +a_{m1}x_{N,1} & +\cdots & +{\color{red}a_{mj}}x_{N,j} & \cdots & +a_{mn}x_{N,n} \\
\hline
z & c_0 & +c_1x_{N,1} & +\cdots & +{\color{red}c_j}x_{N,j} & \cdots & +c_nx_{N,n}
\end{array}
$$

(A) Suppose $x_{N,j}$ is chosen to enter and $x_{B,i}$ is the corresponding leaving variable, then show that the value of $x_{B,k}$ in the next dictionary after pivoting is given by

$$b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right).$$

Also write down for each of the constants b_k, a_{kj}, b_i, a_{ij} whether it is known that constant will be > 0, < 0, ≥ 0, ≤ 0 or nothing may be said about its sign.

(B) Show that if the leaving variable analysis is correct then the value of each basic variable $x_{B,k}$ in the subsequent dictionary is ≥ 0. In other words:

$$b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0\,.$$

In other words, we conclude that starting from a feasible dictionary and pivoting yields another feasible dictionary.

(**Hint:** Split two cases on the sign of a_{kj}. For one case it will be trivially true. For the other, you have to appeal to the leaving variable analysis as to why $x_{B,i}$ was the leaving variable and not $x_{B,k}$).

(C) Using the analysis above, prove that if $x_{B,k}$ and $x_{B,i}$ are both possible leaving variables ($i \neq k$) for $x_{N,j}$ entering, then the subsequent dictionary will be degenerate. (**Hint:** Assume that $x_{B,i}$ is chosen to leave. Show that even though $x_{B,k}$ did not leave the basis, its value in the next dictionary will be 0).

P3 (10 points) Provide examples of dictionaries that satisfy the properties stated below. Try to construct examples that are as small as possible. If no such dictionaries can exist, briefly reason why.

(A) A degenerate dictionary that is also unbounded. Recall that an unbounded dictionary does not have a leaving variable for some choice of an entering variable.

(B) A degenerate dictionary D which upon pivoting yields another degenerate dictionary D', but the objective value strictly increases.

(C) A non-degenerate dictionary D which upon pivoting yields another dictionary D' but the value of the objective function stays the same.

(D) A dictionary that is feasible but upon pivoting yields an infeasible dictionary.

(E) A dictionary that does not have leaving variable (is unbounded) for one choice of entering variable but has a leaving variable for a different choice of an entering variable.

P4 (15 points) Consider the polyhedron below:

$$\begin{array}{rrrcl} -x & +2y & +2z & \leq & 2 \\ 2x & -y & +2z & \leq & 2 \\ x & & & \geq & 0 \\ & y & & \geq & 0 \\ & & z & \geq & 0 \end{array}$$

(A) Compute all the vertices and for each vertex write down if it is degenerate or non-degenerate.
(B) Consider the optimization problem:

$$\begin{array}{rrrrcll} \max & 2x & +3y & -2z & & & \\ \mathsf{s.t} & -x & +2y & +2z & \leq & 2 & \#Slack\ w_1 \\ & 2x & -y & +2z & \leq & 2 & \#Slack\ w_2 \\ & x & & & \geq & 0 & \\ & & y & & \geq & 0 & \\ & & & z & \geq & 0 & \end{array}$$

Write down all the dictionaries corresponding to the degenerate vertices. Use slack variables w_1, w_2 as indicated.
(C) Draw a graph whose nodes are the vertices described in (A) with edges between adjacent vertices.

(D, extra credit) Given a polyhedron P, and for each vertex of the polyhedron, can you write down an objective function that is uniquely maximized only at that vertex and no other vertex of P?

(E, extra credit) For any polyhedron P, the polyhedral graph (also called its skeleton) is one where the nodes form the vertices of the polyhedron, and the edges connect adjacent vertices. Prove that this graph is strongly connected for any P. I.e, given any two vertices $\mathbf{v}_1$ and $\mathbf{v_2}$ there is a path between them in this graph.

(F, extra credit) Prove that the graph in part (E) for a d-dimensional polyhedron P has the property that if any subset of $d-1$ or fewer vertices in the graph are removed, it will still remain strongly connected (This is called Balinski's theorem).

To illustrate this, draw the skeleton of a cube and remove any two vertices from this skeleton. You will notice that there is a path in this graph between any pair of remaining vertices.