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Homework 1

P1.

1. First change the problem into a maximization problem:

$$\text{minimize } 3x_1 - 5x_2 \Rightarrow \text{maximize } -3x_1 + 5x_2$$

2. Change the constraints to $\leq$:

$$4x_1 + x_2 \geq -4 \Rightarrow -4x_1 - x_2 \leq 4$$

$$2x_1 - x_2 \geq -8 \Rightarrow -2x_1 + x_2 \leq 8$$

3. Make sure the variables have non-negativity constraints:

$$x_2 \mapsto x_2^+ - x_2^-$$

$$x_2^+, x_2^- \geq 0$$

Replace x_2 with $x_2^+ - x_2^-$ in the problem.

The problem can now be written in standard form as:

$$\begin{aligned} \textbf{maximize} \quad & -3x_1 + 5x_2^+ - 5x_2^- \\ \textbf{s.t.} \quad & -4x_1 - x_2^+ + x_2^- \leq 4 \\ & -2x_1 + x_2^+ - x_2^- \leq 8 \\ & x_1 + 2x_2^+ - 2x_2^- \leq 4 \\ & x_1, x_2^+, x_2^- \geq 0 \end{aligned}$$

P2.

Slack, initial dictionary, feasibility

I will add the slack variables w_1, w_2, and w_3 to the constraints and let z be the value of the objective function.

Rewriting the problem with the slack variables gives:

$$
\begin{aligned}
\textbf{maximize } z &= -3x_1 + 5x_2^+ - 5x_2^- \\
\textbf{s.t. } w_1 &= 4 + 4x_1 + x_2^+ - x_2^- \\
w_2 &= 8 + 2x_1 - x_2^+ + x_2^- \\
w_3 &= 4 - x_1 - 2x_2^+ + 2x_2^- \\
& x_1, x_2^+, x_2^-, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

The first dictionary of this problem can be written as:

$$
\begin{array}{l}
w_1 = 4 + 4x_1 + x_2^+ - x_2^- \\
w_2 = 8 + 2x_1 - x_2^+ + x_2^- \\
w_3 = 4 - x_1 - 2x_2^+ + 2x_2^- \\
\hline
z = 0 - 3x_1 + 5x_2^+ - 5x_2^-
\end{array}
$$

A solution to this dictionary is feasible if and only if all the right-hand sides are nonnegative as set by the non-negativity constraints.

If we set the non-basic variables x_1, x_2^+, x_2^- to 0 then we have the solutions: $w_1 = 4$, $w_2 = 8$, $w_3 = 4$, and $z = 0$.

Since the non-negativity constraints are respected, this dictionary is feasible.

Initial pivoting

To pivot the dictionary we need to choose a variable from the objective function that if increased the value z of the objective function increases. This is the case because we are trying to maximize the value of the objective function. In this case, we will choose x_2^+ because it is the only variable that will increase the value of z. The other variables all of $-$ in front which means that if we increased those variables, we would decrease the value z of the objective function.

To find the leaving variable we need to solve for x_2^+ in all of the equations of the dictionary where their is a constraint on how much x_2^+can increase. We then choose the variable where x_2^+ has the lowest upper limit. We set all of the other variables to 0.

$$
\begin{aligned}
w_1 &= 4 + 4x_1 + x_2^+ - x_2^- \Rightarrow \text{No constraint on the increase of } x_2^+ \\
w_2 &= 8 + 2x_1 - x_2^+ + x_2^- \Rightarrow x_2^+ \leq 8 \\
w_3 &= 4 - x_1 - 2x_2^+ + 2x_2^- \Rightarrow x_2^+ \leq 2
\end{aligned}
$$

In this case, we choose w_3 as the leaving variable because it gives x_2^+ the lowest upper bound constraint.

Once we have the entering and leaving variables, we need to find the new dictionary after pivoting. To do this we first solve the equation of the leaving variable for the entering variable:

$$
w_3 = 4 - x_1 - 2x_2^+ + 2x_2^- \Rightarrow x_2^+ = 2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2}
$$

We then substitute that equation in for all places where the entering variable occurs in the dictionary:

$$
\begin{aligned}
w_1 &= 4 + 4x_1 + (2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2}) - x_2^- \\
w_2 &= 8 + 2x_1 - (2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2}) + x_2^- \\
x_2^+ &= 2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2} \\
z &= 0 - 3x_1 + 5(2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2}) - 5x_2^-
\end{aligned}
$$

After some algebra, the new dictionary after pivoting looks like:

$$
\begin{array}{l}
w_1 = 6 + \frac{7}{2}x_1 - \frac{w_3}{2} \\
w_2 = 6 + \frac{3}{2}x_1 + \frac{w_3}{2} \\
x_2^+ = 2 - \frac{x_1}{2} + x_2^- - \frac{w_3}{2} \\
\hline
z = 10 - \frac{11}{2}x_1 - \frac{5}{2}w_3
\end{array}
$$

P3.

Linear programming model of the problem

The client needs to invest money in each of investments options in order to maximize their profit. Let x_i be the amount of money invested in the investment option with ID i. Let epu_i be the expected profit per unit for the investment option with ID i. Let pu_i be the price per unit for the investment option with ID i. The objective function for this maximization problem can then be written as:

$$\textbf{maximize} \sum_{i=1}^{15} \frac{epu_i}{pu_i} x_i$$

This says that the total expected profit of all the investments is equal to the sum of the amount of money invested in each investment option divided by the price per unit for each investment option multiplied by the expected profit per unit for each investment option. This is subject to the constraints:

1. $\sum_{i=1}^{15} x_i \leq 10000$
   - Total investment is at most \$10,000.
2. $1500 \leq x_1 + x_4 + x_{10} + x_{13} \leq 3500$
   - The minimum and maximum investment in risk category A.
3. $4500 \leq x_2 + x_5 + x_8 + x_9 + x_{14} \leq 6500$
   - The minimum and maximum investement in risk category B.
4. $1000 \leq x_3 + x_7 + x_{11} + x_{15} \leq 3000$ - The minimum and maximum investement in risk category C.
5. $500 \leq x_6 + x_{12} \leq 2500$
   - The minimum and maximum investement in risk category D.
6. $0 \leq x_1 + x_8 + x_{11} \leq 3000$
   - The minimum and maximum investement in investement market Tech.
7. $0 \leq x_2 + x_3 + x_5 + x_6 + x_7 + x_{15} \leq 4000$
   - The minimum and maximum investement in investement market Finance.
8. $0 \leq x_4 + x_9 + x_{13} \leq 5000$
   - The minimum and maximum investement in investement market PetroChem.

9. $0 \leq x_{10} + x_{12} + x_{14} \leq 7000$
   - The minimum and maximum investement in investement market Automobile.
10. $2000 \leq x_1 + x_2 + x_3 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{13} \leq 10000$
   - The minimum and maximum investement in EcoFriendly.
11. $x_1, ..., x_{15} \geq 0$
   - You cannot invest a negative amount of money in an investment option.

P4.

P5.