CSCI 5654-Fall16: Assignment #2
Robert Werthman

P1.
(A)

Entering variables	Leaving variables
x_2	w_1 or w_5
x_5	w_5
x_6	w_2 or x_3
w_3	w_2 or x_3

(B)
To find the corresponding solutions, we need to solve for x_2 in the w_5 row of the dictionary:

$$x_2 = 2 - \frac{x_1}{2} - 2x_5 + \frac{w_4}{2} - \frac{w_5}{2}$$

This equation is then plugged into all the places x_2 is a nonbasic variable.

Basic variables	Solutions
w_1	0
w_2	2
x_3	4
x_4	1
x_2	2
z	15

Nonbasic variables
x_1
w_5
x_5
x_6
w_3
w_4

Yes, this dictionary is degenerate. The solution b of the basic variable w_1 is 0.

(C)
The next dictionary looks like:

$$\begin{array}{r|cccccccc}
w_1 & 0 & -7x_1 & -7x_5 & +w_4 & -2w_5 & -2x_6 & +2w_3 \\
w_2 & 2 & -x_1 & -x_5 & & & -x_6 & -w_3 \\
x_3 & 4 & -\frac{x_1}{2} & -\frac{w_5}{2} & -\frac{3w_4}{2} & -\frac{w_5}{2} & -x_6 & -w_3 \\
x_4 & 1 & -\frac{x_1}{2} & +x_5 & +\frac{w_4}{2} & +\frac{w_5}{2} & -x_6 & +2w_3 \\
x_2 & 2 & -\frac{x_1}{2} & -2x_5 & +\frac{w_4}{2} & -\frac{w_5}{2} & & \\
\hline
z & 15 & -2x_1 & -3x_5 & -w_4 & -w_5 & +x_6 & +3w_3
\end{array}$$

We can choose betwen x_6 and w_3 to be the entering variable. We should choose w_3 to be the entering variable, because it increases the value of the objective function z the most. Since the leaving variable is automatically chosen based on the which basic variable constrains the entering variable the most, the value of the objective function does not depend on the choice of the leaving variable.

P2.
(A)

$x_{N,j}$ is the entering variable and $x_{B,i}$ is the leaving variable. This means, first, we need to solve the current equation in the dictionary given for $x_{B,i}$ for $x_{N,j}$. We start with the equation:

$$x_{B,i} = b_i + a_{i1}x_{N,1} + \cdots + a_{ij}x_{N,j} + \cdots + a_{in}x_{N,n}$$

Solving for $x_{N,j}$ we get the equation:

$$x_{N,j} = \frac{b_i}{-a_{ij}} + \frac{a_{i1}x_{N,1}}{-a_{ij}} + \cdots + \frac{a_{in}x_{N,n}}{-a_{ij}} + \frac{x_{B,i}}{-a_{ij}}$$

The next step is to take that equation and plug it into any instances of $x_{N,j}$ on the nonbasic side of the dictionary. If we use it in the equation for $x_{B,k}$ then we get the equation:

$$x_{B,k} = b_k + a_{k1}x_{N,1} + \cdots + a_{kj}(\frac{b_i}{-a_{ij}} + \frac{a_{i1}x_{N,1}}{-a_{ij}} + \cdots + \frac{a_{in}x_{N,n}}{-a_{ij}} + \frac{x_{B,i}}{-a_{ij}}) + \cdots + a_{kn}xN, n$$

Since we are trying to find the value of $x_{B,k}$, we set all of the nonbasic variables to 0 in the equation for $x_{B,k}$ yielding:

$$x_{B,k} = b_k + a_{kj}(\frac{b_i}{-a_{ij}})$$

Constant	Sign	Reason
b_k	≥ 0	Dictionary is feasible
a_{kj}	Nothing may be said about its sign	Coefficient could be any value
b_i	≥ 0	Dictionary is feasible
a_{ij}	< 0	Entering variable must be constrained

(B)

We know $b_i \geq 0$ because the dictionary is feasible and $a_{ij} < 0$ because $x_{N,i}$ had to be constrained in order for $x_{N,j}$ to be chosen as the leaving variable. We know then that the value of $x_{N,j} = (\frac{b_i}{-a_{ij}}) \geq 0$ because the dictionary is still feasible. We also know $b_k \geq 0$ because the original dictionary was feasible. What we don't know is the sign of a_{kj}. It could be negative, positive, or the value could be 0.

1. If a_{kj} is negative, then $b_k - a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0$ which could violate the ≥ 0 if $b_k < -a_{kj}\left(\frac{b_i}{-a_{ij}}\right)$. But since we did not choose $x_{B,k}$ as the leaving variable, this means $\frac{b_k}{-a_{kj}} \geq \frac{b_i}{-a_{ij}} \Rightarrow b_k \leq -a_{kj}\frac{b_i}{-a_{ij}}$ and not strictly $< -a_{kj}\frac{b_i}{-a_{ij}}$.
2. If a_{kj} is 0 then $b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0 \Rightarrow b_k \geq 0$
3. If a_{kj} is positive then $b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0$ which, based on the signs of the other constants, is true.

(C)

If $x_{B,k}$ and $x_{B,i}$ are both possilbe leaving variables for $x_{N,j}$ then the value for $x_{N,j}$ in the equation of $x_{B,k}$ equals the value for $x_{N,j}$ in the equation of $x_{B,i}$. This means:

$$\frac{b_k}{-a_{kj}} = \frac{b_i}{-a_{ij}}$$

If we choose $x_{B,i}$ to be the leaving variable instead of $x_{B,k}$ then the value of $x_{B,k}$ in the next dictionary is $b_k + a_{kj}(\frac{b_i}{-a_{ij}})$. Since $\frac{b_k}{-a_{kj}} = \frac{b_i}{-a_{ij}}$ we get:

$$x_{B,k} = b_k + a_{kj}(\frac{b_i}{-a_{ij}}) \Rightarrow x_{B,k} = b_k + a_{kj}(\frac{b_k}{-a_{kj}}) = b_k - b_k = 0$$

Because the value of $x_{B,k} = 0$, the next dictionary will be degenerate.

P3.

(A) A degenerate dictionary that is also unbounded. Recall that an unbounded dictionary does not have a leaving variable for some choice of an entering variable.

$$\begin{array}{r|lll} w_1 & 5 & +2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x2 & -x1 \end{array}$$

(B) A degenerate dictionary D which upon pivoting yields another degenerate dictionary D', but the objective value strictly increases.

$$\begin{array}{r|lll} w_1 & 5 & -2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x2 & -x1 \end{array}$$

(C) A non-degenerate dictionary D which upon pivoting yields another dictionary D' but the value of the objective function stays the same.

This is not possible. The only way the value of the objective function z' of a new dictionary D' remains the same is if $b_i = 0$ in the equation $\frac{b_i}{a_{ij}}$. This can only be the case if D was degenerate.

(D) A dictionary that is feasible but upon pivoting yields an infeasible dictionary.

This is not possible. Once you have a feasible dictionary you pivot to other feasible dictionaries. You cannot pivot to an infeasible dictionary from a feasible dictionary. You can, however, pivot to a degenerate dictionary from a feasible dictionary.

(E) A dictionary that does not have leaving variable (is unbounded) for one choice of entering variable but has a leaving variable for a different choice of an entering variable.

$$\begin{array}{r|lll} w_1 & 5 & +2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x2 & +x1 \end{array}$$

P4 (15 points) Consider the polyhedron below:

$$\begin{array}{llllll} -x & +2y & +2z & \leq & 2 \\ 2x & -y & +2z & \leq & 2 \\ x & & & \geq & 0 \\ & y & & \geq & 0 \\ & & z & \geq & 0 \end{array}$$

(A)
First we need to assign numbers to each constraint and saturate them to be equalities.

$$\begin{array}{rrrcl} -x & +2y & +2z & = & 2 \to 1 \\ 2x & -y & +2z & = & 2 \to 2 \\ x & & & = & 0 \to 3 \\ & y & & = & 0 \to 4 \\ & & z & = & 0 \to 5 \end{array}$$

We can then find the vertices by finding where each face–given by the constraint equations–intersects. In this case we have three variables (3 dimensions) so we need to see where three equations intersect to find a vertex. There will be $\binom{5}{3} = 10$ combinations of equations.

If there are multiple dictionaries that describe a vertex, then that vertex is degenerate. If there is only a single vertex to describe a vertex, then that vertex is non-degenerate. This means if more than one combination of equations gives the same vertex, then that vertex is degenerate. Otherwise, the vertex is not degenerate.

Equations Used	Vertex (x,y,z)	Degenerate or non-degenerate
3, 4, 5	(0,0,0)	non-degenerate
1, 4, 5	(-2,0,0)	non-degenerate
1, 3, 5	(0,1,0)	non-degenerate
1, 3, 4	(0,0,1)	degenerate
2, 4, 5	(1,0,0)	non-degenerate
2, 3, 4	(0,0,1)	degenerate
2, 3, 5	(0,-2,0)	non-degenerate
1, 2, 3	(0,0,1)	degenerate
1, 2, 4	(0,0,1)	degenerate
1, 2, 5	(2,2,0)	non-degenerate

(B) Consider the optimization problem:

$$\begin{array}{rrrrcll} \max & 2x & +3y & -2z & & & \\ \mathsf{s.t} & -x & +2y & +2z & \leq & 2 & \#\mathit{Slack}\ w_1 \\ & 2x & -y & +2z & \leq & 2 & \#\mathit{Slack}\ w_2 \\ & x & & & \geq & 0 & \\ & & y & & \geq & 0 & \\ & & & z & \geq & 0 & \end{array}$$

Write down all the dictionaries corresponding to the degenerate vertices. Use slack variables w_1, w_2 as indicated.
The vertex that is degenerate is $(0, 0, 1)$. There are four dictionaries (four combinations of different equations) that produce that vertex.

Using the equations 1, 3, and 4, we get the dictionary:

$$\begin{array}{r|rrrr} w_2 & 0 & -3x & +3y & +w_1 \\ z & 1 & +\frac{x}{2} & -y & -\frac{w_1}{2} \\ \hline \zeta & -2 & +x & +5y & +w_1 \end{array}$$

Using the equations 2, 3, and 4, we get the dictionary:

$$\begin{array}{r|rrrr} w_1 & 0 & +3x & -3y & +w_2 \\ z & 1 & -x & +\frac{y}{2} & -\frac{w_2}{2} \\ \hline \zeta & -2 & +4x & +2y & +w_2 \end{array}$$

Using the equations 1, 2, and 3, we get the dictionary:

$$\begin{array}{r|rrrr} y & 0 & +x & -\frac{w_1}{3} & +\frac{w_2}{3} \\ z & 1 & -\frac{x}{2} & -\frac{w_1}{6} & -\frac{w_2}{3} \\ \hline \zeta & -2 & +6x & -\frac{2w_1}{3} & +\frac{5w_2}{3} \end{array}$$

Using the equations 1, 2, and 4, we get the dictionary:

$$\begin{array}{r|rrrr} x & 0 & +y & +\frac{w_1}{3} & -\frac{w_2}{3} \\ z & 1 & -\frac{y}{x} & -\frac{w_1}{3} & -\frac{w_2}{2} \\ \hline \zeta & -2 & +6y & +\frac{4w_1}{3} & -\frac{w_2}{3} \end{array}$$

(C) Draw a graph whose nodes are the vertices described in (A) with edges between adjacent vertices.
You can only use the vertices that comply with the constraints which means they are in the feasible region. This leaves the following vertices in the feasible region {(0,0,0), (0,1,0), (0,0,1), (1,0,0), (2,2,0)}.

(D, extra credit) Given a polyhedron P, and for each vertex of the polyhedron, can you write down an objective function that is uniquely maximized only at that vertex and no other vertex of P?

(E, extra credit) For any polyhedron P, the polyhedral graph (also called its skeleton) is one where the nodes form the vertices of the polyhedron, and the edges connect adjacent vertices. Prove that this graph is strongly connected for any P. I.e, given any two vertices $\mathbf{v}_1$ and $\mathbf{v_2}$ there is a path between them in this graph.

(F, extra credit) Prove that the graph in part (E) for a d-dimensional polyhedron P has the property that if any subset of $d-1$ or fewer vertices in the graph are removed, it will still remain strongly connected (This is called Balinski's theorem).

To illustrate this, draw the skeleton of a cube and remove any two vertices from this skeleton. You will notice that there is a path in this graph between any pair of remaining vertices.