**CSCI 5654-Fall16**: Assignment #3 (Reading: Chapter 3, 4.4 and 5 of Vanderbei's book).
**Due Date:** Friday, September 30, 2016 (before class)
**In-class:** Assignment should be submitted on paper – no emails.
**Distance Students:** Assignment may be submitted on paper or by email.

**Your Name:**

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**P1. (15 points)** Consider the following LP below:

$$\begin{array}{rlllll} \max & 2x_1 & +3x_2 & -x_3 & +x_4 & \\ \text{s.t.} & x_1 & -x_2 & +x_3 & & \leq -1 \\ & x_1 & & +x_3 & +x_4 & \leq 0 \\ & & 2x_2 & +x_3 & +3x_4 & \leq 4 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

(A) Write down the dual corresponding to the primal problem above? Add slack variables $z_1, z_2, \ldots$ to the dual problem.

(B) Initialize the primal problem by changing the dual objective function appropriately. Pivot the dual dictionaries and write down the initial primal dictionary that you would obtain in this process ( Pls. change the primal objective to $-x_1 - x_2 - x_3 - x_4$ just to make the grading uniform).

(C) Repeat parts (A) and (B) on the problem below:

$$\begin{array}{rllll} \max & x_1 & -x_2 & +x_3 & \\ \text{s.t.} & x_1 & -2x_2 & & \leq 1 \\ & x_1 & +x_2 & -2x_3 & \leq 1 \\ & -3x_1 & & +6x_3 & \leq -4 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

Interpret the result obtained from the Simplex over the dual. Once again, please use $-x_1-x_2-x_3$ as the changed objective to initialize the problem.

**P2 (15 points)** **(A)** For the LP in problem P1 (A) set up the KKT conditions following the complementary slackness theorem.
**(B)** Now solve the KKT conditions to find dual variables corresponding to the following primal feasible solution?

$$x_1 = 0,\ x_2 = 2,\ x_3 = 0,\ x_4 = 0$$

Is this primal solution optimal?
Do not attempt to use an LP solver for this part. Instead substitute the primal solution into the KKT constraints and solve for dual solutions that are dual feasible and complementary to the primal solution above.

**P3 (10 points)** Consider a standard form LP wherein two different non-degenerate vertices $\mathbf{x}_1,\ \mathbf{x}_2$ are both optimal for the primal. Show using the complementary slackness theorem that any dual optimal solution is degenerate.

(**Hint:** Each vertex saturates precisely $n$ of the constraints. Now, since the solution $\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ is also optimal, consider how many constraints this solution saturates. Apply complementary slackness to derive how many dual variables should be zero in any dual optimal solution).

**P4 (15 points) (A)** Consider a simple LP over a hyper-rectangle:

$$\max\ \mathbf{c}^t\mathbf{x} \text{ s.t. } \ell \leq \mathbf{x} \leq \mathbf{u}$$

Here $\mathbf{c}$ represents the objective, and $\ell, \mathbf{u}$ represent vectors of upper and lower bounds on $\mathbf{x}$.

Write down a linear time algorithm for solving the above LP. (**Hint:** The solution for each variable $x_i$ can be one of two possibilities, what are they?)

**(B)** Using the result in (A) above, write down an algorithm to solve LPs of the form:

$$\begin{array}{rl} \max & \mathbf{c}^t\mathbf{y} \\ \text{s.t.} & \mathbf{y} = A\mathbf{x} \\ & \ell \ \leq \ \mathbf{x} \ \leq \ \mathbf{u} \end{array}$$

(**Hint:** Eliminate $\mathbf{y}$ and use the result in (A) ).

**P5 (20 points)** We will now prove a well-known and useful *theorem of the alternative* called Motzkin transposition theorem using what we have learned so far.

**Theorem [Motzkin 1936]:** The following system of constraints (P) is infeasible

$$\begin{array}{l} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{array}$$

if and only if the system of constraints (D) below is feasible

$$\begin{array}{l} A^t\mathbf{y} \geq 0, \\ \mathbf{b}^t\mathbf{y} < 0 \\ \mathbf{y} \geq 0 \end{array}$$

**(A)** Prove that if (P) is feasible then (D) cannot be feasible. (**Hint:** If $\mathbf{x}, \mathbf{y}$ are simultaneously feasible for (P), (D) respectively then derive a contradiction by applying different sets of inequalities from (P) and (D) to show that $\mathbf{y}^t A\mathbf{x}$ will simultaneously be $< 0$ and $\geq 0$.)

**(B)** Derive the dual for the auxilliary problem

$$\begin{array}{rlll} \max & \mathbf{0}^t\mathbf{x} & -x_0 & \\ \text{s.t.} & A\mathbf{x} & -\mathbf{1}x_0 & \leq \mathbf{b} \\ & \mathbf{x}, & x_0 & \geq 0 \end{array}$$

Note that $\mathbf{1}$ is the column vector of all 1s. Recall that the auxilliary problem above always has an optimal solution, and if (P) is infeasible, then the optimal value of this auxilliary problem is strictly negative. Call it $\beta^* < 0$.

**(C)** Prove using **(B)** that if system (P) is infeasible then system (D) is feasible. (**Hint:** Use strong duality to conclude that the dual problem derived in **(B)** has to have an optimal solution whose value is $\beta^*$. Proceed from there to conclude a solution for system (D). )