

**P1.**  
(A)

Entering variables	Leaving variables
$x_2$	$w_1$ or $w_5$
$x_5$	$w_5$
$x_6$	$w_2$ or $x_3$
$w_3$	$w_2$ or $x_3$

(B)

To find the corresponding solutions, we need to solve for  $x_2$  in the  $w_5$  row of the dictionary:

$$x_2 = 2 - \frac{x_1}{2} - 2x_5 + \frac{w_4}{2} - \frac{w_5}{2}$$

This equation is then plugged into all the places  $x_2$  is a nonbasic variable.

Basic variables	Solutions	Nonbasic variables
$w_1$	0	$x_1$
$w_2$	2	$w_5$
$x_3$	4	$x_5$
$x_4$	1	$x_6$
$x_2$	2	$w_3$
$z$	15	$w_4$

Yes, this dictionary is degenerate. The solution  $b$  of the basic variable  $w_1$  is 0.

(C)

The next dictionary looks like:

$w_1$	0	$-7x_1$	$-7x_5$	$+w_4$	$-2w_5$	$-2x_6$	$+2w_3$
$w_2$	2	$-x_1$	$-x_5$			$-x_6$	$-w_3$
$x_3$	4	$-\frac{x_1}{2}$	$-\frac{w_5}{2}$	$-\frac{3w_4}{2}$	$-\frac{w_5}{2}$	$-x_6$	$-w_3$
$x_4$	1	$-\frac{x_1}{2}$	$+x_5$	$+\frac{w_4}{2}$	$+\frac{w_5}{2}$	$-x_6$	$+2w_3$
$x_2$	2	$-\frac{x_1}{2}$	$-2x_5$	$+\frac{w_4}{2}$	$-\frac{w_5}{2}$		
$z$	15	$-2x_1$	$-3x_5$	$-w_4$	$-w_5$	$+x_6$	$+3w_3$

We can choose between  $x_6$  and  $w_3$  to be the entering variable. We should choose  $w_3$  to be the entering variable, because it increases the value of the objective function  $z$  the most. Since the leaving variable is automatically chosen based on the which basic variable constrains the entering variable the most, the value of the objective function does not depend on the choice of the leaving variable.

**P2.**

(A)

$x_{N,j}$  is the entering variable and  $x_{B,i}$  is the leaving variable. This means, first, we need to solve the current equation in the dictionary given for  $x_{B,i}$  for  $x_{N,j}$ . We start with the equation:

$$x_{B,i} = b_i + a_{i1}x_{N,1} + \cdots + a_{ij}x_{N,j} + \cdots + a_{in}x_{N,n}$$

Solving for  $x_{N,j}$  we get the equation:

$$x_{N,j} = \frac{b_i}{-a_{ij}} + \frac{a_{i1}x_{N,1}}{-a_{ij}} + \cdots + \frac{a_{in}x_{N,n}}{-a_{ij}} + \frac{x_{B,i}}{-a_{ij}}$$

The next step is to take that equation and plug it into any instances of  $x_{N,j}$  on the nonbasic side of the dictionary. If we use it in the equation for  $x_{B,k}$  then we get the equation:

$$x_{B,k} = b_k + a_{k1}x_{N,1} + \cdots + a_{kj}\left(\frac{b_i}{-a_{ij}} + \frac{a_{i1}x_{N,1}}{-a_{ij}} + \cdots + \frac{a_{in}x_{N,n}}{-a_{ij}} + \frac{x_{B,i}}{-a_{ij}}\right) + \cdots + a_{kn}x_{N,n}$$

Since we are trying to find the value of  $x_{B,k}$ , we set all of the nonbasic variables to 0 in the equation for  $x_{B,k}$  yielding:

$$x_{B,k} = b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right)$$

Constant	Sign	Reason
$b_k$	$\geq 0$	Dictionary is feasible
$a_{kj}$	Nothing may be said about its sign	Coefficient could be any value
$b_i$	$\geq 0$	Dictionary is feasible
$a_{ij}$	$< 0$	Entering variable must be constrained

### (B)

We know  $b_i \geq 0$  because the dictionary is feasible and  $a_{ij} < 0$  because  $x_{N,i}$  had to be constrained in order for  $x_{N,j}$  to be chosen as the leaving variable. We know then that the value of  $x_{N,j} = \left(\frac{b_i}{-a_{ij}}\right) \geq 0$  because the dictionary is still feasible. We also know  $b_k \geq 0$  because the original dictionary was feasible. What we don't know is the sign of  $a_{kj}$ . It could be negative, positive, or the value could be 0.

1. If  $a_{kj}$  is negative, then  $b_k - a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0$  which could violate the  $\geq 0$  if  $b_k < -a_{kj}\left(\frac{b_i}{-a_{ij}}\right)$ . But since we did not choose  $x_{B,k}$  as the leaving variable, this means  $\frac{b_k}{-a_{kj}} \geq \frac{b_i}{-a_{ij}} \Rightarrow b_k \leq -a_{kj}\frac{b_i}{-a_{ij}}$  and not strictly  $< -a_{kj}\frac{b_i}{-a_{ij}}$ .
2. If  $a_{kj}$  is 0 then  $b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0 \Rightarrow b_k \geq 0$
3. If  $a_{kj}$  is positive then  $b_k + a_{kj}\left(\frac{b_i}{-a_{ij}}\right) \geq 0$  which, based on the signs of the other constants, is true.

### (C)

If  $x_{B,k}$  and  $x_{B,i}$  are both possible leaving variables for  $x_{N,j}$  then the value for  $x_{N,j}$  in the equation of  $x_{B,k}$  equals the value for  $x_{N,j}$  in the equation of  $x_{B,i}$ . This means:

$$\frac{b_k}{-a_{kj}} = \frac{b_i}{-a_{ij}}$$

If we choose  $x_{B,i}$  to be the leaving variable instead of  $x_{B,k}$  then the value of  $x_{B,k}$  in the next dictionary is  $b_k + a_{kj}(\frac{b_i}{-a_{ij}})$ . Since  $\frac{b_k}{-a_{kj}} = \frac{b_i}{-a_{ij}}$  we get:

$$x_{B,k} = b_k + a_{kj}(\frac{b_i}{-a_{ij}}) \Rightarrow x_{B,k} = b_k + a_{kj}(\frac{b_k}{-a_{kj}}) = b_k - b_k = 0$$

Because the value of  $x_{B,k} = 0$ , the next dictionary will be degenerate.

**P3.**

(A) A degenerate dictionary that is also unbounded. Recall that an unbounded dictionary does not have a leaving variable for some choice of an entering variable.

$$\begin{array}{c|ccc} w_1 & 5 & +2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x_2 & -x_1 \end{array}$$

(B) A degenerate dictionary  $D$  which upon pivoting yields another degenerate dictionary  $D'$ , but the objective value strictly increases.

$$\begin{array}{c|ccc} w_1 & 5 & -2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x_2 & -x_1 \end{array}$$

(C) A non-degenerate dictionary  $D$  which upon pivoting yields another dictionary  $D'$  but the value of the objective function stays the same.

This is not possible. The only way the value of the objective function  $z'$  of a new dictionary  $D'$  remains the same is if  $b_i = 0$  in the equation  $\frac{b_i}{a_{ij}}$ . This can only be the case if  $D$  was degenerate.

(D) A dictionary that is feasible but upon pivoting yields an infeasible dictionary.

This is not possible. Once you have a feasible dictionary you pivot to other feasible dictionaries. You cannot pivot to an infeasible dictionary from a feasible dictionary. You can, however, pivot to a degenerate dictionary from a feasible dictionary.

(E) A dictionary that does not have leaving variable (is unbounded) for one choice of entering variable but has a leaving variable for a different choice of an entering variable.

$$\begin{array}{c|ccc} w_1 & 5 & +2x_2 & -3x_1 \\ w_2 & 0 & & +x_1 \\ \hline z & 5 & +x_2 & +x_1 \end{array}$$

**P4 (15 points)** Consider the polyhedron below:

$$\begin{array}{rcll} -x & +2y & +2z & \leq 2 \\ 2x & -y & +2z & \leq 2 \\ x & & & \geq 0 \\ & y & & \geq 0 \\ & & z & \geq 0 \end{array}$$

- (A) Compute all the vertices and for each vertex write down if it is degenerate or non-degenerate.  
 (B) Consider the optimization problem:

$$\begin{array}{llllll}
 \max & 2x & +3y & -2z & & \\
 \text{s.t} & -x & +2y & +2z & \leq & 2 \quad \#Slack \ w_1 \\
 & 2x & -y & +2z & \leq & 2 \quad \#Slack \ w_2 \\
 & x & & & \geq & 0 \\
 & & y & & \geq & 0 \\
 & & & z & \geq & 0
 \end{array}$$

Write down all the dictionaries corresponding to the degenerate vertices. Use slack variables  $w_1, w_2$  as indicated.

- (C) Draw a graph whose nodes are the vertices described in (A) with edges between adjacent vertices.

(D, **extra credit**) Given a polyhedron  $P$ , and for each vertex of the polyhedron, can you write down an objective function that is uniquely maximized only at that vertex and no other vertex of  $P$ ?

(E, **extra credit**) For any polyhedron  $P$ , the polyhedral graph (also called its skeleton) is one where the nodes form the vertices of the polyhedron, and the edges connect adjacent vertices. Prove that this graph is strongly connected for any  $P$ . I.e, given any two vertices  $\mathbf{v}_1$  and  $\mathbf{v}_2$  there is a path between them in this graph.

(F, **extra credit**) Prove that the graph in part (E) for a  $d$ -dimensional polyhedron  $P$  has the property that if any subset of  $d - 1$  or fewer vertices in the graph are removed, it will still remain strongly connected (This is called Balinski's theorem).

To illustrate this, draw the skeleton of a cube and remove any two vertices from this skeleton. You will notice that there is a path in this graph between any pair of remaining vertices.