CSCI 5654-Fall16: Assignment #3
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P1.
(A)
Assign dual variables y_i to each of the constraint equations of the primal LP except for the non-negativity constraints.

$$
\begin{array}{rrrrrl}
\max & 2x_1 & +3x_2 & -x_3 & +x_4 & \\
\text{s.t. } y_1 \rightarrow & x_1 & -x_2 & +x_3 & & \leq -1 \\
y_2 \rightarrow & x_1 & & +x_3 & +x_4 & \leq 0 \\
y_3 \rightarrow & & 2x_2 & +x_3 & +3x_4 & \leq 4 \\
& x_1, & x_2, & x_3, & x_4 & \geq 0
\end{array}
$$

The dual objective is created from the constants in the b column of the primal LP. The dual constraints are created by the columns of the primal constraints and the coefficients of the primal objective function.

$$
\begin{array}{rrrrl}
\min & -y_1 & & +4y_3 & \\
\text{s.t.} & +y_1 & +y_2 & & \geq 2 \\
& -y_1 & & +2y_3 & \geq 3 \\
& +y_1 & +y_2 & +y_3 & \geq -1 \\
& & +y_2 & +3y_3 & \geq 1 \\
& y_1, & y_2, & y_3 & \geq 0
\end{array}
$$

Changing the dual LP into standard form and adding the slack variables $z_1, z_2, \dots$ we get the following dual problem:

$$
\begin{array}{rrrrrrrrl}
\max & y_1 & & -4y_3 & & & & & \\
\text{s.t.} & -y_1 & -y_2 & & +z_1 & = -2 & & & \\
& +y_1 & & -2y_3 & +z_2 & = -3 & & & \\
& -y_1 & -y_2 & -y_3 & +z_3 & = +1 & & & \\
& & -y_2 & -3y_3 & +z_4 & = -1 & & & \\
& y_1, & y_2, & y_3, & z_1, & z_2, & z_3, & z_4 & \geq 0
\end{array}
$$

(B)
If the primal objective is $\eta = \max -x_1 - x_2 - x_3 - x_4$ then the dual problem in standard form is

$$
\begin{array}{rrrrrrrrl}
\max & y_1 & & -4y_3 & & & & & \\
\text{s.t.} & -y_1 & -y_2 & & +z_1 & = +1 & & & \\
& +y_1 & & -2y_3 & +z_2 & = +1 & & & \\
& -y_1 & -y_2 & -y_3 & +z_3 & = +1 & & & \\
& & -y_2 & -3y_3 & +z_4 & = +1 & & & \\
& y_1, & y_2, & y_3, & z_1, & z_2, & z_3, & z_4 & \geq 0
\end{array}
$$

The dual dictionary is

$$
\begin{array}{r|rrrr}
z_1 & +1 & +y_1 & +y_2 & \\
z_2 & +1 & -y_1 & & +2y_3 \\
z_3 & +1 & +y_1 & +y_2 & +y_3 \\
z_4 & +1 & & +y_2 & +3y_3 \\
\hline
-\xi & 0 & +y_1 & & -4y_3
\end{array}
$$

Since the dual dictionary is feasible and the primal is not we will apply the simplex method to the dual. The only entering variable we can choose is y_1 and the only leaving variable that constrains y_1 is z_2. Solving for y_1 in the z_2 row and subsituting that equation in for all instances of y_1 in the nonbasic variables yields the dictionary

$$
\begin{array}{r|lccc}
z_1 & +2 & +y_2 & +2y_3 & -z_2 \\
y_1 & +1 & & +2y_3 & -z_2 \\
z_3 & +2 & +y_2 & +3y_3 & -z_2 \\
z_4 & +1 & +y_2 & +3y_3 & \\
\hline
-\xi & +1 & & -2y_3 & -z_2
\end{array}
$$

The above dictionary is optimal because there are no entering variables. We now need to convert the dual dictionary above to the corresponding primal dictionary. w_i corresponds to y_i and x_i corresponds to z_i. If the variable is nonbasic in the dual dictionary it is basic in the primal dictionary. The primal dictioary we get is

$$
\begin{array}{r|ccccc}
w_3 & 2 & -2x_1 & -2w_1 & -3x_3 & -3x_4 \\
x_2 & 1 & +x_1 & +1w_1 & +1x_3 & \\
\hline
\eta & -1 & -2x_1 & -1w_1 & -2x_3 & -1x_4
\end{array}
$$

Reinstating the original objective function ζ fpr η we get

$$
\begin{aligned}
\zeta &= 2x_1 + 3x_2 - x_3 + x_4 \\
&= 2x_1 + 3(1 + x_1 + w_1 + x_3) - x_3 + x_4 \\
&= 3 + 5x_1 + 2x_3 + x_4 + 3w_1
\end{aligned}
$$

(C)

The dual problem is

$$
\begin{array}{rcccl}
\min & +y_1 & +y_2 & -4y_3 & \\
\text{s.t.} & +y_1 & +y_2 & -3y_3 & \geq 1 \\
& -2y_1 & +y_2 & & \geq -1 \\
& & -2y_2 & +6y_3 & \geq 1 \\
y_1, & y_2, & y_3 & \geq 0 &
\end{array}
$$

The dual problem in standard form with slack variables $z_1, z_2, \ldots$ and the primal objective function $\eta = -x_1 - x_2 - x_3$ is

$$
\begin{array}{rcccccl}
\max & -y_1 & -y_2 & +4y_3 & & & \\
\text{s.t.} & -y_1 & -y_2 & +3y_3 & +z_1 & & = +1 \\
& +2y_1 & -y_2 & & & +z_2 & = +1 \\
& & +2y_2 & -6y_3 & & +z_3 & = +1 \\
y_1, & y_2, & y_3, & z_1, & z_2, & z_3 & \geq 0
\end{array}
$$

The dual dictionary is then

$$
\begin{array}{r|cccc}
z_1 & +1 & +y_1 & +y_2 & -3y_3 \\
z_2 & +1 & -2y_1 & +y_2 & \\
z_3 & +1 & & -2y_2 & +6y_3 \\
\hline
-\xi & 0 & -y_1 & -y_2 & +4y_3
\end{array}
$$

The only entering variable is y_3 and the only leaving variable is z_1. This produces the new dual dictionary

$$\begin{array}{c|cccc}
y_3 & \frac{1}{3} & +\frac{y_1}{3} & +\frac{y_2}{3} & -\frac{z_1}{3} \\
z_2 & +1 & -2y_1 & +y_2 & \\
z_3 & +3 & +2y_1 & & -2z_1 \\
\hline
-\xi & +\frac{4}{3} & +\frac{y_1}{3} & +\frac{y_2}{3} & -\frac{4z_1}{3}
\end{array}$$

We see that y_2 is an entering variable, but there is no leaving variable for it. This means this dual dictionary is unbounded and therefore the primal LP is infeasible

P2.
(A)
The KKT conditions are

1. $Ax + w = b$ (Primal LP feasibility constraints)
   (a) $x_1 - x_2 + x_3 + w_1 = -1$
   (b) $x_1 + x_3 + x_4 + w_2 = 0$
   (c) $2x_2 + x_3 + 3x_4 + w_3 = 4$
   (d) $x_1, x_2, x_3, x_4, w_1, w_2, w_3 \geq 0$
2. $A^t y - z = c$ (Dual LP feasibility constraints)
   (a) $y_1 + y_2 - z_1 = 2$
   (b) $-y_1 + 2y_3 - z_2 = 3$
   (c) $y_1 + y_2 + y_3 - z_3 = -1$
   (d) $y_2 + 3y_3 - z_4 = 1$
   (e) $y_1, y_2, y_3, z_1, z_2, z_3, z_4 \geq 0$
3. $x, y, w, z \geq 0$ (Nonnegativity constraints)
4. $x_i z_i = 0 \, for \, i = 1, 2, \ldots, n$ (Complementary slackness constraints)
   (a) $x_1 z_1 = 0$
   (b) $x_2 z_2 = 0$
   (c) $x_3 z_3 = 0$
   (d) $x_4 z_4 = 0$
5. $w_j y_j = 0 \, for \, j = 1, 2, \ldots, m$ (Complementary slackness constraints)
   (a) $w_1 y_1 = 0$
   (b) $w_2 y_2 = 0$
   (c) $w_3 y_3 = 0$

(B)
First let's find w_1, w_2, w_3 with the solution

$$x_1 = 0,\ x_2 = 2,\ x_3 = 0,\ x_4 = 0$$

Plugging that solution into the primal feasibility constraints we get

$$w_1 = 1, w_2 = 0, w_3 = 0$$

Using the the complementary slackness constraints from part A, we get

$$y_1 = 0, z_2 = 0$$

The variables that aren't necessarily 0 are

$$y_2, y_3, z_1, z_3, z_4$$

If we eliminate the variables that are 0 from the dual feasiblity constraints, we are left with the equations

$$\begin{aligned} y_2 - z_1 &= 2 \\ 2y_3 &= 3 \\ y_2 + y_3 - z_3 &= -1 \\ y_2 + 3y_3 - z_4 &= 1 \end{aligned}$$

Guessing values for the variables and solving for those remaining equations, we get the dual solution

$$y_1 = 0, y_2 = 4, y_3 = \frac{3}{2}, z_1 = 2, z_2 = 0, z_3 = \frac{13}{2}, z_4 = \frac{15}{2}$$

If we plug in the primal solution into the primal objective function we get $2(0) + 3(2) - (0) + 0 = 6$. If we plug the dual solution into the dual objective function we get $-(0) + 4(\frac{3}{2}) = 6$. Because both the dual and primal solution produce the same objective function value we know that the primal solution is optimal.

P3 (10 points) Consider a standard form LP wherein two different non-degenerate vertices $\mathbf{x}_1,\ \mathbf{x}_2$ are both optimal for the primal. Show using the complementary slackness theorem that any dual optimal solution is degenerate.

(**Hint:** Each vertex saturates precisely n of the constraints. Now, since the solution $\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ is also optimal, consider how many constraints this solution saturates. Apply complementary slackness to derive how many dual variables should be zero in any dual optimal solution).

P4 (15 points) (A) Consider a simple LP over a hyper-rectangle:

$$\max\ \mathbf{c}^t\mathbf{x} \text{ s.t. } \ell \leq \mathbf{x} \leq \mathbf{u}$$

Here $\mathbf{c}$ represents the objective, and $\ell, \mathbf{u}$ represent vectors of upper and lower bounds on $\mathbf{x}$.

Write down a linear time algorithm for solving the above LP. (**Hint:** The solution for each variable x_i can be one of two possibilities, what are they?)

(B) Using the result in (A) above, write down an algorithm to solve LPs of the form:

$$\begin{array}{rl} \max & \mathbf{c}^t\mathbf{y} \\ \mathsf{s.t.} & \mathbf{y} = A\mathbf{x} \\ & \ell \ \leq \ \mathbf{x} \ \leq \ \mathbf{u} \end{array}$$

(**Hint:** Eliminate $\mathbf{y}$ and use the result in (A)).

P5 (20 points) We will now prove a well-known and useful *theorem of the alternative* called Motzkin transposition theorem using what we have learned so far.

Theorem [Motzkin 1936]: The following system of constraints (P) is infeasible

$$\begin{array}{l} A\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{array}$$

if and only if the system of constraints (D) below is feasible

$$\begin{array}{l} A^t\mathbf{y} \geq 0, \\ \mathbf{b}^t\mathbf{y} < 0 \\ \mathbf{y} \geq 0 \end{array}$$

(A)

We need to show y^tAx can be < 0 and ≥ 0 simultaneously.

Because $y \geq 0$

$$\begin{array}{l} Ax \leq b \\ \Rightarrow y^tAx \leq y^tb \\ = b^ty < 0 \end{array}$$

This shows $y^tAx < 0$.

Because $x \geq 0$

$$\begin{array}{l} A^ty \geq 0 \\ \Rightarrow x^tA^ty \geq x^t * 0 \\ \Rightarrow (Ax)^ty \geq 0 \\ \Rightarrow ((Ax)^ty)^t \geq 0 \\ \Rightarrow y^tAx \geq 0 \end{array}$$

This shows $y^tAx \geq 0$.

If both systems are feasible then $y^tAx \geq 0$ and $y^tAx < 0$ which is impossible. This means only one system can be feasible at a time.

(B) Derive the dual for the auxilliary problem

$$\begin{array}{rlll} \max & \mathbf{0}^t\mathbf{x} & -x_0 & \\ \mathsf{s.t.} & A\mathbf{x} & -\mathbf{1}x_0 & \leq \mathbf{b} \\ & \mathbf{x}, & x_0 & \geq 0 \end{array}$$

Note that $\mathbf{1}$ is the column vector of all 1s. Recall that the auxilliary problem above always has an optimal solution, and if (P) is infeasible, then the optimal value of this auxilliary problem is strictly negative. Call it $\beta^* < 0$.
The dual problem of the auxillary problem is

$$\begin{array}{rll} \min & b^t y & \\ \mathsf{s.t.} & A^t y & \geq 0 \\ & -1^t y & \geq -1 \\ & y & \geq 0 \end{array}$$

The optimal value of the auxillary problem is β^* which is < 0.

(C) Prove using **(B)** that if system (P) is infeasible then system (D) is feasible. (**Hint:** Use strong duality to conclude that the dual problem derived in **(B)** has to have an optimal solution whose value is β^*. Proceed from there to conclude a solution for system (D).)
Strong duality theorem says

$$c^t x = b^t y$$

We assume system (P) is infeasible and system (D) is feasible. We then have the system (D) constraints

$$\begin{array}{l} A^t\mathbf{y} \geq 0, \\ \mathbf{b}^t\mathbf{y} < 0 \\ \mathbf{y} \geq 0 \end{array}$$