CSCI 5654-Fall16: Assignment #5
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P1.
(A)
Solve the initial linear equation and then branch on any variables that have a fractional value. Add the constrains of the fractional variables, solve the linear equation, and repeat until the solution is integral and not fractional.

Branches	Solution	Optimal value
	$x = [-5 \quad -5 \quad 1.5 \quad -.5]$	2.5
$x_4 \leq -1$	$x = [-5 \quad -5 \quad 2 \quad -1]$	2
$x_4 \geq 0$	$x = [-2 \quad -4 \quad 0 \quad 0]$	2

The optimal value of the objective function is 2. One solution that leads to this value is:

$$x_1 = 1 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = 1$$

(B)

Branches	Solution	Optimal value
	$x = [1.3333 \quad 1 \quad 1 \quad .6667]$	6.3333
$x_4 \geq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_4 \leq 0$	$x = [0 \quad 1 \quad 1 \quad 0]$	3
$x_1 \leq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_1 \geq 2$	Infeasible	

The optimal value of the objective function is 6. One solution that leads to this value is:

$$x_1 = 1 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = 1$$

P2.

Dictionary # 1

First, we choose x_1 because it is a variable with a fractional solution. We rewrite the equation for x_1 as:

$$0.666667x_5 - 0.333333x_4 + x_1 = 0.666666666667$$

Next we rewrite the above equation in terms of an integer part and a fractional part.

$$(0x_5 - x_4 + x_1) + (0.666667x_5 + 0.777777x_4) = 0 + 0.666666666667$$

The fractional part $(0.666667x_5 + 0.777777x_4) \geq 0.666666666667$. The cutting plane is then given by the equation:

$$(0.666667x_5 + 0.777777x_4) + w_6 = 0.666666666667$$

Dictionary # 2

Equations for variables with fractional solutions:

$$\begin{aligned}
-0.333333x_8 - 0.666667x_9 + 0.333333x_3 + x_4 &= 4.33333333333\\
0.333333x_8 - 0.333333x_9 + 2.666667x_3 + x_5 &= 8.66666666667\\
0.333333x_8 + 0.666667x_9 - 0.333333x_3 + x_1 &= 5.66666666667\\
-0.333333x_8 + 0.333333x_9 - 2.666667x_3 + x_2 &= 1.33333333333
\end{aligned}$$

Equations written with integral and fractional parts:

$$\begin{aligned}
(-x_8 - x_9 + 0x_3 + x_4) + (0.777777x_8 + 0.444443x_9 + 0.333333x_3) &= 4 + .33333333333\\
(0x_8 - x_9 + 2x_3 + x_5) + (0.333333x_8 + 0.777777x_9 + 666667x_3) &= 8 + .66666666667\\
(0x_8 + 0x_9 - x_3 + x_1) + (0.333333x_8 + 0.666667x_9 + 0.777777x_3) &= 5 + .66666666667\\
(-x_8 + 0x_9 - 3x_3 + x_2) + (0.777777x_8 + 0.333333x_9 + 0.444443x_3) &= 1 + .33333333333
\end{aligned}$$

Cutting planes for the above equations:

$$\begin{aligned}
(0.777777x_8 + 0.444443x_9 + 0.333333x_3) + w_{10} &= .33333333333\\
(0.333333x_8 + 0.777777x_9 + 666667x_3) + w_{11} &= .66666666667\\
(0.333333x_8 + 0.666667x_9 + 0.777777x_3) + w_{12} &= .66666666667\\
(0.777777x_8 + 0.333333x_9 + 0.444443x_3) + w_{13} &= .33333333333
\end{aligned}$$

P3.
(A)
Let x_i be node n_i. If $x_i = 1$ then there is a hospital at that node. If $x_i = 0$ then there is no hospital at that node, but there should be at least one other node $x_j = 1$ and the distance to that node $W(i, j)$ should be between 0 and 1.

This is a 0-1 Integer Linear Program given by the following formulation:

$$\begin{array}{lll} \min & \sum_{j=1}^{n}(cost_j * node_j) & \\ \text{s.t.} & & \\ & \sum_{j=1}^{n} I(W(i,j) \leq 1) * n_j & \geq 1 \text{ for all } i = 1...n \\ & n_j & \in \{0, 1\} \end{array}$$

For the objective function, we want to minimize the cost of placing hospitals. The constraint

$$\sum_{j=1}^{n} I(W(i,j) \leq 1) * n_j \geq 1 \text{ for all } i = 1...n$$

says that for each node i we want to make sure that there is a node j that has a hospital $n_j = 1$ and is within 1 hour of it $I(W(i,j) \leq 1) = 1$.

(B)
The code to the solution is at the end of this document. The solution itself is the vector

$$x = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]$$

This means in order to minimize the cost of building the hospitals within the driving time of 1 hour for each node, hospitals should be placed at node 6 and node 8. The total cost of the hospitals ends up being 2.1 million dollars.

P4.

(A) We add the indicator variable $z_1 ... z_n$ to the problem. If $x_i == 0$ then $z_i = 0$ else $z_i = 1$. In this problem we want to minimize the sum of z_i and this occurs best when x_i is 0.

$$\begin{array}{ll} \min & \sum z \\ \text{s.t.} & \\ & \begin{array}{ll} Ax & \leq b \\ x & \leq u * z \\ x & \geq \ell * z \\ x & \leq u \\ x & \geq \ell \\ z & \in \{0, 1\} \end{array} \end{array}$$

The additional constraints:

$$x \leq u * z$$
$$x \geq \ell * z$$

say that if x is negative then z must equal 1, referring to $x \geq \ell * z$. If x is positive then z must be equal to 1, referring to $x \leq u * z$. But if x is 0 then z should be equal to 0 because this is a minimization problem. Write down mixed integer programs that will find the point $\mathbf{x} \in P$ with the smallest number of 0 entries in $\mathbf{x}$.

(B)

We can use the same problem formulation from above except we make it a maximization problem instead of a minimization.

$$\begin{array}{ll} \max & \sum z \\ \text{s.t.} & \\ & \begin{array}{ll} Ax & \leq b \\ x & \leq u * z \\ x & \geq \ell * z \\ x & \leq u \\ x & \geq \ell \\ z & \in \{0, 1\} \end{array} \end{array}$$

The additional constraints:

$$x \leq u * z$$
$$x \geq \ell * z$$

say that if x is negative then z must equal 1, referring to $x \geq \ell * z$. If x is positive then z must be equal to 1, referring to $x \leq u * z$. But if x is 0 then z should be equal to 1 because this is a maximization problem.

(C) Write down a mixed integer program that will search for a solution $\mathbf{x} \in P$ maximizing an objective function $\mathbf{c}^t\mathbf{x}$ such that $\mathbf{x}$ *does not satisfy* $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$, for given $\mathbf{a}, \mathbf{b}$.

In order to not satisfy the inequality $a \leq x \leq b$, x needs to be $< a$ or $> b$.

P5.

We have to find a subset S that contains at least one element in the set S_i for $i = 1, \ldots, k$ and the sum of the elements in S is minimized.

$$\begin{array}{lll} \min & \sum_{i=1}^{n}(i * x_i) & \\ \text{s.t.} & & \\ & \sum_{i=1}^{n}(i * x_{ij}) & \geq 1 \text{ for all } j = 1...k \\ & x_i & \in \{0, 1\} \end{array}$$

$x_i = 1$ if the element i is in the subset S otherwise $x_i = 0$. The constraint

$$\sum_{j=1}^{n}(j * x_{ji}) \geq 1 \text{ for all } i = 1...k$$

says that for each set S_i, the subset S must contain at least one element from S_i. Solving the above example in the $0-1$ ILP we get the solution vector

$$x = [1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

This means the subset $S = \{1, 2\}$ which sums to 3.