

P1.

(A)

Solve the initial linear equation and then branch on any variables that have a fractional value. Add the constraints of the fractional variables, solve the linear equation, and repeat until the solution is integral and not fractional.

Branches	Solution	Optimal value
	$x = [-5 \quad -5 \quad 1.5 \quad -.5]$	2.5
$x_4 \leq -1$	$x = [-5 \quad -5 \quad 2 \quad -1]$	2
$x_4 \geq 0$	$x = [-2 \quad -4 \quad 0 \quad 0]$	2

The optimal value of the objective function is 2. One solution leads to this value is:

$$x_1 = 1 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = 1$$

(B)

Branches	Solution	Optimal value
	$x = [1.3333 \quad 1 \quad 1 \quad .6667]$	6.3333
$x_4 \geq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_4 \leq 0$	$x = [0 \quad 1 \quad 1 \quad 0]$	3
$x_1 \leq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_1 \geq 2$	Infeasible	

The optimal value of the objective function is 6. One solution leads to this value is:

$$x_1 = 1 \quad x_2 = 1 \quad x_3 = 1 \quad x_4 = 1$$

P2.**Dictionary # 1**

First, we choose x_1 because it is a variable with a fractional solution. We rewrite the equation for x_1 as:

$$0.666667x_5 - 0.333333x_4 + x_1 = 0.666666666667$$

Next we rewrite the above equation in terms of an integer part and a fractional part.

$$(0x_5 - x_4 + x_1) + (0.666667x_5 + 0.777777x_4) = 0 + 0.666666666667$$

The fractional part $(0.666667x_5 + 0.777777x_4) \geq 0.666666666667$. The cutting plane is then given by the equation:

$$(0.666667x_5 + 0.777777x_4) + w_6 = 0.666666666667$$

Dictionary # 2

Equations for variables with fractional solutions:

$$\begin{aligned} -0.333333x_8 - 0.666667x_9 + 0.333333x_3 + x_4 &= 4.3333333333 \\ 0.333333x_8 - 0.333333x_9 + 2.666667x_3 + x_5 &= 8.6666666667 \\ 0.333333x_8 + 0.666667x_9 - 0.333333x_3 + x_1 &= 5.6666666667 \\ -0.333333x_8 + 0.333333x_9 - 2.666667x_3 + x_2 &= 1.3333333333 \end{aligned}$$

Equations written with integral and fractional parts:

$$\begin{aligned} (-x_8 - x_9 + 0x_3 + x_4) + (0.777777x_8 + 0.444443x_9 + 0.333333x_3) &= 4 + .3333333333 \\ (0x_8 - x_9 + 2x_3 + x_5) + (0.333333x_8 + 0.777777x_9 + 666667x_3) &= 8 + .6666666667 \\ (0x_8 + 0x_9 - x_3 + x_1) + (0.333333x_8 + 0.666667x_9 + 0.777777x_3) &= 5 + .6666666667 \\ (-x_8 + 0x_9 - 3x_3 + x_2) + (0.777777x_8 + 0.333333x_9 + 0.444443x_3) &= 1 + .3333333333 \end{aligned}$$

Cutting planes for the above equations:

$$\begin{aligned} (0.777777x_8 + 0.444443x_9 + 0.333333x_3) + w_{10} &= .3333333333 \\ (0.333333x_8 + 0.777777x_9 + 666667x_3) + w_{11} &= .6666666667 \\ (0.333333x_8 + 0.666667x_9 + 0.777777x_3) + w_{12} &= .6666666667 \\ (0.777777x_8 + 0.333333x_9 + 0.444443x_3) + w_{13} &= .3333333333 \end{aligned}$$

P3.

(A)

Let x_i be node n_i . If $x_i = 1$ then there is a hospital at that node. If $x_i = 0$ then there is no hospital at that node, but there should be at least one other node $x_j = 1$ and the distance to that node $W(i, j)$ should be between 0 and 1.

This is a 0-1 Integer Linear Program given by the following formulation:

$$\begin{aligned} \min \quad & \sum_{j=1}^n (\text{cost}_j * \text{node}_j) \\ \text{s.t.} \quad & \sum_{j=1}^n I(W(i, j) \leq 1) * n_j \geq 1 \text{ for all } i = 1 \dots n \\ & n_j \in \{0, 1\} \end{aligned}$$

For the objective function, we want to minimize the cost of placing hospitals. The constraint

$$\sum_{j=1}^n I(W(i, j) \leq 1) * n_j \geq 1 \text{ for all } i = 1 \dots n$$

says that for each node i we want to make sure that there is a node j that has a hospital $n_j = 1$ and is within 1 hour of it $I(W(i, j) \leq 1) = 1$.

(B)

The code to the solution is at the end of this document. The solution itself is the vector

$$x = [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1]$$

This means in order to minimize the cost of building the hospitals within the driving time of 1 hour for each node, hospitals should be placed at node 6 and node 8. The total cost of the hospitals ends up being 2.1 million dollars.

P4 (15 points) Consider a polyhedron P given by the constraints

$$A\mathbf{x} \leq \mathbf{b}, \ell \leq \mathbf{x} \leq \mathbf{u}.$$

- Write down mixed integer programs that will find the point $\mathbf{x} \in P$ with the largest number of 0 entries in \mathbf{x} ,
- Write down mixed integer programs that will find the point $\mathbf{x} \in P$ with the smallest number of 0 entries in \mathbf{x} .
- Write down a mixed integer program that will search for a solution $\mathbf{x} \in P$ maximizing an objective function $\mathbf{c}^t \mathbf{x}$ such that \mathbf{x} *does not satisfy* $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$, for given \mathbf{a}, \mathbf{b} .

P5 (10 points) We are given sets of numbers $\langle S_1, \dots, S_k \rangle$ such that each $S_i \subseteq \{1, \dots, n\}$. For example, $n = 10$ and the sets are

$$S_1 : \{1, 3, 6\}, S_2 : \{2, 7, 8\}, S_3 : \{1, 8, 9\}, S_4 : \{1, 6, 5, 3\}.$$

Our goal is to select a subset $S \subseteq \{1, 2, \dots, n\}$ such that $S \cap S_i \neq \emptyset$ for $i = 1, \dots, k$ and the sum of elements in the chosen set S is minimized.

Formulate a 0–1 ILP for the problem for given $n, k, \langle S_1, \dots, S_k \rangle$. Also, solve it for the example above.