CSCI 5654-Fall16: Assignment #5 (Reading: Slides on ILP, Vanderbei Chapter 23).
Due Date: Friday, Nov. 4, 2016 (before class)
In-class: Assignment should be submitted on paper – no emails.
Distance Students: Assignment may be submitted on paper or by email.

Your Name:

P1. (10 points) Solve each of the problems below using a branch-and-bound method. Write down the solution obtained and the enumeration tree obtained. You may use any LP solver of your choice to solve the subproblems. Please do not use a ILP solver directly.

(A)

$$\begin{array}{lllllll}
\max & 2x_1 & -3x_2 & -2x_3 & -x_4 & \\
\text{s.t.} & x_1 & -x_2 & +x_3 & & \leq 5 \\
& x_1 & -2x_2 & -x_3 & +x_4 & \leq 3 \\
& x_1 & -x_2 & -x_3 & -x_4 & \leq -1 \\
& x_1, & x_2, & x_3, & x_4 & \in [-5, 5] \\
& x_1, & x_2, & x_3, & x_4 & \in \mathbb{Z}
\end{array}$$

Hint: Use x_4 as the variable to branch on.

Branches	Solution	Optimal value
	$x = [-5 \quad -5 \quad 1.5 \quad -.5]$	-2.5
$x_4 \leq -1$	$x = [-3.5 \quad -5 \quad 3.5 \quad -1]$	-2
$x_4 \geq 0$	$x = [-2 \quad -4 \quad 3 \quad 0]$	-2
$x_4 \leq -1$, $x_3 \leq 3$	$x = [-4 \quad -5 \quad 3 \quad -1]$	-2
$x_4 \leq -1$, $x_3 \geq 4$	$x = [-4 \quad -5 \quad 4 \quad -2]$	-1
$x_4 \leq -1$, $x_1 \leq -4$	$x = [-4 \quad -5 \quad 3 \quad -1]$	-2
$x_4 \leq -1$, $x_1 \geq -3$	$x = [-3 \quad -4.5 \quad 3.5 \quad -1]$	-1.5
$x_4 \leq -1$, $x_1 \geq -3$, $x_2 \leq -5$	Infeasible	
$x_4 \leq -1$, $x_1 \geq -3$, $x_2 \geq -4$	$x = [-2.5 \quad -4 \quad 3.5 \quad -1]$	-1
$x_4 \leq -1$, $x_1 \geq -3$, $x_3 \leq 3$	$x = [-3 \quad -4 \quad 3 \quad -1]$	-1
$x_4 \leq -1$, $x_1 \geq -3$, $x_3 \geq 4$	$x = [-3 \quad -4 \quad 4 \quad -2]$	-1.7764e-15

(B)

$$\begin{array}{llllll}
\max & 2x_1 & & +3x_3 & +x_4 & \\
\text{s.t.} & x_1 & -x_2 & & +x_4 & \leq 1 \\
& & 2x_2 & & -x_4 & \leq 2 \\
& x_1 & & -x_3 & -2x_4 & \leq -1 \\
& -x_1 & & & +x_4 & \leq 1 \\
& x_1 & & & & \in \{-2, -1, 0, 1, 2\} \\
& & x_2 & & & \in \{-1, 0, 1\} \\
& & & x_3 & & \in \{0, 1\} \\
& & & & x_4 & \in \{-1, 0, 1\}
\end{array}$$

Branches	Solution	Optimal value
	$x = [1.3333 \quad 1 \quad 1 \quad .6667]$	6.3333
$x_4 \geq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_4 \leq 0$	$x = [0 \quad 1 \quad 1 \quad 0]$	3
$x_1 \leq 1$	$x = [1 \quad 1 \quad 1 \quad 1]$	6
$x_1 \geq 2$	Infeasible	

The optimal value of the objective function is 6. Two solutions lead to this value:

$$\begin{array}{cccc} x_1 = 1 & x_2 = 1 & x_3 = 1 & x_4 = 1 \\ x_1 = 0 & x_2 = 1 & x_3 = 1 & x_4 = 0 \end{array}$$

P2. (10 points) Consider the final dictionaries for the LP relaxation of a few ILPs. Assuming all variables are integers, write down all the cutting planes:

Dictionary # 1:

$$\begin{array}{c|cccc} x_1 & 0.666666666667 & -0.666667x_5 & +0.333333x_4 \\ x_2 & 1 & -1x_5 & \\ x_3 & 2 & +4x_5 & -1x_4 \\ \hline z & 1 & -1x_5 & \end{array}$$

Dictionary # 2:

$$\begin{array}{c|cccc} x_4 & 4.33333333333 & +0.333333x_8 & +0.666667x_9 & -0.333333x_3 \\ x_5 & 8.66666666667 & -0.333333x_8 & +0.333333x_9 & -2.666667x_3 \\ x_6 & 10 & & & -1x_3 \\ x_7 & 3 & -3x_8 & +1x_9 & -18x_3 \\ x_1 & 5.66666666667 & -0.333333x_8 & -0.666667x_9 & +0.333333x_3 \\ x_2 & 1.33333333333 & +0.333333x_8 & -0.333333x_9 & +2.666667x_3 \\ \hline z & 7 & & -1x_9 & -2x_3 \end{array}$$

P3 (20 points). Consider the graph below which shows various locations and the driving times between them in hours:

Our goal is to decide whether or not to place a hospital at each node. The following are the cost of building a hospital at various nodes in millions of dollars:

Node	Cost
1	3
2	3
3	1.5
4	1
5	1.2
6	1.3
7	0.9
8	0.8

The following constraints should apply to our placement of hospitals: each node should either have a hospital or be within 1 hour driving distance of a hospital.

For your convenience, the table of pairwise shortest path distances is given as an excel spreadsheet.

(A) Let G be a graph with n nodes and $W(i,j)$ denote the shortest path weight between nodes i and j. Finally let $\mathbf{c}$ be a $n \times 1$ vector of node costs. Write down an integer linear programming formulation of the above problem.

(B) Formulate and solve your ILP for the given example.

P4 (15 points) Consider a polyhedron P given by the constraints

$$A\mathbf{x} \leq \mathbf{b},\ \ell \ \leq \ \mathbf{x} \ \leq \ \mathbf{u}\,.$$

(a) Write down mixed integer programs that will find the point $\mathbf{x} \in P$ with the largest number of 0 entries in $\mathbf{x}$,

(b) Write down mixed integer programs that will find the point $\mathbf{x} \in P$ with the smallest number of 0 entries in $\mathbf{x}$.

(c) Write down a mixed integer program that will search for a solution $\mathbf{x} \in P$ maximizing an objective function $\mathbf{c}^t\mathbf{x}$ such that $\mathbf{x}$ *does not satisfy* $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$, for given $\mathbf{a}, \mathbf{b}$.

P5 (10 points) We are given sets of numbers $\langle S_1, \ldots, S_k \rangle$ such that each $S_i \subseteq \{1, \ldots, n\}$. For example, $n = 10$ and the sets are

$$S_1 : \{1,3,6\},\ S_2 :\ \{2,7,8\},\ S_3 :\ \{1,8,9\},\ S_4 :\ \{1,6,5,3\}\,.$$

Our goal is to select a subset $S \subseteq \{1, 2, \ldots, n\}$ such that $S \cap S_i \neq \emptyset$ for $i = 1, \ldots, k$ and the sum of elements in the chosen set S is minimized.

Formulate a $0-1$ ILP for the problem for given $n, k, \langle S_1, \ldots, S_k \rangle$. Also, solve it for the example above.