CSCI 5654-Fall16: Assignment # 4
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P1.

(A)
Let $\max(2x_1 + 3x_2 - 5x_3,\ x_1,\ x_2,\ 2) \leq t$. We can then form the following linear program:

$$\begin{array}{lllll}
\min & t & & & \\
\text{s.t.} & +2x_1 & +3x_2 & -5x_3 & \leq t \\
 & +2x_1 & -x_2 & +x_3 & \leq t \\
 & x_1, & x_2 & & \leq t
\end{array}$$

(B)
Let $t_1,\ t_2,\ t_3,\ t_4 \geq 0$.
Let $|x_1 + x_2| \leq t_1$, $|x_2 - x_3| \leq t_2$, $|x_3 - x_1| \leq t_3$, $|x_1 + x_2 + x_3| \leq t_4$.
You now have the linear problem

$$\begin{array}{llllll}
\min & +t_1 & +t_2 & +t_3 & +t_4 & \\
\text{s.t.} & +x_1 & +x_2 & & & \leq t_1 \\
 & -x_1 & -x_2 & & & \leq t_1 \\
 & & +x_2 & -x_3 & & \leq t_2 \\
 & & -x_2 & +x_3 & & \leq t_2 \\
 & -x_1 & & +x_3 & & \leq t_3 \\
 & +x_1 & & -x_3 & & \leq t_3 \\
 & +x_1 & +x_2 & +x_3 & & \leq t_3 \\
 & -x_1 & -x_2 & -x_3 & & \leq t_3 \\
 & t_1, & t_2 & t_3 & t_4 & \geq 0
\end{array}$$

(C)
Let $\max(|x_1|,\ |x_2|,\ |x_3|,\ |x_1 + x_2|) \leq t$.
You now have the linear problem

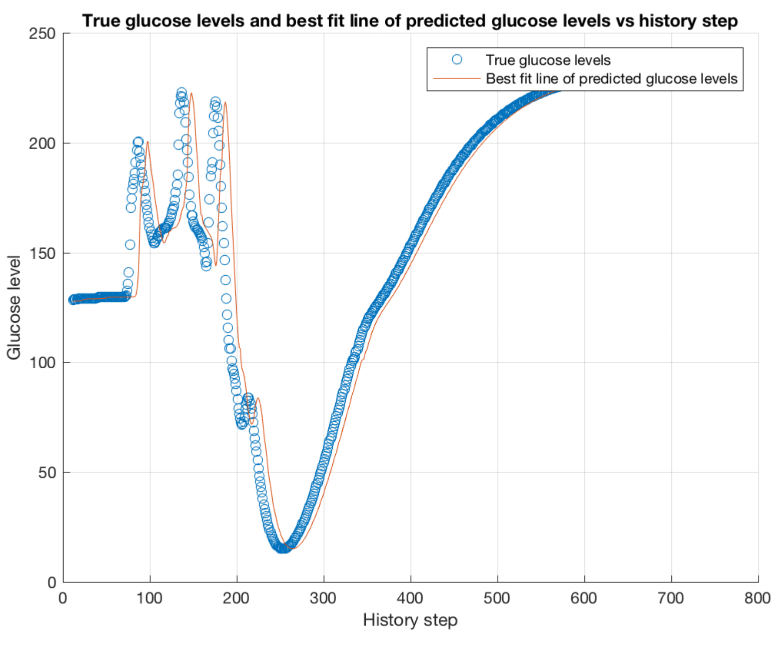
$$\begin{array}{lllll}
\min & t & & & \\
\text{s.t.} & +x_1 & -x_2 & & \leq 5 \\
 & & +x_2 & & \leq 3 \\
 & +x_1 & & & \leq t \\
 & -x_1 & & & \leq t \\
 & & +x_2 & & \leq t \\
 & & -x_2 & & \leq t \\
 & & & +x_3 & \leq t \\
 & & & -x_3 & \leq t \\
 & +x_1 & +x_2 & & \leq t \\
 & -x_1 & -x_2 & & \leq t \\
 & t & & & \geq 0
\end{array}$$

P2.

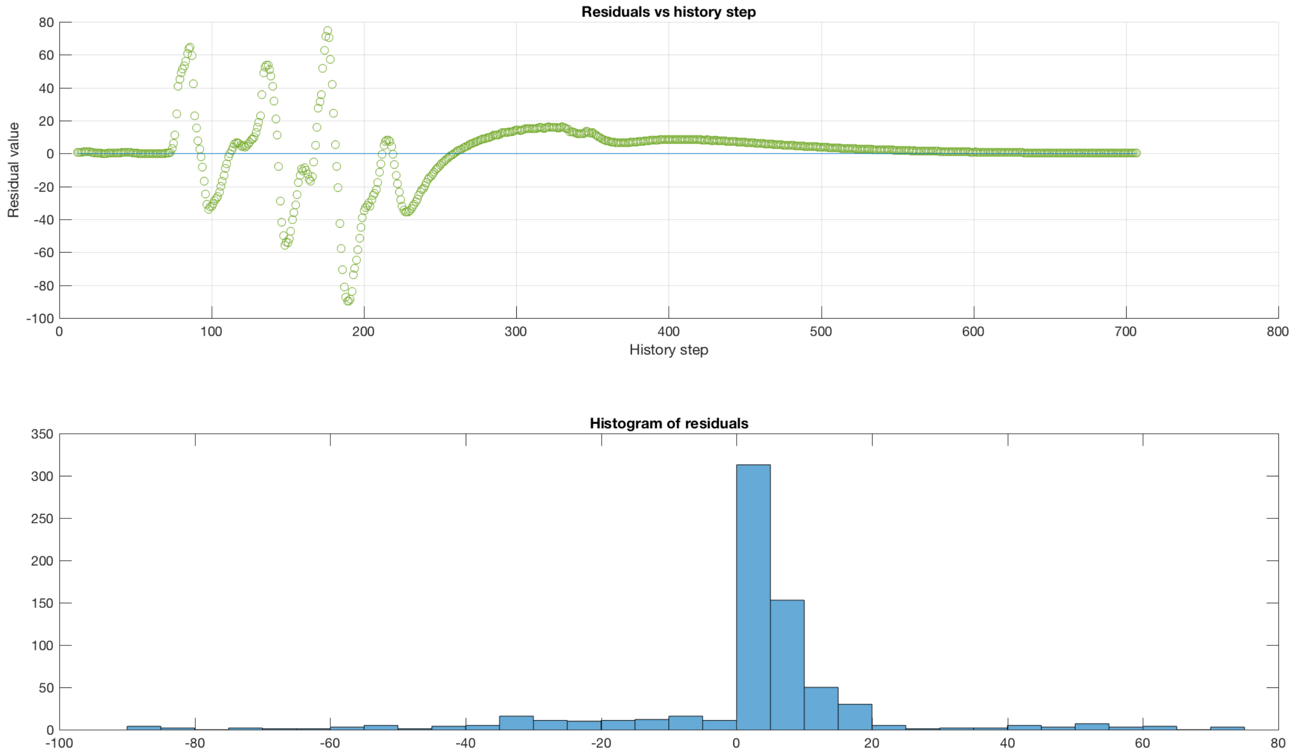
```
options = optimoptions('linprog', 'Algorithm', 'dual-simplex');

% Read in the csv file skipping the first row
M = csvread('insulinGlucose1.csv', 1);

% Insulin input values
u = M(:,1);
% Glucose levels
G = M(:,2);

A = [];
b = [];

[m,~] = size(G(12:end));
for i = 1:m
    b = [b, G(i)];
    b = [b, -G(i)];
end

b = b';

% Initialize the objective function which is the minimum of the sum
% of the absolute values of the residuals.
% Residuals have the form y - Ax.
f = ones(1,17);

% Create the constraint matrix A which is the residuals
% of glucose levels 12 -> 707.
% If u = |x| then u = x and u = -x
[m,~] = size(G);
for t = 11:(m-1)
    a = [G(t) G(t-1) G(t-2) G(t-3) G(t-4) G(t-5) G(t-6) G(t-7) G(t-8) G(t-9) G(t-10)];
    a = horzcat(a, [u(t) u(t-1) u(t-2) u(t-3) u(t-4) u(t-5)]);
    % Append the constraint ax.
    A = [A;a];

    % Append the constraint -ax.
    a = times(a, -1);
    A = [A;a];
end

% Solve for the coefficents.
% x will have an array of the coefficient values when it is solved.
x = linprog(f,A,b);

% Recreate the matrix A without the additional absolute value constraints.
A = [];
for t = 11:(m-1)
    a = [G(t) G(t-1) G(t-2) G(t-3) G(t-4) G(t-5) G(t-6) G(t-7) G(t-8) G(t-9) G(t-10)];
    a = horzcat(a, [u(t) u(t-1) u(t-2) u(t-3) u(t-4) u(t-5)]);
    A = [A;a];
end

% Create an array of the history steps.
t = [];
for i = 12:m
    t = [t, i];
end

% Plot the true glucose levels at each history step t_i.
figure(1);
scatter(t,G(12:end));
hold on;

% Plot the best fit line of the predicted glucose levels at each
% history step t_i.
% x stores the estimated coefficients of the best fit line.
% |x| = sqrt(x^2)
A = (A*x).^2;
A = sqrt(A);
plot(t, A);
grid on;
legend('True glucose levels', 'Best fit line of predicted glucose levels');
title('True glucose levels and best fit line of predicted glucose levels vs history step');
xlabel('History step');
ylabel('Glucose level');
saveas(1, 'predictions.png');

% Plot the residuals.
figure(2);
residuals = [];
trueGlucoseValues = G(12:end);
predictedGlucoseValues = A;
[m,~] = size(predictedGlucoseValues);
for i = 1:m
    residuals = [residuals, trueGlucoseValues(i) - predictedGlucoseValues(i)];
end

subplot(2,1,1);
scatter(t, residuals);
hold on;
plot(t, zeros(m));
grid on;
title('Residuals vs history step');
xlabel('History step');
ylabel('Residual value');

subplot(2,1,2);
```

```
histogram(residuals);
title('Histogram of residuals');
saveas(2, 'residuals.png');
```

Coefficient	Value
a_0	8.704148513061227e-14
a_1	-3.375077994860476e-14
a_2	-1.474376176702208e-13
a_3	2.131628207280301e-13
a_4	-1.243449787580175e-13
a_5	-1.056932319443149e-13
a_6	1.376676550535194e-13
a_7	-3.979039320256561e-13
a_8	3.179678742526448e-13
a_9	6.430411758628907e-13
a_{10}	0.999999999999309
b_0	1.989519660128281e-13
b_1	-2.060573933704291e-13
b_2	1.847411112976261e-13
b_3	-8.526512829121202e-14
b_4	-1.207922650792170e-13
b_5	7.815970093361102e-14



(a) Linear Regression Predictions


Residuals vs history step
Residual value
History step
Histogram of residuals

(b) Residuals

P3 (25 points). Consider the linear programming problem below:

$$
\begin{array}{lrrrrrrl}
\max & -2x_1 & -3x_2 & -x_3 & -x_4 & & +x_6 & \\
\text{s.t.} & x_1 & -x_2 & & & & -x_6 & \leq 3 \\
 & x_1 & & & -x_4 & -x_5 & & \leq -1 \\
 & & & -x_3 & & & -x_6 & \leq -2 \\
 & & -x_2 & -x_3 & +x_4 & & +x_6 & \leq 4 \\
 & x_1 & & +x_3 & & +x_5 & +x_6 & \leq 6 \\
 & -x_1 & & & -x_4 & +x_5 & -x_6 & \leq -2 \\
 & & & & & x_1, \ \ldots, & x_6 & \geq 0
\end{array}
$$

We add slack variables $w_1, w_2, \ldots, w_6$ for the 6 constraints. For your convenience, the problem data is provided separately for easy cut and paste into your python/matlab/other code.

(A) Compute and write out the dictionaries for the following set of basic variables. If no dictionaries exist, say why.

1. $\{x_1, x_2, x_3, w_1, w_2, w_3\}$.

$$
\begin{array}{r|rrrrrrr}
x_1 & +2 & -x_4 & +x_5 & -x_6 & & & +w_6 \\
x_2 & -8 & & +2x_5 & +x_6 & +w_4 & +w_5 & +w_6 \\
x_3 & +4 & +x_4 & -2x_5 & & & -w_5 & -w_6 \\
w_1 & -7 & +x_4 & +x_5 & +3x_6 & +w_4 & +w_5 & \\
w_2 & -3 & +2x_4 & & +x_6 & & & -w_6 \\
w_3 & 2 & +x_4 & -2x_5 & +x_6 & & -w_5 & -w_6 \\
\hline
\zeta & 16 & & -6x_5 & & -3w_4 & -2w_5 & -5w_6
\end{array}
$$

2. $\{x_1, x_2, x_5, w_3, w_5, w_6\}$.

$$
\begin{array}{r|rrrrrrr}
x_1 & -1 & -x_3 & +x_4 & +2x_6 & -w_1 & & +w_4 \\
x_2 & -4 & -x_3 & +x_4 & +x_6 & & & +w_4 \\
x_5 & 0 & -x_3 & & +2x_6 & -w_1 & +w_2 & +w_4 \\
w_3 & -2 & +x_3 & & +x_6 & & & \\
w_5 & 7 & +x_3 & -x_4 & -5x_6 & +2w_1 & -w_2 & -2w_4 \\
w_6 & -3 & & 2x_4 & +x_6 & & -w_2 & \\
\hline
\zeta & 14 & +4x_3 & -6x_4 & -6x_6 & +2w_1 & & -5w_4
\end{array}
$$

3. $\{x_1, x_2, x_6, w_4, w_5, w_6\}$.

$$
\begin{array}{r|rrrrrrr}
x_1 & -1 & & +x_4 & +x_5 & & -w_2 & \\
x_2 & -6 & +x_3 & +x_4 & +x_5 & +w_1 & -w_2 & -w_3 \\
x_6 & 2 & -x_3 & & & & & +w_3 \\
w_4 & -4 & +3x_3 & & +x_5 & +w_1 & -w_2 & -2w_3 \\
w_5 & 5 & & -x_4 & -2x_5 & & +w_2 & -w_3 \\
w_6 & -1 & -x_3 & +2x_4 & & & -w_2 & +w_3 \\
\hline
\zeta & 22 & -5x_3 & -6x_4 & -5x_5 & -3w_1 & +5w_2 & +4w_3
\end{array}
$$

(B) Perform one step of the revised simplex method for the dictionary with the basis:

$$\{x_3, x_4, x_5, w_1, w_2, w_6\}$$

1. Compute the constant column and objective rows of this dictionary.

$$
\begin{array}{r|rrrrrrr}
x_3 & 2 & & & -x_6 & +w_3 & & \\
x_4 & 6 & & +x_2 & -2x_6 & +w_3 & -w_4 & \\
x_5 & 4 & -x_1 & & & -w_3 & & -w_5 \\
w_1 & 3 & -x_1 & +x_2 & +x_6 & & & \\
w_2 & 9 & -2x_1 & +x_2 & -2x_6 & & -w_4 & -w_5 \\
w_6 & 0 & +2x_1 & +x_2 & -x_6 & +2w_3 & -w_4 & +w_5 \\
\hline
\zeta & -8 & -2x_1 & -4x_2 & +4x_6 & -2w_3 & +w_4 &
\end{array}
$$

2. Choose the entering variable with the largest coefficient.

   x_6 will be the entering variable because it has the largest coefficient.

3. Set up the equations to construct the column for the entering variable.

   The equation for the entering variable column a_i is $a_i = -B^{-1}Ne_i$.

4. Choose the leaving variable.

   The leaving variable with be w_6 because it constrains the entering variable the most.

5. Write down the basic variables in the next dictionary.

   The basic variables will be $x_3, x_4, x_5, w_1, w_2, x_6$.

```
A = [+1 -1 0 0 0 -1 +1 0 0 0 0 0;...
     +1 0 0 -1 -1 0 0 +1 0 0 0 0;...
     0 0 -1 0 0 -1 0 0 +1 0 0 0;...
     0 -1 -1 +1 0 +1 0 0 0 +1 0 0;...
     +1 0 +1 0 +1 +1 0 0 0 0 +1 0;...
     -1 0 0 -1 +1 -1 0 0 0 0 0 +1];

b = [3 -1 -2 +4 +6 -2]';

c = [-2 -3 -1 -1 0 +1 0 0 0 0 0 0];

% P3 a1
xB = [1 2 3 7 8 9];
xN = [4 5 6 10 11 12];
B = A(:, xB);
N = A(:, xN);
Binv = inv(B);
% Dictionary constant value column.
Binv*b;
% Dictionary coefficient value rows.
-Binv*N;

cB = c(:, xB);
cN = c(:, xN);
% Objective function constant value.
cB*(Binv*b);
% Object function coefficient values.
cN - cB*(Binv*N);

% P3 a2
xB = [1 2 5 9 11 12];
xN = [3 4 6 7 8 10];

B = A(:, xB);
N = A(:, xN);
Binv = inv(B);
% Dictionary constant value column.
Binv*b;
% Dictionary coefficient value rows.
-Binv*N;

cB = c(:, xB);
cN = c(:, xN);
% Objective function constant value.
cB*(Binv*b);
% Object function coefficient values.
cN - cB*(Binv*N);

% P3 a3
xB = [1 2 6 10 11 12];
xN = [3 4 5 7 8 9];

B = A(:, xB);
N = A(:, xN);
Binv = inv(B);
```

```
% Dictionary constant value column.
Binv*b;
% Dictionary coefficient value rows.
-Binv*N;

cB = c(:, xB);
cN = c(:, xN);
% Objective function constant value.
cB*(Binv*b);
% Object function coefficient values.
cN - cB*(Binv*N);

% p3 b
xB = [3 4 5 7 8 12];
xN = [1 2 6 9 10 11];

B = A(:, xB);
N = A(:, xN);
Binv = inv(B);
% Dictionary constant value column.
Binv*b;
% Dictionary coefficient value rows.
-Binv*N;

cB = c(:, xB);
cN = c(:, xN);
% Objective function constant value.
cB*(Binv*b);
% Object function coefficient values.
cN - cB*(Binv*N);

% Choose the third variable as the entering variable.
% Get the column of the entering variable.
a_i = -Binv*N(:, 3)

% Determine the leaving variable row index by finding the value that
% constraints the entering variable the most.
-(Binv*b)./a_i

% Get the leaving variable row.
a_j = -(Binv*N);
a_j = a_j(6,:)
```

(C) Write down the final dictionary for the problem. First you may want to solve the problem using your favorite Simplex solver and work out what the basic variables should be from your solution.

(D) We wish to now update the objective function to a new function

$$(-2x_1 - 3x_2 - x_3 - x_4 + x_6) + \mu(x_1 + x_2 + x_6)$$

where μ is a parameter. Write down the range of values of μ for which the final dictionary from part (D) continues to remain final as the objective is changed.

(E) Going back to (C), we wish to update the right hand sides of the original problem to

$$\begin{pmatrix} 3 \\ -1 \\ -2 \\ 4 \\ 6 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Calculate the range of values for λ for which, the dictionary in (C) remains feasible.