Project 2

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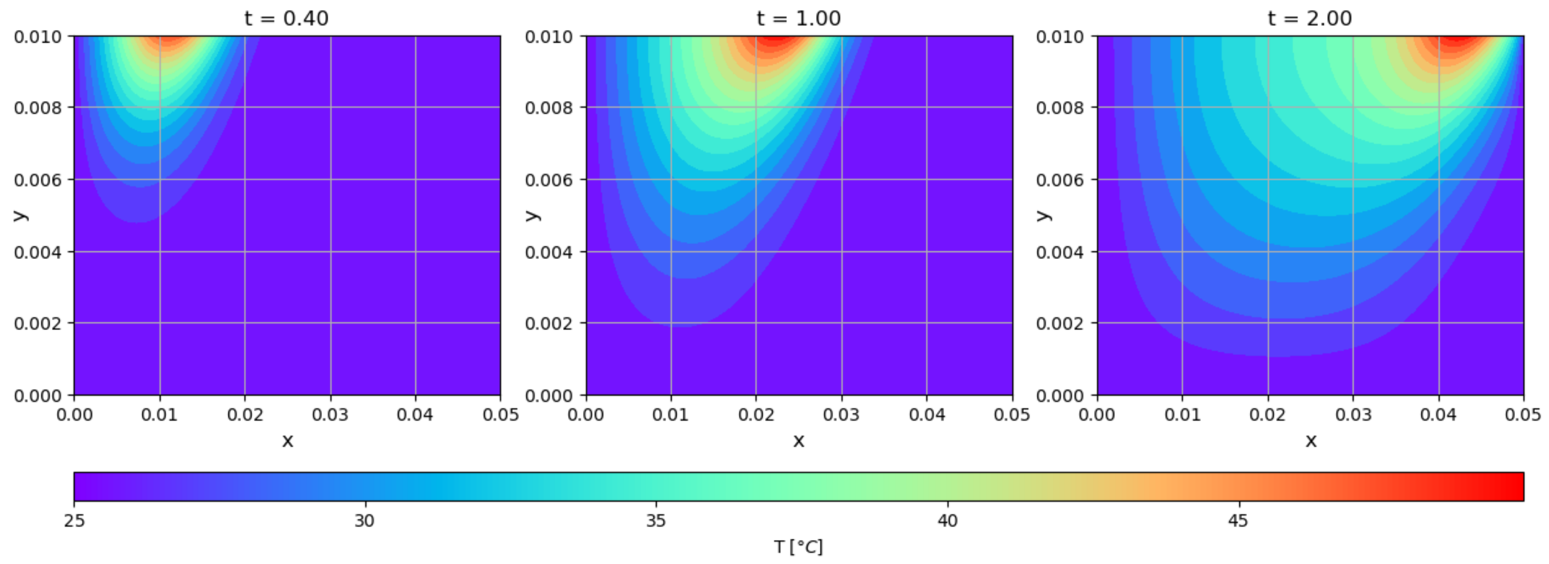
1.1

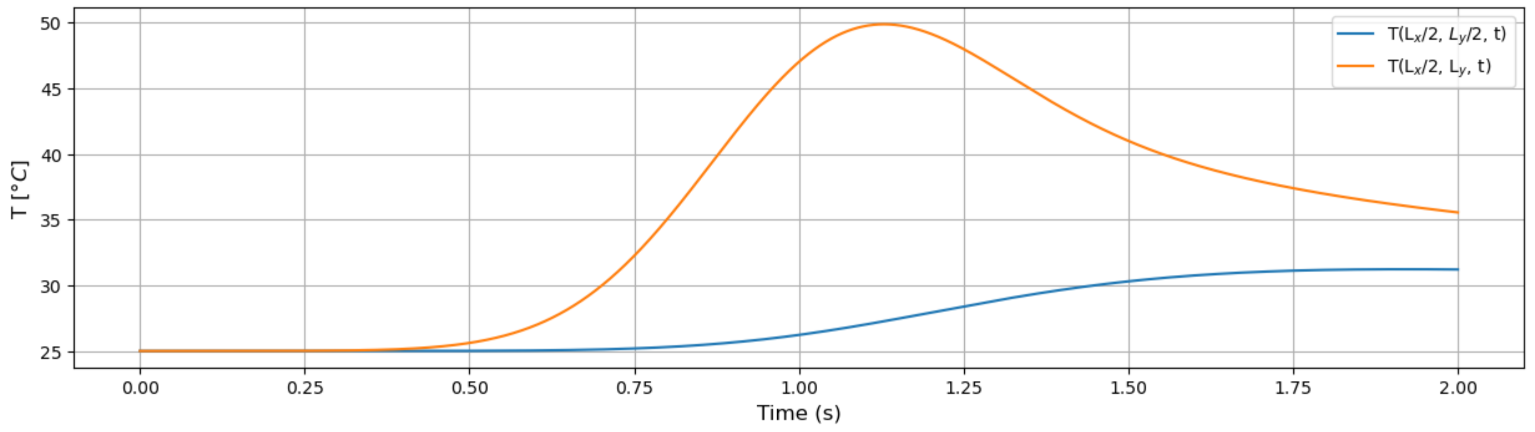
```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.gridspec import GridSpec


def laser_heat_2d(
    coord_lim, N_array, t_end, Nt, rho, c, k, P, w, x_start, x_end, vL, T_bar
):
    xmin, xmax = coord_lim[0, :]
    ymin, ymax = coord_lim[1, :]
    Nx, Ny = N_array

    # Calculating grid spacings and time step
    dx = (xmax - xmin) / Nx
    dy = (ymax - ymin) / Ny
    dt = t_end / Nt

    # Thermal diffusivity
    alpha = k / (rho * c)  # m^2/s

    # Maximum allowable time step for stability
    dt_max = 1 / (2 * alpha * (1 / dx**2 + 1 / dy**2))

    # Grid initialization
    x = np.linspace(xmin, xmax, Nx + 1)
    y = np.linspace(ymin, ymax, Ny + 1)
    t = np.linspace(0, t_end, Nt + 1)

    # Initial condition
    T = np.full((Nx + 1, Ny + 1, Nt + 1), T_bar, dtype=float)

    # Time integration using Forward Euler
    for n in range(Nt):
        # Temperature update for interior points
        # use vectorized 2D Laplacian calculation
        T_xx = (
            T[2 : Nx + 1, 1:Ny, n] - 2 * T[1:Nx, 1:Ny, n] + T[0 : Nx - 1, 1:Ny, n]
        ) / dx**2

        T_yy = (
            T[1:Nx, 2 : Ny + 1, n] - 2 * T[1:Nx, 1:Ny, n] + T[1:Nx, 0 : Ny - 1, n]
        ) / dy**2
        T[1:Nx, 1:Ny, n + 1] = T[1:Nx, 1:Ny, n] + alpha * dt * (T_xx + T_yy)

        # Laser flux on top boundary
        if t[n] <= (x_end - x_start) / vL:
            xL = x_start + vL * t[n]
            qL = (
                P
                / (np.sqrt(2 * np.pi) * w)
                * np.exp(-((x[1:Nx] - xL) ** 2) / (2 * w**2))
            )

        else:  # After laser shuts off
            qL = 0

        T[1:Nx, Ny, n + 1] = T[1:Nx, Ny - 1, n + 1] + dy * qL / k

        # Boundary conditions for sides and bottom
        T[:, 0, n + 1] = T_bar  # Bottom
        T[0, :, n + 1] = T_bar  # Left
        T[Nx, :, n + 1] = T_bar  # Right

    return dt_max, x, y, t, T
```

1.2

```
# Simulation parameters
Nx = 250
Ny = 50
t_end = 2
dt = 5e-4
Nt = int(t_end / dt)
Lx = 0.05  # m
Ly = 0.01  # m
T_bar = 25.0  # C
coord_lim = np.array([[0, Lx], [0, Ly]])

# Material parameters
k = 48  # W/m/C
rho = 7900  # kg/m^3
c = 470  # W/kg C

# Laser parameters
w = 0.004  # m
P = 5000  # W
x_start = 0.1 * Lx  # m
x_end = 0.9 * Lx  # m
vL = 0.02  # m/s

dt_max, x, y, t, T = laser_heat_2d(
    coord_lim, np.array([Nx, Ny]), t_end, Nt, rho, c, k, P, w, x_start, x_end, vL, T_bar
)
```

```python
X, Y = np.meshgrid(x, y)
contour_regions = np.linspace(T_bar, np.max(T.flatten()), 21)
n1, n2, n3 = int(0.2 * Nt), int(0.5 * Nt), int(Nt)
n = [n1, n2, n3]

gs = GridSpec(2, 3, height_ratios=[1, 0.7])

fig = plt.figure(figsize=(15, 10))
ax1 = fig.add_subplot(gs[0, 0])
ax2 = fig.add_subplot(gs[0, 1])
ax3 = fig.add_subplot(gs[0, 2])
axs = [ax1, ax2, ax3]

for i, n in enumerate([n1, n2, n3]):
    axs[i].contourf(X, Y, T[:, :, n].T, levels=contour_regions, cmap="rainbow")
    axs[i].set_xlabel("x", size=12)
    axs[i].set_ylabel("y", labelpad=-10, size=12)
    axs[i].set_title(f"t = {t[n]:.2f}", size=12)
    axs[i].grid()

norm = cm.colors.Normalize(vmin=np.min(T), vmax=np.max(T))
cbar = fig.colorbar(
    cm.ScalarMappable(norm=norm, cmap="rainbow"),
    ax=axs,
    orientation="horizontal",
    pad=0.15,
    aspect=50,
)
cbar.set_label("T $[\degree{C}]$")

center_temp = T[Nx // 2, Ny // 2, :]
top_center_temp = T[Nx // 2, Ny, :]

ax4 = fig.add_subplot(gs[1, :])
ax4.plot(t, center_temp, label="T(L$_{x}$/2, $L_{y}$/2, t)")
ax4.plot(t, top_center_temp, label="T(L$_{x}$/2, L$_{y}$, t)")
ax4.grid()
ax4.legend()
ax4.set_xlabel("Time (s)", size=12)
ax4.set_ylabel("T $[\degree{C}]$", size=12)
pos = ax4.get_position()
ax4.set_position([pos.x0, pos.y0 + 0.05, pos.width, pos.height])
```

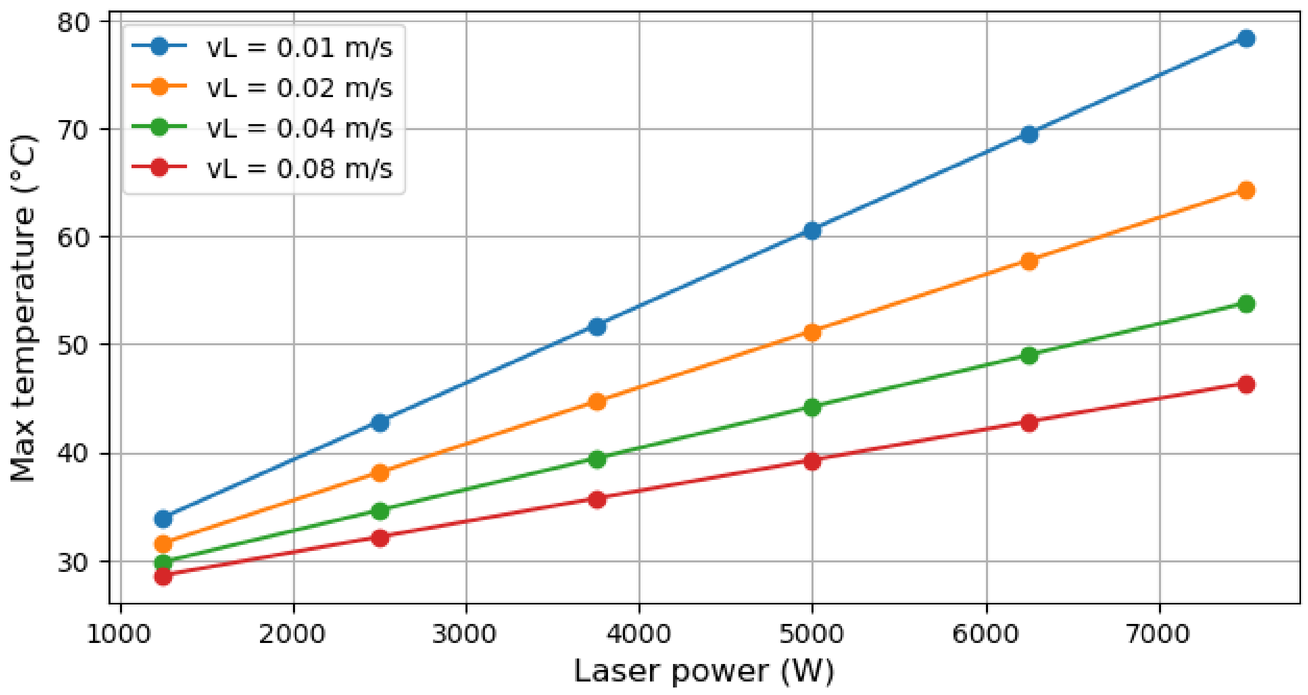




1.3

```
P = np.array([1250.0, 2500.0, 3750.0, 5000.0, 6250.0, 7500.0])
vL = np.array([0.01, 0.02, 0.04, 0.08])
Lx = 0.05  # m
Ly = 0.01  # m
Nx = 100
Ny = 20
dt = 1e-3
t_end = 2
Nt = int(t_end / dt)
T_bar = 25.0  # C
w = 0.004  # m
x_start = 0.1 * Lx  # m
x_end = 0.9 * Lx  # m
coord_lim = np.array([[0, Lx], [0, Ly]])
k = 48  # W/m/C
rho = 7900  # kg/m^3
c = 470  # W/kg C

max_temps = np.zeros((len(vL), len(P)))

for i, v in enumerate(vL):
    for j, p in enumerate(P):
        dt_max, x, y, t, T = laser_heat_2d(
            coord_lim,
            np.array([Nx, Ny]),
            t_end,
            Nt,
            rho,
            c,
            k,
            p,
            w,
            x_start,
            x_end,
            v,
            T_bar,
        )
        max_temps[i, j] = np.max(T)
```

```
fig, ax = plt.subplots(figsize=(8, 4))

for i, v in enumerate(vL):
    ax.plot(P, max_temps[i, :], label=f"vL = {v} m/s", marker="o")

ax.set_xlabel("Laser power (W)", size=12)
ax.set_ylabel("Max temperature ($\degree{C}$)", size=12)
ax.grid()
ax.legend()
plt.show()
```

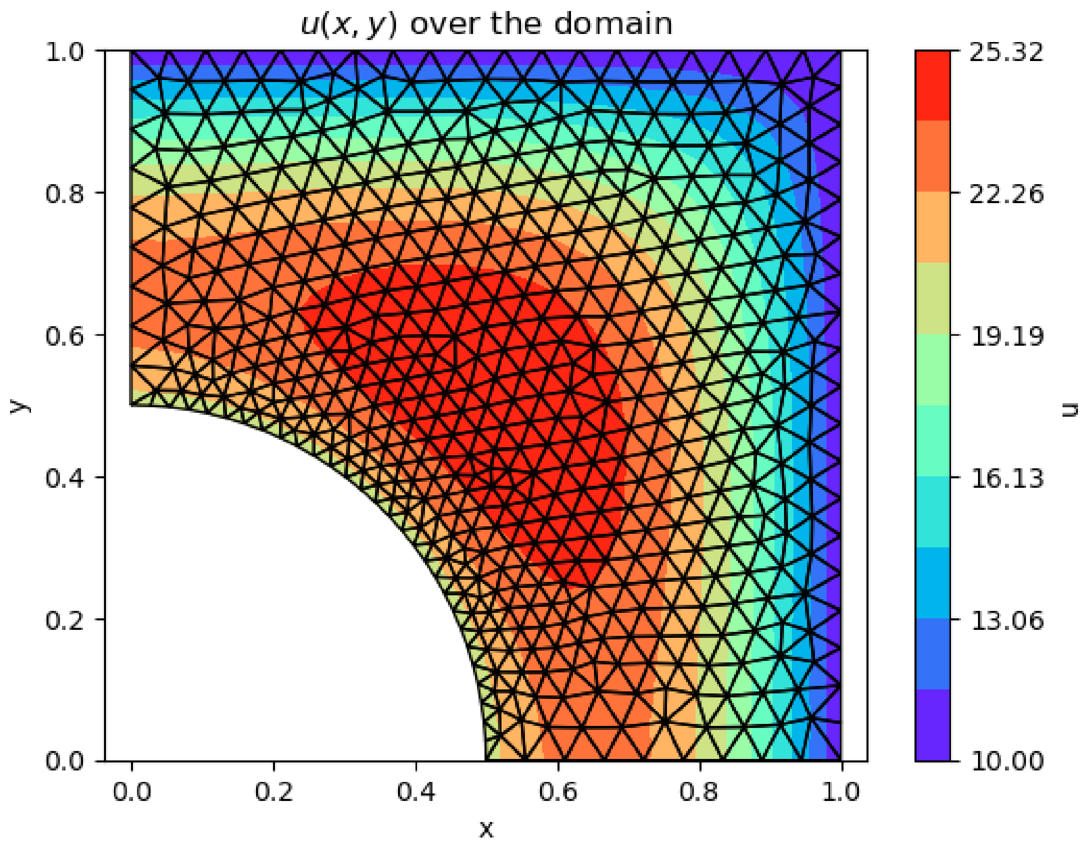


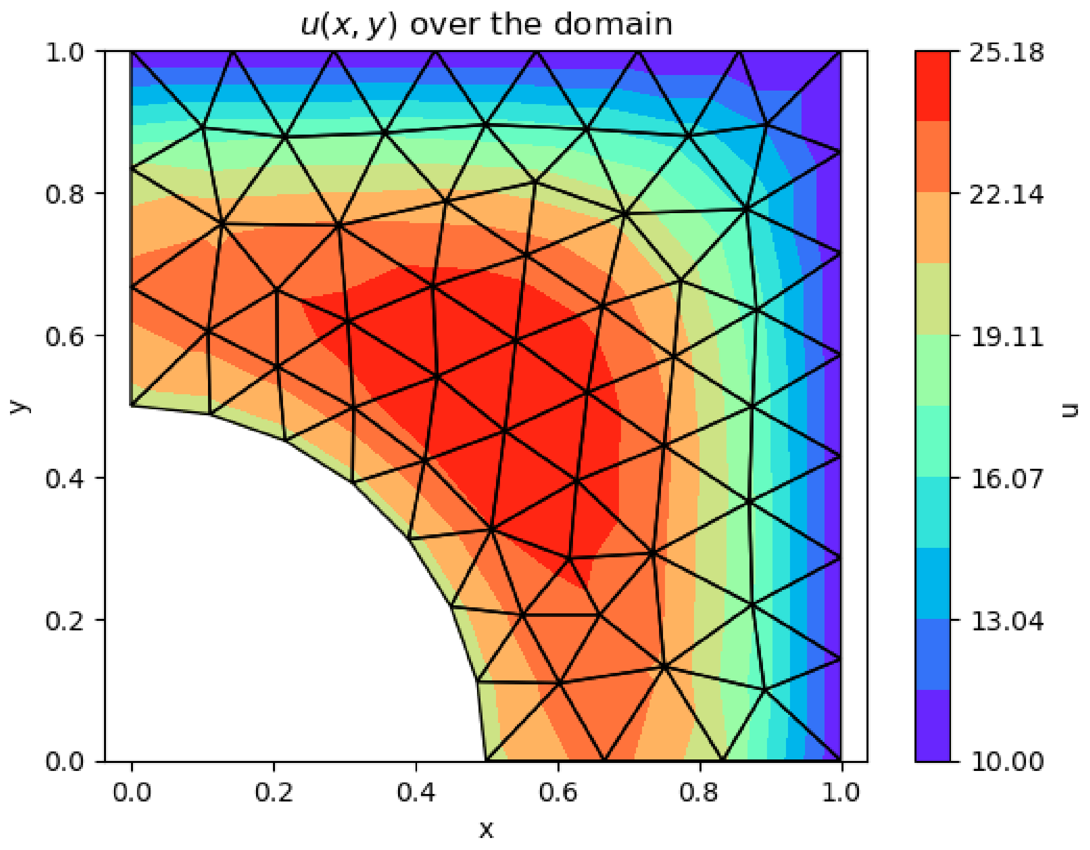
2.1

```python
from utilities import *
from fea_functions import *


def tri3_elem_arrays(nen, xe, ye, k, Q):
    """Compute element stiffness matrix and force vector for 3 node linear triangle with constant body

    Parameters
    ----------
    nen : int
        Number of nodes per element.
    xe : array_like
        Element x coordinates.
    ye : array_like
        Element y coordinates.
    k : float
        Thermal conductivity.
    Q : float
        Constant body heat load.

    Returns
    -------
    ke : array_like
        Element stiffness matrix.
    fe : array_like
        Element force vector.
    """

    y23 = ye[1] - ye[2]
    x32 = xe[2] - xe[1]
    x13 = xe[0] - xe[2]
    x21 = xe[1] - xe[0]
    y13 = ye[0] - ye[2]
    x23 = xe[1] - xe[2]
    y32 = ye[2] - ye[1]
    y21 = ye[1] - ye[0]

    A = 0.5 * np.abs(x13 * y23 - y13 * x23)

    B = np.array(
        [
            [x32, x13, x21],
            [y32, y13, y21],
        ]
    )

    ke = k * B.T @ B / (4 * A)
    fe = Q * A / 3 * np.ones(nen)

    return ke, fe


def tri3_assemble_global_arrays(numnp, numel, nen, p, LM, k, Q):
    """
    Assemble global stiffness matrix and force vector for 3 node linear triangle with constant body he

    Parameters
    ----------
    numnp : int
        Number of nodes in the mesh.
    numel : int
        Number of elements in the mesh.
    nen : int
        Number of nodes per element.
    p : array_like
        Nodal coordinates, array of shape (numnp, 2)
    LM : array_like
        Element connectivity, array of shape (nen, numel)
    k : float
        Thermal conductivity.
    Q : float

    """
    K = np.zeros((numnp, numnp))
    F = np.zeros(numnp)
    for i_elem in range(numel):
        node_indices = LM[:, i_elem] - 1
        xe = p[node_indices, 0]
        ye = p[node_indices, 1]
        ke, fe = tri3_elem_arrays(nen, xe, ye, k, Q)
        for i_local, i_global in enumerate(node_indices):
            F[i_global] += fe[i_local]
            for j_local, j_global in enumerate(node_indices):
                K[i_global, j_global] += ke[i_local, j_local]

    return K, F


def tri3_apply_BCs(nodeFlag, To, Ti, K, F):
    """
    Apply Dirichlet boundary conditions to the global stiffness matrix and force vector.


    Parameters
    ----------
    nodeFlag : array_like
        Boundary condition codes for each node.
    To : float
        Outer boundary temperature.
    Ti : float
        Inner boundary temperature.
    K : array_like
        Global stiffness matrix.
    F : array_like
        Global force vector.

    Returns
    -------
    K : array_like
        Modified global stiffness matrix.
    F : array_like
        Modified global force vector.
    """
    for i, flag in enumerate(nodeFlag):
        if flag == 1:
            K[i, :] = 0
            K[i, i] = 1
            F[i] = To
        elif flag == 2:
            continue
        elif flag == 3:
            K[i, :] = 0
            K[i, i] = 1
            F[i] = Ti

    return K, F


def solve_fea_heat2d(p, LM, nodeFlag, k, Q, To, Ti):
    """
    Solve 2D heat conduction problem using FEA.

    Parameters
    ----------
    p : array_like
        Nodal coordinates, array of shape (numnp, 2)
    LM : array_like
        Element connectivity, array of shape (nen, numel)
    nodeFlag : array_like
        Boundary condition codes for each node, array of shape (numnp,)
    k : float
        Thermal conductivity.
    Q : float
        Constant body heat load.
    To : float
        Outer boundary temperature.
    Ti : float
        Inner boundary temperature.

    Returns
    -------
    T : array_like
        Nodal temperatures, array of shape (numnp,)
    """

    numnp, _ = p.shape
    _, numel = LM.shape
    nen = LM.shape[0]
    K, F = tri3_assemble_global_arrays(numnp, numel, nen, p, LM, k, Q)
    K, F = tri3_apply_BCs(nodeFlag, To, Ti, K, F)
    T = np.linalg.solve(K, F)

    return T
```

2.2

```
filename = "fine.msh"
p, LM, nodeFlag = msh_reader_2D(filename)
k = 1
# Conductivity
Q = 200  # Body heating term
To = 10  # Dirichlet temperature on the straight edges.
Ti = 20  # Dirichlet condition on the arc.
T = solve_fea_heat2d(p, LM, nodeFlag, k, Q, To, Ti)

# Plot the contour of the solution field T
plot_solution_contour(LM.shape[1], p, LM, T)

filename = "coarse.msh"
p, LM, nodeFlag = msh_reader_2D(filename)
T = solve_fea_heat2d(p, LM, nodeFlag, k, Q, To, Ti)
plot_solution_contour(LM.shape[1], p, LM, T)
```





$u(x, y)$ over the domain
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
x
25.18
22.14
19.11
16.07
13.04
10.00
u