Université Libre de Bruxelles

Master Thesis

Some approaches to prevent Rank Reversals in the PROMETHEE methods

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Chapter 1

Introduction to Multi-Criteria Decision Aid

1.1 Introduction to Decision aid

Every day, every human being has to face numerous choices (or decision problems). Most of the humans are rational and therefore try to make the choices leading to the most favorable consequences. All these decision problems appear in specific contexts that will be called the *decision problem reality* or more simply, the problem reality. The contexts of different decision problems can have very different natures [6] : economical, political, industrial, military or even familial or personal.

The decision must not always be taken by a unique person. It can as well be taken by a board of directors, a jury, or even a familly. The entity, composed by one or more people, in charge of the decision will be called the *decision maker*.

Most of the decision problems that we encounter in our daily lifes are straightforward or of relatively small importance (what should we eat tonight? what movie should I watch?). They can generally be solved naturally and in a *qualitative* way [6]. As the problem is simple and the consequences of a non optimal decision are not dramatic, it is sufficient for the decision maker to use its own experience and its assessments of the problem reality in order to choose a solution that seems optimal to him.
These kind of decision problems take most of the time place in a personnal or familial context.

Some problems, on the other hand, require a more in depth reflection (should this old nuclear power plant be closed? should I invest my savings in this particular company?).
The decision problem reality in economical or industrial contexts have become too complex to be apprehensible in a purely qualitative manner [6]. Furthermore, the choice of a good or even of the best solution for these problems can be crucial since the consequences of a bad decision can be disastrous. In such

cases, the decision maker could want to use a *quantitative* approach [6]. In a quantitative approach, a mathematical model needs to be built in order to represent the reality of the decision problem. This mathematical model must forsee and quantify the consequences of each possible decision.

The discipline dealing with the analysis of the decision realities and the elaboration of the decision models is called operational research. As done in [6], the expression *decision aid* will be used as its signification is closer to the problem described.

It should also be noted that in decision aid, the quantitative and the qualitative approach are not mutually exclusive. The decision problem being generally too complex and chaotic to be entirely modelable, the quantitavtive approach must be combined with the qualitative one. The quantitative approach will be used to give some insight of the problem to the decision maker, which will always use his own qualitative perception to use these quantitative insights.

1.2 Mono-criterion approach

1.2.1 Formalisation and properties of monocriterion problems

A monocriterion decision problem can be formulated in the following way [6]:

$$Opt\{f(a_i) \mid a_i \in A\} \tag{1.1}$$

These classical models where the only one used until the end of the sixties and are formalised according to the three following properties [9]:

1. A forms a well-defined set of possible alternatives [9]

The set A of alternitives (also called actions) forms a well-defined set. These are all the possible decisions of the decision maker. This set of alternatives can be finite and enumerable, finite but not enumerable or infinite. Here under are examples of decision problem with these kinds of alternative sets :

- **finite and enumerable:** the decision maker must hire one person from a set of n applicants,
- **finite but not enumerable:** the decision maker is facing a *Travelling Salesman Problem*. As the number of possible path increases exponentially, the number of alternatives often becomes far to large to be enumerated,
- **infinite:** the decision maker must choose the amount of money invested in a project.

If the alternatives are not enumerable, the set must be defined by stating the properties which characterize its elements [10].

The set of alternatives A is also characterized by the following two properties [10]:
A can be either *stable*, if the set does not change during the decision procedure, or *evolutive*, if, on the other hand, it can change during the decision process. There can be different reasons leading to some changes of the alternatives considered like for example an evolving decision reality or the elimination during intermediate steps of the procedure of alternatives that are considered to be not relevant anymore.
A can be either *globalised*, if the decision problem consist in the selection of one unique alternative (we say that the alternatives are mutually exclusive), or *fragmented* if the decision consists in the selection of a subset of one or more alternatives.

2. The real-valued function $f(.)$ correctly reflects the preferences of the decision maker [9]

The preferences of the decision maker are considered to be correclty represented by the unique criterion $f(.)$. This criterion must therefore synthetise on its own all the objectives of the decision maker and the consequences of each of the alternatives [9] [6].

The problem could consist in minimising this criterion (e.g. $f(.)$ represents a cost) or to maximise the criterion (e.g. $f(.)$ represents a profit). For the rest of this section we will consider that $f(.)$ is a criterion to maximise.

$$Max\{f(a_i) \mid a_i \in A\} \tag{1.2}$$

3. The decision problem forms a well-formulated mathematical problem [9]

In mono-criterion decision problems, there exist one or more solutions such that:

$$f(a_i^*) \geq f(a_j), \forall a_j \in A \tag{1.3}$$

Such solutions are said to be optimal solutions of the decision problem. These are entirely determined by the modelling of the problem. The process of finding such an optimal solution is simply a classical optimisation problem, leading to the discovery of some “hidden truth” [9][10]. Indeed, if the decision maker approves the model, he should adopt one of the proposed optimal solutions.

4. Additional properties of the mono-criterion problem

With a mono-criterion problem, more can be done than finding an optimal solution. Indeed, $f(.)$ implies a natural dominance relation (P,I) on every pair of elements of A [6]:

$$\forall a_i, a_j \in A : \begin{cases} f(a_i) & > & f(a_j) & \Leftrightarrow & a_i P a_j \\ f(a_i) & = & f(a_j) & \Leftrightarrow & a_i I a_j \\ f(a_i) & < & f(a_j) & \Leftrightarrow & a_j P a_i \end{cases} \tag{1.4}$$

where P and I respectively stand for preference and indifference.

The preference structure of this relation form a *complete preorder* [10]. All the alternatives can be ranked from the best one to the worst one (complete) eventually with ties between two alternatives (preorder).

The following properties of P and I should also be noted [10]:

$$\forall a_i, a_j \in A \begin{cases} a_i P a_j \Rightarrow a_j \neg P a_i : & \text{P is asymetric} \\ a_i I a_i : & \text{I is reflexive} \\ a_i I a_j \Rightarrow a_j I a_i : & \text{I is symetric} \end{cases} \tag{1.5}$$

1.2.2 Drawback of the mono-criterion approach

Two main drawbacks of the mono-criterion approach are underlined in [6]. The first one is that, if the decision maker wants to take a decision according to several points of view (e.g. cost, suistainability, equity, ...), a single criterion can generally not synthesize the decision problem reality.
Imagine for example that you want to go to the restaurant tonight. You will probably want to go to a restaurant where the food is tasty, but you will probably not be ready to go to a too expensive restaurant. You will probably also prefer a restaurant closer to your place, or maybe close to the metro station. Furthermore, if you have already eaten pasta for breakfast and for lunch, you will probably not choose an Italian restaurant for tonight.
In conclusion, we can see that it is not always realistic to hope to find a unique criterion synthesizing correctly all of the aspects of the decision problem reality.

The second drawback of the mono-criterion approach is that the classic notion of a criterion is only used in order to compare if the evaluation of an alternative a_i is greater, smaller, or equal to the evaluation of an alternative a_j. This information could be misleading.
If we consider again the problem of choosing an optimal restaurant for tonight and we are only focusing on the proximity criterion. A restaurant two streets away will be preferred over a restaurant three streets away in the same way that it would be preferred over a restaurant on the other side of the ocean. This does not make a lot of sense since going to a restaurant on the other side of the ocean is generally quite problematic, but, on the other hand, a difference of only a few meters will generally be considered as neglectable. A restaurant three streets away and a restaurant two streets away would then be equally preferred.

1.3 Multi-Criterion Approach

A multi-criterion decision problem can be modeled in the following way [6]:

$$Opt\ \{f_1(a_i), f_2(a_i), \cdots, f_c(a_i), \cdots, f_k(a_i) | a_i \in A\} \tag{1.6}$$

With A, the set of alternatives, and $f_c(.), c = 1, \cdots, k$, a set of k evaluation criteria which are applications of A on the set of real numbers.

Without loss of generality, we will once again assume that the criteria are to be maximised.

$$\max\ \{f_1(a_i), f_2(a_i), \cdots, f_c(a_i), \cdots, f_k(a_i) | a_i \in A\} \tag{1.7}$$

1.3.1 The set of alternatives A

The definition of the set of alternatives does not change from the mono-criterion case (section 1.2.1).
The set of alternatives must be defined by an enumeration of each of the alternatives (if the alternatives are enumerables). Or, if the set is too large, it must be defined by the properities characterizing the alternatives in the set.
The set of alternatives can be stable or evolutive and globalized or fragmented.

1.3.2 Mathematically ill-defined problem

In opposition to the mono-criterion decision problem, the concept of optimal solution does not make sense anymore in a multi-criterion context. Indeed, due to the usually conflicting nature of the different criteria considered in a decision problem reality, it is generally not possible to find an alternative a_i such that :

$$f_c(a_i) \geq f_c(a_j), \forall a_j \in A, \forall c = 1, \ldots, k \tag{1.8}$$

The problem in this case can not consist anymore in the discovering of some hidden truth. It will consist in finding one or more alternatives that consist in a good compromise on all the criteria. The quality of a compromise is of course strongly dependent on the decision maker's perception of the problem reality.

1.3.3 Dominance relation

The natural dominance relation defined in a mono-criterion context 1.4 is generalized to a multi-criterion context as follow [6]:

$$\forall a_i, a_j \in A : \begin{cases} a_i P a_j & \Leftrightarrow & \begin{cases} f_c(a_i) \geq f_c(a_j) & \forall c = 1 \ldots k \\ \exists h : f_h(a_i) > f_h(a_j) \end{cases} \\ a_i I a_j & \Leftrightarrow & f_c(a_i) = f_c(a_j) \quad \forall c = 1 \ldots k \\ a_i R a_j & \Leftrightarrow & \begin{cases} \exists h : f_h(a_i) > f_h(a_j) \\ \exists h' : f_{h'}(a_i) < f_{h'}(a_j) \end{cases} \end{cases} \tag{1.9}$$

where P and I still stand for preference and indifference and R stands for incomparability. The preference structure of this relation is a *partial preorder* structure [10] as only some subsets of alternatives can be ranked from “best" to “worst" with eventual ties.

As the preference and indifference relation requires unanimity on all the criteria, one can easily convince himself that the dominance relation is generally very poor, and that most pairs of alternatives will be incomparable. The main objective of the multi-criteria decison aid methods will be to enrich this dominance relation by reducing the number of incomparable pairs of alternatives. The final ordering of the alternatives is, as it will be seen here under, not only be dependent on the decision maker, but also on the decision method used to enrich the dominance relation.

1.3.4 Types of multi-criteria decision problems

As shown by B. Roy, there exist different types of decisions problems [6]. Here under are three examples of common problem types.

- *Choice Problems* : these problems consist in choosing one of several alternatives from A which could be considered as the best alternatives.
- *Ordering Problems* : these problems consist in ordering all the alternatives from the worst to the best one.
- *Classifying Problems* : these problems consist in classifying all the alternatives in different categories.

More examples of decision problem types with some applications of these problems can be found in [5].

1.3.5 Types of multi-criteria decision aid methods

Multi-criteria decision aid methods can by divided in three families of methods [6]:

1. Agregating methods (Multi-attribute utility theory)
2. Outranking methods
3. Interactive methods

Agregating methods (Multi-attribute utility theory)

Agregating methods consist in the substitution of a multi-criterion decison problem into a mono-criterion one. This is done by synthesizing all of the k criterion of the multi-criteria problem, into a unique utility function $U(a_i)$:

$$U(a_i) = U[f_1(a_i), \ldots, f_k(a_i)] \tag{1.10}$$

The problem is therefore resumed to :

$$Max\{U(a_i) \mid a_i \in A\} \tag{1.11}$$

A usual way to select an utility function is to build it as a sum of all the criteria evaluations [10]:

$$U(a_i) = \sum_{c=1}^{k} U_c(f_c(a_i)) \tag{1.12}$$

This kind of utility function will constitute the additive model.

The role of the different $U_c(.)$ functions must be at least to normalize all the criteria to get rid of all scaling effects introduced by the measuring scale in which the criteria are expressed. Indeed, no multi-criteria decision method could be considered serious if its results are influenced by the units in which the criteria are expressed.

In a second time, some weight factors w_c can also be introduced to express the relative importance of each criterion according to the decision maker :

$$U(a_i) = \sum_{c=1}^{k} w_c \cdot U_c(f_c(a_i)) \tag{1.13}$$

The effect of these w_c could of course also be obtained by selecting an adequate $U_c(.)$ function but the two are kept distinct here to emphasise their respective meaning.

It is quite obvious that the preference structure of such a problem will be the same as in a mono-criterion context. The dominance relation will therefore form a complete preorder.

This seems at first sight to be fabulous and to solve all the problem introduced by the different criteria. Unfortunately, these methods have some drawbacks. First of all, in such additive models, a very bad evaluation on one criterion can be completely compensated by better performances on the other criteria.
Let's consider once again, the example of having to choose a restaurant for

tonight. Let's suppose we are evaluating the two only possible alternatives, *NY-Burger* and *Local-Burger*, according to four criteria : *taste, price, service and distance from home.*
To get rid of any scaling effect, the $U_c(.)$ functions will assigne to each criterion a score between 0 and 100 according to the preferences of the decision maker.

Suppose, once again, that one of the two alternatives (NY-Burger) is located on the other side of the ocean, and therefore the score given by the decision maker to it's "distance from home" evaluation will be 0.

The evaluation table of the two alternatives could be similar to the following one :

	$U_c(.)$			
Alternatives	Taste	Price	Service	Distance
NY-Burger	100	40	90	0
Local-Burger	50	50	50	50
Weights	0.5	0.2	0.15	0.15

Table 1.1: evaluation table for the restaurant problem using an MAUT method

From this table we can easily see that even if the evaluation on the distance criterion of the first alternative is the worst possible, the total utility function $U(\text{NY-Burger})$ will be equal to $0.5 \cdot 100 + 0.2 \cdot 40 + 0.15 \cdot 90 = 71.5$. This is greater than the utility function $U(\text{Local-Burger})$ which is equal to 0.5.

As we can see, using this additive model, a very good restaurant, but on the other side of the ocean, could be preferred over a median one.

Another problem of the additive model is that not all preferences of a decision maker can be represented.

Let's for example consider the following example proposed by Vincke [10]:

a_i :	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$f_1(a_i)$	1	1	1	2	2	2	3	3	3
$f_2(a_i)$	1	3	5	1	3	5	1	3	5

Table 1.2: evaluation for a potential decision maker

and suppose that the preferences of the decision maker are the following ones :

$$a_9 P a_6 P a_8 P a_5 P a_3 I a_7 P a_2 I a_4 P a_1 \tag{1.14}$$

This global preference can not be represented using an additive function.
Indeed by the definition of the indifference relation in a monocriterion context (section 1.4), and therefore valable for a aggregating method :

$$\begin{aligned} a_2 I a_4 &\Rightarrow U(a_2) = U(a_4) \Rightarrow w_1 U_1(1) + w_2 U_2(3) = w_1 U_1(2) + w_2 U_2(1) \\ a_3 I a_7 &\Rightarrow U(a_3) = U(a_7) \Rightarrow w_1 U_1(1) + w_2 U_2(5) = w_1 U_1(3) + w_2 U_2(1) \end{aligned} \tag{1.15}$$

If we substract the second equation to the first one, we obtain the following equality:

$$w_1U_1(3) + w_2U_2(3) = w_1U_1(2) + w_2U_2(5) \tag{1.16}$$

which is in contradiction with a_6Pa_8.

There exist other models than the additive model, but, as already stated, the additive model is the most commonly used in practice.

The result of the application of a multiple attribute theory method leads to a complete preorder of all the alternatives. This preorder generally contains far more information than the natural dominance relation defined on the initial multi-criteria problem. This abundant quantity of information is due to the theory's strong assumptions (e.g. existence of a function U, additivity) and to the vast amount of information asked to the decision maker (e.g. preference intensities, ...) [10].

Building a complete preorder may seems too radical as the data available with the problem is not always sufficient to compare all alternatives. Furthermore, keeping some alternatives incomparable can give some insights to the decision maker about the conflicting nature of the criteria in a specific multi-criteria problem.

Outranking methods

To deal with the problems mentioned above, B. Roy introduced the concept of *outranking relations*. An outranking relation is a relation aimed at enriching the dominance relation which is considered too poor, but in a realistic way. This is, not as much as with an utility function which is considered too rich to be reliable[6].

B. Roy defines the outranking relation as [10] "a binary relation S defined in A such that a_iSa_j if, given what is known about the decision maker's preference and given the quality of the evaluations of the alternatives and the nature of the problem, there are enough arguments to decide that a_i is at least as good as a_j, while there is no essential reason to refute that statement." [1]

The outranking relations have different mathematical formalization in each outranking method. The outranking relations must not be complete (as already suggested) and must neither be transitive.

The PROMETHEE methods are outranking methods and will be explained in details in the following chapter.

Here under is a brief description of the ELECTRE I method.

ELECTRE I

The outranking relation a_iSa_j, meaning that a_i is at least as good as a_j, defined in the ELECTRE I method is built according to two indices [8]: the *concordance*

[1] Ceci est une citation quasiment mot-à-mot avec quelques adaptations pour rester coherent avec le reste du travail. Par exemple, j'ai remplacé 'actions' par 'alternatives' et 'a' par 'a_i'. Puis je faire des citations mot-à-mot "adaptées"?

and the *discordance* index. These indices are computed as follows.

Given two alternatives a_i and a_j, the set of criteria $C(a_i, a_j)$ is the set composed of all criteria c where the evaluation of a_i is better or equal than the evaluation of a_j.

If we suppose once again that the multi-criteria decision problem is composed of k criteria to maximise, we can define $C(a_i, a_j)$ as :

$$C(a_i, a_j) = \{c \mid f_c(a_i) \geq f_c(a_j)\} \tag{1.17}$$

This set of criteria includes all the criteria in concordance with $a_i S a_j$ [8]. The other criteria, that are not included in $C(a_i, a_j)$, form another set $D(a_i, a_j)$. This set includes all the criteria that are in discordance with $a_i S a_j$.

To be able to use these two sets consistently, some relative importance factor must be assigned to each criterion in order to be able to measure the importance of the two sets.
With these factors w_c, we can compute a concordance index $c(a_i, a_j)$ [8]:

$$c(a_i, a_j) = \frac{1}{W} \sum_{\substack{c \in \\ C(a_i, a_j)}} w_c \quad \text{with } W = \sum_{c=1}^{k} w_c \tag{1.18}$$

and a discordance index $d(a_i, a_j)$ [8]:

$$d(a_i, a_j) = \begin{cases} 0 & \text{if } D(a_i, a_j) = \emptyset \\ \frac{1}{d} \max\limits_{\substack{c \in \\ D(a_i, a_j)}} [f_c(a_j) - f_c(a_i)] & \text{if } D(a_i, a_j) \neq \emptyset \end{cases} \tag{1.19}$$

with

$$d = \max_{c,i,j} \mid f_c(a_i) - f_c(a_j) \mid \tag{1.20}$$

being the maximal difference of two alternatives on any criterion.

The concordance index and the discordance index both have values in the range $[0, 1]$. The concordance index $c(a_i, a_j)$ increases as the number of criteria for which a_i is preferred over a_j increases. It represents the relative importance of the coalition of criteria for which a_i is preferred.
The discordance index increases as the maximal gap between an evaluation of a_j and an evaluation of a_i (where a_j is preferred) increases. This index can be seen as the veto power to reject that a_i outranks a_j.

Using these two indices, the outranking relation S will be built as follow :

$$\forall a_i, a_j \in A, a_i S a_j \Leftrightarrow \begin{cases} c(a_i, a_j) \geq p \\ d(a_i, a_j) \leq q \end{cases} \tag{1.21}$$

with p and q being the concordance and discordance thresholds defined by the decison maker.

Once the outranking relation is built, it must still be exploited. We can represent the problem as a graph having as vertices the alternatives of the problem. The graph will have a directed edge from the vertice a_i to the vertice a_j if and only if $a_i S a_j$. We will therefore denote it as the outranking graph of the decision problem.

The ELECTRE I method was designed for the multi-criteria choice problem (see section 1.3.4). It is aimed at finding a subset N of alternatives such that no alternative in N is outranked by any other alternative in N and all alternatives in $A \setminus N$ are outranked by at least one alternative of N.

$$\begin{cases} \forall a_j \in A \setminus N, \exists a_i \in N : a_i S a_j \\ \forall a_i, a_j \in N : a_i \neg S a_j \end{cases} \tag{1.22}$$

The problem of finding a subset N of alternatives fulfilling these conditions is the problem of finding a kernel of the outranking graph.

Chapter 2

Introduction to the PROMETHEE methods

The PROMETHEE (Preference Ranking Organisation METHod for Enrichment Evaluations) methods are outranking methods that have been developed at the UNIVERSITÉ LIBRE DE BRUXELLES and the VRIJE UNIVERSITEIT BRUSSEL since the beginning of the 80'. Initially proposed by J.P. Brans in 1982, these methods are used to address multi-criteria problems.

As seen in the the preceding chapter (section 1.3), multi-criteria problems (with a finite and enumerable set of alternatives and where the criteria consit in functions to maximise) can be modeled in the following way:

$$max\{f_1(a_i), f_2(a_i), \cdots, f_c(a_i), \cdots, f_k(a_i) | a_i \in A\} \tag{2.1}$$

With A, the set $\{a_1, \cdots, a_n\}$ of n considered alternatives and $f_c(.), c = 1, \cdots, k$, a set of k evaluation criteria which are applications of A on the set of real numbers. The information about each alternative and it's evaluations for each criterion can be summurized in a table :

a	$f_1(.)$	$\dots$	$f_c(.)$	$\dots$	$f_k(.)$
a_1	$f_1(a_1)$	$\dots$	$f_c(a_1)$	$\dots$	$f_k(a_1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_i	$f_1(a_i)$	$\dots$	$f_c(a_i)$	$\dots$	$f_k(a_i)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$f_1(a_n)$	$\dots$	$f_c(a_n)$	$\dots$	$f_k(a_n)$

Table 2.1: example of evaluation table

This raw information is generally not directly usable by the decision maker. The PROMETHEE methods offer some tools to help the decision maker in his choice problem (PROMETHEE I and II) or in his ordering problem (PROMETHEE II) of the alternatives.

These methods are based on the three followings concepts [6] :

1. **Enrichment of the preference structure**
   For each criterion, a preference function is introduced. These preference functions are aimed at indicating the degree at which one alternative is prefered over another alternative for the specific criterion. This is done by taking into account the deviation's amplitude between the evaluations of the two alternatives regarding the concerned criterion. This procedure also allows to get rid of all the scaling effects due to the measuring system used to express the different criteria.
2. **Enrichment of the dominance relationship**
   For each couple of alternatives, a global preference degree of the one over the other will be computed using the previously defined preference functions on all the criteria. Using these global preference degree, a quantitative outranking score is computed.
3. **Decision support**
   An outranking relation is used to rank the alternatives according to their outranking scores. This relation is used to provide some insights to the decision maker.

2.1 Enrichment of the preference structure

First of all, let us introduce the notation $d_c(a_i, a_j)$ that will represent the difference between $f_c(a_i)$ and $f_c(a_j)$:

$$d_c(a_i, a_j) = f_c(a_i) - f_c(a_j) \tag{2.2}$$

The natural dominance relation defined in section 1.3.3 has the following two important drawbacks :

1. **the relation is misleading:**
   Only the sign of the difference between the evaluations is taken into account and not the amplitudes of these differences. A difference of amplitude d_c of 1% between $f_c(a_i)$ and $f_c(a_j)$ should generally not have the same impact as a difference of 200% (see the restaurant choosing problem example on page 4).
2. **the relation is poor:**
   An alternative a_i is dominating another alternative a_j only if there is unanimity on all the criteria. This means that the evaluation for every criterion of the alternative a_i must be at least as good as the evaluations of the corresponding criterion of a_j. This is generally not the case due to the conflicting nature of the criteria (for instance, a smartphone with a better battery will generaly be heavier than one with a poor battery).

To cope with the first disadvantage, the preference structure will be enriched with the new notion of *preference functions*. These preference functions $P_c(a_i, a_j)$ will give the preference degree of alternative a_i over a_j on the criterion c in function of the difference d_c between $f_c(a_i)$ and $f_c(a_j)$:

$$P_c(a_i, a_j) = P_c[d_c(a_i, a_j)] \tag{2.3}$$

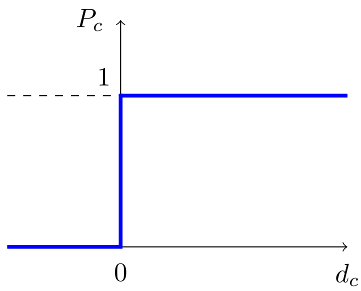
The preference degree will be between 0 and 1 and will be valued as follow [6] :

$$\begin{cases} P_c(a_i, a_j) = 0 & if \quad d_c(a_i, a_j) \leq 0 \quad \text{(no preference)} \\ P_c(a_i, a_j) \approx 0 & if \quad d_c(a_i, a_j) > 0 \quad \text{(weak preference)} \\ P_c(a_i, a_j) \approx 1 & if \quad d_c(a_i, a_j) >> 0 \quad \text{(strong preference)} \\ P_c(a_i, a_j) = 1 & if \quad d_c(a_i, a_j) >>> 0 \quad \text{(strict preference)} \end{cases} \tag{2.4}$$

To be compatible with the meaning of preference, the preference functions must be choosen non decreasing and must be equal to 0 for any negative difference of evaluation ($d_c(a_i, a_j) \leq 0$).

The decision of a preference function satisfying these conditions is let to the decision maker. However, a set of 6 different types are proposed in [6] which should satisfy the majority of the decison makers:

Type 1: the usual criterion

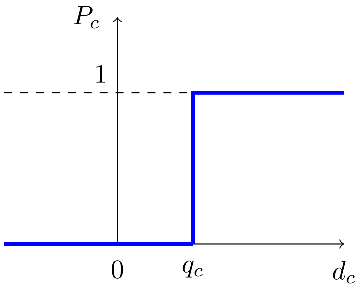


$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq 0 \\ 1 & \text{if} \quad 0 < d_c \end{cases} \tag{2.5}$$

Figure 2.1: Preference function of the usual criterion

This criterion does not imply any extension of the classical notion of preferences. As soon as the evaluation between the two alternatives is not identical, there will be a strict preference for the alternative having the highest evaluation. This criterion is therefore not enriching the preference relation but allows the decision maker to use the criteria in their usual sense, without having to define any parameter.

Type 2: the quasi-criterion

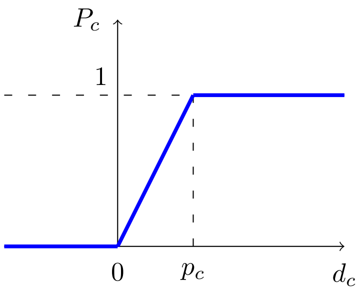


$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq q_c \\ 1 & \text{if} \quad q_c < d_c \end{cases} \tag{2.6}$$

Figure 2.2: Preference function for the quasi-criterion

This criterion adds an indifference threshold q_c. This means that as long as d_c does not exceeds that threshold the difference will be neglected and the two evaluations will be equally prefered.

Type 3: the criterion with linear preference



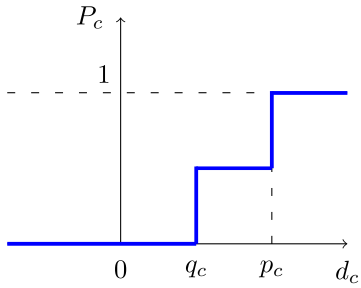
$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq 0 \\ \frac{d_c}{p_c} & \text{if} \quad 0 < d_c < p_c \\ 1 & \text{if} \quad d_c \geq p_c \end{cases} \tag{2.7}$$

Figure 2.3: Preference function for criterion with linear preferences

The preference function for a criterion with linear preferences increases continuously with the difference of the evaluations between 0 and a preference threshold p_c. If the difference is higher than this threshold, then the alternative with the highest evalutation is strictly prefered over the other.

This preference function is the first one that allows the decision maker to express non strict preferences. If the difference of the evaluation lies between 0 and p_c the preference degree will be between $[0, 1]$.

Type 4: the level-criterion

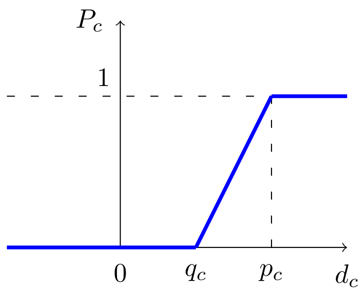


$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq 0 \\ \frac{1}{2} & \text{if} \quad q_c < d_c \leq p_c \\ 1 & \text{if} \quad p_c < d_c \end{cases} \tag{2.8}$$

Figure 2.4: Preference function for level-criterion

This preference function uses two parameters, the indifference treshold q_c and the preference treshold p_c. If the difference of evaluations lies between these two thresholds, then the preference functions is equal to 0.5. This kind of preference function can be usefull for criterions whose evaluations have discrete values (for example a criterion whose values can be “bad”, “good”, “very good”). If the criterion’s evaluations can take more than three values, an intuitive generalisation of this preference function with more than three levels can be used.

Type 5: the criterion with linear preferences and indifference zone

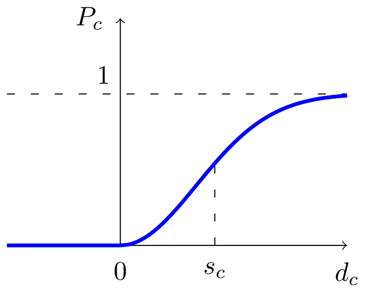


$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq q_c \\ \frac{d_c - q_c}{p_c - q_c} & \text{if} \quad q_c < d_c \leq p_c \\ 1 & \text{if} \quad p_c < d_c \end{cases} \tag{2.9}$$

Figure 2.5: Preference function for the criterion with linear and indifference zone

This preference function is a more general verison of the Type 3 preference function. Here a parameter q_c can be used to define an indifference threshold.

Type 6: the gaussian criterion



$$P_c = \begin{cases} 0 & \text{if} \quad d_c \leq 0 \\ 1 - e^{-d_c^2/2s_c^2} & \text{if} \quad 0 < d_c \end{cases} \tag{2.10}$$

Figure 2.6: Preference function for the gaussian criterion

This preference function leads to a preferene increasing continuously with d_c. No threshold parameter is needed for this function but instead, the parameter s_c must be fixed. This parameter indicates the difference d_c needed to have a mean preference (0.39) of the criterion with the highest evaluation over the other.

It is important to note that at most two parameters (q_c, p_c or s_c) must be defined by the decision maker. As each of these paremeters have a meaning in the real world, their assignation by the decision maker should not be an insurmountable task.
The choice of these parameters and of the preference functions consists in all the information *within a criterion* [2] that must be given by the decision maker. This choice should be done with care as these parameters can have a crucial impact on the final ranking. For example, choosing indifference thresholds too high for a specific criterion will have as consequence that this criterion will not have much impact as two alternatives will generally be equally preferred regarding to this criterion. On the other hand, choosing a preference threshold too low will impoverish the preference structure, as alternatives will generally be strictly preferred the one over the other, and the situation will be similar as when the preferrence was computed only by taking into account the sign of d_c.

2.2 Enrichment of the dominance relationship

2.2.1 Global preference index

After having computed the preference degrees for a pair of alternatives for each criterion, a set of weights w_c must still be defined in order to be able to compute the global preference index $\pi(a_i, a_j)$ of the alternative a_i over the alternative a_j. These weights w_c represent the relative importance of a criterion c compared to the others. The global preference index is given by :

$$\pi(a_i, a_j) = \sum_{c=1}^{k} P_c(a_i, a_j) \cdot w_c \tag{2.11}$$

The choice of the different w_c consists in the information *between criteria* [2] that must be given by the decision maker. Similarly as for the parameters giving information within a criterion, the choice of the w_c is of the highest importance. For example, a criterion with a weight equal to zero would simply have no influence on the global preference index.

As these parameters only consists of multiplicative constants, there is no inconvenient to choose them normalised :

$$\sum_{c=1}^{k} w_c = 1 \tag{2.12}$$

Some relations can be deduced from the equations (2.3) and (2.11) :

$$\pi(a_i, a_i) = 0 \tag{2.13}$$

which states that an alternative is not preferred over itself, and

$$0 \leq \pi(a_i, a_j) \leq 1 \tag{2.14}$$

which holds since the weights are normalised and the P_c are lower or equal to 1.

It should also be clear that [2]:

$$\begin{cases} \pi(a_i, a_j) \approx 0 & \Leftrightarrow \quad \text{weak preference of } a_i \text{ over } a_j \\ \pi(a_i, a_j) \approx 1 & \Leftrightarrow \quad \text{strong preference of } a_i \text{ over } a_j \end{cases} \tag{2.15}$$

2.2.2 Outranking flow

As just seen here above, $\pi(a_i, a_j)$ gives an indication on how a_i is dominating a_j on all the criterions. This information is only related to a pair of alternatives and does not give a sufficient indication of the dominance of a_i in the set A.

Such as for the ELECTRE method (section 1.3.5), an outranking graph can be built. The graph will be a complete digraph on n vertices representing the n possible alternatives. The valued arc going from an alternative a_i to an alternative a_j will be valued $\pi(a_i, a_j)$ [3].
This graph is therefore not a graph directly leading to an ordering, but will, for each pair of alternatives a_i and a_j, give a quantified information of how much a_i outranks a_j and vice-versa.

This graph can be used to compute an *outgoing outranking flow* and an *incoming outranking flow* for each alternative [6].
The outgoing flow is given by :

$$\phi^+(a_i) = \frac{1}{n-1} \sum_{j=1}^{n} \pi(a_i, a_j) \tag{2.16}$$

This flow indicates how much alternative a_i is dominating, or outranking, the remaining $n - 1$ alternatives. The higher this flow, the more a_i should be preferred by the decision maker.

Similarly, the incoming flow is defined by [6]:

$$\phi^-(a_i) = \frac{1}{n-1}\sum_{j=1}^{n}\pi(a_j, a_i) \tag{2.17}$$

This flow indicates how much alternative a_i is outranked by the $n-1$ other alternatives. The lower this flow, the more alternative a_i should be preferred.

One can see in the expressions of the outranking flows that the summations of the preferrence index are divided by $n-1$. This is done to enshure that the outranking flows will be lower or equal to 1 (the summations consist of sums on n terms lower than 1 from which at least one is null by (2.13)).

The netflow score can be obtained by substraction the incoming flow to the outgoing flow :

$$\phi(a_i) = \phi^+(a_i) - \phi^-(a_i) \tag{2.18}$$

2.3 Decision aid

2.3.1 PROMETHEE I

To help the decision maker in his task, the PROMETHEE methods will rank the alternatives according to their outranking flows.

The two complete preorders (P^+,I^+) and (P^-,I^-) obtained with the outranking flows are defined as follow [6]:

$$\forall a_i, a_j \in A \begin{cases} a_i P^+ a_j & \Leftrightarrow & \phi^+(a_i) > \phi^+(a_j) \\ a_i I^+ a_j & \Leftrightarrow & \phi^+(a_i) = \phi^+(a_j) \end{cases} \tag{2.19}$$

$$\forall a_i, a_j \in A \begin{cases} a_i P^- a_j & \Leftrightarrow & \phi^-(a_i) < \phi^-(a_j) \\ a_i I^- a_j & \Leftrightarrow & \phi^-(a_i) = \phi^-(a_j) \end{cases} \tag{2.20}$$

The PROMETHEE I is made by taking the intersection of these two preorders. An alternative a_i is preferred over another alternative a_j according to PROMETHEE I if both its outgoing flow and its incoming flow are better (respectively greater and smaller) than the ones of a_j. If both the flows are equal for the two alternatives, then the alternatives will be indifferent, and if none of these two cases hold, then the alternatives will be said to be incomparable [6]:

$$\forall a_i, a_j \in A : \begin{cases} a_i P a_j & \Leftrightarrow & \begin{cases} a_i P^+ a_j \text{ and } a_i P^- a_j \\ a_i P^+ a_j \text{ and } a_i I^- a_j \\ a_i I^+ a_j \text{ and } a_i P^- a_j \end{cases} \\ a_i I a_j & \Leftrightarrow & a_i I^+ a_j \text{ and } a_i I^- a_j \\ a_i R a_j & \Leftrightarrow & \text{else} \end{cases} \tag{2.21}$$

2.3.2 PROMETHEE II

It can happen that the decision maker wishes to obtain a complete ordering of the alternatives (without any incomparabilities). This can be done using the PROMETHEE II method. This method builds the ranking based on the net flow score of each alternatives :

$$\forall a_i, a_j \in A \begin{cases} a_i P a_j \Leftrightarrow \phi(a_i) > \phi(a_j) \\ a_i I a_j \Leftrightarrow \phi(a_i) = \phi(a_j) \end{cases} \tag{2.22}$$

As it can be seen, using the PROMETHEE II methode, two alternatives can be either prefered the one over the other (P) either indifferent (I). Two alternatives can therefore not be incomparable, leading to a complete preorder of the alternatives.

There exist four other variations of the PROMETHEE methods, named PROMETHEE III until VI. These methods will not be detailed here and the interested reader can refer to [2] for additional information.

There also exists an additional tool, named GAIA (Geometrical Analysis for Interactive Assistance), which is, as its name suggests, a tool aimed at performing an interactive visualisation of the data. Once again, this method will not be detailed here and the interested reader can refer to [2] or [6] for additional information.

Chapter 3

Rank Reversal

The so called *rank reversal* phenomenon is a phenomenon initially highlighted by Belton and Gear in 1983 [1].

The is not one unique definition of the rank reversal phenomenon but there rather exist several definition, which however all share the same idea : "the relative ordering between two alternatives depands of the presence of one or several other alternatives".
Some of the definitions that can be found in the literature [2] are that the relative position of two alternatives can be influenced by the presence of :

- a non discriminating criterion
- a copy of an alternative
- a dominated alternative
- any other alternative

As it can easily be imagined, multi-criteria decision methods based on pairwise comparisons of all the alternatives are susceptible to suffer from this phenomenon.
W. De Keyser and P. Peeters [4] where the first to point out that the PROMETHEE methods suffer from rank reversal occurences.

3.1 Some results about the rank reversal phenomenon in the PROMETHEE methods

3.1.1 Transitive preference matrix and pairwise comparisons

Some important facts have been highlighted by Mareschal et al. [7].

Let's again consider a multi-criteria decision problem consisting in the selection of an alternative from a set A of n alternatives which must maximise k evaluation

functions.
If the multi-criteria decision aid method is based on :

1. the computation of a $n \times n$ matrix P of pairwise preferences whose elements p_{ij} represent the preference of the alternative a_i over a_j
2. the use of this matrix to obtain a complete preorder.

then the method is said to be a *ranking method based on pairwise comparisons.*

The PROMETHEE methods are examples of such methods where the $n \times n$ preference matrix consist in the $n \times n$ evaluations of $\pi(a_i, a_j)$ for all a_i, a_j in A.

Furthermore, a ranking method based on pairwise comparisons is said to be consistant if the ordering obtained with the application of the method on a set of two alternatives is consistant with the preference matrix.
This means that if alternative a_i is preferred over alternative a_j ($p_{ij} > p_{ji}$) then a consistant ranking method based on pairwise comparisons should rank a_i before a_j when this method is applied on the set composed of these two alternatives.

This property seems rather intuitive and necessary for any ranking method based on pairwise comparisons.
By recording how the outrank flow is computed in the PROMETHEE methods (section 2.2.2), one can easily see that the outranking flow will be consistant with any 2×2 preference matrix π.

If a multi-criteria method is based on pairwise comparisons, is consistant and is applied on a preference matrix that is not transitive, then the rank reversals are unavoidable [7].

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To finish

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Example of why it is interessant to have a non transitive preference function,
Use the figure below to show that non transitivity and complete preorders lead to rank reversal

$$\pi = \begin{array}{c} \\ i \\ j \\ m \end{array} \begin{array}{c} \begin{array}{ccc} i & j & m \end{array} \\ \begin{pmatrix} 0 & 0.75 & 0.1 \\ 0.20 & 0 & 0.6 \\ 0.8 & 0.05 & 0 \end{pmatrix} \end{array}$$

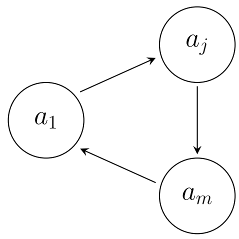

Figure 3.1: Example of non transitive preference matrix

3.1.2 Conditions under which rank reversal can appear in PROMETHEE

Explaination of the different bounds on netflow difference that lead to rank reversals in PROMETHEE II.

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