1 Non monotonicity of the Robust Promethee method

The property of monotonicity for a multicriteria decision aid method states that for any alternatives $a_\alpha, a_\beta \in A$, if a_α is prefered over a_β (with the classical notion of preferences), then a monotone multicriteria decision aid method should always prefer a_α over a_β (or both alternatives should be equally prefered).

The classical notion of preference relation is defined as follow[1] :

$$\forall a_i, a_j \in A : \begin{cases} a_i P a_j & \Leftrightarrow \quad \begin{cases} f_c(a_i) \geq f_c(a_j) \quad \forall c = 1 \ldots k \\ \exists h : f_h(a_i) > f_h(a_j) \end{cases} \\ a_i I a_j & \Leftrightarrow \quad f_c(a_i) = f_c(a_j) \quad \forall c = 1 \ldots k \\ a_i R a_j & \Leftrightarrow \quad \begin{cases} \exists h : f_h(a_i) > f_h(a_j) \\ \exists h' : f_{h'}(a_i) < f_{h'}(a_j) \end{cases} \end{cases} \tag{1}$$

with $a_i P a_b$ indicating the preference of a_i over a_j, $a_i I a_j$ the indifference, and $a_i R a_j$ the incomparability.

With a_α being a random alternative chosen from A and a_α^+ being the same alternative but with an increasement on at least on of the criteria evaluation, one can see that the Promethee II method is monotone :

$$\begin{aligned} \forall a_\alpha, a_\alpha^+ \in A, \forall c = 1 \ldots k : \quad & f_c(a_\alpha^+) \geq f_c(a_\alpha) \\ & \Rightarrow \pi_c(a_\alpha^+, a_i) \geq \pi_c(a_\alpha, a_i) \qquad \forall a_i \in A \\ & \quad \wedge \exists h : \pi_h(a_\alpha^+, a_\alpha) > \pi_h(a_\alpha, a_\alpha^+) \\ & \Rightarrow \phi(a_\alpha^+) > \phi(a_\alpha) \end{aligned} \tag{2}$$

Unfortunately, the Robust Promethee method is not guarenteed to be monotone. The following paragraphs will give an example of the Robust Promethee method where an alternative a_α, which is prefered (in the classical sens (see 1)) over another alternative a_α^-, but which will be ranked after this other alternative.
Consider for example the problem with an evaluation table given in Table 1.

It is well known that the Promethee II methods suffer from rank reversal instances. We can therefore assume that we can find two alternatives a_α and a_β such that there exist a family of subsets $S_{\alpha\beta} \subset A^m$ where $a_\alpha \succ a_\beta$ but such that there also exists another family of subsets $S_{\beta\alpha} \subset A^m$ where $a_\beta \succ a_\alpha$.

Now consider the same problem where two alternatives a_α^- and a'_β have been added (Table 2), with a_β and a'_β being two identical alternatives while a_α and a_α^- are two nearly identical alternatives, a_α^- having a slightly worse evaluation than a_α on at least one criterion.

a	$f_1(.)$	$\ldots$	$f_c(.)$	$\ldots$	$f_k(.)$
a_1	$f_1(a_1)$	$\ldots$	$f_c(a_1)$	$\ldots$	$f_k(a_1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_β	$f_1(a_\beta)$	$\ldots$	$f_c(a_\beta)$	$\ldots$	$f_k(a_\beta)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_α	$f_1(a_\alpha)$	$\ldots$	$f_c(a_\alpha)$	$\ldots$	$f_k(a_\alpha)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$f_1(a_n)$	$\ldots$	$f_c(a_n)$	$\ldots$	$f_k(a_n)$

Table 1: example evaluation table

a	$f_1(.)$	$\ldots$	$f_c(.)$	$\ldots$	$f_k(.)$
a_1	$f_1(a_1)$	$\ldots$	$f_c(a_1)$	$\ldots$	$f_k(a_1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_β	$f_1(a_\beta)$	$\ldots$	$f_c(a_\beta)$	$\ldots$	$f_k(a_\beta)$
a'_β	$f_1(a_\beta)$	$\ldots$	$f_c(a_\beta)$	$\ldots$	$f_k(a_\beta)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_α	$f_1(a_\alpha)$	$\ldots$	$f_c(a_\alpha)$	$\ldots$	$f_k(a_\alpha)$
a^-_α	$f_1(a_\alpha)$	$\ldots$	$< f_c(a_\alpha)$	$\ldots$	$f_k(a_\alpha)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$f_1(a_n)$	$\ldots$	$f_c(a_n)$	$\ldots$	$f_k(a_n)$

Table 2: modified evaluation table of the example

Since a'_β is identical to a_β and a^-_α is arbitrarly similar to a_α, we can expand our definition of the family of sets $S_{\alpha\beta}$ to also contain the sets where $a^-_\alpha \succ a_\beta$ or $a^-_\alpha \succ a'_\beta$ (and similarly for $S_{\beta\alpha}$).

It should not be forgotten that if the Robust Promethee method was monotone, a_α should be prefered to a^-_α. We will show that it is not always the case in our previous example.

Assume that at each of the R iterations of the Robust Promethee method one of the three following cases happen :

- Not both an alternative from a_α and a^-_α and another alternative from a_β and a'_β are selected in the m compaired alternatives.
- a^-_α and one of a_β or a'_β are selected in the set of m alternatives, and this set of alternatives belongs to the family $S_{\alpha\beta}$. This implies that a^-_α will always be ranked before alternatives a_β and a'_β.
- a_α and one of a_β or a'_β are selected in the set of m alternatives, and this set of alternatives belongs to the family $S_{\beta\alpha}$. This implies that a_α will

always be ranked after alternatives a_β and a'_β.

After the R iterations of the method, the probability matrix will be similar to the folowing one :

$$
P = \begin{array}{c} \\ \beta \\ \beta' \\ \\ \alpha \\ \alpha^- \end{array}
\begin{array}{c} \begin{array}{cccc} \beta & \beta' & \alpha & \alpha^- \end{array} \\
\begin{pmatrix} & & 1 & 0 \\ & & 1 & 0 \\ & & & \\ 0 & 0 & & 1 \\ 1 & 1 & 0 & \end{pmatrix} \end{array}
$$

We can see in this matrix that $P_{\alpha\alpha^-} = 1$ and $P_{\alpha^-\alpha} = 0$ (due to the fact that the Promethee II method is monotone). These values could also have had a default value of 0.5 if the two alternatives had not been compaired together at least once during one of the iterations.

If we consider that the diferences between the preferences of a_α over the other alternatives (not shown in the matrix) and the preferences a_α^- over the other alternatives are neglectables, then we can easily see that the net flow score of a_α^- will be greater than the one of a_α :

$$
\begin{aligned}
\phi(a_\alpha^-) - \phi(a_\alpha) &= \sum_{i=1}^{n} \left[(P(a_\alpha^-, a_i) - P(a_i, a_\alpha^-)) - (P(a_\alpha, a_i) - P(a_i, a_\alpha) \right] \\
&= P(a_\alpha^-, a_\beta) + P(a_\alpha^-, a'_\beta) - P(a_\alpha, a_\alpha^-) \\
&\quad - (-P(a_\beta, a_\alpha) - P(a'_\beta, a_\alpha) + P(a_\alpha, a_\alpha^-)) \\
&= (1 + 1 - 1) - (-1 - 1 + 1) \\
&> 0
\end{aligned} \tag{3}
$$

This example shows the non monotonicity of the Robust Promethee method. However, it reposes on possible but unlikely assumptions that the dominated alternative a_α^- is always compared to other alternatives in “favorable” sets and a_α in “unfavorable” ones. It should be verified if this phonomenon happens in practical problems.

References

[1] Jean-Pierre Brans and Yves De Smet. *PROMETHEE Methods*, pages 187–219. Springer New York, New York, NY, 2016.