1 Modeling

We model only one axis of tilt. By physics laws we know that

$$\ddot{x}(t) = \frac{F_{\parallel}(t)}{m}, \tag{1}$$
$$F_{\parallel}(t) = mg\sin\theta(t), \tag{2}$$

where x is the position of the ball along the plane, m its mass, g the gravitational constant and θ the tilt angle. Notice that x and θ depend on time. By combining 1 and 2 we obtain the following model of the unforced dynamics:

$$\ddot{x}(t) = g\sin\theta(t). \tag{3}$$

By including the control input $u(t)$, which sets the angular velocity of the tilt angle, we obtain the final non-linear dynamical model as a system of differential equations:

$$\begin{cases} \ddot{x}(t) = g\sin\theta(t) \\ \dot{\theta}(t) = u(t) \end{cases} . \tag{4}$$

We then translate 4 to a system of first order differential equations:

$$\begin{cases} \dot{x}(t) = v \\ \dot{v}(t) = g\sin\theta(t) \\ \dot{\theta}(t) = u(t) \end{cases} . \tag{5}$$

And finally we bring 5 to the following continuous-time non-linear system in state space form

$$\begin{cases} \dot{\vec{x}}(t) = f(\vec{x}(t)) + Gu(t) \\ y(t) = H\vec{x}(t) \end{cases} , \tag{6}$$

where the state vector $\vec{x}(t)$ and the input/output matrices G, H are

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ v(t) \\ \theta(t) \end{pmatrix} \in \mathbb{R}^3 \qquad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{7}$$

and $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field that maps

$$\begin{pmatrix} x \\ v \\ \theta \end{pmatrix} \mapsto f\left(x, v, \theta\right) = \begin{pmatrix} v \\ g\sin\theta \\ 0 \end{pmatrix} . \tag{8}$$

2 Kalman filter design

Since the system is non-linear, we adopt the technique of linearizing the system around the current working point.

We start by performing the Taylor expansion of 8 around $\bar{x}$ truncated to the first order, which gives

$$f(\vec{x}) \approx f(\bar{x}) + \mathbf{J}_f(\bar{x})(\vec{x} - \bar{x}) \tag{9}$$

so the system in 6 can be approximated as

$$\begin{cases} \dot{\vec{x}}(t) = f(\bar{x}) + \mathbf{J}_f(\bar{x})\vec{x}(t) - \mathbf{J}_f(\bar{x})\bar{x} + Gu(t) \\ y(t) = H\vec{x}(t) \end{cases} \tag{10}$$

which we rewrite in this form

$$\begin{cases} \dot{\vec{x}}(t) = F_{\bar{x}}\vec{x}(t) + G_{\bar{x}} \begin{pmatrix} u(t) \\ 1 \end{pmatrix} \\ y(t) = H\vec{x}(t) \end{cases} \tag{11}$$

where

$$F_{\bar{x}} \coloneqq \mathbf{J}_f(\bar{x}) \qquad\qquad G_{\bar{x}} \coloneqq \left[\begin{array}{c|c} G & f(\bar{x}) - \mathbf{J}_f(\bar{x})\bar{x} \end{array}\right]. \tag{12}$$

In this way we obtain a linear system which also includes the known dynamics around the linearization point (which, in general, is not an equilibrium). This is achieved by an additional column of the new input matrix $G_{\bar{x}}$ which is associated to a secondary constant input equal to 1.

The reason for writing the system in this form is so that we can apply common discretization techniques, designed for systems in state-space form, for obtaining a discrete time linear system starting from the continuous time linear system in 11. This is done automatically using the zero-order-hold (ZOH) discretization method[1] every time the state estimate of the filter changes. The linearization point $\bar{x}$ is chosen as the current state estimate. The standard (linear) Kalman filter can then be applied.

Performing some computations, and with reference to the definition of f in 8, the actual system matrices of the approximated linear system we obtained are

$$F_{\bar{x}} \coloneqq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & g\cos(\bar{\theta}) \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{\bar{x}} \coloneqq \left[\begin{array}{c|c} 0 & 0 \\ 0 & g(\sin\bar{\theta} - \bar{\theta}\cos\bar{\theta}) \\ 1 & 0 \end{array}\right]. \tag{13}$$

The initial covariance matrix of the state estimate P_0 is chosen as diagonal with all entries 10^3, an arbitrarily large number to encode the condition of "no information about the initial state".

[1]The real system is continuous time and we assume that the controller (which drives $u(t)$) holds the value constant during discrete time steps, the ZOH discretization technique then is the perfect one for such scenario.

However, an issue arises regarding the initial estimation of the tilt angle θ under a specific condition: the ball being stationary. In such case $v = 0$ and there are infinitely many values of θ that can satisfy such condition, w.r.t. the dynamics in 8. If the filter converges to an initial estimate of the form $\hat{\theta} = 2\pi k,\ k \in \mathbb{Z}$ there are no issues, however it might happen that the convergence leads to an estimate of the form $\hat{\theta} = \pi + 2\pi k,\ k \in \mathbb{Z}$. While being compatible with the condition $v = 0$, it makes the filter prediction steps produce a non-meaningful result as soon as the tilt angle θ beings to vary (for instance due to $u(t)$) as the predicted velocity will be inverted in sign. Since designing an ad-hoc non-linear non-gaussian filter is outside of this scope[2] this problem is dealt with by clamping, at each discrete-time filter update step, the estimate $\hat{\theta}$ in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

[2]In such case we could encode the initial state estimate for θ with an uniform random variable in the valid mechanical range $T \subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.