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Damped Oscillations with a Smart Cart

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The introduction of the Wireless Smart Cart by PASCO scientific in April 2016 has ushered in a paradigm shift in the design and implementation of low-cost undergraduate physics and engineering laboratory experiments. The use of smartphones in experimental physics is by now widely accepted and documented.[1-5] The smart cart in combination with student-owned smartphones and free apps has opened up a new universe of low-cost experiments that have traditionally required cumbersome and expensive equipment.[6] In this paper we demonstrate the simplicity, convenience, and cost saving achieved by replacing a plethora of traditional laboratory sensors, wires, air tracks, and other equipment clutter with the smart cart and the free SPARKvue app[7] for smartphones by carrying out an experiment on damped oscillations.

Experimental setup for damped oscillations

A magnetic damping accessory, also available from PASCO scientific, was snapped on to the front of the smart cart. The other end of the smart cart was hooked to a spring and the entire assembly could oscillate on an inclined aluminum track. Damped motion (magnetic braking) occurs because of the eddy currents generated in the nonmagnetic but electrically conducting aluminum track. Lenz's law dictates that the motion of the cart will be resisted by a magnetic force proportional to the velocity of the cart.[8,9] The experimental setup is shown in Fig. 1.

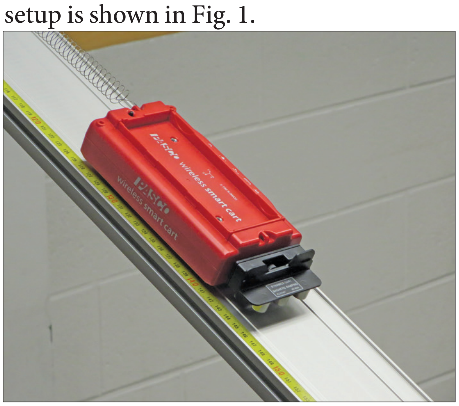

Fig. 1. The Smart Cart with attached spring and magnetic damping accessory on an inclined aluminum track.

The magnetic damping accessory can be simply slid up and down or removed to provide different amounts of magnetic damping, called "viscous damping" for historical reasons. In this manner, one can obtain no magnetic damping (minimal overall damping), underdamping, critical damping, and overdamping of the oscillations of the smart cart. The free SPARKvue app paired up with the smart cart via Bluetooth and recognized the unique identification sticker on the cart. The position and time data (in addition to velocity, acceleration, and a cornucopia of other data), sampled at 50 Hz, were wirelessly transmitted by the smart cart as a CSV file (comma separated values). The data were tabulated in an Excel file and a graph of position vs. time is depicted in Fig. 2.

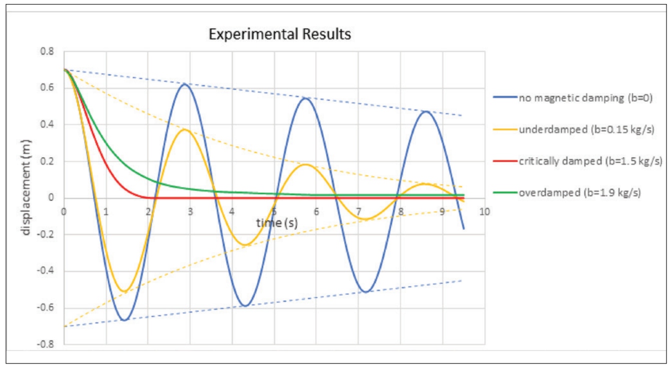

Fig. 2. Experimental position vs. time of the Smart Cart.

Theoretical underpinnings, results, and calculations

The unforced, damped harmonic oscillator is governed by the following differential equation[10-13]:

$$md^2x/dt^2 + b\,dx/dt + kx = 0\,, \qquad (1)$$

where m is the oscillating mass, k is the spring constant, and b is the magnetic (viscous) damping coefficient. The characteristic roots of the differential equation suggest that:

- $b^2 < 4mk$ = underdamping occurs when b is small relative to m and k.
- $b^2 > 4mk$ = overdamping occurs when b is large relative to m and k.
- $b^2 = 4mk$ = critical damping occurs when b is just between under- and overdamping.

In the case of underdamping, the characteristic roots are not real and using the symbol

$$\omega_d \equiv \frac{\sqrt{4mk - b^2}}{2m} \quad \text{for the underdamped frequency,} \qquad (2)$$

the solution to the differential equation is written as

$$x(t) = c_1 e^{-bt/2m}\cos(\omega_d t) + c_2 e^{-bt/2m}\sin(\omega_d t).$$

The constants c_1 and c_2 are determined from the initial position and velocity of the damped oscillator. For underdamped oscillations the displacement equation can be recast in terms of amplitude and phase as

$$x = A_0 \exp(-bt/2m)\cos[(2\pi t/T_d) + \varphi]\,. \qquad (3)$$

The time period, T_d, and angular frequency ω_d, are simply related as $T_d = 2\pi/\omega_d$. The oscillations are pseudo periodic and the amplitude decays exponentially. In the case of overdamping, the two characteristic roots are real, distinct, and negative.

$$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

and the general solution is

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$x(t) = c_1 e^{r1t} + c_2 e^{r2t}$.

An unforced, overdamped harmonic oscillator does not oscillate and the amplitude goes asymptotically to the equilibrium position $x = 0$. In the case of critical damping, $b^2 = 4mk$, the term under the square root is zero, the characteristic polynomial has repeated roots, $-b/2m$, $-b/2m$, and the general solution is

$$x(t) = e^{-bt/2m}(c_1 + c_2 t).$$

Again, there are no oscillations, and the system returns to the equilibrium position quicker than in the overdamped case. Many door springs are actually adjusted to be slightly overdamped as a tradeoff between slower return and quickest possible return, which may slam the door shut on unsuspecting folks!

In our experiment, plugging in the mass of the cart m and the spring constant k from Table I, $b < 1.5$ kg/s = underdamping, $b > 1.5$ kg/s = overdamping, and $b = 1.5$ kg/s = critical damping. For convenience, a dimensionless damping ratio

$$\zeta = b/(4mk)^{1/2} \tag{4}$$

may be defined in terms of the damping ratio ζ, for all values of m and k, $\zeta < 1$ = underdamping, $\zeta > 1$ = overdamping, and $\zeta = 1$ = critical damping.

Table I.

Mass of the smart cart plus magnetic damper	$m = 0.344 \pm 0.0001$ kg
Spring constant	$k = 1.66 \pm 0.001$ N/m
Critical damping	$b_c = 2(mk)^{1/2} = 1.51 \pm 0.001$ kg/s
Undamped period	$T_0 = 2\pi(m/k)^{1/2} = 2.860 \pm 0.001$ s

The natural time period with no damping and a light spring[14] is

$$T_0 = (2\pi)(m/k)^{1/2}. \tag{5}$$

The time period with damping is

$$T_d = T_0 / (1-\zeta^2)^{1/2}. \tag{6}$$

For underdamped oscillations the displacement is (see Eq. [3])

$$x = A_0 \exp(-bt/2m) \cos[(2\pi t/T_d) + \varphi].$$

The exponential decay in the amplitude of oscillations may be measured by a dimensionless parameter

$$\delta \equiv 1/n\,(\ln A_0/A_n). \tag{7}$$

From Eqs. (4)–(7), it can be shown that δ and ζ are related by

$$1/\zeta^2 = 1 + (2\pi/\delta)^2. \tag{8}$$

Referring to Fig. 2, the amplitude of the underdamped oscillations decreases from 0.70 m to 0.18 m when $n = 2$, so $\delta = 0.63$. Thus, we calculate $\zeta = 0.10$ and $b = 0.15$ kg/s. Again, referring to Fig. 2, T_d can be read off as approximately 2.9 s. The equation $T_d = T_0 / (1-\zeta^2)^{1/2}$ also gives 2.9 s. Our smart cart is very smart indeed.

The astute reader will notice that T_d and T_0 are virtually indistinguishable! This is generally true in damped oscillatory motion. The amplitude decreases exponentially but the period of oscillation barely changes. As we push ζ to higher values, the oscillatory behavior vanishes before any change in period can be discerned.

Coulomb damping

Why does the amplitude of the undamped (no magnetic or viscous damping) oscillations decrease slowly? Because of ever-present kinetic friction, also known as Coulomb damping. This decrease in the amplitude is not exponential but linear. From an elementary work-energy consideration, let A_0 be the initial amplitude and A_1 be the amplitude after one oscillation. Now consider $A_{1/2}$, the amplitude after half an oscillation. The loss in potential energy of the spring after half an oscillation is equal to the work done against friction.

$$\begin{aligned} &½\,k(A_0)^2 - ½\,k(A_{1/2})^2 = F_f(A_0 + A_{1/2}). \\ &½\,k(A_0 - A_{1/2})(A_0 + A_{1/2}) = F_f(A_0 + A_{1/2}) \\ &F_f = ½\,k(A_0 - A_{1/2}). \end{aligned} \tag{9}$$

In terms of the amplitude after one oscillation,[15]

$$F_f = (1/4)k(A_0 - A_1), \tag{10}$$

where A_0 is the initial amplitude, A_1 is the amplitude after one oscillation, F_f is the frictional force, and k is the spring constant. Reading off Fig. 3, $F_f = (1/4)(1.66$ N/m$)(0.70$ m $- 0.62$ m$) = 0.033$ N. For a track inclined at an angle θ, the coefficient of kinetic friction

$$\mu_k = F_f / (mg\cos\theta). \tag{11}$$

For $\theta = 30.0°$, we find $\mu_k = 0.01$. We attribute this minimal kinetic friction to rolling friction and friction associated with the axle bearings.[16,17]

To continue this analysis, the average energy loss per half oscillation for magnetic damping is $\Delta En = ½\,k(A_n - A_{n+½})(A_n + A_{n+½})$ compared with $\Delta E_f = F_f(A_n + A_{n+½})$, where n is the half oscillation number, so $\Delta E_f/\Delta En = 2F_f/[k(A_n - A_{n+½})] = 2F_f/[k A_0 e^{-n\varepsilon}(1 - e^{-\varepsilon})]$ with $\varepsilon = \pi b/2(mk)^{1/2}$. Thus, the

Table II.

Magnet Clearance (mm)	Track Inclination θ (degrees)	Terminal Velocity (m/s)	Damping Coefficient b (kg/s)	Damping Ratio ζ	Theoretical Period (s) $T_0/(1-\zeta^2)^{1/2}$	Damping
2.0	30	1.6	1.05	0.695	3.977	$\zeta < 1$ underdamped
1.6	30	1.4	1.20	0.794	4.705	$\zeta < 1$ underdamped
1.0	30	1.2	1.40	0.926	7.598	$\zeta < 1$ underdamped
0.8	30	1.08	1.56	1.032	–	$\zeta \sim 1$ critically damped
0.6	30	1.03	1.64	1.085	–	$\zeta > 1$ overdamped
0.4	30	0.94	1.79	1.185	–	$\zeta > 1$ overdamped
0.2	30	0.90	1.87	1.237	–	$\zeta > 1$ overdamped

energy loss due to friction during the first half oscillation is 2*0.033 /[1.66*(0.7-0.5)] = 0.20 of the total, and this fraction rapidly increases with n. Thus, the differential Eq. (1) does not describe this situation and the full equation including magnetic plus frictional damping has to be employed.[18] As $F_f/F_b = (\mu g \cos\ \theta/2\,\zeta v)(m/k)^{1/2}$, increasing the spring constant reduces the influence of the friction.

Direct determination of the damping coefficient *b*

We also performed an experiment with the smart cart on the track without the spring to determine the damping coefficient b for different distances of the magnetic damping accessory from the surface of the aluminum track. This was done by measuring the terminal velocity of the cart on a track inclined to the horizontal at $\theta = 30^o$. When terminal velocity is attained, the gravitational force down the track must equal the magnetic damping force.

$$mg\sin\theta - F_f = bv$$
$$b = (mg\sin\theta - F_f)\,/\,v. \tag{12}$$

Conclusions

In this paper we demonstrated the simplicity, convenience, and cost saving achieved by replacing a plethora of traditional laboratory sensors, wires, air track, and other equipment clutter with only the inexpensive smart cart, aluminum track, and the free app SPARKvue from PASCO scientific by carrying out an experiment on damped oscillations. The types of experiments that can be performed are limited only by one's imagination. In order to appreciate the complexity and cost that would be involved in this type of experiment using just a smartphone but no smart cart, the reader is invited to review these two references in the *American Journal of Physics*[19] and the *European Journal of Physics*.[20] Another experiment in our physics laboratory where we have achieved tremendous savings in addition to eliminating the complexity and clutter is by replacing the air track, air source, photogates, and the like for a Collisions and Momentum Conservation laboratory setup with two smart carts (red and blue) on an aluminum track. A significant advantage of the smart cart is that the data can be seen in real time rather than only after a download. This is significant pedagogically, in that the students can directly connect the graphical data with the motion they observe. This inexpensive and uncomplicated equipment has been shown to produce precise and versatile experimental data on damped harmonic motion, which can be analyzed with varying levels of sophistication.

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