215A : Algebraic Topology

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1 Foundational Review

1.1 Motivating The Search for a Fundamental Group $\pi_1(X)$

This course is built on two different subjects. You have topology, which tends to be intuitive. For instance, we can easily say a sphere and torus are different simply by the presence of a hole. But this initially lacks precision. So instead, we turn to the precise and algorithmic language of algebra to solve this problem. Specifically, given a topological space, we can establish a fundamental group. We do this in the hopes that equivalent spaces can be identified with the same fundamental group!

Definition 1.1. *The solid ball of radius 1 is defined*

$$D^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\} \quad n \geq 1$$

as well as the sphere of radius 1

$$S^{n-1} := \partial D^n = \{x \in \mathbb{R} : \|x\| = 1\}$$

Note 1.2. *When $n = 0$, define $D^0 = \{0\}$.*

We want to define a magical function that is able to distinguish between the torus and the sphere. Specifically, suppose a function π_1 such that

$$\pi_1(\text{torus}) = \mathbb{Z} \oplus \mathbb{Z}$$

$$\pi_1(S^1) = \mathbb{Z}$$

$$\pi_1(S^2) = \{1\}$$

$$\pi(\text{2-torus } \infty) = \text{(a non-commutative) free group generated by two elements}$$

We need to develop algebraic machinery in order to define such a function.

1.2 Equivalence Relation

Definition 1.3. *Given a space X, we consider equivalence relations $\bowtie$ that follow*

- $x \bowtie x$
- $x \bowtie y \implies y \bowtie x$
- $x \bowtie y$ *and* $y \bowtie z \implies x \bowtie z$

We can prove that this partitions the space, allowing us to consider classes

$$[x] := \{y \in X : x \bowtie y\}$$

1.3 Review of Topology

- Open sets of a topological space X
  - Union of open sets is open
  - Finite intersections of opens sets is open
  - $\emptyset, X$ are open
- Base: We establish a family of open subsets of X such that for all open sets $U \subset X$, and $x \in U$, there exists V within our basis such that
$$x \in V \subset U$$

Definition 1.4. *Given X, Y topological spaces, a function $f : X \to Y$ is continuous if*
$$f^{-1}(U) = \{x \in X : f(x) = U\} \text{ is open}$$
for all open subsets $U \subset Y$

In this course, any $g : X \to Y$ is assumed to be continuous.

Definition 1.5. *X and Y are homeomorphic if there exists $f : X \to Y$ and $g : Y \to X$ such that*
$$f \circ g \equiv id_Y$$
$$g \circ f \equiv id_X$$

Notation: Given any topological space X, Y, we will write $X = Y$ provided X and Y are homeomorphic.

Definition 1.6. *X is compact if for all open covers $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X, there exists a finite subcover*
$$X \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$$

Remark 1.7. *Lebesgue's Covering Lemma: If $X \subset \mathbb{R}^n$ is compact, then for any open cover $\{U_\alpha\}$, we can identify δ such that for all $x \in X$, there exists U_α such that*
$$x + \delta D^n = B_\delta(x) \subset U_\alpha$$

Definition 1.8. *Let $I = [0, 1]$. A path is a continuous function $\phi : I \to X$ such that*
$$x_0 = \phi(0) = \textit{initial point}$$
$$x_1 = \phi(1) = \textit{terminal point}$$
The image of ϕ connects x_0 with x_1.

Proposition 1.9. *If X is a topological space, then we can define an equivalence relation $x \bowtie y$, if and only if there exists a path $\phi : I \to X$ with $\phi(0) = x$ and $\phi(1) = y$.*

Proof.
- It's easy to see for any $x \in X$, the constant function $\phi(t) = x$ is a path from x to itself. So $x \bowtie x$
- Suppose $x \bowtie y$. Then there is a path from x to y, say ϕ. Now define $\psi(t) = \phi(1 - t)$. Then ψ is a path from y to x. So $y \bowtie x$.
- Lastly, suppose $x \bowtie y$ and $y \bowtie z$. Then we must have a path from x to y, say ϕ, and a path from y to z, say ψ. Then we see that we can define a sort of product between the two paths by:
$$\phi \cdot \psi := \begin{cases} \phi(2t) & 0 \leq t \leq \frac{1}{2} \\ \psi(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$
Further, we see that $\phi \cdot \psi$ is a path between x and z. So $x \bowtie z$.

■

Corollary 1.9.1. *The equivalence classes under the equivalence relation of path connectedness are the path connected components of the space X.*

Corollary 1.9.2. *X is path connected if and only if for any $x, y \in X$ can be connected by a path.*

1.4 Review of Groups

We are all familiar with the definition of a group.

Definition 1.10. *$H \subset G$ is a subgroup if $g^{-1}h \in H$ for all $g, h \in H$.*

Definition 1.11. *One action fundamentally linking geometry with algebra is conjugation, in which for any $g, h \in G$*

$$g^{-1}hg$$

is the conjugate.

Definition 1.12. *N is a normal subgroup of G provided N is a subgroup of G and*

$$ghg^{-1} \in N$$

for all $h \in N$. That is, N is a subgroup closed under conjugation.

Definition 1.13. *A function $\phi : G_1 \to G_2$ is a homomorphism for group G_1, G_2 provided*

$$\phi(g_1 g_2) = \phi(g_1) \cdot \phi(g_2)$$

and ϕ is a surjection. We can further describe $Ker(\phi) := \{g \in G_1 : \phi(g) = 1_{G_2}\}$. Such a homomorphism is called an isomorphism provided $Ker(\phi) = \{0\}$.

Remark 1.14. *We will want to make use of an equivalent definition of isomorphism. $\phi : G \to H$ is an isomorphism if and only if $\exists$ a homomorphism $\psi : H \to G$ such that*

$$\phi \circ \psi = id_H$$

$$\psi \circ \phi = id_G$$

Proposition 1.15. *$Ker(\phi)$ is a normal subgroup of G_1.*

Proposition 1.16. *If ϕ is surjective, then $G_2 \cong G_1/Ker(\phi)$.*

Definition 1.17. *The commutator subgroup $K = \langle ghg^{-1}h^{-1} : g, h \in G \rangle$ is a normal subgroup.*

Proposition 1.18. *$G/K :=$ commutative group. It's also the maximally commutative group generated by projecting G over any subgroup of G.*

Example 1.19.

$$\mathbb{Z}$$

$$\mathbb{Z}_2 = \{0, 1\}$$

$$\mathbb{Z}^n = \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n \text{ copies}}$$

Definition 1.20. *Given a set of generators $\{g_\alpha\}_{\alpha \in \mathcal{A}}$, a free group is any finite sequence $a_1, a_2, \ldots, a_k$ such that $a_i = g_\alpha$ or g_α^{-1}*

Remark 1.21. *$\emptyset$ is the unit element of any free group since it is a sequence of length 0.*

Remark 1.22. *We define the product operation of a free group to be*

$$(a_1, \ldots, a_k)(b_1, \ldots, b_k) = a_1 \ldots a_k b_1 \ldots b_k$$

1.5 Operations on Topological Spaces

Definition 1.23. *Given topological spaces X, Y, the* <u>*product*</u>

$$X \times Y := \{(x, y) : x \in X, y \in Y\}$$

equipped with the basis of topology defined to be:

$$\mathcal{B} = \{U \times V : \textit{for any open subsets } U \subset X, V \subset Y\}$$

Example 1.24. $\mathbb{R}^n := \underbrace{\mathbb{R} \times \dots \mathbb{R}}_{n \textit{ times}}$

Example 1.25. *The torus* $\mathbb{T} := S^1 \times S^1$.

Definition 1.26. *Given a topological spaces X, the* <u>*quotient*</u> *of an equivalence relation* $\bowtie$ *is defined*

$$X/\bowtie := \{\textit{set of equivalence classes}\}$$

equipped with the basis of topology defined to be:

$$\mathcal{B} = \{U/\bowtie : \textit{for any open subset } U \subset X\}$$

Note 1.27. *Another way to consider this is by the projection mapping* $\pi : X \to X/\bowtie$

Example 1.28. *Given a topological space X and any subset* $A \subset X$. *Then we can define*

$$x \bowtie y \iff x, y \in A \textit{ or } x = y$$

Then $X/\bowtie = X/A$.

Example 1.29. $S^1 = I/\bowtie$ *where* $0 \bowtie 1$ *and all other points are self-identifiable. Further, we can simply write:*

$$S^1 = I/\{0, 1\}$$

Definition 1.30. *Given a group G and a topological space X, we can define the* <u>*action of G on X*</u> *provided we can identify continuous functions for every* $g \in G$, *such that*

$$g \cdot (h \cdot x) = (gh) \cdot x$$

for any $g, h \in G, x \in X$

Note 1.31. *Group actions induce an equivalence relation on the topological space X by*

$$x \bowtie y \iff \exists g \in G \textit{ such that } y = gx$$

i.e. The equivalence classes are the orbits of our action by G. Quotienting by this equivalence relation is denoted $X/\bowtie = X/G$

Example 1.32. *Given the action of* $\mathbb{Z}$ *onto* $\mathbb{R}$ *by*

$$k \cdot x = x + k$$

Then we see that $S^1 = \mathbb{R}/\mathbb{Z} = [0, 1)$ *represents each equivalence class uniquely with* $0 \bowtie 1$.

This can become a bit of a notational nightmare.

Example 1.33. *Given the action of* $\mathbb{R}$ *onto* $\mathbb{C}$ *by*

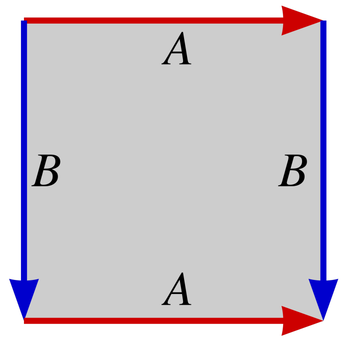
$$x \cdot z = e^{ix} z$$

Then we see that

$$\mathbb{C}/\mathbb{R} \neq \mathbb{C}/\bowtie$$

So one must be careful with this notation if it isn't obvious.

Example 1.34. *The torus can be identified using the polygonal diagram:*



Alternatively, we can identify the torus by defining the relation on I^2

$$x \bowtie y \iff x - y \in \mathbb{Z}^2 \implies \mathbb{T} = I^2/\bowtie$$

These notations can be unified on the following manner:

Example 1.35. *Let* $\mathbb{Z}^2$ *act on* $\mathbb{R}^2$ *by defining*

$$\forall (r, m) \in \mathbb{Z}^2, (x, y) \in \mathbb{R}^2, (r, m) \cdot (x, y) = (x + r, y + m)$$

Then we see that

$$\mathbb{T} = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$$

Therefore, we see that

$$I^2/\bowtie = \mathbb{R}^2/\mathbb{Z}^2$$

Example 1.36. *Observe* $S^{n-1} \subset D^n$*, the* $D^n/S^{n-1} = S^n$*. We can see this by defining the quotient mapping*

$$\begin{aligned} \pi : D^n &\to S^n \subset \mathbb{R}^{n+1} \\ ru &\to (\underbrace{u \sin \pi r}_{\in \mathbb{R}^n}, \cos \pi r) \in \mathbb{R}^{n+1} \end{aligned}$$

where $u \in S^{n-1}$.

2 Homotopy

2.1 Defining Homotopy

Definition 2.1. *Given two functions $f, g : X \to Y$, we say f and g are homotopic, denoted $f \sim g$, provided there exists a continuous function $F : X \times I \to Y$ such that*

$$F(x, 0) = f(x) \quad F(x, 1) = g(x)$$

We call such a function F a deformation of f into g.

Proposition 2.2. *$\sim$ is an equivalence relation on the set of continuous maps $f, g : X \to Y$*

Proof.

- $f \sim f$ obviously since we can consider the deformation $F(x, t) = f(x)$
- $f \sim g \implies g \sim f$ by the deformation $G(x, t) = F(x, 1 - t)$
- $f \sim_F g, g \sim_G h \implies f \sim h$ by the deformation

$$H(x, t) = \begin{cases} F(x, 2t) & t \leq \frac{1}{2} \\ G(x, 2t - 1) & t \geq \frac{1}{2} \end{cases}$$

∎

Proposition 2.3. *Suppose $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. If $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Proof. Let F_t be a homotopy between f_0 and f_1 and G_t be a homotopy between g_0 and g_1. Then $g_0 \circ f_0 \sim g_0 \circ f_1$ via $g_0 \circ F_t$ and $g_0 \circ f_1 \sim g_1 \circ f_1$ via $G_t \circ f_1$. Since homotopy forms an equivalence relation, by transitivity, we have $g_0 \circ f_0 \sim g_1 \circ f_1$ ∎

Example 2.4. *Any two maps $f_0, f_1 : X \to D^n$ are homotopic by the deformation*

$$F_t(x) = (1 - t)f_0(x) + tf_1(x)$$

Example 2.5. *Let $X = S^1, Y = D^2$, and consider*

$$f_0 = i : S^1 \hookrightarrow D^2$$

$$f_1(x) = (0, 0) \in D^2$$

Then by the previous example, we have the deformation

$$F_t(x) = (1 - t)x$$

Example 2.6. *Let $X = S^1, Z = D^2 \setminus \{(\frac{1}{2}, 0)\}$. We again consider*

$$f_0 = i : S^1 \hookrightarrow Z$$

$$f_1(x) = (0, 0) \in Z$$

Then we don't have a way of deforming the spaces into one another by the previous deformation.

Remark 2.7. *Not only the homotopy above breaks down, but we will see that there does not exist a homotopy between $f_0, f_1 : S^1 \to Z$.*

Definition 2.8. *Let $A \subset X$ and $f_0, f_1 : X \to Y$. Then $f_0 \sim f_1$ relative to A provided*

- *$f_0(x) = f_1(x)$ for $x \in A$*
- *$\exists F_t(x) = f_0(x)$ for all $x \in A$.*

Example 2.9. *Given two curves $\gamma, \delta : I \to X$ with $\gamma \sim \delta$, we say that $\gamma \sim \delta$ relative to $\{0,1\} \subset I$ provided that*

$$\phi_t(0) = \gamma(0) = \delta(0)$$

$$\phi_t(1) = \gamma(1) = \delta(1)$$

The initial and terminal points are fixed!

Proposition 2.10. *If $Y = D^n$, or any convex $Y \subset \mathbb{R}^n$, then for any $f_0, f_1 : X \to Y$ that agree on $A \subset X$, and $f_0 \sim f_1$, then $f_0 \sim f_1$ relative to A.*

Definition 2.11. *X, Y are homotopy equivalent provided $\exists f : X \to Y, g : Y \to X$ such that*

$$f \circ g \sim id_Y$$

$$g \circ f \sim id_X$$

Definition 2.12. *If X is homotopy equivalent to a point, then we say X is contractible.*

Definition 2.13. *Suppose $A \subset X$.*

- *$r : X \to A$ is a retraction, or A is a retract of X, if r is continuous and $r(x) = x$ for $x \in A$.*
- *A is a deformation retraction of X if there exists a homotopy $F_t : X \to A$ relative to A where $f_0 = id_X$ and $f_1 : X \to A$ a retraction.*

Example 2.14. *S^{n-1} is a deformation retraction of $D^n \setminus \{0\}$ by taking the deformation:*

$$F_t : D^n \setminus \{O\} \to D^n \setminus \{O\}$$
$$x \to (1-t)x + t\frac{x}{||x||}$$

Remark 2.15. *The function $f : X \to Y$ is the mapping cylinder $X \times I \perp\!\!\!\perp Y/ \bowtie$*

Proposition 2.16. *$A \subset X$ is a deformation retract of X implies A and X are homotopy equivalent.*

Proof. Since $r = \phi_1 : X \to A$ and $i : A \to X$, then we see that $r \circ i \sim id_A$ and $i \circ r \sim id_X$. ∎

Example 2.17. *$id_X : X \to X$ is homotopic to some $r : X \to A$ relative to A, meaning there exists $F : X \times I \to X$ such that if we let $\phi_t(x) = F(x,t)$, then*

$$\phi_0(x) = x \ \forall x \in X$$

$$\phi_1(x) \in A \ \forall x \in X$$

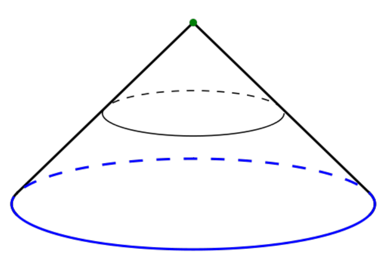
$$\phi_t(z) = z \ \forall z \in A$$

This is known as an elementary contraction.

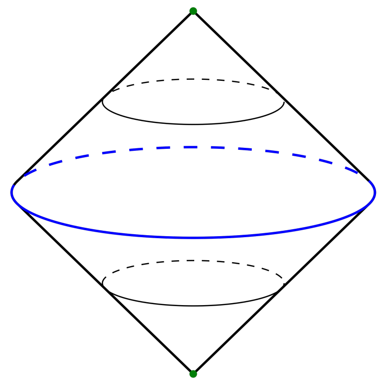
Theorem 2.18 (Hatcher). *If f is a homotopy equivalence between X and Y, then both $X = X \times O$ and Y are deformation retractions of $X \times I \perp\!\!\!\perp Y/ \bowtie$.*

2.2 New Operations from Homotopy

Definition 2.19. *The cone operation from $X \to C(X) = X \times I / X \times \{1\}$*



Definition 2.20. *The* <u>*suspension operation*</u> $S(X) = C(X)/X \times \{O\}$



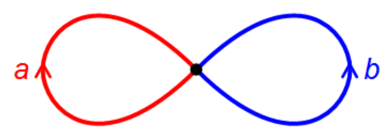
Remark 2.21. *Any function* $f : X \to Y$ *extends naturally to* $Cf : CX \to CY$ *and* $Sf : SX \to SY$.

Definition 2.22. *The* <u>*wedge operation*</u> $X \vee Y$ *at the points* $x_0 \in X, y_0 \in Y$ *is defined as*

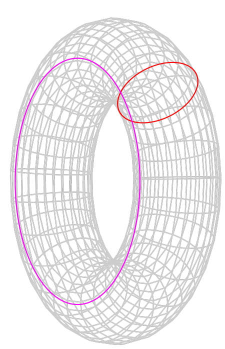
$$X \vee Y = X \amalg Y / \bowtie$$

where $\bowtie$ *is defined by* $x_0 \bowtie y_0$, *and all other points are self-identified.*

Example 2.23. *We can consider the wedge of two circles* $S^1 \vee S^1$



Example 2.24. *We already discussed the torus* $T = S^1 \times S^1$. *If we delete a point* x_0 *from* T, *then* $T \setminus \{x_0\}$ *deformation retracts to a copy of* $S^1 \vee S^1$. *We can see this by choosing some basic circles*



and label them $a, b : I \to T$ *such that they start and stop at the same point*

$$a(0) = a(1) = b(0) = b(1)$$

Then because $T = I^2 / \bowtie$ *where* $x \bowtie y \iff x - y \in \mathbb{Z}^2$. *Then for all* $x \in I^2 \setminus \{x_0\}$, *then there exists* $\lambda_x \geq 1$ *such that*

$$\underbrace{x_0 + \lambda_x(x - x_0)}_{y_x} \in \partial I$$

Then we see that

$$\partial I^2 / \bowtie = a \wedge b$$

2.3 Path Homotopy

Suppose $\gamma, \delta : I \to X$ are paths such that $\gamma(1) = \delta(0)$.

Definition 2.25. *We can define the path product*

$$\gamma \cdot \delta : I \to X$$
$$s \to \begin{cases} \gamma(2s) & s \leq \frac{1}{2} \\ \delta(2s-1) & s \geq \frac{1}{2} \end{cases}$$

Definition 2.26. *Further, we can define the path inverse*

$$\overline{\gamma} = \gamma^{-1}(s) := \gamma(1-s)$$

Definition 2.27. *The constant path on a point $z \in X$ is:*

$$1_z : I \to X$$
$$s \to z$$

Definition 2.28. *Two curves $\gamma, \delta : I \to X$ are fixed point homotopic provided $\gamma \sim \delta$ and*

$$\gamma(0) = \delta(0)$$
$$\gamma(1) = \delta(1)$$

Lemma 2.29. *Consider the curves $\gamma, \delta, \eta : I \to X$. Then*

1. *Suppose $\gamma(0) = y$ and $\gamma(1) = z$. Then*
$$1_y \cdot \gamma \sim \gamma \sim \gamma \cdot 1_z$$
relative to $\{0, 1\}$

2. *$\gamma \cdot \overline{\gamma} \sim 1_y$ and $\overline{\gamma} \cdot \gamma \sim 1_z$ relative to $\{0, 1\}$.*

3. *Suppose a surjective $\Theta : I \to I$ such that $\Theta(0) = 0, \Theta(1) = 1$. Then $\gamma \circ \Theta$ is a reparametrization and $\gamma \circ \Theta \sim \gamma$ relative to $\{0, 1\}$.*

4. *$\gamma(1) = \delta(0), \delta(1) = \eta(0)$, then*
$$(\gamma \cdot \delta) \cdot \eta \sim \gamma \cdot (\delta \cdot \eta)$$
relative to $\{0, 1\}$.

Proof. 1. We need to define ϕ_t such that $\gamma_0 = 1_y \cdot \gamma$ and $\phi_1 = \gamma$. Then

$$\phi_t(s) = y \;\; s \leq \frac{1}{2}(1-t)$$

While implicitly

$$\gamma(s) = \phi_t((1-s)\frac{1}{2}(1-t) + s)$$

2. Define a new curve

$$\sigma^t : I \to X$$
$$s \to \phi(ts)$$

for some $t \in [0, 1]$. Then we can let

$$\phi_t = \sigma_t \cdot \overline{\sigma^t}$$

with $\phi_0 = 1_y$ and $\gamma_1 = \gamma \cdot \overline{\gamma}$.

3. Let $\gamma_0 = \gamma$ and $\gamma_1 = \Theta \cdot \gamma$ with
$$\phi_t(s) = \gamma((1-t)s + t\Theta(s))$$

4. We can identify a reparametrization $\Theta : I \to I$ such that
$$(\gamma \cdot \delta) \cdot \eta = [\gamma \cdot (\delta \cdot \eta)] \circ \Theta$$
But by 3, we know that implies
$$(\gamma \cdot \delta) \cdot \eta \sim \gamma \cdot (\delta \cdot \eta)$$

■

Definition 2.30. *The fundamental group $\pi(X, x_0)$:= the equivalence classes of paths $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = x_0$ under the equivalence relation $\gamma \sim \delta$ relative to $\{0,1\}$. Symbolically, we can write*

$$\pi_1(X, x_0) = \{[\gamma] : \gamma : [0,1] \to X, \gamma(0) = \gamma(1) = x_0, \gamma \bowtie \delta \iff \gamma \sim \delta \textit{ relative to } \{0,1\}\}$$

We can define the operation within the fundamental group over the equivalence classes to be induced from the path operations we defined earlier
$$[\gamma] \cdot [\delta] = [\gamma \cdot \delta]$$
$$[\gamma] \cdot [\overline{\gamma}] = [1_{x_0}]$$

Lemma 2.31. *Suppose X is path connected, and $x_0, y_0 \in X, \eta : [0,1] \to X, \eta(0) = x_0, \eta(1) = y$. Then the mapping*

$$\beta_\eta : \pi_1(X, x_0) \to \pi_1(X, y_0)$$
$$[\gamma] \to [\overline{\eta} \cdot \gamma \cdot \eta]$$

is an isomorphism.

Proof. An easy way to prove this is to find an inverse for β_η. Specifically, $\beta_{\overline{\eta}} = \beta_\eta^{-1}$, which is clearly a homomorphism with
$$\beta_{\overline{\eta}} \circ \beta_\eta[\gamma] = \beta_{\overline{\eta}}(\beta_\eta([\gamma])) = [\eta \cdot (\overline{\eta} \cdot \gamma \cdot \eta) \cdot \overline{\eta}] = [1_{y_0} \cdot \gamma \cdot 1_{x_0}] = [\gamma] \implies \beta_{\overline{\eta}} = id_{\pi_1(X,x_0)}$$
The similar calculation occurs for $\beta_\eta \circ \beta_{\overline{\eta}} = id_{\pi_1(X,y_0)}$. Therefore, β_η is an isomorphism. ■

Remark 2.32. $\pi_1(X, x_0) \cong \pi_1(X, y_0)$ *with the isometry dependent on the path η.*

Theorem 2.33. *If X is path connected, then*

$$\pi(X) = \textit{fundamental group} \cong \pi_1(X, x_0)\ \forall x_0 \in X$$

Remark 2.34. *For some spaces this group is interesting. But for many this fundamental group is trivial.*

Definition 2.35. *X is simply connected if X is path connected and $\pi_1(X) = \{1\}$.*

Example 2.36. *$X = \{z\}$, the space of a single point, then $\pi_1(X) = \{1\}$.*

Example 2.37. *$X \subset \mathbb{R}^n$ a convex set, then for all $x, y \in X$, and for all $t \in [0,1]$, then*

$$(1-t)x + ty \in X$$

So $\pi_1(X) = \{1\}$

Example 2.38. *Since $int(D^n), D^n$, and $\mathbb{R}^n$ are convex, then by the last example we know*

$$\pi_1(int(D^n)) = \pi_1(D^n) = \pi_1(\mathbb{R}^n)$$

Example 2.39. *$S^n \setminus \{x_0\}$ for some $x_0 \in S^n$, then $S^n \cong \mathbb{R}^n$ by stereographic projection, so*

$$\pi_1(S_n \setminus \{x_0\}) = \{1\}$$

Another way of looking at this, we can define a homeomorphism from D^n to $S^n \setminus \{x_0\}$ by

$$f(ru) = (u \sin \pi r, \cos \pi r) \subset \mathbb{R}^n$$

which will serve as a homeomorphism.

Proposition 2.40. *If X is path connected with $x_0 \in X$, and let $A_\alpha, \alpha \in \mathcal{F}$, be an open cover of X such that*

$$X \subset \bigcup_{\alpha \in \mathcal{F}} A_\alpha$$

And let $x_0 \in A_\alpha$ for all α. Lastly, assume for all $\alpha, \beta, A_\alpha \cap A_\beta$ is path-connected. Then for any path $\gamma : [0,1] \to X, \gamma(0) = \gamma(1) = x_0$, then

$$\gamma \sim \delta_1, \ldots, d_r \text{ relative to } \{0,1\}$$

where for all $\delta_i(0) = \delta_i(1) = x_0$ and for all $\delta_i, \exists \alpha_i$ such that $\delta_i \subset A_{\alpha_i}$

Proof. Observe, $\gamma^{-1}(A_\alpha)$ with $\alpha \in \mathcal{F}$ that forms an open covering of $[0,1]$. So there exists a large enough r such that for all $1 \leq i \leq r$, there exists α_i such that

$$\left[\frac{i-1}{r}, \frac{i}{r}\right] \subset \gamma^{-1}(A_{\alpha_i})$$

$$\implies \gamma\left(\left[\frac{i-1}{r}, \frac{i}{r}\right]\right) \subset A_{\alpha_i}$$

we can then restrict our path to:

$$\gamma_i := \gamma|_{[\frac{i-1}{r}, \frac{i}{r}]}$$

Now, for any $2 \leq i \leq r-1$,, there exists a mutually identifiable path $\eta_i : [0,1] \to A_{i-1} \cap A_i$ such that $\eta_{i-1} = x_0, \eta_{i-1}(1) = \gamma(\frac{i-1}{r})$. Therefore, we can define

$$\delta_i = \eta_{i-1}\gamma\eta_i$$

Continuing this process of mutual identification across these intersections, we assume

$$\eta_0 = \eta_r = 1_{x_0}$$

∎

Example 2.41. $\pi_1(S^n) = \{1\}$ *if $n \geq 2$.*

Proof. Take any two points $p, q \in S^n$ such that $p \neq q$. Now define

$$A_1 = S^n \setminus \{p\}$$

$$A_2 = S^n \setminus \{q\}$$

And let $x_0 \in S^n \setminus \{p, q\}$. Then we see that

$$X \subset A_1 \cup A_2$$

so we have an open cover and for x within their intersection, allowing us to apply our previous proposition. Since A_1 and A_2 are simply connected, then any $\gamma \sim 1_{x_0}$. ∎

Definition 2.42. *Given two path-connected topological spaces X, Y, then for any continuous mapping $f : X \to Y$, with $x_0 \in X, f(x_0) = y_0 \in Y$, the* <u>*push-forward*</u> *of f is defined to be:*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\gamma] \to [f \circ \gamma]$$

for any $\gamma : I \to X, \gamma(0) = \gamma(1) = x_0$.

Remark 2.43. *The push-forward is well-defined since for any $\gamma \sim \delta$ relative to $\{0,1\}$ implies $f \circ \gamma \sim f \circ \delta$ relative to $\{0,1\}$.*

Definition 2.44. *A push-forward is a homomorphism provided for any $\gamma, \delta : I \to X, \gamma(0) = \gamma(1) = \delta(0) = \delta(1)$ provided*

$$f_*([\gamma] \cdot [\delta]) = f_*([\gamma]) \cdot f_*([\delta])$$

Definition 2.45. *The push-forward of a composition of the mappings*

$$f : (X, x_0) \to (Y, y_0)$$

$$g : (Y, y_0) \to (Z, z_0)$$

is given by

$$(g \circ f)_* = g_* \circ f_*$$

Proposition 2.46. *Given $A \subset X$ a retraction of X, with $r : X \to A$, then for any base point $x_0 \in A$,*

1. *$r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.*
2. *If r is a deformation retraction, then r_* is an isomorphism.*

Proof. 1. Let $i : A \to X$, with $i(x) = x$. Then $r \circ i = id_A$, and

$$r_* \circ i_* = (id_A)_* = id_{\pi_1(A, x_0)}$$

If $[\gamma] \in \pi_1(A, x_0)$, then

$$[\gamma] = r_*(i_*([\gamma]))$$

2. Let $\phi_t : X \to X$ serve as the homotopy on X with $\phi_0 = id_X, \phi_1 = r$ and $\phi_t(x) = x$ for all $x \in A$. We need to show that r_* is injective. Observe, for any $[\gamma] \in \pi_1(X, x_0)$ with

$$r_*([\gamma]) = 1 \in \pi_1(A, x_0) \implies r \circ \gamma \sim 1_{x_0} \text{ in } A \text{ relative to } \{0,1\}$$

We also know that $\gamma \sim r \circ \gamma$ relative to $\{0,1\}$ via $\phi_t \circ \gamma$. By the transitive relation of homotopy, we know $\gamma \sim 1_{x_0}$ in X.

■

Theorem 2.47. *If X, Y are path-connected topological spaces and $f : X \to Y$ is a homotopic equivalence, then for any base point $x_0 \in X$, $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Remark 2.48. *There exists $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$, possibly with $z_0 = g(y_0) \neq x_0$. (That is, they don't need to be inverses)*

Theorem 2.49. *Consider $h : Z \to Z$, with Z a path-connected topological space and $w_0 \in Z$ some base point. Then if $h \sim id_Z$, then*

$$h_* : \pi_1(Z, w_0) \to \pi_1(X, h(w_0))$$

is an isomorphism.

Proof. Let $\eta(t) = \phi_t(w_0)$ where $h(w_0) = v_0$, $\phi_0 = id_Z$ and $\phi_1 = h$. Recall, $\beta_\eta : \pi_1(Z, w_0) \to \pi_1(Z, v_0)$ is an isomorphism with

- $[\gamma] \in \pi_1(Z, w_0) \implies \beta_\eta([\gamma]) = [\overline{\eta} \cdot \gamma \cdot \eta]$
- By homomorphism,

$$\beta_\eta[\gamma \cdot \delta] = [\overline{\eta} \cdot \gamma \cdot \delta \cdot \eta] = [\overline{\eta} \cdot \gamma \cdot \eta \cdot \overline{\eta} \cdot \delta \cdot \eta] = \beta_h([\gamma]) \cdot \beta_h([\delta])$$

So we prove $h \circ \gamma \sim \overline{\eta}\gamma\eta$ relative to $\{0, 1\}$. So

$$\eta(t) = \phi_t(w_0) \quad \eta^t(s) = \eta(ts)$$

Therefore,

$$\Theta_t := \eta^t \cdot \phi_t \circ \gamma \cdot \overline{\eta^t}$$

is a homotopy which shows

$$\gamma \sim \eta \cdot \phi_1 \circ \gamma \cdot \overline{\eta} = \eta \cdot h \circ \overline{\eta}$$

So we have proved that $\gamma \sim \eta \cdot h \circ \overline{\eta}$ relative to $\{0, 1\}$. Now

$$\overline{\eta} \cdot \gamma\eta \sim \underbrace{\overline{\eta} \cdot \eta}_{id_{v_0}} h \circ \gamma \cdot \underbrace{\overline{\eta} \cdot \eta}_{id_{v_0}}$$

So we get

$$\beta_\eta([\gamma]) = h_*([\gamma])$$
$$\implies h_* = \beta_\eta$$

which is an isomorphism! ■

Corollary 2.49.1. *Given that there exists $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$, possibly with $z_0 = g(y_0) \neq x_0$. Then*

$$f_* \circ g_* = id_{\pi_1(Y)}$$
$$g_* \circ f_* = id_{\pi_1(X)}$$

and moreover,

$$\pi_1(X) \approx \pi_1(Y)$$

Corollary 2.49.2. *If X is homotopy equivalent to a point, then $\pi_1(X) = \{1\}$.*

Proof. **Left as an Exercise** ■

Remark 2.50. *Any closed path starting and ending at a point x_0 can be thought of as maps $\gamma : (S^1, 1) \to (X, x_0)$ since for any $\gamma : [0, 1] \to X$, we can define*

$$\begin{aligned} \gamma_0' : S^1 &\to X \\ e^{2\pi i s} &\to \gamma(s) \end{aligned}$$

3 Covering Spaces

Proposition 3.1. *Suppose X is a path connected topological space. Then $\pi_1(X) = \{1\}$ if and only if for all $x_0, y_0 \in X, \gamma, \delta : [0,1] \to X$ with $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) = \delta(1) = y_0$ then $\gamma \sim \delta$ relative to $\{0,1\}$.*

Proof.

- ($\Leftarrow$) Since $\gamma(0) = \delta(0) = x_0$ and $\gamma(1) = \delta(1) = y_0$, then $\gamma \sim \delta = 1_{x_0} \implies \pi_1(X, x_0) = \{1\}$
- ($\Rightarrow$) Now assume $\pi_1(X) = \{1\}$. Then all homotopies are fixed endpoint. That is,

$$\gamma \sim \gamma \cdot 1_{y_0} \sim \gamma \cdot \overline{\delta} \cdot \delta \sim 1_{x_0} \cdot \delta \sim \delta$$

■

Definition 3.2. *A covering map $p : \tilde{X} \to X$ provided*

- *p is surjective.*
- *For all $x_0 \in X, \exists U \subset X$ open with $x_0 \in U$ such that $p^{-1}(U) = \bigcup_{\alpha \in \mathcal{F}} V_\alpha$ where*

$$V_\alpha \cap V_\beta = \emptyset \ \ \forall \alpha \neq \beta$$

$$p|_{V_\alpha} : V_\alpha \to U \text{ is a homeomorphism for all } V_\alpha$$

Example 3.3. *Define $p : \mathbb{R} \to S^1$ with $p(x) = e^{2\pi i x}$. Then for all $x_0 = e^{2\pi i s} \in S^1$, we let*

$$U = \left\{ e^{2\pi i r} : |r - s| < \frac{1}{2} \right\}$$

Then $p^{-1}(x_0) = \{s + k : k \in \mathbb{Z}\}$

Example 3.4. *Define $p : \mathbb{R}^2 \to S^1 \times S^1 =$ torus by*

$$p(x, z) = (e^{2\pi i x}, e^{2\pi i z})$$

Then for any $w_0 = (e^{2\pi i x_0}, e^{2\pi i y_0}) \in S^1 \times S^1$, then

$$p^{-1}(w_0) = \{(x_0 + k, y_0 + m) : (k, m) \in \mathbb{Z}^2\}$$

This should be familiar because

$$S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2 \text{ (as a group action)}$$

Definition 3.5. $\mathbb{R}P^n := \mathbb{R}^{n+1} \setminus \{0\}/ \bowtie$ *where we define the equivalence relation $x \bowtie y$ if and only if $\exists \lambda \in R \setminus 0$ such that*

$$y = \lambda x$$

Remark 3.6. *Observe, $S^n \subset \mathbb{R}^{n+1} \setminus \{0\}$ and*

$$\mathbb{R}P^n = S^n / \bowtie$$

by the same equivalence relation, which only acts on the antipodal points

$$x = \pm y$$

Example 3.7. *Define $p : S^n \to \mathbb{R}P^n$ by $p(x) = [x] \in \mathbb{R}P^n$, then*

$$p^{-1}(x) = \{x, -x\}$$

3.1 Path Lifting

Lemma 3.8. *Let $p : Y \to X$ a covering map and a base point $x_0 \in X, x_0 \in Y$ an covered open set. Let Z be path connected with a function $f : Z \to X$ that satisfies $f(x_0) = x_0, f(Z) \subset U$. Let $p(y_0 = x_0$ for some $y_0 \in V_0, p|_{V_0} : V_0 \to U_0$ a homeomorphism. Then there exists a unique continuous $\tilde{f} : Z \to Y$ such that*

$$p \circ \tilde{f} = f, \tilde{f}(z_0) = y_0$$

That is, the diagram

$$\begin{tikzcd} Z \arrow[r, "\exists! \tilde{f}"] \arrow[dr, "f"'] & Y \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

Proof.

- Existence: Let $\tilde{f} := p|_{V_0} \cdot f$
- Uniqueness: Now assume that $\tilde{g} : Z \to Y$ such that

$$\tilde{g}(z_0) = y_0, p \circ \tilde{g} = f$$

Then

$$Z = \tilde{g}^{-1}(V_0) \perp\!\!\!\perp \tilde{g}^{-1}(Y \setminus V_0)$$

We claim that $\tilde{g}^{-1}(Y \setminus V_0)$ is open since if $z \in Z, \tilde{g}(z) = y \in V_0$ if and only if $z \in \tilde{g}^{-1}(Y \setminus Y_0)$. So

$$p(y) = p \cdot \tilde{g}(z) = f(z) \in U$$

So there exists V_α such that $p|V_\alpha : V_\alpha \to U$ is a homeomorphism with $V_\alpha \cap V_0 = \emptyset$. So $\tilde{g}(V_\alpha)$ is open with

$$\tilde{g}^{-1}(V_\alpha) \subset \tilde{g}^{-1}(Y \setminus V_0)$$

Therefore, $g^{-1}(V_\alpha)$ is open. Further $\tilde{g}^{-1}(Y \setminus V_0) = \emptyset$ since Z is path connected. So

$$p \circ \tilde{g} = p \circ \tilde{f} = f$$

But since $p|_{V_0}$ is a homeomorphism, then

$$\tilde{g} = \tilde{f}$$

■

Theorem 3.9 (Monodromy, Existence of Path Lifting). *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. If $\gamma : I \to X$ is a path with $\gamma(0) = x_0$, then there exists a unique $\tilde{\gamma} : I \to \tilde{X}$ such that*

1. $\tilde{\gamma}(0) = x_0$
2. $p \circ \tilde{\gamma} = \gamma$

Further, the diagram

$$\begin{tikzcd} I \arrow[r, "\exists! \tilde{\gamma}"] \arrow[dr, "\gamma"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

Proof. There exists $\{U_j\}_{j\in\mathcal{F}}$ a family of evenly covered open $U_j \subset X$ such that $\gamma \subset \bigcup_{j\in\mathcal{F}} U_j$. Lebesgue's lemma tells us that $\{\gamma^{-1}(U_j)\}_{j\in\mathcal{F}}$ serves as a covering of I, specifically,

$$I \subset \bigcup_{j\in\mathcal{F}} \gamma^{-1}(U_j)$$

Further, there exists i, k such that

$$\left[\frac{i-1}{k}, \frac{i}{k}\right] \subset \gamma^{-1}(U_i)$$

Therefore,

$$\gamma\left(\left[\frac{i-1}{k}, \frac{i}{k}\right]\right) \subset U_i$$

By a nod to induction on i, the we see $V_i \subset \tilde{X}$ an open subset, then $p|_{V_i} : V_i \to U_i$ is a homomorphism and $\tilde{\gamma}(\frac{i-1}{k}) \in V_i, \tilde{\gamma}(\frac{0}{k}) = \tilde{x}_0$. ■

Theorem 3.10 (Homotopy Lifting). *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Suppose we have two homotopic paths*

$$\gamma, \delta : I \to X$$

$$\gamma(0) = \delta(0) = x_0$$

$$\gamma(1) = \delta(1) = x_1$$

such that $\gamma \sim \delta$ relative to $\{0, 1\}$. Then there exists two paths

$$\tilde{\gamma}, \tilde{\delta} : I \to \tilde{X}$$

such that

1. $\tilde{\gamma}(0) = \tilde{\delta}(0) = \tilde{x}_0$
2. $\tilde{\gamma}(1) = \tilde{\delta}(1) = \tilde{x}_1$
3. *The following relations are satisfied:*

$$p \circ \tilde{\gamma} = \gamma$$

$$p \circ \tilde{\delta} = \delta$$

$$p(\tilde{x}_0) = x_0$$

$$p(\tilde{x}_1) = x_1$$

Further, we a well defined deformation $F : [0, 1]^2 \to X$, we can identify a unique lifted deformation such that the diagram

$$\begin{tikzcd} I^2 \arrow[r, "\exists!\tilde{\gamma}"] \arrow[dr, "F"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

Proof. Consider the continuous deformation $F : [0, 1]^2 \to X$ with

$$F(s, t) = \phi_t(s)$$

$$F(s, 0) = \gamma(s)$$

$$F(0, t) = x_0$$

$$F(s, 1) = \delta(s)$$

$$F(1, t) = x_1$$

Observe, we can consider $\{U_j\}_{j\in\mathcal{F}}$ an even open covering of X with

$$F([0,1]^2) \subset \bigcap_{j\in\mathcal{J}} U_j$$

Then by Lebesgue's lemma, we can identify a $k \geq 2$ such that for all $\alpha, \beta = 1, \ldots, k$,

$$W_{\alpha,\beta} := [\frac{\alpha-1}{k}, \frac{\alpha}{k}] \times [\frac{\beta-1}{k}, \frac{\beta}{k}]$$

satisfies

$$F(W_{\alpha,\beta}) \subset U_{\alpha,\beta}$$

Then if we define the lexicographic ordering

$$W_{i,j} < W_{\alpha,\beta} \iff i < \alpha \text{ or } i = \alpha \text{ and } j < \beta$$

The for all $\alpha, \beta = 1, \ldots, k$ with $(\alpha, \beta) \neq (1,1)$, then

$$\left(\bigcup_{W_{i,j}<W_{\alpha,\beta}} W_{i,j}\right) \bigcap W_{\alpha,\beta} \neq \emptyset$$

Then we can construct $\tilde{F} : [0,1]^2 \to \tilde{X}$ such that

$$\tilde{F}(0,0) = \tilde{x}_0$$

and $p \circ \tilde{F} = F$ by induction on ¡.

- So first constuct $\tilde{F}_{W_{1,1}}$ and define recursively until defining over all $W_{i,j}$.
- Upon reaching step (α, β), then there exists

  $$z_{\alpha,\beta} \in W_{\alpha,\beta} \cap W_{i,j}$$

  where $W_{i,j} < W_{\alpha,\beta}$. By induction, we know $\tilde{F}(z_{\alpha,\beta}) \in \tilde{X}$. The previous lemma then gives us $\tilde{F}|_{W_{\alpha,\beta}}$

■

Definition 3.11. *A space Y is locally path connected provided there exists a base of path connected open sets.*

Theorem 3.12. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. Suppose Y is a simply connected and locally path connected space. Then for all $f : (Y, y_0) \to (X, x_0)$, then there exists a unique $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that*

$$p \circ \tilde{f} = f$$

In other words, the diagram

$$\begin{tikzcd} Y \arrow[r, "\exists! \tilde{f}"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

Theorem 3.13 (Monodromy, Existence of Path Lifting). *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map on a path connected space X. Then $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality for all $x_0, x_1 \in X$.*

Proof. For all $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x_0$. Let $\tilde{\gamma}_{\tilde{x}} : I \to \tilde{X}$ such that

$$p \circ \tilde{\gamma}_{\tilde{x}} = \gamma$$

and $\tilde{\gamma}_{\tilde{x}}(0) = \tilde{x}$. Where $\gamma : I \to X, \gamma(0) = x_0, \gamma(1) = x_1$. Then

$$\phi : p^{-1}(x_0) \to p^{-1}(x_1) \quad \phi(\tilde{x}) = \tilde{\gamma}_{\tilde{x}}(1)$$
$$\psi : p^{-1}(x_1) \to p^{-1}(x_0) \quad \psi(\tilde{g}) = \tilde{\bar{\gamma}}_{\tilde{y}}(1)$$

Then

$$\psi \circ \phi = id_{p^{-1}(x_0)}$$
$$\phi \circ \psi = id_{p^{-1}(x_1)}$$

And therefore we ϕ is a bijection. ∎

Remark 3.14. *$If \# p^{-1}(x_0) = k \in \mathbb{N}$, then $p^{-1}(x_0)$ is known as k-sheeted covering.*

Definition 3.15. *X is locally path connected if for all $x \in X$ and $S \subset X$ open with $x \in S$. Then there exists $x \in W \subset S$, W open and path connected.*

Exercise 3.16. *Suppose we have a covering map $p : \tilde{X} \to X$ and X is locally path connected. Then $\tilde{X}$ is also locally path connected.*

Theorem 3.17. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map. Let (X, x_0) be path connected, Ysimplyconnected, and let*

$$f : (Z, z_0) \to (X, x_0)$$

continuous function with Z locally path connected. Then there exists a unique continuous $\tilde{f} : (Z, z_0) \to (Y, y_0)$ such that

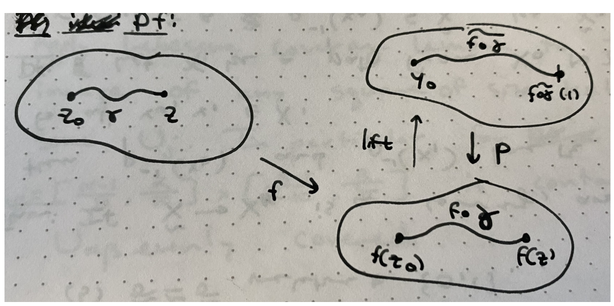
$$p \circ \tilde{f} = f$$

In other words, the diagram

$$\begin{tikzcd} Z \arrow[r, "\exists! \tilde{f}"] \arrow[dr, "f"'] & \tilde{Y} \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

Proof.



For all $q \in Z$, we choose $\tilde{f}(z) = \widetilde{f \circ \gamma}$. Why do we need z simply connected? If not, $\tilde{f}$ is not well-defined, see later example. But if z is simply connected, then any other path δ from z_0 to z is fixed endpoint homotopic to γ, so $f \circ \gamma$ and $f \circ \delta$ are fixed endpoint homotopic, and $\widetilde{f \circ \gamma}$ and $\widetilde{f \circ \delta}$ are fixed endpoint homotopic by homotopy lifting. Then

$$\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \delta}(1)$$

By construction:

- $p \circ \tilde{f} = f$
- $\tilde{f}$ is unique (can check that any $\tilde{f}$ must have the same property)

Since Z is locally path connected, we can show that $\tilde{f}$ is continuous. Specifically, let $S \subset Y$ an open set, and $\tilde{f}(z) - y \in S$. Let $x = p(y) = f(z)$. Then there exists some evenly covered open $U \subset X$ with $p^{-1}(V) \subset S, y \in p^{-1}(V)$ and $p : V \to U$ a homeomorphism. Since Z is path connected, there exists an open neighborhood W of z which is path connected such that $f(W) \subset U$. So for any $q \in W$, we can connect z to q in W by a path τ. So that $f \circ \tau$ gives a path from $f(z)$ to $f(q)$ in U, which is lifted to a path $\widetilde{f \circ \tau}$ from y to $\widetilde{f \circ \tau}(1)$ in S. But that as $\gamma \cdot \tau$ is a path from z_0 to z to q, then $\tilde{f}(q) = \widetilde{f \circ \tau}(1) \in S$. Thus, W is an open neighborhood about z contained within $f^{-1}(S)$. ∎

Example 3.18. *$p : \mathbb{R} \to S^1$ via $p(t) = e^{2\pi i t}$. Then $p(0) = 1 = e^{2\pi i 0}$. Then $\gamma_1(s) = e^{2\pi i s}$ and $\gamma_z(s) = e^{4\pi i s}$ are both loops at $p(0) = 1$, but they get lifted to paths with different endpoints in $\mathbb{R}$.*

Proposition 3.19. *If f is a covering and Y is path connected, then so is $\tilde{f}$.*

Proof. If f is a covering, choose some neighborhood U of x which is evenly covered by both f and p. Then $V_0 = p^{-1}(f(W))$ is homeomorphic to $f(W)$. So $f^{-1}(W_0) = \bigcup_\alpha S_\alpha$ with $f|_{S_\alpha} : S_\alpha \xrightarrow{\sim} f(W)$. This will be the covering! ∎

Remark 3.20. *The notation $\gamma : x \rightsquigarrow y$ means that γ is a path from x to y.*

Theorem 3.21. *Let X be path connected as well as locally path connected. Let*

$$p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$$
$$p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$$

be covering maps with $\tilde{X}_1, \tilde{X}_2$ simply connected. Then there exists a unique homeomorphism $\Phi : \tilde{X}_1 \to \tilde{X}_2$ such that

$$p_2 \circ \Phi = p_1 \quad \Phi(\tilde{X}_1) = \tilde{X}_2$$

That is, the diagram

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, "\exists!\Phi"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

commutes.

Proof. Note that X is locally path connected implies that $\tilde{X}_1, \tilde{X}_2$ are both locally path connected since for sufficiently small neighborhoods in $\tilde{X}_1, \tilde{X}_2$ must be homeomorphic to neighborhoods in X. Then we recall by the previous theorem, we see that

$$\tilde{X}_1 = Z, \tilde{X}_2 = Y$$

gives rise to a continuous $\Phi : \tilde{X}_1 \to \tilde{X}_2$ with $\phi(\tilde{X}_1) = \tilde{X}_2$. Similarly, if we let $\tilde{X}_1 = Y, \tilde{X}_2 = Z$, then by the previous theorem there exists $\Psi : \tilde{X}_2 \to \tilde{X}_1$ with $\Psi(\tilde{x}_2) = \tilde{x}_1$. Since $id_{\tilde{X}_1}$ and $\Psi \circ \Phi$ both satisfy the previous theorem with $Y = Z = X_1$, then

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, bend left, "\Psi\circ\Phi"] \arrow[dr, "p_1"'] & & (\tilde{X}_1, \tilde{x}_1) \arrow[ll, "id_{\tilde{X}_1}"] \arrow[dl, "p_1"] \\
& (X, x_0) &
\end{tikzcd}$$

So the maps are equal. An identical argument establishes that $\Phi \circ \Psi - id_{\tilde{X}_2}$, so that Φ is a homeomorphism. So we can apply the previous theorem to

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, bend left, "\Phi"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[ll, "\Phi'"] \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

which shows that Φ is unique! ■

Remark 3.22. *Two coverings of the same space implies the coverings are equivalent.*

3.2 Universal Covers

Definition 3.23. *If $p : \tilde{X} \to X$ is a covering map such that X is locally path connected and $\tilde{X}$ is simply connected, then $\tilde{X}$ is called a universal cover.*

Proposition 3.24 (Properties of Universal Covers). *1. Universal covers are unique up to unique (pointed) isomorphism. That is, if $\tilde{X}'$ is another universal cover, then there exists a unique pointed isomorphism $\phi : \tilde{X} \xrightarrow{\sim} \tilde{X}'$.*

2. *If $\tilde{X}_1, \tilde{X}_2 \in p^{-1}(x_0)$, then there is a unique homeomorphism $\phi : \tilde{X} \to \tilde{X}$ such that $\phi(\tilde{x}_1) = \tilde{x}_2$ and $p \circ \phi = p$. ϕ here is called a deck transformation.*

3. *If $q : (Y, y_0) \to (X, x_0)$ is any covering of X, then there exists a unique covering $s : (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ such that $q \circ s = p$. That is, the diagram*

$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[rr, "\exists! s"] \arrow[dr, "p"'] & & (Y, y_0) \arrow[dl, "q"] \\ & (X, x_0) & \end{tikzcd}$$

commutes.

Theorem 3.25. *If X is path connected and locally simply connected, then there exists a universal cover!*

Definition 3.26. *Let G be a group acting on Y. Recall that this means that for each $g \in G$, there is a homeomorphism ϕ_g such that $\phi_{gh} = \phi_g \circ \phi_h$ and $\phi_1 = id_Y$. We say that G acts freely if for all $y \in Y$, $g \neq h \in G$, $gy \neq hy$. G acts freely and properly if for all $y \in Y$, there exists an open neighborhood V containing y such that $g \neq h \implies gV \cap hV \neq \emptyset$. This is sometimes called a covering transformation or a properly discontinuous action.*

Remark 3.27. *In the case of Y, $q : Y \to Y/G$ will be a covering.*

Theorem 3.28. *If $p : \tilde{X} \to X$ is a covering map, $\tilde{X}$ simply connected, and X is locally path connected. Then*

1. *$\pi_1(X)$ acts freely and properly on $\tilde{X}$.*

2. *Fixing a basepoint $\tilde{x}_0$, with $p(\tilde{x}_0) = x_0$, then for $[\gamma] \in \pi_1(X, x)$, we can lift the loop γ to the path $\tilde{\gamma}$ in $\tilde{X}$. Then*

$$\phi_{[\gamma]} : \tilde{X} \to \tilde{X}$$

i.e. the action of $[\gamma]$ on $\tilde{X}$, is the unique homeomorphism such that $\phi_{[\gamma]}(\tilde{x}_0) = \tilde{\gamma}(1) \in p^{-1}(x_0)$. Moreover, there is the correspondence

$$\pi_1(X, x_0) \leftrightarrow \text{group } \{\phi_{[\gamma]}\} \leftrightarrow p^{-1}(x_0)$$

Proof. Use the previous two theorems. ■

3.3 Group Actions and Deck Transformations

Theorem 3.29 (Deck Transformation Theorem). *Let G act freely and properly on a simply connected and locally path connected space Y. Then*

1. *For $X = Y/G$, we have $\pi_1(X) \cong G$. That is, G acts on Y as in the previous theorem.*

2. *If $p : Y \to Y/G = X$ is a covering map, with $p(y_0) = x_0$, then*

$$\pi_1(X, x_0) \leftrightarrow p^{-1}(x_0) \sim \{\phi_{[\gamma]} : Y \to Y | [\gamma] \in \pi_1(X)\}$$

where $\phi_{[\gamma]} : Y \to Y$ is the unique homeomorphism such that

$$\phi_{[\gamma]}(y_0) = \tilde{\gamma}(1) \in p^{-1}(x_0)$$

and

$$p \circ \phi_{[\gamma]} = p$$

Remark 3.30. *G is referred to as a deck transform.*

Remark 3.31. *For any $[\gamma] = [\delta] \in \pi_1(X) \iff \tilde{\gamma}(1) = \tilde{\delta}(1)$.*

Theorem 3.32. 1. *$\pi_1(S^1) = \mathbb{Z}$*

2. *For all $n \in \mathbb{Z}$, let w_n be a path $w_n : I \to S^1$ such that*

$$w_n(s) = e^{2\pi i n s}$$

$[w_n] \in \pi_1(S_1, 1)$.

Proof. Let $p : \mathbb{R} \to S^1$ be the covering maps $p(s) = e^{2\pi i s}$. Then the lift

$$\begin{aligned} \tilde{w}_n : I &\to \mathbb{R} \\ s &\to ns \\ 1 &\to n \end{aligned}$$

So $\pi_1(S^1, 1) \cong \mathbb{Z} = p^{-1}(1)$ and we have

$$\begin{aligned} \phi : \mathbb{Z} &\to \pi_1(S^1, 1) \\ n &\to [w_n] \end{aligned}$$

∎

Remark 3.33. *$w_1 : I \to S^1$, $w_1(s) = e^{2\pi i s}$ gives $[w_1]$, a generator of $\pi_1(S^1, 1)$.*

Theorem 3.34. 1. *$\pi(S^1 \times S^1) = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$*

2.

$$\begin{aligned} p : (\mathbb{R}^2, (0,0)) &\to (S^1 \times S^1, (1,1)) \\ (s,t) &\to (e^{2\pi i s}, e^{2\pi i t}) \in S^1 \times S^1 \end{aligned}$$

$S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2 \implies \pi_1(S^1 \times S^1) = \mathbb{Z}^2$ and $[w_1 \times id_1], [id_1 \times w_1]$ generate this group.

Theorem 3.35. *$\mathbb{R}P^n = S^n/\bowtie$, where we define the equivalence relation $\bowtie$ as $x \bowtie \pm x$, for all $n \geq 2$. Since $\mathbb{Z}_2 = \{-1, 1\}$, then*

$$\mathbb{R}P^n = S^n/\mathbb{Z}_2$$

Since $\pi_1(S^n) = \{1\}$, we have $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ by the Deck Transformation Theorem.

Theorem 3.36. *There is no retraction $r : D^2 \to S^1 = \partial D^2$.*

Proof. If there exists such a retraction, then there is a surjection $r_* : \pi_1(D_2, 1) \to \pi_1(S^1, 1)$. But $\pi_1(D^2) = \{1\}, \pi_1(S^1) = \mathbb{Z}$. Contradiction! ∎

Theorem 3.37 (Brouwer Fixed Point)**.** *If $f : D^2 \to D^2$ continuous, then there exists $x \in D^2$ with $f(x) = x$.*

Proof. Suppose $f(x) \neq x$ for all $x \in D^2$. Then for all $x \in D^2$, there exists $\lambda_x > 0$ such that

$$f(x) + (x - f(x))\lambda_x \in \partial D^2$$

Let $r(x) = f(x) + (x - f(x))\lambda_x$ Then $r(x)$ is continuous and $r(x) = x$ for any $x \in S^1$. But $r(x)$ is then homotopic to the identify via straight-line homotopy of each x to $r(x)$. So r is a deformation retraction. Contradiction! ■

Theorem 3.38 (Fundamental Theorem of Algebra)**.** *Let $p : \mathbb{C} \to \mathbb{C}$ be $p(z) = \sum_{i=0}^{n} a_i z^i$ where $a_n \neq 0$ for some $n \geq 1$. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. There exists $R > 1$ such that $|a_n|R^n > \sum_{i=0}^{n-1} |a_j|R^j$. The for all $0 \leq r \leq R$ to consider the path

$$\phi_r : I \to (S^1, 1)$$
$$\frac{p(re^{2\pi is})}{|p(re^{2\pi is})|} \cdot \frac{|p(r)|}{p(r)}$$

Then $\gamma_0(s) = 1$ for all $s \in I$. But $\gamma_R \sim \gamma_0$ relative to $\{0, 1\}$ via $F(s, t) = \gamma_{Rt}(s)$. Now let

$$q_t(s) = a_n R^n e^{2\pi isn} + t \sum_{j=0}^{n-1} a_j e^{2\pi ijs} R^j = a_n z^n + t \sum_{j=0}^{n-1} a_j z^j$$

where $z = Re^{2\pi is}$. Now let

$$\delta_t(s) = \frac{q_t(s)}{|q_t(s)|} \cdot \frac{p(r)}{|p(r)|}$$

Then δ_t is a homotopy from $\delta_1 = \gamma_R$ to $\delta_0(s) = e^{2\pi ins} = w_n(s)$. But the

$$[w_n] = [\delta_0] = [\delta_1] = [\gamma_R] = [\gamma_0] = \text{ constant path } [w_0]$$

Contradiction, since $n \geq 1$! ■

Proposition 3.39. $\pi_1(S^1) = \mathbb{Z}$.

Proposition 3.40. $\pi_1(S^n) = \{1\}$ *for $n \geq 2$.*

Proposition 3.41. $\pi_1(X) = \{1\}$ *if X is contractible.*

Theorem 3.42. *If X_1, X_2 are topological spaces, then $\pi_1(X_1 \times X_2) = \pi_1(X_1) \times \pi_1(X_2)$.*

Proof. (Idea)
Let $X = X_1 \times X_2, x_1 \in X_1, x_2 \in X_2$. Let $(x_1, x_2) \in X$. If $\gamma : I \to X$, then $\gamma = \gamma_1 \times \gamma_2$, where γ_1, γ_2 are continuous. Conversely, if γ_1, γ_2 loops within X_1, X_2, then $\gamma_1 \times \gamma_2$ is a loop in X. ■

Example 3.43. $\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$.

4 Seifert-Van Kampan Theorem

Definition 4.1. *Let $\{G_\alpha\}_{\alpha\in I}$ be a family of groups. Then the free product of the family $\{G_\alpha\}$, denoted $*_{\alpha\in I}G_\alpha$, consists of finite sequences $g_1, g_2, \ldots, g_k$ such that $g_i \in$ some G_α, subject only to the relations with G_α. That is,*

- *If $g_i, g_{i+1} \in G_\beta$, then $g_i g_{i+1}$ can be replaced with $g_i \cdot g_{i+1}$. Or $g_i g_{i+1} \sim g_i \cdot g_{i+1}$.*
- *1_α can be omitted for any G_α, that is, for any β, and any $g \in G_\beta$. $1_\alpha g = g$ or $1_\alpha \sim \epsilon$.*

Theorem 4.2 (Seifert-Van Kampan). *Let X be path connected with $x_0 \in X$. Let $\{A_\alpha\}_{\alpha\in\mathcal{F}}$ be an open cover of X, with $x_0 \in A_\alpha$ for each α and $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected for any $\alpha, \beta, \gamma \in \mathcal{F}$ (not necessarily distinct). Then*

1. *There exists a surjective homomorphism*
$$\psi : *_{\alpha\in\mathcal{F}}\pi_1(A_\alpha, x_0) \to p_1(X, x_0)$$
2. *For all A_α, A_β, let*
$$i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$
$$i_{\beta\alpha} : A_\alpha \cap A_\beta \hookrightarrow A_\beta$$
Then ker(ϕ) is the normal subgroup generated by
$$((i_{\alpha\beta})_*[\gamma])((i_{\beta\alpha})_*[\gamma])$$
for all $\alpha, \beta \in \mathcal{F}, \gamma : I \to A_\alpha \cap A_\beta$. Equivalently, it is the subgroup of the free product with the relations $(i_{\alpha\beta})_*[\gamma] = (i_{\beta\alpha})_*[\gamma]$.

Remark 4.3. *We essentially want $(i_{\alpha\beta})_*[\gamma]$ is equal to $(i_{\beta\alpha})_*[\gamma]$ for any $[\gamma] \in \pi_1(A_\alpha \cap A_\beta, x_0)$*

Proof. Easily readable from Hatcher. ∎

Corollary 4.3.1. *For all $\alpha \neq \beta, A_\alpha \cap A_\beta$ simple connected implies $\pi_1(X, x_0) = *_{\alpha\in\mathcal{F}}\pi_1(A_\alpha, x_0)$*

Example 4.4. *Consider $S^1 \vee S^1$. We recall its definition is*

$$S^1 \vee S^1 = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = 1\}$$

Now, let

$$A_\alpha := \left\{(x, y) \in S^1 \vee S^1 : x < \frac{1}{2}\right\}$$
$$A_\beta := \left\{(x, y) \in S^1 \vee S^1 : x > \frac{-1}{2}\right\}$$

Then we see that $A_\alpha \cap A_\beta$ is contractible since it is homeomorphic to $\times$ and hence it is simple connected. Further, A_α and A_β can both be deformation retracted to S^1. Therefore, by the previous corollary, we see that

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$$

Remark 4.5. *In $*_{\alpha\in\mathcal{F}}\pi_1(A_\alpha, x_0)$, $i_{\alpha\beta *}[\gamma] = i_{\beta\alpha *}[\gamma]$*

Remark 4.6. *If for all $\alpha \neq \beta, \pi_1(A_\alpha \cap A_\beta, x_0) = \{1\}$, then $\pi_1(X, x_0) = *_{\alpha\in\mathcal{F}}\pi_1(A_\alpha, x_0)$*

Example 4.7. *Let $\vee_{\alpha\in\mathcal{F}}S^1_\alpha$ be defined such that for any $x_\alpha \in S^1_\alpha$,*

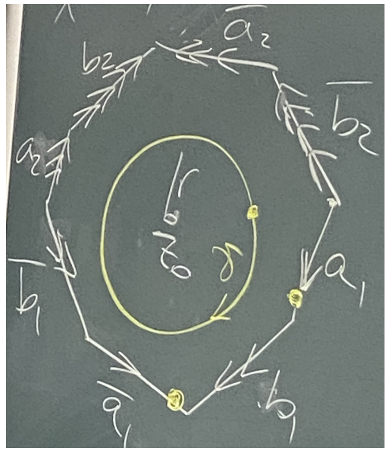
$$X = \amalg_\alpha S^1_\alpha / \bowtie$$

where $x_\alpha \bowtie x_\beta$ for all $\alpha, \beta \in \mathcal{F}$. Then

$$\pi_1(\vee_{\alpha\in\mathcal{F}}S^1_\alpha) = *_{\alpha\in\mathcal{F}}Z$$

with a free group generated by $[\gamma_\alpha]$ with $\gamma_\alpha : I \to S^1_\alpha$ defined by $\gamma_\alpha(x) = e^{2\pi i s}$

Example 4.8. *We can now discuss the oriented surfaces of genus $g \geq 1$. Let $X = P/\bowtie$ where P is a $4g-$polygonal diagram in $\mathbb{R}^2$ shown below:*



Let

$$A_\alpha = P \setminus \{z_0\}, z_0 \in int(P)$$

$$A_\beta = int(P)$$

Then we see that $A_\alpha \cap A_\beta = int(P) \setminus \{z_0\}$. Further,

$$\gamma(s) = z_0 + re^{-2\pi is}$$

with $s \in [0,1]$. Then we see that the deformation retracts to the image of $\gamma \approx rS^1$. We can make this explicity by defining for any $x \in A_\alpha \cap A_\beta = int(P) \setminus \{z_0\}$, then

$$\phi_t(x) = (1-t)x + t(z_0 + r\frac{x - z_0}{||x - z_0||})$$

Further $\pi_1(A_\alpha \cap A_\beta$ is generated by $[\gamma]$. In A_α, we can project the polygon through $g : P/\bowtie \to X$ in $\pi_1(A_\alpha), [\gamma] \sim [\partial P]$. In $A_\beta, \pi_1(A_\beta) = \{1\}$. and A_β is connectible, then

$$i_{\beta\alpha *}[\gamma] \in \pi_1(A_\beta) \text{ in } \{1\}$$

Van Kampen tells us $[\partial P] \sim \{1\}$ in $\pi_1(X)$. So

$$[a_1][b_1][\bar{a}_1][\bar{b}_1] \dots [a_g][b_g][\bar{a}_g][\bar{b}_g] = 1$$

This is the only relation in $\pi_1(A_\alpha) * \pi_1(A_\beta)$ to define $\pi_1(X)$.
How to do this properly with base point.

Let

$$x_0 = \gamma(0) = \gamma(1) \in A_\alpha \cap A_\beta$$

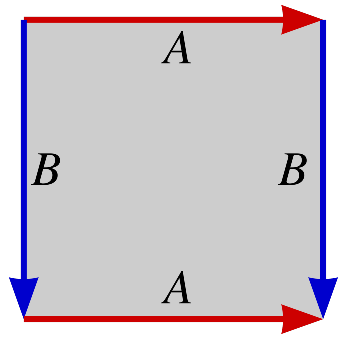
Then we can fix

$$f : S^1 \to \partial P \text{ homeomorphism}$$

We can then carry x_0 to the boundary via the path $\delta : x_0 \sim f(1)$. Then we see that

$$\gamma \sim \underbrace{\bar{\delta} f \delta}_{\partial P}$$

Remark 4.9. *When $g = 1$, then we get the torus polygonal diagram:*



with the relation

$$\mathbb{Z}[a] * \mathbb{Z}[b] / \langle [a][b][\bar{a}][\bar{b}] \rangle$$

where

$$[a][b][\bar{a}][\bar{b}] = 1 \iff [a][b][a]^{-1}[b]^{-1} = 1 \iff [a][b] = [b][a]$$

and therefore,

$$\mathbb{Z}[a] * \mathbb{Z}[b] / \langle [a][b][\bar{a}][\bar{b}] \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

A_α deformation retracts to $a_1 \vee b_1 \vee a_2 \vee b_2 \vee \ldots \vee a_g \vee b_g = Z$. Then Z is the wedge product of $2g$ copies of S^1. In the previous example,

$$\pi_1(A_\alpha) = \mathbb{Z}[a_1] * \mathbb{Z}[b_1] * \ldots * \mathbb{Z}[a_g] * \mathbb{Z}[b_g]$$

5 Cell Complexes

Definition 5.1. *A CW-complex X is the union of X^n, where X^n is the n-skeleton defined inductively on $n \in \mathbb{N}$. Specifically,*

$$X^0 := \text{set of discrete points, possibly } X^0 = \{point\}$$

*Then, given some $X^{n-1}, n \geq 1$, either $X^n = X^{n-1}$ **or** choose some family of $\{D^n_\alpha\}_{\alpha \in \mathcal{F}_n}$ in balls, then $\phi^n_\alpha = S^{n-1}_\alpha : \partial D^n_\alpha \to X^{n-1}$ (referred to as the characteristic maps) are continuous, then*

$$X^n = X^{n-1} \perp\!\!\!\perp (\perp\!\!\!\perp_{\alpha \in \mathcal{F}_n} D^n_\alpha)/ \bowtie$$

where the equivalence relation is defined for any $x \in S^{n-1}_\alpha \bowtie \phi^n_\alpha(x)$. Specifically, $\Phi^n_\alpha : D^n_\alpha \to X$ is homeomorphic to $int(D^n_\alpha)$ and let

$$e^n_\alpha := \Phi^n_\alpha(int D^n_\alpha)$$

and n-cell. Then X^n is the attaching of $e^n_\alpha \in \mathcal{F}_n$ through $\phi^n_\alpha : S^{n-1}_\alpha \to X^{n-1}$. Then we allow

$$X := \bigcup_{n \geq 0} X^n$$

Definition 5.2. *Suppose $X = \bigcup_{n \geq 0} X^n$ be a CW-complex. Then X^n is the quotient topology, where*

- *If $X = X^d$ and $n = d$, then X^d defines X*
- *If X is infinite dimensional, where if $Z \subset X$ is closed if and only if $Z \cap X^n$ is closed for all $n \geq 0$.*

Remark 5.3. *If $Y \subset X$ is a subcomplex, then if Y is closed and if $Y \cap e^n_\alpha \neq \emptyset$, then $e^n_\alpha \subset Y$. Further, $\phi^n_\alpha(D^n) \subset Y$. In this case, Y itself is a CW Complex (Exercise).*

Proposition 5.4. *If $A \subset X$ is a subcomplex, A is closed and $A \cap e^n_\alpha$ yields $e^n_\alpha \subset A$ if and only if $\exists \mathcal{F}^n_A \in \mathcal{F}$ for all n such that*

$$A = \bigcup_n \bigcup_{\alpha \in \mathcal{F}^n_A} \phi^n_\alpha(D^n_\alpha)$$

Proposition 5.5. *If $A, B \subset X$ are subcomplexes, then $A \cap B, A \cup B$ are subcomplexes.*

Proposition 5.6. *If $A \subset X$ is a subcomplex, then A is a CW complex.*

Example 5.7. *Suppose S^n with two cells for $n \geq 1$. Then*

$$X^0 = \{point\} = p, \quad X^0 = \ldots = X^{n-1}$$

$$S^n = D^n/\partial D^n$$

where for an n-dimensional cell,

$$\phi^n_1 : \partial D^n_1 \to \{p\}$$

Moreover,

$$\phi^1_1(1) = \phi^1_1(-1) = p$$

Example 5.8. *We can consider S^n with $2n$ cells. When $n = 1$, then we can split S^2 into the upper and lower hemispheres.*
Insert Image
$S^k \subset S^n$ is a subcomplex for $0 \leq k \leq n$.

Example 5.9. *Let $X = \mathbb{R}P^n = S^n/\bowtie$ where $x \bowtie \pm x$. Then it we consider it over $n+1$ cells, we see*

$$X^0 = \{p\}, n \geq 1$$

$$\begin{aligned} \phi^n : \partial D^n &\to \mathbb{R}P^{n-1} \\ x &\to [x] \end{aligned}$$

insert image
Then we allow $\mathbb{R}P^n = D^n/\bowtie$ where $x \bowtie y, x \neq y \iff x, y \in \partial D^n, x = -y$.

Remark 5.10. *Any differentiable manifold has a CW complex.*

Remark 5.11. *Any real or complex algebraic variety has a CW complex.*

Remark 5.12. *Simplicial complexes are CW Complex Structures.*

Remark 5.13. *Simplicial complexes are CW Complex Structures.*

Example 5.14. *Let $X = Y_1 \cup Y_2 \cup Y_3$ where for all $Y_i = S^1$ and $Y_1 \cap Y_2 = \{z_1\}$ and $Y_2 \cap Y_3 = \{z_2\} \neq \{z_1\}$. Then we have a CW complex with 4 one-dimensional cells. Further*

$$e_\alpha^n \cap X^{n-1} = \emptyset$$

Remark 5.15. *Infinite dimensional CW complexes exhibit strange behavior. Let S_α^1 be an indexed family of circles with $\alpha \in \mathcal{F}$. Then*

$$\bigvee_\alpha S_\alpha^1 = \text{ CW Compex } X^0 = \text{ common point}$$

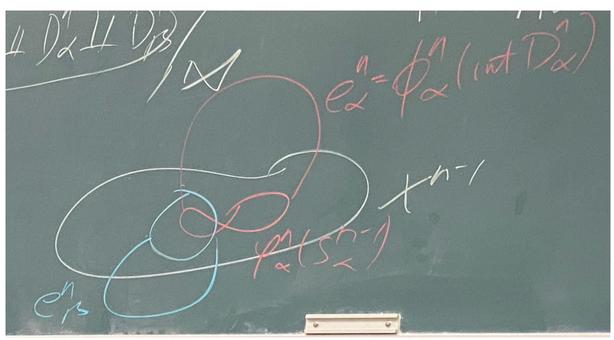
Let $x_\alpha \in S_\alpha^1$. Then we allow $x_\alpha \bowtie$. Take $y_\alpha \in S_\alpha^1 \setminus \{x_\alpha\} \implies y_\alpha \in e_\alpha^1$. Then we see that if $\mathcal{F}$ is infinite, then $\{y_\alpha\}_{\alpha\in\mathcal{F}}$ is a closed CW complex yet may not be locally compact. This should illustrate why we need finiteness.

Definition 5.16. *If $Z \subset X$ is a subcomplex, then if Z is compact if and only if Z is closed and intersects only finitely many cells e_α^n.*

Remark 5.17. *Any CW complex can be recognized as*

$$X^n = X^{n-1} \perp\!\!\!\perp D_\alpha^n \perp\!\!\!\perp D_\beta^n / \bowtie = q$$

where $\phi_\alpha^n : \partial D_\alpha^n = S_\alpha^{n-1} \to X^{n-1}$ and $\phi_\beta^n : \partial D_\beta^n = S_\beta^{n-1} \to X^{n-1}$ are the characteristic maps.



Then if we define

$$\Phi_\alpha^n := q/D_\alpha^n$$

then we see that

$$\Phi_\alpha^n|_{int(D_\alpha^n)}$$

is a homeomorphism.

5.1 Graphs

Definition 5.18. *A 1-dimensional CW complex is called a graph where e_α^1 are referred to as edges and the points of X^0 serves as the vertices.*

Definition 5.19. *A (graph theoretic) simple path of a graph X, consisting of $v_1, \ldots, v_k \in X^0$, is one such that $v_i \neq v_j$ for any $i \neq j$ and for all $1 \leq i \leq k-1$, there exists e_i^1 beginning in v_i and ending in v_{i+1}. A (graph theoretic) simple path is a (graph theoretic) simple circle provided there exists e_k^1 connected v_k to v_1.*

Lemma 5.20. *A 1-dimensional CW complex X^1 is path connected if and only if for all $v, w \in X^1$, there exists a (GT) simple path such that $v = v_1$ and $w = v_k$.*

Lemma 5.21. *For all CW complexes X, X is path connected if and only if X^1 is path connected.*

Definition 5.22. *X^1 is a tree if X^1 is path connected and X does not possess any (graph theoretic) simple circles.*

Lemma 5.23. *X^1 is a tree if and only if for all $v, w \in X^0$, there is a unique graph theoretic simple path connecting v to w.*

Proof. Suppose there exists two graph theoretic simple paths from v to w. Then the two paths share the same initial and terminal points. Further, at some point, the two paths diverge at a point and then converge at a different point. These two archs form a graph theoretic simple circle. This contradicts the fact that X^1 does not posses any simple circles. Therefore, no two paths can exist. ∎

Lemma 5.24. *X^1 is a tree implies X^1 is contractible by a deformation retract to any $v \in X^0$.*

Proof. If we fix $v_0 \in X^0$, then for all $x \in X^1 \setminus \{v_0\}$, there exists a unique injective

$$\begin{aligned}\gamma_x : [0,1] &\to X^1\\ \gamma_x(0) &= \{v_0\}\\ \gamma_x(1) &= X\end{aligned}$$

If $x \notin X^0$, then $x \in e^1_\alpha$ and e^1_α has two endpoints w_1, w_2. We may assume that the unique graph theoretic simple path to w_2 contains w_1. Then upon performing the contraction, each path retracts along themselves until each x is moved to v_0 along γ_x. ∎

Theorem 5.25. *If X^1 is path connected, then there exists a subcomplex $Y \subset X$ such that Y is a tree and $Y^0 = X^0$.*

Remark 5.26. *Y is referred to as a maximal tree. (Existence guaranteed by axiom of choice)*

Definition 5.27. *For any X^1 and $v \in X^0$, then we define N_v to be a (canonical) open neighborhood of v defined as*

1. *$v \in X^0, N_v \subset X^1$ is open where N_v is a deformation retract to v.*
2. *For any $w \neq r$, then $N_v \cap N_w = \emptyset$*
3. *$Y^1 \subset X^1$ a subcomplex implies*

$$N_Y := Y \cup \left(\bigcup_{v \in X^0 \cap Y} N_v \right)$$

Further, N_Y deformation retracts to Y.

Remark 5.28. *$Y^1, Z^1 \subset X^1$ are subcomplexes, then*

$$N_{Y \cap Z} = N_Y \cap N_Z$$

$$N_{Y \cup Z} = N_Y \cup N_Z$$

Theorem 5.29. *Let X^1 be connected, and let $Y^1 \subset X^1$ be a maximal tree. Let $\{e^1_\alpha\}_{\alpha \in \epsilon}$ be the set of 1-cells of $X^1, v_0 \in X^0 = Y^0$ with $e^1_\alpha \cap Y = \emptyset$. Let*

$$\epsilon := \{\alpha \in \mathcal{F} : e^1_\alpha \notin Y\} = \{\text{edges not in } Y\}$$

1. *For any $\alpha \in \epsilon, Z_\alpha = Y \cup e_\alpha$ has a unique graph theoretic simple circle, inducing*

$$[\delta_\alpha] \in \pi_1(Z_\alpha, v_0)$$

2. *$\pi_1(X^1, v_0) = *_{\alpha \in \epsilon} \mathbb{Z}[\delta_\alpha] \implies \pi_1(X) = *_{\alpha \in \epsilon} \mathbb{Z} =$ free group generated by $\{e^1_\alpha\}_{\alpha \in \epsilon}$*

Proof. Let v, w be the endpoints of e^1_α for some $\alpha \in \epsilon$ (it is possible for $v = w$). Since there exists two unique graph theoretic path

$$v_0 = v_1, \dots, v_k = v$$

$$v_0 = w_1, \dots, w_k = w$$

from v_0 to v and w within Y since Y is a tree. Then there exists an index $p \geq 1$ such that the two paths diverge with $v_i = w_i$ if $i \leq p$ and $v_{p+1} \neq w_{p+1}$. Then there exists a graph theoretic simple circle

$$v_p, \dots, v_k, w_m, \dots, w_p$$

which generates a closed path $n_\alpha : v_p \to v_p$. Then $\eta_\alpha = S^1$, and Z_α deformation retracts to η_α as Y in a tree. For for all $x \in Z_\alpha \setminus \eta_\alpha$, there exists a unique

$$\begin{aligned} \gamma_x : I \to Z_\alpha \\ \gamma_x(0) = x \\ \gamma_x \text{ injective} \\ \gamma_x \cap \eta_\alpha = q \in Y^0 \cap \eta_\alpha \end{aligned}$$

which is a deformation retraction of Z_α to η_α along γ_x. So

$$\delta_\alpha : v_0 \rightsquigarrow v_p \xrightarrow{\eta_\alpha} v_p \rightsquigarrow v_0$$

By Van Kampen theorem,

$$A_\alpha = N_{Z_\alpha}, \alpha \in \epsilon$$

Then

$$\pi_1(A_\alpha) = \pi_1(Z_\alpha) = \pi_1(\eta_\alpha) = \pi_1(\text{circle}) = \mathbb{Z}$$

$$\implies \pi_1(A_\alpha, v_0) = \pi_1(Z_\alpha, v_0) = \pi_1(\eta_\alpha, v_0) = \mathbb{Z}[\delta_\alpha]$$

Since $\alpha \neq \beta \implies A_\alpha \cap A_\beta = N_Y$ as $Z_\alpha \cap Z_\beta = Y$, then we see that

$$\pi_1(A_\alpha \cap A_\beta) = \pi_1(\text{tree } Y) = \{1\}$$

since any tree is contractible.

■

Theorem 5.30. *Suppose Y is path connected and for the closed balls $D^n_\alpha, \alpha \in \mathcal{F}$, for some fixed $n \geq 2$. We consider the continuous closed paths*

$$\phi^n_\alpha : \partial D^n_\alpha \to Y$$

Such that we can consider our space X to be identified with

$$X = Y \perp\!\!\!\perp (\perp\!\!\!\perp_{\alpha \in \mathcal{F}} D^n_\alpha) / \bowtie$$

where for any $x \in \partial D^n_\alpha, x \bowtie \phi^n_\alpha(x)$. THEN

1. *When $n \geq 3$, we see that*

$$\pi_1(X) = \pi_1(Y)$$

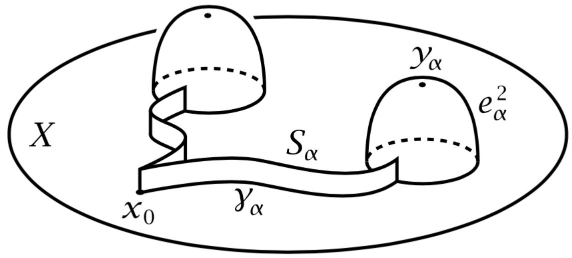
2. *When $n = 2$*

$$\pi_1(X, x_0) = \pi_1(Y, x_0)/H$$

where H is the normal subgroup generated by the classes$\{[\phi^2_\alpha]\}_{\alpha \in \mathcal{F}}$.

More precisely, let $q : Y \perp\!\!\!\perp (\perp\!\!\!\perp_{\alpha \in \mathcal{F}} D^n_\alpha) \to X$ be the quotient map of this equivalence relation, we can define

$$\Phi^n_\alpha := q|_{D^n_\alpha}$$



Lastly, for all $x_0 \in Y, \alpha \in \mathcal{F}$, there is a

$$\gamma_\alpha : I \to Y$$
$$\gamma_\alpha(0) = x_0$$
$$\gamma_\alpha(1) = \phi^n_\alpha(1)$$
$$\partial D^n_x = S^{n-1}_\alpha$$

and

$$\delta_\alpha = \gamma \cdot \phi^2_\alpha \cdot \overline{\gamma_\alpha}$$

Then H is the normal subgroup generated by $[\delta_\alpha]_{\alpha \in \mathcal{F}}$.

Proof. Consult Hatcher. ■

Example 5.31. *On oriented surfaces of genus $g \geq 1$. Then if we allow*

$$X = P/ \bowtie$$

where P is a polygon with $4g$ sides, $g \geq 1$, then

$$\partial P = a_1 b_1 \overline{a_1}\overline{b_1} \dots a_g b_g \overline{a_g}\overline{b_g}$$

Then we see that by the previous theorem,

$$\pi_1(X) = \pi_1(Y)/H = (*_{\alpha=1}\mathbb{Z}[a_\alpha]) * (*_{\alpha=1}\mathbb{Z}[b_\alpha]) / \langle a_1 b_1 \overline{a_1}\overline{b_1} \dots a_g b_g \overline{a_g}\overline{b_g} = 1\rangle$$

where

$$Y = (\bigvee_{\alpha=1} a_\alpha) \vee (\bigvee_{\alpha=1} b_\alpha)$$

Example 5.32. *Now consider a non-orientable surface with genus $g \geq 1$. Let P be a polygon with $2g$ sides. Then we see that*

$$\partial P = a_1 a_1 a_2 a_2 \dots a_g a_g$$

Then by the previous theorem, we know

$$Y = *_{\alpha=1} S^1_\alpha = *_{\alpha=1} a_\alpha$$

Then we see that

$$\pi_1(X) = *^g_{\alpha=1}\mathbb{Z}[a_\alpha]/\langle a_1 a_1 a_2 a_2 \dots a_g a_g = 1\rangle$$

Example 5.33. *When $n = 2$, then we see that*

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}[a_1]/\langle a_1 a_1 = 1\rangle = \mathbb{Z}_2$$

Example 5.34. *When $n \geq 3$, then we see that*

$$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$$

since $X = \mathbb{R}P^n$ is a CW complex with $X^k = \mathbb{R}P^k$.

Theorem 5.35. *If X is any CW complex, then*

$$\pi_1(X) = \pi_1(X^2)$$

Theorem 5.36. *For any group G, then exists a CW complex X^2 such that*

$$\pi_1(X^2) = G$$

Proof. Let $G = *_{\alpha\in\mathcal{F}}\mathbb{Z}[g_\alpha]/N$. Then let

$$X^1 = \bigvee_{\alpha\in\mathcal{F}} S_\alpha, S_\alpha \leftrightarrow g_\alpha$$

Then for each relation among g_α in N, then

$$h_1 = \ldots = h_k, 1 \leq i \leq k$$

for all $h_i = g_{\alpha_i} g_\alpha^{-1}$. Then we define

$$\phi_\beta^2 : S^1 \to X$$
$$\partial S_\beta^1 \to S_{\alpha_1}^1 \ldots S_{\alpha_k}^1$$

■

Proposition 5.37. *For $k \geq 2, k \in \mathbb{Z}$, consider the sequence of unions*

$$X = X_1 \cup X_2 \cup \ldots \cup X_k$$

such that each X_i is homeomorphic to $S^n, n \geq 1$, and each X_i intersects both X_{i-1} and X_{i+1} exactly once. Then

1. *When $n = 1$, then $\pi_1(X) = *_{i=1}^k \mathbb{Z} = *_{i=1}^k \mathbb{Z}[x_i]$*
2. *When $n \geq 2$, then $\pi_1(X) = \{1\}$.*

Proof. We will induce on k. When $k = 1$, the problem is trivial. Now, suppose that the property holds for $k-1$, and consider the complex

$$X' = X_1 \cup \ldots \cup X_{k-1}$$

Then clearly $X = X' \vee S^n$. Then $X' \cap S^n$ is a single point, so by Van Kampen, we see that

$$\pi_1(X) = \pi_1(X') * \pi_1(S^n)$$

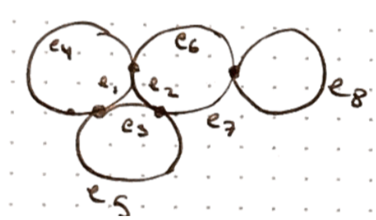
giving the desired result. ■

Proposition 5.38. *Let X be a path connected graph and let Y be a maximal tree in X. Then $\pi_1(X, x)$ is isomorphic to the free group generated by $|X^1| - |T^1|$ elements.*

Example 5.39. *Consider the cell complex defined by*

$$X = X_1 \cup X_2 \cup X_3 \cup X_4$$

such that each $X_i = S^1$ arranged in the following way:



We not only can make this into a CW complex, we can also consider it a graph. Further, it has a maximal tree

$$Y_1 = \{e_1, e_2, e_7\}$$

Therefore, we see that

$$\pi_1(X) = *_{\text{edges not in } Y^1}\mathbb{Z} = *_1^5\mathbb{Z}$$

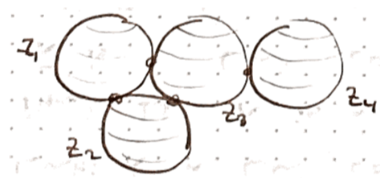
Further, we can set $\epsilon = \{3, 4, 5, 6, 8\}$ in order to get explicit generators:

$$\pi_1(X) = *_{i\in\epsilon}\mathbb{Z}[e_i]$$

Example 5.40. *Consider the cell complex defined by*

$$Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4$$

such that each $Z_i = S^2$ arranged in the following way:



Starting with circles, this can be arranged similar to X by gluing disks to make Z_i/ So we see that $Z^1 = X$ from the previous example! So for each i, we glue $D^2_{i,1}, D^2_{i,2}$ to X_i, giving us the 2-sphere Z_i. This forms a series

$$[e_\delta] \to [1]$$

by either disks attached to give Z_4. For Z_3, we see that e_2, e_6 are already trivial. So

$$[e_6 \cdot e_2 \cdot e_7] \to [1]$$

and since $[e_2 \cdot e_7] \to [1]$, it follows that $[e_6] \to [1]$. So on Z_2, we see that $[e_3 \cdot e_5] \to 1 \implies [e_3] = [e_5]$. Lasly, for $Z_1, [e_4 \cdot e_1] \to [1] \implies [e_1] = 1 = [e_4]$. So all of the generators collapse to the identity except for one, leaving us with

$$\pi_1(Z) = \mathbb{Z}[e_3] = \mathbb{Z}$$

5.2 Some More Facts about CW Complexes

Theorem 5.41. *If X is a CW-complex, every subcomplex $A \subset X$ has an open neighborhood N_A with the following properties:*

1. *N_A deformation retracts to A.*
2. *If B is a subcomplex, $N_A \cup N_B = N_{A\cup B}$ and if $A \cap B \neq \emptyset$, $N_A \cap N_B = N_{A\cap B}$. If $A \cap B\emptyset$, then $N_A \cap N_B = \emptyset$. (We could also just define $N_\emptyset = \emptyset$ to make it more consistent)*

Proof. (Idea) Construct $N_A \cap X^n$ by induction on n. ■

Theorem 5.42. *If X is a CW-complex that locally contractible, then for any $z \in X$, and any neighborhood S of z, there exists an open neighborhood $z \in U \subset S$ such that U deformation retracts to z. That is, every point has an open neighborhood base of contractible sets.*

Remark 5.43. *If X is a CW complex, then X is locally simply connected.*

Theorem 5.44. *If X is a CW-complex, and $p : \tilde{X} \to X$ is a covering map, then $\tilde{X}$ is a CW-complex. Furthermore, if e^n_α is an n-cell in X, then $p^{-1}(e^n_\alpha)$ is a union of n-cells in $\tilde{X}$.*

Theorem 5.45. *If X and Y are CW-complexes, then $X \times Y$ is a CW-complex.*

6 Homotopy Extension

Definition 6.1. *If $A \subset X$, then (X, A) satisfies the* *homotopy extension property* *if given a continuous*

$$g : A \times I \cup X \times \{0\} \to Y$$

Then, there exists $f : X \times I \to Y$ with

$$f|_{X\times\{0\}\cup A\times I} = g$$

Lemma 6.2. *(X, A) with $A \subset X$ has the homotopy extension property if and only if either of the following equivalent conditions holds:*

1. *Given $g_0 : X \to Y$ and a homotopy $g_t : A \times I \to Y$ connecting $g_0|_A$ to $g_1|_A$, then there exists a homotopy $f_t : X \times \{t\} \to Y, t \in I$ such that $f_0 = g_0$ and $f_1|_A = g_1$.*
2. *$A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Remark 6.3. *Can't prescribe $f_1|_{X\setminus A}$.*

Theorem 6.4. *If (X, A) has the homotopy extension property and A is contractible, then the quotient map $g : X \to X/A$ is a homotopy equivalence map.*

In essence, if you contract the "contractible part" of a space, you really shouldn't change the overall homotopy type of the space.

Proof. (Idea:)
We need some map $h : X/A \to X$. Since we know that A is contractible, then we can define

$$g_t : A \times \{t\}$$

where $g_0 = id_A$ and $g_1(x) = x_0 \in A$ for all $x \in A, x_0 \in A$ fixed. By homotopy extension, there exists

$$f_t : X \times \{t\} \to A$$

such that

$$f_0 = id_X \qquad f_1(x) = x_0\ \forall x \in A$$

Then considering the inverse map h, we see that

$$\begin{aligned} h : X/A &\to X \\ [x] &\to f_1(x)\ \ \forall x \in X \end{aligned}$$

Technically one must check that this is continuous, but we'll leave this as an exercise to the already stressed reader. ∎

Theorem 6.5. *If X is a CW complex and A is a subcomplex, then (X, A) satisfies the homotopy extension property.*

7 Returning to Covering Spaces

Theorem 7.1. *If $p : \tilde{X} \to X$ is a covering, $\tilde{X}, X$ path connected spaces. Then*

1. *$p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective with $p(\tilde{x}_0) = x_0$.*
2. *If $\gamma, \delta : I \to X, \gamma(0) = \delta(0) = x_0, \gamma(1) = \delta(1) = x_1$, then*
$$\tilde{\gamma}(1) = \tilde{\delta}(1) \iff [\gamma \cdot \overline{\delta}] \in p_*(\pi_1(X, x_0))$$
3. *$p^{-1}(x_0) \leftrightarrow$ cosets of $\pi_1(X, x_0)$ with respect to $\underbrace{p_*(\pi_1(\tilde{X}_0, \tilde{x}_0))}_{H}$ such that for all $[\delta] \in \pi_1(X, x_0)$*
$$\tilde{\delta}(1) \leftrightarrow H \cdot [\delta]$$
4. *If $\tilde{x}_1 \in p^{-1}(x_0)$ and $\tilde{\eta} \cdot \tilde{x}_0 \rightsquigarrow \tilde{x}_1, [p \circ \tilde{\eta}] = [\eta] \in \pi_1(X, x_0)$, then*
$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = [\overline{\eta}]H[\eta]$$

Theorem 7.2. *If $p : \tilde{X} \to X$ is a covering map, with $\tilde{X}, X$ both path connected, and Z is a path connected and locally path connected space with $p(\tilde{x}_0) = x_0, f : Z \to X, f(z_0) = x_0$, then*

1. *There exists $\tilde{f} : Z \to \tilde{X}$ with $p \circ f = \tilde{f}, \tilde{f}(z_0) = \tilde{x}_0$ and the diagram*
$$\begin{tikzcd} (Z, z_0) \arrow[r, "\exists! \tilde{f}"] \arrow[dr, "f"'] & \tilde{(}\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ & (X, x_0) \end{tikzcd}$$
commutes if and only if we satisfy the <u>lifting criteria:</u>
$$f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$
2. *If the lifting criteria holds, then $\tilde{f}$ is unique.*

Lemma 7.3. *Suppose we have $p : \tilde{X} \to X$ a covering, $X, \tilde{X}$ are path connected and locally path connected, with base point $p(\tilde{x}_0) = x_0$. Then for any $\tilde{y} \in p^{-1}(x_0)$, there exists a continuous $f_{\tilde{y}} : \tilde{X} \to \tilde{X}$ with*
$$f_{\tilde{y}}(\tilde{x}_0) = \tilde{y}$$
and
$$p \circ f_{\tilde{y}} = p$$
if and only if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is a normal subgroup.

Proof. We can realize this lemma using the previous theorem, by stipulating $\tilde{x}_0 = z_0, \tilde{X} = Z$. Then we see that
$$\begin{tikzcd} (\tilde{X}, \tilde{x}_0) \arrow[r, "\exists! \tilde{f}_{\tilde{y}}"] \arrow[dr, "p"'] & (\tilde{X}, \tilde{y}_0) \arrow[d, "p"] \\ & (X, x_0) \end{tikzcd}$$
Then by the lifting criteria, we see that
$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{y}))$$

So we choose $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{y}$ in $\tilde{X}$ such that

$$\gamma = p \circ \tilde{\gamma}$$

Then we see that

$$p_*(\pi_1(\tilde{X}, \tilde{y})) = [\gamma] p_*(\pi_1(\tilde{X}, \tilde{x}_0))[\overline{\gamma}]$$

Therefore, there exists $f_{\tilde{y}}$ for every $\tilde{y} \in p^{-1}(x_0) \implies p^*(\pi(\tilde{X}, \tilde{x}_0))$ coincides with its conjugates and is therefore normal! ■

Remark 7.4. *If $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup, then $f_{\tilde{y}} : \tilde{X} \to \tilde{X}$ is a homeomorphism for every $\tilde{y} \in p^{-1}(x_0)$.*

This is similar to the argument used in our proof of the universal cover.

Theorem 7.5. *If $p : \tilde{X} \to X$ is a covering, with $\tilde{X}, X$ both path connected and locally connected, with base point $p(\tilde{x}_0) = x_0$. If $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is a normal subgroup, then*

$$G = \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

acts freely and properly on $\tilde{X}$ and there exist a natural correspondence such that for every $g \in G, f_g : \tilde{X} \to \tilde{X}$ is a homeomorphism with $G \leftrightarrow p^{-1}(x_0)$ and $g \leftrightarrow f_g(\tilde{x}_0)$.

Theorem 7.6. *If $\tilde{X}$ is both path connected and locally path connected with a group G that acts freely and properly on $\tilde{X}$, then the projection*

$$q : \tilde{X} \to \tilde{X}/G$$

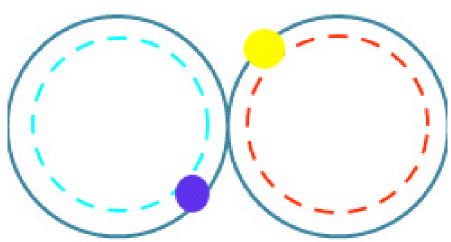
is a covering with

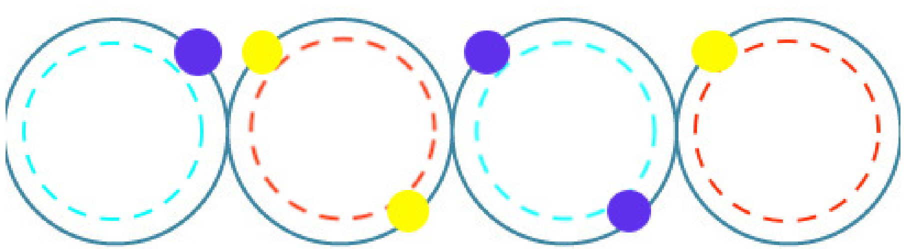
$$\pi_1(X/G) = q_*(\pi_1(\tilde{X})) = G$$

where $q_*(\pi_1(\tilde{X}))$ is normal.

Recall G here is the group of deck transformations.

Example 7.7. *We consider a covering of $S^1 \vee S^1$ by $S^1 \vee S^1 \vee S^1 \vee S^1$*





Specifically, any homeomorphism $f : \tilde{X} \to \tilde{X}$ with $p \circ f = o$ is $f = id_{\tilde{X}}$. This implies that

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

is not a normal subgroup within $\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$.

Theorem 7.8. *If X is a path connected and locally simply connected space, then there exists $p : \tilde{X} \to X$ a covering with $\tilde{X}$ simply connected.*

Remark 7.9. *This tells us that $\tilde{X}$ is a universal cover and is unique.*

Proof. (Idea)
Upon fixing a basepoint $x_0 \in X$, we define

$$\tilde{X} := \{[\gamma] : \text{ where } \gamma : [0,1] \to X \text{ is a path with } \gamma(0) = x_0\}$$

along with the equivalence classes defined

$$[\gamma] := \{\delta : [0,1] \to X, \delta \sim \gamma \text{ relative to } \{0,1\}\}$$

Then we can define $p : \tilde{X} \to X, p([\gamma]) = \gamma(1)$. So we see that $\gamma, \eta : x_0 \rightsquigarrow x$, then $[\gamma] = [\delta] \iff \gamma \sim \delta$ relative to $\{0,1\}$. So

$$p^{-1}(x) = \text{ homotopy classes of paths } x_0 \rightsquigarrow x$$

Now we need to define a topology on this new space. Specifically, let $U \subset X$ containing x be open. Then U is evenly covered if and only if U is simply connected. Finally, $\tilde{X}$ is simply connected since for every $\tilde{x}_0 \in \tilde{X}$, with $p(\tilde{x}_0) = x_0$, then we consider $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{x}_0$ and we can realize

$$\gamma = p \circ \tilde{\gamma}$$

is a closed path $x_0 \rightsquigarrow x_0$. So $\tilde{\gamma}$ is the lift of γ has the same terminal point as constant $\tilde{x}_0$. So $\gamma \sim x_0 \implies \tilde{\gamma} \sim \tilde{x}_0$. ■

7.1 Classification of Covering Spaces

Theorem 7.10. *Let X be a path connected and locally simply connected space and take any base $x_0 \in X$. Then for any subgroup $H \subset \pi_1(X, x_0)$, there exists a unique covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that*

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$$

Moreover, conjugates of H correspond to $p_*(\tilde{X}, \tilde{y}) \to (X, x_0)$ where $\tilde{y} \in p^{-1}(x_0)$.

Remark 7.11. *We have uniqueness since if there exists covering maps $p_i : (\tilde{X}_i, \tilde{x}_i) \to (X, x_0)$ such that for $i = 1, 2$ we have*

$$(p_i)_*(\pi_1(\tilde{X}_i, \tilde{x}_i)) = H$$

then there exists a unique homeomorphism $f : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that

$$p_2 \circ f = p_1$$

Moreover, the diagram

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, "\exists! f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

commutes.

Remark 7.12. *If $y \in p^{-1}(x_0)$ and $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{y}_0$ with $p(\tilde{\gamma}) = \gamma$, then $p_*(\pi_1(\tilde{X}, \tilde{y})) = [\gamma]H[\overline{\gamma}]$.*

Remark 7.13. *To see the existence of the above theorem, we simply allow $\tilde{X}$ to be the equivalence classes of paths $\gamma : [0,1] \to X, \gamma(0) = x_0, \gamma(1) = \delta(1), [\gamma\overline{\delta}] \in H$, specifically where γ and δ are equivalent path.*

Remark 7.14. *In Hatcher, he defines this notion differently. Specifically, let X be path connected, locally path connected and locally simply connected. Then Hatcher's condition states that there exists a basis $\{U_\alpha\}$ of the topology of X such that*

- *All U_α is path connected*
- $\gamma : [0,1] \to U_\alpha, \gamma(0) = \gamma(1) = x_0$, *then* $\exists F : I \times X \to X$ *such that*

$$F(s,0) = \gamma(s) \quad \text{and} \quad F(s,1) = x_0$$

Example 7.15. *Consider a group G with a set of generators $\{g_\alpha\}_{\alpha\in\mathcal{F}}$. Then X_G is a CW-complex where*

$$X_G^{(1)} = \bigvee_{\alpha\in\mathcal{F}} S^1_\alpha$$

Then for all $\alpha \in \mathcal{F}$, we fix

$$\begin{aligned} \delta_\alpha : I &\to S^1_\alpha \\ s &\to e^{2\pi i s} \end{aligned}$$

Then we see that $X_G = X_G^{(2)}$ for each

$$g_{\alpha_1} g_{\alpha_2} \cdots g_{\alpha_k} = 1$$

within our group G. Further, we attach the two-cell

$$\begin{aligned} \phi^1_{\alpha_1,\ldots,\alpha_k} : \partial D^2 &\to X_G^{(1)} \\ s &\to \delta_{\alpha_1} \cdot \delta_{\alpha_2} \cdot \ldots \cdot \delta_{\alpha_k} \end{aligned}$$

So we see that

$$\pi_1(X_G^{(2)}) = \pi_1(X^{(1)})/H = *_{\alpha\in\mathcal{F}} Z[\delta_\alpha]/H$$

where H is the normal group generated by $[\phi_{\alpha_1,\ldots,\alpha_k}] = 1$. So

$$\pi_1(X_G) = G$$

Example 7.16. *We can consider a Cayley digraph Y_G such that $Y_G^{(0)} = G$. That is, two elements h and g are connected by 1-cell edges provided there exists g_α such that*

$$g = hg_\alpha$$

Therefore, if we see that $\#\{g_\alpha\}_{\alpha\in\mathcal{F}} = m$, then we typically say that $Y_G^{(1)}$ is m-regular. Also, for each relation

$$g_{\alpha_1} \cdots g_{\alpha_k} = 1,$$

we attached 2-cells based on each $h \in G$.

7.2 Coverings of Manifolds

Definition 7.17. *X is an n-manifold if there exists an open $\{U_\alpha\}_{\alpha\in\mathcal{F}}$, with $\mathcal{F}$ countable, such that for all U_α, there exists $\phi_\alpha : U_\alpha \to int(D^n)$ homeomorphisms. Further, for any α, β, we maintain a compatibility constraint such that*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

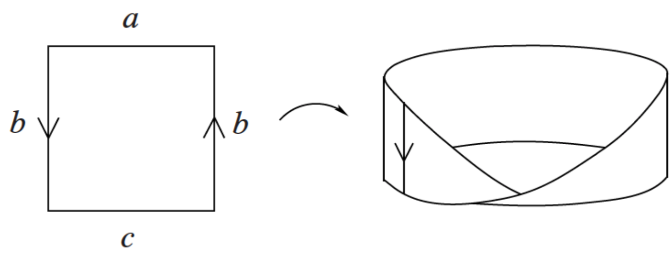
is a diffeomorphism. Such a manifold that satisfies this constraint is a C^1-n-manifold.

Definition 7.18. *X is orientable provided there exists $\{U_\alpha\}_{\alpha\in\mathcal{F}}$ such that*

$$det(\phi_\beta \circ \phi_\alpha^{-1})(x) > 0$$

for all α, β and $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Otherwise, we say X is non-orientable.

Example 7.19. *We consider the non-orientable Mobius band:*



Lemma 7.20. *Let X be a C^1 n-manifold. Then*

1. *With $\gamma : [0,1] \to X$, one can trace along γ the orientation at $\gamma(0)$ in an orientation at $\gamma(1)$.*
2. *$\gamma \sim \delta \implies \gamma$ and δ translate orientation in the same way.*

Theorem 7.21. *Let X be a non-orientable C^1 n-manifold, $x_0 \in X$ and take $H \subset \pi_1(X, x_0)$ such that*

$$H = \{[\gamma] : \gamma \text{ keeps orientation at } x_0\}$$

Then we see that

$$|\pi_1(X, x_0)/H| = 2$$

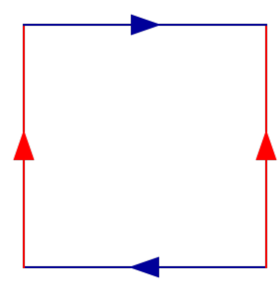
provided there exists $\eta : x_0 \rightsquigarrow x_0$ that changes orientation. Further, if γ, σ change orientation, then

$$[\gamma \cdot \overline{\sigma}] \in H$$

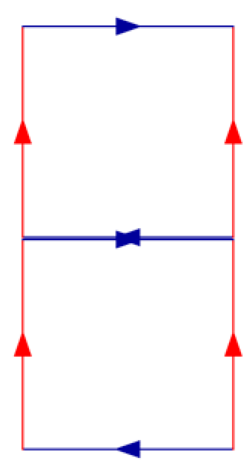
Topologically, this tells us there exists a two sheeted covering $p : \tilde{X} \to X$ and $\tilde{X}$ is an orientable C^1 n-manifold.

Example 7.22. $S^2 \to \mathbb{R}P^2$.

Example 7.23. *Consider the klein bottle.*



Covered by the Toris



Example 7.24. *Universal Covers of Compact 2-Manifolds*

Recall a universal cover $\tilde{X}$ requires the covering to originate from a simply connected space. Then we know there exists a deck transformation G such that

$$X = \tilde{X}/G$$

Then we see when

- $g = 0 \implies \mathbb{C} \cup \{\infty\} = S^2 \to S^2 \implies$ *curvature* $= 1$
- $g = 1 \implies \mathbb{C} = \mathbb{R}^2 \to S^1 \times S^1 =$ *torus* $\implies$ *curvature* $= 0$
- $g \geq 2 \implies int(D^2) \to Mg \implies$ *hyperbolic geometry curvature* $= -1$

This lead many to consider the following hard problem suggested around 1900:

Theorem 7.25 (Poincaré Conjecture)**.** *Every simply connected compact 3-manifolds is homeomorphic to S^3.*

which was finally solved by Perelman in 2002.
This theorem fails to generalize. For example, when $n = 4$, we see that the fundamental groups are the same,

$$\pi_1(S^4) = \{1\} = \pi_1(S^2 \times S^2)$$

but if we consider a generalization of the fundamental group,

$$\pi_2(S^4) = \{1\} \quad \pi_2(S^2 \times S^2) = \mathbb{Z} \times \mathbb{Z}$$

8 Higher Homotopy Groups

Definition 8.1. *Let $n \geq 0$ and consider an arbitrary base point $x_0 \in X$. Then*

$$\pi_n(X, x_0) := \text{ homotopy classes of } \phi : (S^n, \xi_n) \to (X, x_0)$$

where $\xi_n \in S^n$.

Note 8.2. *This object is not necessarily a group for every n.*

Example 8.3. *When $n = 0$, we see that $S^0 = \{\xi_0, z\}$. So any $\phi, \psi : (S^0, \xi_0) \to (X, x_0)$ is homotopic if and only if $\phi(z) = \psi(z)$ are in the same path connected component of X. In other words, $\pi_0(X)$ is the number of path connected components of X.*

Remark 8.4. *For $n \geq 1$, we can write $S^n = D^n/\partial D^n$. This is useful since*

$$\pi_n(X, x_0) := \text{ relative homotopic classes of maps } \{\phi : (D^n, \partial D^n) \to (X, x_0)\}$$

Specficially, $D^n \xrightarrow{\phi}$ and simultaneously $\phi D^n \xrightarrow{\phi} \{x_0\}$. But this is equivalent to

$$\pi_n(X, x_0) := \text{ relative homotopic classes of maps } \{\phi : (I^n, \partial I^n) \to (X, x_0)\}$$

where I is the unit interval.

Proposition 8.5. *When $n \geq 1$, we can define a group structure on $\pi_n(X, x_0)$.*

Proof.

- We have already seen this for the fundamental group $\pi_1(X, x_0)$.
- Let $n \geq 2$ and consider $\phi, \psi : [0, 1] \times I^{n-1} \to (X, x_0)$ to be defined as

$$\phi \cdot \psi(s, x) = \begin{cases} \phi(2s, x) & 0 \leq s \leq \frac{1}{2} \\ \psi(2s - 1, x) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

■

Lemma 8.6.

1. *$\pi_n(X, x_0)$ is a group with operation defined above.*
2. *The unit element is the constant x_0 map.*
3. *When $n \geq 2$, then $\pi_n(X, x_0)$ is commutative.*

Proof. To proof statement (3), we consider a sequence of steps:

- *Claim:* $\psi \cdot \phi$ **Insert proof from picture**

When $n > 2$, we can choose any changes in the first two coordinates. ■

8.1 Action of Paths on $\pi_n(X, x_0)$ with $n \geq 2$

Definition 8.7. *The action of paths in X on $\pi_n(X, x_0)$ for $n \geq 2, x_0, x_1 \in X$ and $\gamma : x_0 \rightsquigarrow x_1$ such that*

$$\gamma : [0, 1] \to X$$

$$\gamma(0) = \gamma(1) = x_1$$

Then $\phi : (S^n, \epsilon_n) \to (X, x_0)$ we can define our function radially by

$$\phi : (D^n, \partial D^n) \to (X, x_0)$$

such that for all $u \in \partial D^2 m$ we see that

$$\gamma\phi(su) := \begin{cases} \phi(2su) & s \leq \frac{1}{2} \\ \gamma(2s - 1) & s \geq \frac{1}{2} \end{cases}$$

Moreover, $[\gamma\phi] \in \pi_n(X, x_1)$ with $x_1 = \gamma(1)$.

Remark 8.8. *Notationally, $\phi \cdot \gamma$ would be more appropriate in order to give a make the group action explicit, with*

$$[(\phi + \psi) \cdot \gamma] = [\phi \cdot \gamma] + [\psi \cdot \gamma]$$

Lemma 8.9.

1. *Given $\gamma : x_0 \rightsquigarrow x_1$ and $\phi : (D^n, \partial D^n) \to (X, x_0)$, then we see that*

$$\overline{\gamma} \cdot (\gamma \cdot \phi) \sim \phi$$

2. *$\gamma \cdot (\phi \cdot \psi) \sim (\gamma \cdot \phi)(\gamma \cdot \psi)$ where $\phi, \psi : (I^n, \partial I^n) \to (X, x_0)$.*

Corollary 8.9.1. *Any path connected space X and any path $\gamma : x_0 \rightsquigarrow x_1$ define an isomorphism*

$$\begin{aligned} \pi_n(X, x_0) &\cong \pi_n(X, x_1) \\ [\phi] &\to [\phi \cdot \gamma] \end{aligned}$$

Remark 8.10. *$\mathbb{Z}\pi_1(X, x_0)$ is a group algebra over $\pi_1(X, x_0)$ which are elements of the form:*

$$\sum_{finite} k_i[\gamma_i]$$

where $k_i \in \mathbb{Z}, [\gamma_i] \in \pi_1(X, x_0)$.

Remark 8.11. *When $n \geq 2, \pi_n(X, x_0)$ is a module over $\mathbb{Z}\pi_1(X, x_0)$ with the ring multiplication action defined as:*

$$(\phi \cdot \gamma) \cdot \delta = \phi \cdot (\gamma \cdot \delta)$$

Definition 8.12. *Given a function $f : (X, x_0) \to (Y, y_0)$ where X and Y are path connected, we define the push forward of f onto the higher homotopy group to be:*

$$\begin{aligned} f_* : \pi_n(X, x_0) &\to \pi_n(Y, y_0) \\ [\gamma] &\to [f \circ \gamma] \end{aligned}$$

Further, this f_* is a homomorphism.

Theorem 8.13. *If $f : (X, x_0) \to (Y, y_0)$, where X and Y are path connected, is a homotopy equivalence (basepoint is irrelevant) then $f_*\pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism.*

Remark 8.14. *There exists $g : (Y, y_0) \to (X, x_0)$ such that*

$$f \circ g \sim id_Y \quad g \circ f \sim id_X$$

Corollary 8.14.1. *If X is contractible, then $\pi_n(X) = \{0\}$ for $n \geq 2$.*

Theorem 8.15. *If $p : \tilde{X} \to X$ is a covering, $p(\tilde{x}_0) = x_0$, $\tilde{X}, X$ path connected, then $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for $n \geq 2$.*

Proof.
- To see that p_* is surjective, since S^n is simply connected, then for any $phi : (S^n, \epsilon_n) \to (X, x_0)$, we know there exists $\tilde{\phi} : (S^n, \epsilon_n) \to (\tilde{X}, \tilde{x}_0)$ such that

$$p \circ \tilde{\phi} = \phi \implies p_*([\tilde{\phi}]) = [\phi]$$

since the diagram

$$\begin{tikzcd} S^n \arrow[r, "\exists! \tilde{\phi}"] \arrow[dr, "\phi"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd}$$

commutes.

- p_* is injective since if we assume $p_*[\gamma] = 0$, then $\tilde{\phi} : (S^n, \epsilon_n) \to (\tilde{X}, \tilde{x}_0)$, then

$$\phi = p \circ \tilde{\phi}$$

Now consider the function $F(z,t) = \phi_t$ to be the homotopy defined by $\phi_t : S^n \to X$ such that $\phi_0 = \phi_1$ and $\phi_1(z) = x_0$ for all $z \in S^n$. Since $S^n \times I$ is simply connected, then $F : S^n \times I \to X$ can be lifted to a homotopy $\tilde{F} : S^n \times I \to \tilde{X}$ where $\tilde{F}(z,1) = \tilde{x}_0$.

■

Theorem 8.16. *$\pi_n(S^1) = 0$ for all $n \geq 2$.*

Proof. Take the typical $\mathbb{R} \to S^1$ to be a covering. Then

$$\pi_n(S^1) = \pi_n(\mathbb{R}) = \{0\}$$

■

Theorem 8.17. *If X is a compact two manifold (i.e. a compact surface), and $X \neq S^2$ nor $\mathbb{R}P^2$, then $\pi_n(X) = 0$ for all $n \geq 2$.*

Proof. Topologically, $\mathbb{R}^2$ is the universal covering space of X, and therefore

$$\pi_n(X) = \pi_n(\mathbb{R}) = \{0\}$$

■

Remark 8.18. *$\pi_n(S^2) = \pi_n(\mathbb{R}P^2)$ since S^2 is the universal covering of $\mathbb{R}P^2$.*

8.2 Higher Homotopies on CW Complexes

Theorem 8.19. *$\pi_n(S^k) = 0$ for all $2 \leq n < k$.*

Proof. Let S^n be represented by the sequence of cells $\{e_0, \ldots, e_n\}$ and let S^k be represented by the sequence of cells $\{d_0, \ldots, d_k\}$. Now suppose we have a function $\phi : S^n \to S^k$ such that

$$\phi(e^0) = d^0$$

Then for any $\psi \sim \phi$, it should follow that $X = S^n, Y = S^k$

$$\psi(X^{(n)}) \subset Y^{(n)} = \{d^0\}$$

■

Lemma 8.20. *For $n \geq 1, \phi : S^n \to X$ is homotopic to a constant map if and only if $\exists f : D^{n+1} \to X$ such that $f|_{S^n} = \phi$.*

Proof.

- ($\Leftarrow$) Suppose there exists such a function $f : D^{n+1} \to X$ with $f|_{\partial D^{n+1}} = \phi$. Then we can define the homotopy, we know that

$$f_t(x) = f(tx) \quad \forall x \in s \in S^{n-1}$$

Notice that $f_0 \equiv$ constant while $f_1 = \phi$

- ($\Rightarrow$) Left as a homework exercise.

■

Theorem 8.21. *For all $n \geq 1, \pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$.*

Remark 8.22. *Not all topologically spaces are expressible as CW complexes.*

Example 8.23. *Let $u = (1,0) \in \mathbb{R}^2$ and $v = (0,1) \in \mathbb{R}^2$. Clearly $\mathbb{R} = \mathbb{R}u + \mathbb{R}v$. Now define*

$$X = \{tu : t \in [0,1]\} \cup \{tv : t \in [0,1]\} \cup \{\frac{1}{n}u + tv : n \in \mathbb{N}, t \in [0,1]\}$$

Notice, there is no small neighborhood of $\frac{1}{2}v$. Therefore, X is not a CW complex.

Remark 8.24. *If X has a CW structure with cells $\{e^n_\alpha\}$ and Y has a CW structure with cells $\{d^m_\beta\}$, then $X \times Y$ has a CW structure with cells $\{e^n_\alpha \times d^m_\beta\}$*

Remark 8.25. *If X has a CW structure with cells $\{e^n_\alpha\}$ and Y has a CW structure with cells $\{d^m_\beta\}$, then $X \times Y$ has a CW structure with cells $\{e^n_\alpha \times d^m_\beta\}$*

Corollary 8.25.1. *If X is a CW complex, then*

$$\pi_n(X) = \pi_n(X^{n+1}) \quad \forall n \geq 1$$

Proof. Let $\phi_0, \phi_1 : (S^n, \epsilon_n) \to (X, x_0), x_0 \in X^0$ and let $\phi_0 \sim \phi_1$ in X relative to x_0. By cellular approximation, there exists

$$\psi_0 \sim \phi_0 \qquad \psi_1 \sim \phi_1$$

relative to ϵ_n such that

$$\psi_0(S^m) \subset X^n \quad \psi_1(S^n) \subset X^m$$

Further, $\psi_0 \sim \psi_1$ relative to ϵ_n in X. So we can identify a homotopy

$$\begin{aligned} F : S^n \times I &\to X \\ F|_{S^n \times \{0\}} &= \phi_0 \\ F|_{S^n \times \{1\}} &= \phi_1 \end{aligned}$$

Then $S^n \times I$ is an $(n+1)$-dimensional CW complex. Let $A = (S^n \times \{0\}) \cup (S^n \times \{1\})$ be a subcomplex. Then by cellular approximation, we have a new homotopy

$$\begin{aligned} G : S^n \times I &\to X^{n+1} \\ G \sim F &\text{ relative to } A \\ G(x,0) &= \phi_0(x) \\ G(x,1) &= \phi_1(x) \end{aligned}$$

■

Theorem 8.26. *If X is a compact surface, S^2 or $\mathbb{R}P^2$, then $\pi_n(X) = \{0\}$ for all $n \geq 2$.*

9 Fiber Bundles and Exact Sequences

9.1 Fiber Bundles

Definition 9.1. *$F \to E \xrightarrow{p} B$ is a fiber bundle if there exists $p : E \to B$ such that for all $b \in B$, there exist open balls $b \in U \subset B$ with $p^{-1}(U)$ homeomorphic to $U \times F$.*

Example 9.2. *Any covering is a fiber bundle if and only if F is a discrete space, in which case*

$$p^{-1}(U) = \amalg_{\alpha} U_{\alpha}$$

Example 9.3. *If E is trivial, then $E = B \times F$.*

Example 9.4. *The Hopf Fibration (1931). Let $\mathbb{R}^4 = \mathbb{C}^2$. Consider the realization*

$$S^3 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2| = 1\}$$

Consider the projection

$$\begin{aligned} p : S^3 &\to S^2 = \mathbb{C} \cup \infty \\ (z_1, z_2) &\to z_1 : z_2 \end{aligned}$$

Then for any $w \in \mathbb{C}$,

$$p^{-1}(w) = \{(z_1|z_2) : |z_1|^2 + |z_2|^2 = 1 \text{ and } z_1 = wz_2\} = \{(wz_2 : z_2) : |z_2|^2 = \frac{1}{1 + |w|^2}\}$$

So $p^{-1}(\mathbb{C})$ is homeomorphic to $\mathbb{C} \times S^1$. So for all $z \in \mathbb{C} \cup \{\infty\}$, $p^{-1}(S^2 \setminus z)$ is homeomorphic to $(S^2 \setminus z) \times S^1$.

Remark 9.5. *Shortly, we'll see*

$$\pi_2(S^2 \times S^1) = \pi_2(S^2) \times \pi_2(S^1) = \mathbb{Z} \times \{0\} = \mathbb{Z}$$

which is a nontrivial fibration, while $\pi_2(S^3) = 0$.

9.2 Exact Sequences

Definition 9.6. *Given A, B, C groups the sequence $A \xrightarrow{\beta} B \xrightarrow{\gamma} C$ with β and γ homomorphisms is exact if and only if $im(\beta) = ker(\gamma)$.*

Example 9.7. *$\{1\} \xrightarrow{\alpha} A \xrightarrow{\beta} B$ is exact implies*

$$ker(\beta) = im(\alpha) = \{1_A\} \implies \beta \text{ is injective}$$

Example 9.8. *$A \xrightarrow{\gamma} B \xrightarrow{\delta} 1$ is exact implies*

$$ker(\delta) = im(\gamma) = \{1_B\} \implies \gamma \text{ is injective}$$

Example 9.9. *$\{1\} \xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \xrightarrow{\delta} 1$ is a short exact sequence which implies*

$$C = B/A$$

Example 9.10. *$\{1\} \to A \xrightarrow{\beta} B \xrightarrow{\gamma} 1$ implies β is an isomorphism.*

9.3 Relative Homotopy

Definition 9.11. *Let $x_0 \in A \subset X$. Then for all n, $\pi_n(X, A, x_0)$ is a set of homotopy classes of*

$$\phi : (D^n, \partial D^n = S^{n-1}, \epsilon_{n-1}) \to (X, A, x_0)$$

such that $\phi \bowtie \psi$ if and only if $\exists \phi_t : (D^n, \partial D^n, \epsilon_{n-1}) \to (X, A, x_0)$ such that

$$\phi_0 = \phi, \phi_t = \psi.$$

Question 9.12. *How do we define a group operation on these homotopy classes?*

Firstly, we cannot define one for $n = 1$. For $n \geq 2$, it is possible to define it on $\phi|_{\partial D}$. Observe, if we consider the hypercubes

$$I^n = \{(s_1, \dots, s_n) \in \mathbb{R}^n : s_i \in [0, 1]\}$$
$$I^{n-1} = \{(s_1, \dots, s_{n-1}, 0) \in \mathbb{R}^n : s_i \in [0, 1]\}$$
$$\mathcal{F}_{n-1} := closure(\partial I^n \setminus I^{n-1})$$

Then we see that

$$\pi_n(X, A, x_0) = \text{ homotopy classes of } \phi : (I^n, I^{n-1}, \mathcal{F}_{n-1}) \to (X, A, x_0)$$

and so we can adapt the same formula as before for its multiplication!

Proposition 9.13. *If $n \geq 2$, then $\pi_n(X, A, x_0)$ is a group with multiplication defined on $\phi|_{\partial D^n}$. When $n \geq 3$, $\pi_n(X, A, x_0)$ is commutative.*

9.4 Long Exact Sequences

Theorem 9.14 (The Long Exact Sequence for Relative Homotopy)**.** *Consider $x_0 \in A \subset X$, and injection $i : A \hookrightarrow X$, and $j : (X, x_0) \to (X, A, x_0)$. Then there exists the long exact sequence for all $n \geq 2$:*

$$\dots \xrightarrow{\partial_*} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \dots$$

Proof. (Idea)
$\phi : (D^n, \partial D^n, \epsilon_{n-1}) \to (X, A, x_0)$ is nullhomotopic in $\pi_n(X, A, x_0)$, so it can be homotopied into $(D^n, \partial D^n, \epsilon_{n-1}) \to \pi_n(A, A, x_0)$:

$$\phi : (D^n, \partial D^n, x_0) \to (X, A, x_0)$$
$$\partial_*[\phi] \to [\phi|_{\partial D}]$$

∎

Lemma 9.15. *$f : S^n \to A$ is homotopic to a constant map if and only if there exists $g : D^{n+1} \to A$ with $g|_{\partial D^{n+1}} = f$.*

Example 9.16. *Let $x_0 \in A \subset X$. If A is contractible, then $\pi_n(X, A, x_0) = \pi_n(X, x_0)$, since by the long exact sequence theorem,*

$$\{1\} = \pi_n(A) \to \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \to \pi_{n-1}(A) = \{1\}$$
$$\implies \{1\} \to \pi_n(X) \to \pi_n(X, A) \to \{1\}$$
$$\implies \pi_n(X) \cong \pi_n(X, A)$$

Example 9.17. *Let $x_0 \in A \subset X$. If A is a deformation retract of X, then $\pi_n(X, A, x_0) = \{1\}$. Observe, we have the long exact sequence*

$$\pi_n(A) \xrightarrow{i_{*,n}} \pi_n(X) \xrightarrow{j_{*,n}} \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A) \xrightarrow{i_{*,n-1}} \pi_{n-1}(X) \to \dots$$

Since $i_{*,n}, i_{*,n-1}$ are isomorphisms, we see that

$$Ker j_{*,n} = Im i_{*,n}(X) \implies Im j_{*,n} = \{1\}$$
$$\implies Im \partial_* = Ker i_{*,n-1} = \{1\}$$
$$\implies Ker \partial_* = \pi_n(X, x_0)$$
$$\implies \pi_n(X, A) = \{1\}$$

Theorem 9.18 (Long Exact Sequence for a Fiber Bundle)**.** *Given $F \to E \xrightarrow{p} B$, then if B and E are path connected, then there exists a sequence*

$$\dots \xrightarrow{\partial_*} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_*} \pi_{n-1}(F, x_0) \xrightarrow{i_*} \pi_{n-1}(E, x_0) \to$$

$$\dots \to \pi_2(B, b_0) \to \pi_1(F, x_0) \to \pi_1(E, x_0) \to \pi_1(B, b_0)$$

Proof. All we need to check is that $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism.

- p_* is surjective since $\phi : (D^n, \partial D^n) \to (B, b_0)$ such that

  $$[\phi] = \pi_n(B, b_0)$$

  As a consequence of Homotopy lifting, we know there exists $\psi : D^n \to E$ such that

  $$p \circ \psi = \phi$$

- π_* is injective: Suppose $\psi : (D^n, \partial D^n, \epsilon_{n-1}) \to (E, F, x_0)$ with $\phi = p \circ \psi \sim b_0$ under a homotopy a homotopy ϕ_t. Then

  $$\psi_1(x) \in F \forall x \in D^n \implies [\psi] = 0$$

■

Remark 9.19. *If $b_0 \in B, b_0 = p(x_0), x_0 \in E$, then we can assume $F = p^{-1}(b_0)$ and $(F, x_0) \hookrightarrow (E, x_0)$.*

Corollary 9.19.1. $\pi_2(S^2) = \mathbb{Z}$

Proof. Since we have $S^1 \to S^3 \to S^2$ by the Hopf Fibration, we see that we have the long exact sequence

$$\{1\} = \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) = \{1\}$$

$$\implies \pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}$$

■

Theorem 9.20 (Freudenthal)**.** $\pi_i(S^n) = \pi_{i+1}(S^{n+1})$ *for $i \leq 2n - 2$.*

Corollary 9.20.1. $\pi_n(S^n) = \mathbb{Z}$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbf{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2{}^2$
S^3	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$
S^4	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{12}$	$\mathbf{Z}_2^2$	$\mathbf{Z}_2^2$	$\mathbf{Z}_{24}\times\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{120}\times\mathbf{Z}_{12}\times\mathbf{Z}_2$	$\mathbf{Z}_{84}\times\mathbf{Z}_2^5$
S^5	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{30}$	$\mathbf{Z}_2$	$\mathbf{Z}_2^3$	$\mathbf{Z}_{72}\times\mathbf{Z}_2$
S^6	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{24}\times\mathbf{Z}_2$	$\mathbf{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}_{120}$	$\mathbf{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24}$	0	0	$\mathbf{Z}_2$	$\mathbf{Z}\times\mathbf{Z}_{120}$

Definition 9.21. *We say that $p : E \to B$ is a fibration with respect to B if whenever $f_0, f_1 : X \to B$ are homotopic via f_t, and there is a lift $g_0 : X \to E$, such that*

$$p \circ g_0 = f_0,$$

then there is a lift

$$g_t : X \to E$$

such that

$$p \circ g_1 = f_1$$

Remark 9.22. *$p : E \to B$ is a fibration if it is a fibration with respect to any X.*

Remark 9.23. *$p : E \to B$ is a Serre fibration if it has this property with respect to ball $X = D^k$ for any $k \geq 1$.*

Proposition 9.24. *Serra fibrations are fibrations with respect to any (X, A) where X is a CW complex and A is a subcomplex.*

Remark 9.25. *$F \to E \xrightarrow{p} B$ is a fiber bundle implies $p : E \to B$ is a Serre Fibration. If B is a CW complex, then $p : E \to B$ is a fibration.*

10 The Cube is a CW complex

Theorem 10.1. *One can represent a CW structure in multiple ways.*

Example 10.2. *A cube is a CW complex.*

Proof. The faces

$$I^n := \{(s_1, \dots, s_n) \in D^n : \forall s_i \in [0,1]\}$$

- $(n-1)$ faces serves as the facets
- *Claim:* There exists $2n$ $(n-1)$-faces.
  If we fix $i \in \{1, \dots, n\}$ and let either $s_i = 1$, then $F_i := \{(s_1, \dots, s_n) \in I^n : s_i = 1\}$ or $s_i = 0 \implies \tilde{F}_i = \{(s_1, \dots, s_n) \in I^n : s_i = 0\}$. This gives us exactly $\{F_1, \dots, F_n, \tilde{F}_1, \dots, \tilde{F}_n\}$ faces each congruent to I^{n-1}.
- Now, for $k = 0, \dots, n-1$ we choose $n-k$ coordinates $\mathcal{F} \subset \{1, \dots, n-1\}$ such that $\#\mathcal{F} = n-k$. Then for each $j \in \mathcal{F}$, we choose $\epsilon_j \in \{0,1\}$. Then

  $$F := \{(s_1, \dots, s_n) \in I^n : s_j = \epsilon_j \text{ for } j \in \mathcal{F}\}$$

  is a k-dimensional face congruent to I^k. In particular, there exist 2^n 0-faces which serve as the vertices.
- The CW structure of $I^n \implies X^k =$ the k cells are the interiors of the k-faces. For each k-face F, there exists a characteristic map

  $$\Phi : I^k \to F$$

  where

  $$\phi = \Phi|_{\partial I^k} : \partial I^l \to X^{k-1}$$

∎

Now we can consider

Definition 10.3. *Let $m \geq 2$. Then the mth <u>subdivision of a cube</u> I^n is*

$$I^n = \bigcup_{\alpha \in \mathcal{F}_{\min}} (x_\alpha + \frac{1}{m} I^n)$$

where $x_\alpha \in \frac{1}{m}\mathbb{Z}^n$, and $x_\alpha + \frac{1}{m} I^n \subset I^n$. This represents a CW-complex with m^n n-cells.

If we consider the relative topology, we see that

$$\phi : (I^n, I^{n-1}, \mathcal{F}_{n-1}) \to (X, A, x_0)$$

Here, then I^{n-1} front facets $= \{(s_1, \dots, s_n) \in I^n : s_n = 0\}$ and

$$\mathcal{F}_{n-1} = \text{ union of the other } 2^{n-1} \text{ facets } = \text{ closure of } \partial I^n \setminus I^{n-1}$$

Theorem 10.4. *Long exact sequences for relative homotopies $x_0 \in A \subset X$, then for $n \geq 2$, we see that*

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{i_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0)$$

with

$$\phi : (D^n, \partial D^n, x_0) \to (X, A, x_0)$$
$$\implies \partial_*[\phi] = [\phi|_{\partial D^n}]$$
$$\implies i : A \to X, \phi(D^n, \partial D^n) \to (X, x_0) \implies \phi \textit{ can be consider } (D^n, \partial D^n, x_0) \to (X, A, x_0)$$

Proof. Why? When need $Im\partial_* = Keri_*$ for $\pi_{n-1}(A, x_0)$. Then

$$\phi : (\partial D^n, x_0) \to (A, x_0)$$

Then $i_*([\phi]) = \{1\} \implies \exists$ homotopy $\phi_t : \partial D^n \to X$ to the constant map $\partial D^n \to x_0$. By lemma, $\exists \tilde{\phi} : D^n \to X$ such that $\tilde{\phi}|_{\partial D^n} = \phi$. By this, we see that $[\phi] = \partial_*([\tilde{\phi}])$. ∎

10.1 Fiber Bundles

Definition 10.5. *Given a sequence $F \to E \xrightarrow{p} B$ where $p : E \to B$ is surjective, and for any $b \in B$, there exists $b \in U \subset B$ an open neighborhood with $p^{-1}(U)$ is homeomorphic to $U \times F$.*

Example 10.6. *Let $E = B \times F$ be the trivial fiber bundle.*

Example 10.7. *In differential geometry, a typical bundle of consideration is the vector bundle.*

Lemma 10.8. *Let $A \subset \partial I^n$ be a subcomplex, and let $f_0, f_1 : I^n \to B$ be homotopic, and let*

$$G : (I^n \times \{1\}) \cup (A \times I) \to B \times F$$

Then there exists $\tilde{G} : I^n \times I \to B \times F$ such that $\tilde{G}|_{I^n \times I \cup A \times I} = G$.

Proof.

- Basis: When $n = 1$, we know that G can be extended to

$$I \times \{1\} \cup \{0\} \times I \cup \{1\} \cup I = \mathcal{F}_1$$

  Say if $0 \notin A$, then we define $\tilde{G}(0, t) = G(0, 1)$.

- Inductive Step: When $n \geq 1$, there exists a retraction

$$r : I^2 \to \mathcal{F}_1$$

  allowing us to define

$$\tilde{G}(s, t) = G(r(s, t))$$

Specifically, then $(n-1)$-facets of I_n, labelled $F_1, \ldots, F_{2n}$, each have $F_i = I^{n-1}$. We already know $\tilde{G}$ on F and on $(A \cap F_1) \times I$. So induction will extend $\tilde{G}$ to $F_1 \times I$. By induction on $j = 1, \ldots, 2n$, we also see that $\tilde{G}$ extends to $F_j \times I$. Therefroe, $\tilde{G}$ has been defined on $I_n \subset \partial I^{n+1}$ with

$$x \in I^n \times I \implies \tilde{G}(x) = \tilde{G}(r_{n+1}(x))$$

■