Definition

metric

Analysis

Definition

norm

Analysis

Definition

convex

Analysis

Let X be a set. A functions $d : X \times X \to \mathbb{R}$ is a metric on X if it satisfies the following conditions:

- $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then a norm on X, denoted by $\|\cdot\|$ if a function $\|\cdot\| : X \to [0, \infty)$ with the following properties

- $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$
- $\|\lambda v\| = |\lambda| \, \|v\|$ with $\lambda \in \mathbb{R}$ or $\mathbb{C}$.
- $\|v + w\| \leq \|v\| + \|w\|$

A subset A of a vector space X is convex if for all $x, y \in A$

$$tx + (1 - t)y \in A$$

for all $t \in [0, 1]$

Definition

sequence

Analysis

Definition

converges

Analysis

Definition

Cauchy

Analysis

A sequence in X is a function $X : \mathbb{N} \to X$, commonly denoted by $\{x_n\}_{n=1}^{\infty}$.

Let $\{x_n\}_{n=1}^{\infty}$ be sequence in X and $x \in X$. Then $\{x_n\}_{n=1}^{\infty}$ converges to x provided for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$d(x, x_n) < \epsilon$$

A sequence in a metric space X is a Cauchy if for every $\epsilon > 0$, there exists N such that

$$d(x_n, x_m) < \epsilon$$

for all $n, m > N$.

Definition

complete

Analysis

Definition

open ball

Analysis

Definition

closed ball

Analysis

A metric space (X, d) is called complete if every Cauchy sequence in (X, d) converges in (X, d).

For all $x \in X, r > 0$, we define the open ball

$$B_r(x) := \{y \in X : d(y, x) < r\}$$

For all $x \in X, r > 0$, we define the closed ball

$$\bar{B}_r(x) := \{y \in X : d(y, x) \leq r\}$$

Definition

open

Analysis

Definition

closed

Analysis

Definition

closure

Analysis

A set $U \subset X$ is open provided for every $x \in U$, we can identify a $r > 0$ such that

$$B_r(x) \subset U$$

A set $F \subset X$ is closed provided $F^c = X \setminus F$ is open.

For any $A \subset X$, the closure of A is the smallest closed subset of X that contains A, denoted by $\bar{A}$. Moreover,

$$\bar{A} := \bigcap_{F \text{ closed } \subset X; A \subset F} F$$

Equivalently,

$$\bar{A} := \{x \in X : \text{ there exists a sequence in } X \text{ that converges to } x\}$$

Definition

dense

Analysis

Definition

seperable

Analysis

Definition

continuous

Analysis

A set $A \subset X$ is <u>dense</u> if $\bar{A} = X$. In other words, for every point $x \in X$, there exists $\{a_n \in A\}_{n=1}^{\infty}$ such that
$$a_n \to x$$

A metric space (X, d) is <u>seperable</u> if it has a countable dense subset.

Let $x_0 \in X$ and $f : X \to Y$ be a function. Then f is <u>continuous</u> at x_0 if for all $\epsilon > 0$, $\exists \delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$$

That is,
$$f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$$

Further, f is continuous on a subset $E \subset X$ provided f is continuous at every point $x \in E$.

Definition

uniformly continuous

Analysis

Definition

upper semi-continuous / lower semi-continuous

Analysis

Definition

isometric embedding

Analysis

A function $f : X \to Y$ is uniformly continuous if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$d_Y(f(x), f(y)) < \epsilon$$

whenever $d_X(x, y) < \delta$.

A function $f : X \to Y$ is upper semi-continuous if for every sequence $\{x_n\} \to x$, we have that

$$\limsup_{n\to\infty} f(x_n) \leq f(x)$$

Similarly, $f : X \to Y$ is lower semi-continuous if for every sequence $\{x_n\} \to x$, we have that

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

Let (X, d) and $(\tilde{X}, \tilde{d})$ be metric spaces. Then an isometric embedding is a one-to-one map $i : X \to \tilde{X}$ such that for all $x, y \in X$

$$\tilde{d}(i(x), i(y)) = d(x, y)$$

Definition

isometric isomorphism

Analysis

Definition

completion of an isometric isomorphism

Analysis

Definition

sequentially compact

Analysis

An isometric isomorphism is a surjective isometric embedding. Moreover, Two Metric Spaces (X, d) and (Y, d') are isometric if there exists a bijection $h : X \to Y$ that preserves the metric.

$(\tilde{X}, \tilde{d})$ is the completion of (X, d) isometric isomorphism if

1. $\exists i : X \to \tilde{X}$ an isometric embedding.
2. $i(X)$ is dense in $\tilde{X}$
3. $\tilde{X}$ is complete

A subset K of a metric space is called sequentially compact if every sequence in K has a convergent subsequence whose limit belongs to K.

Definition

cover

Analysis

Definition

open cover

Analysis

Definition

subcover

Analysis

Let I be an index set. A collection $G = \{G_\alpha : \alpha \in I\}$ of subsets of X is called a cover of a subset $A \subset X$ if

$$A \subset \bigcup_{\alpha \in I} G_\alpha$$

If every G_α in the cover is open, we say that $\{G_\alpha\}$ is an open cover.

A subcover of $G = \{G_\alpha : \alpha \in I\}$ is a collection $\{G_\alpha : \alpha \in I_0\}$ where $I_0 \subset I$.

Definition

finite cover

Analysis

Definition

ϵ-*net*

Analysis

Definition

totally bounded

Analysis

If $G = \{G_\alpha : \alpha \in I\}$ where I is finite, then G is a finite cover.

For all $\epsilon > 0, A \subset X$. An ϵ-net for A is a subset of X of the form

$$\{x_\alpha : \alpha \in I\}$$

such that

$$\{B_\epsilon(x_\alpha) : \alpha \in I\}$$

$A \subset X$ is totally bounded if it has a finite ϵ-net cover for every $\epsilon > 0$. That is, A can be covered by finite many open ϵ-balls for any $\epsilon > 0$.

Definition

support of a function

Analysis

Definition

$C_0(X)$

Analysis

Definition

Bernstein Polynomials

Analysis

The support of a function $f : X \to \mathbb{R}$ is given by

$$supp(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

$$C_c(X) := \{f \in C(X) : supp(f) \text{ is compact}\} \subset C(X)$$

Now define:

$$C_0(X) := \overline{C_c(X)}$$

and equip this set with the $\|\cdot\|_\infty$. We see this set fits within the chain:

$$C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X) \cap B(X)$$

The Bernstein Polynomials. We define a basis of unity:

$$\mathcal{X}_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

Then by the binomial theorem:

$$1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

So for any $f \in C([0,1])$, the Bernstein Polynomial of f is

$$B_n(x; f) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \mathcal{X}_{n,k}(x)$$

Definition

Hausdorff

Analysis

Definition

equibounded

Analysis

Definition

equicontinuous

Analysis

A metric space X space is Hausdorff if for every $x \neq y$, then we can identify open subsets $U_x, V_y \subset X$ such that

$$x \in U_x, y \in V_y \text{ and } U_x \cap V_y = \emptyset$$

We then say A can separate points if we can identify an $a \in A$ "between" two points.

Let $\mathcal{F}$ be a family of functions

$$f : (X, d_X) \to (Y, d_Y)$$

between metric spaces. Then $\mathcal{F}$ is called equibounded provided for all $f \in \mathcal{F}$ there exists $g \in \mathcal{F}$ if

$$\|f\|_\infty \leq M + \|g\|_\infty$$

Let $\mathcal{F}$ be a family of functions

$$f : (X, d_X) \to (Y, d_Y)$$

between metric spaces. Then $\mathcal{F}$ is called equicontinuous at $x \in X$ if for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon, x) > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) \leq \epsilon$$

for all $f \in \mathcal{F}$.

Definition

dimension of a linear space

Analysis

Definition

Schauder Basis

Analysis

Definition

linear mapping

Analysis

The dimension of a linear space is equal to the number of elements that form a (linear) basis for the space.

Let X be a separable Banach space. A Schauder Basis is a sequence $\{x_n\}_{n=1}^{\infty}$ such that for all $x \in X$, there exists a unique sequence $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that

$$x = \sum_{i=1}^{\infty} c_n x_n$$

Let X, Y be linear spaces. Then $T : X \to Y$ is linear provided

$$T(ax + by) = aT(x) + bT(y)$$

for any $x, y \in X, a, b \in \mathbb{F}$.

DEFINITION

$\mathcal{L}(X, Y)$

ANALYSIS

DEFINITION

bounded

ANALYSIS

DEFINITION

$\mathcal{B}(X, Y)$

ANALYSIS

$$\mathcal{L}(X,Y) := \{T : X \to Y : T \text{ is linear } \}$$

Let X, Y be normed, linear spaces. A linear map $T : X \to Y$ is bounded if there exists $M \geq 0$ such that

$$\|Tx\|_Y \leq M \|x\|_X$$

$$\mathcal{B}(X,Y) := \{T : X \to Y : T \text{ is a bounded, linear map from } X \text{ to } Y\}$$

Definition

operator norm

Analysis

Definition

Relation "Stronger" in regards to Norms

Analysis

Definition

equivalent norms

Analysis

Given the space $B(X, Y)$ we can define the operator norm

$$\|T\| = \inf\{M : \forall x \in X,\ \|Tx\| \leq M \|x\|\}$$

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space X. We say $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ if

$$\|x_n\|_2 \to 0 \implies \|x_n\|_1 \to 0$$

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exists $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$$

for all $x \in X$.

DEFINITION

$Ker(T)$

ANALYSIS

DEFINITION

$Range(T)$

ANALYSIS

DEFINITION

well-posed problem

ANALYSIS

$$Ker(T) := \{x \in X : Tx = 0\}$$

It is a subspace of X, and also called the null space of T.

$$Range(T) := \{y \in Y : Tx = y \text{ for some } x \in X\}$$

It is a subspace of Y.

A problem is called <u>well-posed</u> if

- A solution exists
- The solution is unique
- The solution is "stable", i.e. the solution depends continuously on the data.

Definition

open mapping

Analysis

Definition

linear functional

Analysis

Definition

dual space

Analysis

$T : X \to Y$ is an open mapping if TU is open in Y whenever U is open in X.

Let X be a linear space. Then a linear map $f : X \to \mathbb{R}$ is called a linear functional on X.

When X is a normed linear space, its (topological) dual space is

$$X^* := B(X, R)$$

the space of all continuous linear functionals on X.

Definition

bidual

Analysis

Definition

reflexive

Analysis

Definition

weak topology

Analysis

Since X^* is a Banach space, we can consider its dual space

$$(X^*)^* = X^{**}$$

called the <u>bidual</u> of X.

If $X = X^{**}$, then we say that X is <u>reflexive</u>

Let X be a normed, linear space. The <u>weak topology</u> on X is defined as the weakest topology which makes all of the functionals $\phi \in X^*$ continuous. Let $\Phi := \{\phi^{-1}(U) : U \subset \mathbb{R} \text{ open }, \phi \in X^*\}$. Since

$$\bigcup \phi^{-1}(U) = X$$

there exists a unique topology $\mathcal{T}_{\text{weak}}$ on X having Φ as its sub-base. Moreover,

$$\mathcal{T}_{\text{weak}} = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \phi^{-1}(U)$$

DEFINITION

strongly / weakly converging

ANALYSIS

DEFINITION

Weak- Convergence*

ANALYSIS

DEFINITION

inner product space

ANALYSIS

Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence in X^* and $\phi \in X^*$.

1. Strong Convergence: $\phi_n \to \phi$ strongly in X^* provided $\|\phi_n - \phi\|_{X^*} \to 0$
2. Weak Convergence: $\phi_n \rightharpoonup \phi$ weakly in X^* means for all $F \in X^{**}, F(\phi_n) \to F(\phi)$.

Weak-* Convergence $\phi_n \overset{*}{\rightharpoonup} \phi$ in X if $\phi_n(x) \to \phi(x)$ for all $x \in X$.

An inner product space on a complex linear space X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that for all $x, y, z \in X$, and $\lambda, \mu \in \mathbb{C}$:

1. $\langle x, \lambda y + \mu y \rangle = \lambda \langle x, y \rangle + \mu \langle x, y \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Hermitian symmetric)
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

A linear space with an inner product is called an inner product space or a pre-Hilbert space.

Definition

Hilbert Space

Analysis

Definition

orthogonal

Analysis

Definition

orthogonal complement

Analysis

A complete inner product space is called a Hilbert Space. So every Hilbert Space is a Banach Space. A common notation for inner products in Hilbert Space $\langle \cdot, \cdot \rangle$ whereas an inner product space only has $(\cdot, \cdot)$.

If $(x, y) = 0$, then x and y are orthogonal, denoted $x \perp y$. Suppose $M \subset X$. If $x \perp y$ for all $y \in M$, then we say that x is orthogonal to M, and write $x \perp M$.

If M is a subset of an inner product space X, then we can define the orthogonal complement of M by

$$M^{\perp} := \{x \in X : x \perp M\}$$

Definition

orthogonal projection

Analysis

Definition

orthonormal set

Analysis

Definition

isomorphic between inner product spaces

Analysis

A projection $P : \mathcal{H} \to \mathcal{H}$ is called orthogonal if for all $x, y \in \mathcal{H}$,

$$\langle x, Py \rangle = \langle Px, y \rangle$$

Let X be an inner product space. A subset $S = \{e_\alpha : \alpha \in A\} \subset X$ is called orthonormal if it satisfies

1. (Orthogonality) $e_\alpha \perp e_\beta$ for all $\alpha, \beta \in A, \alpha \neq \beta$.
2. (Normalized) $\|e_\alpha\| = 1$ for all $\alpha \in A$

In other words, $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$

Let $(X_1, (\cdot,\cdot)_1)$ and $(X_2, (\cdot,\cdot)_2)$ be two inner product spaces. If there exists an isomorphism $T : X_1 \to X_2$ such that for all $x, y \in X$:

$$(Tx, Ty)_2 = (x, y)_1$$

Then we say that the inner product space X_1, X_2 are isomorphic.

Definition

σ-algebra

Analysis

Definition

algebra

Analysis

Definition

measureable space

Analysis

A <u>σ-algebra</u> on a set X is a collection $\mathscr{A}$ of subets of X such that

1. $\emptyset \in \Sigma$
2. If $A \in \Sigma$, then $A^c \in \Sigma$
3. If $A_1, A_2, \ldots,$ is a countable family of sets in Σ, then

$$\bigcup_{i=1}^{\infty} A_i \in \Sigma$$

An <u>algebra</u> on a set X is a collection $\mathscr{A}$ of subets of X such that

1. $\emptyset \in \Sigma$
2. If $A \in \Sigma$, then $A^c \in \Sigma$
3.

$$\bigcup_{i=1}^{N} A_i \in \Sigma$$

A <u>measureable space</u> $(X, \mathscr{A})$ is a set X and a σ-algebra Σ on X. Elements of Σ are called <u>measurable sets</u>.

Definition

measure

Analysis

Definition

measure space

Analysis

Definition

measure

Analysis

A measure μ on X is a function

$$\mu : \Sigma \to [0, \infty]$$

such that

1. $\mu(\emptyset) = 0$
2. μ is countably additive:

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

   for each countable family of mutually disjoint sets $\{E_n\}_{n=1}^{\infty} \subset \Sigma$.

A measure space $(X, \mathscr{A}, \mu)$ is a set X, a σ-algebra $\mathscr{A}$ on X, and a measure $\mu : \mathscr{A} \to [0, \infty]$.

A measure μ is said to be

- finite if $\mu(X) < \infty$
- σ-finite if there exists a sequence $\{A_i\} \subset \mathscr{A}$ such that $X = \bigcup_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$ for all n.
- probability measure if $\mu(X) = 1$.

Definition

Borel measure

Analysis

Definition

measure zero

Analysis

Definition

complete measure space

Analysis

Given $X, \mathscr{A} = \mathscr{B}(X)$ then Borel σ-algebra. Then any measure μ on $(X, \mathscr{B}(X))$ is called a Borel measure.

Let $(X, \mathscr{A}, \mu)$ be a measure space and $A \subset X$ is said to have measure zero if it is measurable and $\mu(A) = 0$.

A measure space $(X, \mathscr{A}, \mu)$ is complete if every subset of a measure zero set is measurable.

Definition

$$\overline{\mathscr{A}}$$

Analysis

Definition

almost everywhere property

Analysis

Definition

measurable function

Analysis

Given a measure space $(X, \mathscr{A}, \mu)$, we define $\overline{\mathscr{A}}$ to be the σ-algebra generated by

$$\mathscr{A} \bigcup \{\text{subsets of measure zero sets}\}$$

Then

$$\overline{\mathscr{A}} = \{A : \exists E, F \in \mathscr{A} \text{ such that } E \subset A \subset F, \mu(F \setminus F) = 0\}$$

So for all $A \in \bar{\mathscr{A}}$, define

$$\overline{\mu(A)} := \mu(E) = \mu(F)$$

Then the complete measure space $(X, \overline{\mathscr{A}}, \overline{\mu})$ is called the <u>completion</u> of $(X, \mathscr{A}, \mu)$.

A property that hold except on a set of measure zero is said to hold <u>almost everywhere</u> or <u>a.e.</u> for short. If there measure is obvious, then explicitly write $\underline{\mu - a.e.}$.

Let $(X, \mathscr{A})$ be a measurable space. A real-valued function $f : X \to \mathbb{R}$ is a measurable function (with respect to $\mathscr{A}$) if for all $t \in \mathbb{R}$,

$$f^{-1}((t, \infty)) = \{x \in X : f(x) > t\}$$

is measurable.

Definition

$(\mathscr{A}_X, \mathscr{A}_Y)$-measurable

Analysis

Definition

product measure

Analysis

Definition

L^p space

Analysis

A function $f : (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ between two measurable spaces is called $(\mathscr{A}_X, \mathscr{A}_Y)$-measurable if $f^{-1}(\mathscr{A}_Y) \subset \mathscr{A}_X$.

Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two σ-finite measure spaces. For any $E \in \mathscr{A} \times \mathscr{B}$, define

$$(\mu \times \nu)(E) := \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E_y) d\nu(y)$$

$\mu \times \nu$ is a measure on $\mathscr{A} \times \mathscr{B}$, called the product measure.

Let $1 \le p < \infty$ and let $(X, \mathscr{A}, \mu)$ be a measure space. Then

$$\|f\|_{L^p(X)} := \left[\int_X |f|^p \; d\mu\right]^{\frac{1}{p}}$$

Then the L^p space is defined by:

$$L^p(X) := \{f : X \to [-\infty, \infty] : f\mathscr{A} - \text{measurable with } \|f\|_{L^p(X)} < \infty\}$$

Further, $L^p(X)$ consists of equivalence classes. That is, given $f, g \in L^p(X)$, then

$$f \equiv g \iff f = g a.e.$$

DEFINITION

$L^\infty(X)$

ANALYSIS

DEFINITION

$L^{p,\text{weak}}(X)$

ANALYSIS

DEFINITION

distribution function

ANALYSIS

$$L^\infty(X) := \{f : X \to [-\infty, \infty] : f\mathscr{A} - \text{measurable with } \|f\|_{L^\infty(X)} < \infty\}$$

where

$$\|f\|_{L^\infty(X)} := \inf\{M : \mu(\{x \in X : |f| > M\}) = 0\}$$

Given $(X, \mathscr{A}, \mu)$ a measure space with $1 \leq p < \infty$. We define the space

$$L^{p,\text{weak}}(X)$$

as the set of functions $f \in L^{p,\text{weak}}(X)$ provided $f : X \to \mathbb{R}$ is a $\mathscr{A}$-measurable function and

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{C^p}{\lambda^p}$$

for all $\lambda > 0$ and some choice constant C_λ.

The <u>distribution function</u> $m_\mu(\lambda) : \mathbb{R}^+ \to [0, \infty]$ by

$$m_\mu(\lambda) := \mu(\{x \in X : |f(x)| > \lambda\})$$

Definition

$$\|f\|_{L^{p,\text{weak}}(X)}$$

Analysis

Definition

simple function

Analysis

Definition

step function

Analysis

$$\|f\|_{L^{p,\text{weak}}(X)} = \sup_{\lambda>0} \left\{ \lambda m_\mu(\lambda)^{\frac{1}{p}} \right\}$$

A function of the form

$$\phi(x) = \sum_{i=1}^{N} a_i \mathcal{X}_{A_i}(x)$$

with $A_i \subset X$ measurable with $\mu(A_i) < \infty, |a_i| < \infty$ is a simple function.

On the Lebesgue measure space $(X, \mathscr{B}(X), \lambda)$, we define the step function of the form

$$\phi(x) = \sum_{i=1}^{N} a_i \mathcal{X}_{Q_i}(x)$$

where Q_i are pairwise disjoint cubes of the form:

$$Q_i := \prod_{j=1}^{n} [\alpha_j, \alpha_j + \ell)$$

with $\alpha_i, \ell \in \mathbb{R}^+$ and $|a_i| < \infty$.

DEFINITION

$C_c(\mathbb{R}^n)$

ANALYSIS

DEFINITION

bounded linear functional

ANALYSIS

DEFINITION

absolutely continuous

ANALYSIS

$C_c(\mathbb{R}^n)$ = the space of continuous functions of compact support.

A bounded linear functional on $L^p(X)$ is a mapping $\ell : L^p(X) \to \mathbb{R}$ with

$$\ell(af + bg) = a\ell(f) + b\ell(g)$$

and

$$|\ell(f)| \leq C \|f\|_{L^p}$$

for some constant C. Moreover, we define

$$\|\ell\| := \inf\{C : |\ell(f)| \leq C \|f\|_{L^p} \, \forall f \in L^p\}$$

Suppose we have a measurable space $(X, \mathscr{A})$ equipped with two finite measures μ, ν. ν is absolutely continuous with respect to μ, denoted $\nu \ll \mu$, provided

$$\mu(B) = 0 \implies \nu(B) = 0$$

Definition

signed measure

Analysis

Definition

Hardy-Littlewood Maximal Function

Analysis

Definition

$f * g$

Analysis

A function ν on a measurable space $(X, \mathscr{A})$ is a signed measure provided:

1. For any $E \in \mathscr{A}, \nu(E) \in [-\infty, \infty]$
2. $\nu(\emptyset) = 0$
3. ν is countably additive.

Let $f \in L^1_{loc}(\mathbb{R}^n)$ (i.e.

$$\int_K |f| d\mu < \infty$$

for any compact set $K \subset \mathbb{R}^n$), then we can define the Hardy-Littlewoord Maximal Function by

$$\mu f(x) := \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)| \ dy$$

If $f, g : \mathbb{R}^n \to \mathbb{R}$, then

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y) \ dy$$

Definition

approximation identity

Analysis

Definition

$W^{k,p}(\Omega)$

Analysis

Definition

C^1*-boundary*

Analysis

Suppose $K \in L^1$ satisfies the following properties:

1. $\lim_{r \to 0} \int_{|x|>\delta} |K_r(x)| \, dx = 0$
2. $\int K(x) \, dx = 1$

Then K is called an approximation identity.

Given $\Omega \subset \mathbb{R}^n$ open, with $1 \leq p \leq \infty$ and $K \in \mathbb{Z}_{\geq 0}$, then

$W^{k,p}(\Omega) := \{u : \Omega \to \mathbb{R}^n : D^\alpha u \text{ exists } \forall |\alpha| \leq k \text{ and } D^\alpha u \in L^p(\mathbb{R}^n) \, \forall |\alpha| \leq k\}$

Further, we equip this space with the norm:

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}$$

Given a region Ω, a C^1-boundary $\partial\Omega$ is one such that for every $x_0 \in \partial\Omega$, there exists $r > 0$ and an C^1-embedding-function

$$\gamma : \mathbb{R}^{n-1} \to \mathbb{R}^n$$

such that

$$B_r(x_0) \cap \Omega = \{(x_1, \ldots, x_n) \in B_r(x_0) : x_n > \gamma(x_1, \ldots, x_{n-1})\}$$

Definition

boundary value of $u \in W^{1,p}(\Omega)$

Analysis

Definition

Hölder α*-continuous*

Analysis

Definition

$C^{0,\alpha}(\Omega)$

Analysis

The boundary value of $u \in W^{1,p}(\Omega)$ is defined as

$$u|_{\partial\Omega} := \lim_{\epsilon \downarrow 0} Eu * \phi_\epsilon|_{\partial\Omega}$$

where ϕ_ϵ is a standard mollifier.

A function $f : \Omega \to \mathbb{R}$ is Hölder α-continuous provided there exists a constant C such that

$$|f(x) = f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \Omega$.

The space $C^{0,\alpha}(\Omega) = C^\alpha(\Omega) =$ is the space of Hölder α-continuous functions equipped with the finite norm

$$\|f\|_{C^\alpha(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

Definition

Fourier Transform

Analysis

Definition

$H^s(\mathbb{R}^n)$

Analysis

Definition

function

Analysis

Suppose $f \in L^1(\mathbb{R}^n)$. we define the Fourier Transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} \, dx$$

Given $s \in \mathbb{R}$, define the space

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : (\sqrt{1+|\xi|^2})^s \hat{f}(\xi) \in L^2(\mathbb{R})\}$$

- $s > 0 \implies$ fractional differentiation.
- $s < 0 \implies$ fractional integration.

We equip this space with the norm:

$$\|f\|_{H^s(\mathbb{R}^n)} := \left\| (\sqrt{1+|\xi|^2})^s \hat{f}(\xi) \right\|_{L^2(\mathbb{R}^n)}$$

For $0 < \alpha < n$, we define the generalized Riesz Potential

$$(I_\alpha f)(x) := \int_{\mathbb{R}^n} \frac{f(x)}{|x-y|^{n-\alpha}} dy$$

DEFINITION

Space of Schwartz Functions

ANALYSIS

DEFINITION

Weak Convergence / Strong Convergence in $(X, \|\cdot\|)$

ANALYSIS

DEFINITION

spectral radius

ANALYSIS

The Space of Schwartz Functions, denoted $S(\mathbb{R}^n)$, is defined as:

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) : \|u\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u| < \infty \ \forall \alpha, \beta \in \mathbb{Z}^n_{\geq 0}\}$$

where we equip this space with a semi-norm:

$$[f]_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u|$$

Let $(X, \|\cdot\|)$ be a normed linear space. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X. We say

1. x_n converges strongly to x if
$$\|x_n - x\| \to 0$$
denoted $x_n \to x$
2. x_n converges weakly to x if
$$\phi(x_n) \to \phi(x)$$
for all $\phi \in X^*$. Denoted $x_n \rightharpoonup x$.

For any $A \in B(X)$, the number $r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$ is called the spectral radius of A.

Definition

resolvent

Analysis

Definition

spectrum

Analysis

Definition

convex combination

Analysis

Let X be a Banach space and $A \in B(X)$. Then

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$$

For all $\lambda \in \rho(A)$, let

$$R_\lambda = (\lambda I - A)^{-1}$$

be the resolvent of A at λ.

The spectrum of A is the set

$$\sigma(A) := \mathbb{C} \setminus \rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$$

Given $\{x_1, x_2, \dots, x_n\}$ of a linear space X, we define a the set

$$C = \left\{ y = \sum_{k=1}^{n} \lambda_k x_k : \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \right\}$$

each y is called a convex combination of $\{x_1, x_2, \dots, x_n\}$.

Definition

weak convergence in a Hilbert Space

Analysis

Definition

Isomorphic Hilbert Spaces

Analysis

Definition

self-adjoint

Analysis

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$ if

$$\langle x_n, y \rangle \to \langle x, y \rangle$$

for all $y \in \mathcal{H}$.

Two Hilbert spaces are isomorphic if there is a unitary operator between them.

$A \in B(\mathcal{H}, \mathcal{H})$ is self-adjoint if $A = A^*$. That is,

$$\langle x, Ay \rangle = \langle Ax, y \rangle$$

for all $x, y \in \mathcal{H}$

Definition

adjoint operator

Analysis

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. There exists a unique bounded linear operator $A^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle x, Ay \rangle = \langle A^* x, y \rangle$$

for all $x, y \in \mathcal{H}$. A^* is called the adjoint of the operator.