# MAT 207X: Applied Mathematics Prelim Study Guide

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These are meant to be a very quick summary of the content from the 207 series courses. It is certainly in your best interest to do practice problems. This should serve as an aid to remind yourself of the techniques necessary to complete the exam.

## 1 Scalar Autonomous Ordinary Differential Equations

Breaking down the title of this section, we see:

- Scalar $\equiv$ 1-Dimensional
- Autonomous $\equiv$ No explicit "time" / independent variable dependence
- Ordinary $\equiv$ 1 independent variable

In other words, we'll analyze problems of the form:

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(t_0) = x_0 \end{cases}$$

**Question 1.1.** *How do we go about solving such a problem?*

### 1.1 How to Solve SA-ODE's

1. We can solve analytically by finding a closed-form solution
2. Solve graphically using direction fields
3. Use a phase-portrait to draw valuable conclusions about the problem without solving.

Obviously, 207A is mostly concerned with the third approach!

### 1.2 Existence and Uniqueness

**Question 1.2.** *How do we know if an initial-value problem of the form:*

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(t_0) = x_0 \end{cases}$$

*has a solution? Moreover, is that solution unique?*

**Example 1.3.** $f(x) = 2x \implies$ *easy to check this has exactly one solution.*

**Example 1.4.** *Consider* $\begin{cases} f(x) = x^2 \\ x(0) = 1 \end{cases}$. *Solving the equation, we get*

$$x(t) = \frac{1}{1-t}$$

*This has finite time blow up! This solution doesn't exist for all time $t$.*

**Example 1.5.** *Consider* $\begin{cases} f(x) = \sqrt{x} \\ x(0) = 0 \end{cases}$. *Though this can be solved analytically, graphically we notice this in fact has infinite solutions:*

$$x(t) = \begin{cases} 0 & t < t_0 \\ \frac{(t-t_0)^2}{4} & t \geq t_0 \end{cases}$$

*for any $t_0 \in (0, \infty)$.*

Physically, this is problem does exist. If you consider a leaky bucket problem, with water draining, eventually the water drains completely. However, there's no way to take the final state of the bucket, which is now empty, and reversibly solve for the height of the water originally contained. In this case, we say the solution lacks **reversibility.**

So how do we correct this?

**Definition 1.6.** *A function $f(x)$ is* <u>*locally Lipschitz continuous*</u> *if for any $R$, there exists a constant $M$ such that*

$$|f(x) - f(y)| \leq M|x - y|$$

*for $|x| < R, |y| < R$.*

**Theorem 1.7** (Local Existence and Uniqueness)**.** *If $f(x)$ is locally Lipschitz continuous on $\mathbb{R}$, then*

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(t_0) = x_0 \end{cases}$$

*has a unique solution on $\delta_- < t < \delta_+$ for some $\delta_- < 0 < \delta_+$.*

Returning to the first and last case, we see:

- $f(x) = 2x \implies f \in C^1$ and therefore has a unique solution
- $f(x) = \sqrt{x} \implies$ cusp at $x = 0$ so not guaranteed

On the other hand, to consider the second case, we need another theorem.

**Theorem 1.8** (Extension to Existence Theorem)**.** *If $f(x)$ is locally Lipschitz continuous on $\mathbb{R}$, then if there exists a unique solution on an interval*

$$t_0 - \delta_- < t < t_0 + \delta_+$$

*then there must exist a unique solution $x(t)$ on a "maximal interval"*

$$T_- < t < T_+$$

*where $-\infty \leq T_- < t_0 - \delta_-$ and $t_0 + \delta_+ < T_+ \leq \infty$. Futher, if $T_- > -\infty$, then $|x(t)| \to \infty$ as $t \to T_-$. Similarly, if $T_+ < \infty$, then then $|x(t)| \to \infty$ as $t \to T_+$.*

For case 2, we see we in fact have finite-time blow up.

## 1.3 Stability of Steady States

**Definition 1.9.** *Given the initial-value problem:*

$$\begin{cases} \frac{dx}{dt} = f(x) \\ x(t_0) = x_0 \end{cases}$$

*$x^*$ is a steady state if $f(x^*) = 0$.*

**Definition 1.10.** *A steady state $x^*$ is stable if $x(t)$ starts near $x^*$, then it stays near $x^*$. Further, if $x^*$ is not stable, then we say $x^*$ is unstable.*

**Definition 1.11.** *A steady state $x^*$ is asymptotically stable if the solution*

$$|\phi(t; x_0)| \to x^*$$

*as $t \to \infty$ for some $\delta$ such that $|x_0 - x^*| < \delta$.*

**Remark 1.12.** *Though these seem like identical definitions, there is a small battery of examples that show these are different definitions.*

Now, if we consider linearization of $f(x)$, around $x^*$, we see:

$$\dot{x} = f(x) \approx \underbrace{f(x^*)}_{=0} + f'(x)(x - x^*) + O((x - x^*)^2) = f'(x)(x - x^*) + O((x - x^*)^2)$$

Therefore, solving just the lower ordered term equation:

$$\dot{x} = f'(x)(x - x^*)$$

We get:

$$x(t) \approx x^* + (x - x^*) \exp(f'(x^*)t)$$

From this local approximation, we can draw several conclusions:

1. If $f(x^*) < 0$, then $x(t) \to x^* \implies x^*$ is stable!
2. If $f(x^*) > 0$, then $x(t)$ diverges from $x^*$
3. If $f(x^*) = 0$, then we can't draw any conclusions about the behavior purely from the first order linearization.

**Definition 1.13.** *When $f'(x^*) \neq 0$, then the steady state $x^*$ is a hyperbolic steady state. Otherwise, $x^*$ is called degenerate.*

## 1.4 Bifurcation Theory

We seek to analyze first order autonomous ODEs of the form:

$$\dot{x} = f(x, \mu)$$

where $f \in C^\infty$ in both $x$ and $\mu$.
In order for a Bifurcation of fixed points to occur, the following conditions must be met:

$$\begin{cases} f(x^*, \mu_c) = 0 \\ f_x(x^*, \mu_c) = 0 \end{cases}$$

That is, we need the steady state to not be hyperbolic in order for the perturbation $\mu$ to create physical change.

3 Types of Bifurcations:

1. Saddle-Node Bifurcations

   **Example 1.14.** $\dot{x} = \mu - x^2$

2. Transcritical Bifurcation

   **Example 1.15.** $\dot{x} = \mu x - x^2$

3. Pitchfork Bifurcation

   **Example 1.16.** $\dot{x} = -x^3 + \mu x$

   Further pitchfork bifurcations split into two subcases: Supercritical and Subcritical.

Using the implicit function theorem, and the converse, we can establish sufficient, though not entirely necessary conditions for these bifurcations. Most problems will satisfy these conditions so they're pretty handy if you don't feel like drawing a shit ton of graphs.

Sufficient Conditions for Saddle-Node Bifurcation:

$$\begin{cases} f(x^*, \mu_c) = 0 & \text{Steady State} \\ f_x(x^*, \mu_c) = 0 & \text{Degeneracy} \\ f_{xx}(x^*, \mu_c) \neq 0 & \text{Concavity} \\ f_\mu(x^*, \mu_c) \neq 0 & \text{Transversality} \end{cases}$$

Sufficient Conditions for Transcritical Bifurcation:

$$\begin{cases} f(x^*, \mu_c) = 0 & \text{Steady State} \\ f_x(x^*, \mu_c) = 0 & \text{Degeneracy} \\ f_{xx}(x^*, \mu_c) \neq 0 & \text{Concavity} \\ f_\mu(x^*, \mu_c) = 0 & \\ f_{x\mu}(x^*, \mu_c) \neq 0 & \text{Necessary Condition} \end{cases}$$

Sufficient Conditions for Pitchford Bifurcation:

$$\begin{cases} f(x^*, \mu_c) = 0 & \text{Steady State} \\ f_x(x^*, \mu_c) = 0 & \text{Degeneracy} \\ f_\mu(x^*, \mu_c) = 0 & \\ f_{xx}(x^*, \mu_c) = 0 & \\ f_{x\mu}(x^*, \mu_c) \neq 0 & \\ f_{xxx}(x^*, \mu_c) \neq 0 & \end{cases}$$

## 2 Autonomous Linear Systems

**Definition 2.1.** *An autonomous linear system is one of the form:*

$$\dot{x} = A\vec{x}$$

*where $A$ is a matrix.*

## 2.1 Existence of Solution

**Theorem 2.2.** *Solutions of autonomous linear systems are defined for all $t \in \mathbb{R}$*

*Proof.*

$$\begin{aligned}\frac{d}{dt}\|x(t)\|^2 &= \frac{d}{dt}\left(x_1^2(t)+\ldots+x_n^2(t)\right)\\ &= 2\left(x_1(t)x_1'(t)+\ldots+x_n(t)x_n'(t)\right)\\ &= 2(x^TAx)\\ &\leq 2\|x\|\|A\|\|x\| = 2\|A\|\|x\|^2\end{aligned}$$

Therefore,

$$\|x(t)\|^2 \leq e^{\|A\|t^2} \implies \|x(t)\| \leq e^{\|A\|t}$$

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## 2.2 Reduction to Canonical Form

**Theorem 2.3.** *Let $A$ be a $2\times 2$ matrix with real entries. Then there exists a real, invertible matrix $P$ such that*

$$PAP^{-1} = J$$

*where*

$$J = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}, \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}, \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$$

*with $\lambda_1, \lambda_2, \lambda, \alpha, \beta \in \mathbb{R}$.*

*Proof.*

- *Case:* Suppose $A$ has real, distinct eigenvalues $\lambda_1, \lambda_2$
  Then if we let $v_1, v_2$ be the corresponding eigenvectors:

  $$P = \begin{pmatrix}v_1 & v_2\end{pmatrix} \quad \text{with} \quad J = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

- *Case:* Suppose $A$ a double eigenvalue $\lambda$ with two eigenvectors:

  $$P = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{with} \quad J = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

- *Case:* Suppose $A$ a double eigenvalue $\lambda$ with only one eigenvector $v$.
  In this case, we need to solve for the second vector.

  $$(A-\lambda I)v_2 = v_1 = v$$

  Then

  $$P = \begin{pmatrix}v_1 & v_2\end{pmatrix} \quad \text{with} \quad J = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

- *Case:* Suppose $A$ has two complex eigenvalues $\lambda = \alpha \pm \beta i$.
  Letting $\vec{w} = v_1 \pm i\vec{v}_2$, we then write

  $$P = \begin{pmatrix}v_1 & v_2\end{pmatrix} \quad \text{with} \quad J = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$$

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## 2.3 The Fundamental Solution

Given $\dot{x} = Ax$, we convert to new coordinates $\dot{y} = P^{-1}APy = Jy$, and then solve the system of independent equations to get:

$$x(t) = P^{-1}e^{Jt}Px_0$$

Exponentials of the Canonical Systems

- $$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies e^J = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

- $$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \implies e^J = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

- $$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \implies e^J = e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

## 2.4 Linear Stability Analysis

Given a system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

we can linearize the system around a fixed point $(\bar{x}, \bar{y})$. Applying a change of variables:

$$u = x - \bar{x} \qquad v = y - \bar{y}$$

$$\implies \begin{cases} \dot{u} = f(\bar{x}+u, \bar{y}+v) = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \\ \dot{v} = g(\bar{x}+u, \bar{y}+v) = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv) \end{cases}$$

$$\implies \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}}_{\text{Jacobian}} \begin{pmatrix} u \\ v \end{pmatrix}$$

Therefore, we can perform linear stability analysis on Hyperbolic Fixed Points:

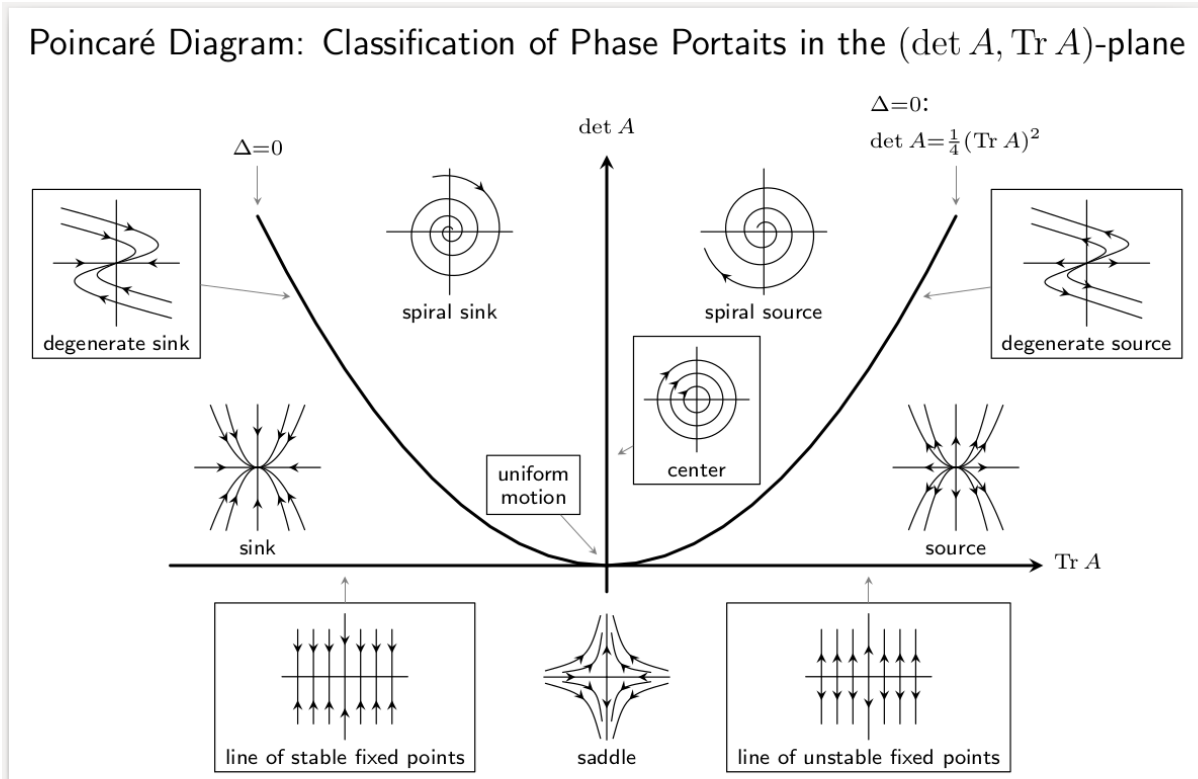
**Algorithm 1:** Linear Stability Analysis

**if** *Eigenvalues are Real* **then**
- **if** *Eigenvalues are Distinct* **then**
  - both negative $\implies$ stable node;
  - both positive $\implies$ unstable node;
  - opposite signs $\implies$ saddle point;
- **end**
- **if** *Equal* **then**
  - **if** $J = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ *where* $c \neq 0$ **then**
    - star node;
  - **end**
  - **if** $J \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ **then**
    - negative $\lambda \implies$ stable degenerate node ;
    - positive $\lambda \implies$ unstable degenerate node ;
  - **end**
- **end**

**else**
- $\implies \lambda = \alpha \pm i\beta$;
- **if** $Re[\lambda] = \alpha = 0$ **then**
  - Center ;
- **else if** $Re[\lambda] = \alpha < 0$ **then**
  - Stable Spiral ;
- **else**
  - Unstable Spiral ;
- **end**

**end**

This classification is identical to the separated regions:




Problems with Linear Stability Analysis
It breaks down when we encounter cases where $Re[\lambda] = 0$. Therefore, we should be skeptical when the analysis yields:

- Centers
- Fixed points with at least one eigenvalue zero.

# 3 Autonomous Nonlinear Systems

We wish to study systems, not necessarily linear, of the form:

$$\begin{cases} \dot{x} = F(x) \\ x \in \mathbb{R}^2, F : \mathbb{R}^2 \to \mathbb{R}^2, F \in C^1. \end{cases}$$

## 3.1 Conservative Systems

A conservative system is characterized by the existence of an invariant function $E(x, y)$ such that

$$E(x(t)) \equiv \text{ constant}$$

along a flow linear solution $x(t)$. Further, trajectories are the level sets of $E(x, y)$. Therefore, we can solve these differential equations by:

1. Find $E(x, y)$
2. Plot the level sets
3. Determine Flow Direction

**Example 3.1.** $\ddot{x} = f(x) \implies \begin{cases} \dot{x} = y \\ \dot{y} = f(x) \end{cases}$ *We can find the first integral:*

$$\int \frac{d^2x}{dt^2}\frac{dx}{dt}dt = \int f(x)\frac{dx}{dt}dt + C$$

$$\implies \frac{1}{2}\int \frac{d}{dt}\left(\left(\frac{dx}{dt}\right)^2\right)dt = \int f(x)\frac{dx}{dt}dt + C$$

$$\implies \frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \int f(x)\frac{dx}{dt}dt + C$$

$$\implies \underbrace{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \int f(x)\frac{dx}{dt}dt}_{\textit{Conserved Quantity}} = C$$

*Define*

$$E(x,y) = \frac{1}{2}y^2 - \int f(x)\ dx = \frac{1}{2}y^2 + \underbrace{V(x)}_{\textit{potential function}}$$

**Definition 3.2.** *Given $\dot{x} = F(x)$, a potential function is a continuously differentiable function $f : D \to \mathbb{R}$ such that for any solution $x(t)$,*

$$f(x(t)) \equiv \ \textit{constant}$$

We can usually obtain this function by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \implies \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

and then integrate to get a potential function.

**Definition 3.3.** *A system is conservative provided that system has a potential function on the whole plane.*

**Example 3.4.**

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

*Attempt to obtain potential function :*

$$\frac{dy}{dx} = \frac{x}{-y} \implies -ydy = xdx \implies x^2 + y^2 = f(x,y)$$

*Since this is defined for all choices $x, y$, this system is conservative.*

**Example 3.5.**

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$$

*Attempt to obtain potential function :*

$$\frac{dy}{dx} = \frac{y}{x} \implies \frac{dy}{y} = \frac{dx}{x} \implies y = Cx \implies f(x,y) = \frac{y}{x}$$

*Notice this is not defined for $x = 0$ so this system is not conservative.*

**Definition 3.6.** *Given $(\dot{x}, \dot{y})$, a system is* <u>*Hamiltonian*</u> *provided there exists an $H$ such that*

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial y} \\ \frac{dy}{dt} = \frac{-\partial H}{\partial x} \end{cases}$$

**Example 3.7.**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases}$$

$$\frac{\partial H}{\partial y} = y \implies H = \frac{y^2}{2} + C(x)$$

$$\frac{\partial H}{\partial x} = x - x^3 \implies H = \frac{x^2}{2} - \frac{x^4}{4} + C(y)$$

*Reconciling the two, we get:*

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}$$

*We could also choose*

$$H(x, y) = y^2 + x^2 - \frac{x^4}{2}$$

Now recall, centers are an issue to detect using linear stability analysis

**Theorem 3.8.** *Nonlinear centers for conservative systems Consider*

$$\dot{x} = f(x) \in C^1(\mathbb{R}^2)$$

*Suppose $f$ is conservative, and $\bar{x}$ is an isolated fixed point (not the limit of a sequence of fixed points). Then if $\bar{x}$ is a local minimum of a potential function $E$, then all trajectories sufficiently closed to $x$ are closed.*

In other words, linear stability analysis on conservative systems works!

**Note 3.9.** *Conservative systems cannot have attracting fixed points. In fact, they cannot have stable or unstable nodes or spirals. Therefore, centers and saddles are all that are possible.*

## 3.2 Linear Stability Analysis on Nonlinear Systems

Let $\dot{x} = f(x)$ be a planar autonomous system.

**Definition 3.10.** *A* <u>*stable equilibrium*</u> *$\bar{x}$ is a point such that for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\left\|x^0 - \bar{x}\right\| < \delta \implies \left\|\phi(t, x^0) - \bar{x}\right\| < \epsilon$$

*where $\phi(t, x^0)$ is a solution going through $x^0$. This also is sometimes called Lyapunov stable. Notice, the solution needs to be "around" $\bar{x}$, but doesn't actually have to go to $\bar{x}$. So this definition includes centers.*

**Definition 3.11.** *A* <u>*unstable equilibrium*</u> *$\bar{x}$ is a point such there exists an $\eta > 0$, such that for all $\delta > 0$, there is an $x^0$ with*

$$\left\|x^0 - \bar{x}\right\| < \delta \text{ and } t_{x_0} > 0 \text{ but } \left\|\phi(t, x^0) - \bar{x}\right\| = \eta$$

**Definition 3.12.** *An* <u>*asymptotically stable equilibrium*</u> *$\bar{x}$ is stable equilibrium such that there exists an $r > 0$ such that*

$$\left\|\phi(t, x^0) - \bar{x}\right\| \to 0 \quad \text{as} \quad t \to +\infty$$

*where $\phi(t, x^0)$ is a solution going through $x^0$ for all $x^0$ satisfying $\left\|x^0 - \bar{x}\right\| < r$. In this definition, our solutions now have to tend to the fixed point as well, thus excluding centers.*

**Theorem 3.13.** *If all the eigenvalues of the coefficient matrix $A$ in the linear system $\dot{x} = Ax$ have negative parts, then the equilibrium point is asymptotically stable and there exists $K, \alpha$ such that*

$$\left\|e^{At}x^0\right\| \leq Ke^{\alpha t}\left\|x^0\right\|$$

*for all $t \geq 0, x^0 \in \mathbb{R}^2$.*

**Corollary 3.13.1.** *If one of the eigenvalues has a positive real part, then the equilibrium is unstable.*

**Definition 3.14.** *The linearization of $f$ at $\bar{x}$ of the system $\dot{x} = f(x)$ is*

$$\dot{x} = Df_{\bar{x}}(x)$$

**Theorem 3.15.** *Let $f$ be a $C^1$ function. If all the eigenvalues of the Jacobian matrix $Df_{\bar{x}}$ have negative real parts, then the equilibrium $\bar{x}$ is asymptotically stable.*

*Proof.* Define $y(t) = x(t) - \bar{x} \implies \dot{y} = f(y + \bar{x})$. by Taylor's Formula,

$$\dot{y} = f(y + \bar{x}) = f(\bar{x}) + Df_{\bar{x}}(y) + g(y)$$

where $g(y)$ satisfies

$$\begin{cases} Dg(0) = 0 \\ g(0) = 0 \end{cases}$$

By the mean value theorem, for all $m$, there exists $\epsilon > 0$ such that

$$\|y\| < \epsilon \implies \|g(y)\| \leq m\|y\|$$

Since $f(\bar{x}) = 0$, then

$$\dot{y} = Df_{\bar{x}}(y) + g(y)$$

We want to show that $y(t) = 0$ is an asymptotically stable solution. Now let $y^0 := y(0)$. Then

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}g(y(s))ds$$

$$\implies \left\|e^{At}\right\| \leq Ke{-}\alpha t$$

by the previous theorem. Further,

$$\|y(t)\| \leq Ke{-}\alpha t\left\|y^0\right\| + \int_0^t Ke^{-\alpha(t-s)}m\|y(s)\|\,ds$$

where $\|y(s)\| \leq \epsilon$ for $0 \leq s \leq t$. Then

$$e^{\alpha t}\|y(t)\| \leq K\left\|y^0\right\| + \int_0^t Kme^{\alpha s}\|y(s)\|\,ds$$

Applying Gronwall's inequality, we see;

$$e^{\alpha t}\|y(t)\| \leq K\left\|y^0\right\| + \int_0^t Kme^{\alpha s}\|y(s)\|\,ds \leq K\left\|y^0\right\|e^{Kmt}$$

$$\implies \|y(t)\| \leq K\left\|y^0\right\|e^{(Km-\alpha)t}$$

Finally, selecting $\delta > 0$ such that $K\delta < \epsilon$, then

$$\left\|y^0\right\| < \delta \implies \|y(t)\| \leq K\delta e^{(Km-\alpha)\delta} < \epsilon e^{(Km-\alpha)\delta} < \epsilon$$

Therefore, $y(t) \to 0$ as $t \to 0 \implies y(t) = 0$ is asymptotically stable. ∎

### 3.3 Polar Coordinates

Two fundamental identities:

$$\begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ \dot{\theta}r^2 = x\dot{y} - y\dot{x} \end{cases}$$

In addition to the obvious identity:

$$x^2 + y^2 = r^2$$

### 3.4 Limit Cycles

**Definition 3.16.** *A limit cycle is an isolated, closed trajectory. Comes in 3 flavors: stable, unstable, and semi-stable.*

**Note 3.17.** *Inherently nonlinear occurrence! Cycles don't count.*

**Example 3.18.**

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}$$

*Has a limit cycle along $r = 1$!*

**Definition 3.19.** *A gradient system is one of the form*

$$\dot{x} = -\nabla V$$

*where $V$ is called a potential function.*

**Theorem 3.20.** *Closed orbits are impossible in gradient systems.*

*Proof.* Suppose we have a closed orbit $\gamma$. Then $\frac{dV}{d\gamma} = 0$. But notice,

$$\int_0^T \frac{dV}{d\gamma}(t)\ dt = \int_0^T \nabla V \cdot \dot{x}\ dt = \int_0^T \dot{x}\dot{x}\ dt = \int_0^T \|\dot{x}\|^2\,dt > 0$$

because $\|\dot{x}\| > 0$ and $\|\dot{x}(t)\|$ is continuous along $t$. Therefore, we have a contradiction! ∎

### 3.5 Liapunov Functions

**Definition 3.21.** *Basin of attraction $:= \{x^0 \in \mathbb{R}^2 : \phi(t, x^0) \to \bar{x}$ as $t \to \infty\}$. It's all the trajectories that tend to the fixed point.*

**Definition 3.22.** *Let $U \subset \mathbb{R}$ be open and contain the origin. $V \in C^1(U)$ is positive definite on $U$ if*

1. $V(0) = 0$
2. $V(x) > 0$ *for all* $x \in U \setminus \{0\}$.

**Lemma 3.23.** *$V$ positive definite on $U$ implies $V$ has a minimum within $U$.*

**Lemma 3.24.** *$V(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ is positive definite if and only if*

$$a > 0 \quad \textit{and} \quad ac - b^2 > 0$$

*Proof.* Suppose $V$ is positive definite. Clearly,

- $V(x, 0) > 0 \implies ax^2 > 0$ for all $x \neq 0 \implies a > 0$
- $V(x, y) > 0 \implies$ no real zeros $\implies$ discriminate negative $\implies (2b^2) - 4ac < 0 \implies ac > b^2$

■

Turning to how solutions of $\dot{x} = f(x)$ cross the level sets of $V$, we see

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x_1}(x(t)) \cdot \dot{x}_1(t) + \frac{\partial V}{\partial x_2}(x(t)) \cdot \dot{x}_2(t) = f(x) \cdot \nabla V = \|f(x)\| \, \|\nabla V\| \cos\theta$$

**Theorem 3.25.** *(Liapunov) Let $\bar{x} = 0$ be an equilibrium point of $\dot{x} = f(x)$ and $V \in C^1(U)$ with $0 \in U$.*

1. *$\dot{V} \leq 0$ for $x \in U \setminus \{0\} \implies$ 0 is stable.*
2. *$\dot{V} < 0$ for $x \in U \setminus \{0\} \implies$ 0 is asymptotically stable.*
3. *$\dot{V} > 0$ for $x \in U \setminus \{0\} \implies$ 0 is unstable.*

*Proof.* Let $\epsilon > 0$ such that $B_\epsilon(\bar{x}) \subset U$. Define

$$m := \min_{x \in \partial B_\epsilon(\bar{x})} \{V(x)\}$$

Since $V$ is positive definite and $\|\bar{x}\| = \epsilon$ is closed and bounded $\implies m > 0$. We then choose $0 < \delta \leq \epsilon$, such

$$\|x\| < \delta \implies 0 < V(x) < m$$

This exists thanks to the mean value theorem between $V(0) = 0$ and $V(\partial B_\epsilon(x)) = m$. So the region $\|x\| < \delta$ is trapping, in which

$$\dot{V}(x(t)) \leq 0 \implies V(x(t)) \leq V(x^0)$$

Therefore, $x(t) \to 0 = \bar{x}$. ■

**Definition 3.26.** *A positive definite function $V$ on an open neighborhood $U$ of the origin is a <u>Liapunov function</u> of $\dot{x} = f(x)$ if*

$$\dot{V}(x) \leq 0 \text{ for all } x \in U \setminus \{0\}$$

*when $\dot{V}(x) < 0$ for all $x$, we add <u>strict</u>.*

## 3.6 Using Liapunov Functions

We can use a <u>strict Liapunov function</u> to show there are no closed orbits.

**Example 3.27.**

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{-g}{\ell} \sin x \end{cases}$$

*Consider $V(x, y) = \frac{1}{2}\ell^2 y^2 + g\ell(1 - \cos x)$*

- *Positive definite with $V(0, 0) = 0$*
- $\dot{V}(x, y) = f(\bar{x}) \cdot \nabla V = \langle y, -\frac{g}{\ell} \sin x \rangle \cdot \langle g\ell \sin x, \ell^2 y \rangle \equiv 0$

*So by the previous theorem, this system is stable but not asymptotically stable. So there could exist a limit cycle.*

**Example 3.28.**

$$\begin{cases} \dot{x} = -x + 4y \\ \dot{y} = -x - y^3 \end{cases}$$

*Recalling $V(x, y) = x^2 + ay^2$ was a strict Liapunov function,*

$$\begin{aligned}\dot{V} &= f(x) \cdot \nabla V \\ &= (-x + 4y)(2x) + (-x - y^3)(2ay) \\ &= -2x^2 - 8xy - 2axy - 2ay^4\end{aligned}$$

*Choosing $a = 4$, we see:*

$$= -2x^2 - 8y^4$$

*So it's negative definite. So by the previous theorem, this system is asymptotically stable and cannot have a limit cycle.*

## 3.7 Dulac's Criterion

**Theorem 3.29.** *Dulac's Criterion Let $\dot{x} = f(x)$ be a continuously differentiable vector field on a simply connected region $R$. If there exists a continuously differentiable function $g$ such that*

$$sign(\nabla \cdot g\dot{x}) \equiv \ constant$$

*Then there are no closed orbits.*

**Corollary 3.29.1.** *If $sign(div(F)) \equiv$ constant, where $\dot{x} = F$, then there are no closed orbits.*

**Example 3.30.**

$$\begin{cases}\dot{x} = x(2 - x - y) \\ \dot{y} = y(4x - x^2 - 3)\end{cases} = \vec{F}$$

*and $D = \mathbb{R}^2_+$. Then*

$$\begin{aligned}div(\vec{F}) &= \frac{\partial}{\partial x}(x(2 - x - y)) + \frac{\partial}{\partial y}(y(4x - x^2 - 3)) \\ &= 2 - 2x - y + 4x - x^2 - 3 \\ &= -1 + 2x - y - x^2\end{aligned}$$

*Inconclusive. Now consider $g(x, y) = \frac{1}{xy}$ which is analytic on chosen $D$. Then*

$$\begin{aligned}div(g(x, y) \cdot \vec{F}) &= \frac{\partial}{\partial x}(\frac{2 - x - y}{y}) + \frac{\partial}{\partial y}(\frac{4x - x^2 - 3}{x}) \\ &= -\frac{1}{y} < 0\end{aligned}$$

*for all $y \in \mathbb{R}^+$. Therefore, there are no closed orbits on the domain $D$.*

## 3.8 Poincare-Bendixson's Theorem

**Theorem 3.31.**

1. *$R$ is closed and bounded region.*
2. *$\dot{x} = f(x)$ is continuously differentiable on an open cover of $R$.*
3. *$R$ does not contain fixed points, although it can surround them.*
4. *$R$ is an invariant region / trapping region.*

*Then $R$ contains a closed orbit! i.e. a limit cycle!*

**Example 3.32.** *Show that the system*

$$\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$$

*has a closed orbit.*

*Converting to polar coordinates:*

$$\begin{cases} \dot{r} = r - r^3(\cos^4 + \sin^4\theta) \\ \dot{\theta} = doesn'tmatter \end{cases}$$

*Notice,* $\frac{1}{2} \leq \cos^4 + \sin^4\theta \leq 1$. *So*

$$r - r^3 \leq \dot{r} \leq r - \frac{r^3}{2}$$

*So when* $r < 1 \implies \dot{r} > 0$ *and when* $r > \sqrt{2} \implies \dot{r} < 0$. *Therefore, the region* $\{1 \leq r \leq \sqrt{2}\}$ *satisfies the conditions of Poincare-Bendixon and proves there's a closed orbit within it.*

### 3.9 Another Mechanism to Finding Limit Cycles

1. Find the fixed point
2. Take the radial vector from fixed point and dot with the field

**Example 3.33.**

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

*Observe,*

$$\begin{aligned} r\dot{r} &= r \cdot f \\ &= x(x - y - x(x^2 + y^2)) + y(x + y - y(x^2 + y^2)) \\ &= x^2 - xy - x^2(x^2 + y^2) + xy + y^2 - y^2(x^2 + y^2) \\ &= r^2 - r^2(r^2) \\ &= r^2 - r^4 \\ &= r^2(1 - r)(1 + r) \end{aligned}$$

*There's a sign change at* $r = 1$. *Therefore, we can surround this* $r = 1$ *to be our trapping region.*

## 4 Bifurcation Theory in 2D

We simply generalize the ideas of Bifurcations from the 1-dimensional case to the 2-dimensional case. This is because most bifurcations in 2-d map to a codimension 1 manifold. Using the bifurcation value $\mu$, solve for the fixed points of the system:

- If two fixed points erupt from nowhere, we have a saddle-node bifurcation
- If a fixed point erupts into three fixed points, you have a pitchfork bifurcation
- If two fixed swap stabilities, you have a transcritical bifurcation
- Lastly, this is the case that doesn't reduce to a codimension 1 manifold.

**Definition 4.1.** *A Hopf bifurcation occurs when the real part of the associated eigenvalue of a fixed point changes from one sign to the next.*

Notice, this bifurcation doesn't involve anymore than one fixed point changing stability.

Often this process involves heavily using the discriminant of the fundamental equation for the Jacobian. So practice these calculations to help speed through the exam.

# 5 Introduction to Variational Problems

## 5.1 Variational Problems in One Independent Variable

Suppose a potential function $U(x, y, z) \in C^2(\mathbb{R}^2)$.
The conservative force field is given by

$$\vec{F} = \nabla U = \langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \rangle$$

Now, suppose a point mass of $m$ moves along a trajectory,

$$\vec{s}(t) = \langle x(t), y(t), z(t) \rangle$$

Then the force associated with the motion of the point is given by

$$\vec{F} = m \frac{d^2 \vec{s}}{dt^2} = m \langle x''(t), y''(t), z''(t) \rangle = \langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \rangle$$

Therefore,

$$m \frac{d^2 \vec{s}}{dt^2} = \nabla U = -\nabla V (\text{by convention})$$

So, taking the line integral along the trace $\Gamma = \{s(t) : t \in \mathbb{R}\}$,

$$U = \int_\Gamma \left( \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right)$$

**Definition 5.1.** *The kinetic energy is the quantity*

$$T = \frac{m}{2} (\ddot{x}(t))^2$$

**Definition 5.2.** *The conventional potential energy is the quantity $V$ such that*

$$F = -\nabla V$$

We can now define the concept of action:

**Definition 5.3.** *The action is the quantity*

$$A := \int_{t_1}^{t_2} \underbrace{(T - V)}_{Lagrangian} \; dt$$

We want to consider all possible paths, and find the one that makes the "most sense". The path of minimized / least action is consider such a path.

**Theorem 5.4.** *Principle of Least Action The path taken is the one that minimizes the integral*

$$A := \int_{t_1}^{t_2} (T - V) \; dt$$

We shall refer to this path as the extremal. Another way of phrasing this idea:

**Theorem 5.5.** *Hamilton's Principle A mechanical system with kinetic energy $T$ and the potential energy $V$ behaves with a times $t_1 \leq t \leq t_2$ for a given initial and end position such that*

$$A := \int_{t_1}^{t_2} L \, dt$$

*where $L := T - V$ assumes a stationary value.*

## 5.2 Fundamental Lemma of the Calculus of Variations

**Theorem 5.6.** *Fundamental Lemma of the Calculus of Variations If $M(x)$ is a continuous function for $x_1 \leq x \leq x_2$ and $\eta(x)$ is any function which is continuously differentiable and $\eta(x_1) = \eta(x_2) = 0$, and if*

$$\int_{x_1}^{x_2} \eta(x) M(x) \, dx = 0$$

*for every $\eta$ chosen, then $M(x) = 0$ for all $x_1 \leq x \leq x_2$.*

*Proof.* Suppose $M(x)$ is not everywhere zero $\implies$ $\exists x_0$ where $|M(x_0)| > 0$. Assume, without loss of generality that $M(x_0) > 0$. Since $M(x)$ is continuous, there must exist $\delta$ such that

$$M(x) > 0 \quad \text{for all} \quad |x - x_0| < \delta$$

Now consider the mollifier

$$\eta(x) = \begin{cases} 0 & |x - x_0| > \delta \\ (x - x_0 + \delta)^2 (x - x_0 - \delta)^2 & |x - x_0| \leq \delta \end{cases}$$

This has the properties:

- zero everywhere except $|x - x_0| < \delta$
- positive in $|x - x_0| < \delta$
- differentiable for all $x_1 \leq x \leq x_1$

Now, by the assumption, we know

$$0 = \int_{x_1}^{x_2} \eta(x) M(x) \, dx$$

But, further, we see that

$$\int_{x_1}^{x_2} \eta(x) M(x) \, dx = \int_{x_0 - \delta}^{x_0 + \delta} \underbrace{(x - x_0 + \delta)^2 (x - x_0 - \delta)^2}_{\text{strictly positive}} \, dx > 0$$

We've reached a contradiction! Therefore, $M(x) \equiv 0$ ■

**Corollary 5.6.1.** *If*

$$\int_{x_1}^{x_2} \eta'(x) M(x) \, dx = 0$$

*for any $\eta(x)$, then $M(x) \equiv$ constant for all $x_1 \leq x \leq x_2$*

*Proof.*

$$\int_{x_1}^{x_2} [M(x) - C] \eta'(x) \, dx = \int_{x_1}^{x_2} M(x) \eta'(x) \, dx - [c\eta]_{x_1}^{x_2} = 0$$

Ensure $\eta \in D^1([x_1, x_2])$. Then by the previous theorem,

$$M(x) - c \equiv 0$$

$$\iff M(x) \equiv c$$

■

## 5.3 Deriving the First Variation

**Note 5.7.** *I'm using the notation in the textbook, which is admittedly an abortion. Feel free to make the notation better. Apparently mathematicians lost interest in making this subject better long ago ... and yet we're being forced to learn it now.*

Given the problem

$$\begin{cases} \min I := \int_{x_1}^{x_2} f(x,y,y')\ dx \\ y(x_1) = y_1, \quad y(x_2) = y_2 \end{cases}$$

**Question 5.8.** *Can we derive a more convenient set of conditions for solvability?*

Consider the variant: $h(x) = \epsilon\eta(x)$ with $\eta(x_1) = \eta(x_2) = 0$

$$\implies \bar{y}_\eta = y + \epsilon\eta$$

Then our integral depends on the variational parameter $\epsilon$

$$J(\epsilon) = \int_{x_1}^{x_2} f(x, y + \epsilon\eta, y' + \epsilon\eta')\ dx$$

We define the total variation $\Delta J := J(\epsilon) - J(0)$.

$$\implies \Delta J = \int_{x_1}^{x_2} [f(x, y+\epsilon\eta, y'+\epsilon\eta') - f(x,y,y')]\ dx$$

Applying the Taylor expansion:

$$\Delta J = \delta J + \frac{1}{2}\delta^2 J + O(\epsilon^3)$$

where

$$\delta J = \frac{dJ(\epsilon)}{d\epsilon}|_{\epsilon=0} \cdot \epsilon = \epsilon \int_{x_1}^{x_2} [f_y(x,y,y')\eta + f_{y'}(x,y,y')\eta']\ dx$$

$$\delta^2 J = \frac{d^2 J(\epsilon)}{d\epsilon^2}|_{\epsilon=0} \cdot \epsilon^2 = \epsilon^2 \int_{x_1}^{x_2} [f_{yy}(x,y,y')\eta^2 + 2f_{yy'}(x,y,y')\eta\eta' + f_{y'y'}(x,y,y')\eta'^2]\ dx$$

First Variation Condition: $J[y] \to \min$ provided $\delta J = 0$ at $y$.

$$\implies \delta J = 0 \implies \int_{x_1}^{x_2} [f_y(x,y,y')\eta + f_{y'}(x,y,y')\eta']\ dx = 0$$

Applying integration by parts:

$$\begin{aligned} \delta J &= \int_{x_1}^{x_2} [f_y(x,y,y')\eta + f_{y'}(x,y,y')\eta']\ dx \\ &= \int_{x_1}^{x_2} [f_y(x,y,y')\eta - \frac{d}{dx}[f_{y'}(x,y,y')]\eta]\ dx + \underbrace{[f_{y'}\eta]_{x_1}^{x_2}}_{=0} \\ &= \int_{x_1}^{x_2} \eta \left( f_y(x,y,y') - \frac{d}{dx}[f_{y'}(x,y,y')] \right)\ dx = 0 \end{aligned}$$

Applying the Fundamental Lemma of Calculus of Variations

$$\implies f_y(x,y,y') - \frac{d}{dx}[f_{y'}(x,y,y') = 0$$

**Definition 5.9.** *The Euler-Lagrange Equation is the equation:*

$$f_y(x,y,y') - \frac{d}{dx}[f_{y'}(x,y,y') = 0$$

Sometimes deriving this equation gets messy when having to deal with all of those partial derivatives of square roots. Here's a simple theorem that could come in handy:

**Theorem 5.10.** *Let $f(y)$ be a given function of $y$. Then the function $y$ that extremizes the integral*

$$\int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\; dx$$

*has the Euler-Lagrange Equation*

$$1+(y')^2 = B[f(y)]^2$$

*where $B \equiv$ constant.*

## 5.4 Degenerate Functionals

- Suppose $f(x, y, y') = M(x, y)$. Then the Euler-Lagrange Equations are

$$\frac{\partial M}{\partial y} = 0$$

- Suppose $f(x, y, y') = f(x, y')$. Then

$$\frac{d}{dx}[\frac{\partial f}{\partial y'}] = 0 \implies \frac{\partial f}{\partial y'} = c$$

- Suppose $f(x, y, y') = f(y, y')$. Then

$$f_y - \frac{d}{dx} f_{y'} = f_y - \underbrace{f_{y'x}}_{=0} - f_{y'y}y' - f_{y'y}y'' = 0$$

  Now, notice:

$$\frac{d}{dx}\left(f - y' f_{y'}\right) = (f_y - f_{y'y}y' - f_{y'y}y'')y' = 0$$
$$\implies f - y' f_{y'} = c$$

**Example 5.11.** *Let $f(y, y') = g(y)\sqrt{1+y'^2}$. Then*

$$f_{y'} = g(y)\frac{y'}{\sqrt{1+y'^2}}$$

$$\sqrt{1+y'^2}g(y) - g(y)\frac{y'^2}{\sqrt{1+y'^2}} = c$$
$$\implies (1+y'^2)g(y) - g(y)(y')^2 = C\sqrt{1+y'^2}$$

*Solving for $g$*

$$(1+y'^2 - y'^2)g(y) = C\sqrt{1+y'^2}$$
$$g(y) = C\sqrt{1+y'^2}$$
$$C'g(y) = \sqrt{1+y'^2}$$
$$B[g(y)]^2 = 1+y'^2$$

*Notice, this prove the previous theorem!*

## 5.5 Higher Order Derivatives for Single-Variable Euler Lagrange Equations

We want to minimize $J[y]$ where

$$J[y] := \int_{x_1}^{x_2} f(x, y, y', y'')\ dx$$

where

$$\begin{cases} y(x_1) = y_1 & y'(x_1) = y_1' \\ y(x_2) = y_2 & y'(x_2) = y_2' \end{cases}$$

Define the variation: $\bar{y} = y + \epsilon\eta$ with

$$\begin{cases} \eta(x_1) = 0 & \eta'(x_1) = 0 \\ \eta(x_2) = 0 & \eta'(x_2) = 0 \end{cases}$$

The total variation is

$$\Delta J = \int_{x_1}^{x_2} [f(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'') - f(x, y, y', y'')]\ dx$$

The first variation is

$$\delta J = \frac{\partial J}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta'' \right)\ dx$$

Integrating by parts,

$$0 = \delta J = \int_{x_1}^{x_2} \eta \left( \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial f}{\partial y''}\right] \right)\ dx$$

By the Fundamental Theorem of Calculus of Variations, we conclude

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial f}{\partial y''}\right] = 0$$

Generalizing, when given

$$J = \int_{x_1}^{x_2} f(x, y, y', y'', \ldots, y^{(n)})\ dx \to \min$$

the Euler-Lagrange Equations are:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial f}{\partial y''}\right] + \ldots + (-1)^n \frac{d^n}{dx^2}\left[\frac{\partial f}{\partial y^n}\right] = 0$$

## 5.6 Single Variable - Multi Function Euler Lagrange Equations

We want to find $\vec{y}$ such that

$$\begin{cases} \int_{x_1}^{x_2} f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\ dx \to \min \\ y_i(x_1) = y_{ia}, y_i(x_2) = y_{i,b} \quad \forall 1 \leq i \leq n \end{cases}$$

Define the variation $\vec{y}_\epsilon = \vec{y} + \epsilon\vec{\eta}$ where $\eta_i(x_1) = \eta_i(x_2) = 0$ for all $1 \leq i \leq n$. The total variation becomes:

$$\Delta J = J[\vec{y}_\epsilon] = J[\vec{y}_0] = \int_{x_1}^{x_2} [f(x, \vec{y} + \epsilon\vec{\eta}, \vec{y}' + \epsilon\vec{\eta}') - f(x, \vec{y}, \vec{y}')]\ dx$$

$$\implies \delta J = \epsilon \int_{x_1}^{x_2} \left( \sum_{i=1}^{n} f_{y_i}\eta_i + f_{y_i'}\eta_i' \right)\ dx$$

Each $\eta_i$ is independent, therefore,

$$\int_{x_1}^{x_2} (f_{y_i}\eta_i + f_{y_i}\eta_i')\ dx$$

Therefore, for each $1 \leq i \leq n$,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left[\frac{\partial f}{\partial y_i'}\right] = 0$$

## 5.7 Euler-Lagrange Equation for 2D Problem

Find the surface $z = z(x, y)$ such that

$$\begin{cases} I = \iint_R f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})\ dx\ dy \to \min \\ z(x, y)|_{(x,y)\in C} = z_0(x, y) \end{cases}$$

Define the variation $z_\epsilon = z + \epsilon\xi$. Then the total variation is:

$$\Delta J = \iint_R \left[ f(x, y, z + \epsilon\xi, \frac{\partial z}{\partial x} + \epsilon\frac{\partial \xi}{\partial x}, \frac{\partial z}{\partial y} + \epsilon\frac{\partial \xi}{\partial y}) - f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) \right]\ dx\ dy$$

$$\begin{aligned} \implies \delta J = \frac{\partial J}{\partial \epsilon}|_{\epsilon=0} &= \iint_R \left( \frac{\partial f}{\partial z}\xi + \frac{\partial f}{\partial z_x}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial z_y}\frac{\partial \xi}{\partial y} \right)\ dx\ dy \\ &= \iint_R \frac{\partial f}{\partial z}\xi dA + \iint_R \left( \frac{\partial f}{\partial z_x}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial z_y}\frac{\partial \xi}{\partial y} \right) dA \\ &= \iint_R \frac{\partial f}{\partial z}\xi dA - \iint_R \xi \left( \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} \right) dA - \underbrace{\oint_C \left( \frac{\partial f}{\partial z_x} dy - \frac{\partial f}{\partial z_y} dx \right)}_{=0} \\ &= \iint_R \xi \left( \frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} \right) dA \end{aligned}$$

Applying the Fundamental Theorem of the Calculus of Variations:

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

## 5.8 Isoperimetric Problem in One independent Variable

We want a curve $y(x)$ such that

$$\begin{cases} J[y] := \int_{x_1}^{x_2} f(x, y, y')\ dx \quad y(x_1) = y_1, \quad y(x_2) = y_2 \\ K[y] := \int_{x_1}^{x_2} g(x, y, y')\ dx = \ell \end{cases}$$

Define the variation:

$$y_\epsilon = y + \epsilon_1\eta_1 = \epsilon_2\eta_2$$

such that

$$\eta_1(x_1) = \eta_1(x_2) = \eta_2(x_1) = \eta_2(x_2) = 0$$

to maximize $J$, we need the Jacobian $\frac{\partial(J,K)}{\partial(\epsilon_1,\epsilon_2)} = 0$

$$\frac{\partial J}{\partial \epsilon_i}|_{\epsilon_1=\epsilon_2=0} = \int_{x_1}^{x_2} \eta_i \left( \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] \right)\ dx$$

$$\frac{\partial K}{\partial \epsilon_i}|_{\epsilon_1=\epsilon_2=0} = \int_{x_1}^{x_2} \eta_i \left( \frac{\partial g}{\partial y} - \frac{d}{dx}\left[\frac{\partial g}{\partial y'}\right] \right)\ dx$$

Therefore, we can relate the two using some constant $\lambda$

$$\int_{x_1}^{x_2} \eta_i \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] - \lambda\left( \frac{\partial g}{\partial y} - \frac{d}{dx}\left[\frac{\partial g}{\partial y'}\right] \right) \right]\ dx$$

Therefore, the Euler-Lagrange Equations are:

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left[\frac{\partial h}{\partial y'}\right] = 0$$

where $h = f + \lambda g$.

### 5.9 Natural Boundary Condition for the One-Dimensional Problem

We want to minimize the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \rightarrow \min$$

through some path $y$. However, no boundary conditions are prescribed. So consider the variation

$$y_\epsilon = y(x, \epsilon) \quad \text{ with } y(x, 0) = y \leftarrow \text{extremal}$$

Let

$$\eta(x, \epsilon) = \frac{\partial y_\epsilon}{\partial \epsilon}$$

Then the first variation is given by

$$\delta J = \frac{dI}{d\epsilon}|_{\epsilon=0} = [\eta f_{y'}]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \left( \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right)\, dx = [\eta f_{y'}]_{x_1}^{x_2}$$

Therefore, for every $\eta$,

$$[\eta f_{y'}]_{x_1}^{x_2} = 0 \implies f_{y'}|_{x_1}^{x_2} = 0$$

The natural boundary conditions are therefore,

$$\frac{\partial f}{\partial y'}|_{x=x_1} = 0 \quad \text{ and } \quad \frac{\partial f}{\partial y'}|_{x=x_2} = 0$$

### 5.10 Natural Boundary Condition for the Two-Dimensional Problem

By the first variation,

$$\begin{aligned}
\delta J = \frac{dI(\epsilon)}{d\epsilon}|_{\epsilon=0} &= \iint_R \xi \left( \frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} \right)\, dA - \oint_C \xi \left( \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy \right) \\
&= -\oint_C \xi \left( \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy \right)
\end{aligned}$$

Therefore, we make the assumption,

$$\oint_C \xi \left( \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy \right) = 0$$

$$\implies \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy = 0$$

Dividing either side by $dx$, we see:

$$\implies \frac{\partial f}{\partial z_y} - \frac{\partial f}{\partial z_x}\frac{dy}{dx} = 0 \quad \text{ along } C$$

## 6 The Separation of Variables Method

Given a PDE

$$f(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0$$

we can make a fundamental assumption that

$$u(x, t) = X(x)T(t)$$

and hope the equation ”separates”.

## 6.1 Three Classic Eigenvalue Problems

1. The Dirichlet Problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

   which has the eigenvalue / eigenfunction pairs

$$\begin{cases} \lambda_n = \frac{n\pi}{L} \\ X_n(x) = \sin(\frac{n\pi x}{L}) \end{cases}$$

2. The Neumann Problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

   which has the eigenvalue / eigenfunction pairs

$$\begin{cases} \lambda_n = \frac{n\pi}{L} \\ X_n(x) = \cos(\frac{n\pi x}{L}) \end{cases}$$

3. The Round Robin Problem

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) - a_0 X(0) = 0 \\ X'(L) - a_0 X(L) = 0 \end{cases}$$

**Example 6.1.**

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L \\ u(0,t) = 0 = u(L,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Assumption: $u(x,t) = X(t)T(t)$
Then substituting this assumption into the equation

$$XT'' = c^2 X'' T$$
$$\implies \frac{-T''}{c^2 T} = \frac{-X''}{X} = \mu$$

Since these are functions of independent variables, it must hold that these expressions are constant! Therefore,

$$\frac{-T''}{c^2 T} = \mu$$
$$\frac{-X''}{X} = \mu$$

This becomes a system of ODEs

$$\begin{cases} X'' + \mu X = 0 \qquad X(0) = X(L) = 0 \\ T'' + c^2 \mu X = 0 \end{cases}$$

Second assumption, we let $\mu = \lambda^2$. Therefore, we can apply the classic eigenvalue problems above:

$$u_n(x,t) = \left( A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L}) \right) \sin(\frac{n\pi x}{L})$$

Therefore, we have a solution of the form by applying superposition over $X$:

$$u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L}) \right) \sin(\frac{n\pi x}{L})$$

Of course further analysis in the Fourier domain is needed to identify the coefficients $A_n, B_n$.

# 7 The Sturm-Liouville Theory

## 7.1 Linear Second-Order Equations

Given a second-order differential equation,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x)$$

We can define a second-order differential operator

$$L := a_0 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_2$$

which transforms the equation into

$$Ly = f$$

Properties of Solutions

- Given $y'' + q(x)y' + r(x)y = 0$, then the solutions are

  $$c_1 y_1 + c_2 y_2$$

  where $y_1, y_2$ are linearly independent.

- Given $y'' + qy' + ry = g$, then the solutions are

  $$y_p + c_1 y_1 + c_2 y_2$$

  where $y_p$ is the particular solution.

- Constant Coefficients Equation: When $q, r$ are constants for the equation

  $$y'' + qy' + ry = 0$$

  Then the characteristic polynomial is

  $$m^2 + qm + r = 0$$

  - $m_1 \neq m_2$

    $$c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

  - $m_1 = m_2 = m$

    $$c_1 e^{mx} + c_2 x e^{mx}$$

  - $m = \alpha \pm i\beta$

    $$(c_1 \cos(\beta x) + c_2 \sin(\beta x))\, e^{\alpha x}$$

- Euler-Cauchy Equation: When $x^2 y'' + axy' + by = 0$. Then the characteristic polynomial is

  $$m^2 + (a-1)m + b = 0$$

– $m_1 \neq m_2$ Real

$$c_1 x^{m_1} + c_2 x^{m_2}$$

– $m_1 = m_2 = m$

$$c_1 x^m + c_2 x^m \log x$$

– $m = \alpha \pm i\beta$

$$\left(c_1 \cos(\beta \ln |x|) + c_2 \sin(\beta \ln |x|)\right) x^\alpha$$

**Theorem 7.1.** *Given $y'' + q(x)y' + r(x)y = 0$ and $q, r$ are analytic functions at some point $x_0$ in an open set $I$, then the general solution can be represented*

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

*which converges on $I$.*

## 7.2 The Wronksian

**Definition 7.2.** *For any $f, g \in C^1$*

$$W[f,g](x) := \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

**Lemma 7.3.** *If $y_1, y_2$ are solutions of the equation*

$$y'' + q(x)y' + r(x)y = 0$$

*on $x \in [a,b]$ with $q \in C$, then either*

1. *$W[y_1, y_2](x) = 0$ for all $x \in [a,b]$*
2. *$W[y_1, y_2](x) \neq 0$ for all $x \in [a,b]$*

*Proof.*

$$\begin{aligned} \frac{d}{dx} W[y_1, y_2](x) &= \frac{d}{dx}[y_1 y_2' - y_2 y_1'] \\ &= y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1'' \\ &= y_1(-q y_2' - r y_2) - y_2(-q y_1' - r y_1) && \text{Substituting in their defining questions} \\ &= -q(y_1 y_2' - y_2 y_1') \\ &= -q W[y_1, y_2](x) \end{aligned}$$

This leaves us with the first order ODE

$$\begin{aligned} W' + qW &= 0 \\ \implies W(x) &= C e^{-\int_a^x q(t)\, dt}, \quad x \in [a,b] \end{aligned}$$

∎

**Theorem 7.4.** *Any two solutions $y_1, y_2$ of*

$$y'' + q(x)y' + r(x)y = 0$$

*are linearly independent if and only if $W[y_1, y_2](x) \neq 0$*

*Proof.* ($\Leftarrow$) Assume $y_1, y_2$ are linearly dependent, then

$$y_2 = \alpha y_1 \implies W[y_1, y_2] = 0$$

($\Rightarrow$) Suppose $W[y_1, y_2](x) = 0$ at any point $x$. Then $W[y_1, y_2](x) = 0$ for all $x$. Then

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \alpha \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

Therefore, $\{y_1, y_2\}$ are linearly dependent. ∎

## 7.3 Lagrange's Identity

**Lemma 7.5.** *Lagrange's Identity If L and M are adjoint differential operators of second order, then there exists $Q(x, y, y', z, z')$ such that*

$$zL[y] - yM[z] = \frac{d}{dx}Q$$

*Proof.*

$$\begin{aligned} zL[y] - yM[z] &= a_0y''z + a_1y'z + a_2yz - y(a_0z)'' + y(a_1z)' - a_2yz \\ &= \frac{d}{dx}\left[a_0y'z - y(a_0z)' + y(a_1z)\right] \end{aligned}$$

∎

**Corollary 7.5.1.** *If L is self-adjoint, then*

$$zL[y] - yL[z] = \frac{d}{dx}[p(x)(y'z - yz')]$$

## 7.4 The Formal Adjoint of a Differential Operator

Start with the general form of a second order linear differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0$$

Define the differential operator $L : L^2(I) \cap C^2(I) \to L^2(I)$

$$L := p(x)\frac{d^2}{dx^2} + q(x)\frac{d}{dx} + r(x)$$

such that the equation above becomes

$$Ly = 0$$

Since $L$ maps $y$ to zero, we can think of $y$ maintaining some orthogonality requirements of the range of $L$. Recall and adjoint of $L$, denoted $L^*$ would be an operator such that

$$\langle Lf, g\rangle = \langle f, L^*g\rangle$$

for all $f, g \in L^2(I) \cap C^2(I)$. Observe, we have a case of integration by parts:

$$\begin{aligned} \langle Lf, g\rangle &= \int_a^b (pf'' + qf' + rf)g\ dx \\ &= \int_a^b pf''g\ dx + \int_a^b qf'g\ dx + \int_a^b rfg\ dx \\ &= [pf'g]_a^b - \int_a^b f'(pg)'\ dx + [pfg]_a^b - \int_a^b f(pg)'\ dx + \int_a^b rfgdx \\ &= \langle f, (pg)'' - (qg') + rg\rangle + [p(f'g - fg') + (q - p')fg]_a^b \end{aligned}$$

Clearly, we want the first quantity on the right side to equal the left, so define:

$$L^* := p\frac{d^2}{dx^2} + (2p' - q)\frac{d}{dx} + (p'' - q' + r)$$

Now, in order for $L^* = L$, we see we arrive at the system of equations:

$$\begin{cases} q = 2p' - q \\ r = p'' - q' + r \end{cases}$$

By the first equation, $2q = 2p' \implies p' = q$. Which doesn't affect $r$. So, we're left with

$$Lf = pf'' + r'f' + rf = (pf')' + rf$$

Therefore, $L$ is self-adjoint if and only if $L$ has the form

$$L := \frac{d}{dx}\left(p\frac{d}{dx}(\cdot)\right) + r$$

Moreover, we see that for an operator $L$ to be self adjoint, then

$$[p(f'g - fg')]_a^b = 0$$

for all $f, g \in C^2(I) \cap L^2(I)$.

## 7.5 Self Adjoint Boundary Value Problem

**Definition 7.6.** *Consider the boundary value problem*

$$\begin{cases} \frac{d}{dx}[Py'] + Qy + \lambda Ry = 0 \\ ay(x_1) + by'(x_2) = 0 \\ cy(x_2) + dy'(x_2) = 0 \end{cases}$$

*This problem is* <u>*self adjoint*</u> *if for any two functions $y, z$ which satisfy the boundary conditions also satisfy*

$$\int_{x_1}^{x_2} (zL[y] - yL[z])\ dx = 0$$

Notice, when $L$ is a self-adjoint operator, then

$$zL[y] - yL[z] = \frac{d}{dx}[p(x)(y'z - yz')]$$

**Example 7.7.** *Prove that*

$$\begin{cases} \frac{d}{dx}(xy') + \lambda xy = 0 \\ |y(0)| < \infty \quad y(1) = 0 \end{cases}$$

*is a self-adjoint BVP.*
*Let $p(x) = x, q(x) = 0, r(x) = x$. Thn*

$$\begin{aligned} \int_{x_1}^{x_2} (zL[y] - yL[z])\ dx &= \int_{x_1}^{x_2} \frac{d}{dx}[p(x)(y'z - yz')]\ dx \\ &= [p(x)(y'z - yz')]_0^1 \\ &= [x(y'z - yz')]_0^1 \\ &= (1)y'(1)z(1) - (0)y(1)z'(1) = 0 \end{aligned}$$

## 7.6 Eigenvalues and Eigenfunctions of Sturm-Liouville Problems

**Theorem 7.8.** *If $L$ is self-adjoint, then the eigenvalues of the equation*

$$Lu + \lambda u = 0$$

*and real AND any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $L^2(a,b)$.*

*Proof.*

- Real Eigenvalues
  Suppose $\lambda$ is an eigenvalue of $-L$. Then there exist $f$ such that

$$Lf + \lambda f = 0$$

$$\implies \lambda \|f\|^2 = \langle \lambda f, f\rangle = -\langle Lf, f\rangle$$

Since $L$ is self-adjoint

$$-\langle Lf, f\rangle = -\langle f, Lf\rangle = \langle f, \lambda f\rangle = \bar{\lambda}\|f\|^2$$

That is,

$$\lambda \|f\|^2 = \bar{\lambda}\|f\|^2$$

Therefore, $\lambda = \bar{\lambda} \implies \lambda$ is real.

- Orthogonal Eigenfunctions
  Suppose $\mu, \lambda$ correspond to $g, f$ are eigenvalue/ eigenfunction pairs of

$$Lu + \lambda u = 0$$

Then

$$\lambda\langle f, g\rangle = -\langle Lf, g\rangle = -\langle f, Lg\rangle = \mu\langle f, g\rangle$$
$$\implies (\lambda - \mu)\langle f, g\rangle = 0$$

When $\lambda \neq 0 \implies \lambda - \mu \neq 0$, so

$$\langle f, g\rangle = 0$$

∎

## 7.7 A Sturm-Loiuville Problem Example

**Example 7.9.**

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u(\pi) = 0 \end{cases}$$

The characteristic equation defined by the ODE is:

$$\implies m^2 + 0m + \lambda = 0$$

$$\implies m = \pm i\sqrt{\lambda}$$

The Eigenvalues need to be real. Therefore,

$$v = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

Applying the first boundary condition,

$$0 = u(0) = c_1\cos(\sqrt{\lambda}0) + c_2\sin(\lambda 0) = c_1 \implies c_1 = 0$$

Applying the second boundary condition,

$$0 = c_2\sin(\sqrt{\lambda}\pi)$$

- *Case:* $c_2 = 0 \implies$ trivial solution
- *Case:* $c_2 \neq 0$. Then

$$\begin{aligned}\sin(\sqrt{\lambda}\pi) = 0 &\implies \sqrt{\lambda}\pi = \pi k \quad \text{for } k \in \mathbb{Z} \\ &\implies \lambda = k^2\end{aligned}$$

Therefore, we have the eigenvalue/ eigensolution pairs:

$$\lambda = k^2, f_k(x) = A\sin(kx), k \in \mathbb{Z}$$

### 7.8 Regular Sturm-Liouville Problem in General Inner Product Space

**Definition 7.10.** *Given a formally self-adjoint differential operator $Lf = (pf')' + qf$ such that*

$$\begin{cases} Lf + wf = 0 \\ B_1(f) + B_2(f) = 0 \end{cases}$$

*with $p > 0, w > 0$, then the boundary value problem is self-adjoint under the inner product defined*

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\ dx = \langle wf, g\rangle = \langle f, \overline{wg}\rangle$$

Specifically, when $p$ does not vanish, the normal Sturm-Liouville problem is referred to as regular.

**Theorem 7.11.** *A regular Sturm-Liouville problem has infinitely many*

$$\lambda_1 < \lambda_2 < \lambda_2 < \dots$$

*eigenvalues for which the problem has nontrivial solution. Moreover,*

$$\lim_{k\to} \lambda_k = \infty$$

### 7.9 Singular Sturm-Liouville Problems

Singular problems tend to require one of the two following situations:

1. $p(x) = 0$ at both $x = a$ and/or $x = b$
   This is to ensure the expression
   $$[p(x)(f'g - fg')]_a^b = 0$$
   $\implies$ the boundary conditions are self-adjoint.
2. Infinite domain
   In order for $u \in L^2_w[-\infty, \infty]$, then it would be necessary for
   $$\sqrt{w}u \to 0 \quad \text{as} \quad |x| \to \infty$$
   However, there is no guarantee that there will be enough eigenfunctions to make an orthonormal basis.

<u>Examples of Singular Sturm-Liouville Problems</u>

**Example 7.12.** *The Legendre Equation*

$$\underbrace{(1 - x^2)}_{vanishes\ at\ x=1,-1} \quad u'' - 2xu' + n(n+1)u = 0$$

**Example 7.13.** *Bessel's Equation*

$$\underbrace{x}_{vanishes\ at\ x=0} \quad u'' - \frac{n^2}{x}u' + \lambda \quad \underbrace{x}_{vanishes\ at\ x=0} \quad u = 0$$

# 8 Fourier Series

For any function $f \in L^2[-\pi, \pi]$, we can represent $f$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

**Theorem 8.1.** *The orthogonal set of functions*

$$\{1, \cos nx, \sin nx : n \in \mathbb{N}\}$$

*is complete in* $L^2(\pi, \pi)$.

The coefficients of this expansion are:

$$\begin{cases} a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx \\ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \end{cases}$$

**Theorem 8.2.** *Dini's Criterion If* $f \in L^1[-\pi, \pi]$ *such that there exists* $\delta > 0$ *such that*

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right|\ dt < \infty$$

*then* $\lim_{N\to\infty} S_N f(x) = f(x)$.

**Theorem 8.3.** *If* $f$ *is sectionally differentiable on an open interval surrounding* $x$*, then*

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}[f(x^+) + f(x^-)]$$

**Note 8.4.**

$$S_N f(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\ dt$$

**Theorem 8.5.** *Riemann-Lebesgue Lemma If* $f(t)$ *is sectionally smooth, in* $a \le t \le b$*, then*

$$\lim_{\lambda\to\infty}\int_a^b f(t)\sin(\lambda t)\ dt = 0$$

**Theorem 8.6.** *Bessel's Inequality If* $f(t)$ *is sectionally smooth and* $2\pi$ *periodic, then*

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\ dx$$

*and equal when* $f$ *is continuous.*

**Theorem 8.7.** *If* $f$ *is continuous on* $[-\pi, \pi]$ *and* $f(-\pi) = f(\pi)$ *and* $f'$ *is sectionally smooth on* $(-\pi, \pi)$*, then*

$$\sum_{n=1}^{\infty}\sqrt{|a_n|^2 + |b_n|^2} \quad \textit{converges}$$

*Proof.* $f'$ is sectionally smooth on $[-\pi, \pi]$, then

$$a_0' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\ dx = \frac{1}{\pi}[f(x)]_{-\pi}^{\pi} = 0$$

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\ dx = [\frac{1}{\pi}f(x)\cos nx]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx = nb_b$$

Further,

$$b_n' = -na_n$$

Now,

$$\begin{aligned} S_N &= \sum_{n=1}^{N}\sqrt{|a_n|^2 + |b_n|^2} \\ &= \sum_{n=1}^{N}\frac{1}{n}\sqrt{|a_n'|^2 + |b_n'|^2} \\ &\leq \left(\sum_{n=1}^{N}\frac{1}{n^2}\sum_{n=1}^{N}(|a_n'|^2 + |b_n'|^2)\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{N}\frac{1}{n^2}\int_{-\pi}^{\pi}|f'(x)|^2\ dx\right)^{\frac{1}{2}} < \infty \end{aligned}$$

∎

# 9 The Legendre Polynomails

## 9.1 Definition and Orthogonality of Legendre Polynomials

The equation

$$(1-x^2)u'' - 2xu' + \lambda u = 0 \quad \text{for} \quad -1 < x < 1$$

can be rewritten

$$\begin{aligned} &u'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}u = 0 \\ \implies\ &u'' - q(x)u' + \lambda r(x)u = 0 \end{aligned}$$

since $q, r$ are analytic in $-1 < x < 1$, then $u$ is also analytic by one of the properties of second order linear differential equations.

**Theorem 9.1.** *Solutions of the Legendre eigenvalue problem are continuous if and only if*

$$\lambda = n(n+1)$$

**Theorem 9.2.** *The Legendre polynomials are orthogonal in $L^2$.*

## 9.2 The Generating Function for the Legendre Polynomials

We can generate the Legendre polynomials by considering the expansion

$$F(x,z) = \frac{1}{\sqrt{1-(2zx - z^2)}} = \sum_{n=0}^{\infty} P_n(x)z^n \quad |x| \leq 1$$

Notice, the function $f$ satisfies the property

$$\begin{aligned}(1-2xz+z^2)\left(\frac{\partial F}{\partial z}\right) &= (1-2xz+z^2)\frac{x-z}{(1-2xz+z^2)^{\frac{3}{2}}}\\ &= \frac{x-z}{(1-2xz+z^2)^{\frac{1}{2}}}\\ &= (x-z)\frac{1}{(1-2xz+z^2)^{\frac{1}{2}}}\\ &= (x-z)F\end{aligned}$$

Now, since $F(x,z)=\sum_{n=0}^{\infty}P_n(x)z^n$, then we see that:

$$(1-2xz+z^2)\left(\frac{\partial F}{\partial z}\right) = (x-z)F$$

$$\iff (1-2xz+z^2)\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n = (x-z)\sum_{n=0}^{\infty}P_n(x)z^n$$

$$\iff \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n-\sum_{n=0}^{\infty}2x(n+1)P_{n+1}(x)z^{n+1}+\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^{n+2} = \sum_{n=0}^{\infty}xP_n(x)z^n-\sum_{n=0}^{\infty}P_n(x)z^{n+1}$$

$$\iff \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n-\sum_{n=0}^{\infty}2x(n+1)P_{n+1}(x)z^{n+1}+\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^{n+2} = \sum_{n=0}^{\infty}xP_n(x)z^n-\sum_{n=0}^{\infty}P_n(x)z^{n+1}$$

$$\iff \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n - \sum_{n=1}^{\infty}2xnP_n(x)z^n + \sum_{n=1}^{\infty}nP_n(x)z^{n+1} = \sum_{n=0}^{\infty}xP_n(x)z^n - \sum_{n=1}^{\infty}P_{n-1}(x)z^n$$

$$\iff \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n - \sum_{n=1}^{\infty}2xnP_n(x)z^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)z^n = \sum_{n=0}^{\infty}xP_n(x)z^n - \sum_{n=1}^{\infty}P_{n-1}(x)z^n$$

For $n\geq 2$, we see:

$$(n+1)P_{n+1}(x)z^n - 2xnP_n(x)z^n + (n-1)P_{n-1}(x)z^n = xP_n(x)z^n - (n-1)P_{n-1}(x)z^n = 0$$

$$\iff (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) = xP_n(x) - P_{n-1}(x) = 0$$

$$\iff (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

Therefore, we have a recursive definition of the Legendre polynomials

$$\begin{cases}(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)\\ P_0(x)=1\\ P_1(x)=x\end{cases}$$

Some elements are:

1. $P_0(x)=1$
2. $P_1(x)=x$
3. $P_2(x)=\frac{1}{2}(3x^2-1)$
4. $P_3(x)=\frac{1}{2}(5x^3-3x)$
5. $P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$
6. ...

### 9.3 Further Properties of Legendre Polynomials

**Lemma 9.3.** *For any $f \in L^2(-1,1)$, we can expand a function in terms of the Legendre polynomials*

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

*where the coefficients can be determined:*

$$c_n := \frac{\langle f, P_n \rangle_{L^2}}{\|P_n\|_{L^2}} = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) f(x)\ dx$$

**Theorem 9.4.** *Rodriquez's Formula*

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \\
&= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n!}{k!(n-k)!} x^{2n02k} \\
&= \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2n-2k)\dots(n+1-2k)}{k(n-k)!} x^{n-2k} \\
&= \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x
\end{aligned}$$

## 10 Bessel's Function

### 10.1 Derivation of Bessel's Function

Given the 2-dimensional wave equation in polar coordinates

$$u_{tt} = c^2 (u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta})$$

Apply the method of separation of variables

$$u(t, r, \theta) = R(r)\Theta(\theta)T(t)$$

Then we have

$$\begin{aligned}
R\Theta T'' &= c^2 (R''\Theta T + \frac{1}{r} R'(r)\Theta(\theta)T(t) + \frac{1}{r^2} R(r)\Theta''(\theta)T(t)) \\
\implies \frac{T''}{T} &= c^2 \left( \frac{R''\Theta}{R} + \frac{1}{rR} R'\Theta + \frac{1}{r^2}\Theta'' \right) = -c^2\mu^2
\end{aligned}$$

We extract the first ODE:

$$T'' + c^2\mu^2 T = 0$$

Further, performing the calculations, we arrive at:

$$\begin{cases} T'' + c^2\mu^2 T = 0 \\ \Theta'' + \nu^2 \Theta = 0 \\ r^2 R'' + rR' + (\mu^2 r^2 - \nu^2) R = 0 \end{cases}$$

Taking the last equation and letting

$$R(r) = f(\mu r) \implies R' = \mu f'(\mu r) \implies R'' = \mu^2 f''(\mu r)$$

which transforms the last equation into

$$r^2\mu^2 f''(\mu r) + r\mu f'(\mu r) + (\mu^2 r^2 - \nu^2) f(\mu r) = 0$$

Lastly, choosing $x = \mu r$, we see:

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$$

This equation is known as Bessel's equation of order $\nu$!

## 10.2 Bessel Function Series Definition (of the First Kind)

Starting with the equation for Bessel's of order $\nu$

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0$$

$$\implies y'' + \underbrace{\frac{1}{x}}\, Q(x) y' + \underbrace{\frac{x^2 - \nu^2}{x^2}}_{R(x)} y = 0$$

Since $Q(x)$ and $R(x)$ are analytic at $x = 0$, we can find a series solution such that

$$y = x^\lambda \sum_{k=0}^{\infty} c_k x^k$$

where $\lambda^2 - \nu^2 = 0$. Subbing this power series into Bessel's equation, we see

$$\implies \sum_{k=0}^{\infty} (k+\lambda)(k+\lambda-1) c_k x^{k+\lambda} + \sum_{k=0}^{\infty} (k+\lambda) c_k x^{k+\lambda} + \sum_{k=0}^{\infty} c_k x^{k+\lambda+2} - \nu^2 \sum_{k=0}^{\infty} c_k x^{k+\lambda} = 0$$

$$\implies \sum_{k=0}^{\infty} c_k \left[ (k+\lambda)^2 x^{k+\lambda} + x^{k+\lambda+2} - \nu^2 x^{k+\lambda} \right]$$

Calculating the coefficients of $x^\lambda, x^{\lambda+1}, \ldots$, we see

$$\begin{cases} \lambda^2 c_0 - \nu^2 c_0 = 0 \\ (\lambda+1)^2 c_1 - \nu^2 c_1 = 0 \\ (\lambda+2)^2 c_2 - \nu^2 c_2 + c_0 \\ \vdots \\ (\lambda+j)^2 - \nu^2 c_j + c_{j-2} = 0 \end{cases}$$

Assumption: $\lambda = \nu$. Then

$$(\lambda+1)^2 c_1 - \nu^2 c_1 = (2\nu+1) c_1 = 0$$

Since $(2\nu + 1) \neq 0$, then we see $c_1 = 0$. Because of this assumption,

$$(\lambda+j)^2 c_j - \nu^2 c_j + c_{j-2} = 0$$

$$(\nu+j)^2 c_j - \nu^2 c_j + c_{j-2} = 0$$

$$2j\nu c_j + j^2 c_j + c_{j-2} = 0$$

$$(2j\nu + j^2) c_j + c_{j-2} = 0$$

$$c_j = \frac{-2c_{j-2}}{j(j+2\nu)}$$

**Note 10.1.** *$c_j = 0$ for all odd values of $j$ since $c_1 = 0$. So we*

So we let $j := 2m$

$$\implies c_{2m} := \frac{-c_{2m-2}}{2^2 m(\nu + m)} \qquad \text{for } m \in \mathbb{N}$$

Therefore, Bessel's Equation yeilds the formal series

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} c_{2m} x^{2m}$$

## 10.3 A Review of the Gamma Function and the Bessel Function

**Definition 10.2.**

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

This function has the characteristic recurrence

$$\begin{cases} \Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 \\ \Gamma(n+1) = n\Gamma(n) = n! \end{cases}$$

Therefore, the Gamma function is a continuous extension of the factorial! For the Bessel function of order $\nu$, the constant $c_0$ wasn't chosen.
The standard choice of $c_0$ is

$$c_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}$$

This gives us an explicit definition for the coefficients of the Bessel function

$$c_{2m} = \frac{(-1)^m}{2^{\nu+2m} m! \Gamma(\nu + m + 1)}$$

And therefore,

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu + m + 1)} \left(\frac{x}{2}\right)^{2m}$$

## 10.4 Identities and Properties of Bessel's Function

**Theorem 10.3.** *Bessel functions $J_\nu$ and $J_{-\nu}$ are linearly independent if and only if $\nu$ is not an integer.*

*Proof.* Suppose $\nu = n \in \mathbb{N}$, then

$$\begin{aligned}
J_{-n}(x) &= \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m - n + 1)} \left(\frac{x}{2}\right)^{2m} \\
&= \left(\frac{x}{2}\right)^{-n} \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(m - n + 1)} \left(\frac{x}{2}\right)^{2m} \text{since } m - n + 1 \le 0 \implies \Gamma(m - n + 1) = \infty \\
&= \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{(m+n)!\Gamma(m + 1)} \left(\frac{x}{2}\right)^{2m+2n} \\
&= (-1)^n \left(\frac{x}{2}\right)^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu + m + 1)} \left(\frac{x}{2}\right)^{2m} \\
&= (-1) J_n(x) \implies \text{ dependent}
\end{aligned}$$

Now suppose $\nu > 0, \nu \notin \mathbb{N}$, then

$$aJ_\nu(x) + bJ_{-\nu}(x) = 0$$

Taking the limit as $x \to 0^+$, we see

$$a \cdot 0 + b|J_{-\nu}(x)| = 0 \implies J_{-\nu} = 0 \text{ or } b = 0$$

However, the first term of the series for $J_{-\nu}$ includes

$$\frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu} \to \infty$$

So

$$|J_{-\nu}(x)| \to \infty \text{ as } x \to 0^+$$

Therefore, $b = 0 \implies a = 0 \implies$ linearly independent! ∎

**Theorem 10.4.**

$$\int_0^x tJ_0(t)\ dt = xJ_1(x) \qquad \textit{for all } x > 0$$

*Proof.*

$$\begin{aligned}
\int_0^t tJ_0(t)\ dt &= \int_0^x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} t^{2m+1}\ dt \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} \int_0^x t^{2m+1}\ dt \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} \frac{x^{2m+2}}{2^{m+2}} \\
&= x\sum_{m=0}^{\infty} \frac{-1}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m+1} \\
&= xJ_1(x)
\end{aligned}$$

∎

**Theorem 10.5.**

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x)$$

*Proof.*

$$\begin{aligned}
\frac{d}{dx}[x^{-\nu}J_\nu(x)] &= \frac{d}{dx}\left[\frac{1}{2^\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m}\right] \\
&= \frac{1}{2^\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\frac{1}{2^{2m}}x^{2m-1} \\
&= \frac{1}{2^\nu}x^{-\nu}x^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+2)}\left(\frac{x}{2}\right)^{2m-1} \\
&= -x^{-\nu}\left(\frac{x}{2}\right)^{\nu+1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+2)}\left(\frac{x}{2}\right)^{2m} \\
&= -x^{-\nu}J_{\nu+1}
\end{aligned}$$

∎

## 10.5 The Generating Function for Bessels Functions

We can generate Bessel functions using

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

Notice, we expand the right side:

$$\begin{aligned} e^{\frac{x}{2}(z-\frac{1}{z})} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{j!k!}\left(\frac{x}{2}\right)^{j+k} z^{j-k} \\ &= \sum_{n=-\infty}^{\infty}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right] z^n && n = j-k \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n \end{aligned}$$

On the other hand, we can use the substitution,

$$z = e^{i\theta} \implies \frac{1}{2}(z - \frac{1}{z}) = i\sin\theta$$

$$\implies e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) z^n = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta}$$

Expanding the left side with Fourier Coefficients

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta} e^{-in\theta} d\theta \\ &\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}\ d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\sin\theta - n\theta)\ d\theta \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos(x\sin\theta - n\theta) d\theta \end{aligned}$$

**Note 10.6.**

$$|J_n(x)| \le \frac{1}{\pi}\cdot\pi \max_{0\le\theta\le\pi, x\in\mathbb{R}} \cos(x\sin\theta - n\theta) = 1$$

Equating the real and imaginary parts of the complex generating function ,

$$\cos(x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\theta)$$

$$\sin(x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\sin(n\theta)$$

Recall because $J_{-n}(x) = (-1)^n J_n(x)$, we see

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{m=0}^{\infty} J_{2m}(x)\cos(2m\theta)$$

$$\sin(x\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m-1}(x)\sin((2m-1)\theta)$$

This gives us the integral representations of Bessel's Functions

$$J_{2m}(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos(2m\theta)\ d\theta$$

$$J_{2m-1}(x) = \frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin((2m-1)\theta)\ d\theta$$

for $n \in \mathbb{N}$.

**Theorem 10.7.**

$$J_0^2(x) + 2\sum_{n=1}^{\infty} J_n^2(x) = 1$$

*Proof.* Observe, by Parseval's Identity,

$$\int_{-\pi}^{\pi}\cos^2(x\sin\theta)\ d\theta = 2\pi J_0^2(x) + 4\pi\sum_{m=1}^{\infty} J_{2m}^2(x)$$

$$\int_{-\pi}^{\pi}\sin^2(x\sin\theta)\ d\theta = 4\pi\sum_{m=1}^{\infty} J_{2m-1}^2(x)$$

Adding these quantities together,

$$2\pi = 2\pi J_0^2(x) + 4\pi\sum_{k=1}^{\infty} J_k^2(x)$$

$$\implies 1 = J_0^2(x) + 2\sum_{k=1}^{\infty} J_k^2(x)$$

∎

# 11 Formulating the Eigenvalue Problem as a Variational Problem

## 11.1 Equivalence of the Sturm-Liouville Problem

Define the functionals

$$I[y] := \int_{x_1}^{x_2}(Py'^2 - Qy^2)\ dx$$

$$C[y] := \int_{x_1}^{x_2} Ry^2\ dx$$

Suppose we want to find a function $y$ that satisfies the boundary-value problem

$$\begin{cases} I[y] \to \min \\ C[y] \equiv \text{ Constant} \end{cases}$$

Rewrite

$$I = \int_{x_1}^{x_2} f(x, y, y')\ dx \quad C = \int_{x_1}^{x_2} g(x, y, y')\ dx$$

Define the variation:

$$y_\epsilon = \bar{y} + \epsilon\eta$$

where $\eta(x_1) = \eta(x_2)$. Then we see that:

$$\begin{aligned}
I(\epsilon) - I(0) &= \int_{x_1}^{x_2} f(x, \bar{y} + \epsilon\eta, \bar{y}' + \epsilon\eta')\ dx - \int_{x_1}^{x_2} f(x, y, y')\ dx \\
&= \int_{x_1}^{x_2} [f(x, \bar{y} + \epsilon\eta, \bar{y}' + \epsilon\eta') - f(x, y, y')]\ dx \\
&= \int_{x_1}^{x_2} \left[\epsilon\eta \frac{\partial f}{\partial y} + \epsilon\eta' \frac{\partial f}{\partial y'}\right]\ dx \\
&= \epsilon \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\ dx + \epsilon \left[\eta \frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} \\
&= \epsilon \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\ dx
\end{aligned}$$

So,

$$\frac{\partial I}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\ dx$$

Similarly,

$$\frac{\partial C}{\partial \epsilon}|_{\epsilon=0} = \int_{x_1}^{x_2} \eta \left[\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right]\ dx$$

Combining terms:

$$\left(\frac{\partial I}{\partial \epsilon}\right)|_{\epsilon=0} - \lambda \left(\frac{\partial C}{\partial \epsilon}\right)|_{\epsilon=0} = \int_{x_1}^{x_2} \eta \left[\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right)\right] dx$$

where $h = f - \lambda g$. By the Fundamental Lemma of Calculus of Variations, since this quantity is zero for all $x_1, x_2$ then we see:

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0$$

Now,

$$h = f - \lambda g = Py'^2 - Qy^2 - \lambda R y^2$$

Then, plugging this into the formula derived as the Euler-Lagrange Equations, we see:

$$\begin{aligned}
0 &= \frac{\partial h}{\partial y} - \frac{d}{dx}\left(\frac{\partial h}{\partial y'}\right) = 0 \\
&= -2Qy - \lambda 2Ry - \frac{d}{dx}[2Py'] \\
\implies & \frac{d}{dx}[Py'] + [Q + \lambda R]y = 0
\end{aligned}$$

which is a self-adjoint differential equation. Further, we can solve for the eigenvalue by:

$$\left(\frac{\partial I}{\partial \epsilon}\right)|_{\epsilon=0} - \lambda \left(\frac{\partial C}{\partial \epsilon}\right)|_{\epsilon=0}$$

$$\implies \frac{\left(\frac{\partial I}{\partial \epsilon}\right)|_{\epsilon=0}}{\left(\frac{\partial C}{\partial \epsilon}\right)|_{\epsilon=0}} = \lambda$$

Therefore, we define the Rayleigh Quotient

$$\frac{\int_{x_1}^{x_2} (Py'^2 - Qy^2)\ dx}{\int_{x_1}^{x_2} Ry^2\ dx} \to \lambda = \min$$

## 11.2 Theorems Discussing this Equivalence

Let $L$ be the self-adjoint operator

$$L(\cdot) := \frac{d}{dx}[P(x)\frac{d}{dx}(\cdot)] + Q(x)(\cdot)$$

with the functions $P, Q$ defined on the interval $[x_1, x_2]$ possessing the properties:

- $P(x)$ differentiable and $P(x) > 0$
- $Q(x) < 0$

Further, let $R(x) > 0$ be continuous along this same domain.
Define the bilinear forms

$$I[y, \eta] := \int_{x_1}^{x_2} (Py'\eta' - Qy\eta)\ dx$$

$$C[y, \eta] := \int_{x_1}^{x_2} Ry\eta\ dx$$

**Theorem 11.1.** *To solve the self-adjoint boundary-value problem*

$$\begin{cases} L[y] + \lambda Ry = 0 \\ y(x_1) = 0 \quad y(x_2) = 0 \end{cases}$$

*is equivalent to finding a sequence* $\{y_i\}_{i=1}^{n}$ *such that*

$$\begin{cases} I[y_n, y_n] \to \min \\ C[y_i, y_n] = \delta_{i,n} \end{cases}$$

We can generalize this theorem as well:

**Theorem 11.2.** *To solve the self-adjoint boundary-value problem*

$$\begin{cases} L[y] + \lambda Ry = 0 \\ \alpha_1 y(x_1) + y'(x_1) = 0 \\ \alpha_2 y(x_2) + y'(x_2) = 0 \end{cases}$$

*Defining the bilinear forms:*

$$I[y, \eta] := \int_{x_1}^{x_2} (Py'\eta' - Qy\eta)\ dx + [\alpha_2 P(x_2)y(x_2)\eta(x_2) - \alpha_1 P(x_1)y(x_1)\eta(x_1)]$$

$$C[y, \eta] := \int_{x_1}^{x_2} Ry\eta\ dx$$

*is equivalent to finding a sequence* $\{y_i\}_{i=1}^{n}$ *such that*

$$\begin{cases} I[y_n, y_n] \to \min \\ C[y_i, y_n] = \delta_{i,n} \end{cases}$$

# 12 Green's Functions

**Definition 12.1.** *Given the self-adjoint operator:*

$$L := p(x)\frac{d^2}{dx^2} + p'\frac{d}{dx} + r$$

*under the boundary conditions:*

$$\begin{cases} \alpha_1 u(x_1) + \alpha_2 u(x_1) = 0, |\alpha_1| + |\alpha_2| > 0 \\ \beta_1 u(x_2) + \beta_2 u(x_2) = 0, |\beta_1| + |\beta_2| > 0 \end{cases}$$

*Then the Green's Function $G : [x_1, x_2] \times [x_1, x_2] \to \mathbb{R}$ is the function that satisfies the properties:*

1. *$G$ is symmetric*

$$G(x, \xi) = G(\xi, x)$$

   *for all $x, \xi \in [x_1, x_2]$.*

2. *$G$ is continuous on $[x_1, x_2] \times [x_1, x_2]$*

3. *$G$ is twice differentiable along $[a, b] \times [a, b] \setminus \{x = \xi\}$, satisfying the equation*

$$p(x)G_{xx}(x, \xi) + p'(x)G_x + r(x)G = 0$$

4. *$\frac{\partial G}{\partial x}$ has a jump discontinuity at $x = \xi$ and*

$$[\frac{\partial G}{\partial \xi}]_{x=\xi} = \frac{1}{p(\xi)}$$

*Further, it follows that the solution of the equation*

$$Lu = f$$

*has the solution*

$$\int_{x_1}^{x_2} G(x, \xi) f(\xi) d\xi = (G * f)(x)$$

## 12.1 How to get the Green's Function

The Long Way: You can successively apply the properties in the definition of the Green's function: Starting with a general solution to a differential equation,

$$G(x, \xi) = \begin{cases} y_1(x, \xi) = Af + Bg & x < \xi \\ y_2(x, \xi) = Cf + Dg & \xi < x \end{cases}$$

This assumption automatically should force the

- Apply Boundary Condition 1 and get $y_1$
- Apply Boundary Condition 2 and get $y_2$
- Apply Continuity Condition: When $x = x', y_1 = y_2$
- Apply Jump Condition:

$$\frac{\partial G(x, \xi)}{\partial x}\Big|_{x=\xi-0}^{x=\xi+0} = \frac{1}{p(\xi)}$$

The Easy Way: I still don't know why the fuck they didn't just teach it like this. Given the operator $L$ and boundary conditions

$$\begin{cases} \alpha_1 u(x_1) + \alpha_2 u'(x_1) = 0 \\ \beta_1 u(x_2) + \beta_2 u'(x_2) = 0 \end{cases}$$

1. Find $v_1$ such that $L[v_1] = 0$ such that

$$v_1(x_1) = \alpha_2 \quad v_1'(x_1) = -\alpha_1$$

2. Find $v_2$ such that $L[v_2] = 0$ such that

$$v_2(x_2) = \beta_2 \quad v_2'(x_2) = -\beta_1$$

3. Calculate

$$c = p(\xi) W[v_1, v_2](\xi)$$

4. Define

$$G(x,\xi) = \begin{cases} \frac{v_1(\xi)v_2(x)}{c} & a \le \xi < x \le b \\ \frac{v_2(\xi)v_1(x)}{c} & a \le x < \xi \le b \end{cases}$$

**Example 12.2.** *Consider*

$$\begin{cases} y'' = f(x) \\ y(0) = y(1) = 0 \end{cases}$$

*What is the Green's function? Find a solution when $f(x) = x^2$.*
*Going through the steps: The defining equation*

$$\begin{cases} y'' = 0 \end{cases}$$

*has two linearly independent solutions*

$$\{1, x\}$$

*So our solutions should be of the form:*

$$u = Ax + B$$

1. *Finding $v_1$, we stipulate:*

$$\begin{cases} v_1(x) = Ax + B \\ v_1(0) = 0 \\ v_1'(0) = 1 \end{cases}$$

   *Therefore,*

$$v_1(x) = x$$

2. *Find $v_2$, we stipulate:*

$$\begin{cases} v_2(x) = Ax + B \\ v_2(1) = 0 \\ v_2'(1) = -1 \end{cases}$$

   *Therefore,*

$$v_2(x) = 1 - x$$

3. *Calculate the value c:*

$$W[y_1,y_2](x) = det \begin{pmatrix} x & 1-x \\ -1 & -1 \end{pmatrix} = -x-1+x = -1$$

*Lastly, calculate c*

$$c(x) = p(x)(2x-1) = -1$$

4. *Construct G*

$$G(x,\xi) = \begin{cases} \frac{v_1(\xi)v_2(x)}{c(\xi)} & \xi < x \\ \frac{v_2(\xi)v_1(x)}{c(\xi)} & \xi > x \end{cases} = \begin{cases} \frac{(\xi)(1-x)}{-1} & \xi < x \\ \frac{(1-\xi)x}{-1} & \xi > x \end{cases} = \begin{cases} \xi(x-1) & \xi < x \\ (\xi-1)x & \xi > x \end{cases}$$

*Now to solve for the case* $f(x) = x^2$, *we see:*

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)f(\xi)\ d\xi \\ &= \int_0^x \xi(x-1)\cdot\xi^2\ d\xi + \int_x^1 (\xi-1)x\cdot\xi^2\ d\xi \\ &= (x-1)\int_0^x \xi^3\ d\xi + x\int_x^1 (\xi^3-\xi^2)d\xi \\ &= (x-1)[\frac{\xi^4}{4}]_0^x + x[\frac{\xi^4}{4} - \frac{\xi^3}{3}]_x^1 \\ &= (x-1)\frac{x^4}{4} + x(\frac{1}{4}-\frac{1}{3}) - x\left(\frac{x^4}{4} - \frac{x^3}{3}\right) \\ &= \frac{1}{12}x^4 - \frac{1}{12}x \end{aligned}$$

# 13 Algebraic Approximations

# 14 Integral Approximations

# 15 Non-Dimensionalization

# 16 Matched Asymptotic Solutions to ODEs

# 17 Solutions with Periodic Assumptions

# 18 Method of Multiple Scales

# 19 The WKB Method