STA207 Discussion 5

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1 Two-way ANOVA

Two-way ANOVA deals with how two factors affect a response variable and whether or not there is an interaction effect between the two factors on the response variable.
Assumptions:

- Normality: The response variable is approximately normally distributed for each group.
- Equal Variances: The variances for each group should be roughly equal.
- Independence: The observations in each group are independent of each other and the observations within groups were obtained by a random sample.

Cell means model:

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}, \epsilon_{ijk} \sim_{iid} \mathcal{N}(0, \sigma^2), i = 1, ..., a, j = 1, ..., b, k = 1, ..., n_{ij}$$

Factor effects model:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \epsilon_{ijk} \sim_{iid} \mathcal{N}(0, \sigma^2), i = 1, ..., a, j = 1, ..., b, k = 1, ..., n_{ij}$$

with constraints:

$$\sum_{i=1}^{a} \alpha_i = 0, \sum_{j=1}^{b} \beta_j = 0$$

$$\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0 \text{ for } \forall j, \quad \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0 \text{ for } \forall i$$

Note that in this case

$$\mu = \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}/ab, \mu_{i.} = \sum_{j=1}^{b} \mu_{ij}/b, \mu_{.j} = \sum_{i=1}^{a} \mu_{ij}/a$$

$$\alpha_i = \mu_{i.} - \mu, \beta_j = \mu_{.j} - \mu, (\alpha\beta)_{ij} = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu$$

Here, μ_{ij} is the expected value of observation in level i of factor A and level j of factor B, μ is the overall mean; α_i is the effect of level $i (i = 1, ..., a)$ of factor A;β_j is the effect of level $j (j = 1, ..., b)$ of factor B;$(\alpha\beta)_{ij}$ is the interaction effect of level i and j;ϵ_{ijk} is the error associated with the kth data point from level i of factor A and level j of factor B.
Why constraints?
For factor effects model, we have $ab + a + b + 2$ parameters to estimate. In order to get unique solutions for LSEs, we need the constraints and solve the following equations:

$$\sum_{k=1}^{n_{ij}} Y_{ijk} = n_{ij}(\hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + (\widehat{\alpha\beta})_{ij}) \text{ for } \forall i, j$$

$$\sum_{i=1}^{a} \hat{\alpha}_i = 0, \sum_{j=1}^{b} \hat{\beta}_j = 0, \sum_{i=1}^{a} (\widehat{\alpha\beta})_{ij} = 0 \text{ for } \forall j, \sum_{j=1}^{b} (\widehat{\alpha\beta})_{ij} = 0 \text{ for } \forall i$$

Suppose **balanced design**, where $n_{11} = n_{12} = ... = n_{ab} = n, N = abn$
Estimations:

$$\hat{\mu}_{ij} = \bar{Y}_{ij.}, \hat{\mu} = \bar{Y}_{...}, \hat{\mu}_{i.} = \bar{Y}_{i..}, \hat{\mu}_{.j} = \bar{Y}_{.j.}$$

$$\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}, \hat{\beta}_j = \bar{Y}_{.j.} - \bar{Y}_{...}, \widehat{(\alpha\beta)}_{ij} = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}$$

Two-way ANOVA table:

Source	SS	df	MS	F test
Main effect A	$SSA = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{Y}_{i..} - \bar{Y}_{...})^2$	$a-1$	$MSA = \frac{SSA}{a-1}$	$\frac{MSA}{MSE}$
Main effect B	$SSB = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{Y}_{.j.} - \bar{Y}_{...})^2$	$b-1$	$MSB = \frac{SSB}{b-1}$	$\frac{MSB}{MSE}$
Interaction effect	$SSAB = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$	$(a-1)(b-1)$	$MSAB = \frac{SSAB}{(a-1)(b-1)}$	$\frac{MSAB}{MSE}$
Within	$SSE = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{Y}_{ijk} - \bar{Y}_{ij.})^2$	$N-ab$	$MSE = \frac{SSE}{N-ab}$	
Total	$SSTO = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (\bar{Y}_{ijk} - \bar{Y}_{...})^2$	$N-1$		

$$SSTO = SSA + SSB + SSAB + SSE$$

$$df(SSTO) = df(SSA) + df(SSB) + df(SSAB) + df(SSE)$$

In imbalanced design, we DO NOT have

$$\hat{\mu} = \bar{Y}_{...}, \hat{\mu}_{i.} = \bar{Y}_{i..}, \hat{\mu}_{.j} = \bar{Y}_{.j.}, SSTO = SSA + SSB + SSAB + SSE$$

2 Practice problem

Derive the least square estimate for two-way ANOVA model in balanced design without interaction effect, i.e. $Y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}$
Solutions:
Loss function:

$$L(\mu, \alpha_i, \beta_j) = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \mu - \alpha_i - \beta_j)^2$$

We'd like to find $\hat{\mu}, \hat{\alpha_i}, \hat{\beta_j}$ so that the loss function is minimized under the constraints $\sum_i \alpha_i = 0, \sum_j \beta_j = 0$.
Take the first derivatives:

$$\frac{\partial L}{\mu} = (-2) \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \mu - \alpha_i - \beta_j) := 0$$

$$\frac{\partial L}{\alpha_i} = (-2) \sum_{j=1}^b \sum_{k=1}^n (Y_{ijk} - \mu - \alpha_i - \beta_j) := 0$$

$$\frac{\partial L}{\beta_j} = (-2) \sum_{i=1}^a \sum_{k=1}^n (Y_{ijk} - \mu - \alpha_i - \beta_j) := 0$$

With constraints $\sum_i \alpha_i = 0, \sum_j \beta_j = 0$, we have the optimal points

$$\mu = \frac{\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n Y_{ijk} - \sum_i \alpha_i - \sum_j \beta_j}{ab} \rightarrow \hat{\mu} = \bar{Y}_{...}$$

$$\alpha_i = \frac{\sum_{j=1}^b \sum_{k=1}^n Y_{ijk} - bn\mu - \sum_j \beta_j}{bn} = \bar{Y}_{i..} - \mu \rightarrow \hat{\alpha}_i = \bar{Y}_{i..} - \hat{\mu}$$

$$\beta_j = \frac{\sum_{i=1}^a \sum_{k=1}^n Y_{ijk} - an\mu - \sum_i \alpha_i}{an} = \bar{Y}_{.j.} - \mu \rightarrow \hat{\beta}_j = \bar{Y}_{.j.} - \hat{\mu}$$

The second derivatives are positive.
Method Of Lagrange Multipliers With Equality Constraints:

Suppose we have the following optimization problem:

$$\textit{Minimize } f(\boldsymbol{x}) \textit{ subject to } g_1(\boldsymbol{x}) = 0, g_2(\boldsymbol{x}) = 0, ..., g_n(\boldsymbol{x}) = 0$$

First construct a function called the Lagrange function:

$$L(\mathbf{x}, \lambda) = f(x) + \lambda_1 g_1(x) + ... + \lambda_n g_n(x)$$

To find the points of local minimum of $f(\mathbf{x})$ subject to the equality constraints, we find the stationary points of the Lagrange function $L(\mathbf{x}, \lambda)$, i.e., we solve the following equations:

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda) = 0$$

$$\frac{\partial L(\mathbf{x}, \lambda)}{\lambda_i} = 0 \textit{ for } \forall i$$

Loss function:

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} (Y_{ijk} - \mu - \alpha_i - \beta_j)^2$$

The goal is to find $\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j$:

$$(\hat{\mu}, \hat{\alpha}_i, \hat{\beta}_j) = \arg \min_{\mu, \alpha_i, \beta_j} \sum_{ijk} (Y_{ijk} - \mu - \alpha_i - \beta_j)^2$$

subject to

$$\sum_{i=1}^{a} \hat{\alpha}_i = 0, \sum_{j=1}^{b} \hat{\beta}_j = 0$$