

# Turbulent Kinetic Energy equation derivation

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1	Turbulent Kinetic Energy Equation in Wave-Following Curvilinear Coordinates (Hara and Sullivan, 2015)	

$$\frac{\partial U_i}{\partial \xi_i} = 0, \quad (1)$$

$$J^{-1} \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j} \left( u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) = 0, \quad (2)$$

$$U_i = J^{-1} u_j \frac{\partial \xi_i}{\partial x_j}. \quad (3)$$

Evaluating  $u_i \times$  Eq.(2), we get

$$\begin{aligned} J^{-1} u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial \xi_j} \left( u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) &= 0, \\ J^{-1} u_i \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j} \left( u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} u_i \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) \\ - \left( u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) \frac{\partial u_i}{\partial \xi_j} &= 0, \end{aligned} \quad (4)$$

Expanding third term in Eq. (4), we get

$$\begin{aligned} \left( u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} \right. \\ \left. + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) \frac{\partial u_i}{\partial \xi_j} = u_i U_j \frac{\partial u_i}{\partial \xi_j} + J^{-1} p \frac{\partial \xi_j}{\partial x_i} \frac{\partial u_i}{\partial \xi_j} + J^{-1} \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} \frac{\partial u_i}{\partial \xi_j} + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \frac{\partial u_i}{\partial \xi_j}, \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j \right) + J^{-1} p \frac{\partial u_i}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial u_i}{\partial x_k} + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik} \left( \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right] + \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \right] \right) \\
&\quad + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik} (S_{ik} + \Omega_{ik}) + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik}^2 + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k} \quad (\text{Using } S_{ik} \Omega_{ik} = 0).
\end{aligned} \tag{5}$$

Using Eq.(5) in Eq.(4), we have

$$\begin{aligned}
&\frac{\partial}{\partial t} \left( J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left( u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} u_i \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \right) \\
&\quad - \left( \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik}^2 + J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k} \right) = 0, \\
&\frac{\partial}{\partial t} \left( J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i (\sigma_{ik} + \sigma_{ik}^{\text{SGS}}) \frac{\partial \xi_j}{\partial x_k} \right) + J^{-1} 2\nu S_{ik}^2 - J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k} = 0, \\
&\frac{\partial}{\partial t} \left( J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left( \frac{1}{2} u_i u_i U_j + p U_j + J^{-1} u_i (\sigma_{ik} + \sigma_{ik}^{\text{SGS}}) \frac{\partial \xi_j}{\partial x_k} \right) + J^{-1} 2\nu S_{ik}^2 - J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k} = 0.
\end{aligned} \tag{6}$$

Applying Reynolds averaging to Eq.(2), and defining  $\tau_{ij}^p = J^{-1} p \partial \xi_j / \partial x_i$ ,  $\tau_{ij}^\nu = J^{-1} \sigma_{ik} \partial \xi_j / \partial x_k$ ,  $\tau_{ij} = \overline{u_i' U_j'}$ ,  $\tau_{ij}^{\text{SGS}} = J^{-1} \sigma_{ik}^{\text{SGS}} \partial \xi_j / \partial x_k$  we have

$$\begin{aligned}
&\overline{J^{-1} \frac{\partial u_i}{\partial t}} + \frac{\partial}{\partial \xi_j} \left( \overline{u_i U_j} + \bar{\tau}_{ij}^p + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}} \right) = 0, \\
&\frac{\partial}{\partial \xi_j} \left( \overline{u_i U_j} + \tau_{ij} + \bar{\tau}_{ij}^p + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}} \right) = 0.
\end{aligned} \tag{7}$$

Evaluating  $\bar{u}_i \times$  Eq. (7), we get the following mean kinetic energy  $E = (1/2) \bar{u}_i \bar{u}_i$  equation,

$$\begin{aligned}
&\bar{u}_i \frac{\partial}{\partial \xi_j} \left( \overline{u_i U_j} + \tau_{ij} + \bar{\tau}_{ij}^p + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}} \right) = 0, \\
&\frac{\partial}{\partial \xi_j} \left( \overline{u_i u_i U_j} + \bar{u}_i \tau_{ij} + \bar{u}_i \bar{\tau}_{ij}^p + \bar{u}_i \bar{\tau}_{ij}^\nu + \bar{u}_i \bar{\tau}_{ij}^{\text{SGS}} \right) - \left( \overline{u_i U_j} + \tau_{ij} + \bar{\tau}_{ij}^p + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}} \right) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
&\frac{\partial}{\partial \xi_j} \left( \frac{1}{2} \bar{u}_i \bar{u}_i U_j + \bar{u}_i (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \overline{u_i J^{-1} p \frac{\partial \xi_j}{\partial x_i}} \right) - \left( \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} + J^{-1} p \frac{\partial \xi_j}{\partial x_i} \frac{\partial \bar{u}_i}{\partial \xi_j} + \bar{\tau}_{ij}^\nu \frac{\partial \bar{u}_i}{\partial \xi_j} + \bar{\tau}_{ij}^{\text{SGS}} \frac{\partial \bar{u}_i}{\partial \xi_j} \right) = 0, \\
&\frac{\partial}{\partial \xi_j} \left( \frac{1}{2} \bar{u}_i \bar{u}_i U_j + \bar{u}_i (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \bar{p} U_j \right) - \left( J^{-1} p \frac{\partial \bar{u}_i}{\partial x_i} + (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} \right) = 0, \\
&\frac{\partial}{\partial \xi_j} \left( \frac{1}{2} \bar{u}_i \bar{u}_i U_j + \bar{u}_i (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \bar{p} U_j \right) - (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0,
\end{aligned}$$

$$\frac{\partial}{\partial \xi_j} (E\bar{U}_j + \bar{u}_i(\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \bar{p}\bar{U}_j) - (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0. \quad (8)$$

Now, let us evaluate  $u_i u_i U_j$

$$\begin{aligned} u_i u_i U_j &= (\bar{u}_i + u'_i)(\bar{u}_i + u'_i)U_j, \\ &= (\bar{u}_i \bar{u}_i + u'_i \bar{u}_i + \bar{u}_i u'_i + u'_i u'_i)(\bar{U}_j + U'_j), \\ &= [(\bar{u}_i \bar{u}_i + u'_i \bar{u}_i + \bar{u}_i u'_i + u'_i u'_i)\bar{U}_j + (\bar{u}_i \bar{u}_i + u'_i \bar{u}_i + \bar{u}_i u'_i + u'_i u'_i)U'_j], \\ &= \bar{u}_i \bar{u}_i \bar{U}_j + u'_i \bar{u}_i \bar{U}_j + \bar{u}_i u'_i \bar{U}_j + u'_i u'_i \bar{U}_j \\ &\quad + \bar{u}_i \bar{u}_i U'_j + u'_i \bar{u}_i U'_j + \bar{u}_i u'_i U'_j + u'_i u'_i U'_j. \end{aligned} \quad (9)$$

Phase averaging or Reynolds averaging of Eq. (9) and defining TKE  $e = (1/2)\overline{u'_i u'_i}$ , we get

$$\begin{aligned} \overline{u_i u_i U_j} &= \bar{u}_i \bar{u}_i \bar{U}_j + \overline{u'_i u'_i \bar{U}_j} + 2\bar{u}_i \tau_{ij} + \overline{u'_i u'_i U'_j}, \\ \frac{1}{2}\overline{u_i u_i U_j} &= \frac{1}{2}\bar{u}_i \bar{u}_i \bar{U}_j + \frac{1}{2}\overline{u'_i u'_i \bar{U}_j} + \bar{u}_i \tau_{ij} + \frac{1}{2}\overline{u'_i u'_i U'_j}, \\ \frac{1}{2}\overline{u_i u_i U_j} &= E\bar{U}_j + e\bar{U}_j + \bar{u}_i \tau_{ij} + \frac{1}{2}\overline{u'_i u'_i U'_j}, \end{aligned} \quad (10)$$

Defining turbulent dissipation rate as  $\epsilon = \overline{2\nu s_{ij} s_{ij}} = 2\nu [\overline{S_{ij} S_{ij}} - \bar{S}_{ij} \bar{S}_{ij}]$  (where,  $s_{ij} = (1/2)[\partial u'_i / \partial x_j + \partial u'_j / \partial x_i]$ ), using Eq.(10) and evaluating (6) – (8), we get

$$\begin{aligned} &\frac{\partial}{\partial \xi_j} \left( \overline{\frac{1}{2} u_i u_i U_j + p \bar{U}_j + \bar{u}_i \tau_{ij}^\nu + \bar{u}_i \tau_{ij}^{\text{SGS}}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - \overline{J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}} \\ &\quad - \frac{\partial}{\partial \xi_j} (E\bar{U}_j + \bar{u}_i(\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \bar{p}\bar{U}_j) + (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\ &\frac{\partial}{\partial \xi_j} \left( (E + e)\bar{U}_j + \bar{u}_i \tau_{ij} + \overline{\frac{1}{2} u'_i u'_i U'_j + p \bar{U}_j + \bar{u}_i \tau_{ij}^\nu + \bar{u}_i \tau_{ij}^{\text{SGS}}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - \overline{J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}} \\ &\quad - \frac{\partial}{\partial \xi_j} (E\bar{U}_j + \bar{u}_i(\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) + \bar{p}\bar{U}_j) + (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\ &\frac{\partial}{\partial \xi_j} \left( e\bar{U}_j + \overline{\frac{1}{2} u'_i u'_i U'_j + p' U'_j + \bar{u}'_i \tau_{ij}^{\nu'} + \bar{u}'_i \tau_{ij}^{\text{SGS}'}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - \overline{J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}} + (\tau_{ij} + \bar{\tau}_{ij}^\nu + \bar{\tau}_{ij}^{\text{SGS}}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\ &\frac{\partial}{\partial \xi_j} \left( e\bar{U}_j + \overline{\frac{1}{2} u'_i u'_i U'_j + p' U'_j + \bar{u}'_i \tau_{ij}^{\nu'} + \bar{u}'_i \tau_{ij}^{\text{SGS}'}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} + \bar{\tau}_{ij}^\nu \frac{\partial \bar{u}_i}{\partial \xi_j} \\ &\quad - \overline{J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial u_i}{\partial x_k}} + \bar{\tau}_{ij}^{\text{SGS}} \frac{\partial \bar{u}_i}{\partial \xi_j} + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\ &\frac{\partial}{\partial \xi_j} \left( e\bar{U}_j + \overline{\frac{1}{2} u'_i u'_i U'_j + p' U'_j + \bar{u}'_i \tau_{ij}^{\nu'} + \bar{u}'_i \tau_{ij}^{\text{SGS}'}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - \overline{J^{-1} 2\nu S_{ik} \frac{\partial \xi_j}{\partial x_k} \frac{\partial \bar{u}_i}{\partial \xi_j}} \\ &\quad - \left( \overline{J^{-1} \sigma_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \frac{\partial u_i}{\partial \xi_j}} - \overline{J^{-1} \bar{\sigma}_{ik}^{\text{SGS}} \frac{\partial \xi_j}{\partial x_k} \frac{\partial \bar{u}_i}{\partial \xi_j}} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\ &\frac{\partial}{\partial \xi_j} \left( e\bar{U}_j + \overline{\frac{1}{2} u'_i u'_i U'_j + p' U'_j + \bar{u}'_i \tau_{ij}^{\nu'} + \bar{u}'_i \tau_{ij}^{\text{SGS}'}} \right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - \overline{J^{-1} 2\nu S_{ik} \frac{\partial \bar{u}_i}{\partial x_k}} \end{aligned}$$

$$\begin{aligned}
& - \left( \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \bar{\tau}_{ij}^{\text{SGS}} \bar{S}_{ij} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
& \frac{\partial}{\partial \xi_j} \left( e \bar{U}_j + \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu'}} + \overline{u'_i \tau_{ij}^{\text{SGS}'}} \right) + J^{-1} 2\nu \overline{S_{ik} S_{ik}} - J^{-1} 2\nu \bar{S}_{ik} \bar{S}_{ik} \\
& - \left( \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \bar{\tau}_{ij}^{\text{SGS}} \bar{S}_{ij} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
& \frac{\partial}{\partial \xi_j} \left( e \bar{U}_j + \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu'}} + \overline{u'_i \tau_{ij}^{\text{SGS}'}} \right) + J^{-1} 2\nu [\overline{S_{ik} S_{ik}} - \bar{S}_{ik} \bar{S}_{ik}] \\
& - \left( \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \bar{\tau}_{ij}^{\text{SGS}} \bar{S}_{ij} \right) + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
& \frac{\partial}{\partial \xi_j} \left( e \bar{U}_j + \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu'}} + \overline{u'_i \tau_{ij}^{\text{SGS}'}} \right) + J^{-1} 2\nu \overline{s_{ik} s_{ik}} - \overline{\tau_{ij}^{\text{SGS}'}} s_{ij} + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
& \frac{\partial}{\partial \xi_j} \left( e \bar{U}_j + \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu'}} + \overline{u'_i \tau_{ij}^{\text{SGS}'}} \right) + J^{-1} \epsilon - \overline{\tau_{ij}^{\text{SGS}'}} s_{ij} + \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} = 0, \\
& \underbrace{\bar{U}_j \frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j} \left( \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu'}} + \overline{u'_i \tau_{ij}^{\text{SGS}'}} \right)}_{\text{Transport}} + \underbrace{J^{-1} \epsilon}_{\text{Viscous dissipation}} - \underbrace{\overline{\tau_{ij}^{\text{SGS}'}} s_{ij}}_{\text{SGS dissipation}} + \underbrace{\tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j}}_{\text{Production}} = 0.
\end{aligned} \tag{11}$$

Applying horizontal averaging to Eq. (11), we get

$$\begin{aligned}
& \frac{\partial}{\partial \xi_3} \left( \langle e \bar{U}_3 \rangle + \left\langle \frac{1}{2} \overline{u'_i u'_i U'_3} \right\rangle + \langle \overline{p' U'_3} \rangle + \langle \overline{u'_i \tau_{i3}^{\nu'}} \rangle + \langle \overline{u'_i \tau_{i3}^{\text{SGS}'}} \rangle \right) + \langle J^{-1} \epsilon \rangle - \langle \overline{\tau_{ij}^{\text{SGS}'}} s_{ij} \rangle + \left\langle \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} \right\rangle = 0, \\
& \frac{\partial}{\partial \zeta} \left( \langle e \bar{W} \rangle + \left\langle \frac{1}{2} \overline{u'_i u'_i W'} \right\rangle + \langle \overline{p' W'} \rangle + \langle \overline{u'_i \tau_{i3}^{\nu'}} \rangle + \langle \overline{u'_i \tau_{i3}^{\text{SGS}'}} \rangle \right) + \langle J^{-1} \epsilon \rangle - \langle \overline{\tau_{ij}^{\text{SGS}'}} s_{ij} \rangle + \left\langle \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} \right\rangle = 0,
\end{aligned}$$

Using

$$\begin{aligned}
\left\langle \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} \right\rangle &= \left\langle (\langle \tau_{ij} \rangle + \tilde{\tau}_{ij}) \left[ \frac{\partial \tilde{u}_i}{\partial \xi_j} + \frac{\partial \langle u_i \rangle}{\partial \xi_j} \right] \right\rangle, \\
&= \left\langle \langle \tau_{ij} \rangle \left[ \frac{\partial \tilde{u}_i}{\partial \xi_j} + \frac{\partial \langle u_i \rangle}{\partial \xi_j} \right] + \tilde{\tau}_{ij} \left[ \frac{\partial \tilde{u}_i}{\partial \xi_j} + \frac{\partial \langle u_i \rangle}{\partial \xi_j} \right] \right\rangle, \\
&= \left\langle \langle \tau_{ij} \rangle \frac{\partial \tilde{u}_i}{\partial \xi_j} + \langle \tau_{ij} \rangle \frac{\partial \langle u_i \rangle}{\partial \xi_j} + \tilde{\tau}_{ij} \frac{\partial \tilde{u}_i}{\partial \xi_j} + \tilde{\tau}_{ij} \frac{\partial \langle u_i \rangle}{\partial \xi_j} \right\rangle, \\
&= \langle \tau_{ij} \rangle \frac{\partial \langle u_i \rangle}{\partial \xi_j} + \left\langle \tilde{\tau}_{ij} \frac{\partial \tilde{u}_i}{\partial \xi_j} \right\rangle,
\end{aligned}$$

and,

$$\langle e \bar{W} \rangle = \langle (\langle e \rangle + \tilde{e})(\langle W \rangle + \tilde{W}) \rangle = \langle (\langle e \rangle + \tilde{e}) \tilde{W} \rangle = \langle \tilde{e} \tilde{W} \rangle,$$

we get the mean TKE equation as follows,

$$\frac{\partial}{\partial \zeta} \left( \langle \tilde{e} \tilde{W} \rangle + \left\langle \frac{1}{2} \overline{u'_i u'_i W'} \right\rangle + \langle \overline{p' W'} \rangle + \langle \overline{u'_i \tau_{i3}^{\nu'}} \rangle + \langle \overline{u'_i \tau_{i3}^{\text{SGS}'}} \rangle \right) + \langle J^{-1} \epsilon \rangle - \langle \overline{\tau_{ij}^{\text{SGS}'}} s_{ij} \rangle + \langle \tau_{ij} \rangle \frac{\partial \langle u_i \rangle}{\partial \xi_j} + \left\langle \tilde{\tau}_{ij} \frac{\partial \tilde{u}_i}{\partial \xi_j} \right\rangle = 0. \tag{12}$$

## 2 Computing TKE budget terms in MATLAB

Vector identities:

$$\begin{aligned}
\overline{u'_i u'_j} &= \overline{(u_i - \bar{u}_i)(u_j - \bar{u}_j)}, \\
&= \overline{u_i u_j - u_i \bar{u}_j - u_j \bar{u}_i + \bar{u}_i \bar{u}_j}, \\
&= \overline{u_i u_j} - 2\bar{u}_i \bar{u}_j + \bar{u}_i \bar{u}_j, \\
\overline{u'_i u'_j} &= \overline{u_i u_j} - \bar{u}_i \bar{u}_j.
\end{aligned}$$

$$\begin{aligned}
\overline{u'_i u'_i U'_j} &= \overline{(u_i - \bar{u}_i)(u_i - \bar{u}_i)(U_j - \bar{U}_j)}, \\
&= \overline{[u_i(u_i - \bar{u}_i) - \bar{u}_i(u_i - \bar{u}_i)](U_j - \bar{U}_j)}, \\
&= \overline{[u_i u_i - u_i \bar{u}_i - \bar{u}_i u_i + \bar{u}_i \bar{u}_i](U_j - \bar{U}_j)}, \\
&= \overline{[u_i u_i - 2u_i \bar{u}_i + \bar{u}_i \bar{u}_i](U_j - \bar{U}_j)}, \\
&= \overline{u_i u_i (U_j - \bar{U}_j) - 2u_i \bar{u}_i (U_j - \bar{U}_j) + \bar{u}_i \bar{u}_i (U_j - \bar{U}_j)}, \\
&= \overline{u_i u_i U_j - u_i u_i \bar{U}_j - 2u_i \bar{u}_i U_j + 2u_i \bar{u}_i \bar{U}_j + \bar{u}_i \bar{u}_i U_j - \bar{u}_i \bar{u}_i \bar{U}_j}, \\
&= \overline{u_i u_i U_j} - \overline{u_i u_i \bar{U}_j} - 2\overline{u_i U_j \bar{u}_i} + 2\overline{u_i \bar{u}_i \bar{U}_j} + \overline{u_i \bar{u}_i U_j} - \overline{u_i \bar{u}_i \bar{U}_j}, \\
&= \overline{u_i u_i U_j} - \overline{u_i u_i \bar{U}_j} - 2\overline{u_i U_j \bar{u}_i} + 2\overline{u_i \bar{u}_i \bar{U}_j}, \\
\overline{u'_i u'_i U'_j} &= \overline{u_i u_i U_j} - \overline{u_i u_i \bar{U}_j} - 2\overline{u_i U_j \bar{u}_i} + 2\overline{u_i \bar{u}_i \bar{U}_j}.
\end{aligned}$$

$$\begin{aligned}
\overline{u_i u_i U_j} &= \overline{(\bar{u}_i + u'_i)(\bar{u}_i + u'_i)(\bar{U}_j + U'_j)}, \\
&= \overline{(\bar{u}_i [\bar{u}_i + u'_i] + u'_i [\bar{u}_i + u'_i])(\bar{U}_j + U'_j)}, \\
&= \overline{(\bar{u}_i \bar{u}_i + \bar{u}_i u'_i + u'_i \bar{u}_i + u'_i u'_i)(\bar{U}_j + U'_j)}, \\
&= \overline{\bar{u}_i \bar{u}_i (\bar{U}_j + U'_j) + \bar{u}_i u'_i (\bar{U}_j + U'_j) + u'_i \bar{u}_i (\bar{U}_j + U'_j) + u'_i u'_i (\bar{U}_j + U'_j)}, \\
&= \overline{\bar{u}_i \bar{u}_i \bar{U}_j + \bar{u}_i \bar{u}_i U'_j + (\bar{u}_i u'_i \bar{U}_j + \bar{u}_i u'_i U'_j) + (u'_i \bar{u}_i \bar{U}_j + u'_i \bar{u}_i U'_j) + (u'_i u'_i \bar{U}_j + u'_i u'_i U'_j)}, \\
&= \overline{\bar{u}_i \bar{u}_i \bar{U}_j} + 2\overline{\bar{u}_i u'_i U'_j} + \overline{u'_i u'_i \bar{U}_j} + \overline{u'_i u'_i U'_j}, \\
&= \overline{\bar{u}_i \bar{u}_i \bar{U}_j} + 2\bar{u}_i [\overline{u_i U_j} - \bar{u}_i \bar{U}_j] + [\overline{u_i u_i} - \bar{u}_i \bar{u}_i] \bar{U}_j + \overline{u'_i u'_i U'_j}, \\
&= \overline{\bar{u}_i \bar{u}_i \bar{U}_j} + 2\bar{u}_i [\overline{u_i U_j} - \bar{u}_i \bar{U}_j] + [\overline{u_i u_i} - \bar{u}_i \bar{u}_i] \bar{U}_j + \overline{u'_i u'_i U'_j}, \\
&= \overline{\bar{u}_i \bar{u}_i \bar{U}_j} + 2\bar{u}_i \overline{u_i U_j} - 2\bar{u}_i \bar{u}_i \bar{U}_j + \overline{u_i u_i} \bar{U}_j - \bar{u}_i \bar{u}_i \bar{U}_j + \overline{u'_i u'_i U'_j}, \\
&= 2\bar{u}_i \overline{u_i U_j} - 2\bar{u}_i \bar{u}_i \bar{U}_j + \overline{u_i u_i} \bar{U}_j + \overline{u'_i u'_i U'_j}, \\
\implies \overline{u'_i u'_i U'_j} &= \overline{u_i u_i U_j} - \overline{u_i u_i \bar{U}_j} - 2\bar{u}_i [\overline{u_i U_j} - \bar{u}_i \bar{U}_j].
\end{aligned} \tag{13}$$

$$\frac{\partial}{\partial \xi_j} \left( e \bar{U}_j + e \bar{U}'_j + p' \bar{U}'_j + \overline{u'_i \tau'_{ij}} \right) + J^{-1} \epsilon + (\tau_{ij} + \bar{\tau}'_{ij}) \frac{\partial \bar{u}_i}{\partial \xi_j} = 0.$$

From Eq. (11), TKE equation is

$$\underbrace{\overline{U_j} \frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j} \left( \frac{1}{2} \overline{u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau'_{ij}} + \overline{u'_i \tau'^{\text{SGS}}_{ij}} \right)}_{\text{Transport}} + \underbrace{J^{-1} \epsilon}_{\text{Viscous dissipation}} - \underbrace{\overline{\tau'^{\text{SGS}}_{ij} s_{ij}}}_{\text{SGS dissipation}} + \underbrace{\tau_{ij} \frac{\partial \overline{u}_i}{\partial \xi_j}}_{\text{Production}} = 0.$$

1. Turbulent Kinetic Energy:

$$e = \frac{1}{2} \overline{u'_i u'_i},$$

$$e = \frac{1}{2} (\overline{u' u'} + \overline{v' v'} + \overline{w' w'}),$$

$$\overline{u' u'} = \overline{u u} - \overline{u} \overline{u}, \quad (14)$$

$$\overline{v' v'} = \overline{v v} - \overline{v} \overline{v}, \quad (15)$$

$$\overline{w' w'} = \overline{w w} - \overline{w} \overline{w}. \quad (16)$$

2. Triple correlation term: From Eq. (13), we have

$$\overline{u'_i u'_i U'_j} = \overline{u_i u_i U_j} - \overline{u_i u_i} \overline{U_j} - 2 \overline{u_i} [\overline{u_i U_j} - \overline{u_i} \overline{U_j}].$$

$$\overline{u_i u_i U_j} = \overline{(u u + v v + w w) U_j},$$

$$\overline{u_i u_i} \overline{U_j} = \overline{(u u + v v + w w)} \overline{U_j},$$

$$\overline{u_i} \overline{u_i U_j} = \overline{u} \overline{u U_j} + \overline{v} \overline{v U_j} + \overline{w} \overline{w U_j},$$

$$\overline{u_i} \overline{u_i} \overline{U_j} = (\overline{u} \overline{u} + \overline{v} \overline{v} + \overline{w} \overline{w}) \overline{U_j}.$$

3. Pressure correlation term:

$$\overline{p' U'_j} = \overline{p U_j} - \overline{p} \overline{U_j}.$$

4. Viscous transport term:

$$\overline{u'_i \tau'^{\nu}_{ij}} = \overline{u_i \tau^{\nu}_{ij}} - \overline{u_i} \overline{\tau^{\nu}_{ij}},$$

$$\overline{u'_i \tau'^{\nu}_{ij}} = \overline{u \tau^{\nu}_{1j}} - \overline{u} \overline{\tau^{\nu}_{1j}} + \overline{v \tau^{\nu}_{2j}} - \overline{v} \overline{\tau^{\nu}_{2j}} + \overline{w \tau^{\nu}_{3j}} - \overline{w} \overline{\tau^{\nu}_{3j}}.$$

For  $j = 1$ ,

$$\overline{u_i \tau^{\nu}_{i1}} = \overline{u \tau^{\nu}_{11}} + \overline{v \tau^{\nu}_{12}} + \overline{w \tau^{\nu}_{13}}.$$

For  $j = 3$ ,

$$\overline{u_i \tau^{\nu}_{i3}} = \overline{u \tau^{\nu}_{13}} + \overline{v \tau^{\nu}_{23}} + \overline{w \tau^{\nu}_{33}}.$$

5. Viscous dissipation term:

$$\epsilon = \overline{2\nu s_{ij} s_{ij}} = 2\nu [\overline{S_{ij} S_{ij}} - \overline{S_{ij}} \overline{S_{ij}}] = 2\nu \left[ \overline{S_{11} S_{11} + S_{22} S_{22} + S_{33} S_{33} + 2(S_{12} S_{12} + S_{13} S_{13} + S_{23} S_{23})} \right. \\ \left. - (\overline{S_{11}} \overline{S_{11}} + \overline{S_{22}} \overline{S_{22}} + \overline{S_{33}} \overline{S_{33}} + 2[\overline{S_{12}} \overline{S_{12}} + \overline{S_{13}} \overline{S_{13}} + \overline{S_{23}} \overline{S_{23}}]) \right].$$

6. SGS dissipation term:

$$\begin{aligned} \overline{\tau_{ij}^{\text{SGS}} s_{ij}} &= \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \overline{\tau_{ij}^{\text{SGS}} \bar{S}_{ij}} = \overline{\tau_{11}^{\text{SGS}} S_{11}} + \overline{\tau_{22}^{\text{SGS}} S_{22}} + \overline{\tau_{33}^{\text{SGS}} S_{33}} + 2 \left( \overline{\tau_{12}^{\text{SGS}} S_{12}} + \overline{\tau_{13}^{\text{SGS}} S_{13}} + \overline{\tau_{23}^{\text{SGS}} S_{23}} \right) \\ &\quad - \left[ \overline{\tau_{11}^{\text{SGS}} \bar{S}_{11}} + \overline{\tau_{22}^{\text{SGS}} \bar{S}_{22}} + \overline{\tau_{33}^{\text{SGS}} \bar{S}_{33}} + 2 \left( \overline{\tau_{12}^{\text{SGS}} \bar{S}_{12}} + \overline{\tau_{13}^{\text{SGS}} \bar{S}_{13}} + \overline{\tau_{23}^{\text{SGS}} \bar{S}_{23}} \right) \right] \end{aligned}$$

7. Production term:

$$\begin{aligned} \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} &= \tau_{i1} \frac{\partial \bar{u}_i}{\partial \xi} + \tau_{i3} \frac{\partial \bar{u}_i}{\partial \zeta}, \\ \tau_{ij} \frac{\partial \bar{u}_i}{\partial \xi_j} &= \tau_{11} \frac{\partial \bar{u}}{\partial \xi} + \tau_{21} \frac{\partial \bar{v}}{\partial \xi} + \tau_{31} \frac{\partial \bar{w}}{\partial \xi} \\ &\quad + \tau_{13} \frac{\partial \bar{u}}{\partial \zeta} + \tau_{23} \frac{\partial \bar{v}}{\partial \zeta} + \tau_{33} \frac{\partial \bar{w}}{\partial \zeta}, \end{aligned}$$

where  $\tau_{ij} = \overline{u'_i U'_j} = \overline{u_i U_j} - \bar{u}_i \bar{U}_j$ .

8. Advection term:

$$\bar{U}_j \frac{\partial e}{\partial \xi_j} = \bar{U} \frac{\partial e}{\partial \xi} + \bar{W} \frac{\partial e}{\partial \zeta}.$$

9. Transport term:

$$\frac{\partial}{\partial \xi_j} \left( \overline{\frac{1}{2} u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau'_{ij}} \right) = \frac{\partial}{\partial \xi} \left( \overline{\frac{1}{2} u'_i u'_i U'} + \overline{p' U'} + \overline{u'_i \tau'_{i1}} \right) + \frac{\partial}{\partial \zeta} \left( \overline{\frac{1}{2} u'_i u'_i W'} + \overline{p' W'} + \overline{u'_i \tau'_{i3}} \right)$$

## References

Tetsu Hara and Peter P. Sullivan. Wave boundary layer turbulence over surface waves in a strongly forced condition. *Journal of Physical Oceanography*, 45(3):868 – 883, 2015. doi: 10.1175/JPO-D-14-0116.1. URL <https://journals.ametsoc.org/view/journals/phoc/45/3/jpo-d-14-0116.1.xml>.