Turbulent Kinetic Energy equation derivation

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Contents

1 Turbulent Kinetic Energy Equation in Wave-Following Curvilinear Coordinates (Hara and Sullivan, 2015)

$$\frac{\partial U_i}{\partial \xi_i} = 0, \tag{1}$$

$$J^{-1}\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j}\left(u_i U_j + J^{-1}p\frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right) = 0, \tag{2}$$

$$U_i = J^{-1}u_j\frac{\partial \xi_i}{\partial x_j}. \tag{3}$$

Evaluating $u_i \times$ Eq.(2), we get

$$J^{-1}u_i\frac{\partial u_i}{\partial t} + u_i\frac{\partial}{\partial \xi_j}\left(u_i U_j + J^{-1}p\frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right) = 0,$$

$$J^{-1}u_i\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j}\left(u_i u_i U_j + J^{-1}u_i p\frac{\partial \xi_j}{\partial x_i} + J^{-1}u_i\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}u_i\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)$$
$$- \left(u_i U_j + J^{-1}p\frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)\frac{\partial u_i}{\partial \xi_j} = 0, \tag{4}$$

Expanding third term in Eq. (4), we get

$$\left(u_i U_j + J^{-1}p\frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k}\right.$$
$$\left. + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)\frac{\partial u_i}{\partial \xi_j} = u_i U_j\frac{\partial u_i}{\partial \xi_j} + J^{-1}p\frac{\partial \xi_j}{\partial x_i}\frac{\partial u_i}{\partial \xi_j} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j} + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j},$$

$$= \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j \right) + J^{-1} p \frac{\partial u_i}{\partial x_i} + J^{-1} \sigma_{ik} \frac{\partial u_i}{\partial x_k} + J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k},$$

$$= \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik} \left(\frac{1}{2} \left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right] + \frac{1}{2} \left[\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \right] \right)$$
$$+ J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k},$$

$$= \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik} \left(S_{ik} + \Omega_{ik} \right) + J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k},$$

$$= \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik}^2 + J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k} \ (\text{Using } S_{ik}\Omega_{ik} = 0).$$

$$(5)$$

Using Eq.(5) in Eq.(4), we have

$$\frac{\partial}{\partial t} \left(J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left(u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i \sigma_{ik} \frac{\partial \xi_j}{\partial x_k} + J^{-1} u_i \sigma_{ik}^{\mathrm{SGS}} \frac{\partial \xi_j}{\partial x_k} \right)$$
$$- \left(\frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j \right) - J^{-1} 2\nu S_{ik}^2 + J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k} \right) = 0,$$

$$\frac{\partial}{\partial t} \left(J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i (\sigma_{ik} + \sigma_{ik}^{\mathrm{SGS}}) \frac{\partial \xi_j}{\partial x_k} \right) + J^{-1} 2\nu S_{ik}^2 - J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k} = 0,$$

$$\frac{\partial}{\partial t} \left(J^{-1} \frac{1}{2} u_i u_i \right) + \frac{\partial}{\partial \xi_j} \left(\frac{1}{2} u_i u_i U_j + p U_j + J^{-1} u_i (\sigma_{ik} + \sigma_{ik}^{\mathrm{SGS}}) \frac{\partial \xi_j}{\partial x_k} \right) + J^{-1} 2\nu S_{ik}^2 - J^{-1} \sigma_{ik}^{\mathrm{SGS}} \frac{\partial u_i}{\partial x_k} = 0.$$

$$(6)$$

Applying Reynolds averaging to Eq.(2), and defining $\tau_{ij}^p = J^{-1} p \partial \xi_j / \partial x_i$, $\tau_{ij}^\nu = J^{-1} \sigma_{ik} \partial \xi_j / \partial x_k$, $\tau_{ij} = \overline{u_i' U_j'}$, $\tau_{ij}^{\mathrm{SGS}} = J^{-1} \sigma_{ik}^{\mathrm{SGS}} \partial \xi_j / \partial x_k$ we have

$$\overline{J^{-1} \frac{\partial u_i}{\partial t}} + \frac{\partial}{\partial \xi_j} \left(\overline{u_i U_j} + \overline{\tau}_{ij}^p + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}} \right) = 0,$$

$$\frac{\partial}{\partial \xi_j} \left(\overline{u}_i \overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^p + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}} \right) = 0. \quad (7)$$

Evaluating $\overline{u}_i \times$ Eq. (7), we get the following mean kinetic energy $E = (1/2)\overline{u}_i \overline{u}_i$ equation,

$$\overline{u}_i \frac{\partial}{\partial \xi_j} \left(\overline{u}_i \overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^p + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}} \right) = 0,$$

$$\frac{\partial}{\partial \xi_j} \left(\overline{u}_i \overline{u}_i \overline{U}_j + \overline{u}_i \tau_{ij} + \overline{u}_i \overline{\tau}_{ij}^p + \overline{u}_i \overline{\tau}_{ij}^\nu + \overline{u}_i \overline{\tau}_{ij}^{\mathrm{SGS}} \right) - \left(\overline{u}_i \overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^p + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}} \right) \frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j} \left(\frac{1}{2} \overline{u}_i \overline{u}_i \overline{U}_j + \overline{u}_i (\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}}) + \overline{u}_i \overline{J^{-1} p \frac{\partial \xi_j}{\partial x_i}} \right) - \left(\tau_{ij} \frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{J^{-1} p \frac{\partial \xi_j}{\partial x_i}} \frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{\tau}_{ij}^\nu \frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{\tau}_{ij}^{\mathrm{SGS}} \frac{\partial \overline{u}_i}{\partial \xi_j} \right) = 0,$$

$$\frac{\partial}{\partial \xi_j} \left(\frac{1}{2} \overline{u}_i \overline{u}_i \overline{U}_j + \overline{u}_i (\tau_{ij} + \overline{\tau}_{ij}^\nu + \tau_{ij}^{\mathrm{SGS}}) + \overline{p} \overline{U}_j \right) - \left(\overline{J^{-1} p \frac{\partial \overline{u}_i}{\partial x_i}} + (\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}}) \frac{\partial \overline{u}_i}{\partial \xi_j} \right) = 0,$$

$$\frac{\partial}{\partial \xi_j} \left(\frac{1}{2} \overline{u}_i \overline{u}_i \overline{U}_j + \overline{u}_i (\tau_{ij} + \overline{\tau}_{ij}^\nu + \tau_{ij}^{\mathrm{SGS}}) + \overline{p} \overline{U}_j \right) - \left(\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\mathrm{SGS}} \right) \frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}}) + \overline{p}\overline{U}_j\right) - \left(\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}}\right)\frac{\partial \overline{u}_i}{\partial \xi_j} = 0.$$

$$(8)$$

Now, let us evaluate $u_i u_i U_j$

$$
\begin{aligned}
u_i u_i U_j &= (\overline{u}_i + u_i')(\overline{u}_i + u_i')U_j, \\
&= (\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')(\overline{U}_j + U_j'), \\
&= [(\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')\overline{U}_j + (\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')U_j'], \\
&= \overline{u}_i\overline{u}_i\overline{U}_j + u_i'\overline{u}_i\overline{U}_j + \overline{u}_i u_i'\overline{U}_j + u_i'u_i'\overline{U}_j \\
&\quad + \overline{u}_i\overline{u}_i U_j' + u_i'\overline{u}_i U_j' + \overline{u}_i u_i' U_j' + u_i'u_i'U_j'.
\end{aligned}
$$

$$(9)$$

Phase averaging or Reynolds averaging of Eq. (9) and defining TKE $e = (1/2)\overline{u_i'u_i'}$, we get

$$
\begin{aligned}
\overline{u_i u_i U_j} &= \overline{u}_i\overline{u}_i\overline{U}_j + \overline{u_i'u_i'}\overline{U}_j + 2\overline{u}_i\tau_{ij} + \overline{u_i'u_i'U_j'}, \\
\frac{1}{2}\overline{u_i u_i U_j} &= \frac{1}{2}\overline{u}_i\overline{u}_i\overline{U}_j + \frac{1}{2}\overline{u_i'u_i'}\overline{U}_j + \overline{u}_i\tau_{ij} + \frac{1}{2}\overline{u_i'u_i'U_j'}, \\
\frac{1}{2}\overline{u_i u_i U_j} &= E\overline{U}_j + e\overline{U}_j + \overline{u}_i\tau_{ij} + \frac{1}{2}\overline{u_i'u_i'U_j'},
\end{aligned}
$$

$$(10)$$

Defining turbulent dissipation rate as $\epsilon = \overline{2\nu s_{ij}s_{ij}} = 2\nu\left[\overline{S_{ij}S_{ij}} - \overline{S}_{ij}\overline{S}_{ij}\right]$ (where, $s_{ij} = (1/2)[\partial u_i'/\partial x_j + \partial u_j'/\partial x_i]$), using Eq.(10) and evaluating $\overline{(6)} - (8)$, we get

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u_i u_i U_j} + \overline{p U_j} + \overline{u_i\tau_{ij}^\nu} + \overline{u_i\tau_{ij}^{\text{SGS}}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial u_i}{\partial x_k}} \\
&\qquad\qquad - \frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}}) + \overline{p}\overline{U}_j\right) + (\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_j}\left((E + e)\overline{U}_j + \overline{u}_i\tau_{ij} + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p U_j} + \overline{u_i\tau_{ij}^\nu} + \overline{u_i\tau_{ij}^{\text{SGS}}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial u_i}{\partial x_k}} \\
&\qquad\qquad - \frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}}) + \overline{p}\overline{U}_j\right) + (\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
\end{aligned}
$$

$$
\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau_{ij}^{\nu\prime}} + \overline{u_i'\tau_{ij}^{\text{SGS}\prime}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial u_i}{\partial x_k}} + (\tau_{ij} + \overline{\tau}_{ij}^\nu + \overline{\tau}_{ij}^{\text{SGS}})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
$$

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau_{ij}^{\nu\prime}} + \overline{u_i'\tau_{ij}^{\text{SGS}\prime}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} + \overline{\tau}_{ij}^\nu\frac{\partial \overline{u}_i}{\partial \xi_j} \\
&\qquad\qquad - \overline{J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial u_i}{\partial x_k}} + \overline{\tau}_{ij}^{\text{SGS}}\frac{\partial \overline{u}_i}{\partial \xi_j} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau_{ij}^{\nu\prime}} + \overline{u_i'\tau_{ij}^{\text{SGS}\prime}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}2\nu S_{ik}\frac{\partial \xi_j}{\partial x_k}\frac{\partial \overline{u}_i}{\partial \xi_j}} \\
&\qquad\qquad - \left(\overline{J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j}} - J^{-1}\overline{\sigma}_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\frac{\partial \overline{u}_i}{\partial \xi_j}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
\end{aligned}
$$

$$
\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau_{ij}^{\nu\prime}} + \overline{u_i'\tau_{ij}^{\text{SGS}\prime}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}2\nu S_{ik}\frac{\partial \overline{u}_i}{\partial x_k}}
$$

$$-\left(\overline{\tau_{ij}^{\mathrm{SGS}}S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}}\overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u'_iu'_iU'_j} + \overline{p'U'_j} + \overline{u'_i\tau_{ij}^{\nu\prime}} + \overline{u'_i\tau_{ij}^{\mathrm{SGS}\prime}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - J^{-1}2\nu \overline{S}_{ik}\overline{S}_{ik}$$
$$-\left(\overline{\tau_{ij}^{\mathrm{SGS}}S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}}\overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u'_iu'_iU'_j} + \overline{p'U'_j} + \overline{u'_i\tau_{ij}^{\nu\prime}} + \overline{u'_i\tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1}2\nu[\overline{S_{ik}S_{ik}} - \overline{S}_{ik}\overline{S}_{ik}]$$
$$-\left(\overline{\tau_{ij}^{\mathrm{SGS}}S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}}\overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u'_iu'_iU'_j} + \overline{p'U'_j} + \overline{u'_i\tau_{ij}^{\nu\prime}} + \overline{u'_i\tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1}2\nu\overline{s_{ik}s_{ik}} - \overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u'_iu'_iU'_j} + \overline{p'U'_j} + \overline{u'_i\tau_{ij}^{\nu\prime}} + \overline{u'_i\tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1}\epsilon - \overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,$$

$$\underbrace{\overline{U}_j\frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u'_iu'_iU'_j} + \overline{p'U'_j} + \overline{u'_i\tau_{ij}^{\nu\prime}} + \overline{u'_i\tau_{ij}^{\mathrm{SGS}\prime}}\right)}_{\text{Transport}} + \underbrace{J^{-1}\epsilon}_{\substack{\text{Viscous dissipation}}} - \underbrace{\overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}}}_{\substack{\text{SGS dissipation}}} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}}_{\substack{\text{Production}}} = 0.$$

(11)

Applying horizontal averaging to Eq. (11), we get

$$\frac{\partial}{\partial \xi_3}\left(\langle e\overline{U}_3\rangle + \left\langle\overline{\frac{1}{2}u'_iu'_iU'_3}\right\rangle + \langle\overline{p'U'_3}\rangle + \langle\overline{u'_i\tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u'_i\tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}}\right\rangle + \left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle = 0,$$

$$\frac{\partial}{\partial \zeta}\left(\langle e\overline{W}\rangle + \left\langle\overline{\frac{1}{2}u'_iu'_iW'}\right\rangle + \langle\overline{p'W'}\rangle + \langle\overline{u'_i\tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u'_i\tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}}\right\rangle + \left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle = 0,$$

Using

$$\left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle = \left\langle(\langle\tau_{ij}\rangle + \widetilde{\tau}_{ij})\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right]\right\rangle,$$
$$= \left\langle\langle\tau_{ij}\rangle\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right] + \widetilde{\tau}_{ij}\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right]\right\rangle,$$
$$= \left\langle\langle\tau_{ij}\rangle\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \widetilde{\tau}_{ij}\frac{\partial\langle u_i\rangle}{\partial \xi_j}\right\rangle,$$
$$= \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \left\langle\widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j}\right\rangle,$$

and,

$$\langle e\overline{W}\rangle = \langle(\langle e\rangle + \widetilde{e})(\langle W\rangle + \widetilde{W})\rangle = \langle(\langle e\rangle + \widetilde{e})\widetilde{W}\rangle = \langle\widetilde{e}\widetilde{W}\rangle,$$

we get the mean TKE equation as follows,

$$\frac{\partial}{\partial \zeta}\left(\langle\widetilde{e}\widetilde{W}\rangle + \left\langle\overline{\frac{1}{2}u'_iu'_iW'}\right\rangle + \langle\overline{p'W'}\rangle + \langle\overline{u'_i\tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u'_i\tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime}s_{ij}}\right\rangle + \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \left\langle\widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j}\right\rangle = 0.$$

(12)

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2   Computing TKE budget terms in MATLAB

Vector identities:

$$\overline{u_i' u_j'} = \overline{(u_i - \overline{u_i})(u_j - \overline{u_j})},$$
$$= \overline{u_i u_j - u_i \overline{u_j} - u_j \overline{u_i} + \overline{u_i}\,\overline{u_j}},$$
$$= \overline{u_i u_j} - 2\overline{u_i}\,\overline{u_j} + \overline{u_i}\,\overline{u_j},$$
$$\overline{u_i' u_j'} = \overline{u_i u_j} - \overline{u_i}\,\overline{u_j}.$$

$$\overline{u_i' u_i' U_j'} = \overline{(u_i - \overline{u_i})(u_i - \overline{u_i})(U_j - \overline{U}_j)},$$
$$= \overline{[u_i(u_i - \overline{u_i}) - \overline{u_i}(u_i - \overline{u_i})](U_j - \overline{U}_j)},$$
$$= \overline{[u_i u_i - u_i \overline{u_i} - \overline{u_i} u_i + \overline{u_i}\,\overline{u_i}](U_j - \overline{U}_j)},$$
$$= \overline{[u_i u_i - 2u_i \overline{u_i} + \overline{u_i}\,\overline{u_i}](U_j - \overline{U}_j)},$$
$$= \overline{u_i u_i(U_j - \overline{U}_j) - 2u_i \overline{u_i}(U_j - \overline{U}_j) + \overline{u_i}\,\overline{u_i}(U_j - \overline{U}_j)},$$
$$= \overline{u_i u_i U_j - u_i u_i \overline{U}_j - 2u_i \overline{u_i} U_j + 2u_i \overline{u_i}\,\overline{U}_j + \overline{u_i}\,\overline{u_i} U_j - \overline{u_i}\,\overline{u_i}\,\overline{U}_j},$$
$$= \overline{u_i u_i U_j} - \overline{u_i u_i}\,\overline{U}_j - 2\overline{u_i U_j}\,\overline{u_i} + 2\overline{u_i}\,\overline{u_i}\,\overline{U}_j + \overline{u_i}\,\overline{u_i}\,\overline{U}_j - \overline{u_i}\,\overline{u_i}\,\overline{U}_j,$$
$$= \overline{u_i u_i U_j} - \overline{u_i u_i}\,\overline{U}_j - 2\overline{u_i U_j}\,\overline{u_i} + 2\overline{u_i}\,\overline{u_i}\,\overline{U}_j,$$
$$\overline{u_i' u_i' U_j'} = \overline{u_i u_i U_j} - \overline{u_i u_i}\,\overline{U}_j - 2\overline{u_i}[\overline{u_i U_j} - \overline{u_i}\,\overline{U}_j].$$

$$\overline{u_i u_i U_j} = \overline{(\overline{u_i} + u_i')(\overline{u_i} + u_i')(\overline{U}_j + U_j')},$$
$$= \overline{(\overline{u_i}[\overline{u_i} + u_i'] + u_i'[\overline{u_i} + u_i'])(\overline{U}_j + U_j')},$$
$$= \overline{(\overline{u_i}\,\overline{u_i} + \overline{u_i} u_i' + u_i' \overline{u_i} + u_i' u_i')(\overline{U}_j + U_j')},$$
$$= \overline{\overline{u_i}\,\overline{u_i}(\overline{U}_j + U_j') + \overline{u_i} u_i'(\overline{U}_j + U_j') + u_i' \overline{u_i}(\overline{U}_j + U_j') + u_i' u_i'(\overline{U}_j + U_j')},$$
$$= \overline{\overline{u_i}\,\overline{u_i}\,\overline{U}_j + \overline{u_i}\,\overline{u_i} U_j' + (\overline{u_i} u_i' \overline{U}_j + \overline{u_i} u_i' U_j') + (u_i' \overline{u_i}\,\overline{U}_j + u_i' \overline{u_i} U_j') + (u_i' u_i' \overline{U}_j + u_i' u_i' U_j')},$$
$$= \overline{u_i}\,\overline{u_i}\,\overline{U}_j + 2\overline{u_i}\,\overline{u_i' U_j'} + \overline{u_i' u_i'}\,\overline{U}_j + \overline{u_i' u_i' U_j'},$$
$$= \overline{u_i}\,\overline{u_i}\,\overline{U}_j + 2\overline{u_i}[\overline{u_i U_j} - \overline{u_i}\,\overline{U}_j] + [\overline{u_i u_i} - \overline{u_i}\,\overline{u_i}]\overline{U}_j + \overline{u_i' u_i' U_j'},$$
$$= \overline{u_i}\,\overline{u_i}\,\overline{U}_j + 2\overline{u_i}[\overline{u_i U_j} - \overline{u_i}\,\overline{U}_j] + [\overline{u_i u_i} - \overline{u_i}\,\overline{u_i}]\overline{U}_j + \overline{u_i' u_i' U_j'},$$
$$= \overline{u_i}\,\overline{u_i}\,\overline{U}_j + 2\overline{u_i}\,\overline{u_i U_j} - 2\overline{u_i}\,\overline{u_i}\,\overline{U}_j + \overline{u_i u_i}\,\overline{U}_j - \overline{u_i}\,\overline{u_i}\,\overline{U}_j + \overline{u_i' u_i' U_j'},$$
$$= 2\overline{u_i}\,\overline{u_i U_j} - 2\overline{u_i}\,\overline{u_i}\,\overline{U}_j + \overline{u_i u_i}\,\overline{U}_j + \overline{u_i' u_i' U_j'},$$
$$\implies \overline{u_i' u_i' U_j'} = \overline{u_i u_i U_j} - \overline{u_i u_i}\,\overline{U}_j - 2\overline{u_i}[\overline{u_i U_j} - \overline{u_i}\,\overline{U}_j]. \tag{13}$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{eU_j'} + \overline{p'U_j'} + \overline{u_i' \tau_{ij}^{\nu'}}\right) + J^{-1}\epsilon + (\tau_{ij} + \overline{\tau}_{ij}^{\nu})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0.$$

From Eq. (11), TKE equation is

$$\underbrace{\overline{U}_j \frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j}\left(\frac{1}{2}\overline{u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\text{SGS}\prime}}\right)}_{\text{Transport}} + \underbrace{J^{-1}\epsilon}_{\text{Viscous dissipation}} - \underbrace{\overline{\tau_{ij}^{\text{SGS}\prime} s_{ij}}}_{\text{SGS dissipation}} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}}_{\text{Production}} = 0.$$

1. Turbulent Kinetic Energy:

$$e = \frac{1}{2}\overline{u_i' u_i'},$$
$$e = \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}),$$
$$\overline{u'u'} = \overline{uu} - \overline{u}\,\overline{u}, \tag{14}$$
$$\overline{v'v'} = \overline{vv} - \overline{v}\,\overline{v}, \tag{15}$$
$$\overline{w'w'} = \overline{ww} - \overline{w}\,\overline{w}. \tag{16}$$

2. Triple correlation term: From Eq. (13), we have

$$\overline{u_i' u_i' U_j'} = \overline{u_i u_i U_j} - \overline{u_i u_i}\overline{U}_j - 2\overline{u}_i[\overline{u_i U_j} - \overline{u_i}\,\overline{U}_j].$$

$$\overline{u_i u_i U_j} = \overline{(uu + vv + ww)U_j},$$
$$\overline{u_i u_i}\overline{U}_j = \overline{(uu + vv + ww)}\,\overline{U}_j,$$
$$\overline{u_i u_i U_j} = \overline{u}\,\overline{uU_j} + \overline{v}\,\overline{vU_j} + \overline{w}\,\overline{wU_j},$$
$$\overline{u_i u_i}\overline{U}_j = (\overline{u}\,\overline{u} + \overline{v}\,\overline{v} + \overline{w}\,\overline{w})\overline{U}_j.$$

3. Pressure correlation term:

$$\overline{p' U_j'} = \overline{pU_j} - \overline{p}\,\overline{U}_j.$$

4. Viscous transport term:

$$\overline{u_i' \tau_{ij}^{\nu\prime}} = \overline{u_i \tau_{ij}^{\nu}} - \overline{u}_i\,\overline{\tau_{ij}^{\nu}},$$
$$\overline{u_i' \tau_{ij}^{\nu\prime}} = \overline{u\tau_{1j}^{\nu}} - \overline{u}\,\overline{\tau_{1j}^{\nu}} + \overline{v\tau_{2j}^{\nu}} - \overline{v}\,\overline{\tau_{2j}^{\nu}} + \overline{w\tau_{3j}^{\nu}} - \overline{w}\,\overline{\tau_{3j}^{\nu}}.$$

For $j = 1$,

$$\overline{u_i \tau_{i1}^{\nu}} = \overline{u\tau_{11}^{\nu}} + \overline{v\tau_{12}^{\nu}} + \overline{w\tau_{13}^{\nu}}.$$

For $j = 3$,

$$\overline{u_i \tau_{i3}^{\nu}} = \overline{u\tau_{13}^{\nu}} + \overline{v\tau_{23}^{\nu}} + \overline{w\tau_{33}^{\nu}}.$$

5. Viscous dissipation term:

$$\epsilon = \overline{2\nu s_{ij} s_{ij}} = 2\nu[\overline{S_{ij}S_{ij}} - \overline{S}_{ij}\overline{S}_{ij}] = 2\nu\left[\overline{S_{11}S_{11} + S_{22}S_{22} + S_{33}S_{33} + 2(S_{12}S_{12} + S_{13}S_{13} + S_{23}S_{23})}\right.$$

$$\left. - (\overline{S}_{11}\overline{S}_{11} + \overline{S}_{22}\overline{S}_{22} + \overline{S}_{33}\overline{S}_{33} + 2[\overline{S}_{12}\overline{S}_{12} + \overline{S}_{13}\overline{S}_{13} + \overline{S}_{23}\overline{S}_{23}])\right].$$

6. SGS dissipation term:

$$\overline{\tau_{ij}^{\text{SGS}\prime} s_{ij}} = \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \overline{\tau_{ij}^{\text{SGS}} \overline{S}_{ij}} = \overline{\tau_{11}^{\text{SGS}} S_{11}} + \overline{\tau_{22}^{\text{SGS}} S_{22}} + \overline{\tau_{33}^{\text{SGS}} S_{33}} + \overline{\tau_{12}^{\text{SGS}} S_{12}} + \overline{\tau_{13}^{\text{SGS}} S_{13}} + \overline{\tau_{23}^{\text{SGS}} S_{23}}$$
$$+ \overline{\tau_{21}^{\text{SGS}} S_{12}} + \overline{\tau_{31}^{\text{SGS}} S_{13}} + \overline{\tau_{32}^{\text{SGS}} S_{23}}$$
$$- \left[\overline{\tau_{11}^{\text{SGS}} \overline{S}_{11}} + \overline{\tau_{22}^{\text{SGS}} \overline{S}_{22}} + \overline{\tau_{33}^{\text{SGS}} \overline{S}_{33}} + \overline{\tau_{12}^{\text{SGS}} \overline{S}_{12}} + \overline{\tau_{13}^{\text{SGS}} \overline{S}_{13}} + \overline{\tau_{23}^{\text{SGS}} \overline{S}_{23}} \right.$$
$$\left. + \overline{\tau_{21}^{\text{SGS}} \overline{S}_{12}} + \overline{\tau_{31}^{\text{SGS}} \overline{S}_{13}} + \overline{\tau_{32}^{\text{SGS}} \overline{S}_{23}} \right]$$

7. Production term:

$$\tau_{ij} \frac{\partial \overline{u}_i}{\partial \xi_j} = \tau_{i1} \frac{\partial \overline{u}_i}{\partial \xi} + \tau_{i3} \frac{\partial \overline{u}_i}{\partial \zeta},$$
$$\tau_{ij} \frac{\partial \overline{u}_i}{\partial \xi_j} = \tau_{11} \frac{\partial \overline{u}}{\partial \xi} + \tau_{21} \frac{\partial \overline{v}}{\partial \xi} + \tau_{31} \frac{\partial \overline{w}}{\partial \xi}$$
$$+ \tau_{13} \frac{\partial \overline{u}}{\partial \zeta} + \tau_{23} \frac{\partial \overline{v}}{\partial \zeta} + \tau_{33} \frac{\partial \overline{w}}{\partial \zeta},$$

where $\tau_{ij} = \overline{u'_i U'_j} = \overline{u_i U_j} - \overline{u}_i \overline{U}_j$.

8. Advection term:

$$\overline{U}_j \frac{\partial e}{\partial \xi_j} = \overline{U} \frac{\partial e}{\partial \xi} + \overline{W} \frac{\partial e}{\partial \zeta}.$$

9. Transport term:

$$\frac{\partial}{\partial \xi_j} \left(\overline{\frac{1}{2} u'_i u'_i U'_j} + \overline{p' U'_j} + \overline{u'_i \tau_{ij}^{\nu\prime}} + \overline{u'_i \tau_{ij}^{\text{SGS}\prime}} \right) = \frac{\partial}{\partial \xi} \left(\overline{\frac{1}{2} u'_i u'_i U'} + \overline{p' U'} + \overline{u'_i \tau_{i1}^{\nu\prime}} + \overline{u'_i \tau_{i1}^{\text{SGS}\prime}} \right)$$
$$+ \frac{\partial}{\partial \zeta} \left(\overline{\frac{1}{2} u'_i u'_i W'} + \overline{p' W'} + \overline{u'_i \tau_{i3}^{\nu\prime}} + \overline{u'_i \tau_{i3}^{\text{SGS}\prime}} \right)$$

References

Tetsu Hara and Peter P. Sullivan. Wave boundary layer turbulence over surface waves in a strongly forced condition. *Journal of Physical Oceanography*, 45(3):868 – 883, 2015. doi: 10.1175/JPO-D-14-0116.1. URL https://journals.ametsoc.org/view/journals/phoc/45/3/jpo-d-14-0116.1.xml.