Turbulent Kinetic Energy equation derivation

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Contents

1 Turbulent Kinetic Energy Equation in Wave-Following Curvilinear Coordinates (Hara and Sullivan, 2015)

$$\frac{\partial U_i}{\partial \xi_i} = 0, \tag{1}$$

$$J^{-1}\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j}\left(u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\right) = 0, \tag{2}$$

$$U_i = J^{-1} u_j \frac{\partial \xi_i}{\partial x_j}. \tag{3}$$

Evaluating $u_i \times$ Eq.(2), we get

$$\begin{aligned}
J^{-1} u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial \xi_j}\left(u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\right) &= 0, \\
J^{-1} u_i \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial \xi_j}\left(u_i u_i U_j + J^{-1} u_i p \frac{\partial \xi_j}{\partial x_i} + J^{-1} u_i \sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1} u_i \sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)& \\
- \left(u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)\frac{\partial u_i}{\partial \xi_j} &= 0,
\end{aligned} \tag{4}$$

Expanding third term in Eq. (4), we get

$$\begin{aligned}
&\left(u_i U_j + J^{-1} p \frac{\partial \xi_j}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k}\right. \\
&\qquad \left. + J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\right)\frac{\partial u_i}{\partial \xi_j} = u_i U_j \frac{\partial u_i}{\partial \xi_j} + J^{-1} p \frac{\partial \xi_j}{\partial x_i}\frac{\partial u_i}{\partial \xi_j} + J^{-1}\sigma_{ik}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j} + J^{-1}\sigma_{ik}^{\text{SGS}}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j},
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j\right) + J^{-1}p\frac{\partial u_i}{\partial x_i} + J^{-1}\sigma_{ik}\frac{\partial u_i}{\partial x_k} + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j\right) - J^{-1}2\nu S_{ik}\left(\frac{1}{2}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right]\right) \\
&\quad + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j\right) - J^{-1}2\nu S_{ik}\left(S_{ik} + \Omega_{ik}\right) + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k}, \\
&= \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j\right) - J^{-1}2\nu S_{ik}^2 + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k} \text{ (Using } S_{ik}\Omega_{ik} = 0).
\end{aligned} \tag{5}
$$

Using Eq.(5) in Eq.(4), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(J^{-1}\frac{1}{2}u_i u_i\right) + \frac{\partial}{\partial \xi_j}\left(u_i u_i U_j + J^{-1}u_i p\frac{\partial \xi_j}{\partial x_i} + J^{-1}u_i\sigma_{ik}\frac{\partial \xi_j}{\partial x_k} + J^{-1}u_i\sigma_{ik}^{\mathrm{SGS}}\frac{\partial \xi_j}{\partial x_k}\right) \\
&\qquad - \left(\frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j\right) - J^{-1}2\nu S_{ik}^2 + J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k}\right) = 0, \\
&\frac{\partial}{\partial t}\left(J^{-1}\frac{1}{2}u_i u_i\right) + \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j + J^{-1}u_i p\frac{\partial \xi_j}{\partial x_i} + J^{-1}u_i(\sigma_{ik} + \sigma_{ik}^{\mathrm{SGS}})\frac{\partial \xi_j}{\partial x_k}\right) + J^{-1}2\nu S_{ik}^2 - J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k} = 0, \\
&\frac{\partial}{\partial t}\left(J^{-1}\frac{1}{2}u_i u_i\right) + \frac{\partial}{\partial \xi_j}\left(\frac{1}{2}u_i u_i U_j + pU_j + J^{-1}u_i(\sigma_{ik} + \sigma_{ik}^{\mathrm{SGS}})\frac{\partial \xi_j}{\partial x_k}\right) + J^{-1}2\nu S_{ik}^2 - J^{-1}\sigma_{ik}^{\mathrm{SGS}}\frac{\partial u_i}{\partial x_k} = 0.
\end{aligned} \tag{6}
$$

Applying Reynolds averaging to Eq.(2), and defining $\tau_{ij}^{p} = J^{-1}p\partial\xi_j/\partial x_i$, $\tau_{ij}^{\nu} = J^{-1}\sigma_{ik}\partial\xi_j/\partial x_k$, $\tau_{ij} = \overline{u_i' U_j'}$, $\tau_{ij}^{\mathrm{SGS}} = J^{-1}\sigma_{ik}^{\mathrm{SGS}}\partial\xi_j/\partial x_k$ we have

$$
\begin{aligned}
\overline{J^{-1}\frac{\partial u_i}{\partial t}} + \frac{\partial}{\partial \xi_j}\left(\overline{u_i U_j} + \overline{\tau}_{ij}^{p} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}\right) &= 0, \\
\frac{\partial}{\partial \xi_j}\left(\overline{u}_i\overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^{p} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}\right) &= 0.
\end{aligned} \tag{7}
$$

Evaluating $\overline{u}_i\times$ Eq. (7), we get the following mean kinetic energy $E = (1/2)\overline{u}_i\overline{u}_i$ equation,

$$
\begin{aligned}
&\overline{u}_i\frac{\partial}{\partial \xi_j}\left(\overline{u}_i\overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^{p} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}\right) = 0, \\
&\frac{\partial}{\partial \xi_j}\left(\overline{u}_i\overline{u}_i\overline{U}_j + \overline{u}_i\tau_{ij} + \overline{u}_i\overline{\tau}_{ij}^{p} + \overline{u}_i\overline{\tau}_{ij}^{\nu} + \overline{u}_i\overline{\tau}_{ij}^{\mathrm{SGS}}\right) - \left(\overline{u}_i\overline{U}_j + \tau_{ij} + \overline{\tau}_{ij}^{p} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}\right)\frac{\partial \overline{u}_i}{\partial \xi_j} = 0, \\
&\frac{\partial}{\partial \xi_j}\left(\frac{1}{2}\overline{u}_i\overline{u}_i\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}) + \overline{u}_i\overline{J^{-1}p\frac{\partial \xi_j}{\partial x_i}}\right) - \left(\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{J^{-1}p\frac{\partial \xi_j}{\partial x_i}}\frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{\tau}_{ij}^{\nu}\frac{\partial \overline{u}_i}{\partial \xi_j} + \overline{\tau}_{ij}^{\mathrm{SGS}}\frac{\partial \overline{u}_i}{\partial \xi_j}\right) = 0, \\
&\frac{\partial}{\partial \xi_j}\left(\frac{1}{2}\overline{u}_i\overline{u}_i\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^{\nu} + \tau_{ij}^{\mathrm{SGS}}) + \overline{p}\overline{U}_j\right) - \left(\overline{J^{-1}p\frac{\partial \overline{u}_i}{\partial x_i}} + (\tau_{ij} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}})\frac{\partial \overline{u}_i}{\partial \xi_j}\right) = 0, \\
&\frac{\partial}{\partial \xi_j}\left(\frac{1}{2}\overline{u}_i\overline{u}_i\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}) + \overline{p}\overline{U}_j\right) - \left(\tau_{ij} + \overline{\tau}_{ij}^{\nu} + \overline{\tau}_{ij}^{\mathrm{SGS}}\right)\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,
\end{aligned}
$$

$$\frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij}) + \overline{p}\overline{U}_j\right) - \left(\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij}\right)\frac{\partial \overline{u}_i}{\partial \xi_j} = 0. \tag{8}$$

Now, let us evaluate $u_i u_i U_j$

$$\begin{aligned}
u_i u_i U_j &= (\overline{u}_i + u_i')(\overline{u}_i + u_i')U_j,\\
&= (\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')(\overline{U}_j + U_j'),\\
&= [(\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')\overline{U}_j + (\overline{u}_i\overline{u}_i + u_i'\overline{u}_i + \overline{u}_i u_i' + u_i'u_i')U_j'],\\
&= \overline{u}_i\overline{u}_i\overline{U}_j + u_i'\overline{u}_i\overline{U}_j + \overline{u}_i u_i'\overline{U}_j + u_i'u_i'\overline{U}_j\\
&\quad + \overline{u}_i\overline{u}_i U_j' + u_i'\overline{u}_i U_j' + \overline{u}_i u_i' U_j' + u_i'u_i'U_j'.
\end{aligned} \tag{9}$$

Phase averaging or Reynolds averaging of Eq. (9) and defining TKE $e = (1/2)\overline{u_i'u_i'}$, we get

$$\begin{aligned}
\overline{u_i u_i U_j} &= \overline{u}_i\overline{u}_i\overline{U}_j + \overline{u_i'u_i'}\overline{U}_j + 2\overline{u}_i\tau_{ij} + \overline{u_i'u_i'U_j'},\\
\frac{1}{2}\overline{u_i u_i U_j} &= \frac{1}{2}\overline{u}_i\overline{u}_i\overline{U}_j + \frac{1}{2}\overline{u_i'u_i'}\overline{U}_j + \overline{u}_i\tau_{ij} + \frac{1}{2}\overline{u_i'u_i'U_j'},\\
\frac{1}{2}\overline{u_i u_i U_j} &= E\overline{U}_j + e\overline{U}_j + \overline{u}_i\tau_{ij} + \frac{1}{2}\overline{u_i'u_i'U_j'},
\end{aligned} \tag{10}$$

Defining turbulent dissipation rate as $\epsilon = \overline{2\nu s_{ij}s_{ij}} = 2\nu\left[\overline{S_{ij}S_{ij}} - \overline{S}_{ij}\overline{S}_{ij}\right]$ (where, $s_{ij} = (1/2)[\partial u_i'/\partial x_j + \partial u_j'/\partial x_i]$), using Eq.(10) and evaluating $\overline{(6)} - (8)$, we get

$$\begin{aligned}
&\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u_i u_i U_j} + \overline{pU_j} + \overline{u_i\tau^{\nu}_{ij}} + \overline{u_i\tau^{\mathrm{SGS}}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma^{\mathrm{SGS}}_{ik}\frac{\partial u_i}{\partial x_k}}\\
&\quad - \frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij}) + \overline{p}\overline{U}_j\right) + (\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left((E+e)\overline{U}_j + \overline{u}_i\tau_{ij} + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{pU_j} + \overline{u_i\tau^{\nu}_{ij}} + \overline{u_i\tau^{\mathrm{SGS}}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma^{\mathrm{SGS}}_{ik}\frac{\partial u_i}{\partial x_k}}\\
&\quad - \frac{\partial}{\partial \xi_j}\left(E\overline{U}_j + \overline{u}_i(\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij}) + \overline{p}\overline{U}_j\right) + (\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau^{\nu\prime}_{ij}} + \overline{u_i'\tau^{\mathrm{SGS}\prime}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}\sigma^{\mathrm{SGS}}_{ik}\frac{\partial u_i}{\partial x_k}} + (\tau_{ij} + \overline{\tau}^{\nu}_{ij} + \overline{\tau}^{\mathrm{SGS}}_{ij})\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau^{\nu\prime}_{ij}} + \overline{u_i'\tau^{\mathrm{SGS}\prime}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} + \overline{\tau}^{\nu}_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\\
&\quad - \overline{J^{-1}\sigma^{\mathrm{SGS}}_{ik}\frac{\partial u_i}{\partial x_k}} + \overline{\tau}^{\mathrm{SGS}}_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau^{\nu\prime}_{ij}} + \overline{u_i'\tau^{\mathrm{SGS}\prime}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}2\nu S_{ik}\frac{\partial \xi_j}{\partial x_k}}\frac{\partial \overline{u}_i}{\partial \xi_j}\\
&\quad - \left(\overline{J^{-1}\sigma^{\mathrm{SGS}}_{ik}\frac{\partial \xi_j}{\partial x_k}\frac{\partial u_i}{\partial \xi_j}} - J^{-1}\overline{\sigma}^{\mathrm{SGS}}_{ik}\frac{\partial \xi_j}{\partial x_k}\frac{\partial \overline{u}_i}{\partial \xi_j}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau^{\nu\prime}_{ij}} + \overline{u_i'\tau^{\mathrm{SGS}\prime}_{ij}}\right) + \overline{J^{-1}2\nu S_{ik}S_{ik}} - \overline{J^{-1}2\nu S_{ik}}\frac{\partial \overline{u}_i}{\partial x_k}
\end{aligned}$$

$$
\begin{aligned}
&-\left(\overline{\tau_{ij}^{\mathrm{SGS}} S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}} \overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\mathrm{SGS}\prime}}\right) + \overline{J^{-1} 2\nu S_{ik} S_{ik}} - J^{-1} 2\nu \overline{S}_{ik}\overline{S}_{ik}\\
&\qquad -\left(\overline{\tau_{ij}^{\mathrm{SGS}} S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}} \overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1} 2\nu [\overline{S_{ik} S_{ik}} - \overline{S}_{ik}\overline{S}_{ik}]\\
&\qquad -\left(\overline{\tau_{ij}^{\mathrm{SGS}} S_{ij}} - \overline{\tau}_{ij}^{\mathrm{SGS}} \overline{S}_{ij}\right) + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1} 2\nu \overline{s_{ik} s_{ik}} - \overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j + \overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\mathrm{SGS}\prime}}\right) + J^{-1}\epsilon - \overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}} + \tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} = 0,\\
&\underbrace{\overline{U}_j\frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\mathrm{SGS}\prime}}\right)}_{\text{Transport}} + \underbrace{J^{-1}\epsilon}_{\text{Viscous dissipation}} - \underbrace{\overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}}}_{\text{SGS dissipation}} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}}_{\text{Production}} = 0.
\end{aligned}
\tag{11}
$$

Applying horizontal averaging to Eq. (11), we get

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_3}\left(\langle e\overline{U}_3\rangle + \left\langle\overline{\frac{1}{2}u_i' u_i' U_3'}\right\rangle + \langle\overline{p' U_3'}\rangle + \langle\overline{u_i' \tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u_i' \tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}}\right\rangle + \left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle = 0,\\
&\frac{\partial}{\partial \zeta}\left(\langle e\overline{W}\rangle + \left\langle\overline{\frac{1}{2}u_i' u_i' W'}\right\rangle + \langle\overline{p' W'}\rangle + \langle\overline{u_i' \tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u_i' \tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}}\right\rangle + \left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle = 0,
\end{aligned}
$$

Using

$$
\begin{aligned}
\left\langle\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}\right\rangle &= \left\langle(\langle\tau_{ij}\rangle + \widetilde{\tau}_{ij})\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right]\right\rangle,\\
&= \left\langle\langle\tau_{ij}\rangle\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right] + \widetilde{\tau}_{ij}\left[\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \frac{\partial\langle u_i\rangle}{\partial \xi_j}\right]\right\rangle,\\
&= \left\langle\langle\tau_{ij}\rangle\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j} + \widetilde{\tau}_{ij}\frac{\partial\langle u_i\rangle}{\partial \xi_j}\right\rangle,\\
&= \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \left\langle\widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j}\right\rangle,
\end{aligned}
$$

and,

$$
\langle e\overline{W}\rangle = \langle(\langle e\rangle + \widetilde{e})(\langle W\rangle + \widetilde{W})\rangle = \langle(\langle e\rangle + \widetilde{e})\widetilde{W}\rangle = \langle\widetilde{e}\widetilde{W}\rangle,
$$

we get the mean TKE equation as follows,

$$
\frac{\partial}{\partial \zeta}\left(\langle\widetilde{e}\widetilde{W}\rangle + \left\langle\overline{\frac{1}{2}u_i' u_i' W'}\right\rangle + \langle\overline{p' W'}\rangle + \langle\overline{u_i' \tau_{i3}^{\nu\prime}}\rangle + \langle\overline{u_i' \tau_{i3}^{\mathrm{SGS}\prime}}\rangle\right) + \langle J^{-1}\epsilon\rangle - \left\langle\overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}}\right\rangle + \langle\tau_{ij}\rangle\frac{\partial\langle u_i\rangle}{\partial \xi_j} + \left\langle\widetilde{\tau}_{ij}\frac{\partial \widetilde{u}_i}{\partial \xi_j}\right\rangle = 0.
\tag{12}
$$

2 Computing TKE budget terms in MATLAB

Vector identities:

$$\begin{aligned}
\overline{u_i'u_j'} &= \overline{(u_i-\overline{u}_i)(u_j-\overline{u}_j)},\\
&= \overline{u_iu_j - u_i\overline{u}_j - u_j\overline{u}_i + \overline{u}_i\overline{u}_j},\\
&= \overline{u_iu_j} - 2\overline{u}_i\overline{u}_j + \overline{u}_i\overline{u}_j,\\
\overline{u_i'u_j'} &= \overline{u_iu_j} - \overline{u}_i\overline{u}_j.
\end{aligned}$$

$$\begin{aligned}
\overline{u_i'u_i'U_j'} &= \overline{(u_i-\overline{u}_i)(u_i-\overline{u}_i)(U_j-\overline{U}_j)},\\
&= \overline{[u_i(u_i-\overline{u}_i)-\overline{u}_i(u_i-\overline{u}_i)](U_j-\overline{U}_j)},\\
&= \overline{[u_iu_i-u_i\overline{u}_i-\overline{u}_iu_i+\overline{u}_i\overline{u}_i](U_j-\overline{U}_j)},\\
&= \overline{[u_iu_i-2u_i\overline{u}_i+\overline{u}_i\overline{u}_i](U_j-\overline{U}_j)},\\
&= \overline{u_iu_i(U_j-\overline{U}_j)-2u_i\overline{u}_i(U_j-\overline{U}_j)+\overline{u}_i\overline{u}_i(U_j-\overline{U}_j)},\\
&= \overline{u_iu_iU_j-u_iu_i\overline{U}_j-2u_i\overline{u}_iU_j+2u_i\overline{u}_i\overline{U}_j+\overline{u}_i\overline{u}_iU_j-\overline{u}_i\overline{u}_i\overline{U}_j},\\
&= \overline{u_iu_iU_j}-\overline{u_iu_i}\overline{U}_j-2\overline{u_iU_j}\overline{u}_i+2\overline{u}_i\overline{u}_i\overline{U}_j+\overline{u}_i\overline{u}_i\overline{U}_j-\overline{u}_i\overline{u}_i\overline{U}_j,\\
&= \overline{u_iu_iU_j}-\overline{u_iu_i}\overline{U}_j-2\overline{u_iU_j}\overline{u}_i+2\overline{u}_i\overline{u}_i\overline{U}_j,\\
\overline{u_i'u_i'U_j'} &= \overline{u_iu_iU_j}-\overline{u_iu_i}\overline{U}_j-2\overline{u}_i[\overline{u_iU_j}-\overline{u}_i\overline{U}_j].
\end{aligned}$$

$$\begin{aligned}
\overline{u_iu_iU_j} &= \overline{(\overline{u}_i+u_i')(\overline{u}_i+u_i')(\overline{U}_j+U_j')},\\
&= \overline{(\overline{u}_i[\overline{u}_i+u_i']+u_i'[\overline{u}_i+u_i'])(\overline{U}_j+U_j')},\\
&= \overline{(\overline{u}_i\overline{u}_i+\overline{u}_iu_i'+u_i'\overline{u}_i+u_i'u_i')(\overline{U}_j+U_j')},\\
&= \overline{\overline{u}_i\overline{u}_i(\overline{U}_j+U_j')+\overline{u}_iu_i'(\overline{U}_j+U_j')+u_i'\overline{u}_i(\overline{U}_j+U_j')+u_i'u_i'(\overline{U}_j+U_j')},\\
&= \overline{\overline{u}_i\overline{u}_i\overline{U}_j+\overline{u}_i\overline{u}_iU_j'+(\overline{u}_iu_i'\overline{U}_j+\overline{u}_iu_i'U_j')+(u_i'\overline{u}_i\overline{U}_j+u_i'\overline{u}_iU_j')+(u_i'u_i'\overline{U}_j+u_i'u_i'U_j')},\\
&= \overline{u}_i\overline{u}_i\overline{U}_j+2\overline{u}_i\overline{u_i'U_j'}+\overline{u_i'u_i'}\overline{U}_j+\overline{u_i'u_i'U_j'},\\
&= \overline{u}_i\overline{u}_i\overline{U}_j+2\overline{u}_i[\overline{u_iU_j}-\overline{u}_i\overline{U}_j]+[\overline{u_iu_i}-\overline{u}_i\overline{u}_i]\overline{U}_j+\overline{u_i'u_i'U_j'},\\
&= \overline{u}_i\overline{u}_i\overline{U}_j+2\overline{u}_i[\overline{u_iU_j}-\overline{u}_i\overline{U}_j]+[\overline{u_iu_i}-\overline{u}_i\overline{u}_i]\overline{U}_j+\overline{u_i'u_i'U_j'},\\
&= \overline{u}_i\overline{u}_i\overline{U}_j+2\overline{u}_i\overline{u_iU_j}-2\overline{u}_i\overline{u}_i\overline{U}_j+\overline{u_iu_i}\overline{U}_j-\overline{u}_i\overline{u}_i\overline{U}_j+\overline{u_i'u_i'U_j'},\\
&= 2\overline{u}_i\overline{u_iU_j}-2\overline{u}_i\overline{u}_i\overline{U}_j+\overline{u_iu_i}\overline{U}_j+\overline{u_i'u_i'U_j'},\\
\implies \overline{u_i'u_i'U_j'} &= \overline{u_iu_iU_j}-\overline{u_iu_i}\overline{U}_j-2\overline{u}_i[\overline{u_iU_j}-\overline{u}_i\overline{U}_j]. \qquad (13)
\end{aligned}$$

$$\frac{\partial}{\partial \xi_j}\left(e\overline{U}_j+\overline{eU_j'}+\overline{p'U_j'}+\overline{u_i'\tau_{ij}^{\nu\prime}}\right)+J^{-1}\epsilon+(\tau_{ij}+\overline{\tau}_{ij}^{\nu})\frac{\partial\overline{u}_i}{\partial\xi_j}=0.$$

From Eq. (11), TKE equation is

$$\underbrace{\overline{U}_j \frac{\partial e}{\partial \xi_j}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u_i'u_i'U_j'} + \overline{p'U_j'} + \overline{u_i'\tau_{ij}^{\nu\prime}} + \overline{u_i'\tau_{ij}^{\mathrm{SGS}\prime}}\right)}_{\text{Transport}} + \underbrace{J^{-1}\epsilon}_{\text{Viscous dissipation}} - \underbrace{\overline{\tau_{ij}^{\mathrm{SGS}\prime} s_{ij}}}_{\text{SGS dissipation}} + \underbrace{\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j}}_{\text{Production}} = 0.$$

1. Turbulent Kinetic Energy:

$$e = \frac{1}{2}\overline{u_i'u_i'},$$

$$e = \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}),$$

$$\overline{u'u'} = \overline{uu} - \overline{u}\ \overline{u}, \tag{14}$$

$$\overline{v'v'} = \overline{vv} - \overline{v}\ \overline{v}, \tag{15}$$

$$\overline{w'w'} = \overline{ww} - \overline{w}\ \overline{w}. \tag{16}$$

2. Triple correlation term: From Eq. (13), we have

$$\overline{u_i'u_i'U_j'} = \overline{u_iu_iU_j} - \overline{u_iu_i}\overline{U}_j - 2\overline{u}_i[\overline{u_iU_j} - \overline{u}_i\overline{U}_j].$$

$$\overline{u_iu_iU_j} = \overline{(uu + vv + ww)U_j},$$

$$\overline{u_iu_i}\overline{U}_j = (\overline{uu} + \overline{vv} + \overline{ww})\overline{U}_j,$$

$$\overline{u}_i\overline{u_iU_j} = \overline{u}\ \overline{uU_j} + \overline{v}\ \overline{vU_j} + \overline{w}\ \overline{wU_j},$$

$$\overline{u}_i\overline{u}_i\overline{U}_j = (\overline{u}\ \overline{u} + \overline{v}\ \overline{v} + \overline{w}\ \overline{w})\overline{U}_j.$$

3. Pressure correlation term:

$$\overline{p'U_j'} = \overline{pU_j} - \overline{p}\overline{U}_j.$$

4. Viscous transport term:

$$\overline{u_i'\tau_{ij}^{\nu\prime}} = \overline{u_i\tau_{ij}^{\nu}} - \overline{u}_i\ \overline{\tau_{ij}^{\nu}},$$

$$\overline{u_i'\tau_{ij}^{\nu\prime}} = \overline{u\tau_{1j}^{\nu}} - \overline{u}\ \overline{\tau_{1j}^{\nu}} + \overline{v\tau_{2j}^{\nu}} - \overline{v}\ \overline{\tau_{2j}^{\nu}} + \overline{w\tau_{3j}^{\nu}} - \overline{w}\ \overline{\tau_{3j}^{\nu}}.$$

For $j = 1$,

$$\overline{u_i\tau_{i1}^{\nu}} = \overline{u\tau_{11}^{\nu}} + \overline{v\tau_{12}^{\nu}} + \overline{w\tau_{13}^{\nu}}.$$

For $j = 3$,

$$\overline{u_i\tau_{i3}^{\nu}} = \overline{u\tau_{13}^{\nu}} + \overline{v\tau_{23}^{\nu}} + \overline{w\tau_{33}^{\nu}}.$$

5. Viscous dissipation term:

$$\epsilon = \overline{2\nu s_{ij}s_{ij}} = 2\nu[\overline{S_{ij}S_{ij}} - \overline{S}_{ij}\overline{S}_{ij}] = 2\nu\Big[\overline{S_{11}S_{11} + S_{22}S_{22} + S_{33}S_{33} + 2(S_{12}S_{12} + S_{13}S_{13} + S_{23}S_{23})}$$
$$- (\overline{S}_{11}\overline{S}_{11} + \overline{S}_{22}\overline{S}_{22} + \overline{S}_{33}\overline{S}_{33} + 2[\overline{S}_{12}\overline{S}_{12} + \overline{S}_{13}\overline{S}_{13} + \overline{S}_{23}\overline{S}_{23}])\Big].$$

6. SGS dissipation term:

$$\overline{\tau_{ij}^{\text{SGS}\prime} s_{ij}} = \overline{\tau_{ij}^{\text{SGS}} S_{ij}} - \overline{\tau_{ij}^{\text{SGS}}}\,\overline{S}_{ij} = \overline{\tau_{11}^{\text{SGS}} S_{11}} + \overline{\tau_{22}^{\text{SGS}} S_{22}} + \overline{\tau_{33}^{\text{SGS}} S_{33}} + 2\left(\overline{\tau_{12}^{\text{SGS}} S_{12}} + \overline{\tau_{13}^{\text{SGS}} S_{13}} + \overline{\tau_{23}^{\text{SGS}} S_{23}}\right)$$
$$- \left[\overline{\tau_{11}^{\text{SGS}}}\,\overline{S}_{11} + \overline{\tau_{22}^{\text{SGS}}}\,\overline{S}_{22} + \overline{\tau_{33}^{\text{SGS}}}\,\overline{S}_{33} + 2\left(\overline{\tau_{12}^{\text{SGS}}}\,\overline{S}_{12} + \overline{\tau_{13}^{\text{SGS}}}\,\overline{S}_{13} + \overline{\tau_{23}^{\text{SGS}}}\,\overline{S}_{23}\right)\right]$$

7. Production term:

$$\begin{aligned}
\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} &= \tau_{i1}\frac{\partial \overline{u}_i}{\partial \xi} + \tau_{i3}\frac{\partial \overline{u}_i}{\partial \zeta}, \\
\tau_{ij}\frac{\partial \overline{u}_i}{\partial \xi_j} &= \quad \tau_{11}\frac{\partial \overline{u}}{\partial \xi} + \tau_{21}\frac{\partial \overline{v}}{\partial \xi} + \tau_{31}\frac{\partial \overline{w}}{\partial \xi} \\
&\quad + \tau_{13}\frac{\partial \overline{u}}{\partial \zeta} + \tau_{23}\frac{\partial \overline{v}}{\partial \zeta} + \tau_{33}\frac{\partial \overline{w}}{\partial \zeta},
\end{aligned}$$

where $\tau_{ij} = \overline{u_i' U_j'} = \overline{u_i U_j} - \overline{u}_i \overline{U}_j$.

8. Advection term:

$$\overline{U}_j \frac{\partial e}{\partial \xi_j} = \overline{U}\frac{\partial e}{\partial \xi} + \overline{W}\frac{\partial e}{\partial \zeta}.$$

9. Transport term:

$$\frac{\partial}{\partial \xi_j}\left(\overline{\frac{1}{2}u_i' u_i' U_j'} + \overline{p' U_j'} + \overline{u_i' \tau_{ij}^{\nu\prime}} + \overline{u_i' \tau_{ij}^{\text{SGS}\prime}}\right) = \frac{\partial}{\partial \xi}\left(\overline{\frac{1}{2}u_i' u_i' U'} + \overline{p' U'} + \overline{u_i' \tau_{i1}^{\nu\prime}} + \overline{u_i' \tau_{i1}^{\text{SGS}\prime}}\right)$$
$$+ \frac{\partial}{\partial \zeta}\left(\overline{\frac{1}{2}u_i' u_i' W'} + \overline{p' W'} + \overline{u_i' \tau_{i3}^{\nu\prime}} + \overline{u_i' \tau_{i3}^{\text{SGS}\prime}}\right)$$

References

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