AMSC808N Final Exam - Problem 1

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All codes for Problems 1 2 are available on ELMS together with this report submission. The chosen language was Python for simplicity. Additionally, one can find the codes for Final exam Problema 1 and 2 at Guilherme's Github

Question 1.

The goal of this question if to find a optimal solution (a^*, b^*) for the objective function

$$f(a,b) = \frac{1}{12} \sum_{j=0}^{5} \left[ReLU(ax_j - b) - g(x_j) \right]^2$$
 (1)

that describes a fitting of $g(x) = 1 - \cos x$ using a ReLU function.

(a) First note that $ReLU(x) = \max(x,0)$ thus for the regions when the argument $ax_j - b < 0$ we don't have any dependence of a and b for the objective function f(a,b). This is precisely the flat region where $\nabla f = 0$.

To find a precise definition for this flat region set we must find $\{a,b\}$ s.t. $ax_j - b < 0 \quad \forall x_j \in [0,\pi/2]$. After some careful thought we can see that the region of stationary points consists of two regions:

$$\{\nabla f = 0\} = \{(a, b) | (a < 0, b/a < 0) \cup (a > 0, b < a\pi/2)\}$$
(2)

it's clear if we plot this (unbounded from below) region

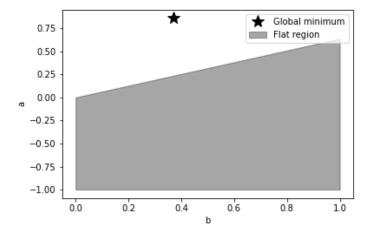


Figure 1: Flat region such that $\nabla f = 0$. Note that it's unbounded for negative values of a and positive values of b. Star denotes the global minimum.

Then, to find the optimal solution (a^*, b^*) we must solve for the minimizer of the objective function

Eq.1. To do this, first note that the solution for a linear least square problem is given by

$$f(x) = ax - b$$

$$a = \frac{\langle yx \rangle - \langle y \rangle \langle x \rangle}{\langle x^2 \rangle - \langle x \rangle^2}$$

$$b = a \langle x \rangle - \langle y \rangle$$
(3)

where $\langle \cdot \rangle$ denotes average over all data and y(x) is the set of points we want to fit using a linear regression f(x).

Now we note that the ReLU function simply cuts off the values of x_j that $ax_j - b < 0$ so effectively we are solving a linear least square problems (as in Eq.3) in those x values that satisfy the cutoff. Finally, we can find the optimal solutions by minimizing the following:

$$(a^*, b^*) = \min_{i=0,\dots 5} f(a_i, b_i), \quad s.t. \begin{cases} a_i = \frac{\sum_{j=i}^5 (x_j - \langle x \rangle)(g(x_j) - \langle g \rangle)}{\sum_{j=i}^5 (x_j - \langle x \rangle)^2} \\ b_i = \frac{1}{6-i} \sum_{j=i}^5 (a_i x_j - g(x_j)) \end{cases}$$
(4)

Using Eq.(4) we find the optimal solution to be $(a^*, b^*) = (0.86, 0.37)$ and we plotted it together with the function $1 - \cos x$.

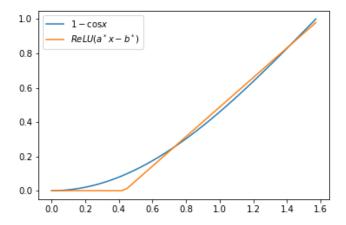


Figure 2: The optimal solution is zero when g(x) is small and approximates kinda good near the endpoints.

(b) To find the smallest stepsize to reach the flat region we need to solve the iteration $(1,0) - \alpha^* \nabla f(1,0) =$ boundary of set $(\nabla f = 0)$. Solving this equation we find $\alpha^* = \frac{1}{\partial_a f(1,0) - \partial_b f(1,0) 2/\pi} = 1.51$. To numerically test this solution I implemented a gradient descent with varius different constant stepsizes $\alpha = [0.1, 1.31, 1.49, 1.51]$ using 1000 iterations.

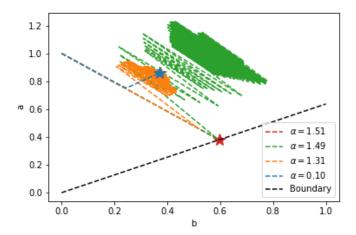


Figure 3: Gradient descent with different stepsizes, black dashed line is the boundary to the flat region. All initial conditions are (1,0) and the final position is denoted by a star. For $\alpha < 1.31 \pm 0.01$ (blue and orange) the method converges to the global minimum (0.86,0.37). Using $\alpha = \alpha^*$ (red) the method reaches the stationary region and stays there. Using $\alpha = 0.99\alpha^*$ (green) the method doesn't reach the flat region and doesn't converge to the global minimum.

From Fig.?? we found numerically that $\alpha < 1.31 \pm 0.01$ is the threshold for convergence to the global minimum. The region $1.31 < \alpha < 1.51$ the method doesn't converge as shown in green. For larger values $\alpha \geq 1.51$ the method reaches the flat region in the first step and stays there.

(c) Then we implemented the stochastic gradient descent with batch size of 1 point. For stepsize strategy I used $\alpha_k = 1.5/2^{k/50}$ to test the convergence to the global minimum. As shown in the next result we see clearly the convergence even if $\alpha = 1.5$ the Gradient Descent doesn't converge.

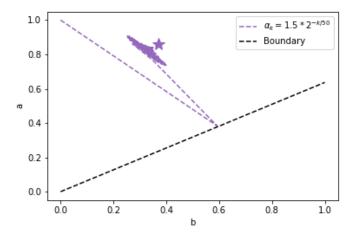


Figure 4: Using stochastic gradient descent the method converges using a exponential decrease $\alpha_k = 1.5/2^{k/50}$, every 50 steps the stepsize get smaller by a factor of 1/2.