

A Global Stability Analysis for Delayed Neural Fields

{Georgios.Detorakis, Antoine.Chaillot, Ihab.Haidar}@lss.supelec.fr

Introduction

The main aim of this work is to provide a global stability analysis for the delayed neural field equations with a generic type of activation functions and connection structures. Neural fields are integro-differential equations used mostly in describing biological phenomena at the level of neural populations. They can be considered as a generalization of the Wilson & Cowan model [4] since they provide information in time and space, by modeling the neuronal populations involved as a continuum. Delayed neural fields additionally take into account the spreading of action potentials along the neural axons within a population or from a population to another. So far, several works have addressed the stability of neural fields. In [2] and [3] the authors proposed a Lyapunov functional for Amari's neural fields, without axonal delays. In [1] a delayed neural field is linearized around an equilibrium and a stability analysis is conducted using a Krasovskii-Lyapunov functional, thus providing conditions for the local stability of neural fields.

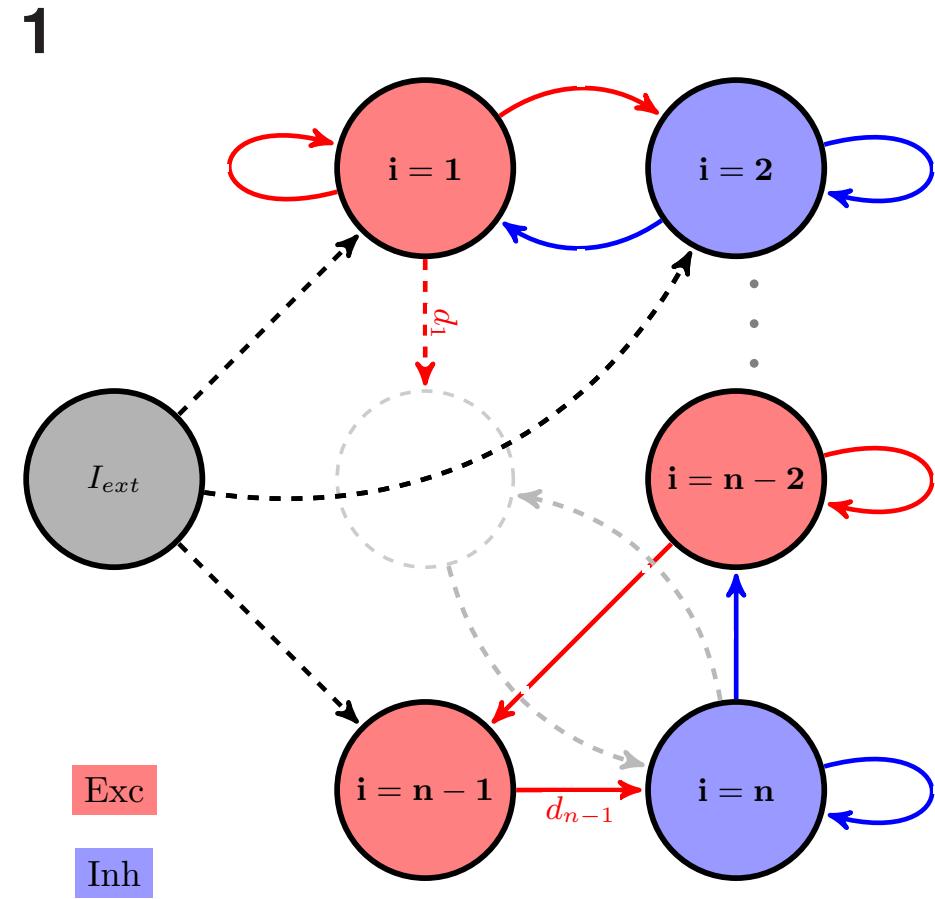
In this work, we rely on Razumikhin (see pg. 127, [6]) theorem to propose a sufficient condition for the delayed neural fields to have a unique equilibrium configuration to which all solutions asymptotically converge. This condition is explicit and involves the Lipschitz constants of the activation functions, the axonal delays, and a measure of the synaptic strength from one population to another. Moreover, we perform some numerical simulations to illustrate our theoretical findings and to check the conservativeness of the proposed condition.

The Neural Field Model

A delayed neural field model of n populations of $k \in \{1, 2, 3\}$ dimensions is given by

$$\tau_i \frac{\partial U_i}{\partial t}(r, t) = -U_i(r, t) + \int_{\Omega} \sum_{j=1}^n W_{ij}(r, r') S_j(U_j(r', t - d_j(r, r'))) dr' + I_i^{\text{ext}} \quad (1)$$

where $i \in \{1, \dots, n\}$, $\Omega \subset \mathbb{R}^k$ is a bounded set over which the populations are distributed and, for each population $i \in \{1, \dots, n\}$, $U_i(r, t)$ represents the mean neuronal activity at position r and at time t , τ_i is a time constant, I_i^{ext} is the external input, $d_{ij}(r, r')$ is the time needed for the information of the population j at location r' to reach population i at location r , $W_{ij}(r, r')$ describes the synaptic strength between location r' in population j and location r in population i , and $S_j : \mathbb{R}^n \rightarrow \mathbb{R}$ is the activation function of neural populations. We assume that this function is Lipschitz continuous. A schematic view of n neural populations is illustrated in Figure 1. We can have excitatory (red discs) populations and inhibitory ones (blue discs) and they can be interconnected or self-connected.



Sufficient Conditions for Stability

As we explained in the introduction the main goal of this work is to provide a global stability analysis for Eq. (1). We first provide some notations and then we put forward two theorems that provide a sufficient condition for global stability.

Notation 1 A function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- \mathcal{K} if it is continuous, strictly increasing and zero at zero. It is of class- \mathcal{K}_{∞} if it is of class- \mathcal{K} and unbounded. A function g is of class- \mathcal{L} if it is continuous, non-increasing and tends to zero as its argument tends to infinity. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ belongs to class- \mathcal{KL} if, given any $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and given any $s \geq 0$, $\beta(s, \cdot) \in \mathcal{L}$.

Theorem 1 Let $\mathcal{V} \in C^1(\mathbb{R} \rightarrow \mathbb{R}_+)$ be such that the following implication holds true for some function $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $p(s) > s$ for each $s > 0$:

$$\max_{\theta \in [-r, 0]} \{\mathcal{V}(t + \theta)\} \leq p(\mathcal{V}(t)) \Rightarrow \dot{\mathcal{V}}(t) \leq -\alpha(\mathcal{V}(t)), \quad \forall t \geq t_0 \geq 0$$

where α is a class- \mathcal{K} function. Then there exists a function $\beta \in \mathcal{KL}$ such that

$$\mathcal{V}(t) \leq \beta(\max_{\theta \in [-r, 0]} \mathcal{V}(t_0 + \theta), t - t_0), \quad \forall t \geq t_0 \geq 0.$$

The proof of Theorem 1 follows the proof of Theorem 1 of Teel [7].

Theorem 2 Let L_j be the Lipschitz constant of the firing rate function S_j of j -th population and assume that,

$$\underbrace{n \sum_{i,j=1}^n \int_{\Omega} \int_{\Omega} L_j^2 W_{ij}^2(r, r') dr' dr}_{\Xi} < 1 \quad (2)$$

then if Eq. (1) has an equilibrium then this equilibrium is unique and all the solutions converge to it.

We do not provide the entire proof of Theorem 2 here. Instead, we give a brief sketch of the proof. Hence, we start from a Lyapunov function defined as

$$\mathcal{V}(t) = \frac{1}{2} \sum_{i=1}^n \int_{\Omega} \tau_i v_i^2(r, t) dr, \quad (3)$$

where $v_i(r, t) := u_i(r, t) - u_i^*(r)$, where $u_i^*(r)$ is an equilibrium configuration of Eq. (1). Then we compute the time derivative $\dot{\mathcal{V}}(t)$. An expression of $\mathcal{V}(t)$ and $\mathcal{V}(t - d_j)$ appears in our calculations. In order to bound from above those expressions, we apply twice the inequality of Cauchy-Schwarz in \mathbb{R} and in L^2 and then we use the inequality, $ab \leq \frac{1}{2}(a^2 + b^2)$. The hint is the observation that for each j it holds that $\int_{\Omega} v_j^2(r', t - d_j(r, r')) dr' \leq \mathcal{V}(t - d_j(r, r')) \leq \max_{\theta \in [-r, 0]} \{\mathcal{V}(t + \theta)\}$. Putting all these together, we conclude that Theorem 2 holds true when inequality (2) is satisfied.

Physical Interpretation

Condition of Theorem 2 (inequality (2)), is similar to the one found by in [1], [8] and [9] and implies that if the synaptic connections of neural populations are not too strong, or the slope of the firing rate function is not too steep then the system described by Eq. (1) is stable and converges to a unique equilibrium point. When the system is not stable, it is able to express a rich repertoire of behavior. The fact that the slope of the firing rate function is too steep, points out that the neurons of a specific neural population trigger spikes in a rapid way. The interplay between synaptic coupling and stability, as it has been suggested by inequality (2) could explain some pathological conditions, such as epileptic seizures or beta oscillations in Parkinson's Disease.

Numerical Simulations

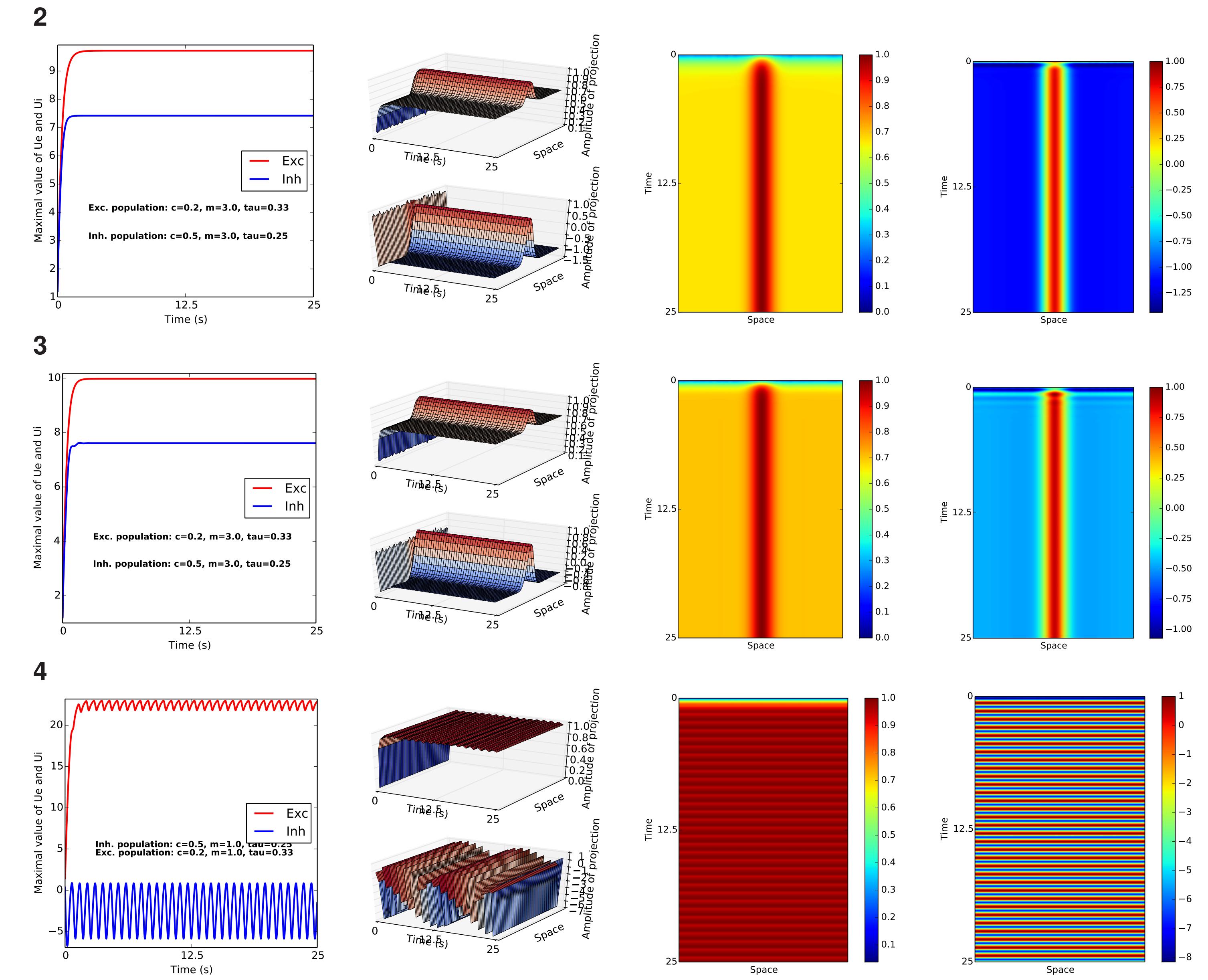
We solve numerically Eq. (1) for $n = 2$ (i.e. we have two interconnected populations) and $k = 2$ (i.e. two-dimensional populations). We choose as firing rate function a sigmoid given by

$$S_i(x) = \frac{1}{1 + \exp(-\mu_i x)}, \quad i = 1, 2 \quad (4)$$

for which the Lipschitz constant $L_i = \frac{\mu_i}{4}$. As synaptic connections function, we choose a Gaussian function such as

$$W_{ij}(r, r') = \frac{K_{ij}}{\sigma_{ij} \sqrt{2\pi}} \exp\left(-\frac{(|r - r'| - c)^2}{2\sigma_{ij}^2}\right), \quad i, j = 1, 2. \quad (5)$$

Then we discretize the domain $\Omega = [-1, 1] \times [-1, 1]$ using 64×64 nodes. In order to solve numerically the discretized problem we follow the algorithm proposed by Hutt and Rougier [5], which is very fast since it exploits the speed of FFT. The results are pointed in Figure 2, 3, , 4. In each Figure, on the left panel it is illustrated the maximal value of solution u_1 (blue line) and u_2 (red line), on the center is depicted the sampling evolution of the projection of the two solutions over time, and on the right panel is presented the projection of the solutions, u_1 and u_2 , respectively. Figure 2 shows the evolution of Eq. (1), when the parameters have been chosen to satisfy the condition (2) of Theorem 2 ($\Xi = 0.151 < 1$). It is apparent that the system converges to the equilibrium. In Figure 3, it is depicted a case where the conditions of Theorem 2 are violated ($\Xi = 1.36 > 1$) but still the system converges to an equilibrium. And finally, Figure 4 illustrates a non-stable case: The condition (2) of Theorem 2 does not hold true ($\Xi = 136.128$) and an oscillatory pattern of the solution of Eq. (1) emerges. This illustrates the possible conservativeness of condition (2).



Conclusions

In this work we introduce a theorem for global asymptotic stability of Eq. (1) relying on Razumikhin theorem. We first, adapted Theorem 1 of Teel [7] and then based on that adapted version of Razumikhin theorem we constructed our theorem about global stability. The direct impact of Theorem 2 is a sufficient condition for global stability. We proved here that if $\Xi < 1$ holds true then Eq. (1) converges globally to a unique equilibrium. We have shown using numerical simulations that when the condition holds true then our two-coupled populations system converge to an equilibrium configuration. When the condition is violated then system described by Eq. (1) is capable of expressing different types of behaviors.

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