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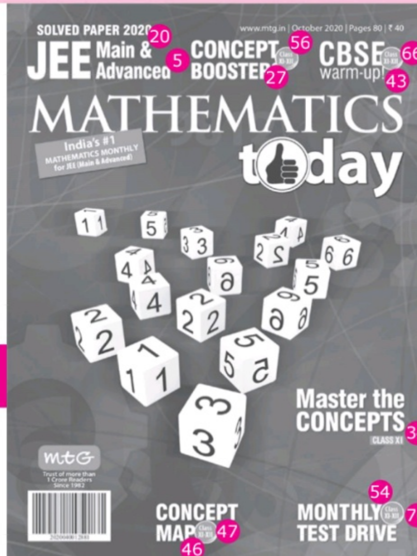
CONTENTS



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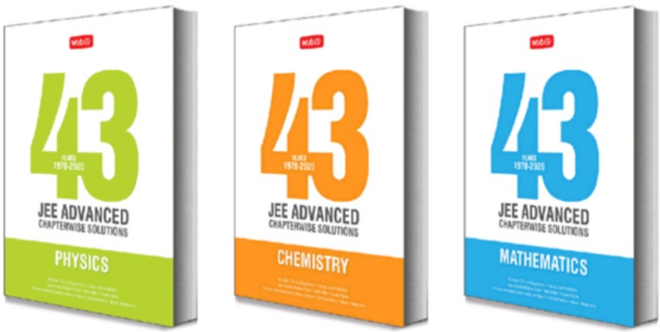

43
JEE ADVANCED
CHAPTERWISE SOLUTIONS
PHYSICS
43
JEE ADVANCED
CHAPTERWISE SOLUTIONS
CHEMISTRY
43
JEE ADVANCED
CHAPTERWISE SOLUTIONS
MATHEMATICS

*ALOK KUMAR

PAPER-1

SECTION 1 (Maximum Marks : 18)

- *This section contains SIX (06) questions.*
- *Each question has FOUR options. ONLY ONE of these four options is the correct answer.*
- *For each question, choose the correct option corresponding to the correct answer.*
- *Answer to each question will be evaluated according to the following marking scheme:*

  Full Marks : *+3 If ONLY the correct option is chosen.*

  Zero Marks : *0 If none of the options is chosen (i.e. the question is unanswered).*

  Negative Marks : *–1 In all other cases.*

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of
$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is

(a) 0 (b) 8000 (c) 8080 (d) 16000

2. If the function $f : R \to R$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is TRUE?

(a) f is one-one, but NOT onto
(b) f is onto, but NOT one-one
(c) f is BOTH one-one and onto
(d) f is NEITHER one-one NOR onto

3. Let the functions $f : R \to R$ and $g : R \to R$ be defined by $f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$.

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

(a) $(2-\sqrt{3})+\frac{1}{2}(e-e^{-1})$ (b) $(2+\sqrt{3})+\frac{1}{2}(e-e^{-1})$

(c) $(2-\sqrt{3})+\frac{1}{2}(e+e^{-1})$ (d) $(2+\sqrt{3})+\frac{1}{2}(e+e^{-1})$

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{2}{5}$

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are 2/3 and 1/3, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(a) $\frac{40}{81}$ (b) $\frac{20}{81}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

6. Consider all rectangles lying in the region

$$\left\{(x, y)\in R\times R : 0\le x\le \frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(a) $\frac{3\pi}{2}$ (b) π (c) $\frac{\pi}{2\sqrt{3}}$ (d) $\frac{\pi\sqrt{3}}{2}$

SECTION 2 (Maximum Marks : 24)

- *This section contains SIX (06) questions.*
- *Each question has FOUR options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).*
- *For each question, choose the option(s) corresponding to (all) the correct answer(s).*
- *Answer to each question will be evaluated according to the following marking scheme:*

  Full Marks : *+4 If only (all) the correct option(s) is (are) chosen.*

  Partial Marks : *+3 If all the four options are correct but ONLY three options are chosen.*

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

Partial Marks : **+ 2** ***If three or more options are correct but ONLY two options are chosen, both of which are correct.***

Partial Marks : **+ 1** ***If two or more options are correct but ONLY one option is chosen and it is a correct option.***

Zero Marks : **0** ***If none of the options is chosen (i.e. the question is unanswered).***

Negative Marks : **–1** ***In all other cases.***

7. Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g : R \to R$ be an arbitrary function. Let $fg : R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

(a) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$.

(b) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$.

(c) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$.

(d) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$.

8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statements is/are ALWAYS TRUE?

(a) $M = I$ (b) $\det M = 1$

(c) $M^2 = I$ (d) $(\text{adj } M)^2 = I$

9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(a) $\left|z+\frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(b) $|z| \le 2$ for all $z \in S$

(c) $\left|z+\frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

10. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z}$, then which of the following statements is/are TRUE?

(a) $2Y = X + Z$ (b) $Y = X + Z$

(c) $\tan\frac{X}{2}=\frac{x}{y+z}$ (d) $x^2 + z^2 - y^2 = xz$

11. Let L_1 and L_2 be the following straight lines.

$L_1:\frac{x-1}{1}=\frac{y}{-1}=\frac{z-1}{3}$ and $L_2:\frac{x-1}{-3}=\frac{y}{-1}=\frac{z-1}{1}$.

Suppose the straight line $L:\frac{x-\alpha}{l}=\frac{y-1}{m}=\frac{z-\gamma}{-2}$ lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(a) $\alpha - \gamma = 3$ (b) $l + m = 2$

(c) $\alpha - \gamma = 1$ (d) $l + m = 0$

12. Which of the following inequalities is/are TRUE?

(a) $\int_0^1 x\cos x\,dx \ge \frac{3}{8}$ (b) $\int_0^1 x\sin x\,dx \ge \frac{3}{10}$

(c) $\int_0^1 x^2\cos x\,dx \ge \frac{1}{2}$ (d) $\int_0^1 x^2\sin x\,dx \ge \frac{2}{9}$

SECTION 3 (Maximum Marks : 24)

- ***This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.***
- ***For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.***
- ***Answer to each question will be evaluated according to the following marking scheme:***

Full Marks : **+4** ***If ONLY the correct numerical value is entered.***

Zero Marks : **0** ***In all other cases.***

13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____.

14. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$

holds for some positive integer n, is _____.

15. Let $f: [0, 2] \to R$ be the function defined by

$$f(x)=(3-\sin(2\pi x))\sin\left(\pi x-\frac{\pi}{4}\right)-\sin\left(3\pi x+\frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \ge 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____.

16. In a triangle PQR, let $\vec{a}=\overline{QR}$, $\vec{b}=\overline{RP}$ and $\vec{c}=\overline{PQ}$.

If $|\vec{a}|=3, |\vec{b}|=4$ and $\dfrac{\vec{a}\cdot(\vec{c}-\vec{b})}{\vec{c}\cdot(\vec{a}-\vec{b})}=\dfrac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$,

then value of $|\vec{a}\times\vec{b}|^2$ is _____.

17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by $S=\{(x^2-1)^2(a_0+a_1x+a_2x^2+a_3x^3) : a_0, a_1, a_2, a_3 \in R\}$. For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'}+m_{f''})$, where $f \in S$, is _____.

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit $\lim\limits_{x\to 0^+}\dfrac{(1-x)^{1/x}-e^{-1}}{x^a}$ is equal to a nonzero real, is _____.

PAPER-2

SECTION 1 (Maximum Marks : 18)

- *This section contains SIX (06) questions.*
- *The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 TO 9, BOTH INCLUSIVE.*
- *For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.*
- *Answer to each question will be evaluated according to the following marking scheme:*

Full Marks :	*+3*	*If ONLY the correct integer is entered.*
Zero Marks :	*0*	*If the question is unanswered.*
Negative Marks :	*–1*	*In all other cases.*

1. For a complex number z, let $\mathrm{Re}(z)$ denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4-|z|^4=4iz^2$, where $i=\sqrt{-1}$. Then the minimum possible value of $|z_1-z_2|^2$, where $z_1, z_2 \in S$ with $\mathrm{Re}(z_1)>0$ and $\mathrm{Re}(z_2)<0$, is _____

2. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is NOT less than 0.95, is _____

3. Let O be the centre of the circle $x^2+y^2=r^2$, where $r>\dfrac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x+4y=5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x+2y=4$, then the value of r is _____

4. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18, then the value of the determinant of A is _____

5. Let the functions $f:(-1,1)\to R$ and $g:(-1,1)\to(-1,1)$ be defined by $f(x)=|2x-1|+|2x+1|$ and $g(x)=x-[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Let $fog:(-1,1)\to R$ be the composite function defined by $(fog)(x)=f(g(x))$. Suppose c is the number of points in the interval $(-1, 1)$ at which fog is NOT continuous, and suppose d is the number of points in the interval $(-1, 1)$ at which fog is NOT differentiable. Then the value of $c+d$ is _____

6. The value of the limit

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(\sin 3x+\sin x)}{\left(2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{3x}{2}\right)}$$

is _____

SECTION 2 (Maximum Marks : 24)

- *This section contains SIX (06) questions.*
- *Each question has FOUR options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).*
- *For each question, choose the option(s) corresponding to (all) the correct answer(s).*
- *Answer to each question will be evaluated according to the following marking scheme:*

Full Marks :	*+4*	*If only (all) the correct option(s) is (are) chosen.*
Partial Marks :	*+3*	*If all the four options are correct but ONLY three options are chosen.*
Partial Marks :	*+2*	*If three or more options are correct but ONLY two options are chosen, both of which are correct.*
Partial Marks :	*+1*	*If two or more options are correct but ONLY one option is chosen and it is a correct option.*
Zero Marks :	*0*	*If none of the options is chosen (i.e. the question is unanswered).*
Negative Marks :	*–2*	*In all other cases.*

7. Let b be a nonzero real number. Suppose $f: R \to R$ is a differentiable function such that $f(0) = 1$. If the derivative f' of f satisfies the equation $f'(x) = \dfrac{f(x)}{b^2 + x^2}$ for all $x \in R$, then which of the following statements is/are TRUE?

(a) If $b > 0$, then f is an increasing function
(b) If $b < 0$, then f is a decreasing function
(c) $f(x)f(-x)=1$ for all $x \in R$
(d) $f(x) - f(-x)=0$ for all $x \in R$

8. Let a and b be positive real numbers such that $a > 1$ and $b < a$. Let P be a point in the first quadrant that lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point $(1, 0)$, and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

(a) $1 < e < \sqrt{2}$ (b) $\sqrt{2} < e < 2$
(c) $\Delta = a^4$ (d) $\Delta = b^4$

9. Let $f: R \to R$ and $g: R \to R$ be functions satisfying $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$. If $\lim_{x \to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(a) f is differentiable at every $x \in R$
(b) If $g(0)=1$, then g is differentiable at every $x \in R$
(c) The derivative $f'(1)$ is equal to 1
(d) The derivative $f'(0)$ is equal to 1

10. Let $\alpha, \beta, \gamma, \delta$ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point $(3, 2, -1)$ is the mirror image of the point $(1, 0, -1)$ with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/are TRUE?

(a) $\alpha + \beta = 2$ (b) $\delta - \gamma = 3$
(c) $\delta + \beta = 4$ (d) $\alpha + \beta + \gamma = \delta$

11. Let a and b be positive real numbers. Suppose $\overline{PQ} = a\hat{i} + b\hat{j}$ and $\overline{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram $PQRS$. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overline{PQ}$ and $\overline{PS}$, respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram $PQRS$ is 8, then which of the following statements is/are TRUE?

(a) $a + b = 4$ (b) $a - b = 2$
(c) The length of the diagonal PR of the parallelogram $PQRS$ is 4
(d) $\vec{w}$ is an angle bisector of the vectors $\overline{PQ}$ and $\overline{PS}$

12. For nonnegative integers s and r, let

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$$

For positive integers m and n, let

$$g(m, n) = \sum_{p=0}^{m+n} \frac{f(m, n, p)}{\binom{n+p}{p}}$$

where for any nonnegative integer p,

$$f(m, n, p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}.$$

Then which of the following statements is/are TRUE?

(a) $g(m, n) = g(n, m)$ for all positive integers m, n
(b) $g(m, n + 1) = g(m + 1, n)$ for all positive integers m, n
(c) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n
(d) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

SECTION 3 (Maximum Marks : 24)

- ***This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.***
- ***For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.***
- ***Answer to each question will be evaluated according to the following marking scheme:***
  Full Marks : +4 If ONLY the correct numerical value is entered.
  Zero Marks : 0 In all other cases.

13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that no two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is ______.

14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is ______.

15. Two fair dice, each with faces numbered 1,2,3,4,5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till

the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of $14p$ is _____.

16. Let the function $f: [0, 1] \to R$ be defined by

$f(x) = \dfrac{4^x}{4^x + 2}$. Then the value of

$f\left(\dfrac{1}{40}\right) + f\left(\dfrac{2}{40}\right) + f\left(\dfrac{3}{40}\right) + ... + f\left(\dfrac{39}{40}\right) - f\left(\dfrac{1}{2}\right)$

is _____.

17. Let $f: R \to R$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.

If $F: [0, \pi] \to R$ is defined by $F(x) = \int_0^x f(t)dt$, and if $\int_0^{\pi} (f'(x) + F(x))\cos x\, dx = 2$, then the value of $f(0)$ is _____.

18. Let the function $f: (0, \pi) \to R$ be defined by $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$. Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots, \lambda_r\pi\}$, where $0 < \lambda_1 < \ldots < \lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is _____.

SOLUTIONS

PAPER-1

1. (d) : Let $E = ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$
$= a\{ca - c^2 + da - d^2\} + b\{cb - c^2 + db - d^2\}$
$= a\{a(c + d) - (c^2 + d^2)\} + b\{b(c + d) - (c^2 + d^2)\}$
$= a^2(c + d) - a(c^2 + d^2) + b^2(c + d) - b(c^2 + d^2)$
$= (a^2 + b^2)(c + d) - (a + b)(c^2 + d^2)$
$= \{(a + b)^2 - 2ab\}(c + d) - (a + b)\{(c + d)^2 - 2cd\}$
$= (c + d)(a + b)^2 - 2ab(c + d) - (a + b)(c + d)^2 + 2cd(a + b)$

From given equations, we have $a + b = -20$, $ab = -2020$ and $c + d = 20$, $cd = 2020$

$\therefore \quad E = 20(-20)^2 - 2(-2020)(20) + 20(20)^2 + 2(2020)(-20)$
$= 2(20)(20)^2 = 16000$ **[Rating : Easy]**

2. (c) : $f(x) = |x|(x - \sin x)$

$$f(x) = \begin{cases} x^2 - x\sin x, & x \geq 0 \\ -x^2 + x\sin x, & x < 0 \end{cases}$$

i.e., $$f(x) = \begin{cases} x^2 - x\sin x, & x \geq 0 \\ -x^2 + x\sin x, & x < 0 \end{cases}$$

f is continuous as it is a product of two continuous functions.

$$f(-\infty) = \lim_{x \to -\infty} (-x^2 + x\sin x)$$
$$= \lim_{x \to -\infty} (-x^2)\left(1 - \frac{\sin x}{x}\right) = -\infty$$
$$f(\infty) = \lim_{x \to \infty} (x^2 - x\sin x) = \lim_{x \to \infty} (x^2)\left(1 - \frac{\sin x}{x}\right) = \infty$$

So, the range of f is R. Then f is ONTO.

Again, $f'(x) = \begin{cases} 2x - x\cos x - \sin x, & x > 0 \\ -2x + \sin x + x\cos x, & x < 0 \end{cases}$

Note that, $2x - x\cos x - \sin x = \underbrace{(x - \sin x)}_{\geq 0} + \underbrace{x(1 - \cos x)}_{\geq 0}$

Thus, $f' \geq 0 \ \forall\ x \in (0, \infty)$

Similarly, $f' \geq 0 \ \forall\ x \in (-\infty, 0)$ as f' is even because f is odd.

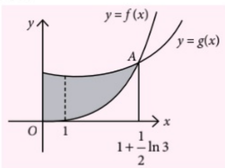
Thus, f is ONE-ONE. **[Rating : Difficult]**

3. (a) : $f(x) = e^{x-1} - e^{-|x-1|}$

$$f(x) = \begin{cases} 0, & x \leq 1 \\ e^{x-1} - e^{1-x}, & x > 1 \end{cases}$$

Also, $g(x) = \dfrac{1}{2}(e^{x-1} + e^{1-x})$

The graph of f and g can be plotted in the first quadrant as given below.



Point of intersection, A is given by $f(x) = g(x)$

$\Rightarrow \quad e^{x-1} - e^{1-x} = \dfrac{1}{2}\left(e^{x-1} + e^{1-x}\right)$

$\Rightarrow \quad e^{x-1} = 3e^{1-x} \quad \Rightarrow \quad (e^{x-1})^2 = 3$

$\Rightarrow \quad e^{2x-2} = 3 \quad \Rightarrow \quad 2x - 2 = \ln 3$

$\Rightarrow \quad x = 1 + \dfrac{1}{2}\ln 3$

$$\therefore \quad \text{Required area} = \int_0^1 \frac{1}{2}(e^{x-1} + e^{1-x})dx$$
$$+ \int_1^{1+\frac{1}{2}\ln 3} \left[\frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x})\right]dx$$

$$=\frac{1}{2}\left[(e^{x-1}-e^{1-x})\right]_0^1-\left[\left(\frac{1}{2}e^{x-1}+\frac{3}{2}e^{1-x}\right)\right]_1^{1+\frac{1}{2}\ln 3}$$

$$=\frac{1}{2}\left(0-\frac{1}{e}+e\right)-\frac{1}{2}\left[e^{\ln\sqrt{3}}+3e^{-\ln\sqrt{3}}-4\right]$$

$$=\frac{1}{2}\left(e-\frac{1}{e}\right)-\frac{1}{2}\left(\sqrt{3}+\frac{3}{\sqrt{3}}-4\right)$$

$$=\frac{1}{2}\left(e-\frac{1}{e}\right)-\frac{1}{2}(2\sqrt{3}-4)\ =\frac{1}{2}\left(e-\frac{1}{e}\right)-(\sqrt{3}-2)$$

$$=2-\sqrt{3}+\frac{1}{2}\left(e-\frac{1}{e}\right)$$

[Rating : Medium]

4. (a) : We have, $y^2 = 4\lambda x$

End point of latus rectum can be taken as $P \equiv (\lambda, 2\lambda)$.

Now, $y^2 = 4\lambda x$

$$\Rightarrow\ 2y\frac{dy}{dx}=4\lambda \Rightarrow \frac{dy}{dx}=\frac{2\lambda}{y}=\left[\frac{dy}{dx}\right]_{(\lambda,2\lambda)}=\frac{2\lambda}{2\lambda}=1$$

Slope of tangent at P to the parabola = 1

$\therefore$ Slope of tangent at P to the ellipse = -1

Also, we have, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow\ \frac{dy}{dx}=-\frac{2x/a^2}{2y/b^2}=-\frac{xb^2}{ya^2}$$

$$\therefore\ \left[\frac{dy}{dx}\right]_{(\lambda,2\lambda)}=-\frac{\lambda}{2\lambda}\cdot\frac{b^2}{a^2}=-1$$

$$\Rightarrow\ \frac{a^2}{b^2}=\frac{1}{2}$$

$$\therefore\ e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

[Rating : Easy]

5. (b) : Given, $P(\text{Head}) = 2/3$ for coin C_1

$P(\text{Head}) = 1/3$ for coin C_2

For r heads, $P(x = r) = {}^2C_r\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{2-r}$ for C_1

For r heads, $P(x = r) = {}^2C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{2-r}$ for C_2

For coin C_1	For coin C_2
$P(x = 0) = 1/9$	$P(x = 0) = 4/9$
$P(x = 1) = 4/9$	$P(x = 1) = 4/9$
$P(x = 2) = 4/9$	$P(x = 2) = 1/9$

For $x^2 - \alpha x + \beta$, to have real and equal roots, $\alpha^2 - 4\beta = 0$

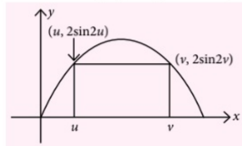
$\Rightarrow\ \alpha^2 = 4\beta$

So, $(\alpha, \beta) = (0, 0)$ or $(2, 1)$

$$\therefore\ \text{Required probability} = \frac{1}{9}\cdot\frac{4}{9}+\frac{4}{9}\cdot\frac{4}{9}=\frac{4}{81}+\frac{16}{81}=\frac{20}{81}$$

[Rating : Medium]

6. (c) : Let's draw the graph of $y = 2\sin 2x$, $x \in (0, \pi/2)$



Now, $2\sin 2u = 2\sin 2v$

$\Rightarrow\ 2u + 2v = \pi\ \Rightarrow\ v=\frac{\pi}{2}-u$

$\therefore$ Perimeter, $P = 2\{(v - u) + 2\sin 2u\}$

$$=2\left\{\frac{\pi}{2}-2u+2\sin 2u\right\}$$

$$\Rightarrow\ \frac{dP}{du}=2[-2+4\cos 2u]$$

$$\Rightarrow\ \frac{d^2P}{du^2}=2[-8\sin 2u]=-16\sin 2u$$

Now, $\frac{dP}{du}=0 \Rightarrow \cos 2u=\frac{1}{2}$

$\therefore\ 2u = \pi/3\ \Rightarrow\ u = \pi/6$

Note that $\frac{d^2P}{du^2}<0$ for $u=\frac{\pi}{6}$

So, perimeter is maximum for $u = \pi/6$

Area of rectangle having maximum perimeter

$$= (v - u) \times 2\sin 2u = \left(\frac{\pi}{2}-\frac{\pi}{3}\right)\cdot 2\sin\frac{\pi}{3}$$

$$=\frac{\pi}{6}\cdot\sqrt{3}=\frac{\pi}{2\sqrt{3}}$$

[Rating : Medium]

7. (a, c) : $f(x) = x^3 - x^2 + (x - 1)\sin x$

$\Rightarrow\ f'(x) = 3x^2 - 2x + (x - 1)\cos x + \sin x$

Let $fg(x) = f(x)\cdot g(x) = h(x)$

So, $h(x) = (x^3 - x^2 + (x - 1)\sin x)g(x)$

As f is differentiable, so g is differentiable, then the product fg will become differentiable.

So, option (c) is correct.

Again, $h(x) = f(x)\, g(x)$, $f(1) = 0$

$$\therefore\ h'(1^+)=\lim_{\theta\to 0}\frac{h(1+\theta)-h(1)}{\theta}=\lim_{\theta\to 0}\frac{f(1+\theta)g(1+\theta)-0}{\theta}$$

$$=\lim_{\theta\to 0}\frac{f(1+\theta)-f(1)}{\theta}\,g(1+\theta)$$

$$=f'(1)\lim_{\theta\to 0}g(1+\theta)$$

Similarly, $h'(1^-) = \lim\limits_{\theta\to 0}\dfrac{h(1-\theta)-h(1)}{-\theta}$

$$= \lim_{\theta\to 0}\frac{f(1-\theta)g(1-\theta)-0}{-\theta}$$

$$= \lim_{\theta\to 0}\frac{f(1-\theta)-f(1)}{-\theta}\cdot g(1-\theta)$$

$$= f'(1)\lim_{\theta\to 0} g(1-\theta)$$

Since, g is continuous.

$\therefore \quad \lim\limits_{\theta\to 0} g(1-\theta) = \lim\limits_{\theta\to 0} g(1+\theta) = g(1)$

So, $h'(1^+) = h'(1^-) = f'(1)\, g(1)$

$\therefore \quad h(x)$ is differentiable at $x = 1$

So, option (a) is correct. **[Rating : Medium]**

8. (b, c, d) : Given, $M^{-1} = \text{adj}(\text{adj}M)$ ...(i)

We know, $M\,\text{adj}M = (\det M)I$

and $(\text{adj}M)\,\text{adj}(\text{adj}M) = \det(\text{adj}M)I$

Multiplying (i) with adj$M$, we get

$(\text{adj}M)M^{-1} = \text{adj}M\,\text{adj}(\text{adj}M) = \det(\text{adj}M)I = (\det M)^2 I$

$\Rightarrow \quad (\text{adj}M) = (\det M)^2\, M$ ...(ii)

$\Rightarrow \quad \det(\text{adj}M) = \det((\det M)^2 M)$

$\Rightarrow \quad (\det M)^2 = (\det M)^6\cdot\det M \Rightarrow (\det M)^2 = (\det M)^7$

As $\det M \neq 0$, we have, $\det M = 1$

Using (ii), $\text{adj}M = M$

$\Rightarrow \quad M\,\text{adj}M = M^2 \Rightarrow (\det M)I = M^2 \Rightarrow M^2 = I$

So, $\text{adj}M = M$

$\Rightarrow \quad (\text{adj}M)^2 = M^2 = I$ **[Rating : Difficult]**

9. (b, c) : $|z^2 + z + 1| = 1$

$$\Rightarrow \quad \left|\left(z+\frac{1}{2}\right)^2+\frac{3}{4}\right| = 1$$

$$\text{Now, } 1 = \left|\left(z+\frac{1}{2}\right)^2+\frac{3}{4}\right| \le \left|\left(z+\frac{1}{z}\right)^2\right|+\frac{3}{4}$$

[Using triangle inequality]

$$\Rightarrow \quad \left|\left(z+\frac{1}{2}\right)\right|^2 \ge \frac{1}{4}. \quad \text{Hence, } \left|z+\frac{1}{2}\right| \ge \frac{1}{2}$$

Again, $|z^2 + z + 1| = 1 \ge ||z^2 + z| - 1|$

$\Rightarrow \quad |z^2 + z| - 1 \le 1$

$\Rightarrow \quad |z^2 + z| \le 2$ ...(i)

Again, $||z|^2 - |z|| \le |z^2 + z| \le 2$ (From (i))

Let $|z| = k$ so $|k^2 - k| \le 2$, which gives $k^2 - k \le 2$

$\Rightarrow \quad k^2 - k - 2 \le 0 \Rightarrow (k-2)(k+1) \le 0$. So, $k \le 2$

Again, $|z^2 + z + 1| = 1 \Rightarrow z^2 + z + 1 = e^{i\theta}$, $\theta \in (-\pi, \pi)$

$\Rightarrow \quad z^2 + z + 1 = \cos\theta + i\sin\theta,\ \theta\in(-\pi, \pi]$

We can find infinite value of z for different values of θ.

[Rating : Difficult]

10. (b, c) : We know that, in triangle ABC,

$$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$$

$$\text{Given, } \tan\frac{X}{2}+\tan\frac{Z}{2} = \frac{2y}{x+y+z}$$

$$\Rightarrow \quad \frac{\Delta}{s(s-x)}+\frac{\Delta}{s(s-z)} = \frac{2y}{x+y+z}$$

$$\Rightarrow \quad \frac{\Delta}{s(s-x)(s-z)}\{2s-(x+z)\} = \frac{2y}{x+y+z}$$

$$\Rightarrow \quad \frac{\Delta y}{s(s-x)(s-z)} = \frac{2y}{x+y+z}$$

$$\Rightarrow \quad \frac{\Delta}{s(s-x)(s-z)} = \frac{2}{x+y+z} \quad (\because y \neq 0)$$

$$\Rightarrow \quad \frac{\Delta}{s}\cdot\frac{1}{(s-x)(s-z)} = \frac{1}{s} \Rightarrow \Delta = (s-x)(s-z)$$

$$\text{Also, } \tan\frac{Y}{2} = \frac{(s-x)(s-z)}{\Delta} = 1 \Rightarrow Y = \frac{\pi}{2}$$

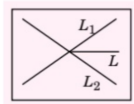
Thus, $Y = X + Z$

$$\text{Also, } \tan\frac{X}{2} = \frac{\Delta}{s(s-x)} = \frac{\frac{1}{2}xz}{\frac{(y+z+x)}{4}(y+z-x)}$$

$$= \frac{2xz}{y^2+z^2+2yz-x^2} = \frac{2xz}{2z^2+2yz} = \frac{x}{y+z}$$

[Rating : Medium]

11. (a, b) : $L_1 : \dfrac{x-1}{1} = \dfrac{y}{-1} = \dfrac{z-1}{3}$



and $L_2 : \dfrac{x-1}{-3} = \dfrac{y}{-1} = \dfrac{z-1}{1}$

The point of intersection of the lines L_1 and L_2 is $(1, 0, 1)$

L_1 has the direction vector, $\hat{i} - \hat{j} + 3\hat{k}$

L_2 has the direction vector, $-3\hat{i} - \hat{j} + \hat{k}$

Acute angle bisector of L_1 and L_2, $\vec{r}$ is given by

$$\vec{r} = \hat{i}+\hat{k}+t\left(\frac{\hat{i}-\hat{j}+3\hat{k}-3\hat{i}-\hat{j}+\hat{k}}{\sqrt{11}}\right)$$

$= \hat{i}+\hat{k}+s(\hat{i}+\hat{j}-2\hat{k})$, s is another constant

Line L is given to be $\dfrac{x-\alpha}{l} = \dfrac{y-1}{m} = \dfrac{z-\gamma}{-2}$

So, $\dfrac{l}{1} = \dfrac{m}{1} = \dfrac{-2}{-2} \Rightarrow l = 1,\ m = 1$

Again, $\dfrac{1-\alpha}{l} = \dfrac{0-1}{m} = \dfrac{1-\gamma}{-2}$

$$\Rightarrow \quad \frac{1-\alpha}{1}=\frac{-1}{1}=\frac{1-\gamma}{-2}. \Rightarrow \gamma = -1, \alpha = 2$$

[Rating : Easy]

12. (a, b, d) : We know that

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\ldots\ldots$$

and $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots\ldots$

So, $\cos x \geq 1-\frac{x^2}{2}$ and $\sin x \geq x-\frac{x^3}{3!}$

(a) $\int_0^1 x\cos x\,dx \geq \int_0^1 x\left(1-\frac{x^2}{2}\right)dx$

$$=\left[\frac{x^2}{2}-\frac{x^4}{8}\right]_0^1=\frac{1}{2}-\frac{1}{8}=\frac{3}{8}$$

(b) $\int_0^1 x\sin x\,dx \geq \int_0^1 x\left(x-\frac{x^3}{6}\right)dx$

$$=\left[\frac{x^3}{3}-\frac{x^5}{30}\right]_0^1=\frac{1}{3}-\frac{1}{30}=\frac{1}{3}\left(1-\frac{1}{10}\right)=\frac{9}{30}=\frac{3}{10}$$

(c) $\int_0^1 x^2\cos x\,dx \leq \int_0^1 x^2 dx$ [since $\cos x \leq 1$]

$$=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

(d) $\int_0^1 x^2\sin x\,dx \geq \int_0^1 x^2\left(x-\frac{x^3}{6}\right)dx=\left[\frac{x^4}{4}-\frac{x^6}{36}\right]_0^1$

$$=\frac{1}{4}-\frac{1}{36}=\frac{1}{4}\left(1-\frac{1}{9}\right)=\frac{8}{4\times 9}=\frac{2}{9}$$

[Rating : Difficult]

13. (8) : Using A.M. $\geq$ G.M. inequality, we have

$$\frac{3^{y_1}+3^{y_2}+3^{y_3}}{3}\geq\sqrt[3]{3^{y_1}\cdot 3^{y_2}\cdot 3^{y_3}}$$

$$\Rightarrow \quad 3^{y_1}+3^{y_2}+3^{y_3}\geq 3\cdot\sqrt[3]{3^{y_1+y_2+y_3}}$$

$$=3\cdot\sqrt[3]{3^9}=3\cdot 3^3=3^4$$

$\Rightarrow \quad \log_3(3^{y_1}+3^{y_2}+3^{y_3})\geq 4$

$\therefore \quad m = 4.$

Again using A.M. $\geq$ G.M., we have

$$\frac{x_1+x_2+x_3}{3}\geq\sqrt[3]{x_1x_2x_3} \quad \Rightarrow \quad x_1x_2x_3 \leq 27 = 3^3$$

$\Rightarrow \quad \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3$

$\therefore \quad M = 3$

Now, $\log_2(m^3) + \log_3(M^2) = 3\log_2 m + 2\log_3 M$

$= 3\log_2 4 + 2\log_3 3 = 6 + 2 = 8$ **[Rating : Easy]**

14. (1) : $2\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$

$\Rightarrow \quad 2\cdot\frac{n}{2}(2c+(n-1)\cdot 2)=c\left(\frac{2^n-1}{2-1}\right)$

$\Rightarrow \quad n(2c + 2n - 2) = c(2^n - 1)$

$\Rightarrow \quad c(2^n - 2n - 1) = n(2n - 2)$

$\Rightarrow \quad c=\frac{2n(n-1)}{2^n-2n-1}\geq 1$ ...(i)

[As, c must be positive integer and $n \geq 2$ as for $n = 1$, $c = 0$, not possible]

$\Rightarrow \quad 2n(n - 1) \geq 2^n - 2n - 1$

$\Rightarrow \quad 2^n \leq 2n^2 - 2n + 2n + 1 \Rightarrow 2^n \leq 2n^2 + 1$

The above inequality holds for $n \leq 6$.

Taking $n = 2, 3, 4, 5, 6$ using (i), we get

For $n=2$, $c=\frac{2\cdot 2\cdot 1}{4-4-1}=-$ve, discarded

For $n=3$, $c=\frac{2\cdot 3\cdot 2}{8-6-1}=12$

For $n=4$, $c=\frac{2\cdot 4\cdot 3}{16-8-1}=\frac{24}{7}$, not an integer

For $n=5$, $c=\frac{2\cdot 5\cdot 4}{32-10-1}=\frac{40}{21}$, not an integer

For $n=6$, $c=\frac{2\cdot 6\cdot 5}{64-12-1}=\frac{60}{51}$, not an integer

Thus, $c = 12$ is the only possible value.

Hence, the number of values of c is 1.

[Rating : Medium]

15. (1) : $f(x)=(3-\sin 2\pi x)\sin\left(\pi x-\frac{\pi}{4}\right)$

$$-\sin\left(3\pi x+\frac{\pi}{4}\right)$$

Let $\pi x-\frac{\pi}{4}=t \quad \Rightarrow \quad \pi x=\frac{\pi}{4}+t$

$\therefore \quad g(t)=\left(3-\sin\left(\frac{\pi}{2}+2t\right)\right)\sin t-\sin\left(\frac{3\pi}{4}+3t+\frac{\pi}{4}\right)$

$= (3 - \cos 2t)\sin t - \sin(\pi + 3t)$

$= (3 - \cos 2t)\sin t + \sin 3t$

$f(x) \geq 0$ *i.e.* $g(t) \geq 0$

$\Rightarrow \quad (3 - \cos 2t)\sin t + \sin 3t \geq 0$

$\Rightarrow \quad \sin t\{3 - \cos 2t + 3 - 4\sin^2 t\} \geq 0$

$\Rightarrow \quad \sin t\{6 - \cos 2t - 2(1 - \cos 2t)\} \geq 0$

$\Rightarrow \quad \sin t(4 + \cos 2t) \geq 0 \Rightarrow \sin t \geq 0$ ...(i)

As $t=\pi x-\frac{\pi}{4}$ for $x\in[0,2] \Rightarrow t\in\left[-\frac{\pi}{4},\frac{7\pi}{4}\right]$

So, (i) gives $0 \leq t \leq \pi$

$$\Rightarrow \quad 0 \le \left(\pi x - \frac{\pi}{4}\right) \le \pi \quad \Rightarrow \quad 0 \le \left(x - \frac{1}{4}\right) \le 1$$

$$\Rightarrow \quad \frac{1}{4} \le x \le \frac{5}{4} \quad \Rightarrow \quad x \in \left[\frac{1}{4}, \frac{5}{4}\right]$$

$$\therefore \quad \beta - \alpha = \frac{5}{4} - \frac{1}{4} = 1$$

[Rating : Medium]

16. (108) : We have, $|\vec{a}| = 3, |\vec{b}| = 4$

Also, $\dfrac{\vec{a}\cdot(\vec{c}-\vec{b})}{\vec{c}\cdot(\vec{a}-\vec{b})} = \dfrac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$

$$\Rightarrow \quad \frac{-(\vec{b}+\vec{c})(\vec{c}-\vec{b})}{-(\vec{a}+\vec{b})(\vec{a}-\vec{b})} = \frac{3}{7} \qquad [\because \vec{a}+\vec{b}+\vec{c}=0]$$

$$\Rightarrow \quad \frac{|\vec{b}|^2-|\vec{c}|^2}{|\vec{b}|^2-|\vec{a}|^2} = \frac{3}{7} \quad \Rightarrow \quad \frac{16-|\vec{c}|^2}{16-9} = \frac{3}{7}$$

$$\Rightarrow \quad 16 - |\vec{c}|^2 = 3 \quad \Rightarrow \quad |\vec{c}|^2 = 13$$

Now, $\vec{a}+\vec{b} = -\vec{c} \quad \Rightarrow \quad |\vec{a}+\vec{b}|^2 = |-\vec{c}|^2$

$$\Rightarrow \quad |a|^2 + |b|^2 + 2\vec{a}\cdot\vec{b} = |c|^2 \Rightarrow 3^2 + 4^2 + 2\vec{a}\cdot\vec{b} = 13$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -12 \quad \Rightarrow \quad \vec{a}\cdot\vec{b} = -6$$

Now, $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$$= 3^2\cdot 4^2 - 6^2 = 144 - 36 = 108$$

[Rating : Easy]

17. (5) : Let $(x) = (x^2 - 1)^2 P(x)$, where

$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

As $f(-1) = f(1) = 0$

$\therefore \quad \exists\, r \in (-1, 1)$ such that $f'(r) = 0$

(By Rolle's theorem)

Again $f'(x) = 0$ has two obvious roots -1 and 1

Thus, $f'(x) = 0$ has 3 roots *i.e.*, $-1, 1, r$; $-1 < r < 1$

Also, $f''(x) = 0$ will have at least two roots, say s and t such that $-1 < s < r < t < 1$ (By Rolle's theorem)

Hence, min $(m_{f'}) = 3$, min $(m_{f''}) = 2$

The minimum possible value of $m_{f'} + m_{f''}$ is 5.

[Rating : Medium]

18. (1) : $\displaystyle\lim_{x\to 0^+} \frac{(1-x)^{1/x} - e^{-1}}{x^a}$

$$= \lim_{x\to 0^+} \frac{e^{\frac{1}{x}\ln(1-x)} - \frac{1}{e}}{x^a} = \lim_{x\to 0^+} \frac{e^{1+\frac{1}{x}\ln(1-x)}\cdot\frac{1}{e} - \frac{1}{e}}{x^a}$$

$$= \frac{1}{e}\lim_{x\to 0^+} \frac{e^{1+\frac{\ln(1-x)}{x}} - 1}{x^a}$$

$$= \frac{1}{e}\lim_{x\to 0^+} \frac{1+\frac{\ln(1-x)}{x}}{x^a} \qquad \text{(Neglecting higher powers)}$$

$$= \frac{1}{e}\lim_{x\to 0^+} \frac{\ln(1-x)+x}{x^{a+1}}$$

$$= \frac{1}{e}\lim_{x\to 0^+} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right)+x}{x^{a+1}}$$

$$= \frac{1}{e}\lim_{x\to 0^+} \frac{\left(-\frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots\right)}{x^{a+1}}$$

For non-zero limit, $a + 1 = 2 \quad \therefore \; a = 1$

[Rating : Medium]

PAPER-2

1. (8) : $z^4 - |z|^4 = 4iz^2$

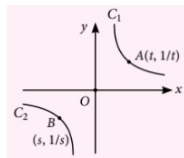
$$\Rightarrow \quad z^4 - (z\bar{z})^2 = 4iz^2 \quad \Rightarrow \quad z^2\,|z^2 - \bar{z}^2| = 4iz^2$$

$\Rightarrow \quad z^2 = 0$ or $z^2 - \bar{z}^2 = 4i$

$\Rightarrow \quad (z+\bar{z})(z-\bar{z}) = 4i$

$\Rightarrow \quad 2x(2iy) = 4i$

$\therefore \quad xy = 1$



The minimun distance occur along with normal.

Let $(t, 1/t)$ be a point on C_1.

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{t^2}$$

Slope of normal $= t^2 \;\; (t > 0)$

Similarly, let $(s, 1/s)$ be a point on C_2, $(s < 0)$.

$\therefore \quad$ Slope of normal $= s^2$

$$\text{Slope of } AB = \frac{\frac{1}{t} - \frac{1}{s}}{t - s} = -\frac{1}{ts}$$

From $t^2 = s^2 = -\dfrac{1}{ts}$, we get $t = 1, s = -1$

Hence, the shortest distance $= \sqrt{(1+1)^2 + (1+1)^2} = \sqrt{8}$

$\therefore \quad |z_1 - z_2|^2 = 8$

[Rating : Medium]

2. (6) : We have,

$$p = 0.75 = \frac{3}{4} \Rightarrow q = \frac{1}{4}$$

$$\text{Let } P(X = r) = {}^nC_r\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{n-r}$$

We have $P((X \ge 3) \ge 0.95$

$\Rightarrow \quad 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\} \ge 0.95$

$$\Rightarrow \; 0.05 \ge {}^nC_0\left(\frac{1}{4}\right)^n + {}^nC_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n-1} + {}^nC_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2}$$

$$\Rightarrow \frac{1}{20} \geq \frac{1}{4^n} + \frac{3n}{4^n} + \frac{n(n-1)}{2} \cdot \frac{9}{4^n}$$

$$\Rightarrow \frac{4^n}{20} \geq 1 + 3n + \frac{9n^2 - 9n}{2}$$

$$\Rightarrow \frac{4^n}{20} \geq \frac{2 + 6n + 9n^2 - 9n}{2} \quad \Rightarrow \frac{4^n}{10} \geq 9n^2 - 3n + 2$$

$\Rightarrow 2^{2n-1} \geq 5(9n^2 - 3n + 2)$

Let $f(x) = 2^{2n-1}$, $g(x) = 5(9n^2 - 3n + 2)$

If $n = 4, f(4) = 2^7 = 128$

and $g(4) = 5(9 \cdot 16 - 3 \cdot 4 + 2) = 5(144 - 12 + 2)$
$= 5(132 + 2) = 5 \cdot 134 = 670$

If $n = 5$, $f(5) = 2^9 = 512$

and $g(5) = 5(9 \cdot 25 - 3 \cdot 5 + 2) = 5(225 - 15 + 2)$
$= 5 \cdot 212 = 1060$

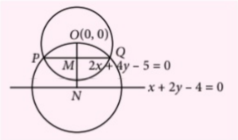
If $n = 6$, $f(6) = 2^{11} = 2048$

and $g(6) = 5(9 \cdot 36 - 3 \cdot 6 + 2) = 5(324 - 18 + 2)$
$= 5 \cdot 308 = 1540$

Hence, the minimum value of n is 6.

[Rating : Difficult]

3. (2) :



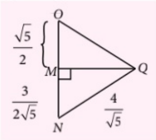
Note that the lines are parallel.

Let the perpendiculars from O to the lines be OM and ON.

$$OM = \left|\frac{5}{\sqrt{20}}\right| = \frac{\sqrt{5}}{2}; \quad ON = \frac{4}{\sqrt{5}}$$

N is the centre of the circle passing through O, P and Q.

So, $NQ = NO = \dfrac{4}{\sqrt{5}}$



Now, we draw ΔQON.

$MN = ON - OM$

$$= \frac{4}{\sqrt{5}} - \frac{\sqrt{5}}{2} = \frac{8-5}{2\sqrt{5}} = \frac{3}{2\sqrt{5}}$$

We have, $OQ^2 - OM^2 = NQ^2 - NM^2$ $(= MQ^2)$

$$\Rightarrow OQ^2 - \frac{5}{4} = \frac{16}{5} - \frac{9}{20}$$

$$\Rightarrow OQ^2 = \frac{16}{5} + \frac{5}{4} - \frac{9}{20} = \frac{64 + 25 - 9}{20} = \frac{80}{20} = 4$$

$\therefore \quad OQ = 2. \quad \therefore \quad r = 2$ **[Rating : Medium]**

4. (5) : Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $tr(A) = 3 = a + d$

and $\det(A) = ad - bc$

$$\text{So, } A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}$$

$$A^3 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^3+abc+bc(a+d) & \alpha \\ \beta & bc(a+d)+d(d^2+bc) \end{bmatrix},$$

where α and β don't required in trace.

$\therefore \quad \text{tr}(A^3) = a^3 + abc + bca + bcd + bca + bcd + d^3 + bcd$
$= a^3 + d^3 + 3(abc + bcd)$
$= a^3 + d^3 + 3bc(a + d)$
$= (a + d)^3 - 3ad(a + d) + 3bc(a + d)$
$= (a + d)^3 - 3(a + d)(ad - bc)$

$\Rightarrow \quad -18 = 3^3 - 3 \cdot 3 \det(A)$

$\Rightarrow \quad -18 = 27 - 9 \det(A) \Rightarrow -2 = 3 - \det(A)$

$\therefore \quad \det(A) = 5$ **[Rating : Medium]**

5. (4) : We have, $f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x] = \{x\}$

$\therefore \quad f(g(x)) = f(\{x\}) = |2\{x\} - 1| + |2\{x\} + 1|$

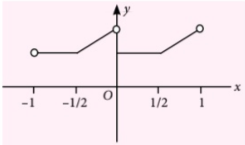
$\Rightarrow \quad f(g(x)) = 2\{x\} - 1 + 2\{x\} + 1 = 4\{x\}$, if $\{x\} > \dfrac{1}{2}$

and $f(g(x)) = 1 - 2\{x\} + 2\{x\} + 1 = 2$, if $\{x\} \leq \dfrac{1}{2}$

Now as $x \in (-1, 1)$, we have

$$\{x\} \leq \frac{1}{2} \Rightarrow x \in \left(-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right]$$

$$\{x\} > \frac{1}{2} \Rightarrow x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, 1\right)$$



Note that –1 and 1 are not in the domain.

So the point where $f(g(x))$ is discontinuous is $x = 0$.

$\therefore \quad c = 1$

$f(g(x))$ is not differentiable at 3 points : $-\dfrac{1}{2}, 0, \dfrac{1}{2}$.

$\therefore \quad d = 3$

Thus $c + d = 4$ **[Rating : Difficult]**

6. (8) :

$$\lim_{x\to\pi/2}\frac{4\sqrt{2}(\sin 3x+\sin x)}{\left(2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{3x}{2}\right)}$$

$$=\lim_{x\to\pi/2}\frac{4\sqrt{2}\cdot 2\sin 2x\cos x}{2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}-\cos\frac{3x}{2}-\sqrt{2}(1+\cos 2x)}$$

$$=\lim_{x\to\pi/2}\frac{8\sqrt{2}\sin 2x\cos x}{2\sin 2x\sin\frac{3x}{2}-2\sin 2x\sin\frac{x}{2}-\sqrt{2}\cdot 2\cos^2 x}$$

$$=\lim_{x\to\pi/2}\frac{8\sqrt{2}\sin 2x\cos x}{2\sin 2x\left(\sin\frac{3x}{2}-\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2 x}$$

$$=\lim_{x\to\pi/2}\frac{16\sqrt{2}\sin x\cos^2 x}{4\sin x\cos x\left(\sin\frac{3x}{2}-\sin\frac{x}{2}\right)-2\sqrt{2}\cos^2 x}$$

$$=\lim_{x\to\pi/2}\frac{16\sqrt{2}\sin x\cos^2 x}{4\sin x\cos x\left(2\sin\frac{x}{2}\cos x\right)-2\sqrt{2}\cos^2 x}$$

$$=\lim_{x\to\pi/2}\frac{16\sqrt{2}\sin x}{8\sin x\sin\frac{x}{2}-2\sqrt{2}}=\frac{16\sqrt{2}}{4\sqrt{2}-2\sqrt{2}}=\frac{16\sqrt{2}}{2\sqrt{2}}=8$$

[Rating : Medium]

7. (a, c) : We have, $f'(x)=\frac{f(x)}{b^2+x^2}$

$$\Rightarrow\quad \frac{f'(x)}{f(x)}=\frac{1}{b^2+x^2}$$

On integration, $\ln|f(x)|=\int\frac{dx}{b^2+x^2}$

$$\Rightarrow\quad \ln|f(x)|=\frac{1}{b}\tan^{-1}\frac{x}{b}+c$$

As $f(0) = 1$, so we have $c = 0$.

$$\therefore\ \ln|f(x)|=\frac{1}{b}\tan^{-1}\frac{x}{b}\ \Rightarrow\ f(x)=\pm e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}$$

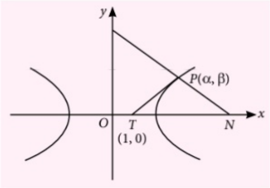
But $f(0) = 1$. $\Rightarrow\ f(x)=e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}$

$$\therefore\quad f(x)\,f(-x)=e^{\frac{1}{b}\tan^{-1}(x/b)}\cdot e^{\frac{1}{b}\tan^{-1}(-x/b)}$$

$$=e^{\frac{1}{b}\left(\tan^{-1}(x/b)+\tan^{-1}(-x/b)\right)}=e^{\frac{1}{b}\times 0}=e^0=1$$

For $b>0$, $f(x)=e^{\frac{1}{b}\tan^{-1}(x/b)}$ is increasing and $\tan^{-1}(x/b)$ is an increasing function. **[Rating : Medium]**

8. (a, d) :



Let $P(\alpha, \beta)$ be the point at which normal makes equal intercepts on the axes.

$\therefore$ Slope of normal $= -1$

Equation of tangent is $\frac{x\alpha}{a^2}-\frac{y\beta}{b^2}=1$

$(1, 0)$ lies on the tangent.

$$\therefore\quad \frac{\alpha}{a^2}=1\ \Rightarrow\ \alpha=a^2$$

Slope of tangent $=\frac{\alpha/a^2}{\beta/b^2}=1$

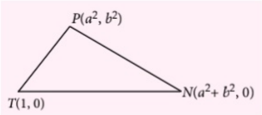
$$\therefore\ \frac{\beta}{b^2}=1\ \Rightarrow\ \beta=b^2$$

Then $P\equiv(a^2, b^2)$

The normal at P is $\frac{a^2}{\alpha}x+\frac{b^2}{\beta}y=a^2+b^2$

N is given by setting $y = 0$.

$$\therefore\quad x=\frac{(a^2+b^2)\alpha}{a^2}=a^2+b^2$$



$$ar(\Delta PTN)=\Delta=\frac{1}{2}(a^2+b^2-1)b^2$$

$$=\frac{1}{2}\cdot 2b^2\cdot b^2=b^4\quad\left(\begin{array}{l}\because\ \text{Slope of } PT=1\\ \Rightarrow\ \frac{b^2}{a^2-1}=1\ \Rightarrow\ b^2=a^2-1\end{array}\right)$$

$$\therefore\quad e^2=1+\frac{b^2}{a^2}=1+\frac{a^2-1}{a^2}=2-\frac{1}{a^2}$$

$$\Rightarrow\ e^2<2\qquad\therefore\ e<\sqrt{2}$$

As hyperbola has $e > 1$, so we have

$1<e<\sqrt{2}$. **[Rating : Medium]**

9. (a, b, d) : We have, $f(x + y) = f(x) + f(y) + f(x)\,f(y)$ ... (i)

and $f(x) = x\,g(x)$ ... (ii)

Also, $\lim\limits_{x\to 0} g(x) = 1$

Now, $f(0) = 0$ (from (i))

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+f(x)f(h)-f(x)-f(0)-f(x)f(0)}{h}$$ [Using (i)]

$$= \lim_{h\to 0}\frac{f(h)-f(0)+f(x)[f(h)-f(0)]}{h}$$

$= f'(0) + f(x)\,f'(0)$

Now, $g(x) = \dfrac{f(x)}{x}, x \neq 0$ [Using (ii)]

Then $1 = \lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x} = f'(0)$

Thus, $f'(0) = 1$.

$\therefore\ f'(x) = 1 + f(x) \Rightarrow 1 + f(x) = e^x$

$\Rightarrow f(x) = e^x - 1 \Rightarrow f'(x) = e^x$

Thus, f is differentiable at every $x \in R$.

$\therefore\ g(x) = \dfrac{e^x-1}{x}, x \neq 0$

Now, $\lim\limits_{x\to 0} g(x) = 1$ and if $g(0) = 1$, then we have g continuous on R.

$$g(x) = \begin{cases} \dfrac{e^x-1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

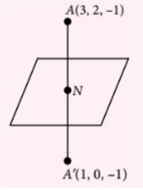
$$g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{\frac{e^h-1}{h}-1}{h}$$

$$= \lim_{h\to 0}\frac{e^h-1-h}{h^2} = \lim_{h\to 0}\frac{1+h+\frac{h^2}{2}+\ldots-1-h}{h^2} = \frac{1}{2}$$

As g is differentiable at 0 and also g is differentiable at $x \neq 0$, then $g(x)$ is differentiable on R.

[Rating : Difficult]

10. (a, b, c) :



We see that N is the midpoint of AA'.

$\therefore\ N \equiv (2, 1, -1)$

Since, N lies on the plane $\alpha x + \beta y + \gamma z = \delta$

$\therefore\ 2\alpha + \beta - \gamma = \delta$

Normal to the plane is $\langle 2, 2, 0\rangle$ *i.e*, $(1, 1, 0)$

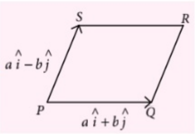
As $\alpha + \gamma = 1$, so we have $\alpha = 1, \gamma = 0$.

Then $\beta = 1$

So, $\delta = 2\alpha + \beta - \gamma = 2\cdot 1 + 1 - 0 = 3$

[Rating : Easy]

11. (a, c) :



Projection vector of $\vec{w} = \hat{i} + \hat{j}$ along $\overline{PQ}$ is

$\vec{u} = \left((\hat{i}+\hat{j})\cdot\dfrac{a\hat{i}+b\hat{j}}{\sqrt{a^2+b^2}}\right)\cdot\hat{e}_1$, where $\hat{e}_1$ is unit vector along $\overline{PQ}$.

Similarly,

$\vec{v} = \left((\hat{i}+\hat{j})\cdot\dfrac{a\hat{i}-b\hat{j}}{\sqrt{a^2+b^2}}\right)\cdot\hat{e}_2$, where $\hat{e}_2$ is unit vector along $\overline{PS}$

$|\vec{u}| = \dfrac{|a+b|}{\sqrt{a^2+b^2}},\ |\vec{v}| = \dfrac{|a-b|}{\sqrt{a^2+b^2}}$

$|\vec{u}| + |\vec{v}| = |\vec{w}| \Rightarrow \dfrac{|a+b|+|a-b|}{\sqrt{a^2+b^2}} = \sqrt{2}$

$\Rightarrow (a + b)^2 + (a - b)^2 + 2|a + b||a - b| = 2(a^2 + b^2)$

$\therefore\ a = b$

Area of parallelogram $= |(a\hat{i}+b\hat{j})\times(a\hat{i}-b\hat{j})| = 2ab$

Thus, $2ab = 8 \Rightarrow 2a^2 = 8 \Rightarrow a = 2$

Hence, $b = 2$

Length of diagonal $= |2a\hat{i}| = 4$

[Rating : Medium]

Monthly Test Drive-5 CLASS XII ANSWER KEY

1. (a)	**2.** (d)	**3.** (c)	**4.** (b)	**5.** (a)
6. (a)	**7.** (a, d)	**8.** (a, b, c)	**9.** (b)	**10.** (a,d)
11. (c,d)	**12.** (a, b, c, d)	**13.** (a, c)	**14.** (a)	**15.** (b)
16. (d)	**17.** (3)	**18.** (3)	**19.** (5)	**20.** (1)

12. (a, b, d) : We have,

$$f(m,n,p)=\sum_{i=0}^{p} {}^mC_i \cdot {}^{n+i}C_p \cdot {}^{p+n}C_{p-i}$$

$$=\sum_{i=0}^{p} {}^mC_i \cdot \frac{(n+i)!}{p!\,(n+i-p)!}\cdot\frac{(p+n)!}{(p-i)!\,(n+i)!}$$

$$=\sum_{i=0}^{p} {}^mC_i \cdot \frac{(p+n)!}{p!(n+i-p)!\,(p-i)!}$$

$$=\sum_{i=0}^{p} {}^mC_i \cdot \frac{(p+n)!}{p!(n)!}\cdot\frac{n!}{(n-p+i)!(p-i)!}$$

$$=\sum_{i=0}^{p} {}^mC_i \cdot {}^{n+p}C_p\, {}^nC_{p-i} = {}^{n+p}C_p \sum_{i=0}^{p} {}^mC_i \cdot {}^nC_{p-i}$$

$$= {}^{n+p}C_p \cdot {}^{m+n}C_p \text{ (Using Vandermonde's identity)}$$

Now, $g(m,n)=\sum_{p=0}^{m+n}\frac{f(m,n,p)}{{}^{n+p}C_p}=\sum_{p=0}^{m+n} {}^{m+n}C_p = 2^{m+n}$

$g(m, n)$ is symmetric in m and n.

$\therefore\ g(m, n) = g(n, m)$

Also, $g(m, n + 1) = 2^{m+n+1} = 2^{(m+1)+n}$

$= g(m + 1, n)$

$g(2m, 2n) = 2^{2m+2n} = (2^{m+n})^2 = (g(m, n))^2$

[Rating : Difficult]

13. (495) : Let the days be arranged like

$$\underbrace{1\ \ 2}_{x_1}\ \ 3\ \ \underbrace{4\ \ 5}_{x_2}\ \ 6\ \ 7 \ldots\ldots\ldots\ldots\ 14\ \ 15$$

$$\underbrace{\quad}_{x_1}\ A\ \underbrace{\quad}_{x_2}\ B\underbrace{\quad}_{x_3}C\underbrace{\quad}_{x_4}D\ \underbrace{\quad}_{x_5}$$

Let A, B, C, D be the days on which the engineer has the visit.

Let x_1 be the first day before A, x_2 between A and B and so on.

We have, $x_1 + x_2 + x_3 + x_4 + x_5 = 11$ ... (i)

Also, $x_1, x_5 \geq 0$. But $x_2, x_3, x_4 \geq 1$ ... (ii)

We want to find the non-negative integral solution of (i) subject to (ii).

So the number of solutions $= {}^{(11-3)+5-1}C_{5-1}$

$$= {}^{12}C_4 = \frac{12\cdot 11\cdot 10\cdot 9}{24} = 495$$

[Rating : Medium]

14. (1080) : We can have 4 groups of person such that 2 groups contain 1 person each and other 2 groups contain 2 persons each.

i.e., $6 = 2 + 2 + 1 + 1$

Number of ways to form the groups

$$=\frac{6!}{2!\,2!\,1!\,1!}\times\frac{1}{2!}\cdot\frac{1}{2!}=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot2\cdot2\cdot2}=45$$

They can be assigned 4 rooms in 4! ways.

So the number of ways $= 45 \times 4! = 45 \times 24 = 1080$

[Rating : Easy]

15. (8) : Sum = prime *i.e.*, 2, 3, 5, 7, 11

$P(S=2)=\frac{1}{36}$; $P(S=3)=\frac{2}{36}$; $P(S=5)=\frac{4}{36}$;

$P(S=7)=\frac{6}{36}$; $P(S=11)=\frac{2}{36}$

Similarly, sum = perfect square *i.e.*, 4, 9

$P(S=4)=\frac{3}{36}$; $P(S=9)=\frac{4}{36}$

$$P(S=\text{prime})=\frac{1+2+4+6+2}{36}=\frac{15}{36}=q$$

$$P(S=\text{perfect square})=\frac{3+4}{36}=\frac{7}{36}=t$$

$$P(S=\text{neither square nor prime})=1-\frac{7}{36}-\frac{15}{36}=\frac{14}{36}=s$$

$$P(S=\text{odd perfect square } i.e., 9)=\frac{4}{36}=r$$

Now, P(sum is perfect square before sum is a prime)

$= t + st + s^2t + \ldots = t(1 + s + s^2 + \ldots)$

P(sum is odd perfect square before prime)

$= r + sr + s^2r + \ldots = r(1 + s + s^2 + \ldots)$

$$\therefore\ P\left(\frac{\text{sum is odd perfect square}}{\text{sum is perfect square before prime}}\right)$$

$$\Rightarrow\ p=\frac{r(1+s+s^2+\ldots)}{t(1+s+s^2+\ldots)}=\frac{r}{t}=\frac{4/36}{7/36}=\frac{4}{7}$$

$\therefore\ 14p = 8$

[Rating : Difficult]

16. (19) : We have, $f(x)=\frac{4^x}{4^x+2}$

$$\therefore\ f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}$$

$$f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}$$

$$=\frac{4^x}{4^x+2}+\frac{2}{4^x+2}=\frac{4^x+2}{4^x+2}=1$$

Now, $f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+f\left(\frac{3}{40}\right)+\ldots+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)$

$$=\left(f\left(\frac{1}{40}\right)+f\left(\frac{39}{40}\right)\right)+\left(f\left(\frac{2}{40}\right)+f\left(\frac{38}{40}\right)\right)+\ldots\ldots$$

$$+\left(f\left(\frac{19}{40}\right)+f\left(\frac{21}{40}\right)\right)+f\left(\frac{20}{40}\right)-f\left(\frac{1}{2}\right)$$

$$= \underbrace{(1+1+1+...+1)}_{19 \text{ terms}} + f\left(\frac{1}{2}\right) - f\left(\frac{1}{2}\right) = 19$$

[Rating : Easy]

17. (4) : We have, $F(x) = \int_0^x f(t)\,dt \Rightarrow F'(x) = f(x)$

Also, $\int_0^\pi (f'(x) + F(x))\cos x\,dx = 2$

Let, $I = \int_0^\pi f'(x)\cos x\,dx$

$$= \left[\cos x\, f(x)\right]_0^\pi - \int_0^\pi (-\sin x) f(x)\,dx$$

$$= -f(\pi) - f(0) + \int_0^\pi \sin x\, f(x)\,dx$$

$$= 6 - f(0) + \int_0^\pi \sin x \cdot F'(x)\,dx$$

$$= 6 - f(0) + [\sin x\, F(x)]_0^\pi - \int_0^\pi \cos x\, F(x)\,dx$$

$$= 6 - f(0) + 0 - \int_0^\pi \cos x\, F(x)\,dx$$

$$\Rightarrow \int_0^\pi (f'(x) + F(x))\cos x\,dx = 6 - f(0)$$

$\Rightarrow 2 = 6 - f(0) \Rightarrow f(0) = 4$ **[Rating : Difficult]**

18. (0.5) : $f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$
$= 1 + \sin 2\theta + (1 - \sin 2\theta)^2$

Let $1 - \sin 2\theta = t$. As $\theta \in (0, \pi)$, $\sin 2\theta \in [-1, 1]$

$\therefore\ t \in [0, 2]$

$\therefore\ g(t) = 1 + 1 - t + t^2 = t^2 - t + 2$

Now, $g(t)$ has local minima at $t = 1/2$

$$\therefore\ 1 - \sin 2\theta = \frac{1}{2} \Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}. \quad \text{As } 2\theta \in (0, 2\pi)$$

$$\therefore\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

So, $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$

Thus, $\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = 0.5$

[Rating : Medium]

◈◈

SOLVED PAPER

Held on
5th Sep, Morning Shift

JEE MAIN 2020

1. If $y = y(x)$ is the solution of the differential equation $\frac{5+e^x}{2+y}\cdot\frac{dy}{dx}+e^x=0$ satisfying $y(0) = 1$, then a value of $y(\log_e 13)$ is
(a) 1 (b) -1 (c) 0 (d) 2

2. The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$ is
(a) $\frac{5}{9}$ (b) $\frac{25}{81}$ (c) $\frac{5}{27}$ (d) $\frac{25}{9}$

3. The negation of Boolean expression $x \leftrightarrow \sim y$ is equivalent to
(a) $(x \wedge y) \vee (\sim x \wedge \sim y)$ (b) $(x \wedge y) \wedge (\sim x \vee \sim y)$
(c) $(x \wedge \sim y) \vee (\sim x \wedge y)$ (d) $(\sim x \wedge y) \vee (\sim x \wedge \sim y)$

4. The mean and variance of 7 observations are 8 and 16, respectively. If five observations are 2, 4, 10, 12, 14, then the absolute difference of the remaining two observations is
(a) 1 (b) 4 (c) 2 (d) 3

5. If $2^{10} + 2^9\cdot3^1 + 2^8\cdot3^2 + ... + 2\cdot3^9 + 3^{10} = S - 2^{11}$, then S is equal to
(a) $3^{11} - 2^{12}$ (b) 3^{11}
(c) $\frac{3^{11}}{2}+2^{10}$ (d) $2\cdot3^{11}$

6. If $3^{2\sin2\alpha - 1}$, 14 and $3^{4 - 2\sin2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P. is
(a) 66 (b) 81 (c) 65 (d) 78

7. If the volume of a parallelopiped, whose coterminus edges are given by the vectors $\vec{a}=\hat{i}+\hat{j}+n\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-n\hat{k}$ and $\vec{c}=\hat{i}+n\hat{j}+3\hat{k}$ $(n \geq 0)$, is 158 cu. units, then
(a) $\vec{a}\cdot\vec{c}=17$ (b) $\vec{b}\cdot\vec{c}=10$
(c) $n=7$ (d) $n=9$

8. If S is the sum of the first 10 terms of the series
$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{21}\right)+...,$$
then $\tan(S)$ is equal to
(a) $\frac{5}{6}$ (b) $\frac{5}{11}$ (c) $-\frac{6}{5}$ (d) $\frac{10}{11}$

9. If the four complex numbers $z, \bar{z}, \bar{z}-2\mathrm{Re}(\bar{z})$ and $z - 2\mathrm{Re}(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to
(a) $4\sqrt{2}$ (b) 4 (c) $2\sqrt{2}$ (d) 2

10. A survey shows that 73% of the persons working in an office like coffee, whereas 65% like tea. If x denotes the percentage of them, who like both coffee and tea, then x cannot be
(a) 63 (b) 36 (c) 54 (d) 38

11. If the co-ordinates of two points A and B are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ respectively and P is any point on the conic, $9x^2 + 16y^2 = 144$, then $PA + PB$ is equal to
(a) 16 (b) 8 (c) 6 (d) 9

12. If the point P on the curve, $4x^2 + 5y^2 = 20$ is farthest from the point Q (0, -4), then PQ^2 is equal to
(a) 36 (b) 48 (c) 21 (d) 29

13. Let $\lambda \in R$. The system of linear equations
$2x_1 - 4x_2 + \lambda x_3 = 1$
$x_1 - 6x_2 + x_3 = 2$
$\lambda x_1 - 10x_2 + 4x_3 = 3$
is inconsistent for
(a) exactly one negative value of λ
(b) exactly one positive value of λ
(c) every value of λ
(d) exactly two values of λ

14. If the minimum and the maximum values of the function $f:\left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to R$, defined by

$$f(\theta)=\begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 1 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$$

are m and M respectively, then the ordered pair (m, M) is equal to

(a) $(0, 2\sqrt{2})$ (b) $(-4, 0)$
(c) $(-4, 4)$ (d) $(0, 4)$

15. If (a, b, c) is the image of the point $(1, 2, -3)$ in the line, $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$, then $a + b + c$ is equal to

(a) 2 (b) -1 (c) 3 (d) 1

16. If the function $f(x)=\begin{cases} k_1(x-\pi)^2-1, & x \le \pi \\ k_2 \cos x, & x > \pi \end{cases}$ is twice differentiable, then the ordered pair (k_1, k_2) is equal to

(a) $\left(\frac{1}{2}, 1\right)$ (b) $(1, 0)$
(c) $\left(\frac{1}{2}, -1\right)$ (d) $(1, 1)$

17. If the common tangent to the parabolas, $y^2 = 4x$ and $x^2 = 4y$ also touches the circle, $x^2 + y^2 = c^2$, then c is equal to

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

18. If α is the positive root of the equation $p(x) = x^2 - x - 2 = 0$, then $\lim_{x\to\alpha^+}\frac{\sqrt{1-\cos(p(x))}}{x+\alpha-4}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

19. If $\int(e^{2x}+2e^x-e^{-x}-1)\, e^{(e^x+e^{-x})}\, dx$ $= g(x)\, e^{(e^x+e^{-x})}+c$, where c is a constant of integration, then $g(0)$ is equal to

(a) e (b) e^2 (c) 1 (d) 2

20. The value of $\int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}dx$ is

(a) $\pi/4$ (b) π (c) $\pi/2$ (d) $3\pi/2$

21. Let $f(x)=x\cdot\left[\frac{x}{2}\right]$, for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ______.

22. If the line, $2x - y + 3 = 0$ is at a distance $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ from the lines $4x - 2y + \alpha = 0$ and $6x - 3y + \beta = 0$, respectively, then the sum of all possible values of α and β is ________.

23. The number of words, with or without meaning, that can be formed by taking 4 letters at a time from the letters of the word 'SYLLABUS' such that two letters are distinct and two letters are alike, is ________.

24. The natural number m, for which the coefficient of x in the binomial expansion of $\left(x^m+\frac{1}{x^2}\right)^{22}$ is 1540, is ________.

25. Four fair dice are thrown independently 27 times. Then the expected number of times, at least two dice show up a three or a five, is ______.

SOLUTIONS

1. **(b)** : The given differential equation can be written as

$$\left(\frac{5+e^x}{2+y}\right)\frac{dy}{dx}=-e^x$$

$$\Rightarrow \int\frac{dy}{2+y}=\int\frac{-e^x}{e^x+5}dx$$

$\Rightarrow \log(2 + y) = -\log(e^x + 5) + \log C$
$\Rightarrow (2 + y)(e^x + 5) = C$
As, $y(0) = 1$ $\therefore$ $(2 + 1)(1 + 5) = C$ $\Rightarrow$ $C = 18$

$$\therefore\ y+2=\frac{18}{e^x+5}$$

At $x = \log_e 13$, $y+2=\frac{18}{13+5}=1 \Rightarrow y = -1$

2. **(b)** : We have, $9x^2 - 18|x| + 5 = 0$
$\Rightarrow 9|x|^2 - 15|x| - 3|x| + 5 = 0$ $[\because x^2 = |x|^2]$
$\Rightarrow 3|x|\,(3|x| - 5) - 1(3|x| - 5) = 0$
$\Rightarrow (3|x| - 1)(3|x| - 5) = 0$
$\Rightarrow |x|=\frac{1}{3}, \frac{5}{3} \Rightarrow x=\pm\frac{1}{3}, \pm\frac{5}{3}$

Product of roots $=\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{5}{3}\right)=\frac{25}{81}$

3. (a) : Since $x \leftrightarrow y \equiv (x \to y) \wedge (y \to x)$

$\therefore\ x \leftrightarrow \sim y \equiv (x \to \sim y) \wedge (\sim y \to x)$

$\equiv (\sim x \vee \sim y) \wedge (y \vee x)\,[\because x \to y \equiv \sim x \vee y]$

Hence, $\sim (x \leftrightarrow \sim y) \equiv (x \wedge y) \vee (\sim x \wedge \sim y)$

4. (c) : Let the remaining two observations be x and y.

Given, Mean $(\bar{x}) = 8$

$$\Rightarrow \frac{2+4+10+12+14+x+y}{7} = 8$$

$\Rightarrow 42 + x + y = 56 \Rightarrow x + y = 14$ ...(i)

Also, variance $(\sigma^2) = \dfrac{\sum x_i^2}{n} - \left(\dfrac{\sum x_i}{n}\right)^2$

$$\Rightarrow 16 = \frac{4+16+100+144+196+x^2+y^2}{7} - \left(\frac{56}{7}\right)^2$$

$$\Rightarrow 16 = \frac{460+x^2+y^2}{7} - 8^2 \Rightarrow 16+64 = \frac{460+x^2+y^2}{7}$$

$\Rightarrow 560 = 460 + x^2 + y^2$

$\Rightarrow x^2 + y^2 = 100$ ... (ii)

Squaring (i), we get

$x^2 + y^2 + 2xy = 196 \Rightarrow 100 + 2xy = 196$ [Using (ii)]

$\Rightarrow xy = 48$

Now, $(x - y)^2 = (x + y)^2 - 4xy = 196 - 4 \times 48 = 4$

$\therefore\ (x - y)^2 = 4 \Rightarrow x - y = \pm 2$

$\therefore\ |x - y| = 2$

5. (b) : Let, $S' = 2^{10} + 2^9 \cdot 3 + 2^8 \cdot 3^2 + \ldots + 2 \cdot 3^9 + 3^{10}$, which is a G.P.

Here, $a = 2^{10}$, $r = 3/2$, $n = 11$

$$\therefore\ S' = 2^{10}\frac{\left(\left(\frac{3}{2}\right)^{11} - 1\right)}{\frac{3}{2} - 1} \qquad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}\right]$$

$$= 2^{11}\left(\frac{3^{11}}{2^{11}} - 1\right) = 3^{11} - 2^{11}$$

Since, $S' = S - 2^{11} \Rightarrow 3^{11} - 2^{11} = S - 2^{11}$

$\Rightarrow S = 3^{11}$

6. (a) : Since, the given terms are first three terms of an A.P.

$\therefore\ 2 \times 14 = 3^{2\sin 2\alpha - 1} + 3^{4 - 2\sin 2\alpha}$

Putting, $3^{2\sin 2\alpha} = x$, we get

$$28 = \frac{x}{3} + \frac{81}{x} \Rightarrow 28 = \frac{x^2 + 243}{3x}$$

$\Rightarrow x^2 - 84x + 243 = 0 \Rightarrow x^2 - 81x - 3x + 243 = 0$

$\Rightarrow x(x - 81) - 3(x - 81) = 0 \Rightarrow (x - 3)(x - 81) = 0$

$\Rightarrow x = 3, 81$

$\therefore\ 3^{2\sin 2\alpha} = 3$ or 3^4

$\Rightarrow 2\sin 2\alpha = 1$ or 4

$\Rightarrow \sin 2\alpha = \dfrac{1}{2}$ or 2 (rejected)

First term, $a = 3^{2\sin 2\alpha - 1} = 3^{2\cdot\frac{1}{2} - 1} = 1$

Second term $= 14$

$\therefore$ Common difference, $d = 13$

Now, 6^{th} term, $t_6 = a + 5d = 1 + 5(13) = 66$

7. (b) : Volume of a parallelopiped, $V = |[abc]|$

$$\therefore\ \begin{vmatrix} 1 & 1 & n \\ 2 & 4 & -n \\ 1 & n & 3 \end{vmatrix} = \pm 158$$

$\Rightarrow 1(12 + n^2) - 1(6 + n) + n(2n - 4) = \pm 158$

$\Rightarrow 3n^2 - 5n + 6 = \pm 158$

$\Rightarrow 3n^2 - 5n - 152 = 0$ or $3n^2 - 5n + 164 = 0$

$\Rightarrow 3n^2 + 19n - 24n - 152 = 0$

or imaginary roots, so rejected.

$\Rightarrow n(3n + 19) - 8(3n + 19) = 0$

$\Rightarrow (3n + 19)(n - 8) = 0$

$\Rightarrow n = 8\ [\because n \geq 0]$

Now, $\vec{a} \cdot \vec{c} = 1 + n + 3n = 1 + 4n = 33$

$\vec{b} \cdot \vec{c} = 2 + 4n - 3n = 2 + n = 10$

8. (a) : $S = \tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}\left(\dfrac{1}{13}\right) + \ldots$

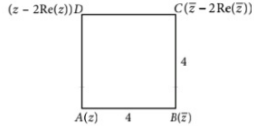
$$= \tan^{-1}\left(\frac{2-1}{1+1\cdot 2}\right) + \tan^{-1}\left(\frac{3-2}{1+2\cdot 3}\right) + \tan^{-1}\left(\frac{4-3}{1+3\cdot 4}\right) + \ldots + \tan^{-1}\left(\frac{11-10}{1+10\cdot 11}\right)$$

$= (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) + \ldots + (\tan^{-1}11 - \tan^{-1}10)$

$= \tan^{-1}11 - \tan^{-1}1 = \tan^{-1}\left(\dfrac{11-1}{1+11}\right) = \tan^{-1}\left(\dfrac{10}{12}\right)$

$\Rightarrow \tan S = \dfrac{10}{12} = \dfrac{5}{6}$

9. (c) : Let $z = x + iy$

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Length of side = 4

$AB = 4 \Rightarrow |z - \bar{z}| = 4 \Rightarrow |2iy| = 4 \Rightarrow |y| = 2$

Also, $BC = 4 \Rightarrow (\bar{z} - (\bar{z} - 2\mathrm{Re}(\bar{z}))) = 4$

$\Rightarrow |2x| = 4 \Rightarrow |x| = 2$

$\therefore |z| = \sqrt{x^2 + y^2} = \sqrt{8} = 2\sqrt{2}$

10. (b) : Let C and T denotes the set of persons like coffee and tea respectively.

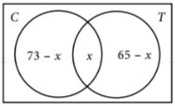
$n(C) = 73$, $n(T) = 65$, $n(C \cap T) = x$

Now, $n(C \cup T) \le 100$

$\Rightarrow n(C) + n(T) - n(C \cap T) \le 100$

$\Rightarrow 73 + 65 - x \le 100$

$\Rightarrow x \ge 38$



Also, $73 - x \ge 0 \Rightarrow x \le 73$

and $65 - x \ge 0 \Rightarrow x \le 65$

$\therefore 38 \le x \le 65.$

Hence, $x = 36$ is not possible.

11. (b) : We have, $\frac{x^2}{16} + \frac{y^2}{9} = 1$, *i.e.*, $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$

which is an ellipse where $a = 4$, $b = 3$.

$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

So, A and B are foci.

$\Rightarrow PA + PB = 2a = 2 \times 4 = 8$

12. (a) : We have, $\frac{x^2}{5} + \frac{y^2}{4} = 1$

Let P is $(\sqrt{5}\cos\theta, 2\sin\theta)$.

Now, $(PQ)^2 = (\sqrt{5}\cos\theta)^2 + (2\sin\theta + 4)^2$

$= 5\cos^2\theta + (4\sin^2\theta + 16 + 16\sin\theta)$

$= \cos^2\theta + 4(\cos^2\theta + \sin^2\theta) + 16 + 16\sin\theta$

$= \cos^2\theta + 16\sin\theta + 20$

$= -\sin^2\theta + 16\sin\theta + 21 = 85 - (\sin\theta - 8)^2$

It will be maximum, when $\sin\theta = 1$

$\therefore (PQ)^2_{max} = 85 - 49 = 36$

13. (a) : $D = \begin{vmatrix} 2 & -4 & \lambda \\ 1 & -6 & 1 \\ \lambda & -10 & 4 \end{vmatrix}$

$= 2(-24 + 10) + 4(4 - \lambda) + \lambda(-10 + 6\lambda)$

$= -28 + 16 - 4\lambda - 10\lambda + 6\lambda^2 = 6\lambda^2 - 14\lambda - 12$

$= 2(3\lambda^2 - 7\lambda - 6) = 2(\lambda - 3)(3\lambda + 2)$

$D_1 = \begin{vmatrix} 1 & -4 & \lambda \\ 2 & -6 & 1 \\ 3 & -10 & 4 \end{vmatrix} = 1(-24 + 10) + 4(8 - 3) + \lambda(-20 + 18)$

$= -14 + 20 - 2\lambda = -2\lambda + 6 = -2(\lambda - 3)$

$D_2 = \begin{vmatrix} 2 & 1 & \lambda \\ 1 & 2 & 1 \\ \lambda & 3 & 4 \end{vmatrix} = 2(8 - 3) - 1(4 - \lambda) + \lambda(3 - 2\lambda)$

$= 10 - 4 + \lambda + 3\lambda - 2\lambda^2 = -2(\lambda^2 - 2\lambda - 3)$

$= -2(\lambda + 1)(\lambda - 3)$

$D_3 = \begin{vmatrix} 2 & -4 & 1 \\ 1 & -6 & 2 \\ \lambda & -10 & 3 \end{vmatrix} = 2(-18 + 20) + 4(3 - 2\lambda) + 1(-10 + 6\lambda)$

$= 4 + 12 - 8\lambda - 10 + 6\lambda = 6 - 2\lambda = -2(\lambda - 3)$

When $\lambda = 3$, $D = D_1 = D_2 = D_3 = 0$

$\Rightarrow$ Infinitely many solutions.

When $\lambda = -2/3$, $D = 0$ but none of D_1, D_2, D_3 is zero.

So, for $\lambda = -\frac{2}{3}$ we have inconsistent system.

14. (b) : We have, $f(\theta) = \begin{vmatrix} -\sin^2\theta & -1 - \sin^2\theta & 1 \\ -\cos^2\theta & -1 - \cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$

$= \begin{vmatrix} -\sin^2\theta & -1 - \sin^2\theta & 0 \\ -\cos^2\theta & -1 - \cos^2\theta & 0 \\ 12 & 10 & -4 \end{vmatrix}$

[Applying $C_3 \to C_3 - (C_1 - C_2)$]

$= -4[(1 + \cos^2\theta)\sin^2\theta - \cos^2\theta(1 + \sin^2\theta)]$

$= -4[\sin^2\theta + \sin^2\theta\cos^2\theta - \cos^2\theta - \cos^2\theta\sin^2\theta]$

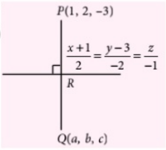
$= 4\cos 2\theta$

Now, $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \Rightarrow 2\theta \in \left[\frac{\pi}{2}, \pi\right]$ $\therefore f(\theta) \in [-4, 0]$

$\therefore (m, M) = (-4, 0)$

15. (a) : Given equation of line is

$\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1} = \lambda$ (say)



Let point R is

$(2\lambda - 1, -2\lambda + 3, -\lambda)$

Direction ratios of PR

$\langle 2\lambda - 1 - 1, -2\lambda + 3 - 2, -\lambda + 3 \rangle$

i.e., $\langle 2\lambda - 2, -2\lambda + 1, -\lambda + 3 \rangle$

Since, PR is perpendicular to line.

$\therefore 2(2\lambda - 2) + (-2)(-2\lambda + 1) + (-1)(-\lambda + 3) = 0$

$\Rightarrow 4\lambda - 4 + 4\lambda - 2 + \lambda - 3 = 0$

$\Rightarrow 9\lambda - 9 = 0 \Rightarrow \lambda = 1$

$\therefore$ Point R is $(1, 1, -1)$

$\because$ R is the mid-point of PQ.

So, $\frac{a+1}{2}=1 \Rightarrow a=1;\ \frac{b+2}{2}=1 \Rightarrow b=0;$

$\frac{c-3}{2}=-1 \Rightarrow c=1$

$\therefore$ $a + b + c = 2$

16. (a) : Since $f(x)$ is differentiable so it will be continuous also.

$\therefore$ $f(\pi^-) = f(\pi) = f(\pi^+) \Rightarrow -1 = -k_2 \Rightarrow k_2 = 1$

Now, $f'(x)=\begin{cases} 2k_1(x-\pi)\ ;\ x\le\pi \\ -k_2\sin x\ ;\ x>\pi \end{cases}$

$\therefore$ $f'(\pi^-) = f'(\pi^+) = 0$

Now, $f''(x)=\begin{cases} 2k_1\ ;\ x\le\pi \\ -k_2\cos x\ ;\ x>\pi \end{cases}$

$\therefore$ $f''(\pi^-) = f''(\pi^+)$

$\Rightarrow$ $2k_1 = k_2 \Rightarrow k_1=\frac{1}{2}$

$\therefore$ $(k_1, k_2)=\left(\frac{1}{2}, 1\right)$

17. (b) : Tangent to $y^2 = 4x$ is $y=mx+\frac{1}{m}$

and tangent to $x^2 = 4y$ is $y = mx - m^2$

$\therefore$ $\frac{1}{m}=-m^2 \Rightarrow m^3=-1 \Rightarrow m=-1$

$\therefore$ Common tangent is $y = -x - 1 \Rightarrow x + y + 1 = 0$

Since, it touches the circle $x^2 + y^2 = c^2$

$\therefore$ $c=\left|\frac{0+0+1}{\sqrt{1+1}}\right|=\frac{1}{\sqrt{2}}$

18. (b) : We have, $p(x) = 0$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = -1, 2 \quad \therefore \alpha = 2$

Now, $\lim\limits_{x\to\alpha^+}\frac{\sqrt{1-\cos(p(x))}}{x+\alpha-4}$

$$= \lim_{x\to 2^+}\frac{\sqrt{1-\cos(x^2-x-2)}}{x-2} = \lim_{x\to 2^+}\frac{\sqrt{2\sin^2\left(\frac{x^2-x-2}{2}\right)}}{x-2}$$

$$= \lim_{x\to 2^+}\frac{\sqrt{2}\sin\left(\frac{x^2-x-2}{2}\right)}{x-2}$$

$$= \lim_{x\to 2^+}\frac{\sqrt{2}\sin\left(\frac{x^2-x-2}{2}\right)}{\left(\frac{x^2-x-2}{2}\right)}\cdot\frac{\left(\frac{x^2-x-2}{2}\right)}{(x-2)}$$

$$= \lim_{x\to 2^+}\frac{1}{\sqrt{2}}\frac{(x-2)(x+1)}{(x-2)} = \lim_{x\to 2^+}\frac{1}{\sqrt{2}}(x+1)=\frac{3}{\sqrt{2}}$$

19. (d) : Let $I=\int(e^{2x}+2e^x-e^{-x}-1)e^{e^x+e^{-x}}dx$

$$=\int(e^{2x}+e^x-1)e^{e^x+e^{-x}}dx+\int(e^x-e^{-x})e^{e^x+e^{-x}}dx$$

$$=\int(e^x+1-e^{-x})e^{e^x+e^{-x}+x}dx+\int(e^x-e^{-x})e^{e^x+e^{-x}}dx$$

$= I_1 + I_2$

Let $I_1=\int(e^x+1-e^{-x})e^{e^x+e^{-x}+x}dx$

Put $e^x + e^{-x} + x = t$

$\Rightarrow$ $(e^x - e^{-x} + 1)dx = dt$

$\therefore$ $I_1=\int e^t dt=e^t+c_1=e^{e^x+e^{-x}+x}+c_1$

Let $I_2=\int(e^x-e^{-x})e^{e^x+e^{-x}}dx$

Put $e^x + e^{-x} = u \Rightarrow (e^x - e^{-x})dx = du$

$\therefore$ $I_2=\int e^u du=e^u+c_2=e^{e^x+e^{-x}}+c_2$

So, $I=e^{e^x+e^{-x}+x}+e^{e^x+e^{-x}}+c$, where $c=c_1+c_2$

$=e^{e^x+e^{-x}}(e^x+1)+c=g(x)e^{e^x+e^{-x}}+c$

$\therefore$ $g(x) = e^x + 1$

Thus, $g(0) = e^0 + 1 = 1 + 1 = 2$

20. (c) : Let, $I=\int\limits_{-\pi/2}^{\pi/2}\frac{1}{1+e^{\sin x}}dx$ ...(i)

$$\Rightarrow I=\int_{-\pi/2}^{\pi/2}\frac{1}{1+e^{-\sin x}}dx \quad \left[\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$$

$$\Rightarrow I=\int_{-\pi/2}^{\pi/2}\frac{e^{\sin x}}{1+e^{\sin x}}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_{-\pi/2}^{\pi/2}\frac{1+e^{\sin x}}{1+e^{\sin x}}dx=\int_{-\pi/2}^{\pi/2}1dx=\pi \quad \therefore I=\frac{\pi}{2}$$

21. (8) : We have, $x \in (-10, 10)$

$\Rightarrow \frac{x}{2} \in (-5, 5)$

$\Rightarrow \left[\frac{x}{2}\right]$ can be −4, −3, −2, −1, 0, 1, 2, 3, 4

Also, we have $f(x) = x \cdot \left[\frac{x}{2}\right]$

Now, at $x = 0$, $f(0) = f(0^+) = f(0^-)$
$\Rightarrow f(x)$ is continuous at $x = 0$.
Hence, $f(x)$ will be discontinuous at integral point *i.e.*, ±4, ±3, ±2, ±1 *i.e.*, at 8 points $f(x)$ will be discontinuous.

22. (30) : Given lines are

$$2x - y + 3 = 0 \quad \text{... (i)}$$

$$4x - 2y + \alpha = 0 \Rightarrow 2x - y + \frac{\alpha}{2} = 0 \quad \text{... (ii)}$$

and $6x - 3y + \beta = 0 \Rightarrow 2x - y + \frac{\beta}{3} = 0$ ... (iii)

Since these lines are parallel.
$\therefore$ Distance between (i) and (ii) is given by

$$\frac{\left|\frac{\alpha}{2} - 3\right|}{\sqrt{2^2 + 1^2}} = \frac{1}{\sqrt{5}} \Rightarrow |\alpha - 6| = 2$$

$\Rightarrow \alpha - 6 = \pm 2 \Rightarrow \alpha = 8, 4$
Also, distance between (i) and (iii) is given by

$$\frac{\left|\frac{\beta}{3} - 3\right|}{\sqrt{2^2 + 1^2}} = \frac{2}{\sqrt{5}} \Rightarrow |\beta - 9| = 6$$

$\Rightarrow \beta - 9 = \pm 6$
$\Rightarrow \beta = 15, 3$
Sum of all the values of α and $\beta = 8 + 4 + 15 + 3 = 30$

23. (240) : The word SYLLABUS consists of
2 S, 2 L, 1 Y, 1 A, 1 B, 1 U

Required number of ways $= {}^2C_1 \times {}^5C_2 \times \frac{4!}{2!}$

$$= 2 \times 10 \times 12 = 240$$

24. (13) : We have, $\left(x^m + \frac{1}{x^2}\right)^{22}$

$$T_{r+1} = {}^{22}C_r (x^m)^{22-r} \left(\frac{1}{x^2}\right)^r = {}^{22}C_r\, x^{22m - mr - 2r}$$

Since, ${}^{22}C_3 = {}^{22}C_{19} = 1540$
$\therefore r = 3$ or 19
Now, for coeff. of x, $22m - mr - 2r = 1$

$$\Rightarrow m = \frac{2r+1}{22-r}$$

At $r = 3$, $m = \frac{7}{19} \notin N$

At $r = 19$, $m = \frac{39}{3} = 13 \in N$

$\therefore m = 13$

25. (11) : Probability of showing 3 or 5, $p = \frac{2}{6} = \frac{1}{3}$

$\therefore$ Probability of not showing 3 or 5, $q = 1 - \frac{1}{3} = \frac{2}{3}$

Experiment is performed with 4 dice independently.
$\therefore P$(At least 2 dice shows up 3 or 5)

$$= {}^4C_4\, p^4 + {}^4C_3\, p^3 q + {}^4C_2\, p^2 q^2$$

$$= \left(\frac{1}{3}\right)^4 + 4\left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right) + 6\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$

$$= \frac{1}{81} + \frac{8}{81} + \frac{24}{81} = \frac{33}{81}$$

$\because$ Experiment performed 27 times.

$\therefore E(X) = XP(X) = \frac{33}{81} \times 27 = 11$

CONCEPT BOOSTERS

Sets and Relations

This column is aimed for students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

SETS

A set is well defined class or collection of objects.

- **Roster method or Listing method :** In this method, a set is described by listing elements, separated by commas, within braces {}. **For ex :** The set of vowels of English alphabet may be described as $\{a, e, i, o, u\}$.
- **Set-builder method or Rule method :** In this method, a set is described by a characterizing property $P(x)$ of its elements x. In such a case, the set is described by $\{x : P(x) \text{ holds}\}$ or $\{x \mid P(x) \text{ holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'.
  For ex : The set $A = \{0, 1, 4, 9, 16, ...\}$ can be written as $A = \{x^2 : x \in Z\}$.

TYPES OF SETS

- **Singleton set :** A set consisting of a single element is called a singleton set. The set {5} is a singleton set.
- **Null set or Empty set :** The set which contains no element at all is called the null set. This set is also called the 'empty set' or the 'void set'. It is denoted by the symbol ϕ or {}.
- **Finite set :** A set is called a finite set if it is either void set or its elements can be listed (counted) by a certain natural number.
  Cardinal number of a finite set : The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$ or $O(A)$.
- **Infinite set :** A set whose elements cannot be listed (counted) by a certain natural number (n) is called an infinite set.
- **Equivalent sets :** Two finite sets A and B are equivalent, if their cardinal numbers are same *i.e.* $n(A) = n(B)$.
  If $A = \{1, 3, 5, 7\}$; $B = \{10, 12, 14, 16\}$ then A and B are equivalent sets, as $O(A) = O(B) = 4$.
- **Equal sets :** Two sets A and B are said to be equal iff every element of A is an element of B and also every element of B is an element of A. Symbolically, $A = B$ if $x \in A \Leftrightarrow x \in B$.
  If $A = \{2, 3, 5, 6\}$ and $B = \{6, 5, 3, 2\}$, then $A = B$ because each element of A is an element of B and vice-versa.
  Note : Equal sets are always equivalent but equivalent sets need not to be equal sets.

SUBSETS (SET INCLUSION)

Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.
If A is subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".
Thus, $A \subseteq B$ *i.e.*, $a \in A \Rightarrow a \in B$.
Note : The total number of subsets of a finite set containing n elements is 2^n.

- **Proper and improper subsets :** If A is a subset of B and $A \neq B$, then A is a proper subset of B. We write this as $A \subset B$.
  The null set ϕ is subset of every set and every set is subset of itself, *i.e.*, $\phi \subset A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. It should be noted that ϕ has only one subset ϕ, which is improper.

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All other subsets of A are called its proper subsets.
Let $A = \{1, 2\}$. Then A has ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$ as its subsets out of which ϕ and $\{1, 2\}$ are improper and $\{1\}$ and $\{2\}$ are proper subsets.

UNIVERSAL SET

A set that contains all sets in a given context is called the universal set.
It should be noted that universal set is not unique.

POWER SET

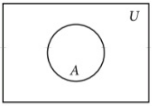
If S is any set, then the set of all the subsets of S is called the power set of S.
The power set of S is denoted by $P(S)$. Symbolically, $P(S) = \{T : T \subseteq S\}$. Obviously ϕ and S are both elements of $P(S)$.
Let $S = \{a, b, c\}$, then $P(S) = \{\phi\ , \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
Note: Power set of a given set is always non-empty.

DISJOINT SETS

Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be non-intersecting or non-overlapping sets.

VENN DIAGRAMS

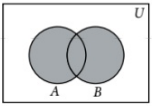
The combination of rectangles and circles are called Venn diagrams.
The universal set is usually represented by a rectangle and its subsets by circles.



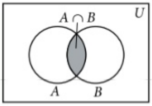
Note : If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. Two disjoints sets are represented by two non-intersecting circles.

OPERATIONS ON SETS

- **Union of sets :** Let A and B be two sets. The union of A and B is the set of all elements which are either in set A or in set B. We denote the union of A and B by $A \cup B$ (read as "A union B"). Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Shaded portion in the given figure represents $A \cup B$.

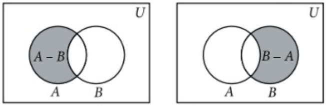


- **Intersection of sets :** Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.



The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B").
Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

- **Difference of sets :** Let A and B be two sets. The difference of A and B written as $A - B$, is the set of all those elements of A which do not belong to B.

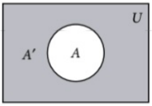


Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$
Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A
$\therefore \quad B - A = \{x : x \in B \text{ and } x \notin A\}$.
Note : For three sets A, B and C,
$(A - B) - C \neq A - (B - C)$.

- **Symmetric difference of two sets :** Let A and B be two sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$.

COMPLEMENT OF A SET

Let U be the universal set and let A be a set such that $A \subset U$. Then, the complement of A with respect to U is denoted by A' or A^c or $U - A$ and is defined the set of all those elements of U which are not in A. Thus, $A' = \{x \in U : x \notin A\}$.



LAWS OF ALGEBRA OF SETS

- **Idempotent laws :** For any set A, we have
(i) $A \cup A = A$
(ii) $A \cap A = A$
- **Identity laws :** For any set A, we have
(i) $A \cup \phi = A$
(ii) $A \cap U = A$
i.e., ϕ and U are identity elements for union and intersection respectively.
- **Commutative laws :** For any two sets A and B, we have
(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
(iii) $A \Delta B = B \Delta A$
- **Associative laws :** If A, B and C are any three sets, then
(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

i.e., union, intersection and symmetric difference of two sets are associative.

- **Distributive laws :** If A, B and C are any three sets, then

  (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  i.e., union and intersection are distributive over intersection and union respectively.

- **De-Morgan's laws :** If A, B and C are any three sets, then

  (i) $(A \cup B)' = A' \cap B'$

  (ii) $(A \cap B)' = A' \cup B'$

  (iii) $A - (B \cap C) = (A - B) \cup (A - C)$

  (iv) $A - (B \cup C) = (A - B) \cap (A - C)$

- If A and B are any two sets, then

  (i) $A - B = A \cap B'$

  (ii) $B - A = B \cap A'$

  (iii) $A - B = A \Leftrightarrow A \cap B = \phi$

  (iv) $(A - B) \cup B = A \cup B$

  (v) $(A - B) \cap B = \phi$

  (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

  (vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

- If A, B and C are any three sets, then

  (i) $A \cap (B - C) = (A \cap B) - (A \cap C)$

  (ii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

SOME IMPORTANT RESULTS

If A, B and C are finite sets and U be the finite universal set, then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets.
- $n(A - B) = n(A) - n(A \cap B)$

  i.e., $n(A - B) + n(A \cap B) = n(A)$
- $n(A \Delta B)$ = Number of elements which belong to exactly one of A or $B = n((A - B) \cup (B - A))$

  $= n(A - B) + n(B - A)$

  [$\because$ $(A - B)$ and $(B - A)$ are disjoint sets]

  $= n(A) - n(A \cap B) + n(B) - n(A \cap B)$

  $= n(A) + n(B) - 2n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- Number of elements in exactly two of the sets A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
- Number of elements in exactly one of the sets $A, B, C = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$
- $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.

Thus, $A \times B = [(a, b) : a \in A \text{ and } b \in B]$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

Let $A = \{a, b, c\}$ and $B = \{p, q\}$.

Then, $A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$

Also, $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$

Note :

- Cartesian product of two sets is not commutative (in general).
- Let A and B two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

IMPORTANT THEOREMS ON CARTESIAN PRODUCT OF SETS

For any three sets A, B and C, we have

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- $(A \times B) \times C \neq A \times (B \times C)$
- $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
- $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

If A and B are any two non-empty sets, then $A \times B = B \times A \Leftrightarrow A = B$

- If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
- If $A \subseteq B$, then $A \times C \subseteq B \times C$, for any set C.
- If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$
- For any sets A, B, C and D, we have

  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

RELATIONS

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B *i.e.*, every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as $a\ R\ b$.

Total number of relations : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations, the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$

Range $(R) = \{b : (a, b) \in R\}$.

Note :

- The whole set B is called co-domain of R.
- Range $\subseteq$ Co-domain

PROBLEMS

Single Correct Answer Type

1. Which of the following is the empty set ?
(a) $\{x : x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x : x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x : x$ is a real number and $x^2 - 9 = 0\}$
(d) $\{x : x$ is a real number and $x^2 = x + 2\}$

2. If a set A has n elements, then the total number of subsets of A is
(a) n (b) n^2 (c) 2^n (d) $2n$

3. In a town of 10,000 families, it was found that 40% families buy newspaper X, 20% buy newspaper Y and 10% families buy newspaper Z, 5% families buy both X and Y, 3% buy both Y and Z and 4% buy both X and Z. If 2% families buy all the three newspapers, then number of families which buy newspaper X only is
(a) 3100 (b) 3300 (c) 2900 (d) 1400

4. In a city, 20 percent of the population travels by car, 50 percent travels by bus and 10 percent travels by both car and bus. Then persons travelling by car or bus is
(a) 80 percent (b) 40 percent
(c) 60 percent (d) 70 percent

5. In a class of 55 students, the number of students studying different subjects are 23 in Mathematics, 24 in Physics, 19 in Chemistry, 12 in Mathematics and Physics, 9 in Mathematics and Chemistry, 7 in Physics and Chemistry and 4 in all the three subjects. The number of students who have taken exactly one subject is
(a) 18 (b) 20
(c) 12 (d) none of these

6. If A, B and C are any three sets, then $A \times (B \cup C)$ is equal to
(a) $(A \times B) \cup (A \times C)$ (b) $(A \cup B) \times (A \cup C)$
(c) $(A \times B) \cap (A \times C)$ (d) none of these

7. If $A = \{2, 4, 5\}$, $B = \{7, 8, 9\}$, then $n(A \times B)$ is equal to
(a) 6 (b) 9 (c) 3 (d) 0

8. The smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$ is
(a) $\{2, 3, 5\}$ (b) $\{3, 5, 9\}$
(c) $\{1, 2, 5, 9\}$ (d) none of these

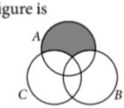
9. If A and B are two sets, then $A \cup B = A \cap B$ iff
(a) $A \subseteq B$ (b) $B \subseteq A$
(c) $A = B$ (d) none of these

10. If $A = \{2, 3, 4, 8, 10\}$, $B = \{3, 4, 5, 10, 12\}$ and $C = \{4, 5, 6, 12, 14\}$, then $(A \cap B) \cup (A \cap C)$ is equal to
(a) $\{3, 4, 10\}$ (b) $\{2, 8, 10\}$
(c) $\{4, 5, 6\}$ (d) $\{3, 5, 14\}$

11. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to
(a) A (b) B
(c) ϕ (d) none of these

12. If $N_a = \{an : n \in N\}$, then $N_5 \cap N_7 =$
(a) N_7 (b) N_{10} (c) N_{35} (d) N_5

13. The shaded region in the given figure is
(a) $A \cap (B \cup C)$
(b) $A \cup (B \cap C)$
(c) $A \cap (B - C)$
(d) $A - (B \cup C)$



14. Let U be the universal set and $A \cup B \cup C = U$. Then $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to
(a) $A \cup B \cup C$ (b) $A \cup (B \cap C)$
(c) $A \cap B \cap C$ (d) $A \cap (B \cup C)$

15. In a battle, 70% of the combatants lost one eye, 80% an ear, 75% an arm, 85% a leg, x% lost all the four limbs. The minimum value of x is
(a) 10 (b) 12
(c) 15 (d) none of these

16. Out of 800 boys in a school, 224 played Cricket, 240 played Hockey and 336 played Basketball. Of the total, 64 played both Basketball and Hockey; 80 played Cricket and Basketball and 40 played Cricket and Hockey; 24 played all the three games. The number of boys who did not play any game is
(a) 128 (b) 216
(c) 240 (d) 160

17. In a class of 100 students, 55 students have passed in Mathematics and 67 students have passed in Physics. Then the number of students who have passed in Physics only is
(a) 22 (b) 33
(c) 10 (d) 45

18. If A, B and C are any three sets, then $A - (B \cap C)$ is equal to
(a) $(A - B) \cup (A - C)$ (b) $(A - B) \cap (A - C)$
(c) $(A - B) \cup C$ (d) $(A - B) \cap C$

19. If A, B, C are three sets, then $A \cap (B \cup C)$ is equal to
(a) $(A \cup B) \cap (A \cup C)$ (b) $(A \cap B) \cup (A \cap C)$
(c) $(A \cup B) \cup (A \cup C)$ (d) none of these

20. In a class of 30 pupils, 12 take English, 16 take Physics and 18 take History. If all the 30 pupils take at least one subject and no one takes all three then the number of pupils taking exactly 2 subjects is
(a) 16 (b) 6 (c) 8 (d) 20

21. A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics–100, Physics–70, Chemistry–40; Mathematics and Physics–30, Mathematics and Chemistry–28, Physics and Chemistry–23, Mathematics, Physics and Chemistry–18. How many students have offered Mathematics alone?
(a) 35 (b) 48 (c) 60 (d) 22

22. Given $n(U) = 20$, $n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 4$, where U is the universal set, A and B are subsets of U, then $n((A \cup B)^c) =$
(a) 17 (b) 9 (c) 11 (d) 3

23. The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs b by $3\}$, is given by
(a) $\{(1, 4), (2, 5), (3, 6), ...\}$
(b) $\{(4, 1), (5, 2), (6, 3), ...\}$
(c) Both (a) and (b)
(d) none of these

24. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is
(a) 2^{mn} (b) $2^{mn} - 1$
(c) $2mn$ (d) m^n

25. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
(a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(c) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
(d) none of these

26. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is
(a) 2^9 (b) 9^2 (c) 3^2 (d) 2^{9-1}

27. The power set of $A = \{\phi, \{\phi\}\}$
(a) $\{\phi, \{\phi\}, \{\{\phi\}\}\}$
(b) $\{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
(c) $\{\{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$
(d) none of these

28. $(A \cup B) \cap (A \cup B^C)$ equals
(a) A (b) B
(c) $A \cap B'$ (d) $A \cup B'$

29. In a survey of 100 persons, it was found that 28 read magazine X, 30 read magazine Y, 42 read magazine Z, 8 read magazines X and Y, 10 read magazines X and Z, 5 read magazines Y and Z and 3 read all the three magazines. Then, number of persons who read none of the three magazines and magazine Z only respectively are
(a) 20, 30 (b) 30, 20 (c) 25, 35 (d) 25, 40

30. Which of the following statements is true?
(a) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
(b) $A \subseteq B \subseteq C \Rightarrow A \subseteq C$
(c) $A \subseteq \phi \Rightarrow A = \phi$ (d) all of these

SOLUTIONS

1. (b) : Since, $x^2 + 1 = 0$, gives $x^2 = -1 \Rightarrow x = \pm i$
$\therefore$ x is not real but x is real (given)
$\therefore$ No value of x is possible.

2. (c) : Number of subsets of A having n elements $= 2^n$

3. (b) : Let sets A, B and C represent families who buy newspaper X, Y and Z respectively.
$n(A) = 40\%$ of $10000 = 4000$
$n(B) = 20\%$ of $10000 = 2000$
$n(C) = 10\%$ of $10000 = 1000$
$n(A \cap B) = 5\%$ of $10000 = 500$
$n(B \cap C) = 3\%$ of $10000 = 300$
$n(C \cap A) = 4\%$ of $10000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10000 = 200$
We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$
$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300$

Monthly Test Drive-5 CLASS XI ANSWER KEY

1. (d)	2. (a)	3. (c)	4. (b)	5. (a)
6. (b)	7. (c, d)	8. (c)	9. (a, b, c)	10. (c, d)
11. (b)	12. (a, b, c)	13. (b, c, d)	14. (b)	15. (b)
16. (c)	17. (6)	18. (9)	19. (3)	20. (6)

4. (c) : Let B and C represents the set of population who travel by bus and car respectively.
Given, $n(C) = 20$, $n(B) = 50$, $n(C \cap B) = 10$
Now, $n(C \cup B) = n(C) + n(B) - n(C \cap B)$
$= 20 + 50 - 10 = 60$
Hence, required number of persons = 60%

5. (d) : Let M, P and C represent the students studying Mathematics, Physics and Chemistry respectively.
$n(M) = 23$, $n(P) = 24$, $n(C) = 19$,
$n(M \cap P) = 12$, $n(M \cap C) = 9$, $n(P \cap C) = 7$,
$n(M \cap P \cap C) = 4$
We have to find $n(M \cap P' \cap C')$, $n(P \cap M' \cap C')$, $n(C \cap M' \cap P')$
Now, $n(M \cap P' \cap C') = n[M \cap (P \cup C)']$
$= n(M) - n(M \cap (P \cup C))$
$= n(M) - n[(M \cap P) \cup (M \cap C)]$
$= n(M) - n(M \cap P) - n(M \cap C) + n(M \cap P \cap C)$
$= 23 - 12 - 9 + 4 = 27 - 21 = 6$
$n(P \cap M' \cap C') = n[P \cap (M \cup C)']$
$= n(P) - n[P \cap (M \cup C)]$
$= n(P) - n[(P \cap M) \cup (P \cap C)]$
$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$
$= 24 - 12 - 7 + 4 = 9$
Similarly, $n(C \cap M' \cap P') = n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M)$
$= 19 - 7 - 9 + 4 = 23 - 16 = 7$
Thus, the number of students who have taken exactly one subject $= 6 + 9 + 7 = 22$

6. (a)

7. (b) : $A \times B = \{(2, 7), (2, 8), (2, 9), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9)\}$
$\therefore \quad n(A \times B) = n(A) \cdot n(B) = 3 \times 3 = 9$

8. (b) : Given, $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$
Hence, $A = \{3, 5, 9\}$.

9. (c) : Let $x \in A \Rightarrow x \in A \cup B$ $[\because A \subseteq A \cup B]$
$\Rightarrow \quad x \in A \cap B$ $[\because A \cup B = A \cap B]$
$\Rightarrow \quad x \in A$ and $x \in B \Rightarrow x \in B \Rightarrow A \subseteq B$
Similarly, $x \in B \Rightarrow x \in A \Rightarrow B \subseteq A$
Now, $A \subseteq B, B \subseteq A \Rightarrow A = B$

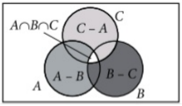
10. (a) : $A \cap B = \{2, 3, 4, 8, 10\} \cap \{3, 4, 5, 10, 12\}$
$= \{3, 4, 10\}$
Also, $A \cap C = \{4\}$
$\therefore \quad (A \cap B) \cup (A \cap C) = \{3, 4, 10\}$

11. (c) : $A \cap (A \cup B)' = A \cap (A' \cap B')$
(De-Morgan's law)
$= (A \cap A') \cap B'$ (By associative law)
$= \phi \cap B' = \phi$

12. (c) : $N_5 \cap N_7 = N_{35}$
$[\because$ 5 and 7 are relatively prime numbers$]$

13. (d)

14. (c) : From Venn diagram,
it is clear that,
$\{(A - B) \cup (B - C) \cup (C - A)\}'$
$= A \cap B \cap C$



15. (a) : Minimum value of
$x = 100 - (30 + 20 + 25 + 15) = 100 - 90 = 10$

16. (d) : Let C, H and B represent the set of boys who played Cricket, Hockey and Basketball respectively.
$\therefore \quad n(C) = 224$, $n(H) = 240$, $n(B) = 336$,
$n(H \cap B) = 64$, $n(B \cap C) = 80$,
$n(H \cap C) = 40$, $n(C \cap H \cap B) = 24$
Number of boys who did not play any game
$= n(C^c \cap H^c \cap B^c) = n[(C \cup H \cup B)^c]$
$= n(U) - n(C \cup H \cup B)$
$= 800 - [n(C) + n(H) + n(B) - n(H \cap C) - n(H \cap B) - n(C \cap B) + n(C \cap H \cap B)]$
$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24] = 160.$

17. (d) : Let M and P represent the set of students who passed in Mathematics and Physics respectively.
$\therefore \quad n(M) = 55$, $n(P) = 67$, $n(M \cup P) = 100$
Now, $n(M \cup P) = n(M) + n(P) - n(M \cap P)$
$\Rightarrow \quad 100 = 55 + 67 - n(M \cap P)$
$\Rightarrow \quad n(M \cap P) = 122 - 100 = 22$
Now, $n(P \text{ only}) = n(P) - n(M \cap P) = 67 - 22 = 45$

18. (a) : From De-morgan's law, we have
$A - (B \cap C) = (A - B) \cup (A - C)$

19. (b) : From distributive law, we have
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

20. (a) : Let E, P and H represent the set of pupils who take English, Physics and History respectively.
$\therefore \quad n(E) = 12$, $n(P) = 16$, $n(H) = 18$
and $n(E \cup P \cup H) = 30$
$\because n(E \cup P \cup H) = n(E) + n(P) + n(H) - n(E \cap P) - n(P \cap H) - n(E \cap H) + n(E \cap P \cap H)$

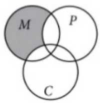
$\therefore \quad n(E \cap P) + n(P \cap H) + n(E \cap H) = 16$

Now, number of pupils taking exactly two subjects

$= n(E \cap P) + n(P \cap H) + n(E \cap H) - 3n(E \cap P \cap H)$

$= 16 - 0 = 16$

21. (c) : $n(M \text{ alone})$



$= n(M) - n(M \cap C) - n(M \cap P) + n(M \cap P \cap C)$

$= 100 - 28 - 30 + 18$

$= 60$

22. (d) : $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 12 + 9 - 4 = 17$

Now, $n((A \cup B)^c) = n(U) - n(A \cup B) = 20 - 17 = 3$

23. (c) : $R = \{(a, b) : a, b \in N, |a - b| = 3\}$

$= \{(4, 1), (5, 2), (6, 3), ...\}$

or $\{(1, 4), (2, 5), (3, 6) ...\}$.

24. (a) : Number of relations from set A to set B having m and n elements respectively is 2^{mn}.

25. (d) : Here $R = \{(x, y) : |x^2 - y^2| < 16\}$

and given $A = \{1, 2, 3, 4, 5\}$

$\therefore \quad R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

26. (a) : Here, $A = \{2, 4, 6\}$; $B = \{2, 3, 5\}$

$\therefore \quad A \times B$ contains $3 \times 3 = 9$ elements.

Hence, number of relations from A to $B = 2^9$

27. (b) : $P(A) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

28. (a)

29. (a) : Let A, B and C represent the set of persons who read the magazine X, Y and Z respectively.

$\therefore \quad n(A) = 28, n(B) = 30, n(C) = 42, n(A \cap B) = 8, n(A \cap C) = 10, n(B \cap C) = 5, n(A \cap B \cap C) = 3$

Since, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$= 100 - 23 + 3 = 80$

$\therefore$ Number of persons who read none of the three magazines $= 100 - 80 = 20$

Also, $n(\text{read magazine } Z \text{ only})$

$= n(C) - (n(A \cap C) + n(B \cap C) - n(A \cap B \cap C))$

$= 42 - 10 - 5 + 3 = 30$

30. (d)

Master the Concepts

OF TRIGONOMETRY

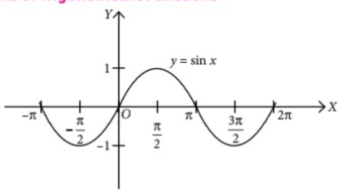
Class XI

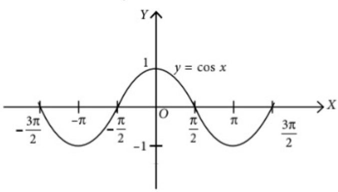
TRIGONOMETRIC FUNCTIONS (CIRCULAR FUNCTIONS)

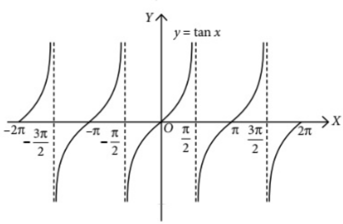
Function	Domain	Range
sin A	R	$[-1, 1]$
cos A	R	$[-1, 1]$
tan A	$R-\left\{\frac{(2n+1)}{2}\pi, n \in I\right\}$	$R = (-\infty, \infty)$
cosec A	$R - \{n\pi, n \in \mathrm{I}\}$	$(-\infty, -1] \cup [1, \infty)$
sec A	$R-\left\{\frac{(2n+1)\pi}{2}, n \in I\right\}$	$(-\infty, -1] \cup [1, \infty)$
cot A	$R - \{n\pi, n \in \mathrm{I}\}$	$(-\infty, \infty)$

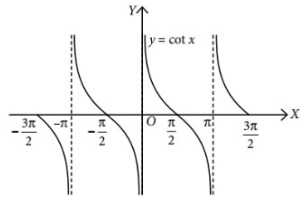
Note : $|\sin A| \le 1$, $|\cos A| \le 1$, $\sec A \ge 1$, or $\sec A \le -1$ and $\operatorname{cosec} A \ge 1$, or $\operatorname{cosec} A \le -1$

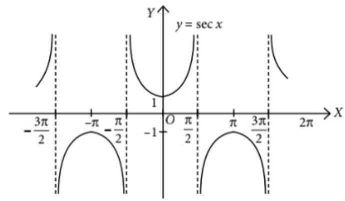
Graphs of Trigonometric Functions

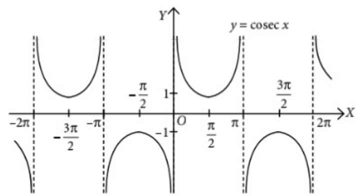






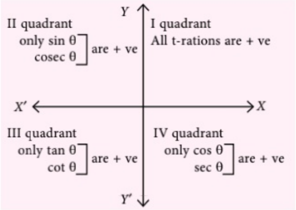






Sign of Trigonometric Functions in Different Quadrants

A system of rectangular coordinate axes divide a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The signs of the trigonometric ratios in the four quadrants are shown below



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Sine, Cosine and Tangent of some angles ≤ 90°

Trigonometric Ratios/ Functions	0°	15°	18°	30°	36°
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$
	37°	45°	53°	60°	90°
sin	≈3/5	$\frac{1}{\sqrt{2}}$	≈4/5	$\frac{\sqrt{3}}{2}$	1
cos	≈4/5	$\frac{1}{\sqrt{2}}$	≈3/5	$\frac{1}{2}$	0
tan	≈3/4	1	≈4/3	$\sqrt{3}$	Not defined

Some Basic Formulae

(a) $\sin^2 A + \cos^2 A = 1$

(b) $\sec^2 A - \tan^2 A = 1$

(c) $\operatorname{cosec}^2 A - \cot^2 A = 1$

(d) $\sin A \operatorname{cosec} A = \tan A \cot A = \cos A \sec A = 1$

Trigonometric Functions of Compound, Multiple and Sub-Multiple Angles

- Trigonometric functions of the algebraic sum/ difference of two angles can be expressed as trigonometric functions of separate angles.

1. $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$
2. $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$
3. $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A . \tan B}$
4. $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

- Trigonometric functions of multiples of an angles can be expressed as trigonometric functions of the angle.

1. $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$
2. $\cos 2A = \cos^2 A - \sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
$$= 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
4. $\sin 3A = 3\sin A - 4\sin^3 A$
5. $\cos 3A = 4\cos^3 A - 3\cos A$
6. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

- Trigonometric functions of half of an angle can be expressed as trigonometric functions of the complete angle.

1. $\sin\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{2}}$ 2. $\cos\dfrac{A}{2} = \sqrt{\dfrac{1+\cos A}{2}}$
3. $\tan\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A} = \dfrac{\sin A}{1+\cos A}$

Transformation Formulae Sum to Product

1. $\sin C + \sin D = 2\sin\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$
2. $\sin C - \sin D = 2\cos\dfrac{C+D}{2}.\sin\dfrac{C-D}{2}$
3. $\cos C + \cos D = 2\cos\dfrac{C+D}{2}.\cos\dfrac{C-D}{2}$
4. $\cos C - \cos D = 2\sin\dfrac{C+D}{2}.\sin\dfrac{D-C}{2}$

Note : (i) $\sin C + \cos D = \sin C + \sin\left(\dfrac{\pi}{2} - D\right)$
$$= 2\sin\frac{C+\frac{\pi}{2}-D}{2}.\cos\frac{C-\frac{\pi}{2}+D}{2}$$

(ii) $\tan C + \tan D = \dfrac{\sin C}{\cos C} + \dfrac{\sin D}{\cos D} = \dfrac{\sin(C+D)}{\cos C \cdot \cos D}$

Product to Sum

1. $\sin A \cdot \cos B = \dfrac{1}{2}\{\sin(A+B) + \sin(A-B)\}$
2. $\sin A \cdot \sin B = \dfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$
3. $\cos A \cdot \cos B = \dfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}$
4. $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$
5. $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$

Some Important Conversions

- $\sin^2 A = \dfrac{1}{2}(1-\cos 2A)$
- $\cos^2 A = \dfrac{1}{2}(1+\cos 2A)$
- $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$
- $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$
- $\cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

- $\cos A \cdot \cos 2A \cdot \cos 2^2 A \cdot \cos 2^3 A ......\cos 2^{n-1} A$

$$= \begin{cases} \dfrac{\sin 2^n A}{2^n \sin A} & \text{, if } A \neq n\pi \\ 1 & \text{, if } A = 2n\pi \\ -1 & \text{, if } A = (2n+1)\pi \end{cases}$$

- $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + ...$

$$+ \cos(\alpha + (n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left\{\frac{2\alpha + (n-1)\beta}{2}\right\}$$

- $\sin(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 ...\cos A_n (S_1 - S_3 + S_5 - S_7 + ...)$
- $\cos(A_1 + A_2 + ... + A_n) = \cos A_1 \cos A_2 ...\cos A_n (1 - S_2 + S_4 - S_6 ...)$
- $\tan(A_1 + A_2 + ... + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}$,

where,

$S_1 = \tan A_1 + \tan A_2 + ... + \tan A_n$ = The sum of the tangents of the separate angles.

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + ...$ = The sum of the tangents taken two at a time.

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + ...$ = The sum of tangents taken three at a time, and so on.

Note : If $A_1 = A_2 = ... = A_n = A$, then $S_1 = n \tan A$, $S_2 = {}^nC_2 \tan^2 A$. $S_3 = {}^nC_3 \tan^3 A$, ...

Conditional Identity

If the angles, A, B and C satisfy a given relation, we can establish many interesting identities connecting the trigonometric functions of these angles.

For example, if $A + B + C = \pi$, then

1. $\sin(B + C) = \sin A$, $\cos B = -\cos(C + A)$
2. $\cos(A + B) = -\cos C$, $\sin C = \sin(A + B)$
3. $\tan(C + A) = -\tan B$, $\cot A = -\cot(B + C)$
4. $\cos\dfrac{A+B}{2} = \sin\dfrac{C}{2}, \cos\dfrac{C}{2} = \sin\dfrac{A+B}{2}$
5. $\sin\dfrac{C+A}{2} = \cos\dfrac{B}{2}, \sin\dfrac{A}{2} = \cos\dfrac{B+C}{2}$
6. $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}, \tan\dfrac{B}{2} = \cot\dfrac{C+A}{2}$

These identities, can be proved by using the properties of complementary and supplementary angles.

Some important Identities

If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$
3. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$
4. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$
5. $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$
7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$
8. $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$
9. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

TRIGONOMETRIC EQUATION

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

Solution of Trigonometric Equation

A solution of a trigonometric equation is the value of the unknown angle that satisfies the equation.

E.g., If $\sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}, ...$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature). These solutions can be classified as :

(i) Principal solution (ii) General solution

(i) Principal Solution

The solutions of a trigonometric equation which lie in the interval $(-\pi, \pi]$ are called principal solutions.

(ii) General Solution

The expression which gives all solutions of a trigonometric equation is called its general solution.

Some trigonometric equations with their general solutions

Trigonometric equation	General solution
$\sin\theta = 0$	$\theta = n\pi$
$\cos\theta = 0$	$\theta = (n\pi + \pi/2) = (2n + 1)\pi/2$
$\tan\theta = 0$	$\theta = n\pi$
$\sin\theta = 1$	$\theta = 2n\pi + \pi/2 = (4n + 1)\pi/2$
$\cos\theta = 1$	$\theta = 2n\pi$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n \alpha$, where $\alpha \in [-\pi/2, \pi/2]$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha$, where $\alpha \in [-\pi/2, \pi/2]$
$\sin^2\theta = \sin^2\alpha$	$\theta = n\pi \pm \alpha$

$\cos^2\theta = \cos^2\alpha$	$\theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha$	$\theta = n\pi \pm \alpha$

where $n \in Z$ (the set of integers).

PROBLEMS

Single Option Correct Type

1. If $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, $\tan\theta_4$ are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x\cos\beta - \sin\beta = 0$ then the value of $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) =$

(a) $\sin\beta$ (b) $\cos\beta$
(c) $\tan\beta$ (d) $\cot\beta$

2. The set of values of x satisfying the equation

$$2^{\tan\left(x-\frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0 \text{ is}$$

(a) An empty set
(b) A singleton set
(c) A set containing two elements
(d) An infinite set

3. The value of $\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right)$ is equal to

(a) $\frac{n-1}{2}$ (b) $\frac{n}{2}-2$ (c) $\frac{n}{2}-1$ (d) $\frac{1}{2}-n$

4. The value of $\cot 7\frac{1^\circ}{2} + \tan 67\frac{1^\circ}{2} - \cot 67\frac{1^\circ}{2} - \tan 7\frac{1^\circ}{2}$

(a) A rational number (b) Irrational number
(c) $2(3+2\sqrt{3})$ (d) $2(3-\sqrt{3})$

5. If $\tan\alpha = \frac{x^2-x}{x^2-x+1}$ and $\tan\beta = \frac{1}{2x^2-2x+1}$ $(x \neq 0, 1)$, where $0<\alpha,\beta<\frac{\pi}{2}$, then $\tan(\alpha+\beta)$ has the value equal to

(a) 1 (b) –1 (c) 2 (d) $\frac{3}{4}$

6. If $\tan\frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\alpha}{2}$ then $\cos\alpha =$

(a) $\frac{1-e\cos\theta}{\cos\theta+e}$ (b) $\frac{1+e\cos\theta}{\cos\theta+e}$

(c) $\frac{\cos\theta-e}{1-e\cos\theta}$ (d) $\frac{1-e\cos\theta}{\cos\theta-e}$

7. If ϕ is eliminated from the equations $x = a\cos(\phi - \alpha)$ and $y = b\cos(\phi - \beta)$, then $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha+\beta)$ is equal to

(a) $\cos^2(\alpha - \beta)$ (b) $\sin^2(\alpha - \beta)$
(c) $\sec^2(\alpha - \beta)$ (d) $\text{cosec}^2(\alpha - \beta)$

8. The general solution of the trigonometric equation $\tan x + \tan 2x + \tan 3x = \tan x \cdot \tan 2x \cdot \tan 3x$ is

(a) $x = n\pi$ (b) $n\pi \pm \frac{\pi}{3}$

(c) $x = 2n\pi$ (d) $x = \frac{n\pi}{3}$, where $n \in I$

9. If $\alpha, \beta, \gamma, \delta$ are first 4 positive solutions of $\sin x = \frac{1}{4}$ with $\alpha < \beta < \gamma < \delta$, then value of $\sin\frac{\delta}{2} + 2\sin\frac{\gamma}{2} + 3\sin\frac{\beta}{2}$ $+ 4\sin\frac{\alpha}{2}$ is

(a) $\sqrt{5}$ (b) $2\sqrt{5}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

10. The value of x that satisifies the relation $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots \infty$, is

(a) $2\cos 36°$ (b) $2\cos 144°$
(c) $2\sin 18°$ (d) None of these

More Than One Option(s) Correct Type

11. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then

(a) $u_{\min}^2 = (a+b)^2$ (b) $u_{\max}^2 = 2(a^2+b^2)$

(c) $u_{\max}^2 - u_{\min}^2 = (a-b)^2$ (d) $u_{\max} - u_{\min} = 0$

12. It is given that $\sin\beta = \frac{4}{5}$ & $0 < \beta < \pi$, then the value of $\frac{\sqrt{3}\sin(\alpha+\beta) - \frac{2}{\cos\frac{\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$ is

(a) Independent of α for all β in $(0, \pi/2)$

(b) $\frac{5}{\sqrt{3}}$ for $\tan\beta > 0$

(c) $\frac{\sqrt{3}(7+24\cot\alpha)}{15}$ for $\tan\beta < 0$

(d) None of these

13. If $\sin t + \cot t = \frac{1}{5}$ then $\tan\frac{t}{2}$ is equal to

(a) –1 (b) –1/3
(c) 2 (d) –1/6

14. The values of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ are}$$

(a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$

15. For $0<\theta<\frac{\pi}{2}$, the solution(s) of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2} \text{ is/are}$$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

Numerical Value Type

16. Let $S=\left\{x\in(-\pi,\pi): x\neq 0, \pm\frac{\pi}{2}\right\}$

The sum of all distinct solutions of the equation $\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$ in the set S is equal to ______.

17. The positive integer value of $n > 3$ satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)}=\frac{1}{\sin\left(\frac{2\pi}{n}\right)}+\frac{1}{\sin\left(\frac{3\pi}{n}\right)} \text{ is ______.}$$

18. In triangle ABC, $3\sin A + 4\cos B = 6$ and $4\sin B + 3\cos A = 1$, then the measure of angle C (in degree) is ______.

19. If $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan k°$. then, k is equal to ______.

20. If a unique ordered pair (x, y) of real numbers with $0<x<\frac{\pi}{2}$ satisfy the relation $\frac{(\sin x)^{2y}}{(\cos x)^{\frac{y^2}{2}}}+\frac{(\cos x)^{2y}}{(\sin x)^{\frac{y^2}{2}}}$ $= \sin 2x$ the value of y is ______.

21. If the sum of all x in the interval $[0, 2\pi]$ which satisfy the equation $3\cot^2 x + 8\cot x + 3 = 0$ is equal to $k\pi$, then the value of k is ______.

22. Let $\{a_n\}$ be the sequence of the real numbers defined by $a_1 = t$ and $a_{n+1} = 4a_n(1 - a_n)$ for $n \geq 1$. If number of distinct values of t for which $a_{1998} = 0$ is equal to $2^k + 1$, then the value of k is ______.

SOLUTIONS

1. **(d)** : From the given equation, we have

$S_1 = \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4 = \sin 2\beta$

$S_2 = \sum \tan\theta_1\tan\theta_2 = \cos 2\beta$

$S_3 = \sum \tan\theta_1\tan\theta_2\tan\theta_3 = \cos\beta$

$S_4 = \sum \tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 = -\sin\beta$

Now, $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{S_1-S_3}{1-S_2+S_4}$

$$=\frac{\sin 2\beta-\cos\beta}{1-\cos 2\beta-\sin\beta}=\frac{\cos\beta(2\sin\beta-1)}{\sin\beta(2\sin\beta-1)}=\cot\beta$$

2. **(a)** : We have, $2^{\tan\left(x-\frac{\pi}{4}\right)}-2(0.25)^{\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}}+1=0$

Consider, $\frac{\sin^2\left(x-\frac{\pi}{4}\right)}{\cos 2x}=\frac{1}{2}\frac{\left(1-\cos 2\left(x-\frac{\pi}{4}\right)\right)}{\cos 2x}$

$$=\frac{1}{2}\frac{(1-\sin 2x)}{\cos 2x}=\frac{1}{2}\tan\left(\frac{\pi}{4}-x\right)$$

Now, take $\tan\left(x-\frac{\pi}{4}\right)=t$, then given expression would be $2^t-2\left(\frac{1}{4}\right)^{t/2}+1=0$

$\Rightarrow 2^t-2\left(\frac{1}{2}\right)^t+1=0$

$(2^t)^2 + (2)^t - 2 = 0$

$(2^t + 2)(2^t - 1) = 0$

$\Rightarrow 2^t = -2$ (Not possible) or $2^t = 1$

$\Rightarrow t = 0$

$\therefore \tan\left(x-\frac{\pi}{4}\right)=0 \Rightarrow x=n\pi+\frac{\pi}{4}, n\in Z$

But in equation $\frac{1}{\cos 2x}$ does not exist for $x = n\pi + \frac{\pi}{4}$, therefore no value of x exist.

3. **(c)** : Given, sum $=\frac{1}{2}\sum_{r=1}^{n-1}\left(1+\cos\frac{2r\pi}{n}\right)$

$$=\frac{1}{2}(n-1)+\frac{1}{2}\left\{\cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+......+\cos\frac{(2n-2)\pi}{n}\right\}$$

$$=\frac{1}{2}(n-1)+\frac{1}{2}\left\{\frac{\sin(n-1)\frac{2\pi}{2n}}{\sin\frac{2\pi}{n\cdot 2}}.\cos\left\{\frac{2\left(\frac{2\pi}{n}\right)+(n-2)\frac{2\pi}{n}}{2}\right\}\right\}$$

$$=\frac{1}{2}(n-1)+\frac{1}{2}\left\{\frac{\sin\left(\pi-\frac{\pi}{n}\right).\cos\pi}{\sin\left(\frac{\pi}{n}\right)}\right\}$$

$$=\frac{1}{2}(n-1)+\frac{1}{2}\left\{\frac{\left(\sin\frac{\pi}{n}\right)(-1)}{\sin\left(\frac{\pi}{n}\right)}\right\}=\frac{1}{2}(n-1)-\frac{1}{2}=\frac{n}{2}-1$$

4. (b) : Let $\cot 7\frac{1}{2}^\circ+\tan 67\frac{1}{2}^\circ-\cot 67\frac{1}{2}^\circ-\tan 7\frac{1}{2}^\circ$

$=\cot\frac{A}{2}+\tan\frac{B}{2}-\cot\frac{B}{2}-\tan\frac{A}{2}$, where $A=15^\circ$ and $B = 135^\circ$.

$$=\frac{1-\tan^2\frac{A}{2}}{\tan\frac{A}{2}}+\frac{\tan^2\frac{B}{2}-1}{\tan\frac{B}{2}}$$

$= 2\cot A - 2\cot B = 2(\cot 15^\circ - \cot 135^\circ)$

$=2(2+\sqrt{3}+1)=2(3+\sqrt{3})$, which is an irrational number.

5. (a) : We have, $\tan\alpha=\frac{x^2-x}{x^2-x+1}$ and $\tan\beta=\frac{1}{2x^2-2x+1}$

$$\Rightarrow 2(x^2-x)+1=\frac{1}{\tan\beta} \Rightarrow x^2-x=\left(\frac{1-\tan\beta}{\tan\beta}\right)\frac{1}{2}$$

$$\therefore \tan\alpha=\frac{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)}{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)+1}=\frac{\frac{1}{2}\left(\frac{1-\tan\beta}{\tan\beta}\right)}{\frac{1}{2}\left(\frac{1+\tan\beta}{\tan\beta}\right)}$$

$\Rightarrow \tan\alpha=\frac{1-\tan\beta}{1+\tan\beta} \Rightarrow \tan\alpha + \tan\alpha\tan\beta = 1 - \tan\beta$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1 \Rightarrow \tan(\alpha+\beta)=1$$

6. (c) : We have, $\tan^2\frac{\theta}{2}=\frac{1-e}{1+e}\tan^2\frac{\alpha}{2}$

$$\Rightarrow \frac{1}{\tan^2\frac{\theta}{2}}=\frac{1+e}{1-e}\frac{\cos^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}}$$

Applying componendo and dividendo, we get

$$\cos\theta=\frac{(1+e)\cos^2\frac{\alpha}{2}-(1-e)\sin^2\frac{\alpha}{2}}{(1+e)\cos^2\frac{\alpha}{2}+(1-e)\sin^2\frac{\alpha}{2}}$$

$$\Rightarrow \cos\theta=\frac{\cos\alpha+e}{1+e\cos\alpha}$$

$\Rightarrow \cos\theta + e\cos\theta\cos\alpha = \cos\alpha + e$

$\Rightarrow \cos\theta - e = (1 - e\cos\theta)\cos\alpha$

$$\Rightarrow \cos\alpha=\frac{\cos\theta-e}{1-e\cos\theta}$$

7. (b) : Given, $x = a\cos(\phi - \alpha)$ and $y = b\cos(\phi - \beta)$

$$\Rightarrow \cos^{-1}\left(\frac{x}{a}\right)=\phi-\alpha \text{ and } \cos^{-1}\left(\frac{y}{b}\right)=\phi-\beta$$

On subtracting (i) from (ii), we get

$$\alpha-\beta=\cos^{-1}\left(\frac{x}{a}\right)-\cos^{-1}\left(\frac{y}{b}\right)$$

$$=\cos^{-1}\left(\frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)$$

$$\Rightarrow \cos(\alpha-\beta)=\frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}}.\sqrt{1-\frac{y^2}{b^2}}$$

$$\Rightarrow \cos(\alpha-\beta)=\frac{xy}{ab}+\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}}$$

$$\Rightarrow \left(\cos(\alpha-\beta)-\frac{xy}{ab}\right)^2=\left(\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}}\right)^2$$

$$\Rightarrow \cos^2(\alpha-\beta)+\frac{x^2y^2}{a^2b^2}-2\cos(\alpha-\beta)\times\frac{xy}{ab}$$

$$=1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)=1-\cos^2(\alpha-\beta)$$

$= \sin^2(\alpha - \beta)$

8. (d) : Given,

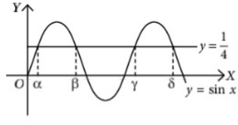
$\tan x + \tan 2x + \tan 3x = \tan x\cdot\tan 2x\cdot\tan 3x$

$$\Rightarrow \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}=-\tan 3x$$

$\Rightarrow \tan 3x = -\tan 3x \Rightarrow \tan 3x = 0$

$\Rightarrow 3x = n\pi \Rightarrow x=\frac{n\pi}{3}, n\in I$

9. (a) : Clearly, $\beta = \pi - \alpha$, $\gamma = 2\pi + \alpha$, $\delta = 3\pi - \alpha$



Now, given expression is equal to

$$\sin\left(\frac{3\pi}{2}-\frac{\alpha}{2}\right)+2\sin\left(\pi+\frac{\alpha}{2}\right)+3\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)+4\sin\frac{\alpha}{2}$$

$$=-\cos\frac{\alpha}{2}-2\sin\frac{\alpha}{2}+3\cos\frac{\alpha}{2}+4\sin\frac{\alpha}{2}$$

$$=2\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)=2\sqrt{1+\sin\alpha}=\sqrt{5}$$

10. (c) : Given, $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots \infty$

$\Rightarrow \quad x = 1 - x(1 - x + x^2 - x^3 + x^4 \ldots\ldots\infty)$

$\Rightarrow \quad x = 1 - x \cdot x \quad \Rightarrow \quad x^2 + x - 1 = 0$

$\Rightarrow \quad x=\dfrac{-1\pm\sqrt{5}}{2}$

But for infinite G.P., $|x| < 1$, therefore required value of $x=\dfrac{-1+\sqrt{5}}{2}=2\sin 18°$

11. (a, b, c) : Clearly $u^2 = a^2 + b^2$

$$+2\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)(a^2\sin^2\theta+b^2\cos^2\theta)}$$

$\Rightarrow \quad u^2 = a^2 + b^2$

$$+2\sqrt{a^2b^2(\cos^4\theta+\sin^4\theta)+(a^4+b^4)(\sin^2\theta\cos^2\theta)}$$

$\Rightarrow \quad u^2=a^2+b^2+\sqrt{4a^2b^2+(a^2-b^2)^2\sin^2 2\theta}$

$\Rightarrow \quad u_{\max}^2=2(a^2+b^2),\ u_{\min}^2=(a+b)^2$

$\Rightarrow \quad u_{\max}^2-u_{\min}^2=(a-b)^2$

12. (a,b,c) : Consider, $\dfrac{\sqrt{3}\sin(\alpha+\beta)-\dfrac{2}{\cos\frac{\pi}{6}}\cos(\alpha+\beta)}{\sin\alpha}$

$$=\left(\frac{\sqrt{3}\times\frac{\sqrt{3}}{2}\sin(\alpha+\beta)-2\cos(\alpha+\beta)}{\sin\alpha}\right)\frac{2}{\sqrt{3}}$$

$$=\frac{2}{\sqrt{3}}\left[\frac{\frac{3}{2}(\sin\alpha\cos\beta+\cos\alpha\sin\beta)-2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}{\sin\alpha}\right]$$

$$=\frac{2}{\sqrt{3}}\left[\frac{3}{2}\cos\beta+\frac{3}{2}\cot\alpha\sin\beta-2\cot\alpha\cos\beta+2\sin\beta\right]$$

Since, $\sin\beta = \dfrac{4}{5}$, therefore $\cos\beta=\pm\dfrac{3}{5}$.

If $\cos\beta=\dfrac{3}{5}$, *i.e.*, $\tan\beta>0$, *i.e.*, $\beta\in\left(0,\dfrac{\pi}{2}\right)$, then

R.H.S.

$$=\frac{2}{\sqrt{3}}\left[\frac{3}{2}\times\frac{3}{5}+\frac{3}{2}\cot\alpha\times\frac{4}{5}-2\cot\alpha\times\frac{3}{5}+2\times\frac{4}{5}\right]$$

$$=\frac{2}{\sqrt{3}}\left[\frac{9}{10}+\frac{8}{5}\right]=\frac{5}{\sqrt{3}}$$

If $\cos\beta=-\dfrac{3}{5}$, *i.e.*, $\tan\beta<0$, *i.e.*, $\beta\in\left(\dfrac{\pi}{2},\pi\right)$, then

R.H.S.

$$=\frac{2}{\sqrt{3}}\left[\frac{3}{2}\left(-\frac{3}{5}\right)+\frac{3}{2}\cot\alpha\times\frac{4}{5}+2\cot\alpha\times\frac{3}{5}+2\times\frac{4}{5}\right]$$

$$=\frac{2}{\sqrt{3}}\left[\frac{12}{5}\cot\alpha+\frac{7}{10}\right]=\left[\frac{24\cot\alpha+7}{15}\right]\sqrt{3}$$

13. (b, c) : Given, $\sin t+\cot t=\dfrac{1}{5}$

$$\Rightarrow \quad \frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}}+\frac{1-\tan^2\frac{t}{2}}{1+\tan^2\frac{t}{2}}=\frac{1}{5}$$

Let $\tan\dfrac{t}{2}=k$, then we get

$5(2k + 1 - k^2) = 1 + k^2$

$\Rightarrow \quad 6k^2 - 10k - 4 = 0 \Rightarrow 3k^2 - 5k - 2 = 0$

$\Rightarrow \quad (3k + 1)(k - 2) = 0$

$\therefore \quad k=-\dfrac{1}{3},2 \quad\Rightarrow\quad \tan\dfrac{t}{2}=\dfrac{-1}{3},2$

14. (a, c) : Given,

$$\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin4\theta\end{vmatrix}=0$$

Applying $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get

$$\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin4\theta\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}=0$$

Applying $C_1 \to C_1 + C_2$, we get

$$\begin{vmatrix}2 & \cos^2\theta & 4\sin4\theta\\ 0 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}=0$$

$\Rightarrow \quad 2 + 4\sin 4\theta = 0$

$\Rightarrow \quad \sin4\theta=\dfrac{-1}{2}=\sin\left(-\dfrac{\pi}{6}\right)$

$$\Rightarrow\ 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), n \in Z.$$

$$\Rightarrow\ \theta = \frac{n\pi}{4} + (-1)^{n+1}\left(\frac{\pi}{24}\right)$$

Clearly, $\theta = \frac{7\pi}{24}, \frac{11\pi}{24}$ are two values of θ lying between 0 and $\frac{\pi}{2}$.

15. (c,d): $\sum_{m=1}^{6} \text{cosec}\left[\theta + \frac{m\pi}{4} - \frac{\pi}{4}\right]\text{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$

$$\Rightarrow\ \sum_{m=1}^{6}\sqrt{2}\times\frac{\sin\left[\left(\theta+\frac{m\pi}{4}\right)-\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)\right]}{\sin\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow\ \sum_{m=1}^{6}\sqrt{2}\times\frac{\sin\left(\theta+\frac{m\pi}{4}\right)\cos\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right) - \cos\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)}{\sin\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow\ \sum_{m-1}^{6}\sqrt{2}\times\left[\cot\left(\theta+\frac{m\pi}{4}-\frac{\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right] = 4\sqrt{2}$$

$$\Rightarrow\ \sqrt{2}\left[\cot\theta - \cot\left(\theta+\frac{3\pi}{2}\right)\right] = 4\sqrt{2}$$

$$\Rightarrow\ \cot\theta - \cot\left(\frac{3\pi}{2}+\theta\right) = 4$$

$\Rightarrow\ \cot\theta + \tan\theta = 4$

$\Rightarrow\ \tan^2\theta - 4\tan\theta + 1 = 0$

$$\Rightarrow\ \tan\theta = \frac{4\pm\sqrt{16-4}}{2}$$

$$\Rightarrow\ \tan\theta = 2\pm\sqrt{3} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12} \text{ in } 0<\theta<\frac{\pi}{2}.$$

16. (0) : $\sqrt{3}\sec x + \text{cosec}\, x + 2(\tan x - \cot x) = 0$

$$\Rightarrow\ \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x} + 2\left[\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right] = 0$$

$$\Rightarrow\ \sqrt{3}\sin x + \cos x + 2[\sin^2 x - \cos^2 x] = 0$$

$$\Rightarrow\ \sqrt{3}\sin x + \cos x = 2\cos 2x$$

$$\Rightarrow\ \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos 2x$$

$$\Rightarrow\ \sin\frac{\pi}{3}\sin x + \cos\frac{\pi}{3}\cos x = \cos 2x$$

$$\Rightarrow\ \cos\left(x - \frac{\pi}{3}\right) = \cos 2x$$

$$\Rightarrow\ x - \frac{\pi}{3} = 2n\pi \pm 2x, n \in Z.$$

$$\Rightarrow\ x = -\left(2n\pi + \frac{\pi}{3}\right) \text{ or } x = \frac{2n\pi}{3} + \frac{\pi}{9}$$

$$\Rightarrow\ x = -\frac{\pi}{3}, \frac{\pi}{9}, -\frac{5\pi}{9}, \frac{7\pi}{9}$$

Sum = 0

17. (7) : Given, $n \in$ Integer and $n > 3$.

Also, we have, $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$

$$\Rightarrow\ \frac{1}{\sin\frac{\pi}{n}} - \frac{1}{\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow\ \frac{\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}}{\sin\frac{\pi}{n}.\sin\frac{3\pi}{n}} = \frac{1}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow\ 2\cos\left(\frac{2\pi}{n}\right)\cdot\sin\frac{\pi}{n} = \frac{\sin\frac{\pi}{n}\cdot\sin\frac{3\pi}{n}}{\sin\frac{2\pi}{n}}$$

$$\Rightarrow\ \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \ \Rightarrow\ \frac{7\pi}{n} = \pi \ \Rightarrow\ n = 7$$

18. (30) : On squaring the two given equations and adding the results, we get

$24(\sin A\cos B + \cos A\sin B) = 12$,

or $\sin(A+B) = \frac{1}{2}$. Now, as $C = 180° - A - B$, we have

$\sin C = \sin(A + B) = \frac{1}{2}$, implying that either $C = 30°$ or $C = 150°$. But if $C = 150°$, $A < 30°$

So $3\sin A + 4\cos B < \frac{3}{2} + 4 < 6$, a contradiction.

Hence the answer is $C = 30°$.

19. (9) : We know that, $\cos 3x = 4\cos^3 x - 3\cos x$,

$$\Rightarrow\ 4\cos^2 x - 3 = \frac{\cos 3x}{\cos x}\ \ \forall\ x \neq (2k+1)\cdot 90°, k \in Z.$$

Thus

$$(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \frac{\cos 27°}{\cos 9°} \cdot \frac{\cos 81°}{\cos 27°} = \frac{\cos 81°}{\cos 9°}$$

$$= \frac{\sin 9°}{\cos 9°} = \tan 9° \Rightarrow k = 9$$

20. (2) : The arithmetic-geometric means inequality gives

$$\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} \geq 2(\sin x \cos x)^{y - y^2/4}$$

Now, if follows that

$$2\sin x \cos x = \sin 2x \geq 2(\sin x \cos x)^{y - y^2/4},$$

and because $\sin x \cos x < 1$, it follows that $1 \leq y - y^2/4$, or $(1 - y/2)^2 \leq 0$.

It follows that all the equalities hold; that is $y = 2$ and $\sin x = \cos x$, and so there is a unique solution $(x, y) = \left(\frac{\pi}{4}, 2\right)$.

21. (5) : Consider the quadratic equation $3u^2 + 8u + 3 = 0$.

The roots of the above equation are $u_1 = \frac{-8 + 2\sqrt{7}}{6}$ and $u_2 = \frac{-8 - 2\sqrt{7}}{6}$. Both roots are real, and their product $u_1 u_2$ is equal to 1.

Now, as $y = \cot x$ is a bijection from the interval $(0, \pi)$ to the real numbers, there is a unique pair of numbers $x_{1,1}$ and $x_{2,1}$ with $0 < x_{1,1}, x_{2,1} < \pi$ such that $\cot x_{1,1} = u_1$ and $\cot x_{2,1} = u_2$. Since, u_1, u_2 are negative, $\frac{\pi}{2} < x_{1,1}, x_{2,1} < \pi$, and so $\pi < x_{1,1} + x_{2,1} < 2\pi$. Because $\cot x \tan x = 1$ and both $\tan x$ and $\cot x$ have period π, it follows that

$$1 = \cot x \tan x = \cot x \cot\left(\frac{\pi}{2} - x\right) = \cot x \cot\left(\frac{3\pi}{2} - x\right)$$

$= \cot x_{1,1} \cot x_{2,1}$.

Therefore, $x_{1,1} + x_{2,1} = \frac{3\pi}{2}$. Likewise, in the interval $(\pi, 2\pi)$, there is a unique pair of numbers $x_{1,2}$ and $x_{2,2}$ satisfying the conditions of the problem with $x_{1,2} + x_{2,2} = \frac{7\pi}{2}$. Thus the answer to the problem is $x_{1,1} + x_{2,1} + x_{1,2} + x_{2,2} = 5\pi$.

22. (1996) : Let $f(x) = 4x(1 - x) = 1 - (2x - 1)^2$. Observe that if $0 \leq f(x) \leq 1$, then $0 \leq x \leq 1$. Hence, if $a_{1998} = 0$, then we must have $0 \leq t \leq 1$. Now choose $0 \leq \theta \leq \frac{\pi}{2}$ such that $\sin\theta = \sqrt{t}$. Observe that for any $\phi \in R$,

$f(\sin^2\phi) = 4\sin^2\phi(1 - \sin^2\phi) = 4\sin^2\phi\cos^2\phi = \sin^2 2\phi$.

Now, as $a_1 = \sin^2\theta$, it follows that

$a_2 = \sin^2 2\theta$, $a_3 = \sin^2 4\theta$, ..., $a_{1998} = \sin^2 2^{1997}\theta$.

Therefore, $a_{1998} = 0$ if and only if $\sin 2^{1997}\theta = 0$, that is, if $\theta = \frac{m\pi}{2^{1997}}$ for some integers m. So the values of t for which $a_{1998} = 0$ are $\sin^2(m\pi/2^{1997})$, where $m \in Z$. Thus, we get $2^{1996} + 1$ such values of t, namely, $\sin^2(m\pi/2^{1997})$ for $m = 0, 1, 2, ..., 2^{1996}$.

CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 5 Sequences and Series

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long answer-**II** type questions carrying **6** marks each.

SECTION-A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Find the 440$^{\text{th}}$ and 441$^{\text{st}}$ terms of the sequence given by

$$t_n = \begin{cases} \dfrac{n}{\dfrac{n}{44}-1}, & \text{if } n \text{ is not the square of a natural number.} \\ 2.7, & \text{if } n \text{ is the square of a natural number.} \end{cases}$$

(a) $\frac{440}{9}$, 2.7 (b) $\frac{9}{440}$, 2.7
(c) 440, 9 (d) 9, 3.7

2. What is the 20$^{\text{th}}$ term of the sequence defined by $a_n = (n - 1)(2 - n)(3 + n)$?
(a) 7866 (b) –7866 (c) 4598 (d) –4598

3. If a, b, c, d, e are in A.P., then find the value of $a - 4b + 6c - 4d + e$.
(a) $3b$ (b) 0 (c) $4a$ (d) $6c$

4. The income of a person is ₹ 300000 in the first year and he receives an increase of ₹ 10000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.
(a) ₹ 7800000 (b) ₹ 7900000
(c) ₹ 7700000 (d) ₹ 790000

5. The product of three numbers in an A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.
(a) 1, 5, 7 (b) 2, 6, 8
(c) 2, 8, 14 (d) 4, 8, 10

6. Which term of the progression 18, –12, 8, ... is $\frac{512}{729}$?
(a) 10$^{\text{th}}$ (b) 8$^{\text{th}}$ (c) 9$^{\text{th}}$ (d) 12$^{\text{th}}$

7. The third term of a G.P. is 42. Find the product of its first five terms.
(a) 42 (b) $(42)^5$ (c) 98 (d) $(25)^5$

8. Find the sum to infinity for $5, \frac{5}{3}, \frac{5}{9}, \ldots\ldots$
(a) 7.5 (b) 15 (c) 7 (d) 17

9. The sum to infinity of the series

$1+2\left(1-\frac{1}{n}\right)+3\left(1-\frac{1}{n}\right)^2+.....$ where $n \in N$, is given by

(a) $n(n-1)$ (b) $n\left(1-\frac{1}{n}\right)^2$

(c) n^2 (d) $\left(\frac{n-1}{n}\right)^2$

10. $(2n+1)$ G.M.'s are inserted between 4 and 2916. Then, the $(n+1)^{\text{th}}$ G.M. is equal to

(a) 36 (b) 54 (c) 108 (d) 324

(Q. 11-Q.15) Fill in the blanks.

11. The sum of odd integers from 1 to 2001 is ______.

OR

The values of 'k' for which $-\frac{2}{7}, k, -\frac{7}{2}$ are in G.P. is ______.

12. The minimum value of expression $3^x + 3^{1-x}, x \in R$, is ______.

13. The n^{th} term of the sequence $2, 2\sqrt{2}, 4, ...$ is 8, then the value of n is ______.

14. If a, b and c are positive numbers in A.P. such that their product is 64, then the minimum value of b is equal to ______.

15. The 15^{th} term of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+...$, is ______.

OR

A quadratic equation such that the A.M. between the roots of the equation is 10 and the G.M. is 6, is ______.

(Q. 16-Q.20) Answer the following questions.

16. If the fifth term of a G.P. is 81 and second term is 24. Find the common ratio.

OR

Find the 15^{th} term of the G.P. $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, ...$

17. Evaluate $7^{1/2} \times 7^{1/4} \times 7^{1/8} \times ...$ to infinite terms.

18. Divide 69 into three parts which are in A.P. and the product of the two smaller parts is 483.

19. At the end of each year the value of a certain machine has depreciated by 20% of its value at the beginning of that year. If its initial value was ₹ 1250, find the value at the end of 5 years.

20. A sequence is defined by $t_n = n(n-1)(n-2)$. Show that the first two terms of the sequence are zero and the rest of the terms are positive.

SECTION - B

21. Find the number of terms of a geometric sequence $\{a_n\}$ if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

22. A man repays a loan of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan?

23. If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of of n.

24. Find the 20^{th} term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... + (n$ terms).

OR

If the third term of an A.P. is 18 and the seventh term is 30, then find the series.

25. Which term of the sequence $25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}$, ... is the first negative term?

26. Find the sum to infinity of the series

$$\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+...\infty$$

OR

The seventh term of a G.P. is 8 times the fourth term. Find the G.P. when its 5^{th} term is 48.

SECTION - C

27. The 2^{nd}, 31^{st} and last term of A.P. are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. Find the first term and the number of terms.

28. Find the least number of terms of the series $19 + 18\frac{1}{5} + 17\frac{2}{5} + ...$ whose sum is negative. Also calculate the exact sum.

OR

If $(p+q)^{\text{th}}$ term of a G.P. is m and $(p-q)^{\text{th}}$ term is n, show that p^{th} term is $\sqrt{mn}$ and 9^{th} term is $m\left(\frac{n}{m}\right)^{p/2q}$.

CONCEPT MAP

STATISTICS

Class XI

STATISTICS

Measures of Dispersion

Mean Deviation

It is the arithmetic mean of the absolute values of deviations about some point (mean or median or mode).

$$\text{Mean Deviation} = \frac{\text{Sum of Deviations}}{\text{Number of Observations}}$$

For Ungrouped Data

Let $x_1, x_2, \ldots, x_n$ be n observations, then mean deviation about mean is given by $\text{M.D.}(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$;

Mean deviation about median is given by,

$$\text{M.D.}\ (M) = \frac{1}{n}\sum_{i=1}^{n}|x_i - M|$$

For Grouped Data

Let $x_1, x_2, \ldots, x_n$ be a set of n observations occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively, then mean deviation about mean is given by $\text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - \bar{x}|$;

Mean deviation about median is given by

$$\text{M.D.}(M) = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - M|$$

Here, x_i are the mid-points of classes and $N = \sum_{i=1}^{n} f_i$ = Sum of frequencies

Shortcut Method

- **About mean :** $\text{M.D.}\ (\bar{x}) = \frac{1}{N}\cdot\sum_{i=1}^{n} f_i\,|x_i - \bar{x}|$, where mean, $\bar{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{N}\times h$, a is assumed mean, $d_i = \frac{x_i - a}{h}$ and h = size of interval.
- **About median :** $\text{M.D.}(M) = \frac{1}{N}\sum_{i=1}^{n} f_i\left|x_i - M\right|$, where median, $M = l + \frac{\frac{N}{2} - cf}{f}\times h$,

  N is sum of frequencies, l, f, h and cf are respectively the lower limit, the frequency of the median class, the width of the class interval and the cumulative frequency of the class just preceding the median class.

Analysis of Frequency Distributions

If the given data has mean $(\bar{x})$ and standard deviation (σ), then

Coefficient of variation (C.V.) $= \frac{\sigma}{\bar{x}}\times 100, \bar{x} \neq 0$

The data whose C.V. is less is said to be more consistent.

Variance and Standard Deviation

Mean of the squares of the deviations from mean is called variance and is denoted by σ^2.

The positive square root of variance is known as standard deviation. It is denoted by σ.

For ungrouped data	$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$
For grouped data	$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i\,(x_i - \bar{x})^2}$
Shortcut method	$\sigma = \frac{h}{N}\left[\sqrt{N\sum_{i=1}^{n} f_i u_i^2 - \left(\sum_{i=1}^{n} f_i u_i\right)^2}\right]$ where $u_i = \frac{x_i - a}{h}$, a = assumed mean, h = width of class-intervals

Properties of Standard Deviation

- S.D. is independent of change of origin.
- S.D. is not independent of change of scale.

Combined Variance of Two Series

If n_1, n_2 are the number of elements, $\bar{x}_1$, $\bar{x}_2$ are the means and σ_1, σ_2 are the standard deviations of two series respectively, then variance of combined series is

$$\sigma^2 = (\text{S.D.})^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

where $d_1 = \bar{x}_1 - \bar{x}$, $d_2 = \bar{x}_2 - \bar{x}$ and $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

MATRICES

CONCEPT MAP

Class XII

Matrix

A set of mn numbers arranged in the form of a rectangular array of m rows and n columns is called a matrix.

Order of a Matrix

The number of rows and columns that a matrix has is called its order.

Types of Matrices

- Row Matrix : Matrix having only one row.
- Column Matrix : Matrix having only one column.
- Square Matrix : Matrix having equal number of rows and columns.
- Diagonal Matrix : A square matrix is called diagonal matrix, if all its non-diagonal elements are zero. The diagonal elements may or may not be zero.
- Zero Matrix : Matrix whose each and every element is zero.
- Identity Matrix : A diagonal matrix whose all diagonal elements is equal to 1.
- Involutary Matrix : $A^2 = I$
- Orthogonal Matrix : $AA^T = A^TA = I$
- Idempotent Matrix : $A^2 = A$
- Unitary Matrix : $AA^\theta = A^\theta A = I$
- Symmetric Matrix : $A^T = A$
- Skew-Symmetric Matrix : $A^T = -A$

Trace of a Matrix

Sum of principal diagonal elements of a square matrix A is called trace of a matrix.

- $tr(A) + tr(B) = tr(A + B)$
- $tr(kA) = k \cdot tr(A)$

Transpose of a Matrix

Transpose of a matrix is obtained by interchanging rows and columns. If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$

- $(A')' = A$
- $(kA)' = kA'$
- $(AB)' = B'A'$
- $(A \pm B)' = A' \pm B'$
- $(ABC)' = C'\,B'\,A'$

Comparable Matrices

Two matrices are said to be comparable if they have same orders.

Equal Matrices : Two comparable matrices A and B are said to be equal iff all of their corresponding elements are equal.

Inverse of a Matrix

A unique matrix B for A such that $AB = BA = I$.

- Inverse of a square matrix, if it exists, is unique.
- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = A^{-1}/k$
- $(AB)^{-1} = B^{-1}\,A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

Elementary Operations

- $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- $R_i \to kR_i$ or $C_i \to kC_i$
- $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Operations on Matrices

Addition and Subtraction	Multiplication	Scalar Multiplication
$A \pm B = C$ *i.e.*, $[a_{ij}]_{m \times n} \pm [b_{ij}]_{m \times n}$ $= [a_{ij} \pm b_{ij}]_{m \times n} = [c_{ij}]_{m \times n}$	$A_{m \times k} \times B_{k \times q} = C_{m \times q}$ *i.e.*, $\left[\sum_{r=1}^{k} a_{ir} b_{rj}\right] = [c_{ij}]$	$kA = B$ *i.e.*, $k[a_{ij}]_{m \times n} = [ka_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$
Properties		
• $A + B = B + A$ • $A + (B + C) = (A + B) + C$ • Additive inverse of $A = -A$ • Additive identity $= O$	• AB exist $\not\Rightarrow BA$ exists • AB may or may not be equal to BA • $(AB)C = A(BC)$ • $I_m \times A_{m \times m} = A_{m \times m} = A_{m \times m} \times I_m$ • $A(B + C) = AB + AC$; $(B + C)\,A = BA + CA$	• $k(A + B) = kA + kB$ • $(k + m)A = kA + mA$

29. Let a, b, c, d, e be five real numbers such that a, b, c are in A.P.; b, c, d are in G.P., $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. If $a = 2$ and $e = 18$, find all possible values of b, c and d.

30. 21 students are to be divided into three groups forming an A.P. If the sum of the squares of number of students in each group is 155. Find the number of students in each group.

31. Find the least value of n for which the sum $1 + 3 + 3^2 + ...$ to n terms is greater than 7000.

OR

If x, y, z are positive, then find the minimum value of $x^{\log y - \log z} + y^{\log z - \log x} + z^{\log x - \log y}$.

32. The arithmetic mean between two positive numbers a and b, where $a > b$, is twice their geometric mean. Prove that $a : b = (2+\sqrt{3}) : (2-\sqrt{3})$.

SECTION-D

33. (i) If there are $(2n + 1)$ terms in an A.P., prove that the sum of odd terms and the sum of even terms bear the ratio $(n + 1) : n$.

(ii) If the m^{th} term of an A.P. is $(1/n)$ and its n^{th} term is $(1/m)$, then show that the sum of mn terms is $\frac{1}{2}(mn + 1)$.

OR

The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

34. If a^2, b^2, c^2 are in A.P., then prove that the following are also in A.P.

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

35. Let x be the A.M. and y, z be two G.M.s between any two positive numbers. Then, prove that $\frac{y^3+z^3}{xyz} = 2$.

OR

One side of an equilateral triangle is 18 cm. The mid points of its sides are joined to form another triangle whose mid points, in term, are joined to form further another triangle and so on up to infinity. Find the sum of the (i) perimeters of all the triangle (ii) area of all the triangles.

36. (i) If a, b, c, d are in G.P., prove that
$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$

(ii) The p^{th}, q^{th} and r^{th} terms of an A.P. as well as those of a G.P. are a, b, c, respectively, prove that $a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$.

SOLUTIONS

1. **(a) :** For $n = 441$, which is the square of natural number 21, we have $t_n = 2.7$

$\therefore \quad t_{441} = 2.7$

For $n = 440$, which is not the square of a natural number, we have $t_n = \frac{n}{\frac{n}{44}-1} \quad \therefore t_{440} = \frac{440}{\frac{440}{44}-1} = \frac{440}{9}$

2. **(b) :** Putting $n = 20$, we obtain

$a_{20} = (20 - 1)(2 - 20)(3 + 20) = -7866$

3. **(b) :** Given, a, b, c, d and e are in A.P.

Let D be the common difference of A.P.

Then, $b - a = c - b = d - c = e - d = D$ ...(i)

Now, consider $a - 4b + 6c - 4d + e$

$= a - b - 3b + 3c + 3c - 3d - d + e$

$= -(b - a) + 3(c - b) - 3(d - c) + (e - d)$

$= -D + 3D - 3D + D$ [Using (i)]

$= 0$

4. **(b) :** Here, we have an A.P. with $a = 300000$, $d = 10000$ and $n = 20$.

$\therefore \quad S_{20} = \frac{20}{2}[600000 + 19 \times 10000] = 7900000$

5. **(c) :** Let the three numbers be $a - d, a, a + d$ $(d > 0)$

Now, $(a - d)a(a + d) = 224 \Rightarrow a(a^2 - d^2) = 224$ ...(i)

Also, $a + d = 7(a - d) \Rightarrow d = \frac{3a}{4}$

Substituting this value of d in (i), we get

$a\left(a^2 - \frac{9a^2}{16}\right) = 224 \Rightarrow a = 8$ and $d = \frac{3}{4} \times 8 = 6$

Hence, the three numbers are 2, 8, 14.

6. **(c) :** Let $a_1 = 18, a_2 = -12, a_3 = 8$

Here, $\frac{a_2}{a_1} = \frac{a_3}{a_2} = -\frac{2}{3}$

Thus, this sequence is a G.P., where $a = 18, r = -\frac{2}{3}$

$\therefore \quad \frac{512}{729} = ar^{n-1} \Rightarrow \frac{2^9}{3^6} = 18\left(\frac{-2}{3}\right)^{n-1}$

$\Rightarrow \frac{2^9}{3^6 \times 3^2 \times 2} = \left(\frac{-2}{3}\right)^{n-1}$

$$\Rightarrow \quad \frac{2^8}{3^8}=\left(\frac{-2}{3}\right)^{n-1} \Rightarrow \left(\frac{-2}{3}\right)^8=\left(\frac{-2}{3}\right)^{n-1}$$

$\Rightarrow \quad n-1=8 \Rightarrow n=9$

Hence, 9th term of the progression is $\frac{512}{729}$.

7. (b) : Let a be the first term and r be the common ratio. Then, $a_3 = 42 \Rightarrow ar^2 = 42$ ...(i)

$\therefore$ Product of first five terms $= a_1a_2a_3a_4a_5$

$= a(ar)(ar^2)(ar^3)(ar^4) = a^5r^{10} = (ar^2)^5$

$= (42)^5$[Using(i)]

8. (a) : Here, $a = 5,\ r=\frac{1}{3}$

$$\therefore \quad S_\infty=\frac{a}{1-r}=\frac{5}{1-\frac{1}{3}}=\frac{15}{2}=7.5$$

9. (c) : Let $S = 1+2\left(1-\frac{1}{n}\right)+3\left(1-\frac{1}{n}\right)^2+.....$ ...(i)

$\therefore \quad \left(1-\frac{1}{n}\right)S=\left(1-\frac{1}{n}\right)+2\left(1-\frac{1}{n}\right)^2+.....$ ...(ii)

(i) – (ii) gives, $\frac{S}{n}=1+\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)^2+.....\infty$

$$=\frac{1}{1-\left(1-\frac{1}{n}\right)}=n$$

$\Rightarrow \quad S = n^2$

10. (c) : $(n + 1)^{\text{th}}$ G.M. = G.M. of 4 and 2916

$= \sqrt{4\times 2916} = 108$

11. Odd integers from 1 to 2001 are

1, 3, 5, ..., 2001.

First term $(a) = 1$

Common difference $= T_2 - T_1 = 3 - 1 = 2$

Last term $(l) = a + (n - 1)d$

$\Rightarrow \quad 2001 = 1+ (n -1)2$

$\Rightarrow \quad 2001 - 1 = 2n - 2$

$\Rightarrow \quad 2002 = 2n \Rightarrow n = 1001$

$\therefore \quad$ Sum of odd terms $=\frac{n}{2}[2a+(n-1)\,d]$

$$=\frac{1001}{2}[2\times 1+(1001-1)\times 2]$$

$$=\frac{1001}{2}[2+2000]=\frac{1001\times 2002}{2}=1002001$$

OR

As, $-\frac{2}{7}, k, -\frac{7}{2}$ are in G.P.

$\therefore \quad k^2=-\left(\frac{2}{7}\right)\times-\left(\frac{7}{2}\right)=1 \ \Rightarrow\ k=\pm 1$

12. We know that, A.M. ≥ G.M. for positive numbers.

Therefore, $\frac{3^x+3^{1-x}}{2}\geq\sqrt{3^x\cdot 3^{1-x}} \Rightarrow 3^x + 3^{1-x} \geq 2\sqrt{3}$

Hence, the minimum value is $2\sqrt{3}$.

13. Let $a_1 = 2,\ a_2 = 2\sqrt{2},\ a_3 = 4$

Here, $\frac{a_2}{a_1}=\frac{a_3}{a_2}=\sqrt{2}$

Thus, this sequence is a G.P., where $a = 2,\ r = \sqrt{2}$

$\therefore \quad 8 = ar^{n-1} \Rightarrow 8 = 2(\sqrt{2})^{n-1} \Rightarrow 2^3 = 2^{\frac{n+1}{2}}$

$\Rightarrow \quad \frac{n+1}{2}=3 \Rightarrow n=5$

Hence, 5th term of the sequence is 8.

14. Given, a and c are positive numbers.

$\therefore \quad \frac{a+c}{2}\geq\sqrt{ac}$

$\Rightarrow \quad \frac{2b}{2}\geq\sqrt{\frac{64}{b}}$ ($\because$ a, b, c are in A.P. $\therefore$ $a + c = 2b$ and $abc = 64$ (Given))

$\Rightarrow \quad b\geq\frac{8}{b^{1/2}}\Rightarrow b^{3/2}\geq 2^3\Rightarrow b^{1/2}\geq 2\Rightarrow b\geq 4$

Hence, minimum value of b is 4.

15. The given series is a geometric series with first term $a = 1$ and common ratio $r=-\frac{1}{2}$

Hence, $a_{15}=ar^{15-1}=1\times\left(-\frac{1}{2}\right)^{15-1}=(-1)^{14}\times 2^{-14}=2^{-14}$

OR

Let α and β be the roots of a quadratic equation.

Then by the given conditions,

$\frac{\alpha+\beta}{2}=10$ and $\sqrt{\alpha\beta}=6$

$\Rightarrow \quad \alpha + \beta = 20$ and $\alpha\beta = 36$

Hence, the quadratic equation whose roots are α and β is $x^2 - 20x + 36 = 0$.

16. Here, $a_5 = 81$ and $a_2 = 24$

Let r be the common ratio of G.P.

$\therefore\ ar^4 = 81$ $\quad [\because\ a_n = ar^{n-1}]$ ...(i)

and $ar = 24$ ...(ii)

Dividing (i) by (ii), we get

$$\frac{ar^4}{ar}=\frac{81}{24}=\frac{27}{8} \Rightarrow r^3=\left(\frac{3}{2}\right)^3 \Rightarrow r=\frac{3}{2}$$

OR

n^{th} term of a G.P. is given by $a_n = ar^{n-1}$

Here, $a = \frac{3}{2}, r = \frac{1}{2}$ and $n = 15$

$\therefore \quad a_{15} = 15^{th}$ term of the G.P. $= \frac{3}{2}\left(\frac{1}{2}\right)^{15-1} = \frac{3}{2^{15}}$

17. We have, $7^{1/2} \times 7^{1/4} \times 7^{1/8} \times ...$ to infinite terms

$= 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+... \text{ to } \infty \text{ terms}} = 7^{\frac{1/2}{1-1/2}}$

$[\because \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$ is an infinite G.P. with $a = \frac{1}{2}$ and $r = \frac{1}{2}]$

$= 7^1 = 7$

18. Let the three parts of 69 be $a - d, a, a + d$ $(d > 0)$

Then, $(a - d) + a + (a + d) = 69 \Rightarrow 3a = 69 \Rightarrow a = 23$

Given, $a(a - d) = 483$

$\therefore \quad 23(23 - d) = 483 \Rightarrow 23 - d = 21 \Rightarrow d = 2$

$\therefore \quad$ Three parts are 21, 23, 25.

19. After each year the value of the machine is 80% of its value the previous year so at the end of 5 years the machine will depreciate as many times as 5.

Hence, we have to find the 6^{th} term of the G.P. whose first term a_1 is 1250 and common ratio r is 0.8.

Hence, value at the end of 5 years $= a_6 = a_1 r^5$

$= 1250\,(0.8)^5 = ₹409.6$

20. We have,

$t_n = n(n - 1)(n - 2)$

$\therefore \quad t_1 = 1(1 - 1)(1 - 2) = 1 \cdot 0 \cdot (-1) = 0$

and $t_2 = 2(2 - 1)(2 - 2) = 0$

For $n > 2$, n, $n - 1$ and $n - 2$ are all positive, therefore, $n(n - 1)(n - 2)$ is also positive.

Hence t_n is positive for $n > 2$.

Thus, we see that the first two terms of the sequence are zero and the rest of the terms are positive.

21. Let r be the common ratio of G.P., then

$a_n = a_1 r^{n-1} \quad \Rightarrow \quad 96 = 3 \cdot r^{n-1} \quad \Rightarrow \quad r^{n-1} = 32 \quad$ ...(i)

Now, $S_n = \frac{a(r^n - 1)}{r - 1} \Rightarrow 189 = \frac{3(r^n - 1)}{r - 1}$

$\Rightarrow \quad 63(r - 1) = r^n - 1 \quad \Rightarrow 63r - 62 = r \cdot r^{n-1}$

$\Rightarrow \quad 63r - 62 = r \times 32 \Rightarrow 31r = 62 \Rightarrow r = 2$

Substituting this value of r in (i), we get

$2^{n-1} = 32 = 2^5 \quad \Rightarrow \quad n - 1 = 5 \quad \Rightarrow \quad n = 6$

Hence, the number of terms = 6

22. Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

Sum of the amounts = 3250 [Given]

$\Rightarrow \quad \frac{n}{2}\{2 \times 20 + (n - 1) \times 15\} = 3250$

$\Rightarrow \quad 3n^2 + 5n - 1300 = 0$

$\Rightarrow \quad (n - 20)(3n + 65) = 0 \Rightarrow n = 20 \quad \left[\because n \neq \frac{-65}{3}\right]$

Thus, the loan is cleared in 20 months.

23. Let $A_1, A_2, ..., A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A.P. ; 20, $A_1, A_2, ..., A_n$, 80. Then, $d = \frac{80 - 20}{n + 1} = \frac{60}{n + 1}$

Now, $A_1 = 20 + d \Rightarrow A_1 = 20 + \frac{60}{n + 1} = 20\left(\frac{n + 4}{n + 1}\right)$

and, $A_n = 20 + nd \Rightarrow A_n = 20 + \frac{60n}{n + 1} = 20\left(\frac{4n + 1}{n + 1}\right)$

It is given that $\frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{\frac{20(n + 4)}{n + 1}}{\frac{20(4n + 1)}{n + 1}} = \frac{1}{3} \Rightarrow \frac{n + 4}{4n + 1} = \frac{1}{3}$

$\Rightarrow \quad 4n + 1 = 3n + 12 \Rightarrow n = 11$

24. First factor of the terms are 2, 4, 6, ...

$\therefore \quad$ First factor of n^{th} term $= 2n$ ...(i)

Second factor of the terms are 4, 6, 8, ...

$\therefore \quad$ Second factor of n^{th} term $= 4 + (n - 1)\,2$

$= 2n + 2$ ...(ii)

$\therefore \quad n^{th}$ term of the given series

$= 2n \times (2n + 2) = 4n(n + 1)$

$\therefore \quad$ Putting $n = 20$,

20^{th} term of the given series $= 4 \times 20 \times (20 + 1)$

$= 80 \times 21 = 1680$

OR

Let a be the first term and d be the common difference of the given A.P.

Given, $t_3 = 18$, *i.e.*, $a + 2d = 18$ ...(i)

and $t_7 = 30$, *i.e.*, $a + 6d = 30$ ...(ii)

Subtracting (i) from (ii), we get $4d = 12 \Rightarrow d = 3$

Substituting this value of d in (i), we get

$a + 2 \cdot 3 = 18 \Rightarrow a = 12$

$\therefore \quad t_n = 12 + (n - 1)\,3 = 3n + 9$

Putting $n = 1, 2, 3, 4, ...$, the A.P. is 12, 15, 18, 21,...

Therefore, the required series is

$12 + 15 + 18 + 21 + ...$

25. The given sequence $25, 24\frac{1}{4}, 23\frac{1}{2}, 22\frac{3}{4}, ...$ is an A.P. with common difference $d = -\frac{3}{4}$ and first term $a = 25$.

Let n^{th} term of the given A.P. be the first negative term, then $a_n < 0 \Rightarrow 25 + (n - 1)\left(-\frac{3}{4}\right) < 0$

$\Rightarrow \quad \frac{103}{4}-\frac{3n}{4}<0 \Rightarrow 103-3n<0 \Rightarrow 103<3n$

$\Rightarrow \quad 3n>103 \Rightarrow n>\frac{103}{3}$ *i.e.*, $n>34\frac{1}{3}$.

Since 35 is the least natural number satisfying $n>34\frac{1}{3}$

$\Rightarrow \quad n=35$

Hence, 35$^{\text{th}}$ term of the given sequence is the first negative term.

26. The series may be written as

$$\left(\frac{1}{7}+\frac{1}{7^3}+\frac{1}{7^5}+\ldots\infty\right)+\left(\frac{2}{7^2}+\frac{2}{7^4}+\frac{2}{7^6}+\ldots\infty\right)$$

$$=\frac{\frac{1}{7}}{\left(1-\frac{1}{7^2}\right)}+\frac{\frac{2}{7^2}}{\left(1-\frac{1}{7^2}\right)} \qquad \left[\text{Using } S_\infty=\frac{a}{1-r}\right]$$

$$=\frac{\frac{1}{7}}{\frac{48}{49}}+\frac{\frac{2}{49}}{\frac{48}{49}}=\frac{7}{48}+\frac{2}{48}=\frac{9}{48}$$

OR

We have, $t_7=8t_4 \Rightarrow ar^6=8ar^3$, where a and r are the first term and common ratio respectively of the G.P.

$\Rightarrow \quad r^3=8=2^3 \;\therefore\; r=2$

Also, $t_5=48 \Rightarrow ar^4=48 \Rightarrow a(2)^4=48 \Rightarrow 16a=48$

$\Rightarrow \quad a=\frac{48}{16}=3$

Hence, the required G.P is 3, 6, 12, 24, ...

27. Let a be the first term and d be the common difference of the A.P.

Given, $t_2=7\frac{3}{4} \Rightarrow a+d=\frac{31}{4}$ ...(i)

and $t_{31}=\frac{1}{2} \Rightarrow a+30d=\frac{1}{2}$ ...(ii)

Subtracting (i) from (ii), we get

$$29d=\frac{1}{2}-\frac{31}{4}=-\frac{29}{4} \quad \therefore \; d=-\frac{1}{4}$$

Putting the value of d in (i), we get

$$a-\frac{1}{4}=\frac{31}{4} \Rightarrow a=\frac{31}{4}+\frac{1}{4}=\frac{32}{4}=8$$

Let the number of terms be n so that $t_n=-\frac{13}{2}$

$\Rightarrow \quad a+(n-1)d=-\frac{13}{2} \Rightarrow 8+(n-1)\left(-\frac{1}{4}\right)=-\frac{13}{2}$

$\Rightarrow \quad 8-\frac{n}{4}+\frac{1}{4}=-\frac{13}{2} \Rightarrow 32-n+1=-26 \Rightarrow n=59$

Hence, first term is 8 and number of terms are 59.

28. The given series $19+18\frac{1}{5}+17\frac{2}{5}+\ldots$ is an arithmetic series with first term $a=19$ and common difference $d=-\frac{4}{5}$

Let the required number of terms be n.

According to given condition, sum to n terms < 0

$\Rightarrow \quad \frac{n}{2}\left[2\times 19+(n-1)\left(-\frac{4}{5}\right)\right]<0 \Rightarrow n\left[19-\frac{2}{5}(n-1)\right]<0$

$\Rightarrow \quad \frac{n}{5}(97-2n)<0 \Rightarrow -\frac{2}{5}n\left(n-\frac{97}{2}\right)<0$

$\Rightarrow \quad n\left(n-\frac{97}{2}\right)>0 \Rightarrow n<0$ or $n>\frac{97}{2}$.

As n (number of terms) is a positive integer, therefore, the least value of $n=49$.

Then, sum $=\frac{49}{2}\left[2\times 19+(49-1)\left(-\frac{4}{5}\right)\right]=-\frac{49}{5}=-9\frac{4}{5}$

OR

Let a be the first term and r be the common ratio of the G.P.

Given, $a_{p+q}=m$ and $a_{p-q}=n$

$\Rightarrow \quad ar^{p+q-1}=m$ and $ar^{p-q-1}=n$

$\Rightarrow \quad mn=a^2r^{p+q-1+p-q-1}=a^2r^{2p-2}$

$\Rightarrow \quad mn=(ar^{p-1})^2=(a_p)^2 \Rightarrow a_p=\sqrt{mn}$.

$\Rightarrow \quad p^{\text{th}}$ term $=\sqrt{mn}$.

Also, $\frac{m}{n}=\frac{ar^{p+q-1}}{ar^{p-q-1}}=r^{2q}$ ...(i)

$$\therefore \quad q^{\text{th}} \text{ term}=ar^{q-1}=\frac{ar^{p+q-1}}{r^p}=\frac{m}{\left(\left(\frac{m}{n}\right)^{\frac{1}{2q}}\right)^p}=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$$

29. Given, a, b, c are in A.P. $\Rightarrow \; b=\frac{a+c}{2}$ ...(i)

b, c, d are in G.P. $\Rightarrow c^2=bd$ ...(ii)

and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. $\Rightarrow \; \frac{2}{d}=\frac{1}{c}+\frac{1}{e}=\frac{e+c}{ce}$

$\Rightarrow \quad d=\frac{2ce}{c+e}$ ...(iii)

From (ii), $c^2=bd=\frac{a+c}{2}\cdot\frac{2ce}{c+e}=\frac{(a+c)ce}{c+e}$

$\Rightarrow \quad c=\frac{(a+c)e}{c+e} \Rightarrow c^2+ce=ae+ce \Rightarrow c^2=ae$ ...(iv)

Given, $a=2, e=18 \quad \therefore \quad$ From (iv) $c^2=36 \Rightarrow c=\pm 6$

From (i), $b=\frac{a+c}{2}=\frac{2\pm 6}{2}=4,-2$

From (ii), $d = \frac{c^2}{b} = \frac{36}{4}$ or $\frac{36}{-2} = 9$ or -18

Thus, $c = 6, b = 4, d = 9$ or $c = -6, b = -2, d = -18$.

30. Let number of students in three group be $a - d, a, a + d$.

Given that, $a - d + a + a + d = 21 \Rightarrow 3a = 21 \Rightarrow a = 7$

Also, $(a - d)^2 + a^2 + (a + d)^2 = 155$

$\Rightarrow 3a^2 + 2d^2 = 155 \Rightarrow 2d^2 = 155 - 147 = 8$

$\Rightarrow d^2 = 4 \Rightarrow d = \pm 2$

If $d = 2$, then number of students in each group are 5, 7, 9.

If $d = -2$, then number of students in each group are 9, 7, 5.

31. We have, $S_n = 1 + 3 + 3^2 + \ldots$ to n terms

$$\Rightarrow S_n = 1.\left(\frac{3^n - 1}{3-1}\right) = \frac{3^n - 1}{2}$$

Now, $S_n > 7000$

$$\Rightarrow \frac{3^n - 1}{2} > 7000 \Rightarrow 3^n - 1 > 14000$$

$$\Rightarrow 3^n > 14001 \Rightarrow n \log 3 > \log 14001$$

$$\Rightarrow n > \frac{\log 14001}{\log 3} \Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

OR

Refer to answer 109, Pg-197, of MTG CBSE Champion Mathematics Class 11.

32. Let A be the A.M. and G be the G.M. between a and b.

Then, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Given, $A = 2G \Rightarrow \frac{a+b}{2} = 2\sqrt{ab} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{1} \text{ [By componendo and dividendo]}$$

$$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{3}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$$

$$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad \text{[By componendo and dividendo]}$$

$$\Rightarrow \frac{a}{b} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 = \frac{4+2\sqrt{3}}{4-2\sqrt{3}} \Rightarrow \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

33. **(i)** Let a be the first term and d be the common difference of the given A.P.

Then, $a_k = a + (k - 1)d$ ...(i)

Let S_1 & S_2 denote the sum of all odd terms and the sum of all even terms respectively, then

$$\therefore S_1 = a_1 + a_3 + a_5 + \ldots + a_{2n+1} = \frac{(n+1)}{2}\{a_1 + a_{2n+1}\}$$

$$= \frac{(n+1)}{2}\{a + a + (2n + 1 - 1)d\} = (n + 1)(a + nd)$$

and $S_2 = a_2 + a_4 + a_6 + \ldots + a_{2n} = \frac{n}{2}[a_2 + a_{2n}]$

$$= \frac{n}{2}[(a + d) + (a + (2n - 1)d)] = n(a + nd)$$

$$\therefore \frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{n+1}{n}$$

Hence, the required ratio is $(n + 1) : n$.

(ii) Let a be the first term and d be the common difference of the given A.P.

Then, $a_m = \frac{1}{n}$ and $a_n = \frac{1}{m}$ [Given]

Now, $a_m = \frac{1}{n} \Rightarrow a + (m - 1)d = \frac{1}{n}$ ...(i)

and $a_n = \frac{1}{m} \Rightarrow a + (n - 1)d = \frac{1}{m}$ ...(ii)

On subtracting (ii) from (i), we get

$$(m - n)d = \left(\frac{1}{n} - \frac{1}{m}\right) = \frac{(m-n)}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \Rightarrow a = \left\{\frac{1}{n} - \frac{(m-1)}{mn}\right\} = \frac{1}{mn}$$

Thus, $a = \frac{1}{mn}$ and $d = \frac{1}{mn}$

$$\therefore S_{mn} = \frac{mn}{2}\{2a + (mn - 1)d\}$$

$$= \frac{mn}{2}\left\{\frac{2}{mn} + \frac{(mn-1)}{mn}\right\} \quad \left[\because a = \frac{1}{mn} \text{ and } d = \frac{1}{mn}\right]$$

$$= \frac{1}{2}(mn + 1)$$

Hence, the sum of mn terms is $\frac{1}{2}(mn + 1)$.

OR

Let the numbers in G.P. be a, ar, ar^2. It is given that the sum of these numbers is 56.

$\therefore a + ar + ar^2 = 56 \Rightarrow a + ar^2 = 56 - ar$ ...(i)

It is also given that $a - 1$, $ar - 7$ and $ar^2 - 21$ are in A.P.

$\therefore 2(ar - 7) = (a - 1) + (ar^2 - 21) \Rightarrow 2ar = a + ar^2 - 8$

$\Rightarrow a + ar^2 = 2ar + 8$ ...(ii)

From (i) and (ii), we have

$2ar + 8 = 56 - ar \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}$ ...(iii)

Putting $r = \dfrac{16}{a}$ in (i), we get

$a + 16 + \dfrac{256}{a} = 56 \Rightarrow a^2 - 40a + 256 = 0$

$\Rightarrow (a - 32)(a - 8) = 0 \Rightarrow a = 8, 32$

Putting $a = 8$ in $r = \dfrac{16}{a}$, we get $r = \dfrac{16}{8} = 2$

Putting $a = 32$ in $r = \dfrac{16}{a}$, we get $r = \dfrac{16}{32} = \dfrac{1}{2}$

When $a = 8$ and $r = 2$, we obtain 8, 16 and 32 as the numbers of G.P.

When $a = 32$ and $r = \dfrac{1}{2}$, we obtain 32, 16, 8 as the numbers of G.P.

Hence, the numbers are 8, 16 and 32 or 32, 16 and 8.

34. **(i)** It is given that a^2, b^2, c^2 are in A.P.

$\therefore \quad b^2 - a^2 = c^2 - b^2$

$\Rightarrow (b - a)(b + a) = (c - b)(c + b) \Rightarrow \dfrac{b-a}{b+c} = \dfrac{c-b}{a+b}$

$\Rightarrow \dfrac{(b+c)-(a+c)}{b+c} = \dfrac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \dfrac{(b+c)-(a+c)}{(a+c)(b+c)} = \dfrac{(c+a)-(b+a)}{(a+b)(a+c)}$

$\left[\text{Multiplying both side by } \dfrac{1}{a+c}\right]$

$\Rightarrow \dfrac{1}{a+c} - \dfrac{1}{b+c} = \dfrac{1}{a+b} - \dfrac{1}{a+c}$

$\Rightarrow \dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

(ii) It is given that a^2, b^2, c^2 are in A.P., then from (i) part $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P.

$\Rightarrow \dfrac{a+b+c}{b+c}, \dfrac{a+b+c}{c+a}, \dfrac{a+b+c}{a+b}$ are also in A.P.

$\Rightarrow 1+\dfrac{a}{b+c}, 1+\dfrac{b}{c+a}, 1+\dfrac{c}{a+b}$ are in A.P.

$\Rightarrow \dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.

35. Let a and b be two positive numbers.

Then x = A.M. of a and $b \Rightarrow x = \dfrac{a+b}{2}$ ...(i)

It is given that y and z are two geometric means between a and b.

Then a, y, z, b is a G.P. with common ratio $\left(\dfrac{b}{a}\right)^{\frac{1}{2+1}} = \left(\dfrac{b}{a}\right)^{1/3}$

$\therefore \quad y = ar \Rightarrow y = a\left(\dfrac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3}a^{2/3}$

$z = ar^2 \Rightarrow z = a\left(\dfrac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3}a^{1/3}$

$\therefore \quad y^3 + z^3 = (b^{1/3}a^{2/3})^3 + (b^{2/3}a^{1/3})^3 = ba^2 + b^2a = ab(a + b)$

and $yz = (b^{1/3}a^{2/3})(b^{2/3}a^{1/3}) = ab$

$\therefore \quad y^3 + z^3 = yz(a + b) \Rightarrow y^3 + z^3 = yz(2x)$ [Using(i)]

$\Rightarrow \dfrac{y^3+z^3}{xyz} = 2$

OR

Refer to answer 159, Pg-no. 206-207 of MTG CBSE Champion Mathematics Class 11.

36. **(i)** Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore$ L.H.S. $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^4r^2(1 + r^2 + r^4)^2$ ...(i)

And, R.H.S. $= (ab + bc + cd)^2$

$= (a^2r + a^2r^3 + a^2r^5)^2 = a^4r^2(1 + r^2 + r^4)^2$ ...(ii)

Hence, from (i) and (ii), have

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

(ii) Let x be the first term and d be the common difference of an A.P. Then,

$x + (p - 1)d = a$ ...(i) $x + (q - 1)d = b$ ...(ii)

$x + (r - 1)d = c$ ...(iii)

On subtracting (ii) from (i), we get

$a - b = (p - q)d$ ...(iv)

On subtracting (iii) from (ii), we get

and $b - c = (q - r)d$ ...(v)

Now, let A be the first term and R be the common ratio of the G.P. Then,

$AR^{p-1} = a$ ...(vi) $AR^{q-1} = b$ ...(vii) and $AR^{r-1} = c$ ...(viii)

On dividing (vi) by (vii) and (vii) by (viii), we get

$\Rightarrow \dfrac{a}{b} = R^{p-q}$ and $\dfrac{b}{c} = R^{q-r} \Rightarrow \left(\dfrac{a}{b}\right)^{\frac{1}{p-q}} = \left(\dfrac{b}{c}\right)^{\frac{1}{q-r}} = R$

$\Rightarrow \left(\dfrac{a}{b}\right)^{\frac{d}{a-b}} = \left(\dfrac{b}{c}\right)^{\frac{d}{b-c}}$ [Using (iv) and (v)]

$\Rightarrow \left(\dfrac{a}{b}\right)^{b-c} = \left(\dfrac{b}{c}\right)^{a-b} \Rightarrow \dfrac{a^{b-c}}{b^{b-c}} = \dfrac{b^{a-b}}{c^{a-b}}$

$\Rightarrow a^{b-c} \cdot c^{a-b} = b^{a-b+b-c} = b^{-c+a}$

$\Rightarrow a^{b-c} \cdot c^{a-b} = \dfrac{1}{b^{c-a}} \Rightarrow a^{b-c} \cdot b^{c-a} \cdot c^{a-b} = 1$

MONTHLY TEST DRIVE

XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Time Taken : 60 Min.**

Series 5 : Binomial Theorem

Only One Option Correct Type

1. If each of the coefficient in the expansion of $(a + cx)^{-2}$ is positive, then quadratic equation $ax^2 + bx + c = 0$ has
   (a) real and positive roots.
   (b) real and negative roots.
   (c) imaginary roots.
   (d) real roots with opposite signs.

2. If $f(n) = \sum_{s=1}^{n}\sum_{r=s}^{n} {}^nC_r\,{}^rC_s$, then $f(3) =$
   (a) 19 (b) 18
   (c) 21 (d) none of these

3. If in the expansion of $\left(2^{\frac{1}{3}}+\frac{1}{3^{1/3}}\right)^n$, the ratio of the 7^{th} term from the beginning to the 7^{th} term from the end is 1 : 6, then $n =$
   (a) 7 (b) 8 (c) 9 (d) 10

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is
   (a) 120 (b) 210 (c) 310 (d) 4

5. The remainder when 17^{20} is divided by 81 is equal to
   (a) 46 (b) 35
   (c) 47 (d) none of these

6. The coefficient of x^2y in the expansion of $(1 + x + 2y)^5$ is
   (a) 20 (b) 60 (c) 35 (d) 40

One or More Than One Option(s) Correct Type

7. Which of the following is true for the expansion of $(2 - 2x + x^2)^9$?
   (a) Number of distinct terms is 10.
   (b) Coefficient of x^4 is 97.
   (c) Sum of coefficients is 1.
   (d) Number of distinct terms is 55.

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is
   (a) $2^{21} - 2^{10}$ (b) $2^{20} - 2^9$
   (c) $2^{20} - 2^{10}$ (d) $2^{21} - 2^{11}$

9. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then

(a) $a_1 = 20$ (b) $a_2 = 210$
(c) $a_4 = 8085$ (d) $a_{20} = 2^2 \cdot 3^7 \cdot 7$

10. Which of the following statements is/are true?
(a) $5^{2n} + 1$ is divisible by 13 if n is even.
(b) $5^{2n} - 1$ is divisible by 13 if n is odd.
(c) $5^{2n} - 1$ is divisible by 13 if n is even.
(d) $5^{2n} + 1$ is divisible by 13 if n is odd.

11. If the coefficients of x^{-2} and x^{-4} in the expansion of $\left(x^{\frac{1}{3}} + \frac{1}{2x^{\frac{1}{3}}}\right)^{18}$, $(x > 0)$, are m and n respectively, then $\frac{m}{n}$ is equal to
(a) 27 (b) 182 (c) $\frac{5}{4}$ (d) $\frac{4}{5}$

12. If $S = C_0^2 + \frac{C_1^2}{C_0} + 2\cdot\frac{C_2^2}{C_1} + 3\cdot\frac{C_3^2}{C_2} + \ldots + n\cdot\frac{C_n^2}{C_{n-1}}$ where $C_r = {}^nC_r$, then
(a) $(n + 2)$ divides $S + n$
(b) 2^{n-1} divides $S + n$
(c) S is even if n is even
(d) S is even for all n

13. If the coefficient of x^8 in the expansion of $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}\right)^2$ is $\frac{1}{M}$, then divisor(s) of M is/are
(a) 2 (b) 3 (c) 5 (d) 7

Comprehension Type

If n is a positive integer, then
$(x_1 + x_2 + x_3 + \ldots + x_m)^n$
$$= \sum \frac{n!}{n_1!\,n_2!\,n_3!\ldots n_m!} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \ldots\ldots x_m^{n_m}$$
where $n_1, n_2, n_3, \ldots\ldots, n_m$ are all non-negative integers subject to the condition $n_1 + n_2 + n_3 + \ldots. + n_m = n$.

14. The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ is
(a) 40 (b) 60 (c) 80 (d) 100

15. If coefficient of x^{20} in $(1 - x + x^2)^{20}$ and $(1 + x - x^2)^{20}$ are respectively a and b, then
(a) $a = b$ (b) $a > b$
(c) $a < b$ (d) $a + b = 0$

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	$\binom{10}{10}+\binom{11}{10}+\binom{12}{10}+\ldots+\binom{20}{10}$	1.	$\binom{19}{9}$
Q.	$11\cdot\binom{10}{10}+10\cdot\binom{11}{10}+9\cdot\binom{12}{10}+\ldots+1\cdot\binom{20}{10}$	2.	$\binom{20}{9}$
R.	$\binom{10}{0}\binom{10}{1}+\binom{10}{1}\binom{10}{2}+\binom{10}{2}\binom{10}{3}+\ldots+\binom{10}{9}\binom{10}{10}$	3.	$\binom{21}{10}$
S.	The number of non-negative integral solutions of $x_1 + x_2 + x_3 + \ldots + x_{10} = 10$ is	4.	$\binom{22}{10}$

	P	Q	R	S
(a)	1	3	4	2
(b)	3	2	4	1
(c)	3	4	2	1
(d)	1	3	2	4

Integer Answer Type

17. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n equals _______.

18. If the sum of the two middle coefficients of $(1 + x)^9$ is xyz then $x + y + z$ equals _______.

19. The sum of the last 4 digits of the natural number 3^{100} is _______.

20. The value of remainder when $(27)^{999}$ is divided by 7 is _______.

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Class XII

Inverse Trigonometric Functions

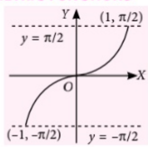
This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

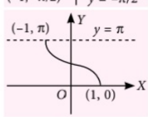
The inverse of a function $f: A \to B$ exists if f is one-one onto *i.e.*, a bijection and is given by $f(x) = y$ $\Rightarrow$ $f^{-1}(y) = x$.

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

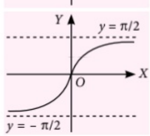
- Graph of $y = \sin^{-1}x$



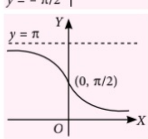
- Graph of $y = \cos^{-1}x$



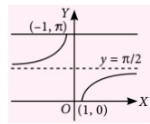
- Graph of $y = \tan^{-1}x$



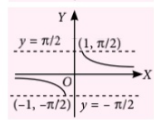
- Graph of $y = \cot^{-1}x$



- Graph of $y = \sec^{-1}x$



- Graph of $y = \text{cosec}^{-1}x$



DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Functions	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

PRINCIPAL VALUES OF INVERSE TRIGONOMETRIC FUNCTIONS

Principal value for $x \geq 0$	Principal value for $x < 0$
$0 \leq \sin^{-1} x \leq \frac{\pi}{2}$	$-\frac{\pi}{2} \leq \sin^{-1} x < 0$

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$0 \le \cos^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cos^{-1} x \le \pi$
$0 \le \tan^{-1} x < \frac{\pi}{2}$	$-\frac{\pi}{2} < \tan^{-1} x < 0$
$0 < \cot^{-1} x \le \frac{\pi}{2}$	$\frac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x < \frac{\pi}{2}$	$\frac{\pi}{2} < \sec^{-1} x \le \pi$
$0 < \mathrm{cosec}^{-1} x \le \frac{\pi}{2}$	$-\frac{\pi}{2} \le \mathrm{cosec}^{-1} x < 0$

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

- $\sin^{-1}(\sin x) = x, \forall -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
- $\cos^{-1}(\cos x) = x, \forall\ 0 \le x \le \pi$
- $\tan^{-1}(\tan x) = x, \forall -\frac{\pi}{2} < x < \frac{\pi}{2}$
- $\cot^{-1}(\cot x) = x, \forall\ 0 < x < \pi$
- $\sec^{-1}(\sec x) = x, \forall\ x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
- $\mathrm{cosec}^{-1}(\mathrm{cosec}\, x) = x, \forall\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
- $\sin(\sin^{-1}x) = x, \forall -1 \le x \le 1$
- $\cos(\cos^{-1}x) = x, \forall -1 \le x \le 1$
- $\tan(\tan^{-1}x) = x, \forall -\infty < x < \infty$
- $\cot(\cot^{-1}x) = x, \forall -\infty < x < \infty$
- $\sec(\sec^{-1}x) = x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\mathrm{cosec}(\mathrm{cosec}^{-1}x) = x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall -1 \le x \le 1$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall\ x \in R$
- $\sec^{-1} x + \mathrm{cosec}^{-1} x = \frac{\pi}{2}, \ \forall\ x \le -1 \text{ or } x \ge 1$
- $\sin^{-1}(-x) = -\sin^{-1}x \ , \forall -1 \le x \le 1$
- $\cos^{-1}(-x) = \pi - \cos^{-1}x, \forall -1 \le x \le 1$
- $\tan^{-1}(-x) = -\tan^{-1}x, \forall -\infty < x < \infty$
- $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall -\infty < x < \infty$
- $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall\ |x| \ge 1$
- $\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x, \forall\ |x| \ge 1$
- $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$
  $= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$
  $= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \mathrm{cosec}^{-1}\left(\frac{1}{x}\right)$
- $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$
  $= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{x}\right)$
  $= \mathrm{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
- $\tan^{-1} x = \sec^{-1}\sqrt{1+x^2} = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
  $= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \mathrm{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \cot^{-1}\left(\frac{1}{x}\right)$
- $\sin^{-1}\left(\frac{1}{x}\right) = \mathrm{cosec}^{-1}x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$
- $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$
- $\tan^{-1}x + \tan^{-1}y$
  $$= \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \forall\ xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \forall\ x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \forall\ x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$
- $\tan^{-1}x - \tan^{-1}y$
  $$= \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \forall\ xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \forall\ x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \forall\ x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$
- $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right),$
  $\forall\ x, y, z \ge 0$
- If $x_1, x_2, x_3, ..., x_n \in R$, then $\tan^{-1}x_1 + \tan^{-1}x_2 + ... + \tan^{-1} x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + ...}\right)$
  where S_k denotes the sum of the products of $x_1, x_2, ..., x_n$ taken k at a time.
- $\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy-1}{y+x}\right)$

- $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\dfrac{xy+1}{y-x}\right)$
- $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), \forall -1 \le x, y \le 1, \\ \qquad x^2+y^2 \le 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}), \forall 0 < x, y \le 1, \\ \qquad x^2+y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, \\ \qquad \forall -1 \le x, y < 0, x^2+y^2 > 1 \end{cases}$$

- $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}), \forall -1 \le x, y \le 1, \\ \qquad x^2+y^2 \le 1 \\ \pi - \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}), \forall 0 < x \le 1, \\ \qquad -1 \le y < 0,\ x^2+y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, \forall -1 \le x < 0, \\ \qquad 0 < y \le 1, x^2+y^2 > 1 \end{cases}$$

- $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}), \forall -1 \le x, y \le 1, \\ \qquad x+y \ge 0 \\ 2\pi - \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}), \forall -1 \le x, \\ \qquad y \le 1, x+y \le 0 \end{cases}$$

- $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}), \forall x \ge 0, y \ge 0, y \ge x \\ -\cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}), \forall x \ge 0, y \ge 0, y \le x \end{cases}$$

- $$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), \quad \forall -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), \forall \frac{1}{\sqrt{2}} < x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}), \forall -1 \le x < \frac{-1}{\sqrt{2}} \end{cases}$$

- $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), \forall -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), \forall \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), \forall -1 \le x < -\frac{1}{2} \end{cases}$$

- $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1), \forall 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), \forall -1 \le x \le 0 \end{cases}$$

- $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), \forall \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), \forall -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), \forall -1 \le x \le -\frac{1}{2} \end{cases}$$

- $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), \forall -1 \le x \le 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), \forall x < -1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), \forall x > 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), \forall x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right), \forall x < -1 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), \forall |x| < 1 \\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \forall 0 \le x < \infty \\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \forall -\infty < x \le 0 \end{cases}$$

- $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \forall -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \forall x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), \forall x < -\frac{1}{\sqrt{3}} \end{cases}$$

PROBLEMS

Single Correct Answer Type

1. $\tan[\sec^{-1}\sqrt{1+x^2}] =$

(a) $\dfrac{1}{x}$ (b) x

(c) $\dfrac{1}{\sqrt{1+x^2}}$ (d) $\dfrac{x}{\sqrt{1+x^2}}$

2. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]=$

(a) $\frac{\pi}{4}-\frac{x}{2}$ (b) $\frac{\pi}{4}+\frac{x}{2}$

(c) $\frac{x}{2}$ (d) $\frac{\pi}{4}-x$

3. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}=$

(a) $\frac{\pi}{2}+\operatorname{cosec}^{-1}x$ (b) $\frac{\pi}{2}+\sec^{-1}x$

(c) $\operatorname{cosec}^{-1}x$ (d) $\sec^{-1}x$

4. $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=$

(a) 5 (b) 13 (c) 15 (d) 6

5. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then x is equal to

(a) 1 (b) 0 (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

6. If $\sin^{-1}x=\theta+\beta$ and $\sin^{-1}y=\theta-\beta$, then $1+xy=$

(a) $\sin^2\theta+\sin^2\beta$ (b) $\sin^2\theta+\cos^2\beta$

(c) $\cos^2\theta+\cos^2\beta$ (d) $\cos^2\theta+\sin^2\beta$

7. If $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x$, then x is equal to

(a) 0 (b) $\frac{\sqrt{5}-4\sqrt{2}}{9}$

(c) $\frac{\sqrt{5}+4\sqrt{2}}{9}$ (d) $\frac{\pi}{2}$

8. $\tan(\cos^{-1}x)$ is equal to

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{1+x^2}$

(c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\sqrt{1-x^2}$

9. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right)$, $0\le x\le 1$ are

(a) $0,\pi$ (b) $0,\frac{\pi}{4}$ (c) $-\frac{\pi}{4},\frac{\pi}{4}$ (d) $\frac{\pi}{4},\frac{\pi}{2}$

10. The value of x which satisfies the equation $\tan^{-1}x=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(a) 3 (b) -3 (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

11. If $\theta=\sin^{-1}[\sin(-600°)]$, then one of the possible value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{-2\pi}{3}$

12. $\sin[\cot^{-1}(\cos\tan^{-1}x)]=$

(a) $\frac{x}{\sqrt{x^2+2}}$ (b) $\frac{x}{\sqrt{x^2+1}}$

(c) $\frac{1}{\sqrt{x^2+2}}$ (d) $\sqrt{\frac{x^2+1}{x^2+2}}$

13. If $\sin(\cot^{-1}(x+1))=\cos(\tan^{-1}x)$, then $x=$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{9}{4}$

14. $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=$

(a) 90° (b) 60° (c) 45° (d) $\tan^{-1}2$

15. $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]=$

(a) 6/17 (b) 17/6 (c) 7/16 (d) 16/7

16. If $\cot^{-1}\frac{3}{4}+\sin^{-1}\frac{5}{13}=\sin^{-1}(k)$, then $k=$

(a) $\frac{63}{65}$ (b) $\frac{12}{13}$ (c) $\frac{65}{68}$ (d) $\frac{5}{12}$

17. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then

(a) $x^2+y^2+z^2+xyz=0$

(b) $x^2+y^2+z^2+2xyz=0$

(c) $x^2+y^2+z^2+xyz=1$

(d) $x^2+y^2+z^2+2xyz=1$

18. $\cos\left[2\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}\right]=$

(a) $\frac{2\sqrt{6}}{5}$ (b) $-\frac{2\sqrt{6}}{5}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

19. $\cot^{-1}\frac{xy+1}{x-y}+\cot^{-1}\frac{yz+1}{y-z}+\cot^{-1}\frac{zx+1}{z-x}=$

(a) 0

(b) 1

(c) $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z$

(d) None of these

20. If $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$, then $x^2=$

(a) $2\sqrt{3}a$ (b) $\sqrt{3}a$ (c) $2\sqrt{3}a^2$ (d) $\sqrt{3}a^2$

21. $\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}=$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 0

22. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$, then $x=$

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$

(c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

23. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

24. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then $\cos^{-1}x+\cos^{-1}y=$

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

25. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $-\frac{3\pi}{4}$

26. If $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then $x=$

(a) 4 (b) 5 (c) 1 (d) 3

27. If $x^2+y^2+z^2=r^2$, then

$\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan\left(\frac{zx}{yr}\right)=$

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

28. If $a<\frac{1}{32}$, then the number of solution of $(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3$ is

(a) 0 (b) 1 (c) 2 (d) Infinite

29. If $\tan(x+y)=33$ and $x=\tan^{-1}3$, then y will be

(a) 0.3 (b) $\tan^{-1}(1.3)$

(c) $\tan^{-1}(0.3)$ (d) $\tan^{-1}\left(\frac{1}{18}\right)$

30. $\tan^{-1}\left(\frac{1}{11}\right)+\tan^{-1}\left(\frac{2}{12}\right)=$

(a) $\tan^{-1}\left(\frac{33}{132}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}\left(\frac{132}{33}\right)$ (d) $\tan^{-1}\left(\frac{17}{65}\right)$

31. If $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$, then $x=$

(a) $\sqrt{2}$ (b) 3 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

32. If $4\sin^{-1}x+\cos^{-1}x=\pi$, then x is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

Multiple Correct Answer Type

33. $2\cot^{-1}7+\cos^{-1}\frac{3}{5}$ is equal to

(a) $\cot^{-1}\left(\frac{44}{117}\right)$ (b) $\operatorname{cosec}^{-1}\left(\frac{125}{117}\right)$

(c) $\tan^{-1}\left(\frac{4}{117}\right)$ (d) $\cos^{-1}\left(\frac{44}{125}\right)$

34. If $0<x<1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to

(a) $\frac{1}{2}\cos^{-1}x$ (b) $\cos^{-1}\sqrt{\frac{1+x}{2}}$

(c) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (d) None of these

35. If the equation $\sin^{-1}(x^2+x+1)+\cos^{-1}(\lambda x+1)=\pi/2$ has exactly two distinct solutions, then value(s) of λ can be

(a) 0 (b) 1

(c) -1 (d) 2

36. If α is a real number for which $f(x)=\ln\cos^{-1}x$ is defined, then a possible value of $[\alpha]$ is (where [.] denotes greatest integer function)

(a) 0 (b) 1

(c) -1 (d) -2

37. If $f(x)=\sin^{-1}x+\cos^{-1}x$, then $\pi/2$ is equal to

(a) $f\left(-\frac{1}{2}\right)$ (b) $f(k^2-2k+3),\ k\in R$

(c) $f\left(\frac{1}{1+k^2}\right),\ k\in R$ (d) $f(-2)$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.

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38. **Statement-1 :** The number of solutions of the equation $\sin x+\cos x=\sin^{-1}x+\cos^{-1}x$ is zero.

Statement-2 : $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\ \forall x\in[-1,1]$ and the maximum value of $(\sin x+\cos x)$ is $\sqrt{2}$ for $x\in R$.

22. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$, then $x=$

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$

(c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

23. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

24. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y =$

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

25. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $-\frac{3\pi}{4}$

26. If $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then $x=$

(a) 4 (b) 5 (c) 1 (d) 3

27. If $x^2 + y^2 + z^2 = r^2$, then

$\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan\left(\frac{zx}{yr}\right)=$

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

28. If $a<\frac{1}{32}$, then the number of solution of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$ is

(a) 0 (b) 1 (c) 2 (d) Infinite

29. If $\tan(x + y) = 33$ and $x = \tan^{-1}3$, then y will be

(a) 0.3 (b) $\tan^{-1}(1.3)$

(c) $\tan^{-1}(0.3)$ (d) $\tan^{-1}\left(\frac{1}{18}\right)$

30. $\tan^{-1}\left(\frac{1}{11}\right)+\tan^{-1}\left(\frac{2}{12}\right)=$

(a) $\tan^{-1}\left(\frac{33}{132}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}\left(\frac{132}{33}\right)$ (d) $\tan^{-1}\left(\frac{17}{65}\right)$

31. If $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$, then $x=$

(a) $\sqrt{2}$ (b) 3 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

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(c) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (d) None of these

35. If the equation $\sin^{-1}(x^2+x+1)+\cos^{-1}(\lambda x+1)=\pi/2$ has exactly two distinct solutions, then value(s) of λ can be

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36. If α is a real number for which $f(x) = \ln \cos^{-1}x$ is defined, then a possible value of $[\alpha]$ is (where [.] denotes greatest integer function)

(a) 0 (b) 1

(c) −1 (d) −2

37. If $f(x) = \sin^{-1}x + \cos^{-1}x$, then $\pi/2$ is equal to

(a) $f\left(-\frac{1}{2}\right)$ (b) $f(k^2 - 2k + 3),\ k \in R$

(c) $f\left(\frac{1}{1+k^2}\right),\ k\in R$ (d) $f(-2)$

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

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Statement-2 : $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\ \forall x\in[-1,1]$ and the maximum value of $(\sin x + \cos x)$ is $\sqrt{2}$ for $x \in R$.

39. **Statement-1 :** If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi, n \in N$, then $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^3$

Statement-2 : $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}, \forall x \in [-1, 1]$

40. **Statement-1 :** If $\alpha \in (-\pi/2, 0)$, then the value of $2\tan^{-1}(\text{cosec}\alpha) + \tan^{-1}(2\sin\alpha \sec^2\alpha)$ is $-\pi$.

Statement-2 : If $x < 0$, then $\tan^{-1}x + \tan^{-1}\frac{1}{x} = -\frac{\pi}{2}$

41. **Statement-1 :** $\sec^{-1} 5 < \tan^{-1} 7$

Statement-2 : $\sec^{-1}x < \tan^{-1}x$, if $x \ge 1$ and $\sec^{-1}x > \tan^{-1}x$ if $x \le -1$ and $\tan^{-1}x_1 > \tan^{-1}x_2$ if $x_1 > x_2$.

42. **Statement-1 :** $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement-2 : $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall\ x, y \in (0, 1)$.

Comprehension Type

Paragraph for Question No. 43 to 45

It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$ where $x > 0$ and $B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$, where $t \in \left[0, \frac{1}{\sqrt{2}}\right]$ and $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \le x \le 1$ and

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for all $x \in R$.

43. The interval in which A lies is

(a) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right)$ (b) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right)$

(c) $\left[\frac{\pi^3}{40}, \frac{\pi^3}{10}\right)$ (d) None of these

44. The maximum value of B is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{16}$

(c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{6}$

45. If least value of A is λ and maximum value of B is μ, then $\cot^{-1}\cot\left(\frac{\lambda - \mu\pi}{\mu}\right) =$

(a) $\frac{\pi}{8}$ (b) $-\frac{\pi}{8}$

(c) $\frac{7\pi}{8}$ (d) $-\frac{7\pi}{8}$

Matrix-Match Type

46. If $[\cdot]$ represents greatest integer function, then match the following.

	Column-I		Column-II
A.	If $f(x) = \sin^{-1} x$ and $\lim_{x\to(1/2)^+} f(3x-4x^3) = a - 3 \lim_{x\to(1/2)^+} f(x)$, then $[a] =$	p.	2
B.	If $f(x) = \tan^{-1} g(x)$, where $g(x) = \frac{3x - x^3}{1-3x^2}$ and $\lim_{h\to 0}\frac{f(a+3h)-f(a)}{3h} = \frac{3}{1+a^2}$, when $\frac{-1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$, then $\left[\lim_{h\to 0}\frac{f\left(\frac{1}{2}+6h\right)-f\left(\frac{1}{2}\right)}{6h}\right] =$	q.	3
C.	If $\cos^{-1}(4x^3 - 3x) = a + b\cos^{-1}x$ for $-1 < x < \frac{-1}{2}$, then $[a + b + 2] =$	r.	-3
D.	If $f(x) = \cos^{-1}(4x^3 - 3x)$ and $\lim_{x\to(1/2)^+} f'(x) = a$ and $\lim_{x\to(1/2)^-} f'(x) = b$, then $[a + b - 3] =$	s.	-2

Numerical Value Type

47. The number of ordered pairs (x, y) satisfying the system of equations $(\cos^{-1}x)^2 + \sin^{-1}y = 1$ and $\cos^{-1}x + (\sin^{-1}y)^2 = 1$ is (are)

48. If $S = \sum_{r=1}^{50} \tan^{-1}\left(\frac{2r}{2+r^2+r^4}\right) = \tan^{-1}\left(\frac{\lambda}{\mu}\right)$, where $\left(\frac{\lambda}{\mu}\right)$ is in simplest form, then $\mu - \lambda =$

49. Let $P = \tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right] + \tan^{-1}\left[\frac{\tan\alpha}{4}\right]$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, then $\frac{P}{\alpha}$ equals

50. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then $|x|$ equals

SOLUTIONS

1. (b) : $\tan\left(\sec^{-1}\sqrt{1+x^2}\right) = \tan\left(\sec^{-1}\sqrt{1+\tan^2\theta}\right)$ (Putting $x = \tan\theta$)

$= \tan(\sec^{-1}(\sec\theta)) = \tan\theta = x$

2. (a) : $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]=\tan^{-1}\left[\frac{\sin(\pi/2-x)}{1+\cos(\pi/2-x)}\right]$

$=\tan^{-1}\left[\frac{2\sin(\pi/4-x/2)\cos(\pi/4-x/2)}{2\cos^2(\pi/4-x/2)}\right]$

$=\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{\pi}{4}-\frac{x}{2}$

3. (c) : $\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\text{cosec}^2\theta-1}}$ (Putting x = cosecθ)

$=\tan^{-1}\left(\frac{1}{\cot\theta}\right)=\theta=\text{cosec}^{-1}x$

4. (c) : Let $\tan^{-1} 2 = \alpha \Rightarrow \tan\alpha = 2$
and $\cot^{-1} 3 = \beta \Rightarrow \cot\beta = 3$
Now, $\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3)$
$= \sec^2\alpha + \text{cosec}^2\beta = 1 + \tan^2\alpha + 1 + \cot^2\beta$
$= 2 + (2)^2 + (3)^2 = 15$

5. (d) : We have, $\sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\frac{1}{5}=\frac{\pi}{2}-\cos^{-1}x=\sin^{-1}x \quad\Rightarrow\quad x=\frac{1}{5}$

6. (b) : Obviously $x = \sin(\theta + \beta)$ and $y = \sin(\theta - \beta)$
$\therefore \quad 1 + xy = 1 + \sin(\theta + \beta)\sin(\theta - \beta)$
$= 1 + \sin^2\theta - \sin^2\beta = \sin^2\theta + \cos^2\beta$

7. (c) : We have, $\sin^{-1}x=\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}$

$=\sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}}+\frac{2}{3}\sqrt{1-\frac{1}{9}}\right]=\sin^{-1}\left[\frac{\sqrt{5}+4\sqrt{2}}{9}\right]$

8. (a) : Let $\cos^{-1}x = \theta$. Then $x = \cos\theta$

$\Rightarrow \quad \tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{x^2}-1}=\frac{\sqrt{1-x^2}}{x}$

$\therefore \quad \tan(\cos^{-1}x)=\tan\theta=\frac{\sqrt{1-x^2}}{x}$

9. (b) : We have, $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}-\tan^{-1}x$

Since $0\le x\le 1\Rightarrow 0\le\tan^{-1}x\le\frac{\pi}{4}$

$\Rightarrow \quad 0\ge-\tan^{-1}x\ge\frac{-\pi}{4}\Rightarrow\frac{\pi}{4}\ge\frac{\pi}{4}-\tan^{-1}x\ge 0$

$\Rightarrow \quad \frac{\pi}{4}\ge\tan^{-1}\left(\frac{1-x}{1+x}\right)\ge 0$

10. (a) : Given, $\tan^{-1}x=\sin^{-1}\left[\frac{3}{\sqrt{10}}\right]$

$\Rightarrow \quad x=\tan\left\{\sin^{-1}\left[\frac{3}{\sqrt{10}}\right]\right\}=\tan\{\tan^{-1}3\}\Rightarrow x=3$

11. (a) : $\theta = \sin^{-1}[\sin(-600°)]$
$\Rightarrow \quad \theta = \sin^{-1}[-(\sin 240°)] = \sin^{-1}[-\sin(180° + 60°)]$

$\Rightarrow \quad \theta=\sin^{-1}(\sin 60°)=\sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right]=\frac{\pi}{3}$

12. (d)

13. (a) : $\sin[\cot^{-1}(x+1)]=\sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right)$

$=\frac{1}{\sqrt{x^2+2x+2}}$ ...(i)

and $\cos(\tan^{-1}x)=\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}$ ...(ii)

From (i) and (ii), $x=-\frac{1}{2}$

14. (d) : $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\left(\frac{1}{2}\right)$

$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{3}{4}=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{3}{4}}{1-\frac{1}{2}\times\frac{3}{4}}\right)=\tan^{-1}(2)$

15. (b) : $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]$

$=\tan\left[\tan^{-1}\frac{\sqrt{\left(1-\frac{16}{25}\right)}}{\frac{4}{5}}+\tan^{-1}\frac{2}{3}\right]$

$=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)=\tan\left(\tan^{-1}\frac{17}{6}\right)=\frac{17}{6}$

16. (a) : Let $\cot^{-1}\frac{3}{4}=\theta\Rightarrow\cot\theta=\frac{3}{4}$

and $\sin\theta=\frac{1}{\sqrt{1+\cot^2\theta}}=\frac{1}{\sqrt{1+(9/16)}}=\frac{4}{5}$

Hence, $\cot^{-1}\frac{3}{4}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}$

$=\sin^{-1}\left[\frac{4}{5}.\sqrt{1-\frac{25}{169}}+\frac{5}{13}.\sqrt{1-\frac{16}{25}}\right]=\sin^{-1}\frac{63}{65}$

17. (d) : Given that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$
$\Rightarrow \quad \cos^{-1}(x) + \cos^{-1}(y) = \cos^{-1}(-1) - \cos^{-1}(z)$

$\Rightarrow \quad \cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})=\cos^{-1}\{(-1)(z)\}$

$\Rightarrow \quad (xy+z)=\sqrt{(1-x^2)(1-y^2)}$

Squaring both sides, we get $x^2 + y^2 + z^2 + 2xyz = 1$.

Alternate Method

Put $x = y = z = \frac{1}{2}$, so that $\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\pi$

Obviously option (d) holds for these values of x, y and z.

18. (b) : $\cos\left(\cos^{-1}\frac{1}{5}+\sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}\right)$

$=\cos\left(\frac{\pi}{2}+\cos^{-1}\frac{1}{5}\right)$

$=-\sin\left(\cos^{-1}\frac{1}{5}\right)=-\sin\left(\sin^{-1}\sqrt{\frac{24}{25}}\right)=-\frac{2\sqrt{6}}{5}$

19. (a) : $\cot^{-1}\frac{xy+1}{x-y}+\cot^{-1}\frac{yz+1}{y-z}+\cot^{-1}\frac{zx+1}{z-x}$

$= \cot^{-1}y - \cot^{-1}x + \cot^{-1}z - \cot^{-1}y + \cot^{-1}x - \cot^{-1}z = 0$

20. (c) : We have, $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$

$\Rightarrow \tan^{-1}\left(\frac{\frac{a+x}{a}+\frac{a-x}{a}}{1-\frac{a+x}{a}.\frac{a-x}{a}}\right)=\frac{\pi}{6}$

$\Rightarrow \frac{2a^2}{x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}} \Rightarrow x^2=2\sqrt{3}a^2$

21. (b) : $\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}$

$=\tan^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$=\tan^{-1}(\cot 2\theta)+\cos^{-1}(\cos 2\theta)=\frac{\pi}{2}-2\theta+2\theta=\frac{\pi}{2}$

22. (d) : $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$

$\Rightarrow \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\sin^{-1}\left(\frac{2\tan\phi}{1+\tan^2\phi}\right)=2\tan^{-1}x$

[Putting $a = \tan\theta$ and $b = \tan\phi$]

$\Rightarrow \sin^{-1}\sin(2\theta) + \sin^{-1}\sin(2\phi) = 2\tan^{-1}x$

$\Rightarrow 2(\theta + \phi) = 2\tan^{-1}x$

Hence, $x = \tan(\theta + \phi) \Rightarrow x=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{a+b}{1-ab}$

23. (c) : $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$=2\tan^{-1}\left[\frac{\frac{2}{5}}{1-\frac{1}{25}}\right]+\tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}.\frac{1}{70}}\right]$

$=2\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{-29}{6931}\right)$

$=\tan^{-1}\left[\frac{\frac{5}{6}}{1-\frac{25}{144}}\right]-\tan^{-1}\left(\frac{29}{6931}\right)$

$=\tan^{-1}\frac{120}{119}-\tan^{-1}\frac{1}{239}=\tan^{-1}(1)=\frac{\pi}{4}$

24. (b) : $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$

$\Rightarrow \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x+\cos^{-1}y=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$

25. (c) : $\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

$=\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{1-y/x}{1+y/x}\right)$

$=\tan^{-1}\frac{x}{y}-\left(\tan^{-1}1-\tan^{-1}\frac{y}{x}\right)=\tan^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}-\frac{\pi}{4}$

$=\tan^{-1}\frac{x}{y}+\cot^{-1}\frac{x}{y}-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$

26. (d) : $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{x}{5}=\frac{\pi}{2}-\sin^{-1}\frac{4}{5} \Rightarrow \sin^{-1}\frac{x}{5}=\cos^{-1}\frac{4}{5}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right)=\sin^{-1}\frac{3}{5} \Rightarrow x=3$

27. (b) : $\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)$

$=\tan^{-1}\left[\frac{\frac{xy}{zr}+\frac{yz}{xr}+\frac{xz}{yr}-\frac{xyz}{r^3}}{1-\left(\frac{x^2+y^2+z^2}{r^2}\right)}\right]=\tan^{-1}\infty=\frac{\pi}{2}$

28. (a) : We know, $\frac{\pi^3}{32}\le(\sin^{-1}x)^3+(\cos^{-1}x)^3\le\frac{7\pi^3}{8}$

Here $a<\frac{1}{32}$. So, number of solution is zero.

29. (c) : $x + y = \tan^{-1} 33$

$\Rightarrow y = \tan^{-1} 33 - \tan^{-1}3$ [$\because x = \tan^{-1}3$]

$\Rightarrow y=\tan^{-1}\left(\frac{33-3}{1+99}\right)=\tan^{-1}\left(\frac{30}{100}\right) \Rightarrow y=\tan^{-1}(0.3)$

30. (d) : $\tan^{-1}\left(\frac{1}{11}\right)+\tan^{-1}\left(\frac{2}{12}\right)$

$$= \tan^{-1}\left(\frac{12+22}{12\times 11-2}\right) = \tan^{-1}\left(\frac{17}{65}\right)$$

31. (c) : The given equation may be written as

$$\tan^{-1}x + \cot^{-1}x + \cot^{-1}x = \frac{2\pi}{3}$$

$$\Rightarrow \cot^{-1}x = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6} \Rightarrow x = \sqrt{3}$$

32. (b) : We have, $4\sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow 3\sin^{-1}x + \sin^{-1}x + \cos^{-1}x = \pi$

$$\Rightarrow 3\sin^{-1}x = \pi - \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow \sin^{-1}x = \pi/6 \Rightarrow x = \sin\frac{\pi}{6} = \frac{1}{2}$$

33. (a, b, d) : $2\cot^{-1}7 + \cos^{-1}\frac{3}{5} = \cos^{-1}\left(\frac{44}{125}\right)$

34. (a,b,c): For $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$, put $x = \cos\theta, 0 < \theta < \frac{\pi}{2}$

35. (a) : $x^2 + x + 1 = \lambda x + 1, -1 \le x^2 + x + 1 \le 1$

$x = 0$ or $x = \lambda - 1, x^2 + x \le 0, x^2 + x + 2 \ge 0$

$\Rightarrow -1 \le x \le 0$ and $\forall\, x \in R$

$\because\; x = 0$ is one solution

$-1 \le x < 0 \Rightarrow -1 \le \lambda - 1 < 0 \Rightarrow 0 \le \lambda < 1$

36. (a, c) : Domain of $f(x) = \ln\, \cos^{-1}x$ is $x \in [-1, 1)$

$\therefore\; [\alpha] = -1$ or 0

37. (a, c) : $f(x) = \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \forall -1 \le x \le 1$

$$\because\; -\frac{1}{2} \in [-1,1] \Rightarrow f\left(-\frac{1}{2}\right) = \frac{\pi}{2}$$

$$0 < \frac{1}{1+k^2} \le 1 \forall k \in R \Rightarrow f\left(\frac{1}{1+k^2}\right) = \frac{\pi}{2}$$

$k^2 - 2k + 3 = (k-1)^2 + 2 \ge 2\ \forall\ k \in R$

38. (a) : The maximum value of $(\sin x + \cos x)$ is $\sqrt{2}$ while $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\ \forall\ x \in [-1, 1]$ so the equation $\sin x + \cos x = \sin^{-1}x + \cos^{-1}x$ has no solution.

39. (a) : $\sum_{i=1}^{2n} \sin^{-1}x_i = n\pi$ is possible if

$x_1 = x_2 = ... = x_{2n} = 1$

$$\sum_{i=1}^{n} x_i = 1+1+1+......n \text{ terms} = n$$

$$\sum_{i=1}^{n} x_i^2 = 1^2 + 1^2 + 1^2 + ...n \text{ terms} = n.$$

$$\sum_{i=1}^{n} x_i^3 = 1^3 + 1^3 + 1^3 + ...n \text{ terms} = n$$

40. (a) : Let $\sin\alpha = t \in (-1, 0)$

So, $2\tan^{-1}\left(\frac{1}{t}\right) + \tan^{-1}\left(\frac{2t}{1-t^2}\right)$

$$= 2\left[\tan^{-1}\left(\frac{1}{t}\right) + \tan^{-1}(t)\right] = 2\left(-\frac{\pi}{2}\right) = -\pi$$

41. (a) : $\tan^{-1}x$ is an increasing function.

Also, $\sec^{-1}x = \tan^{-1}\left(\sqrt{x^2-1}\right)$, for $x \ge 1$

Now, $\sec^{-1}x < \tan^{-1}x$ if $\tan^{-1}\left(\sqrt{x^2-1}\right) < \tan^{-1}x$

i.e., $\sqrt{x^2-1} < x$ *i.e.,* $x^2 - 1 < x^2$, which is true.

42. (a) : $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$

$$\left[\because\; x > y, \frac{x}{\sqrt{1-x^2}} > x\right]$$

$\therefore$ Statement -2 is true.

We know that $e < \pi \Rightarrow \frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

So, by statement-2

$$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$$

$\therefore$ Statement-1 is true.

43. (b) : $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$

$\Rightarrow A = (\tan^{-1}x + \cot^{-1}x)^3 - 3\tan^{-1}x\cot^{-1}x\,(\tan^{-1}x + \cot^{-1}x)$

$$\Rightarrow A = \left(\frac{\pi}{2}\right)^3 - 3\tan^{-1}x\cot^{-1}x \cdot \frac{\pi}{2}$$

$$\Rightarrow A = \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\tan^{-1}x - \frac{\pi}{4}\right)^2$$

As $x > 0\ \therefore\ \frac{\pi^3}{32} \le A < \frac{\pi^3}{8}$

44. (c) : $B = (\sin^{-1}t)^2 + (\cos^{-1}t)^2$

$\Rightarrow B = (\sin^{-1}t + \cos^{-1}t)^2 - 2\sin^{-1}t\cos^{-1}t$

$$\Rightarrow B = \frac{\pi^2}{4} - 2\sin^{-1}t\left(\frac{\pi}{2} - \sin^{-1}t\right)$$

$$\Rightarrow B = \frac{\pi^2}{8} + 2\left(\sin^{-1}t - \frac{\pi}{4}\right)^2$$

$$\Rightarrow B_{\max} = \frac{\pi^2}{8} + 2.\frac{\pi^2}{16} = \frac{\pi^2}{4}$$

45. (a) : We have, $\lambda = \frac{\pi^3}{32}$ and $\mu = \frac{\pi^2}{4}$

$$\therefore\; \frac{\lambda}{\mu} = \frac{\pi}{8} \text{ and } \frac{\lambda - \mu\pi}{\mu} = \frac{\pi}{8} - \pi = \frac{-7\pi}{8}$$

$$\therefore\; \cot^{-1}\cot\left(\frac{\lambda-\mu\pi}{\mu}\right) = \cot^{-1}\cot\left(-\frac{7\pi}{8}\right) = \frac{\pi}{8}$$

46. $A \to q; B \to p; C \to s; D \to r$

(A) $\sin^{-1}(3x-4x^3)=\pi-3\sin^{-1}x$, if $\frac{1}{2}<x\le 1$

$\therefore \lim\limits_{x\to(1/2)^+} f(3x-4x^3)=\lim\limits_{x\to(1/2)^+}(\pi-3\sin^{-1}x)$

$=\pi-3\lim\limits_{x\to(1/2)^+}\sin^{-1}x$

$\therefore\ a=\pi \Rightarrow [a]=3$

(B) $f(x)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=3\tan^{-1}x$,

when $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

If $-\frac{1}{\sqrt{3}}<a<\frac{1}{\sqrt{3}}$,

then $\lim\limits_{h\to 0}\frac{f(a+3h)-f(a)}{3h}=\frac{3}{1+a^2} \Rightarrow f'(a)=\frac{3}{1+a^2}$

Now, $\lim\limits_{h\to 0}\frac{f\left(\frac{1}{2}+6h\right)-f\left(\frac{1}{2}\right)}{6h}=f'(1/2)=\frac{12}{5}$

$\therefore$ Required value = 2

(C) $\cos^{-1}(4x^3-3x)=\cos^{-1}(\cos 3\theta)=3\theta-2\pi$ $[\because 2\pi/3<\theta<\pi]$

$=-2\pi+3\cos^{-1}x$

$\therefore\ [a+b+2]=[-2\pi+3+2]=-2$

(D) $f(x)=\cos^{-1}(4x^3-3x)=\cos^{-1}(\cos 3\theta)$

$$=\begin{cases}3\theta, & 0<\theta<\frac{\pi}{3}\\ 2\pi-3\theta, & \frac{\pi}{3}<\theta<\frac{\pi}{2}\end{cases}=\begin{cases}3\cos^{-1}x, & \frac{1}{2}<x<1\\ 2\pi-3\cos^{-1}x, & 0<x<\frac{1}{2}\end{cases}$$

$$\therefore\ f'(x)=\begin{cases}\frac{-3}{\sqrt{1-x^2}}, & \frac{1}{2}<x<1\\ \frac{3}{\sqrt{1-x^2}}, & 0<x<\frac{1}{2}\end{cases}$$

$a=\lim\limits_{x\to(1/2)^+}f'(x)=-2\sqrt{3}$, $b=\lim\limits_{x\to(1/2)^-}f'(x)=2\sqrt{3}$

$\therefore\ [a+b-3]=-3$

47. (3) : Let $a=\cos^{-1}x$, $b=\sin^{-1}y$

$\therefore\ a^2+b=1$ and $a+b^2=1$

$\Rightarrow a=b$ or $a=1, b=0$ or $a=0, b=1$

If $a=b$, then $a^2+a-1=0 \Rightarrow a=\frac{-1\pm\sqrt{5}}{2}$

$a\in[0,\pi], b\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right] \Rightarrow a=b=\frac{\sqrt{5}-1}{2}$

$\Rightarrow (x,y)=\left(\cos\left(\frac{\sqrt{5}-1}{2}\right), \sin\left(\frac{\sqrt{5}-1}{2}\right)\right)$

$a=1, b=0 \Rightarrow (x,y)=(\cos 1, 0)$

$a=0, b=1 \Rightarrow (x,y)=(1, \sin 1)$

48. (1) : $S=\sum\limits_{r=1}^{50}(\tan^{-1}(1+r+r^2)-\tan^{-1}(1-r+r^2))$

$=\tan^{-1}(2551)-\tan^{-1}1=\tan^{-1}\left(\frac{2550}{2552}\right)=\tan^{-1}\left(\frac{1275}{1276}\right)$

$\therefore\ \lambda=1275, \mu=1276 \Rightarrow \mu-\lambda=1$

49. (1) : $P=\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\frac{\tan\alpha}{4}\right]$

$=\tan^{-1}\left(\frac{6\tan\alpha}{8+2\tan^2\alpha}\right)+\tan^{-1}\left(\frac{\tan\alpha}{4}\right)$

$=\tan^{-1}\left(\frac{\frac{3\tan\alpha}{4+\tan^2\alpha}+\frac{\tan\alpha}{4}}{1-\frac{3\tan^2\alpha}{16+4\tan^2\alpha}}\right)$ $\left[\because \frac{3\tan^2\alpha}{16+4\tan^2\alpha}<1\right]$

$=\tan^{-1}(\tan\alpha)=\alpha$

50. (1) : $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1}x+\cot^{-1}x)^2-2\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{5\pi^2}{8}$

$\Rightarrow \frac{\pi^2}{4}-\pi\tan^{-1}x+2(\tan^{-1}x)^2=\frac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0$

$\therefore\ \tan^{-1}x=\frac{3\pi}{4}, -\frac{\pi}{4}$

$\Rightarrow \tan^{-1}x=-\frac{\pi}{4}$ $\left[\because \tan^{-1}x\ne\frac{3\pi}{4}\right]$

$\therefore\ x=-1$ is the solution.

CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 5 — Integrals and Application of Integrals

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Evaluate :
$\int (3\sin x - 2\cos x + 4\sec^2 x - 5\,\text{cosec}^2 x)\,dx$
(a) $-3\cos x - 2\sin x + 4\tan x + 5\cot x + C$
(b) $3\cos x + 2\sin x + 4\tan x + 5\cot x + C$
(c) $3\cos x + 2\sin x - 4\tan x - 5\cot x + C$
(d) $-3\cos x - 2\sin x - 4\tan x - 5\cot x + C$

2. $\int x e^{x^2} dx$ is equal to
(a) $-\dfrac{e^{x^2}}{2}+C$ (b) $\dfrac{e^{x^2}}{2}+C$
(c) $\dfrac{e^{x}}{2}+C$ (d) $-\dfrac{e^{x}}{2}+C$

3. Evaluate : $\int \cos^3 x\, e^{\log \sin x} dx$
(a) $\dfrac{\cos^4 x}{4}+C$ (b) $-\dfrac{\cos^4 x}{4}+C$
(c) $\dfrac{\cos^4 x}{4x}+C$ (d) None of these

4. $\int \dfrac{a}{(1+x^2)\tan^{-1} x}\,dx =$
(a) $a\log|\tan^{-1}x| + C$ (b) $\dfrac{a}{2}(\tan^{-1}x)^2+C$
(c) $a\log(1+x^2)+C$ (d) None of these

5. $\int_1^3 x^2 \log x\, dx =$
(a) $3\log 3 - \dfrac{26}{3}$ (b) $3\log 3 - \dfrac{26}{9}$
(c) $9\log 3 - \dfrac{26}{3}$ (d) $9\log 3 - \dfrac{26}{9}$

6. Evaluate : $\int_0^1 \left\{e^x + \sin\dfrac{\pi x}{4}\right\} dx$
(a) $1-\dfrac{4}{\pi}+\dfrac{2\sqrt{2}}{\pi}$ (b) $1+\dfrac{2}{\pi}-\dfrac{2\sqrt{2}}{\pi}$
(c) $1+\dfrac{4}{\pi}-\dfrac{2\sqrt{2}}{\pi}$ (d) None of these

7. Evaluate : $\int \frac{dx}{x+\sqrt{x}}$

(a) $\log(\sqrt{x}+1)+C$ (b) $\log(x+1)+C$

(c) $2\log(\sqrt{x}+1)+C$ (d) None of these

8. Evaluate : $\int (2x+1)\sqrt{x^2+x+1}dx$

(a) $\frac{1}{3}(x^2+x+1)^3+C$ (b) $\frac{1}{2}(x^2+x+1)^2+C$

(c) $\frac{2}{3}(x^2+x+1)^{3/2}+C$

(d) None of these.

9. Evaluate: $\int_0^{\pi/2} e^x(\sin x-\cos x)dx$

(a) -1 (b) 1

(c) 0 (d) 2

10. If $f(2-a)=f(2+a)\ \forall\ a \in R$, then $\int_{2-a}^{2+a} f(x)\,dx$ is equal to

(a) $2\int_2^{2+a} f(x)\,dx$ (b) $2\int_0^a f(x)\,dx$

(c) $2\int_0^2 f(x)\,dx$ (d) None of these

(Q. 11-Q.15) Fill in the blanks.

11. If $I=\int \sqrt{1+\sin 2x}dx$, then value of I is ______ .

OR

If $I=\int \frac{1+\tan x}{x+\log\sec x}dx$, then value of I is ______ .

12. Value of the integral $\int 9^{\log_3 x}dx$ is ______ .

13. Value of the integral $\int \sec^4 x\tan x\,dx$ is ______ .

OR

Value of the integral $\int_{-\pi/4}^{\pi/4} x^3\sin^4 x dx$ is ______ .

14. Value of the integral $\int \frac{2x\tan^{-1}(x^2)}{1+x^4}dx$ is ______ .

15. Value of the integral $\int \frac{\sin x}{1+\cos x}dx$ is ______ .

(Q.16-Q.20) Answer the following questions.

16. Evaluate : $\int \frac{4x-3}{\sqrt[3]{2x^2-3x+7}}dx$

OR

Find the value of $\int_1^2 |x-3|\,dx$.

17. Evaluate : $\int \frac{x^3-x^2+x-1}{x-1}dx$

18. Evaluate : $\int \left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

19. Find the area bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

20. Find the area bounded by the curve $y=\sec^2 x$, $y=0$ and $|x|=\frac{\pi}{3}$.

SECTION - B

21. Evaluate : $\int_{-1}^{1} 5x^4\sqrt{x^5+1}dx$

22. Show that $\int_0^1 \log\left(\frac{1}{x}-1\right)dx=0$.

OR

Evaluate : $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}dx$

23. Evaluate : $\int \frac{dx}{5-8x-x^2}$

24. Using integration, find the area of the region bounded by the line $2y + x = 8$, the x-axis and the lines $x = 3$ and $x = 5$.

OR

Evaluate : $\int_0^1 \frac{xe^x}{(x+1)^2}dx$

25. Find the value of $\int e^{-x}\operatorname{cosec}^2(2e^{-x}+5)dx$.

26. Evaluate : $\int \frac{\cos 2x+2\sin^2 x}{\cos^2 x}dx$

SECTION - C

27. Evaluate : $\int \left[\log(\log x)+\frac{1}{(\log x)^2}\right]dx$

28. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

OR

Find the area bounded by the curves $y = \cos x$ and $y = \sin x$ between the lines $x = 0$ and $x = 3\pi/2$.

29. Evaluate : $\int_1^4 \{|x-1|+|x-2|+|x-4|\}dx$

30. Evaluate : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

31. Evaluate : $\int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

OR

Evaluate : $\int (2x+3)\sqrt{x^2+4x+3}\,dx$

32. Find the area of the region bounded by the curve $y = x^2 - 6x + 5$, the x-axis and the lines $x = 2$, $x = 4$.

SECTION-D

33. Evaluate : $\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}dx$

OR

Evaluate : $\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx$

34. Using integration, find the area of the triangle ABC whose vertices are $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.

35. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.

OR

Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2x - x^2$, $y = 2^x$.

36. Evaluate : $\int \sqrt{\tan x}\,dx$

SOLUTIONS

1. **(a) :** Let

$I = \int (3\sin x - 2\cos x + 4\sec^2 x - 5\ \text{cosec}^2 x)\ dx$

$\Rightarrow\ I = 3\int \sin x\,dx - 2\int \cos x\,dx + 4\int \sec^2 x\,dx$
$\qquad - 5\int \text{cosec}^2 x\,dx$

$\Rightarrow\ I = -3\cos x - 2\sin x + 4\tan x + 5\cot x + C$

2. **(b) :** Let $I = \int xe^{x^2}dx$

Put $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$

$\therefore\ I = \frac{1}{2}\int e^t dt = \frac{e^t}{2} + C = \frac{e^{x^2}}{2} + C$

3. **(b) :** Let, $I = \int \cos^3 x\, e^{\log \sin x}dx = \int \cos^3 x \sin x\,dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$\therefore\ I = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$

4. **(a) :** Let $I = a\int \frac{dx}{(1+x^2)\tan^{-1} x}$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore\ I = a\int \frac{dt}{t} = a\log|t| + C = a\log|\tan^{-1}x| + C$

5. **(d) :** $\int_1^3 x^2 \log x\,dx$

$= \left[(\log x)\left(\frac{x^3}{3}\right)\right]_1^3 - \int_1^3 \frac{1}{x}\cdot\frac{x^3}{3}dx$

$= 9\log 3 - 0 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^3 = 9\log 3 - \frac{26}{9}$

6. **(d) :** Let $I = \int_0^1 \left\{e^x + \sin\frac{\pi x}{4}\right\}dx$

$= \left[e^x\right]_0^1 + \frac{4}{\pi}\left[-\cos\frac{\pi}{4}x\right]_0^1 = e - 1 - \frac{4}{\sqrt{2}\pi} + \frac{4}{\pi}$

7. **(c) :** Let $I = \int \frac{dx}{x+\sqrt{x}} \Rightarrow I = \int \frac{dx}{(\sqrt{x}+1)\sqrt{x}}$

Put $(\sqrt{x}+1) = t \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\therefore\ I = \int \frac{2dt}{t} = 2\log|t| + C = 2\log(\sqrt{x}+1) + C$

8. **(c) :** Let $I = \int (2x+1)\sqrt{x^2+x+1}\,dx$

Put $x^2 + x + 1 = t \Rightarrow (2x + 1)dx = dt$

$\therefore\ I = \int \sqrt{t}\,dt = \frac{2}{3}t^{3/2} + C = \frac{2}{3}(x^2+x+1)^{3/2} + C$

9. **(b) :** Let $I = \int_0^{\pi/2} e^x(\sin x - \cos x)dx$

$= -\int_0^{\pi/2} e^x\{\cos x + (-\sin x)\}dx$

$\therefore\ I = -[e^x \cos x\,dx]_0^{\pi/2}$

$[\because \int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C]$

$= -(e^{\pi/2} \times 0) + (e^0 \times 1) = 1$

10. **(a) :** $\because\ f(2 - a) = f(2 + a)$

$\therefore$ Function is symmetrical about the line $x = 2$

Then, $\int_{2-a}^{2+a} f(x)dx = 2\int_2^{2+a} f(x)dx$

11. We have, $I = \int \sqrt{1+\sin 2x}\,dx$

$\therefore\ I = \int \sqrt{(\cos x+\sin x)^2}\,dx$

$= \int (\cos x + \sin x)dx = \sin x - \cos x + C$

OR

We have, $I=\int \frac{1+\tan x}{x+\log \sec x}dx$

Put $x + \log \sec x = t \Rightarrow (1 + \tan x)dx = dt$

$\therefore \quad I=\int \frac{dt}{t}=\log|t|+C=\log|x+\log \sec x|+C$

12. We have, $\int 9^{\log_3 x}dx=\int (3^2)^{\log_3 x}dx=\int 3^{2\log_3 x}dx$

$=\int 3^{\log_3 x^2}dx=\int x^2 dx=\frac{1}{3}x^3+C.$

13. Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$\therefore \quad \int \sec^4 x \tan x\, dx=\int (\sec x)^3 \sec x \tan x\, dx=\int t^3\, dt$

$=\frac{t^4}{4}+C=\frac{1}{4}\sec^4 x+C.$

OR

Let $f(x) = x^3 \sin^4 x$

Then, $f(-x) = (-x)^3 \sin^4(-x) = -x^3(-\sin x)^4 = -f(x)$

$\therefore \quad f(x)$ is an odd function.

So, $\int_{-\pi/4}^{\pi/4} f(x)dx=0$.

14. Let $I=\int \frac{2x\tan^{-1}(x^2)}{1+x^4}dx$

Put $\tan^{-1}(x^2) = t \quad \Rightarrow \quad \frac{2x}{1+x^4}dx=dt$

$\therefore \quad I=\int t dt=\frac{t^2}{2}+C=\frac{\left[\tan^{-1}(x^2)\right]^2}{2}+C$

15. Let $I=\int \frac{\sin x}{1+\cos x}dx$

Put $\cos x + 1 = t \quad \Rightarrow \quad -\sin x\, dx = dt$

$\therefore \quad I=\int \frac{-dt}{t}=-\log|1+\cos x|+C$

16. Put $2x^2 - 3x + 7 = t \Rightarrow (4x - 3)\, dx = dt$

$\therefore \quad \int \frac{4x-3}{\sqrt[3]{2x^2-3x+7}}dx=\int \frac{1}{t^{1/3}}dt=\int t^{-1/3}dt=\frac{t^{2/3}}{\frac{2}{3}}+C$

$=\frac{3}{2}(2x^2-3x+7)^{2/3}+C.$

OR

We have, $\int_1^2 |x-3|\,dx=\int_1^2 -(x-3)dx$

$[\because\ 1 < x < 2 \Rightarrow x - 3 < 0]$

$=-\left[\frac{x^2}{2}-3x\right]_1^2=-\left[\left(\frac{2^2}{2}-3\times 2\right)-\left(\frac{1}{2}-3\right)\right]=\frac{3}{2}.$

17. Let $I=\int \frac{x^3-x^2+x-1}{x-1}dx$

$=\int \frac{x^2(x-1)+1(x-1)}{x-1}=\int \frac{(x^2+1)(x-1)}{x-1}dx$

$=\int (x^2+1)dx=\frac{1}{3}x^3+x+C$

18. We have $\int\left(5x^3+2x^{-5}-7x+\frac{1}{\sqrt{x}}+\frac{5}{x}\right)dx$

$=5\int x^3dx+2\int x^{-5}dx-7\int xdx+\int x^{-1/2}dx+5\int \frac{1}{x}dx$

$=5\cdot\frac{x^4}{4}+2\cdot\frac{x^{-4}}{(-4)}-7\cdot\frac{x^2}{2}+\frac{x^{1/2}}{(1/2)}+5\log|x|+C$

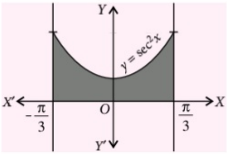
$=\frac{5x^4}{4}-\frac{1}{2x^4}-\frac{7x^2}{2}+2\sqrt{x}+5\log|x|+C$

19. Here, $a^2 = 4$ and $b^2 = 9$.

We know that, area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq. units.

$\therefore$ Required area $= \pi \times 3 \times 2 = 6\pi$ sq. units.

20. We have, $y = \sec^2 x$ and $y = 0$ and $x=\frac{\pi}{3},-\frac{\pi}{3}$



Required area = area of shaded region

$=\int_{-\pi/3}^{\pi/3} \sec^2 x\, dx=[\tan x]_{-\pi/3}^{\pi/3}=2\sqrt{3}$ sq. units

21. Let $I=\int_{-1}^{1} 5x^4\sqrt{x^5+1}dx$

Put $x^5 + 1 = t \Rightarrow 5x^4dx = dt$

When $x = -1$, $t = 0$ and when $x = 1$, $t = 2$

$\therefore \quad I=\int_0^2 (t)^{1/2}dt=\frac{2}{3}\left[(t)^{3/2}\right]_0^2=\frac{4\sqrt{2}}{3}$

22. Let $I=\int_0^1 f(x)dx$, where

$f(x)=\log\left(\frac{1}{x}-1\right)=\log\left(\frac{1-x}{x}\right)$

Then, $f(1-x) = \log\left(\frac{1-(1-x)}{1-x}\right) = \log\frac{x}{1-x}$

$= \log\left[\left(\frac{1-x}{x}\right)^{-1}\right] = -\log\left(\frac{1-x}{x}\right) = -f(x)$

$\because \quad f(1-x) = -f(x) \quad \therefore \int_0^1 f(x)dx = 0.$

OR

We have, $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}dx$

$= \int\left(\frac{1}{\sqrt{2x+3}+\sqrt{2x-3}} \times \frac{\sqrt{2x+3}-\sqrt{2x-3}}{\sqrt{2x+3}-\sqrt{2x-3}}\right)dx$

$= \int \frac{\sqrt{2x+3}-\sqrt{2x-3}}{(2x+3)-(2x-3)}dx = \frac{1}{6}\int((2x+3)^{1/2}-(2x-3)^{1/2})dx$

$= \frac{1}{6}\left(\frac{(2x+3)^{3/2}}{\frac{3}{2}.2} - \frac{(2x-3)^{3/2}}{\frac{3}{2}.2}\right) + C$

$= \frac{1}{18}[(2x+3)^{3/2} - (2x-3)^{3/2}] + C.$

23. Let $I = \int \frac{dx}{5-8x-x^2} = \int \frac{dx}{5+16-16-8x-x^2}$

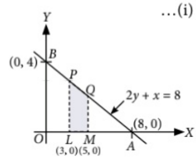
$= \int \frac{dx}{21-(x+4)^2} = \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$

$= \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$

24. The given line is

$2y + x = 8 \Rightarrow y = 4 - \frac{1}{2}x$ ...(i)

The area between the line $y = 4 - \frac{1}{2}x$, and the x-axis between $x = 3$ and $x = 5$ is represented by region $PLMQ$.



Now, required area $= \int_3^5 y\,dx = \int_3^5\left(4 - \frac{1}{2}x\right)dx$

$= \left[4x - \frac{1}{4}x^2\right]_3^5 = \left[4(5) - \frac{25}{4}\right] - \left[12 - \frac{9}{4}\right]$

$= 8 - 4 = 4$ sq. units.

OR

$\frac{x e^x}{(x+1)^2} = \frac{(x+1)-1}{(x+1)^2}e^x = \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)e^x$

Let $f(x) = \frac{1}{x+1}$, then $f'(x) = -\frac{1}{(x+1)^2}$

Thus, the given integral is of the form $\int(f(x)+f'(x))e^x dx$

$\therefore \quad \int_0^1 \frac{x e^x}{(x+1)^2}dx = \int_0^1\left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right)e^x\,dx = \left[\frac{1}{x+1}\cdot e^x\right]_0^1$

$= \frac{1}{2}e^1 - \frac{1}{1}e^0 = \frac{1}{2}e - 1.$

25. Let $I = \int e^{-x}\operatorname{cosec}^2(2e^{-x}+5)dx$

Put $2e^{-x} + 5 = t \Rightarrow 2e^{-x}(-1)dx = dt \quad \Rightarrow e^{-x}dx = -\frac{1}{2}dt$

$\therefore \quad I = \int \operatorname{cosec}^2 t \cdot \left(-\frac{1}{2}dt\right) = -\frac{1}{2}(-\cot t) + C$

$= \frac{1}{2}\cot(2e^{-x}+5) + C.$

26. Let $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx$

$= \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}dx$

$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + C$

27. Let $I = \int\left(\log(\log x) + \frac{1}{(\log x)^2}\right)dx$

Put $\log x = t$, then $x = e^t$ and $dx = e^t dt$

$\therefore \quad I = \int\left(\log t + \frac{1}{t^2}\right)e^t dt = \int e^t\left(\log t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt$

Consider, $f(t) = \log t - \frac{1}{t} \Rightarrow f'(t) = \frac{1}{t} + \frac{1}{t^2}$

Thus, the given integrand is in the form

$\int e^t(f(t)+f'(t))dt$

$\therefore \quad I = \int e^t\left(\log t + \frac{1}{t^2}\right)dt = e^t\left(\log t - \frac{1}{t}\right) + C$

$\Rightarrow \quad I = x\left(\log(\log x) - \frac{1}{\log x}\right) + C$

28. Given circles are $x^2 + y^2 = 1$ ...(i)

It has centre at (0, 0) and radius = 1

and $(x-1)^2 + y^2 = 1.$ ...(ii)

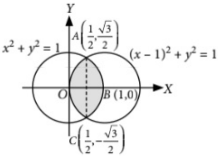
It has centre at (1, 0) and radius = 1

Subtracting (ii) from (i), we get

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

From (i), when $x = \frac{1}{2}$,

$y^2 = \frac{3}{4} \quad \therefore\ y = \pm\frac{\sqrt{3}}{2}$.



Now, required area
= area $(OABCO)$
= 2 area $(OABO)$

$$= 2\int_0^{1/2}\sqrt{1-(x-1)^2}\,dx + 2\int_{1/2}^{1}\sqrt{1-x^2}\,dx$$

$$= 2\left[\frac{(x-1)}{2}\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2} + 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{1/2}^{1}$$

$$= \left[\left(\frac{1}{2}-1\right)\sqrt{1-\frac{1}{4}} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right] + \left[(0+\sin^{-1}1) - \frac{1}{2}\sqrt{1-\frac{1}{4}} - \sin^{-1}\frac{1}{2}\right]$$

$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units.}$$

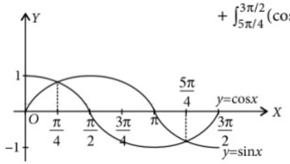
OR

We have, $y = \cos x$ ... (i), $0 \le x \le \frac{3\pi}{2}$

and $y = \sin x$ ... (ii), $0 \le x \le \frac{3\pi}{2}$

From (i) and (ii), points of intersections between $\left[0, \frac{3\pi}{2}\right]$ are $\frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\text{Area} = \int_0^{\pi/4}(\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)\,dx + \int_{5\pi/4}^{3\pi/2}(\cos x - \sin x)\,dx$$



$$= \left|(\sin x + \cos x)\right|_0^{\pi/4} - \left|(\cos x + \sin x)\right|_{\pi/4}^{5\pi/4} + \left|(\sin x + \cos x)\right|_{5\pi/4}^{3\pi/2}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right) - \left[\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right] + \left(\left(-1 + \frac{1}{\sqrt{2}}\right) + \left(0 + \frac{1}{\sqrt{2}}\right)\right)$$

$= (4\sqrt{2} - 2)$ sq. units.

29. Let $I = \int_1^4 (|x-1| + |x-2| + |x-4|)\,dx$

Also, let $f(x) = |x-1| + |x-2| + |x-4|$
We have three critical points $x = 1, 2, 4$.

$$f(x) = \begin{cases}(x-1)-(x-2)-(x-4), & \text{if } 1 \le x < 2\\ (x-1)+(x-2)-(x-4), & \text{if } 2 \le x < 4\end{cases}$$

$$\therefore\ f(x) = \begin{cases}-x+5, & \text{if } 1 \le x < 2\\ x+1, & \text{if } 2 \le x < 4\end{cases}$$

$$\therefore\ I = \int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$= \int_1^2 (-x+5)\,dx + \int_2^4 (x+1)\,dx = \left[-\frac{x^2}{2} + 5x\right]_1^2 + \left[\frac{x^2}{2} + x\right]_2^4$$

$$= \left(-\frac{4}{2} + 10\right) - \left(-\frac{1}{2} + 5\right) + \left(\frac{16}{2} + 4\right) - \left(\frac{4}{2} + 2\right)$$

$$= 8 - \frac{9}{2} + 12 - 4 = 16 - \frac{9}{2} = \frac{23}{2}$$

30. Let $I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\,d\theta$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}\,d\theta$$

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$$\therefore\ I = \int \frac{1}{(4+t^2)(1+4t^2)}\,dt$$

Consider, $\frac{1}{(4+t^2)(1+4t^2)} = \frac{At+B}{4+t^2} + \frac{Ct+D}{1+4t^2}$

(Using partial fraction)

$1 = (At + B)(1 + 4t^2) + (Ct + D)(4 + t^2)$
$= At + B + 4At^3 + 4Bt^2 + 4Ct + Ct^3 + 4D + Dt^2$
$= (4A + C)t^3 + (4B + D)t^2 + (A + 4C)t + (B + 4D)$

Comparing coeff. of like powers, we get

$4A + C = 0$ ...(i)
$4B + D = 0$ ...(ii)
$A + 4C = 0$ ...(iii)
$B + 4D = 1$ ...(iv)

Solving (i) & (iii), we get $A = 0$ and $C = 0$

Solving (ii) & (iv), we get $B = \frac{-1}{15}$ and $D = \frac{4}{15}$

$$\therefore\ \frac{1}{(4+t^2)(1+4t^2)} = \frac{-1/15}{4+t^2} + \frac{4/15}{1+4t^2}$$

$$\therefore\ I = -\frac{1}{15}\int\frac{1}{4+t^2}\,dt + \frac{4}{15}\times\frac{1}{4}\int\frac{1}{\frac{1}{4}+t^2}\,dt$$

$$= -\frac{1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + \frac{1}{15}\times\frac{1}{1/2}\tan^{-1}\left(\frac{t}{1/2}\right) + C$$

$$= -\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right)+\frac{2}{15}\tan^{-1}(2t)+C$$

$$= \frac{2}{15}\tan^{-1}(2\sin\theta)-\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right)+C$$

31. Let $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Then, $dx = (-2\alpha\cos\theta\sin\theta + 2\beta\sin\theta\cos\theta)\,d\theta$

$= 2(\beta - \alpha)\sin\theta\cos\theta\,d\theta$

When $x = \alpha$, we have, $\alpha = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow \alpha(1 - \cos^2\theta) = \beta\sin^2\theta \Rightarrow \alpha\sin^2\theta = \beta\sin^2\theta$

$\Rightarrow (\alpha - \beta)\sin^2\theta = 0 \quad \therefore \theta = 0$

When $x = \beta$, we have, $\beta = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow \beta\cos^2\theta = \alpha\cos^2\theta \Rightarrow (\beta - \alpha)\cos^2\theta = 0$

$\Rightarrow \cos^2\theta = 0 \quad \therefore \theta = \pi/2$

Now, $I = \displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

$$= \int_0^{\pi/2}\frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}\,d\theta$$

$$= \int_0^{\pi/2}\frac{2(\beta-\alpha)\sin\theta\cos\theta}{\sqrt{(\beta-\alpha)\sin^2\theta(\beta-\alpha)\cos^2\theta}}\,d\theta$$

$$= 2\int_0^{\pi/2}d\theta = 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2}-0\right] = \pi\,.$$

OR

Let $2x + 3 = A\dfrac{d}{dx}(x^2 + 4x + 3) + B = A(2x + 4) + B$

On comparing coefficient of x and constant term, we get

$2A = 2$ and $4A + B = 3 \Rightarrow A = 1, B = -1$

Let $I = \int(2x+3)\sqrt{x^2+4x+3}\,dx$

$= \int(2x+4)\sqrt{x^2+4x+3}\,dx - \int\sqrt{x^2+4x+3}\,dx$

$= I_1 - I_2$ (say) ...(i)

For I_1, put $x^2 + 4x + 3 = t \Rightarrow (2x + 4)dx = dt$.

$\therefore \quad I_1 = \int\sqrt{t}\,dt = \dfrac{t^{3/2}}{\frac{3}{2}}+C_1 = \dfrac{2}{3}(x^2+4x+3)^{3/2}+C_1$ ...(ii)

and $I_2 = \int\sqrt{x^2+4x+3}\,dx = \int\sqrt{(x+2)^2-1^2}\,dx$

$= \int\sqrt{u^2-1^2}\,du$ (By putting $x + 2 = u \Rightarrow dx = du$)

$$= \frac{u}{2}\sqrt{u^2-1^2}-\frac{1^2}{2}\log|u+\sqrt{u^2-1^2}|+C_2$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+3}-\frac{1}{2}\log|x+2+\sqrt{x^2+4x+3}|+C_2$$

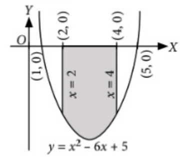
...(iii)

From (i), (ii) and (iii), we get

$$I = \frac{2}{3}(x^2+4x+3)^{3/2}-\frac{1}{2}(x+2)\sqrt{x^2+4x+3}+\frac{1}{2}\log|x+2+\sqrt{x^2+4x+3}|+C.$$

32. The given curve is $y = x^2 - 6x + 5$.

$\Rightarrow y - (-4) = (x-3)^2$, which represents an upward parabola with vertex at $(3, -4)$.



$\therefore$ Required area

$$= \left|\int_2^4 y\,dx\right| = \left|\int_2^4 (x^2-6x+5)\,dx\right|$$

$$= \left|\left[\frac{x^3}{3}-6\cdot\frac{x^2}{2}+5x\right]_2^4\right| = \left|\left(\frac{64}{3}-48+20\right)-\left(\frac{8}{3}-12+10\right)\right|$$

$= 22/3$ sq. units.

33. Let $I = \displaystyle\int_0^{\pi/2}\frac{\cos x}{1+\cos x+\sin x}\,dx$ ...(i)

Then, $I = \displaystyle\int_0^{\pi/2}\frac{\cos(\pi/2-x)}{1+\cos(\pi/2-x)+\sin(\pi/2-x)}\,dx$

$\Rightarrow \quad I = \displaystyle\int_0^{\pi/2}\frac{\sin x}{1+\sin x+\cos x}\,dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2}\frac{\cos x+\sin x}{1+\sin x+\cos x}\,dx = \int_0^{\pi/2}\left\{1-\frac{1}{1+\sin x+\cos x}\right\}dx$$

$$= \int_0^{\pi/2}1\cdot dx - \int_0^{\pi/2}\frac{1}{1+\sin x+\cos x}\,dx$$

$$= [x]_0^{\pi/2} - \int_0^{\pi/2}\frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}\,dx$$

$$= \frac{\pi}{2}-\int_0^{\pi/2}\frac{\sec^2 x/2}{2+2\tan x/2}\,dx = \frac{\pi}{2}-\int_0^1\frac{2dz}{2+2z}$$

(where, $z = \tan\dfrac{x}{2}$)

$$= \frac{\pi}{2}-[\log(1+z)]_0^1$$

$\therefore \quad 2I = \dfrac{\pi}{2} - \log 2 \Rightarrow I = \dfrac{\pi}{4}-\dfrac{1}{2}\log 2.$

OR

Let $I = \displaystyle\int\frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx = \int\frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}\,dx$

$= \displaystyle\int\frac{x}{(x^2+1)(x+1)}\,dx + 2\int\frac{dx}{(1+x^2)^2} = I_1 + I_2$ ...(i)

For I_1, let $\dfrac{x}{(1+x)(1+x^2)} = \dfrac{A}{1+x}+\dfrac{Bx+C}{1+x^2}$

$\therefore \quad x = A(1 + x^2) + (Bx + C)(1 + x)$

Putting $x = -1$, we get $A = -\frac{1}{2}$

Putting $x = 0$, we get, $0 = A + C \Rightarrow C = -A = \frac{1}{2}$

Putting $x = 1$, we get $1 = 2A + 2B + 2C \Rightarrow B = \frac{1}{2}$

$$\therefore \quad I_1 = \int\left(-\frac{1}{2(1+x)}+\frac{\frac{1}{2}x+\frac{1}{2}}{1+x^2}\right)dx$$

$$= -\frac{1}{2}\log|1+x|+\frac{1}{2}\int\frac{x}{1+x^2}dx+\frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= -\frac{1}{2}\log|1+x|+\frac{1}{4}\log(1+x^2)+\frac{1}{2}\tan^{-1}x+C_1 \quad \ldots\text{(ii)}$$

Now, $I_2 = \int\frac{2dx}{(1+x^2)^2}$, put $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

Then, $I_2 = \int\frac{2\sec^2\theta}{(1+\tan^2\theta)^2}d\theta$

$$= \int 2\cos^2\theta\, d\theta = \int(1+\cos 2\theta)d\theta$$

$$= \left[\theta+\frac{\sin 2\theta}{2}\right]+C_2 = [\theta + \sin\theta\cos\theta] + C_2$$

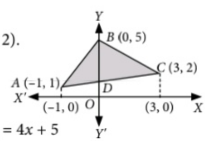
$$= \left[\tan^{-1}x+\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right] + C_2$$

$$= \tan^{-1}x+\frac{x}{(1+x^2)}+C_2 \quad \ldots\text{(iii)}$$

Now from (i), (ii) and (iii), we get

$$I = \frac{3}{2}\tan^{-1}x - \frac{1}{2}\log|1+x| + \frac{1}{4}\log(1+x^2) + \frac{x}{1+x^2} + C.$$

34. Given points are $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.



Equation of AB is

$$y - 1 = \frac{5-1}{0+1}(x+1)$$

$\Rightarrow \ y - 1 = 4(x + 1)$ or $y_1 = 4x + 5$

Equation of BC is $y - 5 = \frac{5-2}{0-3}(x - 0)$

or $\ y_2 = 5 - x$

Equation of AC is $y - 1 = \frac{2-1}{3+1}(x + 1)$

or $\ y_3 = \frac{x+5}{4}$

Now, area of ΔABC = area $ABDA$ + area $DBCD$

$$= \int_{-1}^{0}(y_1 - y_3)dx + \int_0^3(y_2 - y_3)dx$$

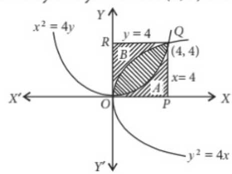
$$= \int_{-1}^{0}\left[4x+5-\frac{x+5}{4}\right]dx+\int_0^3\left[5-x-\frac{x+5}{4}\right]dx$$

$$= [2x^2+5x]_{-1}^{0}-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{0}+\left[5x-\frac{x^2}{2}\right]_0^3-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_0^3$$

$$= [0-(2-5)]-\frac{1}{4}\left[0-\left(\frac{1}{2}-5\right)\right]+\left[\left(15-\frac{9}{2}\right)-0\right] -\frac{1}{4}\left[\left(\frac{9}{2}+15\right)-0\right]$$

$$= 3-\frac{9}{8}+\frac{21}{2}-\frac{39}{8}=3+\frac{21}{2}-6=\frac{15}{2} \text{ sq. units.}$$

35. The rough sketch of parabolas $y^2 = 4x$ and $x^2 = 4y$ is as shown in the figure. The intersection points of the parabolas $x^2 = 4y$ and $y^2 = 4x$ are (0, 0) and (4, 4).



The area of region $OAQBO$

$$= \int_0^4 (y_1 - y_2)dx,$$

where $y_1 = 2\sqrt{x}$ and $y_2 = \frac{x^2}{4}$

$$\therefore \quad \text{Area} = \int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx = \left[2\times\frac{2}{3}x^{3/2}-\frac{x^3}{12}\right]_0^4$$

$$= \frac{4}{3}(4)^{3/2}-\frac{64}{12}=\frac{32}{3}-\frac{16}{3}=\frac{16}{3} \quad \ldots\text{(i)}$$

Again the area of region $OPQAO$

$$= \int_0^4 y_2 dx = \int_0^4\frac{x^2}{4}dx = \left[\frac{x^3}{12}\right]_0^4 = \frac{16}{3} \quad \ldots\text{(ii)}$$

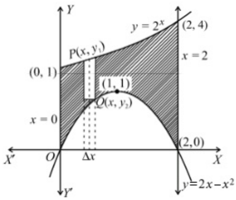
Similarly, the area of region $OBQRO$

$$= \int_0^4 x\,dy = \int_0^4\frac{y^2}{4}dy = \left[\frac{y^3}{12}\right]_0^4 = \frac{16}{3} \quad \ldots\text{(iii)}$$

From (i), (ii) and (iii), it is concluded that the area of region $OAQBO$ = area of region $OPQAO$ = area of region $OBQRO$.

OR

The equation $y = 2x - x^2$ represents a parabola opening downwards having vertex at (1, 1) and cutting x-axis at (0, 0) and (2, 0).



The equation $y = 2^x$ represents the exponential curve as shown in figure. Lines $x = 0$ and $x = 2$ are shown in figure. The area bounded by these curves is shaded in figure. We slice the shaded region into vertical strips.

So, required area $= \int_0^2 (y_1 - y_2)\,dx = \int_0^2 (2^x - 2x + x^2)\,dx$

$$= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2$$

$$= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2} = \left(\frac{3}{\log 2} - \frac{4}{3}\right) \text{sq. units}$$

36. Let $I = \int \sqrt{\tan x}\,dx$

Put $\tan x = t^2$, then $\sec^2 x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t\,dt}{\sec^2 x} = \frac{2t\,dt}{1+\tan^2 x} = \frac{2t\,dt}{1+t^4}$$

$$\therefore \quad I = \int \frac{t(2t\,dt)}{1+t^4} = \int \frac{2t^2}{1+t^4}\,dt$$

$$\Rightarrow \quad I = \int \frac{t^2+t^2}{1+t^4}\,dt = \int \frac{(t^2+1)+(t^2-1)}{(t^4+1)}\,dt$$

$$\Rightarrow \quad I = \int \frac{t^2+1}{t^4+1}\,dt + \int \frac{t^2-1}{t^4+1}\,dt$$

$$\Rightarrow \quad I = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt + \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$\Rightarrow \quad I = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2 - 2}\,dt + \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + 2}\,dt$$

Put $t + \frac{1}{t} = v$, then $\left(1 - \frac{1}{t^2}\right)dt = dv$ in first integral and $t - \frac{1}{t} = u$, then $\left(1 + \frac{1}{t^2}\right)dt = du$ in second integral.

$$\therefore \quad I = \int \frac{dv}{v^2 - \left(\sqrt{2}\right)^2} + \int \frac{du}{u^2 + \left(\sqrt{2}\right)^2}$$

$$I = \frac{1}{2\sqrt{2}} \log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| + \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}\right] + C$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1}\right| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C$$

MONTHLY TEST DRIVE

XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 5 : Application of Derivatives

Time Taken : 60 Min.

Only One Option Correct Type

1. The function $f(x) = x\sqrt{(ax-x^2)}$, $a > 0$
   (a) increases on the interval $(0, 3a/4)$
   (b) decreases on the interval $(0, 3a/4)$
   (c) increases on the interval $(3a/4, a)$
   (d) none of these

2. If the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 2$ has a positive point of maximum, then $a > 0$ lies in
   (a) $(0, \infty)$ (b) $(1, 7)$ (c) $(3, \infty)$ (d) $\left(3, \frac{29}{7}\right]$

3. x and y are the sides of two squares such that $y = x - x^2$. The rate of change of area of the second square with respect to that of the first square is
   (a) $2x^2 + 3x + 1$ (b) $3x^2 + 2x - 1$
   (c) $2x^2 - 3x + 1$ (d) $3x^2 + 2x + 1$

4. Let $f(x)$ be a quadratic expression which is positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x,
   (a) $g(x) < 0$ (b) $g(x) > 0$
   (c) $g(x) = 0$ (d) none of these

5. The value of parameter 'a' so that the line $(3 - a)x + ay + (a^2 - 1) = 0$ is normal to the curve $xy = 1$, may lie in the interval
   (a) $(-\infty, 0) \cup (3, \infty)$ (b) $(1, 3)$
   (c) $(-3, 3)$ (d) none of these

6. The minimum and maximum value of
   $f(x) = \sin(\cos x) + \cos(\sin x)\ \forall -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ are respectively
   (a) $\cos 1$ and $1 + \sin 1$
   (b) $\sin 1$ and $1 + \cos 1$
   (c) $\cos 1$ and $\cos\left(\frac{1}{\sqrt{2}}\right)+\sin\left(\frac{1}{\sqrt{2}}\right)$
   (d) none of these

One or More Than One Option(s) Correct Type

7. Let $f: R \to (0, \infty)$ and $g: R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0, f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then
   (a) f has a local minimum at $x = 2$
   (b) f has a local maximum at $x = 2$
   (c) $f''(2) > f(2)$
   (d) $f(x) - f''(x) = 0$ for at least one $x \in R$

8. Let $f(x)$ be a differentiable function and $f(a) = f(b) = 0$ $(a < b)$, then which of the following can be true in the interval (a, b)?
   (a) $f(x) + f'(x) = 0$ has at least one root.
   (b) $f(x) - f'(x) = 0$ has at least one root.
   (c) $f(x) \cdot f'(x) = 0$ has at least one root.
   (d) none of these

9. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is
   (a) increasing on $[0, \infty)$
   (b) decreasing on $[0, \infty)$
   (c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$
   (d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

10. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, (where $\alpha^2 + \beta^2 = 61$), then the value of a is
    (a) -30 (b) 10 (c) 20 (d) 30

11. Let $f''(x) > 0\ \forall\ x \in R$ and $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$. Then,
    (a) g attains maximum at $x = \pm 1$
    (b) g attains minimum at $x = \pm 1$
    (c) g attains maximum at $x = 0$
    (d) g attains minimum at $x = \pm 2$

12. The interval to which b may belong so that the function $f(x)=\left(1-\frac{\sqrt{(21-4b-b^2)}}{b+1}\right)x^3+5x+\sqrt{6}$ is increasing at every point of its domain is

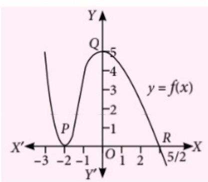
(a) $[-7, -1)$ (b) $[-6, -2]$
(c) $[2, 2.5]$ (d) $[2, 3]$

13. Which of the following statements is true for the curve $x = 2a\sin t + a\sin t\cos^2 t$ and $y = -a\cos^3 t$?

(a) Normal is inclined at an angle $\frac{\pi}{2}+t$ with x-axis.
(b) Normal is inclined at an angle t with x-axis.
(c) Portion of normal contained between the co-ordinate axes is equal to $2a$.
(d) Portion of normal contained between the co-ordinate axes is equal to $4a$.

Comprehension Type

A polynomial of degree three is shown in the given figure and at Q its gradient is 3. Then,



14. The polynomial $f(x)$ is given by

(a) $-\frac{1}{2}x^3-\frac{3}{4}x^2+3x+5$

(b) $-\frac{1}{2}x^3+3x+5$

(c) $-\frac{1}{2}x^3-\frac{3}{4}x^2+5$ (d) $-\frac{1}{2}x^3+\frac{3}{4}x^2-3x+5$

15. The equation of normal at R is

(a) $6x - 81y - 15 = 0$
(b) $8x - 81y - 20 = 0$
(c) $2x - 81y - 5 = 0$
(d) $4x - 81y - 10 = 0$

Matrix Match Type

16. Match the following:

	Column-I	Column-II	
P.	An open box with square base of side x and height y has a given surface area. Its volume is maximum, if $\frac{x}{y}$ =	1.	24
Q.	A cylindrical can with lid having base radius x and height y has a given surface area. Its volume is maximum, if $\frac{6y}{x}$ =	2.	3
R.	Curves $x^{2/3}+y^{2/3}=c^{2/3}$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may touch, if $\frac{3c}{a+b}$ is equal to	3.	2
S.	Circular plate is expanded by heat from radius 6 cm to 6.06 cm. If approximate increase in area is $\frac{3k\pi}{100}$, then k equals	4.	12

	P	Q	R	S
(a)	1	3	4	2
(b)	2	3	1	4
(c)	4	2	3	1
(d)	3	4	2	1

Integer Answer Type

17. The maximum possible integral value of $\frac{\beta-\alpha}{\tan^{-1}\beta-\tan^{-1}\alpha}$, where $0<\alpha<\beta<\sqrt{3}$, is ______.

18. If $y = a\ln x + bx^2 + x$ has its extreme values at $x = -1$, 2 then $2(a + b)$ is equal to ______.

19. If the rate of decrease of $\frac{x^2}{2}-2x+5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease is non-zero) ______.

20. The chord of the curve $y = -a^2x^2 + 5ax - 4$ touches the curve $y=\frac{1}{1-x}$ at the point $x = 2$ and is bisected by that point. Then the value of a is ______. ◈◈

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SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

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