Notes on 1-HITs

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In type theory, a data type is described by its introduction rule (the constructors) and its elimination rule (the induction principle) along with a computation rule describing how these two rules interact.

Natural numbers example?

4.1 Induction principle

Given a functor $\mathsf{F} : \mathsf{Type} \to \mathsf{Type}$ given as a container $\mathsf{F} :\equiv \mathsf{S} \triangleleft \mathsf{P}$, the W-type W is defined as having the following introduction rule / constructor:

$$\mathsf{c} : \mathsf{F}\,\mathsf{W} \to \mathsf{W}$$

as well as an elimination rule / induction principle:

$$\begin{aligned}
&\mathsf{ind} : \\
&\qquad (\mathsf{B} : \mathsf{W} \to \mathsf{Type}) \\
&\qquad (\mathsf{m} : (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \to \Box\,\mathsf{F}\,\mathsf{B}\,\mathsf{x} \to \mathsf{B}\,(\mathsf{c}\,\mathsf{x})) \\
&\qquad (\mathsf{x} : \mathsf{W}) \\
&\quad \to \mathsf{B}\,\mathsf{x}
\end{aligned}$$

with computation rule:

$$\begin{aligned}
&\mathsf{ind}\text{-}\beta_0 : \\
&\qquad (\mathsf{B} : \mathsf{W} \to \mathsf{Type}) \\
&\qquad (\mathsf{m} : (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \to \Box\,\mathsf{F}\,\mathsf{B}\,\mathsf{x} \to \mathsf{B}\,(\mathsf{c}\,\mathsf{x})) \\
&\qquad (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \\
&\quad \to \mathsf{ind}\,\mathsf{B}\,\mathsf{m}\,(\mathsf{c}\,\mathsf{x}) = \mathsf{m}\,\mathsf{x}\,(\Box\text{-}\mathsf{lift}\,\mathsf{F}\,(\mathsf{ind}\,\mathsf{B}\,\mathsf{m})\,\mathsf{x})
\end{aligned}$$

4.2 Algebras

The type of *F-algebras*, or simply *algebras*, can be defined as follows:

$$\mathsf{Alg} :\equiv (\mathsf{X} : \mathsf{Type}) \times (\theta : \mathsf{F}\,\mathsf{X} \to \mathsf{X})$$

where the type morphisms is defined as follows:

$$\begin{aligned}&\mathsf{Alg\text{-}hom} : \mathsf{Alg} \to \mathsf{Alg} \to \mathsf{Type}\\&\mathsf{Alg\text{-}hom}\,(\mathsf{X},\theta)\,(\mathsf{Y},\rho) :\equiv\\&\qquad(\mathsf{f} : \mathsf{X} \to \mathsf{Y})\\&\quad\times (\mathsf{f}\text{-}\beta_0 : (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{f}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{f}\,\mathsf{x}))\end{aligned}$$

The witness of commutativity has suggestively been named $\mathsf{f}\text{-}\beta_0$ as this gives us the β-rule for the recursion and induction principles.

Something about having $\mathsf{f}_0 : \mathsf{f} \circ \theta \equiv \rho \circ \mathsf{F}\,\mathsf{f}$ instead. We need function extensionality either way, but this way it makes the arguments later on a bit easier. Also, the dependent versions don't work as pointfree as these ones

4.2.1 Homotopy initial algebras

We call an algebra (X,θ) *homotopy initial* if it has the property that for every algebra (Y,ρ), $\mathsf{Alg\text{-}hom}\,(\mathsf{X},\theta)\,(\mathsf{Y},\rho)$ is contractible, i.e.:

$$\begin{aligned}&\mathsf{is\text{-}initial} : \mathsf{Alg} \to \mathsf{Type}\\&\mathsf{is\text{-}initial}\,\theta :\equiv (\rho : \mathsf{Alg}) \to \mathsf{is\text{-}contr}\,(\mathsf{Alg\text{-}hom}\,\theta\,\rho)\\&\qquad\qquad\quad \equiv (\rho : \mathsf{Alg}) \to (\mathsf{f} : \mathsf{Alg\text{-}hom}\,\theta\,\rho) \times ((\mathsf{g} : \mathsf{Alg\text{-}hom}\,\theta\,\rho) \to \mathsf{f} = \mathsf{g})\end{aligned}$$

4.2.2 Equality of algebra morphisms

As we see in the definition of homotopy initiality, we need to be able to talk about equality of algebra morphisms. Given algebras (X,θ) and (Y,ρ) and morphisms $(\mathsf{f},\mathsf{f}\text{-}\beta_0)$ and $(\mathsf{g},\mathsf{g}\text{-}\beta_0)$ between them, we know that, by equality on Σ-types, the following holds:

$$\begin{aligned}&\ (\mathsf{f},\mathsf{f}\text{-}\beta_0) = (\mathsf{g},\mathsf{g}\text{-}\beta_0)\\&\simeq (\mathsf{p} : \mathsf{f} = \mathsf{g})\\&\times (\mathsf{p}\text{-}\beta_0 : \mathsf{transport}\,(\lambda\,\mathsf{h} \circ (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{h}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{h}\,\mathsf{x}))\,\mathsf{f}\text{-}\beta_0 = \mathsf{g}\text{-}\beta_0)\end{aligned}$$

We not only need to show that the functions f and g are equal, but also that their β-laws are in some sense compatible with eachother, respecting the equality $\mathsf{f} = \mathsf{g}$.

As it turns out, the above is equivalent to something which is more convenient in subsequent proofs:

$$\mathsf{transport}\,(\lambda\,\mathsf{h} \to (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{h}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{h}\,\mathsf{x}))\,\mathsf{p}\,\mathsf{f}\text{-}\beta_0 = \mathsf{g}\text{-}\beta_0$$

$\simeq$ { transport over Π-types }
$(\lambda\ x \to \mathsf{transport}\ (\lambda\ h \to h\ (\theta\ x)\ =\ \rho\ (F\ h\ x))\ p\ (f\text{-}\beta_0\ x))\ =\ g\text{-}\beta_0$
$\simeq$ { function extensionality }
$((x\ :\ A) \to \mathsf{transport}\ (\lambda\ h \to h\ (\theta\ x)\ =\ \rho\ (F\ h\ x))\ p\ (f\text{-}\beta_0\ x)\ =\ g\text{-}\beta_0\ x)$
$\simeq$ { transporting over equalities }
$!\ (\mathsf{ap}\ (\lambda\ h \to h\ (\theta\ x))\ p) \cdot f\text{-}\beta_0\ x \cdot \mathsf{ap}\ (\lambda\ h \to \rho\ (F\ h\ x))\ p\ =\ g\text{-}\beta_0\ x$
$\simeq$ { path algebra }
$f\text{-}\beta_0\ x \cdot \mathsf{ap}\ (\lambda\ h \to \rho\ (F\ h\ x))\ p\ =\ \mathsf{ap}\ (\lambda\ h \to h\ (\theta\ x))\ p \cdot g\text{-}\beta_0\ x$

The last equation tells us that showing that two algebra morphisms are equal is somewhat like giving an algebra morphism from the witness of the β-law for f to that of g, as we can see if we draw the corresponding diagram:

comm diag

4.3 Initiality implies induction

The induction principle tells us that for a family $B\ :\ W\ \to\ \mathsf{Type}$ and a motive $m\ :\ (x\ :\ F\ W) \to \Box\ F\ B\ x \to B\ (c\ x)$, we get a dependent function $\mathsf{ind}\ :\ (x\ :\ W)\ \to\ B\ x$ along with a computation rule.

Note that m, along with c, can be seen as a morphism between the families (W, B) and $(F\ W, \Box\ F\ B)$.

Another way to say this is that given a function $p\ :\ B\ \to\ W$ for some $B\ :\ \mathsf{Type}$ and that we want $\theta\ :\ F\ B\ \to\ B$ such that θ, along with c, becomes a morphism $Fp\ \to\ p$ in the arrow category. This is equivalent to asking for an algebra (B, θ) along with an algebra morphism $(B, \theta)\ \to\ (W, c)$.

Give arguments why section induction is correct

Show that initiality implies section induction

4.4 Induction implies initiality

We want to show that, given $T\ :\ \mathsf{Type}$ and $c\ :\ F\ T\ \to\ T$ that satisfies the induction principle, the algebra (T, c) is initial, i.e. for any algebra (X, θ), $\mathsf{Alg\text{-}hom}\ (T, c)\ (X, \theta) \simeq 1$. We will first show that we have an algebra morphism $f\ :\ (T, c)\ \to\ (X, \theta)$ and will then show that this algebra morphism is unique.

4.4.1 Existence

We can use the induction principle to produce the algebra morphism we want:

$$f\ :\ T\ \to\ X$$
$$f\ :\equiv\ \mathsf{ind}\ (\lambda\ x\ \to\ X)\ (\lambda\ (s, \_)\ t\ \to\ \theta\ (s, t))$$

The computation rule is then given directly by the computation rule for the induction rule:

$$\mathsf{f}_0 \ : \ (\mathsf{x} \ : \ \mathsf{F}\ \mathsf{T}) \ \to \ \mathsf{f}\ (\theta\ \mathsf{x}) \ = \ \theta\ (\mathsf{F}\ \mathsf{f}\ \mathsf{x})$$
$$\mathsf{f}_0 \ :\equiv \ \mathsf{ind}\text{-}\beta$$

4.4.2 Uniqueness

Assuming that we have an algebra morphism $(\mathsf{g}, \mathsf{g}_0) \ : \ (\mathsf{T}, \mathsf{c}) \ \to \ (\mathsf{X}, \theta)$, we need to show that $(\mathsf{f}, \mathsf{f}_0) \ = \ (\mathsf{g}, \mathsf{g}_0)$. Showing that the $\mathsf{f} \ = \ \mathsf{g}$ can be done by induction, using the motive $\lambda\ \mathsf{x} \ \to \ \mathsf{f}\ \mathsf{x} \ = \ \mathsf{g}\ \mathsf{x}$. The induction step is then proven as follows:

$$\begin{array}{l}
\mathsf{step} \ : \ (\mathsf{x} \ : \ [\![\ \mathsf{F}\ ]\!]_0\ \mathsf{T}) \to \Box\ \mathsf{F}\ \mathsf{f{=}g\text{-}B}\ \mathsf{x} \to \mathsf{f{=}g\text{-}B}\ (\mathsf{c}\ \mathsf{x}) \\
\mathsf{step}\ \mathsf{x}\ \mathsf{u} \ :\equiv \\
\quad \mathsf{f}\ (\mathsf{c}\ \mathsf{x}) \\
\qquad =\{\ \mathsf{f}_0\ \mathsf{x}\ \} \\
\quad \theta\ ([\![\ \mathsf{F}\ ]\!]_1\ \mathsf{f}\ \mathsf{x}) \\
\qquad =\{\ \mathsf{ap}\ \theta\ (\mathsf{ind\text{-}hyp}\ \mathsf{x}\ \mathsf{u})\ \} \\
\quad \theta\ ([\![\ \mathsf{F}\ ]\!]_1\ \mathsf{g}\ \mathsf{x}) \\
\qquad =\{\ !\ \ (\mathsf{g}_0\ \mathsf{x})\ \} \\
\quad \mathsf{g}\ (\mathsf{c}\ \mathsf{x}) \ \blacksquare
\end{array}$$

Explain ind-hyp

We can define the witness of $\mathsf{f} \ = \ \mathsf{g}$ as follows:

$$\mathsf{p} \ : \ (\mathsf{x} \ : \ \mathsf{T}) \ \to \ \mathsf{f}\ \mathsf{x} \ = \ \mathsf{g}\ \mathsf{x}$$
$$\mathsf{p} \ :\equiv \ \mathsf{ind}\ (\lambda\ \mathsf{x} \ \to \ \mathsf{f}\ \mathsf{x} \ = \ \mathsf{g}\ \mathsf{x})\ \mathsf{step}$$

which comes with the computation rule:

$$\mathsf{p}\text{-}\beta_0 \ : \ (\mathsf{x} \ : \ \mathsf{FT}) \ \to \ \mathsf{p}\ (\mathsf{c}\ \mathsf{x}) \ = \ \mathsf{f}\text{-}\beta_0\ \mathsf{x} \cdot \mathsf{ap}\ \theta\ (\mathsf{ind\text{-}hyp}\ \mathsf{x}\ (\Box\text{-}\mathsf{lift}\ \mathsf{F}\ \mathsf{p}\ \mathsf{x})) \cdot\ !\ \ (\mathsf{g}\text{-}\beta_0\ \mathsf{x})$$

say that $\lambda{=}$ is fun ext

We now need to show that our witness p satisfies the "computation rule" $(\mathsf{x} \ : \ \mathsf{FT}) \ \to \ \mathsf{f}\text{-}\beta_0\ \mathsf{x} \cdot \mathsf{ap}\ (\lambda\ \mathsf{h} \to \rho\ (\mathsf{F}\ \mathsf{h}\ \mathsf{x}))\ \mathsf{p} \ = \ \mathsf{ap}\ (\lambda\ \mathsf{h} \to \mathsf{h}\ (\theta\ \mathsf{x}))\ \mathsf{p} \cdot \mathsf{g}\text{-}\beta_0\ \mathsf{x}$. Let $\mathsf{x} \ : \ \mathsf{FT}$, then we can calculate:

$$\begin{array}{l}
\mathsf{f}\text{-}\beta_0\ \mathsf{x} \cdot \mathsf{ap}\ (\lambda\ \mathsf{h} \to \theta\ (\mathsf{F}\ \mathsf{h}\ \mathsf{x}))\ (\lambda{=}\ \mathsf{p}) \\
\qquad = \ \{\mathsf{ap\ magic}\} \\
\mathsf{f}\text{-}\beta_0\ \mathsf{x} \cdot \mathsf{ap}\ \theta\ (\mathsf{ind\text{-}hyp}\ \mathsf{x}\ (\Box\text{-}\mathsf{lift}\ \mathsf{F}\ \mathsf{p}\ \mathsf{x})) \\
\qquad = \ \{\mathsf{symmetry\ is\ involutive}\} \\
\mathsf{f}\text{-}\beta_0\ \mathsf{x} \cdot \mathsf{ap}\ \theta\ (\mathsf{ind\text{-}hyp}\ \mathsf{x}\ (\Box\text{-}\mathsf{lift}\ \mathsf{F}\ \mathsf{p}\ \mathsf{x})) \cdot\ !\ \ (\mathsf{g}\text{-}\beta_0\ \mathsf{x}) \cdot \mathsf{g}\text{-}\beta_0\ \mathsf{x} \\
\qquad = \ \{\mathsf{computation\ rule\ for\ p}\} \\
\mathsf{p}\ (\mathsf{c}\ \mathsf{x}) \cdot \mathsf{g}\text{-}\beta_0\ \mathsf{x}
\end{array}$$

$$= \{\textsf{computation rule for } \lambda{=}\}$$
$$\mathsf{ap}\ (\lambda\ \mathsf{h} \to \mathsf{h}\ (\mathsf{c}\ \mathsf{x}))\ (\lambda{=}\ \mathsf{p}) \cdot \mathsf{g}\text{-}\beta_0\ \mathsf{x}$$

We have now established initiality of (T, c).

5 Homotopy initiality for restricted 1-HITs

To establish initiality for restricted 1-HITs , we proceed along the same lines as we have previously for ordinary W-types. We first define the induction principle and the algebras, and show that induction implies initiality and the other way around.

5.1 Induction principle

To get the induction principle of restricted 1-HITs , we need to take that of the ordinary W-types and extend it with the new (path) constructor: we have to add a method to ind and the appropriate computation rule for that.

Consider the method for the 0-constructor:

$$\mathsf{m}_0 : (\mathsf{x} : \mathsf{F}_0\ \mathsf{X}) \to \square\ \mathsf{F}_0\ \mathsf{B}\ \mathsf{x} \to \mathsf{B}\ (\mathsf{c}_0\ \mathsf{x})$$

We can read this as showing that if for all subterms of $\mathsf{c}_0\ \mathsf{x}$ B holds, B also holds for $\mathsf{c}_0\ \mathsf{x}$ itself. Since we are dealing with a 0-constructor, we don't have any issues with the target of the constructor: the target functor is the identity functor. For the motive of the 1-constructor, we want something like:

$$\mathsf{m}_1 : (\mathsf{x} : \mathsf{F}_1\ \mathsf{X}) \to \square\ \mathsf{F}_1\ \mathsf{B}\ \mathsf{x} \to \square\ \mathsf{G}\ (\mathsf{c}_1\ \mathsf{x})$$

We need to figure out what the lifting of B to the target functor G is.

5.2 Algebras

$$\begin{aligned}
&\mathsf{Alg} :\equiv \\
&\quad\ (\mathsf{X} : \mathsf{Type}) \\
&\quad \times (\theta_0 : \mathsf{F}_0\ \mathsf{X} \to \mathsf{X}) \\
&\quad \times (\theta_1 : (\mathsf{x} : \mathsf{F}_1\ \mathsf{X}) \to \mathsf{l}\ (\theta_0 * \mathsf{x}) = \mathsf{r}\ (\theta_0 * \mathsf{x}))
\end{aligned}$$

θ_0 is an F_0-algebra and θ_1 can be seen as a dependent dialgebra by defining the functor G_1:

$$\begin{aligned}
&\mathsf{G}_1 : \textstyle\int \mathsf{F}_0\text{-}\mathsf{Algebra}\ \mathsf{F}_1 \to \mathsf{Type} \\
&\mathsf{G}_1\ ((\mathsf{X}, \theta), \mathsf{x}) :\equiv (\mathsf{l}\ (\theta_0 * \mathsf{x}) = \mathsf{r}\ (\theta_0 * \mathsf{x}))
\end{aligned}$$

Its hom-types can be defined as follows:

$$
\begin{array}{l}
\mathsf{Alg\text{-}hom} : \mathsf{Alg} \to \mathsf{Alg} \to \mathsf{Type} \\
\mathsf{Alg\text{-}hom}\,(\mathsf{X}, \theta)\,(\mathsf{Y}, \rho) :\equiv \\
\quad\;\; (\mathsf{f} : \mathsf{X} \to \mathsf{Y}) \\
\quad \times\, (\mathsf{f}\text{-}\beta_0 : (\mathsf{x} : \mathsf{F}_0\,\mathsf{X}) \to \mathsf{f}\;(\theta_0\,\mathsf{x}) = \rho_0\,(\mathsf{F}_0\,\mathsf{f}\,\mathsf{x})) \\
\quad \times\, (\mathsf{f}\text{-}\beta_1 : (\mathsf{x} : \mathsf{F}_1\,\mathsf{X}) \to \mathsf{G}_1 \times \mathsf{f}, \mathsf{f}_0\;(\theta_1\,\mathsf{x}) = \rho_1\,(\mathsf{F}_1\,\mathsf{f}\,\mathsf{x}))
\end{array}
$$

Initiality for these algebras is defined in the same way as before.

5.2.1 Equality of algebra morphisms

todo

5.3 Initiality implies induction

5.4 Induction implies initiality

todo

5.4.1 Existence

5.4.2 Uniqueness