Notes on 1-HITs

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1 Introduction

2 Describing higher inductive types

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In type theory, a data type is described by its introduction rule (the constructors) and its elimination rule (the induction principle) along with a computation rule describing how these two rules interact.

Natural numbers example?

4.1 Induction principle

Given a functor $\mathsf{F} : \mathsf{Type} \to \mathsf{Type}$ given as a container $\mathsf{F} :\equiv \mathsf{S} \triangleleft \mathsf{P}$, the W-type W is defined as having the following introduction rule / constructor:

$$\mathsf{c} : \mathsf{F}\,\mathsf{W} \to \mathsf{W}$$

as well as an elimination rule / induction principle:

$$\begin{aligned}
&\mathsf{ind} : \\
&\qquad (\mathsf{B} : \mathsf{W} \to \mathsf{Type}) \\
&\qquad (\mathsf{m} : (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \to \square\,\mathsf{F}\,\mathsf{B}\,\mathsf{x} \to \mathsf{B}\,(\mathsf{c}\,\mathsf{x})) \\
&\qquad (\mathsf{x} : \mathsf{W}) \\
&\quad \to \mathsf{B}\,\mathsf{x}
\end{aligned}$$

with computation rule:

$$\begin{aligned}
&\mathsf{ind}\text{-}\beta_0 : \\
&\qquad (\mathsf{B} : \mathsf{W} \to \mathsf{Type}) \\
&\qquad (\mathsf{m} : (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \to \square\,\mathsf{F}\,\mathsf{B}\,\mathsf{x} \to \mathsf{B}\,(\mathsf{c}\,\mathsf{x})) \\
&\qquad (\mathsf{x} : \mathsf{F}\,\mathsf{W}) \\
&\quad \to \mathsf{ind}\,\mathsf{B}\,\mathsf{m}\,(\mathsf{c}\,\mathsf{x}) = \mathsf{m}\,\mathsf{x}\,(\square\text{-}\mathsf{lift}\,\mathsf{F}\,(\mathsf{ind}\,\mathsf{B}\,\mathsf{m})\,\mathsf{x})
\end{aligned}$$

4.2 Algebras

The type of *F-algebras*, or simply *algebras*, can be defined as follows:

$$\mathsf{Alg} :\equiv (\mathsf{X} : \mathsf{Type}) \times (\theta : \mathsf{F}\,\mathsf{X} \to \mathsf{X})$$

where the type morphisms is defined as follows:

$$\begin{aligned}
&\mathsf{Alg\text{-}hom} : \mathsf{Alg} \to \mathsf{Alg} \to \mathsf{Type}\\
&\mathsf{Alg\text{-}hom}\ (\mathsf{X}, \theta)\ (\mathsf{Y}, \rho) :\equiv\\
&\qquad (\mathsf{f} : \mathsf{X} \to \mathsf{Y})\\
&\quad \times (\mathsf{f}\text{-}\beta_0 : (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{f}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{f}\,\mathsf{x}))
\end{aligned}$$

The witness of commutativity has suggestively been named $\mathsf{f}\text{-}\beta_0$ as this gives us the β-rule for the recursion and induction principles.

Something about having $\mathsf{f}_0 : \mathsf{f} \circ \theta \equiv \rho \circ \mathsf{F}\ \mathsf{f}$ instead. We need function extensionality either way, but this way it makes the arguments later on a bit easier. Also, the dependent versions don't work as pointfree as these ones

4.2.1 Homotopy initial algebras

We call an algebra (X, θ) *homotopy initial* if it has the property that for every algebra (Y, ρ), $\mathsf{Alg\text{-}hom}\ (\mathsf{X}, \theta)\ (\mathsf{Y}, \rho)$ is contractible, i.e.:

$$\begin{aligned}
&\mathsf{is\text{-}initial} : \mathsf{Alg} \to \mathsf{Type}\\
&\mathsf{is\text{-}initial}\ \theta :\equiv (\rho : \mathsf{Alg}) \to \mathsf{is\text{-}contr}\ (\mathsf{Alg\text{-}hom}\ \theta\ \rho)\\
&\qquad\qquad\quad \equiv (\rho : \mathsf{Alg}) \to (\mathsf{f} : \mathsf{Alg\text{-}hom}\ \theta\ \rho) \times ((\mathsf{g} : \mathsf{Alg\text{-}hom}\ \theta\ \rho) \to \mathsf{f} = \mathsf{g})
\end{aligned}$$

4.2.2 Equality of algebra morphisms

As we see in the definition of homotopy initiality, we need to be able to talk about equality of algebra morphisms. Given algebras (X, θ) and (Y, ρ) and morphisms $(\mathsf{f}, \mathsf{f}\text{-}\beta_0)$ and $(\mathsf{g}, \mathsf{g}\text{-}\beta_0)$ between them, we know that, by equality on Σ-types, the following holds:

$$\begin{aligned}
&\ \ (\mathsf{f}, \mathsf{f}\text{-}\beta_0) = (\mathsf{g}, \mathsf{g}\text{-}\beta_0)\\
&\simeq (\mathsf{p} : \mathsf{f} = \mathsf{g})\\
&\times (\mathsf{p}\text{-}\beta_0 : \mathsf{transport}\ (\lambda\,\mathsf{h} \to (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{h}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{h}\,\mathsf{x}))\ \mathsf{f}\text{-}\beta_0 = \mathsf{g}\text{-}\beta_0)
\end{aligned}$$

We not only need to show that the functions f and g are equal, but also that their β-laws are in some sense compatible with eachother, respecting the equality $\mathsf{f} = \mathsf{g}$.

As it turns out, the above is equivalent to something which is more convenient in subsequent proofs:

$$\mathsf{transport}\ (\lambda\,\mathsf{h} \to (\mathsf{x} : \mathsf{F}\,\mathsf{X}) \to \mathsf{h}\,(\theta\,\mathsf{x}) = \rho\,(\mathsf{F}\,\mathsf{h}\,\mathsf{x}))\ \mathsf{p}\ \mathsf{f}\text{-}\beta_0 = \mathsf{g}\text{-}\beta_0$$

$\simeq$ { transport over Π-types }

$(\lambda\, x \to \mathsf{transport}\ (\lambda\, h \to h\ (\theta\ x)\ =\ \rho\ (F\ h\ x))\ p\ (f\text{-}\beta_0\ x))\ =\ g\text{-}\beta_0$

$\simeq$ { function extensionality }

$((x\ :\ A) \to \mathsf{transport}\ (\lambda\, h \to h\ (\theta\ x)\ =\ \rho\ (F\ h\ x))\ p\ (f\text{-}\beta_0\ x)\ =\ g\text{-}\beta_0\ x)$

$\simeq$ { transporting over equalities }

$!\ (\mathsf{ap}\ (\lambda\, h \to h\ (\theta\ x))\ p) \cdot f\text{-}\beta_0\ x \cdot \mathsf{ap}\ (\lambda\, h \to \rho\ (F\ h\ x))\ p\ =\ g\text{-}\beta_0\ x$

$\simeq$ { path algebra }

$f\text{-}\beta_0\ x \cdot \mathsf{ap}\ (\lambda\, h \to \rho\ (F\ h\ x))\ p\ =\ \mathsf{ap}\ (\lambda\, h \to h\ (\theta\ x))\ p \cdot g\text{-}\beta_0\ x$

The last equation tells us that showing that two algebra morphisms are equal is somewhat like giving an algebra morphism from the witness of the β-law for f to that of g, as we can see if we draw the corresponding diagram:

comm diag

4.3 Initiality implies induction

The induction principle tells us that for a family $B\ :\ W\ \to\ \mathsf{Type}$ and a motive $m\ :\ (x\ :\ F\ W) \to \Box\ F\ B\ x \to B\ (c\ x)$, we get a dependent function $\mathsf{ind}\ :\ (x\ :\ W)\ \to\ B\ x$ along with a computation rule.

Note that m, along with c, can be seen as a morphism between the families (W, B) and $(F\ W, \Box\ F\ B)$.

Another way to say this is that given a function $p\ :\ B\ \to\ W$ for some $B\ :\ \mathsf{Type}$ and that we want $\theta\ :\ F\ B\ \to\ B$ such that θ, along with c, becomes a morphism $Fp\ \to\ p$ in the arrow category. This is equivalent to asking for an algebra (B, θ) along with an algebra morphism $(B, \theta)\ \to\ (W, c)$.

4.4 Induction implies initiality

5 Homotopy initiality for restricted 1-HITs