Computing an plotting the evolute of a curve

This project was done for Differential Geometry Course taught by Prof. Gabriel Pripoaie at students from second year of University Spiru-Haret, Faculty of Mathematics and Computer Science.

1 Introduction

Definition 1 *Let $c : I \to E_n$ be a parametrized curve. If the following vectors are linear independents $c^{(1)}(t), c^{(2)}(t), ..., c^{(n-1)}(t)$ $\forall t \in I$ we consider c is in general position.*

Proposition 2 *Let $c : I \to E_n$ be a curve in general position and let $\{e_1, ..., e_n\}$ be the Frenet mark associated to the curve. We have the following relations:*

$$\begin{aligned} e_i'(t) &= \sum_{j=1}^{n} a_{ij}(t) e_j(t) \\ a_{ij}(t) + a_{ji}(t) &= 0, \forall i, j \in \{1, ..., n\} \\ a_{ij}(t) &= 0, if \;\; j > i + 1 \end{aligned}$$

Definition 3 *Let $c : I \to E_n$ be a curve in general position and let $\{e_1, ..., e_n\}$ be the Frenet mark associated to the curve. We call the following function $K_i : I \to R$ the curvatures of c curve in the point $c(t)$ and they are defined by the following relation*

$$\begin{aligned} K_i(t) &= \frac{a_{ii+1}(t)}{\|c'(t)\|}, i \in \{1, ..., n-1\} \\ a_{ii+1}(t) &= \langle e_i'(t), e_{i+1}'(t) \rangle \end{aligned}$$

Definition 4 *Let consider a curve $c : I \to E_2$ and let be $t_0 \in I$ so that $c'(t_0) \neq 0$ and $K_1(t_0) \neq 0$. If a circle has three confronted points in $c(t_0)$ with the image of the curve c then it is called osculator circle of the curve in the point $c(t_0)$.*

Definition 5 *Let $c : t \in I \to c(t) = (x(t), y(t)) \in E_2$ a regulated curve with $K_1(t) \neq 0, \forall t \in I$. The osculator circle of the curve c in any point $c(t)$ has the center $\Omega = (X(t), Y(t))$. The curve $\Gamma : t \in I \to \Gamma(t) = (X(t), Y(t)) \in E_2$ is called evolute of the curve c. Where X(t) and Y(t) are defined by*

$$\begin{aligned} X(t) &= x(t) - \frac{y'(t)(x'^2(t) + y'^2(t))}{x'(t)y^{(2)}(t) - x^{(2)}(t)y'(t)} \\ Y(t) &= y(t) + \frac{x'(t)(x'^2(t) + y'^2(t))}{x'(t)y^{(2)}(t) - x^{(2)}(t)y'(t)} \end{aligned}$$

2 Implementation

The implementation is done in MAPLE because I have there differentiation, solving and plotting of data.

Algorithm 6

- compute $x'(t)y^{(2)}(t) - x^{(2)}(t)y'(t)$ and find it's solutions. This points will be eliminated from the plot, because the function isn't define in those points.
- compute $K_1(t) = a_{ii+1}(t)/\left\|c'(t)\right\|$.
- compute $c'(t) = 0$ and find it's solutions. This points will be also eliminated from the plot.
- compute analytical $X(t)$ and $Y(t)$
- plot with blue the evolute and with red the curve
- mark the singularity points where the curve is zero

Exercise 7 *Compute and plot the evolute for cicloide.*

Solution:
The cicloide is given by the following curve

$$c : I \subset R \to E_2, a > 0 \ where\, c(t) = (a * (t - \sin(t)), a * (1 - \cos(t))$$

Let be $a = 2$ and $I = [-2\pi, 5\pi]$.

In this case we have $K = 4(\cos(t) - 1)\left(|-2 + 2\cos(t)|^2 + 4\,|\sin(t)|^2\right)^{-3/2}$ and the parametric coordinates of osculator circle are $(2t + 2\sin(t), -2 + 2\cos(t))$.