Fundamentals of Deterministic Digital Signal Processing

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Overview

Review of Digital Signal Processing Concepts

- *Clerical work before we get started*
- *Signals & Systems*: LTI, sampling, discrete signals, quantization
- *Transforms*: $\mathcal{Z}$-transform, discrete-time Fourier transform, fast Fourier transform
- *Frequency Response & Filters*: transfer functions, FIR, IIR, filter design

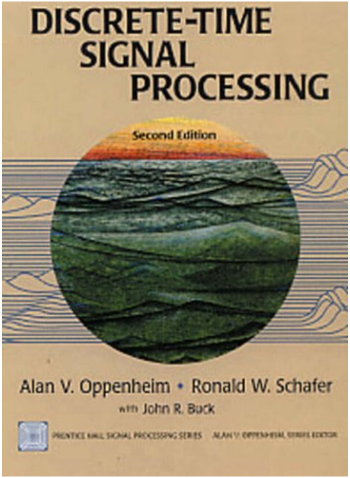
Random Signals

- *Probability & Statistics Review*: random variables, mean, expectations, variance
- *Random Processes*: Bernoulli

Examples & Homework

- We are going to do several examples, some of which do not have their solutions in the slides, and there are homework problems that are due in two weeks.

Logistics



ECES631
Fund. of Deterministic DSP

Instructor
Dr. Gail Rosen (gailr@ece.drexel.edu)

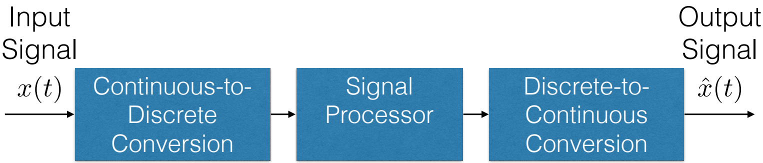
Office Hours
See Syllabus.

Text
Oppenheim, Schafer & Buck, "Discrete-Time Signal Processing," 3rd Ed.

Other Stuff

- Course materials are available on BBLearn.
- Homework #1 is posted. Due in 2 weeks.

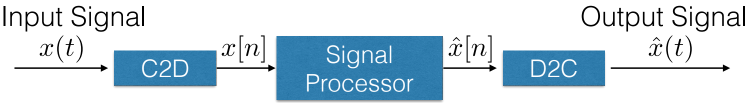
What is DSP?



Digital Signal Processing

- **Digital**
  - Method to represent a quantity, a phenomenon or an event
  - Why Digital?
- **Signal**
  - What is a signal?
  - What are we interested in?
- **Processing**
  - What kind of processing do we need to perform?
  - What special effects do we need to look out for?

What is DSP?



Digital Signal Processing

- What is a digital signal? Its just a sequence of numbers that can be represented as

$$x = \{x[n]\}, \qquad -\infty < n < \infty$$

- $x[n]$ is sampled from an analog signal

$$x[n] = x(nT_s) \qquad -\infty < n < \infty$$

where T_s is the sampling period, which is the reciprocal of the sampling rate (f_s).

Unit and Impulse Sequences

The discrete unit step ($u[n]$), and impulse sequences $\delta[n]$ are among the most commonly utilized sequences in DSP. Why?

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} = \sum_{k=-\infty}^{\infty} \delta[n]$$

Exponential Sequences

The exponential sequences is important for representing and analyzing linear time-invariant discrete-time systems

$$x[n] = A\alpha^n$$

where if $A, \alpha \in \mathbb{R}$ then $x[n] \in \mathbb{R}$.

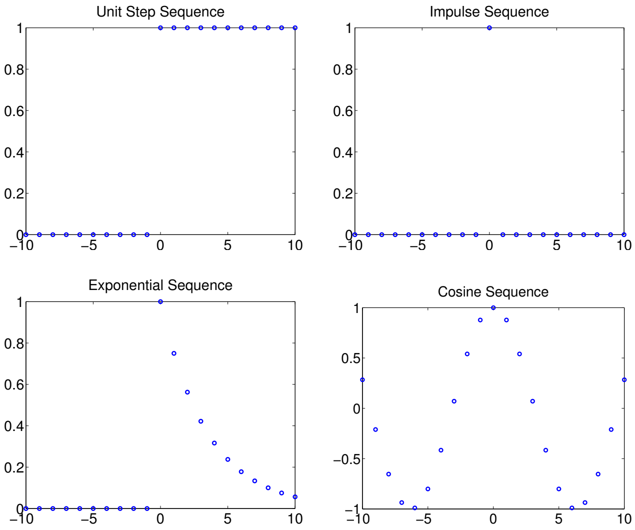
Euler's Identities

Never forget!

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}, \quad \sin(\omega n) = \frac{e^{j\omega n} - e^{-j\omega n}}{j2}$$

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n), \quad e^{-j\omega n} = \cos(\omega n) - j\sin(\omega n)$$

What do they look like?



Linear Systems

What is a linear system?

A class *of linear systems* is defined by the property of superposition. Let T be an operation, and $y_1[n]$ and $y_2[n]$ be the system responses of a system when $x_1[n]$ and $x_2[n]$ are the inputs, respectively. Then the system is linear if, and only if:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \quad \text{(additivity)}$$

and

$$T\{\alpha x[n]\} = \alpha T\{x[n]\} = \alpha y[n] \quad \text{(homogenity)}$$

where α is an arbitrary constant.

Questions

Is an accumulator system given by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

a linear system?

What is time-invariance?

A system is said to be time-invariant if a delay on the input sequence results in an equal delay of the output sequence. That is, if $\hat{x}[n] = x[n - n_0]$ then $\hat{y}[n] = y[n - n_0]$.

The Accumulator as a Time-Invariant System

Define $x_i[n] = x[n - n_0]$ and let

$$y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Next, we have

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k - n_0]$$

Substituting the change of variables for $k_1 = k - n_0$ into the sum gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]$$

Other Important Stuff

Causality
A system is casual if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only of the input sequence values for $n \leq n_0$. Is $y[n] = x[n+1] - x[n]$ causal? How about $y[n] = \log(x[|n| - n_0])$?

Stability
A system is bounded-input bounded-output (BIBO) stable if and only if every bounded input sequence results in a bounded output sequence.

Everything else
Seriously, read Chapter 2 of the text book!

Linear Time-Invariant Systems

What are they?

- As the name states, these systems are linear and time-invariant. They are one of the most important components to the field of digital signal processing.
- An LTI system can be completely characterized by its impulse response. That is, $x[n] = \delta[n]$.

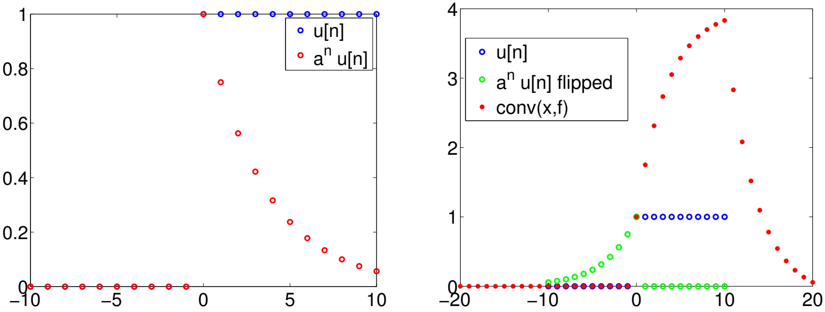
Convolution

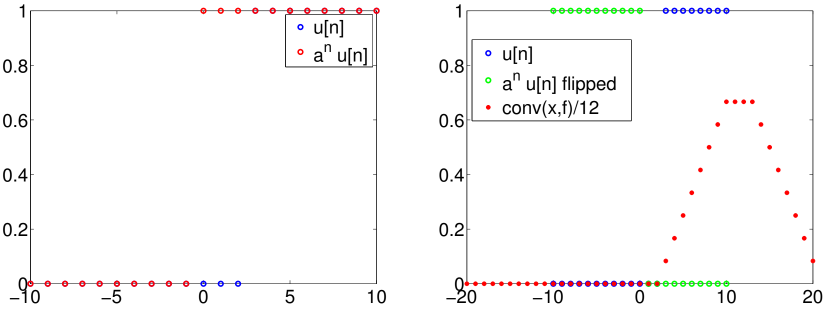
- The convolution of two sequences x and h is defined by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \star h[n]$$

- The importance of the equation shown above cannot be overstated enough

Convolution Example




u[n]
a^n u[n]
1
0.8
0.6
0.4
0.2
0
–10
–5
0
5
10
u[n]
a^n u[n] flipped
conv(x,f)/12
1
0.8
0.6
0.4
0.2
0
–20
–10
0
10
20

Properties of LTI Systems

Some Properties

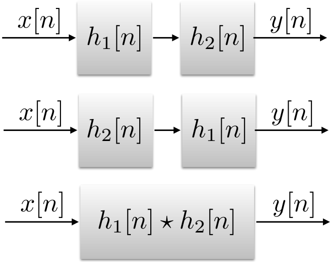
- An LTI system can be completely characterized by its impulse response.
- Convolution is commutative, that is, $x[n] \star h[n] = h[n] \star x[n]$

$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] \star x[n]$$

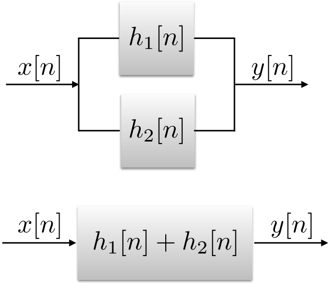
- For an LTI system with $h[n] = h_1[n] + h_2[n]$, we have

$$x[n] \star (h_1[n] + h_2[n]) = h_1[n] \star x[n] + h_2[n] \star x[n]$$

Equivalence of LTI Systems



$$h_1[n] \star h_2[n] = h_2[n] \star h_1[n]$$



LTI Examples: Moving Average

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

Linear?: Yes

$$\frac{1}{L}\sum_{k=0}^{L-1}(ax_1[n-k] + bx_2[n-k]) = a\left(\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right) + b\left(\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right)$$

Time-Invariant?: Yes

$$\frac{1}{L}\sum_{k=0}^{L-1} x[n-k-n_0] = \frac{1}{L}\sum_{k=0}^{L-1} x[(n-n_0)-k] = y[n-n_0]$$

Casual?: Yes

$$y[n] = \frac{1}{L}\left(x[n] + x[n-1] + \ldots + x[n-L+1]\right)$$

BIBO?: Yes

$$|y[n]| = \left|\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right| \leq \frac{1}{K}\sum_{k=0}^{L-1} |x[n-k]| \leq B_x$$

$$y[n] = x[Mn]$$

Linear?: Yes

$$ax_1[Mn] + bx_2[Mn] = ay_1[n] + by_2[n]$$

Time-Invariant?: No

$$y_1[n] = x_1[Mn] = x[Mn - n_0] \neq y[n - n_0] = x[M(n - n_0)]$$

Casual?: No

$$y[-1] = x[M], \text{but } y[1] = x[M]$$

BIBO?: Yes

$$|y[n]| = |x[Mn]| \leq B_x$$

Ideal Delay

$$h[n] = \delta[n - n_0]$$

Moving Average

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$
$$= \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

Forward Difference

$$h[n] = \delta[n+1] - \delta[n]$$

Backward Difference

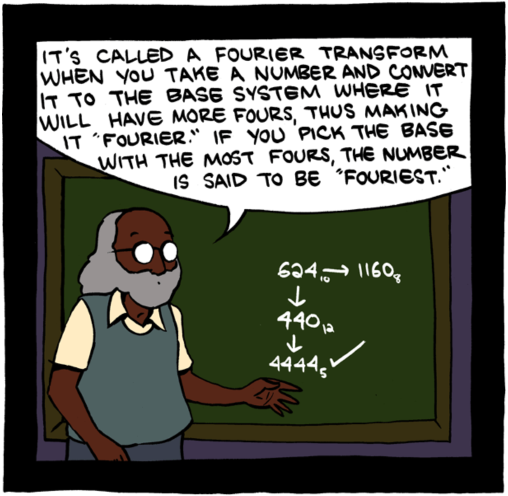
$$h[n] = \delta[n] - \delta[n-1]$$

Accumulator

$$h[n] = \sum_{k=-M_1}^{M_2} \delta[k]$$
$$= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = u[n]$$

⋆Constant Coefficient Linear Difference⋆

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{m=0}^{M-1} b_m x[n-m]$$



Teaching math was way more fun after tenure.

Frequency Domain Representations of Discrete Signals

LTI Systems

- LTI systems can be written as a weighted sum of delayed impulse response coefficients. Until this point, we have only considered time domain representations of signals.
- Recall a sinusoid can be written as a complex exponential functions, and as it turns out, complex exponential sequences are eigenfunctions of a LTI system.

Eigenfunctions for LTI Systems

To demonstrate the eigenfunction property of LTI systems, let $x[n] = \mathrm{e}^{j\omega n}$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[n]\mathrm{e}^{-j\omega(n-k)} = \mathrm{e}^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-j\omega k}\right)$$

If $H(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-j\omega k}$, then

$$y[n] = H(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}$$

Thus, $\mathrm{e}^{j\omega n}$ is an eigenfunction with a corresponding eigenvalue $H(\mathrm{e}^{j\omega})$. Furthermore, for convenience, we may write $H(\mathrm{e}^{j\omega})$ as $|H(\mathrm{e}^{j\omega})|\mathrm{e}^{j\angle H(\mathrm{e}^{j\omega})}$.

Frequency Domain Example

Question

Find the frequency response of a moving averaging system given by:

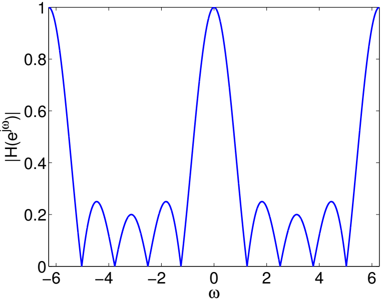
$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

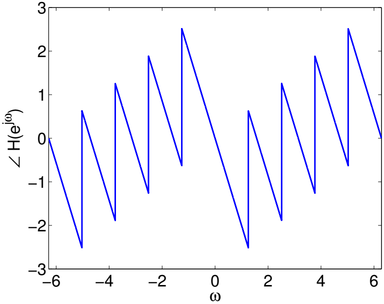
Frequency Domain Example

Therefore, the frequency response is given by

$$
\begin{aligned}
H(\mathrm{e}^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} \mathrm{e}^{-j\omega n} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{\mathrm{e}^{j\omega M_1} - \mathrm{e}^{-j\omega(M_2+1)}}{1 - \mathrm{e}^{-j\omega}} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{\mathrm{e}^{j\omega(M_1+M_2+1)/2} - \mathrm{e}^{-j\omega(M_1+M_2+1)/2}}{1 - \mathrm{e}^{-j\omega}} \mathrm{e}^{-j\omega(M_2-M_1+1)/2} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{\mathrm{e}^{j\omega(M_1+M_2+1)/2} - \mathrm{e}^{-j\omega(M_1+M_2+1)/2}}{\mathrm{e}^{j\omega/2} - \mathrm{e}^{-j\omega/2}} \mathrm{e}^{-j\omega(M_2-M_1)/2} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{\sin\left(\omega(M_1 + M_2 + 1)/2\right)}{\sin(\omega/2)} \mathrm{e}^{-j\omega(M_2-M_1)/2}
\end{aligned}
$$

Frequency Domain Example





Frequency Response on the Ideal Delay

Let us examine the simple system $y[n] = x[n - n_0]$ where n_0 is a fixed integer. If we consider $x[n] = e^{j\omega n}$ as an input to the system, then:

$$y[n] = e^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) = H(e^{j\omega})e^{j\omega n} = e^{j\omega(n-n_0)} = e^{j\omega n}e^{-j\omega n_0}$$

Thus for any value of ω, we obtain an output that is the input multiplied by a complex constant. The frequency response is given by:

$$H(e^{j\omega}) = e^{-j\omega n_0}$$

What is $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$?

Example

Frequency Response on the Ideal Delay

Let us examine the simple system $y[n] = x[n - n_0]$ where n_0 is a fixed integer. If we consider $x[n] = \mathrm{e}^{j\omega n}$ as an input to the system, then:

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}\mathrm{e}^{j\phi}\mathrm{e}^{j\omega_0 n} + \frac{A}{2}\mathrm{e}^{-j\phi}\mathrm{e}^{-j\omega_0 n} = x_1[n] + x_2[n]$$

If $y[n] = h[n] \star x[n]$, then

$$\begin{aligned} y[n] &= \frac{A}{2}\left(H(\mathrm{e}^{j\omega_0})\mathrm{e}^{j\phi}\mathrm{e}^{j\omega_0 n} + H(\mathrm{e}^{-j\omega_0})\mathrm{e}^{-j\phi}\mathrm{e}^{-j\omega_0 n}\right) \\ &= A|H(\mathrm{e}^{j\omega_0})|\cos(\omega_0 n + \phi + \angle H(\mathrm{e}^{j\omega_0})) \end{aligned}$$

For the simple example of an ideal delay we have $\angle H(\mathrm{e}^{j\omega_0} = -\omega_0 n_0$, then

$$y[n] = A\cos(\omega_0 n + \phi + -\omega_0 n_0) = A\cos(\omega_0(n - n_0) + \phi)$$

Historical Note: The 1889 World's Fair in Paris, France

And Who Made the List?

List of the 72 names on the Eiffel Tower

From Wikipedia, the free encyclopedia

FOURIER	Joseph Fourier	mathematician	NE13	

THÉORIE

DE

LA CHALEUR.

Et ignem regunt numeri. PLATO.

CHAPITRE PREMIER.

INTRODUCTION.

SECTION PREMIÈRE.

Exposition de l'objet de cet ouvrage.

ART. 1er.

THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

The Discrete-Time Fourier Transform

Discrete Fourier-Time Transform (Analysis)

The frequency spectrum for some signal $x[n]$ can be represented by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- For convenience, we write $\angle X(e^{j\omega}) \in [\pm\pi]$.

Inverse Discrete-Time Fourier Transform (Synthesis)

Any discrete sequence can be represented by:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega$$

Example

Absolute Summability for a Suddenly-Applied Exponential

Let $x[n] = a^n u[n]$. The Discrete-Time Fourier transform of this sequence is

$$X(\mathrm{e}^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-j\omega n} = \sum_{n=0}^{\infty} a^n \mathrm{e}^{-j\omega n} = \sum_{n=0}^{\infty} \left(a\mathrm{e}^{-j\omega}\right)^n$$
$$= \frac{1}{1 - a\mathrm{e}^{-j\omega}}$$

if $|a\mathrm{e}^{-j\omega}| < 1$ or $a < 1$. Clearly, the condition $a < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty$$

again, only if $a < 1$.

Low Pass Filter

Let us deterring the impulse response of an *ideal* low-pass filter. The frequency response is given by:

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Then

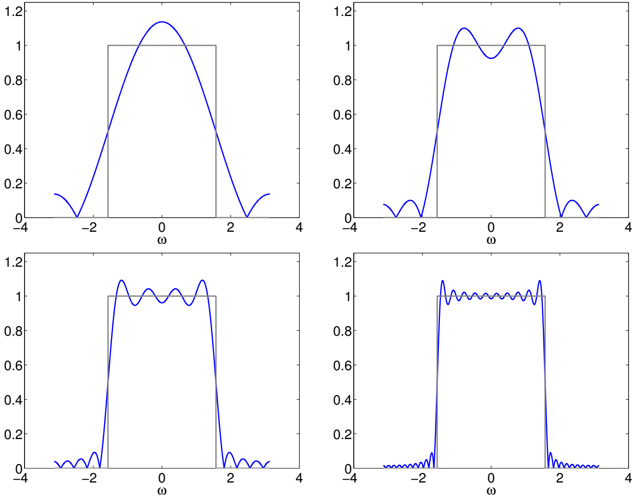
$$h_{\text{lp}}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\text{lp}}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{1}{j2\pi n}\left[e^{j\omega n}\right]_{\omega=-\omega_c}^{\omega=\omega_c}$$

$$= \frac{1}{j2\pi n}\left(e^{j\omega n} - e^{-j\omega n}\right) = \frac{\sin(\omega_c n)}{\pi n}$$

Notes

- The impulse sequence $h_{\text{lp}}[n]$ is not zero for $n < 0$, and $h_{\text{lp}}[n]$ is not absolutely summable.
- **Super Important Tip**: See Table 2.1 on page 56 of DTSP.

Example of Gibbs Phenomenon



Parseval's Theorem

General Idea

- Parseval's theorem provides a convenient way to compute the energy of a signal in the time-domain or frequency domain. In physics, this theorem is commonly referred to as the the Plancherel theorem.

Derivation

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\mathrm{e}^{j\omega})|^2\mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}X(\mathrm{e}^{j\omega})X^*(\mathrm{e}^{j\omega})\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-j\omega n}\right)\left(\sum_{n'=-\infty}^{\infty}x^*[n']\mathrm{e}^{j\omega n'}\right)\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}x[n]\sum_{n'=-\infty}^{\infty}x^*[n']\mathrm{e}^{j\omega(n'-n)}\mathrm{d}\omega \\
&= \sum_{n=-\infty}^{\infty}x[n]\sum_{n'=-\infty}^{\infty}x^*[n']\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{j\omega(n'-n)}\mathrm{d}\omega}_{\begin{cases}1 & n=n' \\ 0 & \text{otherwise}\end{cases}} = \sum_{n=-\infty}^{\infty}x[n]x^*[n] = \sum_{n=-\infty}^{\infty}|x[n]|^2
\end{aligned}$$

Frequency Differentiation

Derivation

Lets us look at what happens when we take the derivative of the DTFT of a sequence $x[n]$.

$$\frac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-j\omega n}\right\} = \sum_{n=-\infty}^{\infty} x[n]\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\mathrm{e}^{-j\omega n}\right\} = -j\sum_{n=-\infty}^{\infty} nx[n]\mathrm{e}^{-j\omega n}$$

Therefore,

$$nx[n] \leftrightarrow j\frac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega}$$

Example

Find the DTFT of $na^n u[n]$.

Properties of the Discrete-Time Fourier Transform

A Few other use properties ($x[n] \leftrightarrow X(\mathrm{e}^{j\omega})$)

- If $y[n] = h[n] \star x[n]$ then $Y(\mathrm{e}^{j\omega}) = H(\mathrm{e}^{j\omega})X(\mathrm{e}^{j\omega})$. This is the *Convolution Theorem*.
- Frequency Differentiation

$$nx[n] \leftrightarrow j\frac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega}$$

- Time Shifting

$$x[n-n_0] \leftrightarrow \mathrm{e}^{j\omega n_0}X(\mathrm{e}^{j\omega})$$

- Frequency Shifting

$$\mathrm{e}^{j\omega_0 n}x[n] \leftrightarrow X\left(\mathrm{e}^{j(\omega-\omega_0)}\right)$$

- Convolutions

$$x[n] \star y[n] \leftrightarrow X(\mathrm{e}^{j\omega})Y(\mathrm{e}^{j\omega})$$

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{j\theta})Y(\mathrm{e}^{j(\omega-\theta)})\mathrm{d}\theta$$

Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{j\omega})(1 - be^{j\omega})}$$

for $a \neq b$. Using the equation for the inverse discrete-time Fourier transform leads to an integral that is difficult to evaluate. However, we can use partial fraction expansion to put $X(e^{j\omega})$ in a form where we can use tables of transforms.

$$X(e^{j\omega}) = \frac{\frac{a}{a-b}}{1 - ae^{j\omega}} + \frac{\frac{b}{a-b}}{1 - be^{j\omega}}$$

After of examining tables with discrete-time Fourier transform leads to

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

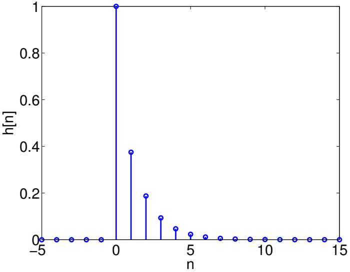
Does this look unfamiliar? Read up on partial fraction expansion.

Example

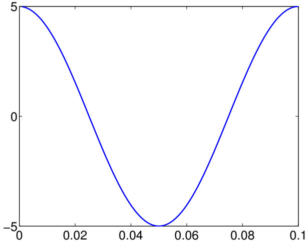
Determining the Impulse Response

Given that $y[n] = h[n] \star x[n]$, find $h[n]$ when

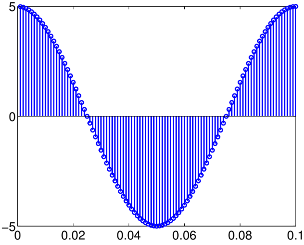
$$y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$$



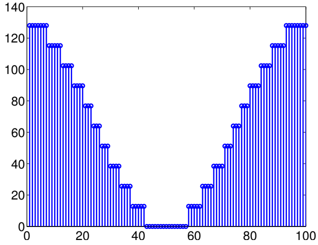
Sampling



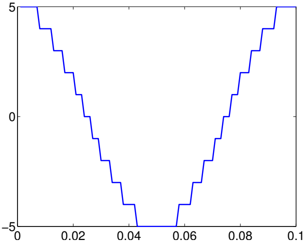
(a) $x(t)$



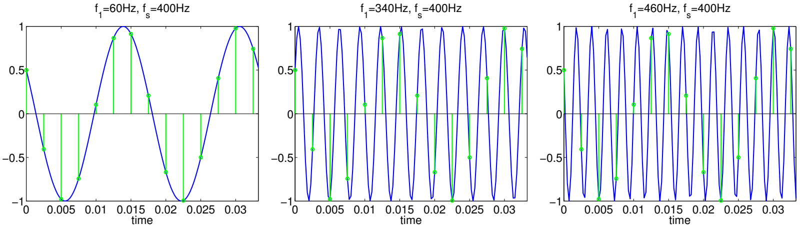
(b) sampled $x(t)$



(c) quantized signal



(d) $\hat{x}(t)$

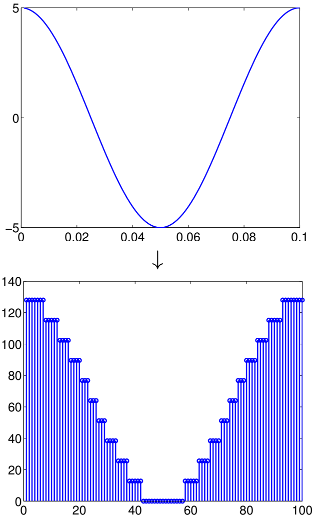


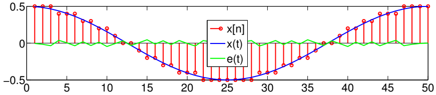
The Sampling Theorem

- A bandlimited signal can be reconstructed exactly from samples taken with sampling frequency

$$\frac{1}{T} = f_s \geq 2f_{\max}$$

- Question: If I sampled a signal x which is a cosine with $f = 34,723,487$Hz at a rate of $f_s = 1,234$Hz, where will it show up in the spectrum? Are we seeing folding?

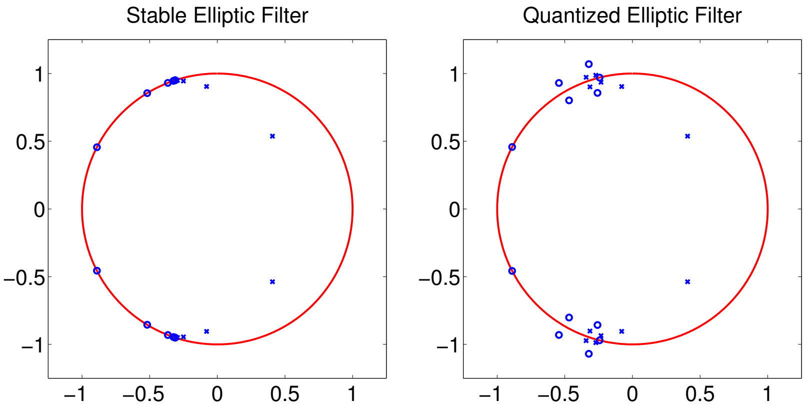




Quantization

- Truncating a continuous signal's discrete representation to a finite set of values.
  - for example, a signal $x(t) \in (\pm 5V)$ is quantized with a resolution of $\frac{10V}{128}$
  - a form of compression
- Quantization can be non-uniform for the range of $x(t)$. For example, sampling of voiced speech.
- Quantizing coefficients can have adverse effects to a systems response. Can you think of an example?

Pole-Zero plots of an elliptic filter before and after have the coefficients a_k and b_k quantized.



Quantization & the Elliptic Filter

Frequency and phase response of an elliptic filter before and after have the coefficients a_k and b_k quantized.