Fundamentals of Deterministic Digital Signal Processing

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Overview

Review of Digital Signal Processing Concepts

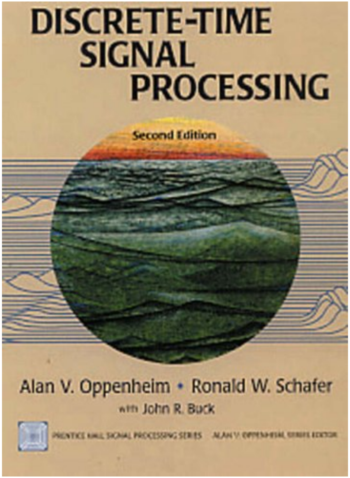
- *Clerical work before we get started*
- *Signals & Systems*: LTI, sampling, discrete signals, quantization
- *Transforms*: $\mathcal{Z}$-transform, discrete-time Fourier transform, fast Fourier transform
- *Frequency Response & Filters*: transfer functions, FIR, IIR, filter design

Random Signals

- *Probability & Statistics Review*: random variables, mean, expectations, variance
- *Random Processes*: Bernoulli

Examples & Homework

- We are going to do several examples, some of which do not have their solutions in the slides, and there are homework problems that are due in two weeks.



ECES631
Fund. of Deterministic DSP

Instructor
Dr. Gail Rosen (gailr@ece.drexel.edu)

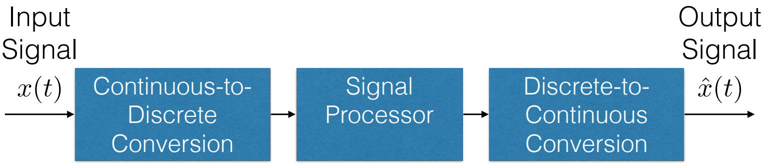
Office Hours
See Syllabus.

Text
Oppenheim, Schafer & Buck, "Discrete-Time Signal Processing," 3rd Ed.

Other Stuff

- Course materials are available on BBLearn.
- Homework #1 is posted. Due in 2 weeks.

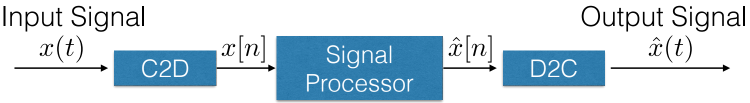
What is DSP?



Digital Signal Processing

- **Digital**
  - Method to represent a quantity, a phenomenon or an event
  - Why Digital?
- **Signal**
  - What is a signal?
  - What are we interested in?
- **Processing**
  - What kind of processing do we need to perform?
  - What special effects do we need to look out for?

What is DSP?



Digital Signal Processing

- What is a digital signal? Its just a sequence of numbers that can be represented as

$$x = \{x[n]\}, \qquad -\infty < n < \infty$$

- $x[n]$ is sampled from an analog signal

$$x[n] = x(nT_s) \qquad -\infty < n < \infty$$

where T_s is the sampling period, which is the reciprocal of the sampling rate (f_s).

Common sequences and operations

Unit and Impulse Sequences

The discrete unit step ($u[n]$), and impulse sequences $\delta[n]$ are among the most commonly utilized sequences in DSP. Why?

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \qquad u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} = \sum_{k=-\infty}^{\infty} \delta[n]$$

Exponential Sequences

The exponential sequences is important for representing and analyzing linear time-invariant discrete-time systems

$$x[n] = A\alpha^n$$

where if $A, \alpha \in \mathbb{R}$ then $x[n] \in \mathbb{R}$.

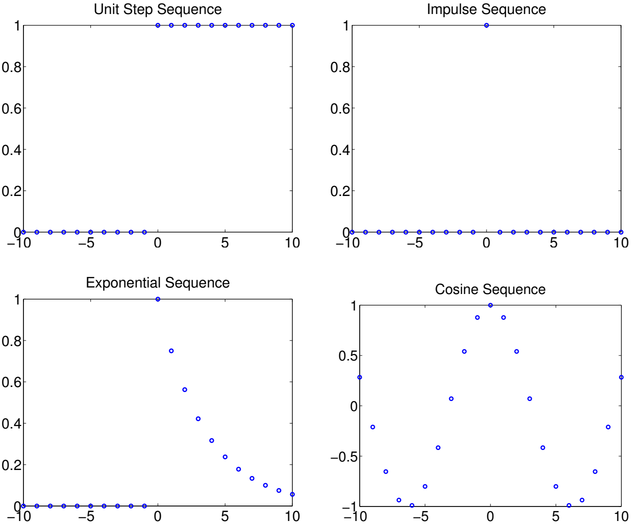
Euler's Identities

Never forget!

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}, \quad \sin(\omega n) = \frac{e^{j\omega n} - e^{-j\omega n}}{j2}$$

$$e^{j\omega n} = \cos(\omega n) + j\sin(\omega n), \quad e^{-j\omega n} = \cos(\omega n) - j\sin(\omega n)$$

What do they look like?



What is a linear system?

A class *of linear systems* is defined by the property of superposition. Let T be an operation, and $y_1[n]$ and $y_2[n]$ be the system responses of a system when $x_1[n]$ and $x_2[n]$ are the inputs, respectively. Then the system is linear if, and only if:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \quad \text{(additivity)}$$

and

$$T\{\alpha x[n]\} = \alpha T\{x[n]\} = \alpha y[n] \quad \text{(homogenity)}$$

where α is an arbitrary constant.

Questions

Is an accumulator system given by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

a linear system?

Time-Invariant Systems

What is time-invariance?

A system is said to be time-invariant if a delay on the input sequence results in an equal delay of the output sequence. That is, if $\hat{x}[n] = x[n - n_0]$ then $\hat{y}[n] = y[n - n_0]$.

The Accumulator as a Time-Invariant System

Define $x_i[n] = x[n - n_0]$ and let

$$y[n - n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

Next, we have

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] = \sum_{k=-\infty}^{n} x[k - n_0]$$

Substituting the change of variables for $k_1 = k - n_0$ into the sum gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]$$

Other Important Stuff

Causality
A system is casual if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only of the input sequence values for $n \leq n_0$. Is $y[n] = x[n+1] - x[n]$ causal? How about $y[n] = \log(x[|n| - n_0])$?

Stability
A system is bounded-input bounded-output (BIBO) stable if and only if every bounded input sequence results in a bounded output sequence.

Everything else
Seriously, read Chapter 2 of the text book!

What are they?

- As the name states, these systems are linear and time-invariant. They are one of the most important components to the field of digital signal processing.
- An LTI system can be completely characterized by its impulse response. That is, $x[n] = \delta[n]$.

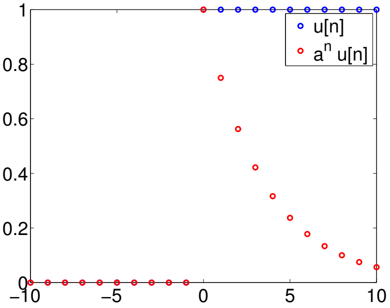
Convolution

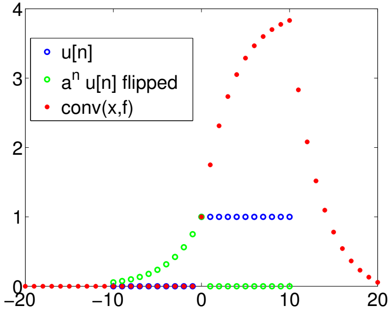
- The convolution of two sequences x and h is defined by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] \star h[n]$$

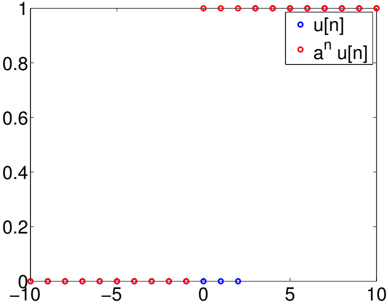
- The importance of the equation shown above cannot be overstated enough

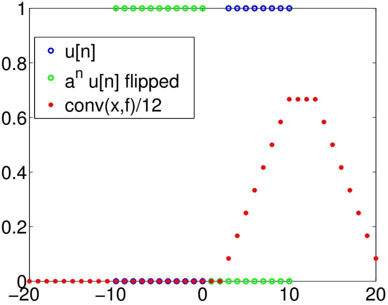
Convolution Example





Convolution Example





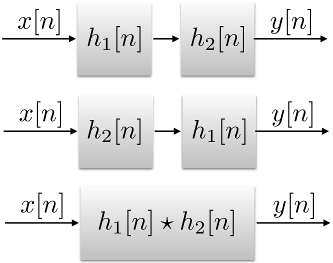
Some Properties

- An LTI system can be completely characterized by its impulse response.
- Convolution is commutative, that is, $x[n] \star h[n] = h[n] \star x[n]$

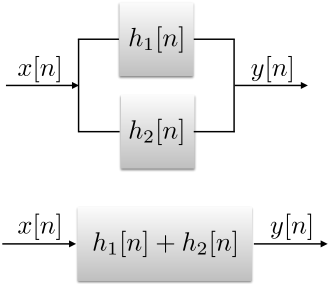
$$y[n] = \sum_{m=\infty}^{-\infty} x[n-m]h[m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] \star x[n]$$

- For an LTI system with $h[n] = h_1[n] + h_2[n]$, we have

$$x[n] \star (h_1[n] + h_2[n]) = h_1[n] \star x[n] + h_2[n] \star x[n]$$



$$h_1[n] \star h_2[n] = h_2[n] \star h_1[n]$$



$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

Linear?: Yes

$$\frac{1}{L}\sum_{k=0}^{L-1}(ax_1[n-k] + bx_2[n-k]) = a\left(\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right) + b\left(\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right)$$

Time-Invariant?: Yes

$$\frac{1}{L}\sum_{k=0}^{L-1} x[n-k-n_0] = \frac{1}{L}\sum_{k=0}^{L-1} x[(n-n_0)-k] = y[n-n_0]$$

Casual?: Yes

$$y[n] = \frac{1}{L}\left(x[n] + x[n-1] + \ldots + x[n-L+1]\right)$$

BIBO?: Yes

$$|y[n]| = \left|\frac{1}{L}\sum_{k=0}^{L-1} x[n-k]\right| \leq \frac{1}{K}\sum_{k=0}^{L-1} |x[n-k]| \leq B_x$$

LTI Examples: Downsampler

$$y[n] = x[Mn]$$

Linear?: Yes

$$ax_1[Mn] + bx_2[Mn] = ay_1[n] + by_2[n]$$

Time-Invariant?: No

$$y_1[n] = x_1[Mn] = x[Mn - n_0] \neq y[n - n_0] = x[M(n - n_0)]$$

Casual?: No

$$y[-1] = x[M], \text{but } y[1] = x[M]$$

BIBO?: Yes

$$|y[n]| = |x[Mn]| \leq B_x$$

Ideal Delay

$$h[n] = \delta[n - n_0]$$

Moving Average

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$
$$= \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

Forward Difference

$$h[n] = \delta[n+1] - \delta[n]$$

Backward Difference

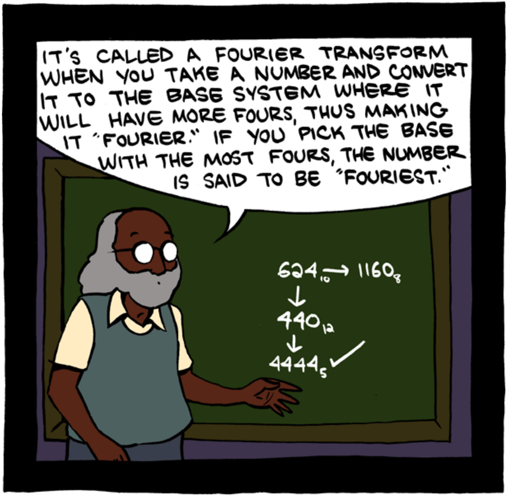
$$h[n] = \delta[n] - \delta[n-1]$$

Accumulator

$$h[n] = \sum_{k=-M_1}^{M_2} \delta[k]$$
$$= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = u[n]$$

⋆Constant Coefficient Linear Difference⋆

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{m=0}^{M-1} b_m x[n-m]$$



Teaching math was way more fun after tenure.

LTI Systems

- LTI systems can be written as a weighted sum of delayed impulse response coefficients. Until this point, we have only considered time domain representations of signals.
- Recall a sinusoid can be written as a complex exponential functions, and as it turns out, complex exponential sequences are eigenfunctions of a LTI system.

Eigenfunctions for LTI Systems

To demonstrate the eigenfunction property of LTI systems, let $x[n] = \mathrm{e}^{j\omega n}$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]\mathrm{e}^{-j\omega(n-k)} = \mathrm{e}^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[n-k]\mathrm{e}^{-j\omega k}\right)$$

If $H(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\mathrm{e}^{-j\omega k}$, then

$$y[n] = H(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}$$

Thus, $\mathrm{e}^{j\omega n}$ is an eigenfunction with a corresponding eigenvalue $H(\mathrm{e}^{j\omega})$. Furthermore, for convenience, we may write $H(\mathrm{e}^{j\omega})$ as $|H(\mathrm{e}^{j\omega})|\mathrm{e}^{j\angle H(\mathrm{e}^{j\omega})}$.

Frequency Domain Example

Question

Find the frequency response of a moving averaging system given by:

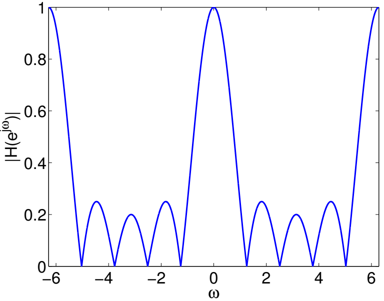
$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

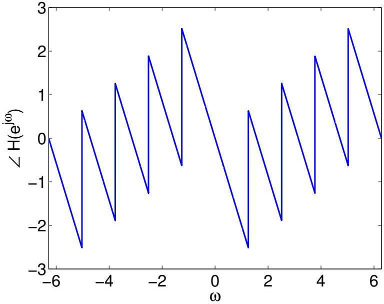
Frequency Domain Example

Therefore, the frequency response is given by

$$
\begin{aligned}
H(e^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{1 - e^{-j\omega}} e^{-j\omega(M_2-M_1+1)/2} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-j\omega(M_2-M_1)/2} \\
&= \frac{1}{M_1 + M_2 + 1} \frac{\sin\left(\omega(M_1 + M_2 + 1)/2\right)}{\sin(\omega/2)} e^{-j\omega(M_2-M_1)/2}
\end{aligned}
$$

Frequency Domain Example





Historical Note: The 1889 World's Fair in Paris, France

List of the 72 names on the Eiffel Tower

From Wikipedia, the free encyclopedia

FOURIER	Joseph Fourier	mathematician	NE13	

THÉORIE

DE

LA CHALEUR.

Et ignem regunt numeri. PLATO.

CHAPITRE PREMIER.

INTRODUCTION.

SECTION PREMIÈRE.

Exposition de l'objet de cet ouvrage.

ART. 1er.

THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

Discrete Fourier-Time Transform (Analysis)

The frequency spectrum for some signal $x[n]$ can be represented by:

$$X(\mathrm{e}^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-j\omega n}$$

- For convenience, we write $\angle X(\mathrm{e}^{j\omega}) \in [\pm\pi]$.

Inverse Discrete-Time Fourier Transform (Synthesis)

Any discrete sequence can be represented by:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}\mathrm{d}\omega$$

Example

Absolute Summability for a Suddenly-Applied Exponential

Let $x[n] = a^n u[n]$. The Discrete-Time Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$
$$= \frac{1}{1 - ae^{-j\omega}}$$

if $|ae^{-j\omega}| < 1$ or $a < 1$. Clearly, the condition $a < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty$$

again, only if $a < 1$.

Example

Low Pass Filter

Let us deterring the impulse response of an *ideal* low-pass filter. The frequency response is given by:

$$H_{\mathrm{lp}}(\mathrm{e}^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

Then

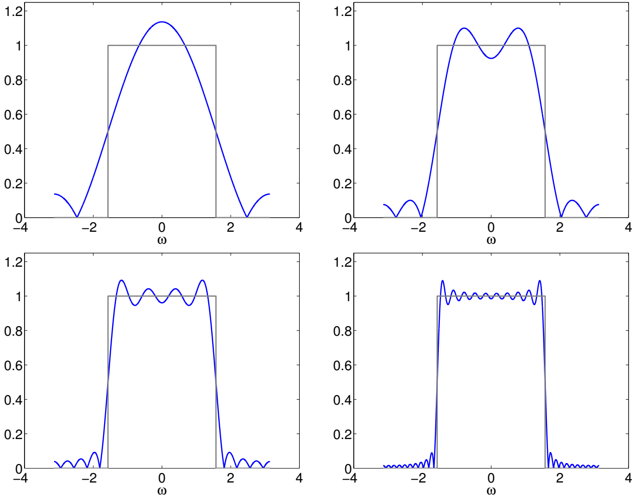
$$h_{\mathrm{lp}}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{\mathrm{lp}}(\mathrm{e}^{j\omega}) \mathrm{e}^{j\omega n} \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \mathrm{e}^{j\omega n} \mathrm{d}\omega = \frac{1}{j2\pi n} \left[\mathrm{e}^{j\omega n}\right]_{\omega=-\omega_c}^{\omega=\omega_c}$$

$$= \frac{1}{j2\pi n} \left(\mathrm{e}^{j\omega n} - \mathrm{e}^{-j\omega n}\right) = \frac{\sin(\omega_c n)}{\pi n}$$

Notes

- The impulse sequence $h_{\mathrm{lp}}[n]$ is not zero for $n < 0$, and $h_{\mathrm{lp}}[n]$ is not absolutely summable.
- **Super Important Tip**: See Table 2.1 on page 56 of DTSP.

Example of Gibbs Phenomenon



Parseval's Theorem

General Idea

- Parseval's theorem provides a convenient way to compute the energy of a signal in the time-domain or frequency domain. In physics, this theorem is commonly referred to as the the Plancherel theorem.

Derivation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\mathrm{e}^{j\omega})|^2\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}X(\mathrm{e}^{j\omega})X^*(\mathrm{e}^{j\omega})\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}x[n]\mathrm{e}^{-j\omega n}\right)\left(\sum_{n'=-\infty}^{\infty}x^*[n']\mathrm{e}^{j\omega n'}\right)\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}x[n]\sum_{n'=-\infty}^{\infty}x^*[n']\mathrm{e}^{j\omega(n'-n)}\mathrm{d}\omega$$

$$= \sum_{n=-\infty}^{\infty}x[n]\sum_{n'=-\infty}^{\infty}x^*[n']\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{j\omega(n'-n)}\mathrm{d}\omega}_{\begin{cases}1 & n=n'\\ 0 & \text{otherwise}\end{cases}} = \sum_{n=-\infty}^{\infty}x[n]x^*[n] = \sum_{n=-\infty}^{\infty}|x[n]|^2$$

Properties of the Discrete-Time Fourier Transform

A Few other use properties ($x[n] \leftrightarrow X(\mathrm{e}^{j\omega})$)

- If $y[n] = h[n] \star x[n]$ then $Y(\mathrm{e}^{j\omega}) = H(\mathrm{e}^{j\omega})X(\mathrm{e}^{j\omega})$. This is the *Convolution Theorem*.
- Frequency Differentiation

$$nx[n] \leftrightarrow j\frac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega}$$

- Time Shifting

$$x[n - n_0] \leftrightarrow \mathrm{e}^{j\omega n_0}X(\mathrm{e}^{j\omega})$$

- Frequency Shifting

$$\mathrm{e}^{j\omega_0 n}x[n] \leftrightarrow X\left(\mathrm{e}^{j(\omega-\omega_0)}\right)$$

Discrete-Time Fourier Transforms using Tables

Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{j\omega})\,(1 - be^{j\omega})}$$

for $a \neq b$. Using the equation for the inverse discrete-time Fourier transform leads to an integral that is difficult to evaluate. However, we can use partial fraction expansion to put $X(e^{j\omega})$ in a form where we can use tables of transforms.

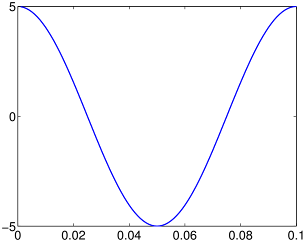
$$X(e^{j\omega}) = \frac{\frac{a}{a-b}}{1 - ae^{j\omega}} + \frac{\frac{b}{a-b}}{1 - be^{j\omega}}$$

After of examining tables with discrete-time Fourier transform leads to

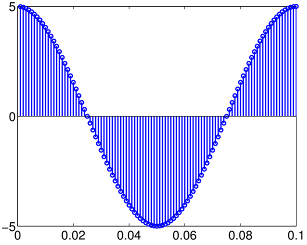
$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

Does this look unfamiliar? Read up on partial fraction expansion.

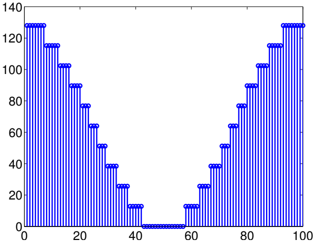
Sampling



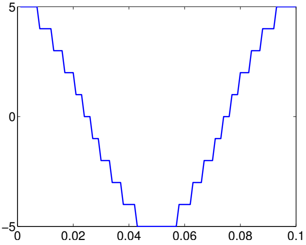
(a) $x(t)$



(b) sampled $x(t)$

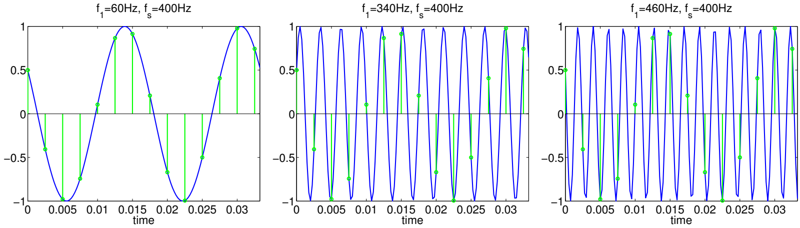


(c) quantized signal



(d) $\hat{x}(t)$

Aliasing



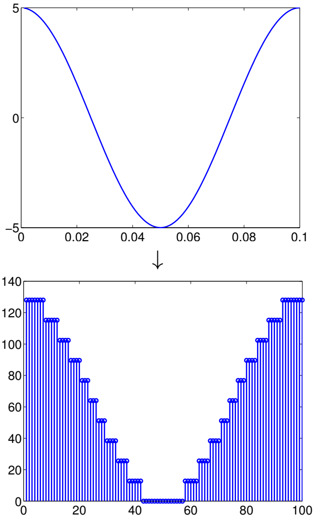
The Sampling Theorem

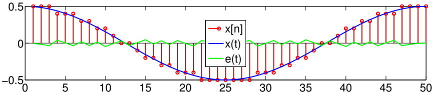
- A bandlimited signal can be reconstructed exactly from samples taken with sampling frequency

$$\frac{1}{T} = f_s \geq 2f_{\max}$$

- Question: If I sampled a signal x which is a cosine with $f = 34,723,487$Hz at a rate of $f_s = 1,234$Hz, where will it show up in the spectrum? Are we seeing folding?

Quantization

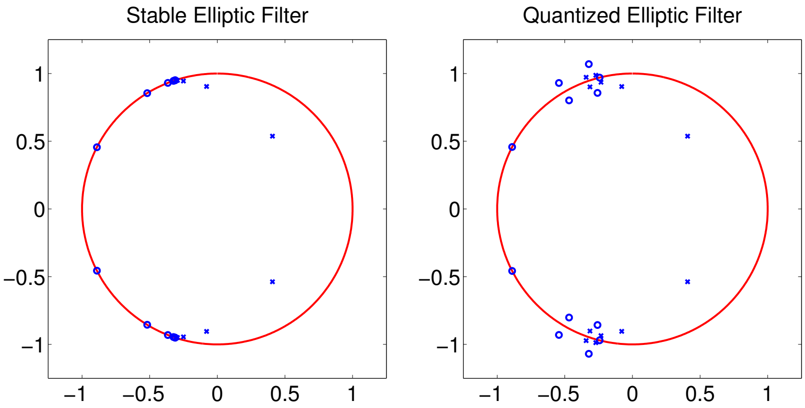




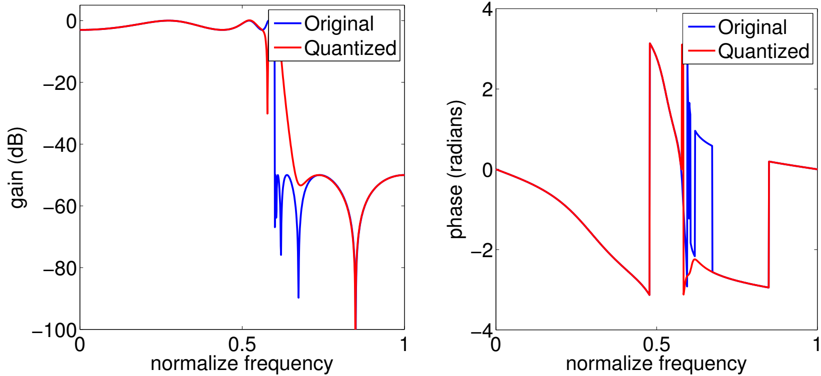
Quantization

- Truncating a continuous signal's discrete representation to a finite set of values.
  - for example, a signal $x(t) \in (\pm 5V)$ is quantized with a resolution of $\frac{10V}{128}$
  - a form of compression
- Quantization can be non-uniform for the range of $x(t)$. For example, sampling of voiced speech.
- Quantizing coefficients can have adverse effects to a systems response. Can you think of an example?

Pole-Zero plots of an elliptic filter before and after have the coefficients a_k and b_k quantized.



Frequency and phase response of an elliptic filter before and after have the coefficients a_k and b_k quantized.



What's this all about?

- The Discrete Fourier Transform is a sampled version of the Discrete-Time Fourier Transform with the sampling happening in the frequency domain. We let $\hat{\omega} = \frac{2\pi k}{N}$ then the analysis equation becomes

$$X[k] = \sum_{n=0}^{N-1} x[n] \mathrm{e}^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

and the synthesis equation is given by

$$x[k] = \sum_{k=0}^{N-1} X[n] \mathrm{e}^{j\frac{2\pi kn}{N}} = \sum_{k=0}^{N-1} X[n] W_N^{-kn}$$

Clever Manipulations of the Discrete Fourier Transform

Let us take a moment and begin by decomposing the Discrete Fourier Transform into the sum of the even and odd terms of $n \in [N-1]$.

$$
\begin{aligned}
X[k] = \sum_{n=0}^{N-1} x[n] \mathrm{e}^{-j\frac{2\pi kn}{N}} &= \sum_{n=0}^{(N/2)-1} x[2n] \mathrm{e}^{-j\frac{2\pi k(2n)}{N}} + \sum_{n=0}^{(N/2)-1} x[2n+1] \mathrm{e}^{-j\frac{2\pi k(2n+1)}{N}} \\
&= \sum_{n=0}^{(N/2)-1} x[2n] \mathrm{e}^{-j\frac{2\pi k(2n)}{N}} + \mathrm{e}^{-j\frac{2\pi k}{N}} \sum_{n=0}^{(N/2)-1} x[2n+1] \mathrm{e}^{-j\frac{2\pi k(2n)}{N}} \\
&= \sum_{n=0}^{(N/2)-1} x[2n] W_N^{2kn} + W_N^k \sum_{n=0}^{(N/2)-1} x[2n+1] W_N^{2kn} \\
&= \sum_{n=0}^{(N/2)-1} x[2n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{(N/2)-1} x[2n+1] W_{N/2}^{kn}
\end{aligned}
$$

Let us take a moment to examine the 2nd half of the transform, that is, $\hat{k} = k + N/2$.

Continuing along,

$$X[k+N/2] = \sum_{n=0}^{(N/2)-1} x[2n] W_{N/2}^{(k+N/2)n} + W_N^{k+N/2} \sum_{n=0}^{(N/2)-1} x[2n+1] W_{N/2}^{(k+N/2)n}$$

After some examination, we see that
$W_{N/2}^{(k+N/2)n} = W_{N/2}^{kn} W_{N/2}^{nN/2} = W_{N/2}^{kn} \mathrm{e}^{-j2\pi n 2N/(2N)} = W_{N/2}^{kn}$. Also, $W_N^{k+N/2} = -W_N^k$. Then

$$X[k+N/2] = \sum_{n=0}^{(N/2)-1} x[2n] W_{N/2}^{kn} - W_N^k \sum_{n=0}^{(N/2)-1} x[2n+1] W_{N/2}^{kn}$$

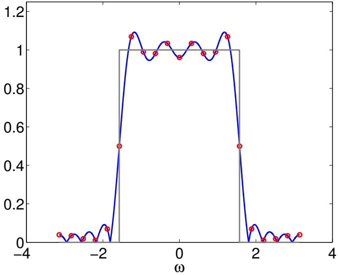
and recall that

$$X[k] = \sum_{n=0}^{(N/2)-1} x[2n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{(N/2)-1} x[2n+1] W_{N/2}^{kn}$$

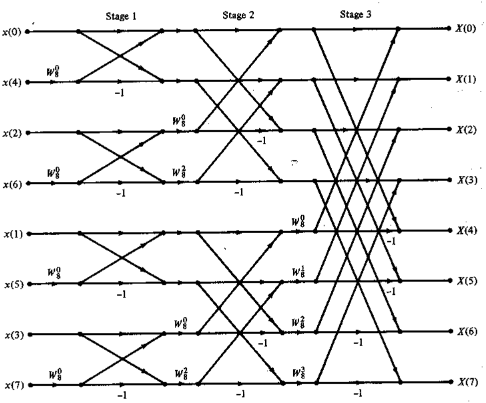
Okay... What did we just show?

Decimation-in-Time FFT

- We derived the basis for a radix-2 decimation in time FFT algorithm. The terms W_N^k are called “twiddle” factors.
- Most implementations of the sequence that we are taking the FFT has a length that is of length 2^K. Why the power of 2 length assumption?
    - What if the sequence is not of length 2^K? Zero padding?
    - Does zero padding give you more frequency resolution?



FFT in one picture



What is the Z-transform: In words

- The Z-transform is a little more general than the discrete-time Fourier transform. In fact, the discrete-time Fourier transform is a special can case of the Z-transform. The DTFT is the Z-transform being evaluated on the unit circle.
- The Z-transform of a sequence $x[n]$ must have its region of convergence within the unit circle for the DTFT to exist.
- The Z-transform is the discrete cousin of the Laplace transform.

What is the Z-transform: In equations

- The Z-transform is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

  where $z \in \mathbb{C}$.
- The set of values for z where the Laurent series converges is called the *region of convergence*.

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} \leq \infty$$

Z-transform

- The Z-transform can *generally* be written as a ratio of polynomials of z.

$$H(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{\sum_{j=0}^{M-1} a_j z^{-j}} = \frac{\prod_{k=0}^{N-1}(1 - \beta_k z^{-1})}{\prod_{j=0}^{M-1}(1 - \alpha_j z^{-1})}$$

  The roots of $P(z)$ and $Q(z)$ are called the *zeros* and *poles*, respectively.
- Poles could occur at $z = 0$ or $|z| = \infty$. The locations of the poles are very important for determining the region of convergence.
- If the ROC does not include the unit circle then the DTFT does not exist

Z-transform (Example)

Consider the signal $x[n] = a^n u[n]$. Because it is non-zero for only $n \geq 0$, this is an example of a right-sided sequence. By definition,

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we need

$$\sum_{n=0}^{\infty} |az^{-1}|^n \leq \infty$$

Thus the ROC is all the values for z where $|az^{-1}| < 1$ or $|a| < |z|$. Then

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

only if $|a| < |z|$.

$\mathcal{Z}$-transform (Example)

Consider the signal $x[n] = -a^n u[-n-1]$. Imagine looking at the previous signal in a mirror. Because it is non-zero for only $n \leq -1$, this is an example of a left-sided sequence. By definition,

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

If $a^{-1}z < 1$ or $|a| > |z|$. Then

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

only if $|a| > |z|$.

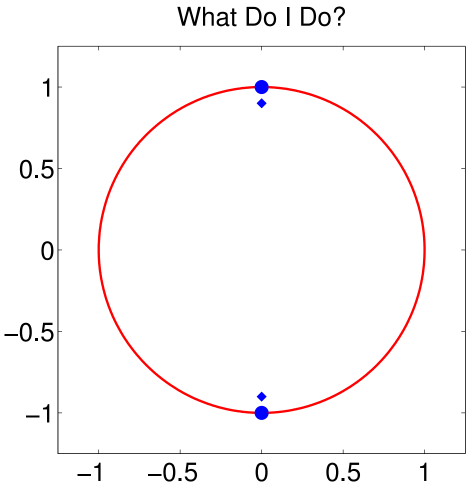
Note that $|a| < 1$, the sequence $-a^n u[-n-1]$ grows exponentially as $n \to -\infty$, and thus, the DTFT does not exist.

A Couple of Notes

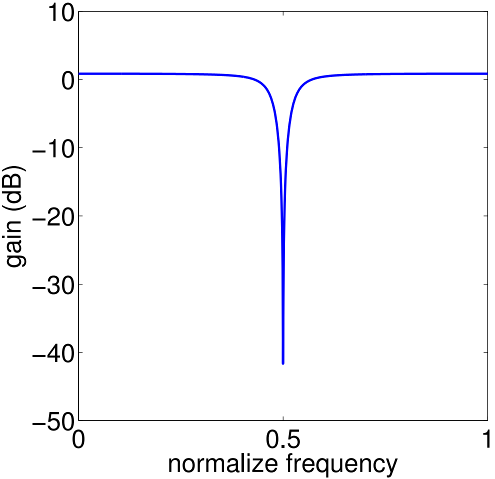
More Observations About the Z-transform

- If $x[n]$ has a Z-transform $X(z)$, and $x[n]$ is a sum of two terms. The ROC will be the intersection of the two individual ROCs.
- Obvious, I know, but the Z-transform is a linear operator.

Poles and Zeros of a System



There are poles close to the zeros, which have a magnitude just a little bit smaller than 1.


10
0
−10
−20
−30
−40
−50
gain (dB)
0
0.5
1
normalize frequency

$\mathcal{Z}$-transform (Example)

Question

Given the system

$$H(z) = e^{z^{-1}}$$

What is $h[n]$? What is the ROC? Is the system BIBO stable?

Solution

Directly apply the Lorentz series expansion to $H(z)$

$$\begin{aligned} H(z) = e^{z^{-1}} &= 1 + z^{-1} + \frac{1}{2!}z^{-2} + \cdots \\ &= \sum_{n=0}^{\infty} h[n]u[n] \end{aligned}$$

Thus, $h[n] = \frac{1}{n!}u[n]$. Is the system BIBO stable? Apply the ratio test to see if the series converges, diverges, or neither.

$$\begin{aligned} \lim_{n\to\infty} \frac{h[n+1]}{h[n]} = \lim_{n\to\infty} \frac{n!}{(n+1)!} &= \lim_{n\to\infty} \frac{n!}{(n+1)n!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} = 0 \end{aligned}$$

Therefore, $H(z)$ is BIBO stable. Examining $H(z)$, we see there are no poles or zeros. Only an essential singularity. ROC is all z except @ 0.