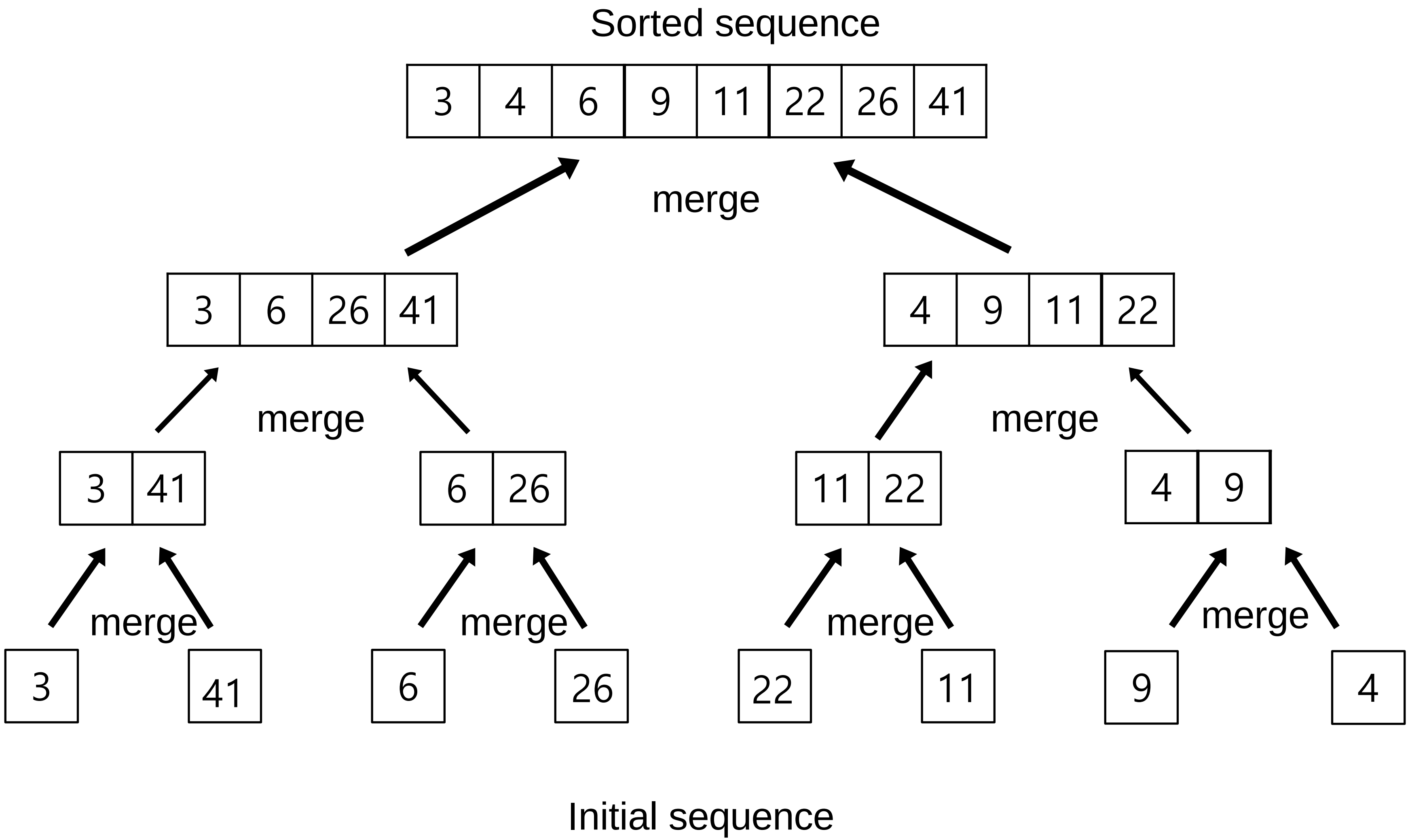
1. Using Figure 2.4 (in the text book) as a model, illustrate the operation of merge sort (ascending order) on the array A = <3, 41, 6, 26, 22, 11, 9, 4>



2. Consider sorting *n* numbers stored in array A by first finding the largest element of A and exchanging it with the element in A[1]. Then find the second largest element of A, and exchange it with A[2]. Continue in this manner for the first n-1 elements of A.

a. Write pseudocode for this algorithm, which is known as **selection sort**.

$selection\ sort(A)$

$for\ i \leftarrow 1\ \ to\ \ n-1$

$do\ \ largest \leftarrow i$

$for\ j \leftarrow i+1\ \ to\ \ n$

$if\ A[j] >\ \ A[largest]$

$do\ largest\ \ \leftarrow j$

$Exchange\ A[i]\ \ \leftrightarrow A[largest]$

b. Why does it need to run for only the first n-1 elements, rather than for all n elements?

Selection sort finds the maximum value among unsorted data.

Thereafter, the first data and data that meet the condition are exchanged.

If this method is repeatedly executed up to n-1 times, the nth data is automatically sorted

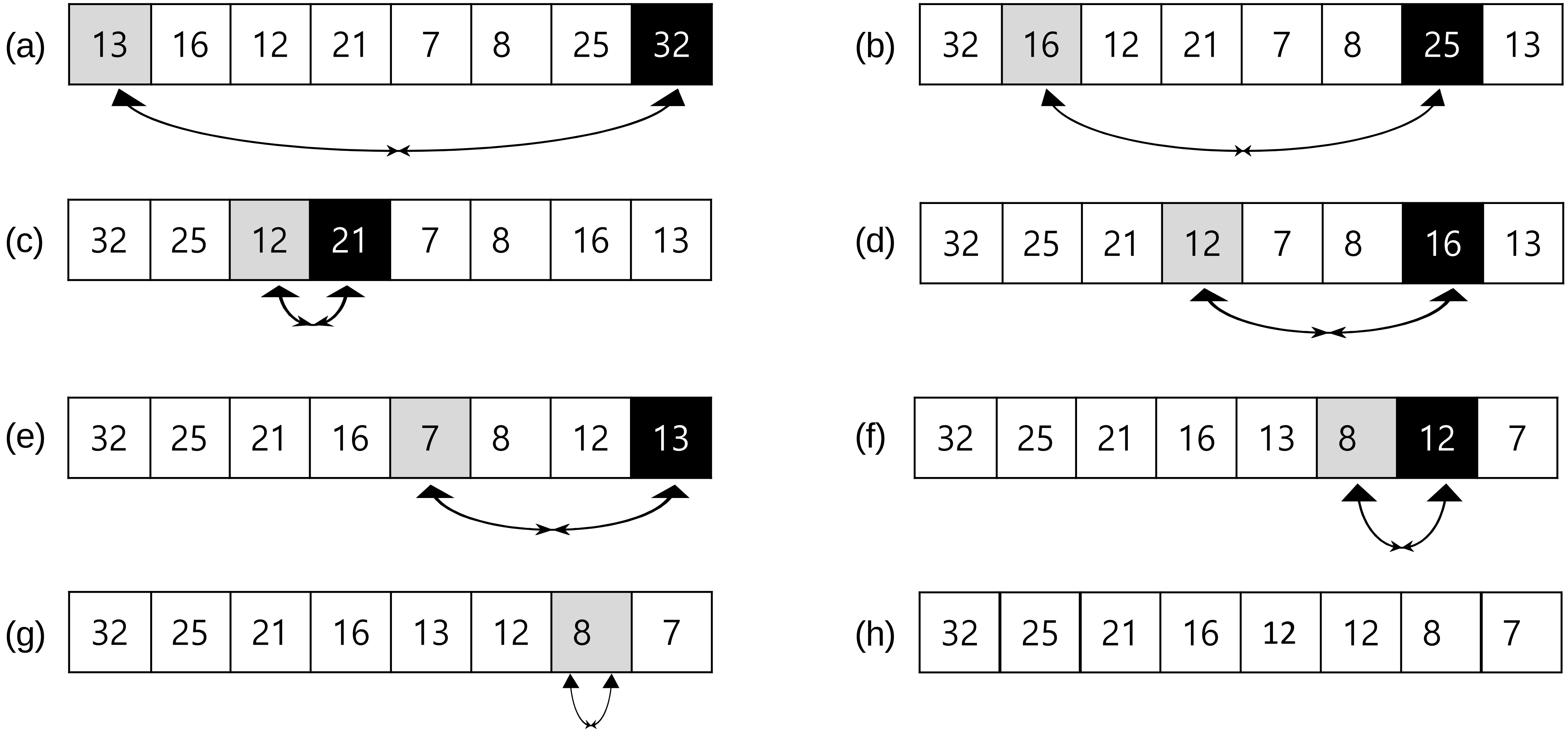
because it becomes the minimum value.

For this reason, selection sort is executed up to n-1 times.

c. Give the best-case and worst-case running times of selection sort in θ-notation.

Both the best-case and worst-case execution times are $\theta(n^2)$.

d. Using Figure 2.2 as models, illustrate the operation of the selection sort on the array A =
<13, 16, 12, 21, 7, 8, 25, 32>.



3. Express the following functions in terms of θ-notation.

a. $2n^3 + n^2 + 1$

$since\ 2n^3 + n^2 + 1 < 2n^3 + n^3 + n^3 = 4n^3\ for\ all\ n \geq 1$

$we\ may\ take\ C_1 = 4\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$

$2n^3 + n^2 + 1 = O(n^3)$ .... (1)

$since\ 2n^3 + n^2 + 1 \geq 2n^3\ for\ all\ n \geq 1$

$we\ may\ take\ C_2 = 2\ and\ N_2 = 1\ in\ the\ definition\ and\ conclude\ that$

$2n^3 + n^2 + 1 = \Omega(n^3)$ .... (2)

$Therefore, 2n^3 + n^2 + 1 = \theta(n^3)\ is\ established\ for\ (1)\ and\ (2).$

3. Express the following functions in terms of θ-notation.

b. $n^2 + 2n + lgn$

$$since\ n^2 + 2n + lgn \leq n^2 + 2n^2 + n^2 = 4n^2 \quad for\ all\ n \geq 1$$

$$we\ may\ take\ C_1 = 4\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$$

$$n^2 + 2n + lgn = O(n^2) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

$$since\ n^2 + 2n + lgn \geq n^2 \quad for\ all\ n \geq 1$$

$$we\ may\ take\ C_1 = 1\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$$

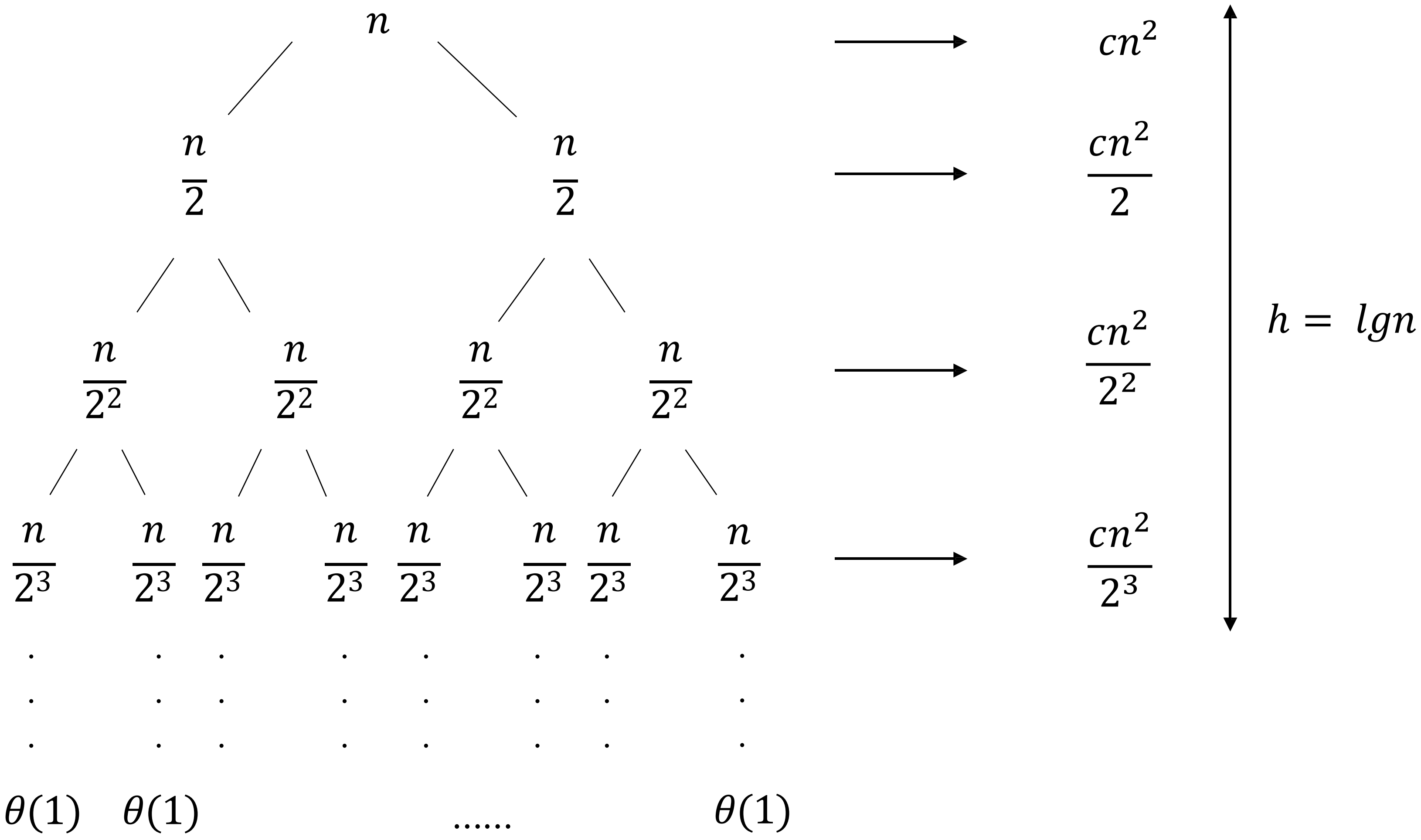
$$n^2 + 2n + lgn = \Omega(n^2) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

$$Therefore, n^2 + 2n + lgn = \theta(n^2)\ is\ established\ for\ (1)\ and\ (2).$$

4. Draw the recursion tree for $T(n) = 2T\left(\frac{n}{2}\right) + cn^2$ where, c is constant.

Provide a good asymptotic upper bound (O-notation). Also, verify your bound by the substitution method.

Recursion tree



※ Total

sum of all level:

$$c\left(n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + \frac{n^2}{2^3} + \dots\right)$$
$$= 2cn^2$$

$$\therefore T(n) = 2cn^2$$

For k is *big enough*

$$2cn^2 \le kn^2,$$

the upper bound of $T(n)$ *is*

$$T(n) = O(n^2)$$

< substitution method >

$T(n) = 2T\left(\frac{n}{2}\right) + cn^2$ $\qquad T(1) = 1$

$Guess\ \ \mathrm{T(n)} = O(n^2)$

$Assume\ that\ T(x) \le k\,x^2 \quad for\ 2 < x < n\ , \quad k > 0 \qquad \leftarrow\ Induction\ Hypothesis\ (I.H.)$

$Prove\ T(n) \le kn^2 \quad by\ induction$

$-$ ***Basis step***

$T(n) = \theta(1)\ \ for\ all\ \ n < n_0$

$for\ 1 \le n \le n_0 \qquad \theta(1) \le kn^2$

$pick\ k\ big\ enough$

− *Inductive step*

$$T(n) = 2T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + cn^2$$

$$\le 2k\left(\left[\frac{n}{2}\right]\right)^2 + cn^2$$

$$\le 2\mathrm{k}\left(\frac{n}{2}\right)^2 + cn^2 \qquad \left(\because \left(\frac{n}{2}\right) \ge \left\lfloor\frac{n}{2}\right\rfloor\right)$$

$$= \frac{k}{2}n^2 + cn^2$$

$$= \left(\frac{k}{2} + c\right)n^2$$

$$= \left(k - \frac{k}{2} + c\right)n^2$$

$$= kn^2 - \left(\frac{k}{2} - c\right) \longleftarrow \text{desired - residual}$$

$$= kn^2 - \left(\frac{k}{2} - c\right) \le kn^2$$

$$\therefore \mathrm{k} \ge 2\mathrm{c}$$

Therefore, $T(x) \le kx^2$ is true.
Thus, by the principle of mathematical induction, the claim holds.

5. Express the following functions in terms of θ-notation.

a. $2n^2 + 2n + 5lgn$

$since\ 2n^2+2n+5lgn < 2n^2 + 2n^2 + 5n^2 = 9n^2\ for\ all\ n \geq 1$

$we\ may\ take\ C_1 = 9\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$

$2n^2 + 2n + 5lgn\ = O(n^2)$ .... (1)

$since\ 2n^2 + 2n + 5lgn \geq 2n^2\ for\ all\ n \geq 1$

$we\ may\ take\ C_2 = 2\ and\ N_2 = 1\ in\ the\ definition\ and\ conclude\ that$

$2n^2 + 2n + 5lgn\ = \Omega(n^2)$ .... (2)

$Therefore, 2n^2 + 2n + 5lgn = \theta(\ n^2\ )\ is\ established\ for\ \ (1)\ and\ (2)$.

5. Express the following functions in terms of θ-notation.

b. $n^3 + 3n + 10$

$since\ n^3 + 3n + 10 \leq n^3 + 3n^3 + 10n^3 = 14n^3 \quad for\ all\ n \geq 1$

$we\ may\ take\ C_1 = 14\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$

$n^3 + 3n + 10 = O(n^3)$ .... (1)

$since\ n^3 + 3n + 10 \geq n^3 \quad for\ all\ n \geq 1$

$we\ may\ take\ C_1 = 1\ and\ N_1 = 1\ in\ the\ definition\ and\ conclude\ that$

$n^3 + 3n + 10 = \Omega(n^3)$ .... (2)

$Therefore, n^3 + 3n + 10 = \theta(n^3)$ is $established\ for\ (1)\ and\ (2).$

6. Prove the following sum by mathematical induction.

$$\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \qquad for\ n > 0$$

$-$ ***Basis step***

$$n = 1, \quad \sum_{i=1}^{1} i^2 = (1+1)(2+1)/6$$

$$\therefore \sum_{i=1}^{1} i^2 = 1^2 = 1$$

$\therefore$ $\sigma_{i=1}^{1}\, i^2 = (1+1)(2+1)/6 \quad for\ n > 0$ is true.

$-$ ***Inductive step***

$$n = k, \quad \sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6 \qquad \leftarrow \quad Induction\ Hypothesis\ (I.H.) \qquad \text{(1)}$$

$$n = k+1, \quad \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (\mathrm{k}+1)^2 \quad = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)\{k(2k+1) + 6(k+1)\}}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} \quad = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\therefore \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6} \qquad \text{(2)}$$

$\therefore$ (2) is true when (1) is true.

$$\therefore \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \quad \text{for n} > 0 \ \ \text{Theorem is proved}$$

7. Use the master method to give tight asymptotic bounds for the following recurrences.

a. $T(n) = 9T\left(\frac{n}{3}\right) + n$

$a = 9, \;\; b = 3, f(n) = n$

$O(n^{\log_3 9-\mathcal{E}}) = n \quad for\ some\ constart \;\; \mathcal{E} = 1$

$\therefore T(n) = \Theta(n^2)$ using Case 1

b. $T(n) = 9T\left(\frac{n}{3}\right) + n^2$

$a = 9, \;\; b = 3, f(n) = n^2$

$\Theta(n^{\log_3 9}) = n^2$

$\therefore T(n) = \Theta(n^2 \lg n)$ using Case 2

c. $T(n) = 9T\left(\frac{n}{3}\right) + n^3$

$a = 9, \;\; b = 3, f(n) = n^3$

$\Omega(n^{\log_3 9+\mathcal{E}}) = n^3 \;\; for\ some\ constart \;\; \mathcal{E} = 1$

$\therefore T(n) = \Theta(n^3)$ using Case 3