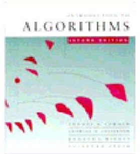
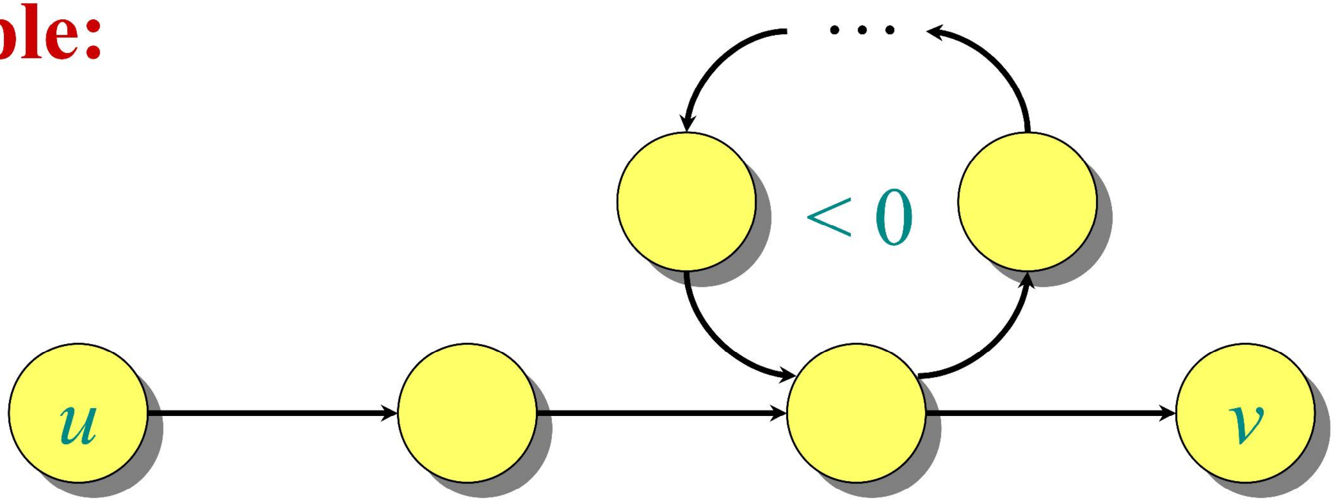


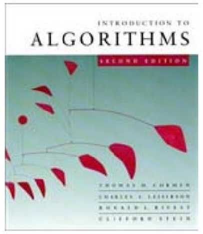
Shortest Paths 2

Negative-weight cycles

Recall: If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

Example:





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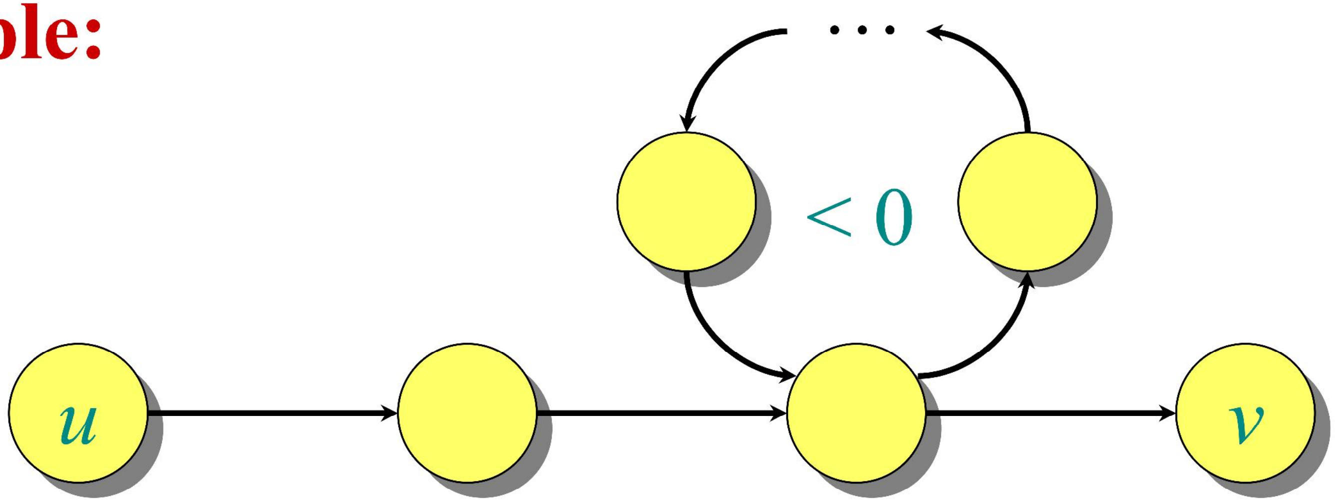
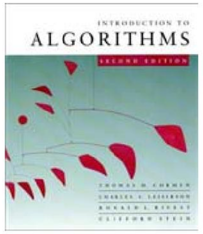


Bellman-Ford algorithm: Finds all shortest-path lengths from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.



Bellman-Ford algorithm

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$

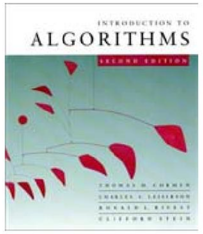
initialization

for $i \leftarrow 1$ **to** $|V| - 1$
    do for each edge $(u, v) \in E$
        do if $d[v] > d[u] + w(u, v)$
            then $d[v] \leftarrow d[u] + w(u, v)$

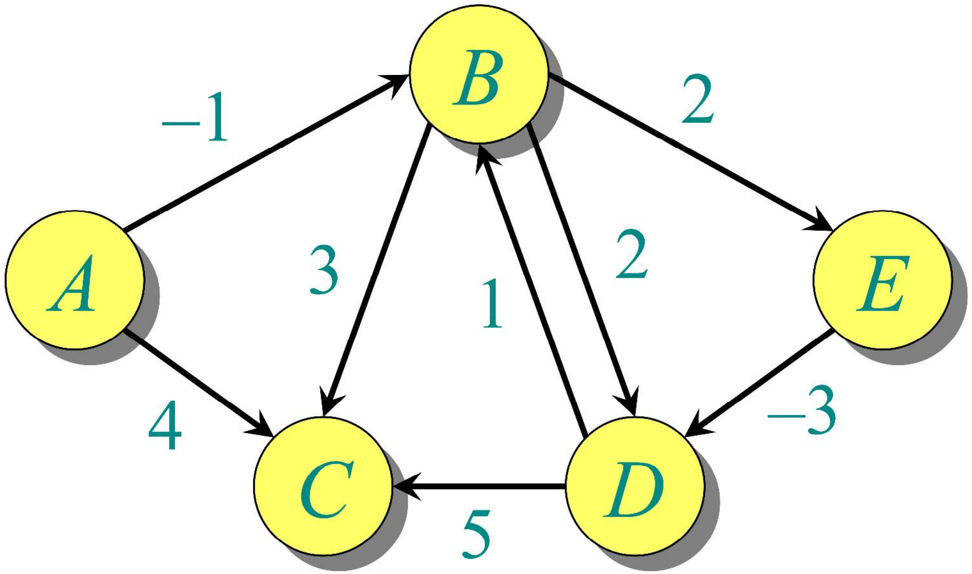
relaxation step

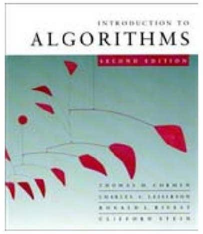
for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
        then report that a negative-weight cycle exists

At the end, $d[v] = \delta(s, v)$, if no negative-weight cycles.
Time $= O(VE)$.

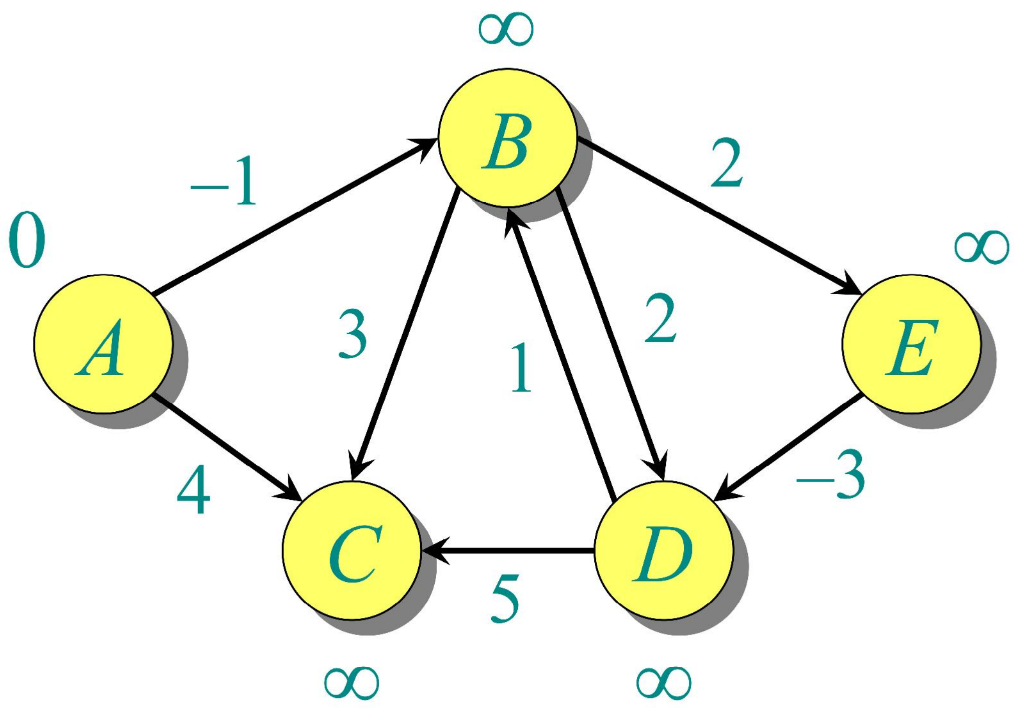


Example of Bellman-Ford

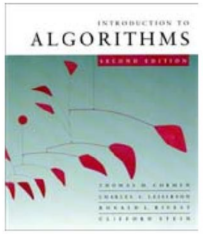




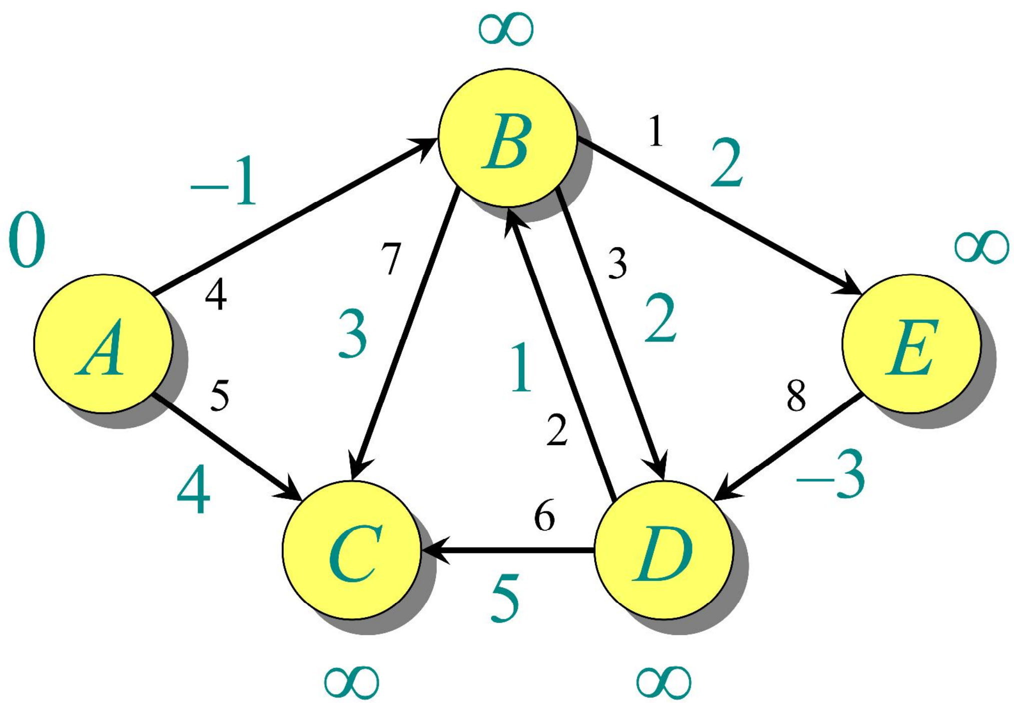
Example of Bellman-Ford



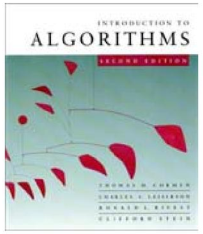
Initialization.



Example of Bellman-Ford



Order of edge relaxation.



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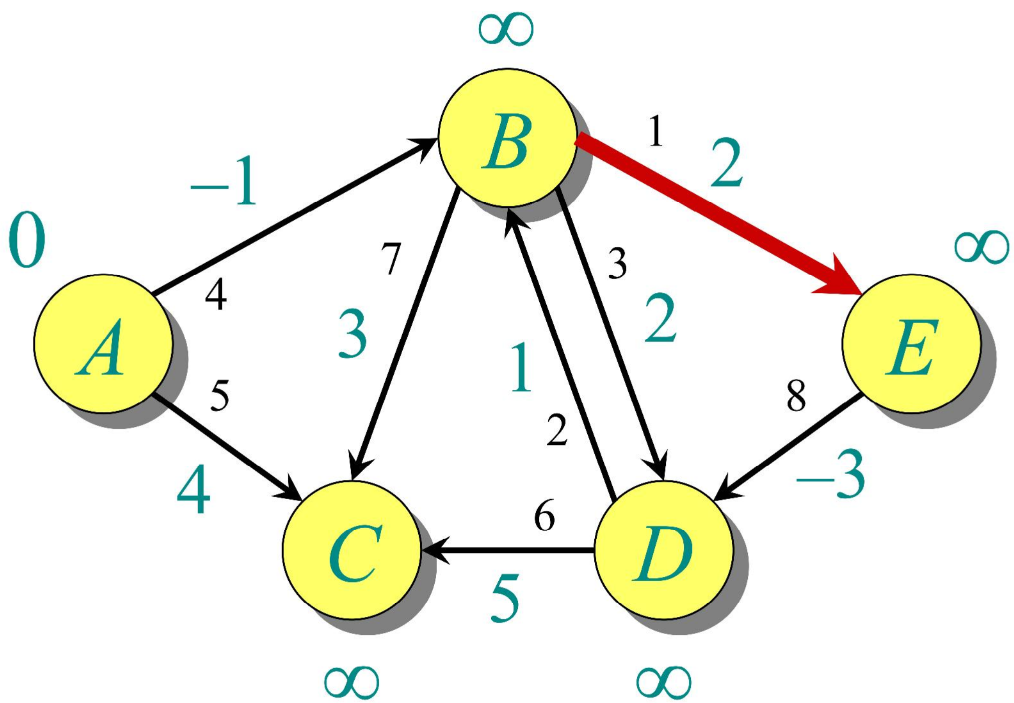

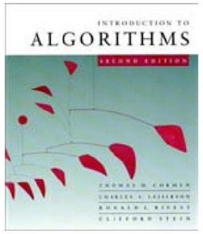

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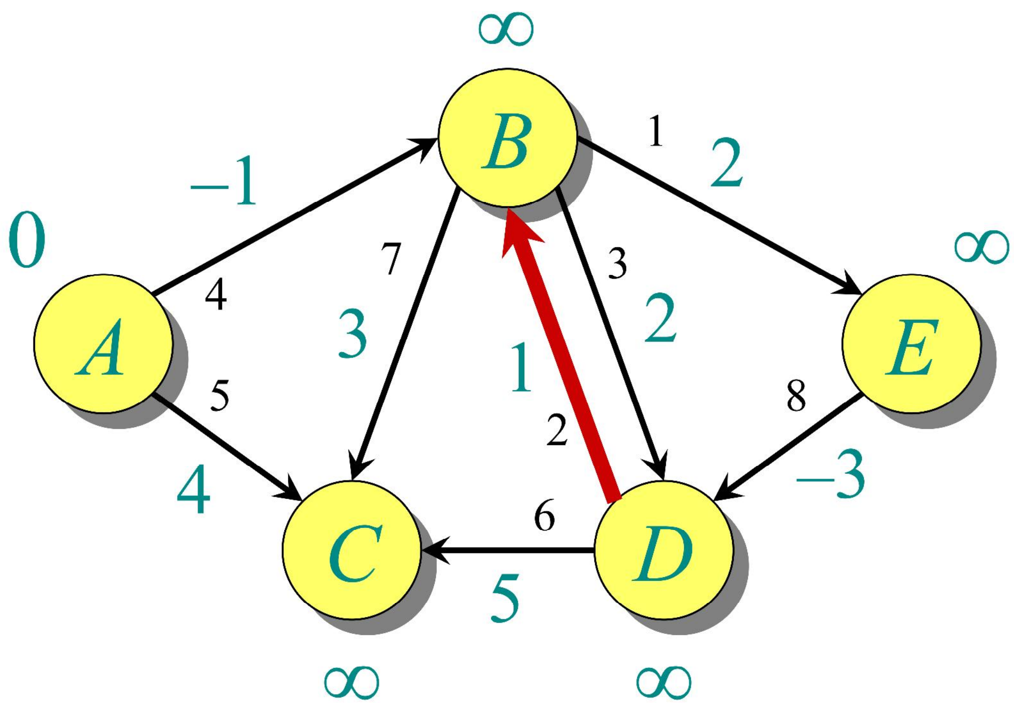

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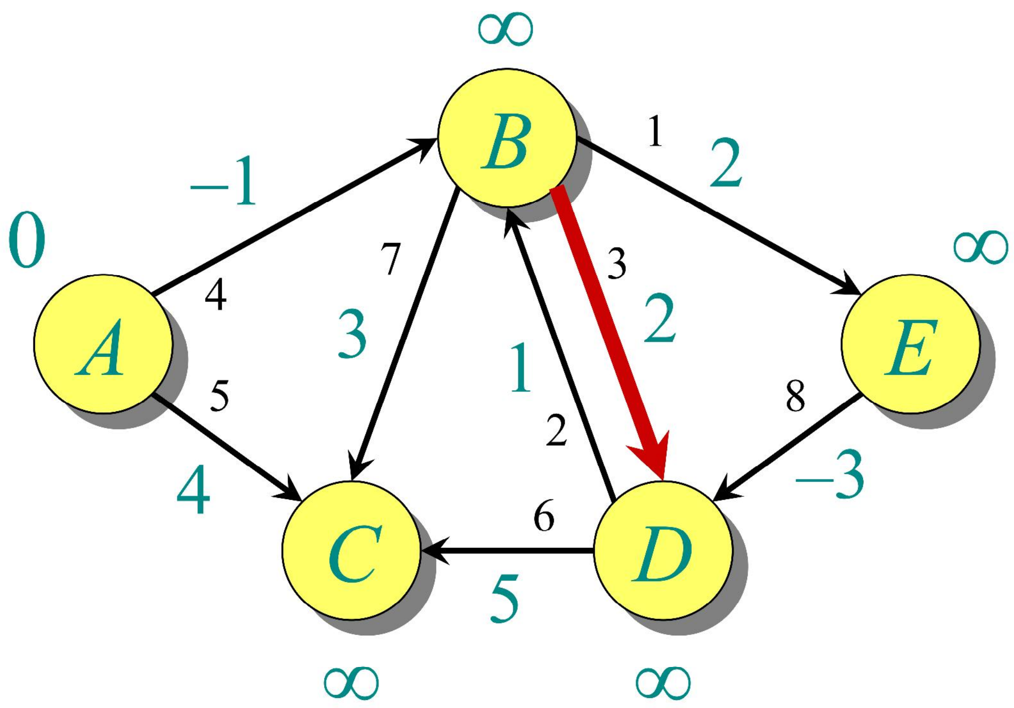

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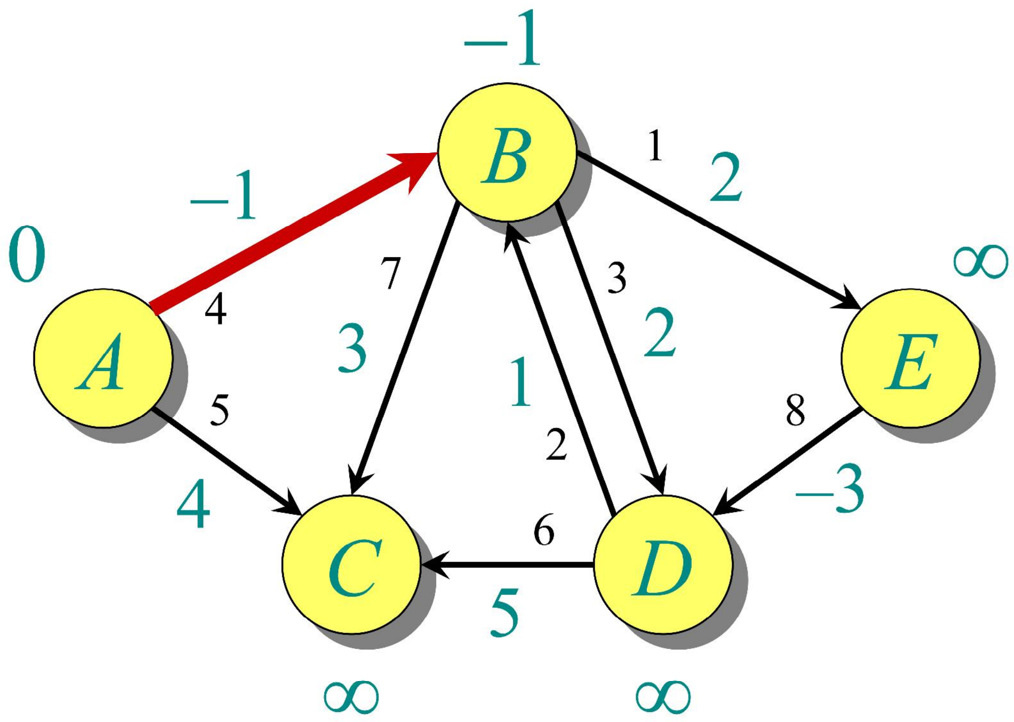

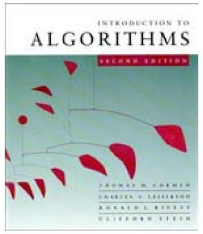

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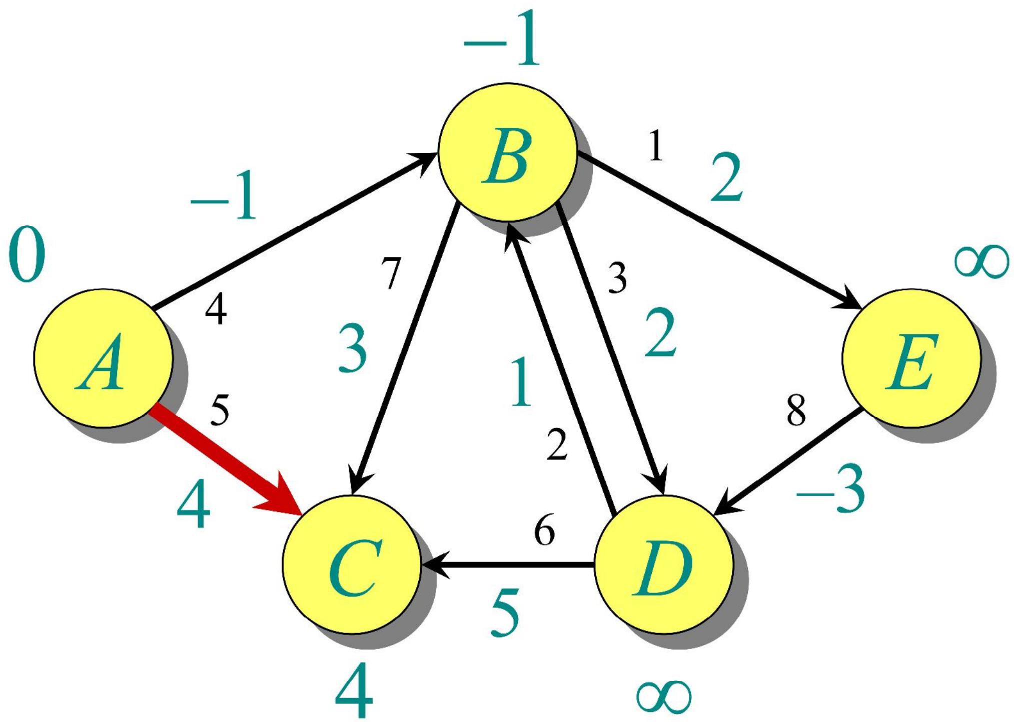

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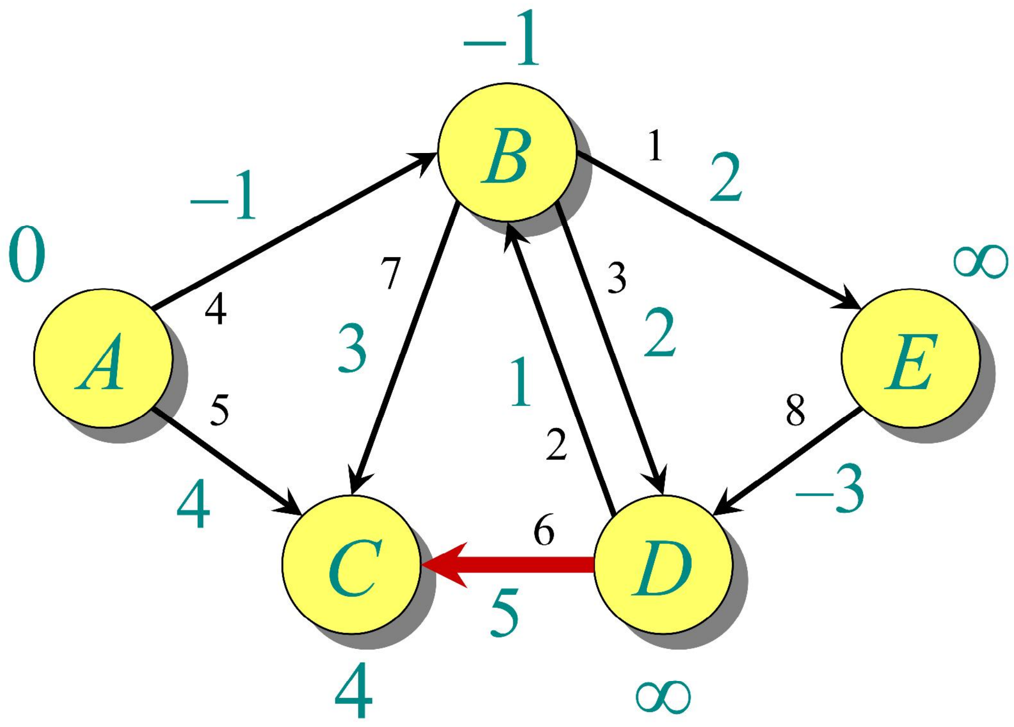

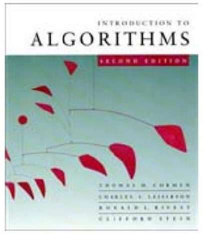

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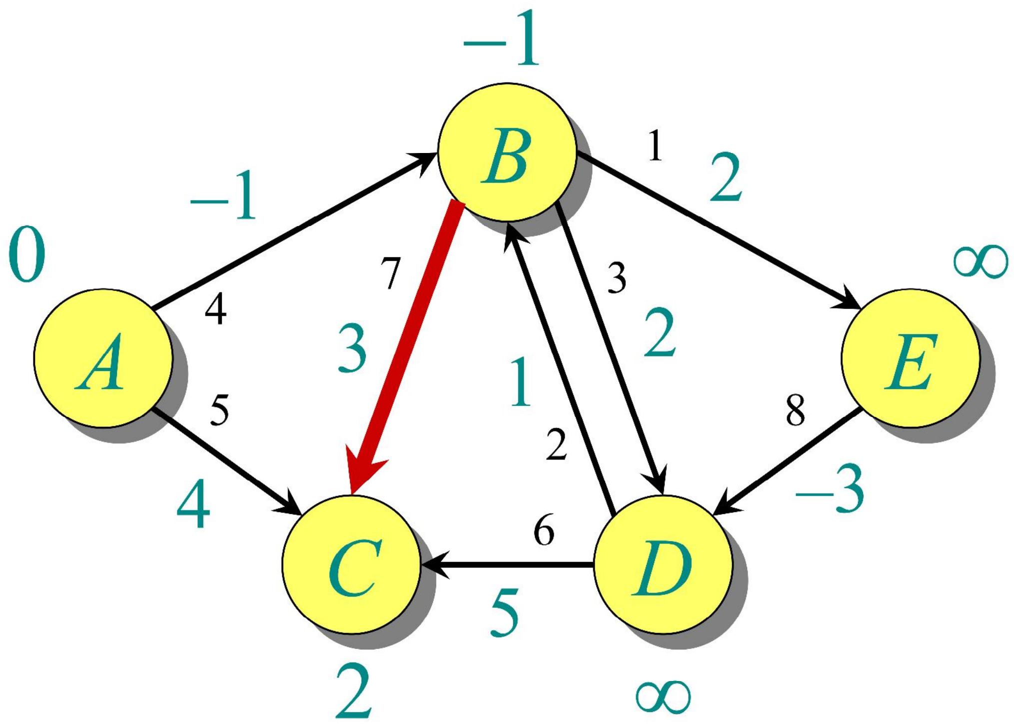

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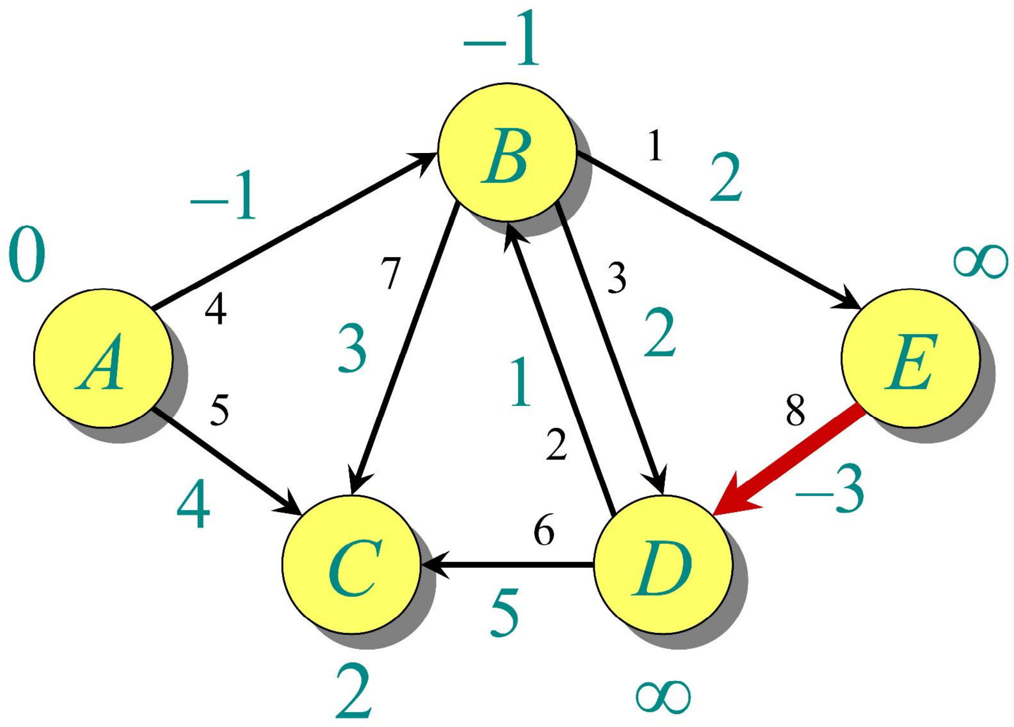

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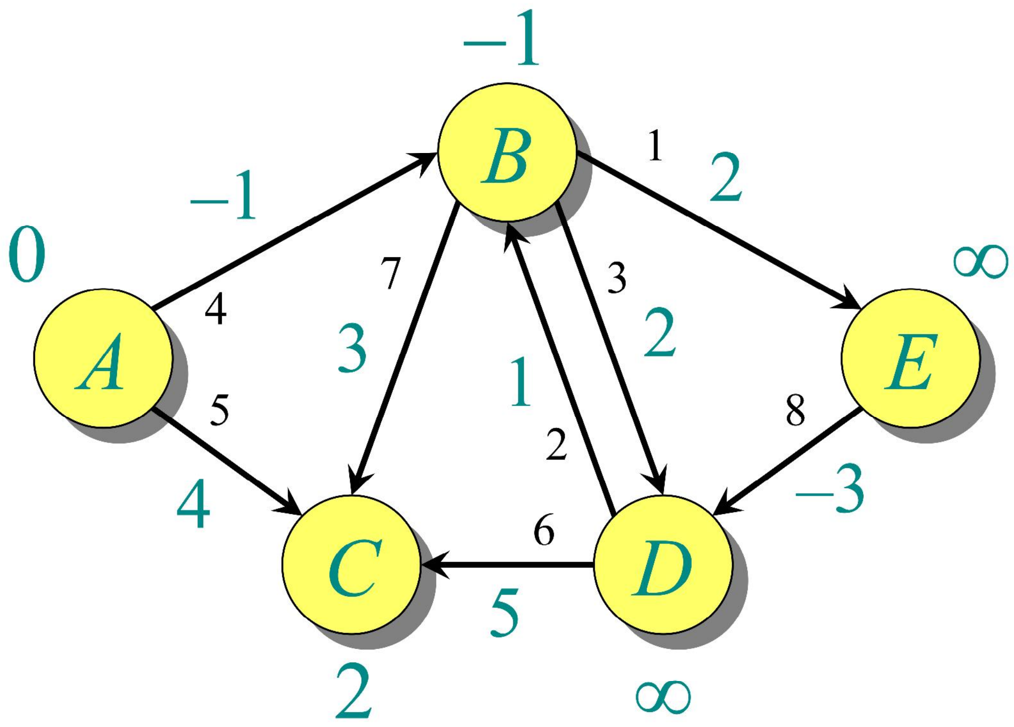
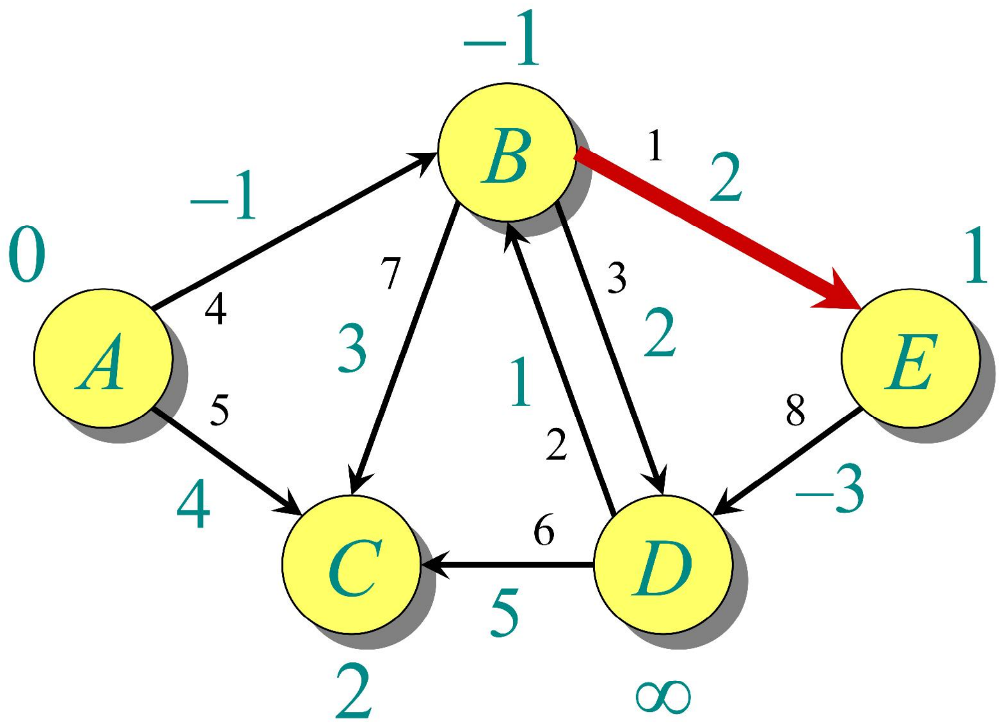
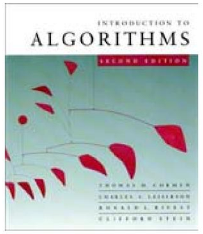


End of pass 1.

Example of Bellman-Ford





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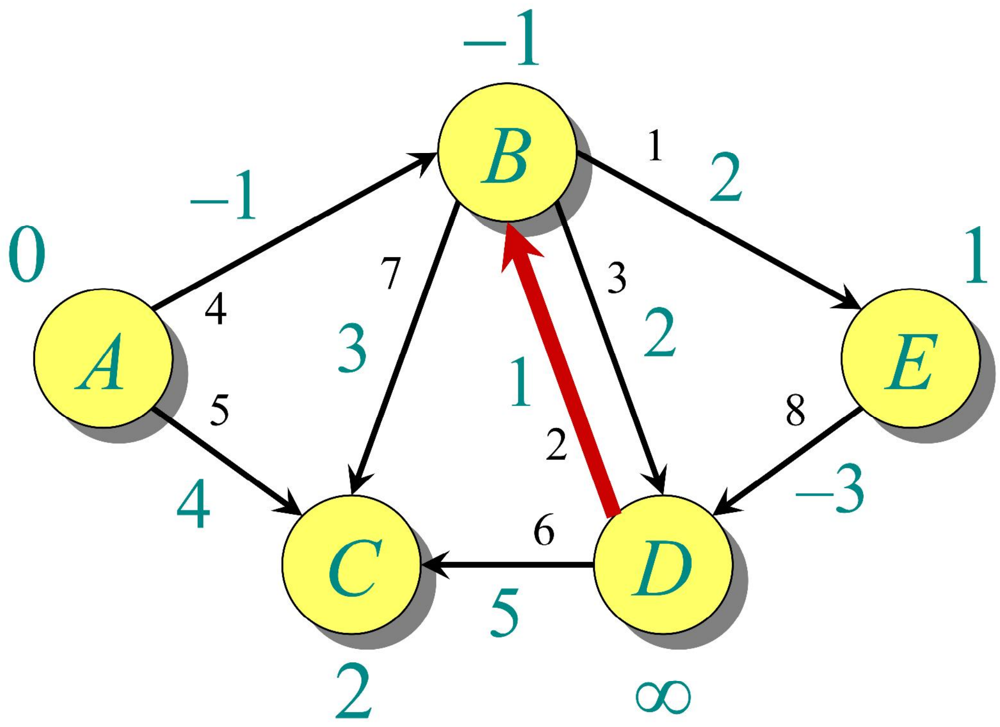

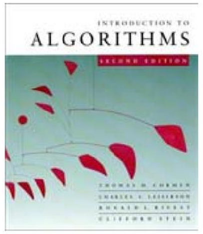

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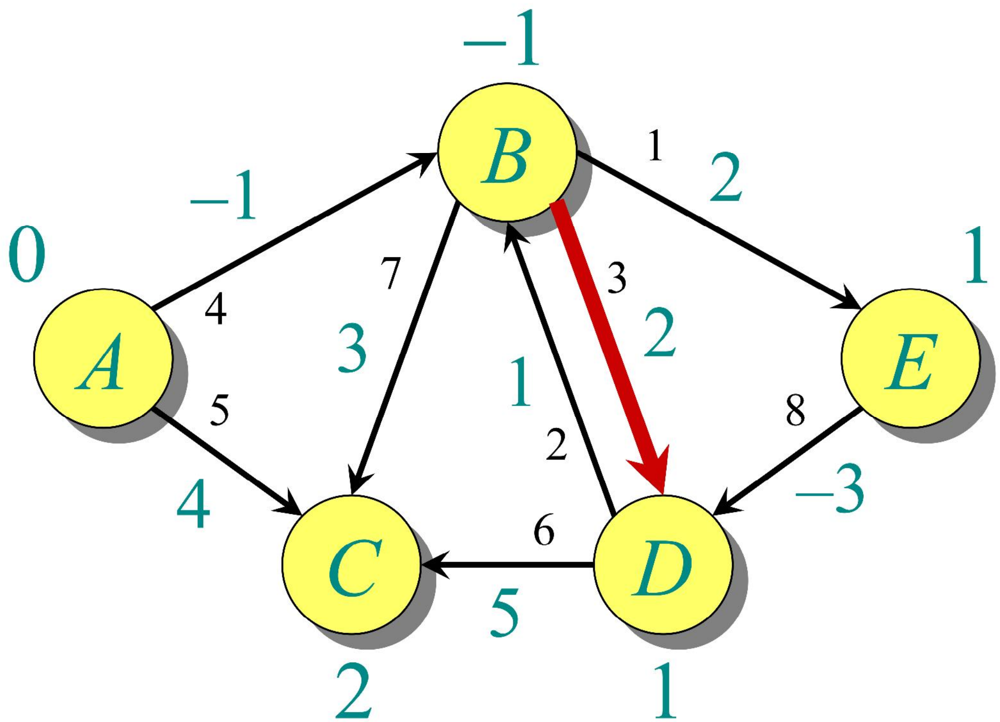

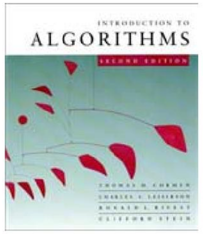

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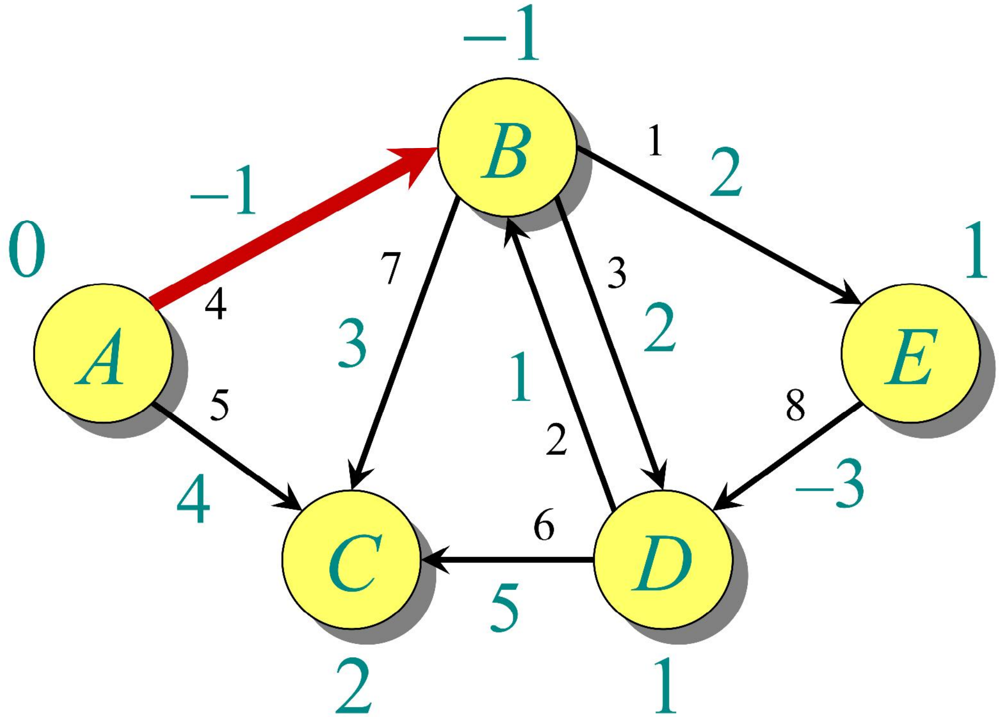

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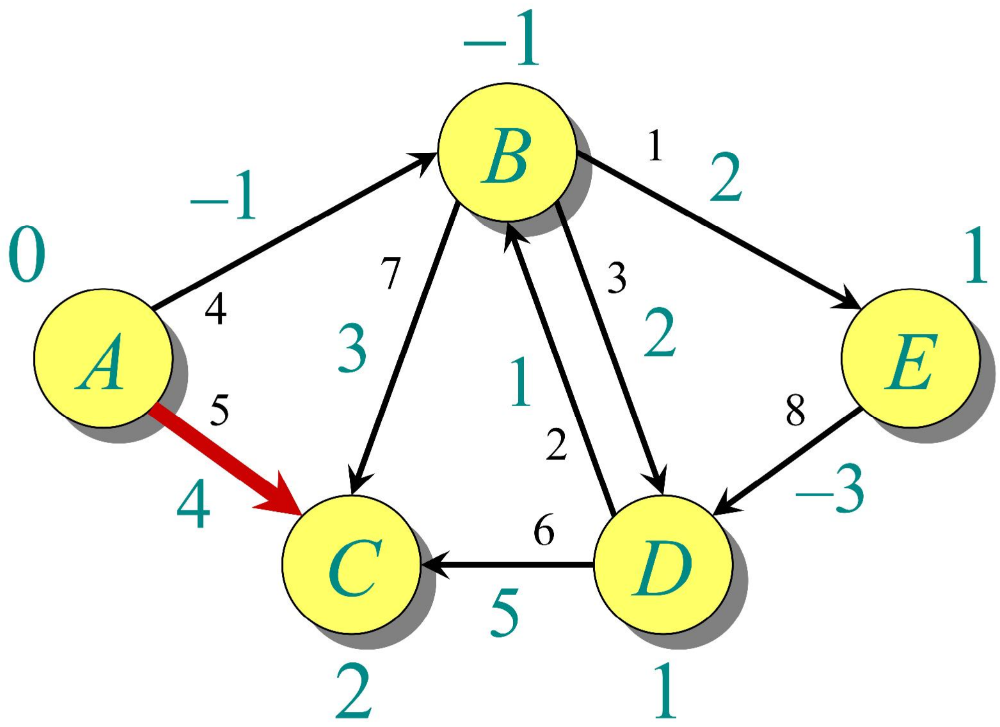

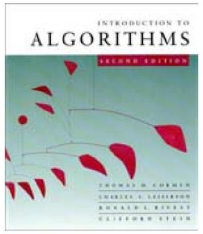

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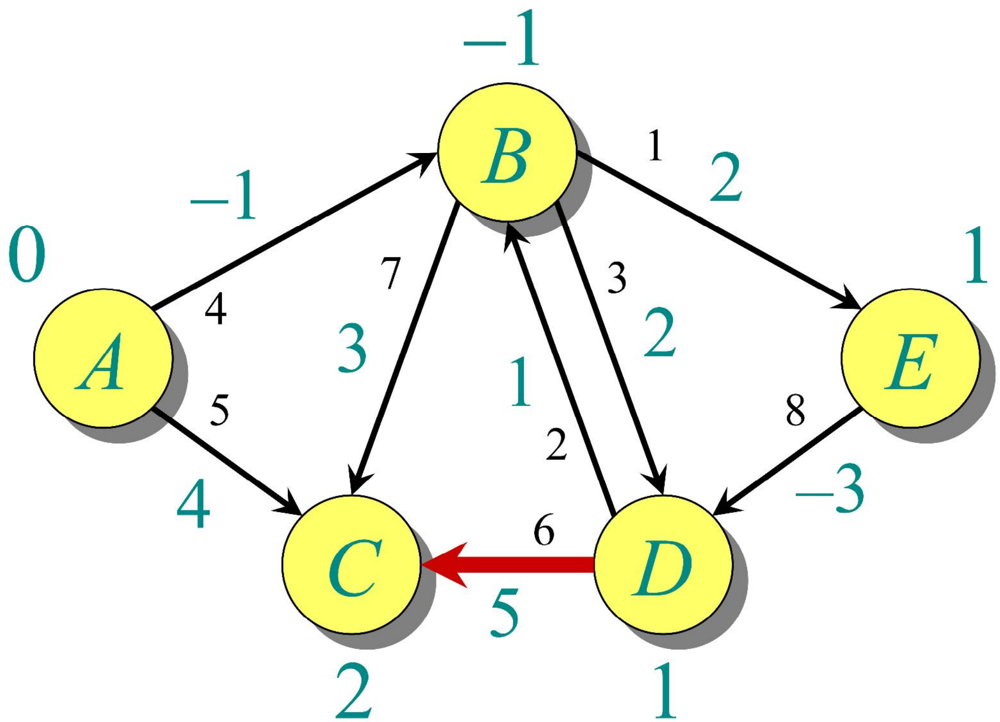

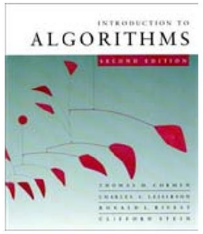

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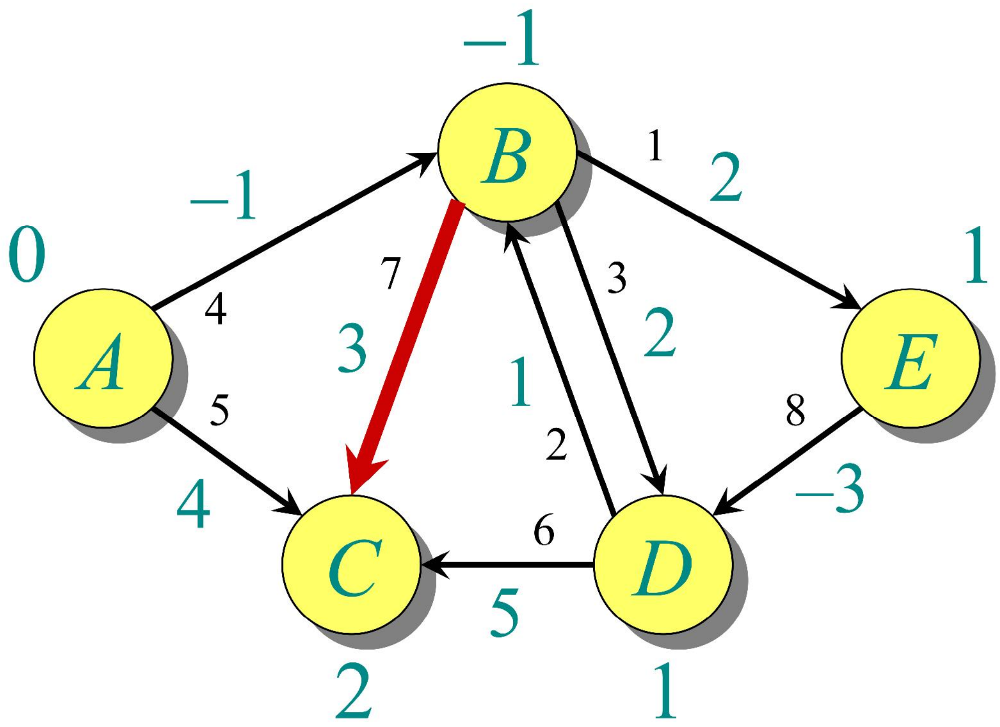

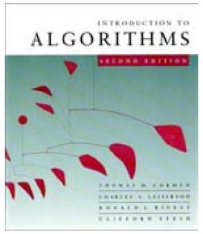

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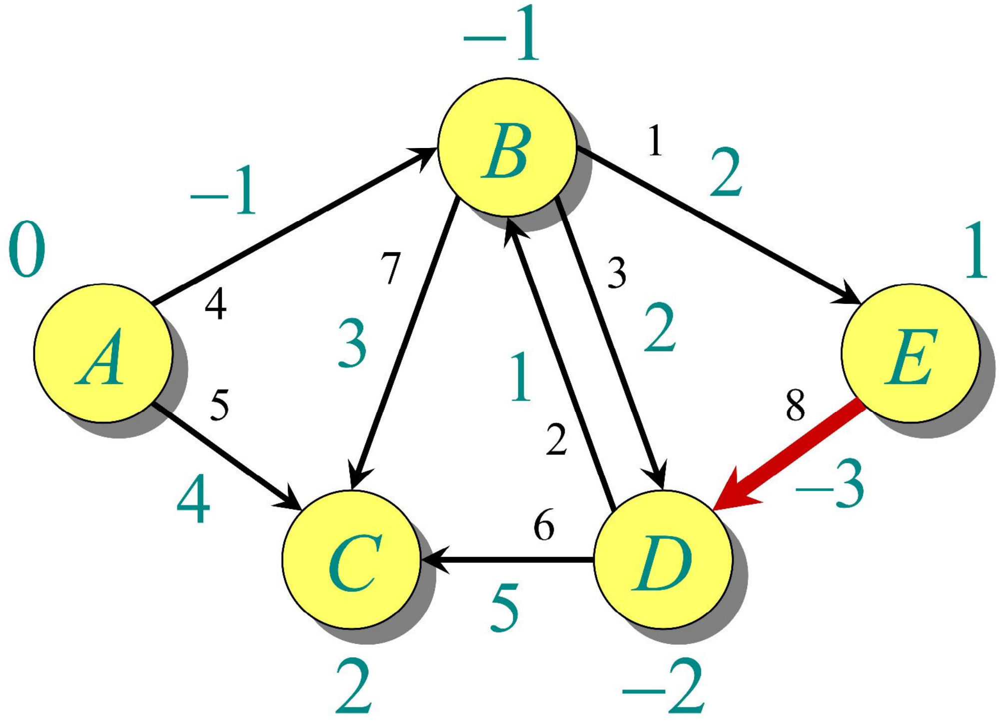

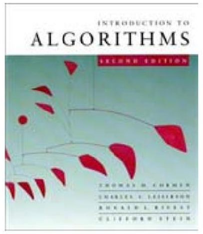

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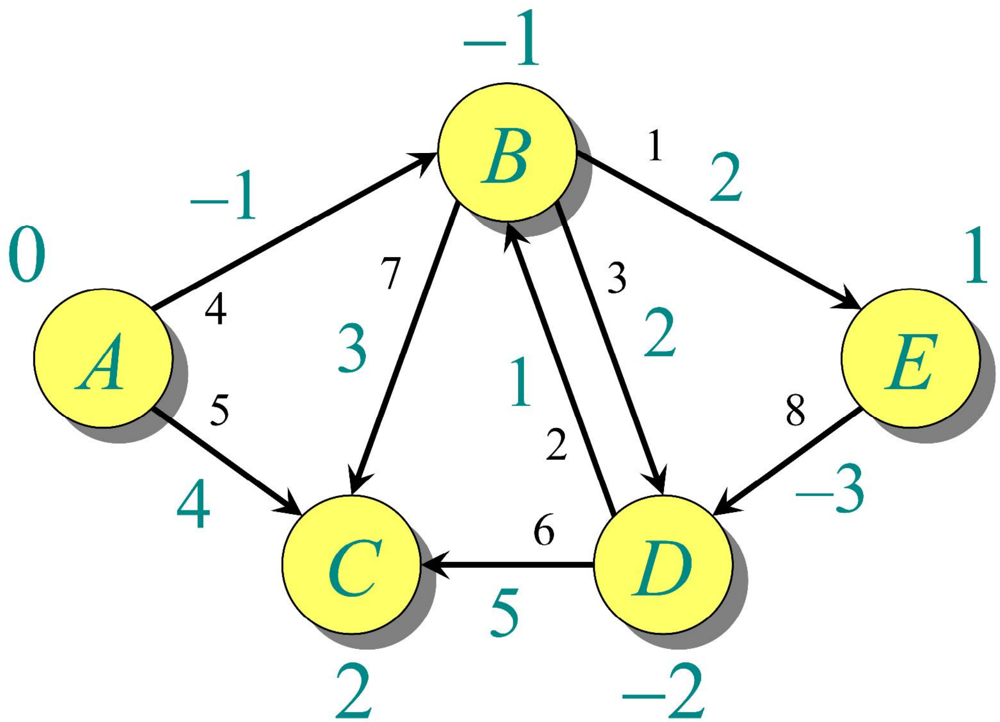
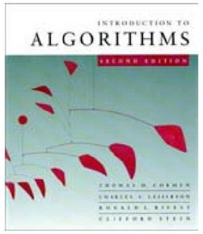


End of pass 2 (and 3 and 4).



Correctness

Theorem. If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

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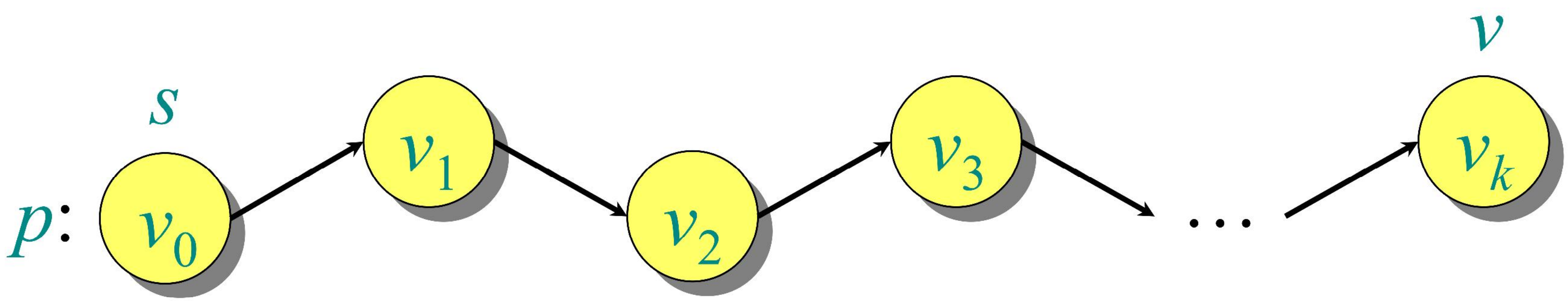
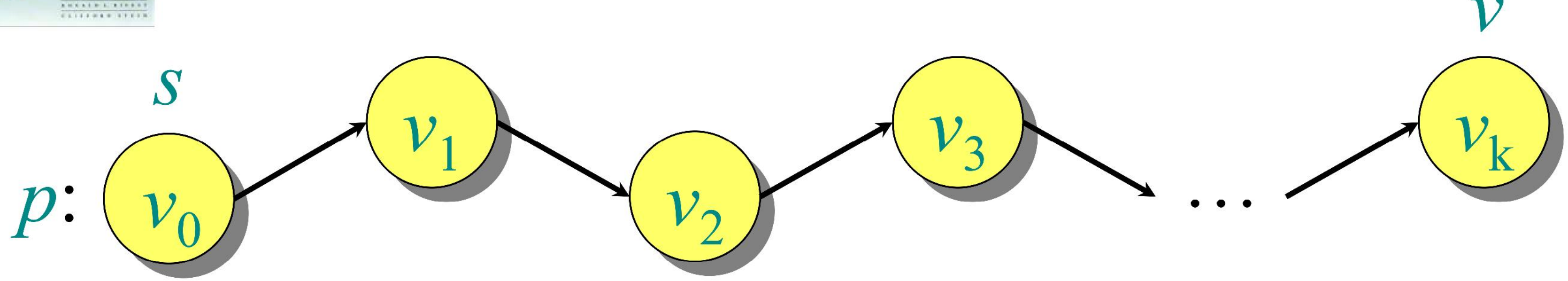
Proof. Let $v \in V$ be any vertex, and consider a shortest path p from s to v with the minimum number of edges.



Since p is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$

Correctness (continued)

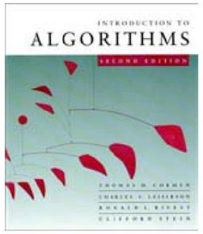


Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[v_0]$ is unchanged by subsequent relaxations (because of the lemma from Lecture 14 that $d[v] \geq \delta(s, v)$).

- After 1 pass through E, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through E, we have $d[v_2] = \delta(s, v_2)$.

  ⋮
- After k passes through E, we have $d[v_k] = \delta(s, v_k)$.

Since G contains no negative-weight cycles, p is simple. Longest simple path has $\leq |V| - 1$ edges. ◻



Detection of negative-weight cycles

Corollary. If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s. ∎