A Short Course in Computational Finance

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Chapter 1

Generating Random Variables

1.1 The Distribution Function

Let $f : [a, b] \to \mathbb{R}$

$$f(x) > 0 \quad \forall x \in [a, b]$$
$$\int_a^b f(x)\, dx = 1$$

We relay on the intuitive concept of **random**, and we say that X is distributed according to f if

$$\mathbb{P}(X \in [a, m]) = \int_a^m f(x)\, dx \quad a \leq m \leq b$$

Once f is chosen, a and b are defined by f so we will use the notation:

$$\mathbb{P}(X < m) = \int_a^m f(x)\, dx$$

and the constraint

$$a \leq m \leq b$$

is underrstood.
The most important distribution is $f = 1, \quad f : [0, 1] \to [0, 1]$. It is so important that it deserves a name of its own, and we shall call it u (a mnemonic for uniform). We assume we have some tool that once we call it, gives back to us exactly such a random variable From here on we are on our own.
More on u:

$$\mathbb{P}(X \in [l_1, l_2]) = \int_{l_1}^{l_2} dx = l_2 - l_1 \quad 0 \leq l_1 \leq l_2 \leq 1.$$

Theorem 1.1.1. *Let*

$$U = \int_a^X f(s)ds$$

then

Proof. Let

$$\Phi(x) := \int_a^x f(s)ds,$$

therefore

$$X = \Phi^{-1}(U)$$

and

$$\mathbb{P}(X < \Lambda) = \mathbb{P}(\Phi^{-1}(U) < \Lambda) = \mathbb{P}(U < \Phi(\Lambda))$$

$\square$

1.2 How to approximate an integral

Let

$$X_i \stackrel{d}{=} f$$

$$Y = \frac{1}{N}\sum_{i=1}^{N} G(X_i)$$

$$\mathbb{E}[Y] = \frac{1}{N}\sum_{i=1}^{N}\int_a^b G(x_i)f(x_i)dx_i = \int_a^b G(s)f(s)ds =: \mathbb{E}_f[G]$$

$$\begin{aligned} Y^2 &= \frac{1}{N^2}\sum_{i,j=1}^{N} G(X_i)G(X_j) \\ &= \frac{1}{N^2}\sum_{i=1}^{N} G^2(X_i) + \frac{1}{N^2}\sum_{i\neq j=1}^{N} G(X_i)G(X_j) \end{aligned}$$

$$\begin{aligned} \mathbb{E}[Y^2] &= \frac{1}{N}\int_a^b G^2(s)f(s)ds + \frac{N(N-1)}{N^2}\left(\int_a^b G(s)f(s)ds\right)^2 \\ &= \frac{1}{N}\int_a^b G^2(s)f(s)ds + \left(1-\frac{1}{N}\right)\left(\mathbb{E}_u[G]\right)^2 \end{aligned}$$

finally:

$$\begin{aligned}\mathbb{E}\left[(Y-\mathbb{E}_f[G])^2\right] &= \frac{1}{N}\int_a^b G^2(s)f(s)ds - \frac{1}{N}\left(\int_a^b G(s)f(s)ds\right)^2 \\ &= \frac{1}{N}\left[\int_a^b G^2(s)f(s)ds - \left(\int_a^b G(s)f(s)ds\right)^2\right] = \frac{\sigma_G^2}{N}\end{aligned}$$

If we ask for

$$\sqrt{\mathbb{E}\left[(Y-\mathbb{E}_f[G])^2\right]} < \epsilon,$$

we must consider to use at least

$$N \geq \frac{\sigma_G^2}{\epsilon^2}.$$

1.2.1 A Word of Caution

Most of the time we care for the percentage error, that is the quantity:

$$\frac{\sqrt{\mathbb{E}\left[(Y-\mathbb{E}_f[G])^2\right]}}{\mathbb{E}[\,Y\,]}.$$

There are cases where $\mathbb{E}[\,Y\,] = 0$, so we stick with the absolute error and leave it to the reader to handle the more general situation.

1.3 Examples

1.3.1 Linear Density

Let

$$f(x) = Zx, \quad x \in [1,2]$$

Since

$$\int_1^2 x\,dx = \left.\frac{x^2}{2}\right|_1^2 = \frac{3}{2},$$

we have a proper normalisation setting $Z = 2/3$.
To generate such a RV we need to solve for X, where X is given by:

$$U = \frac{2}{3}\int_1^X x\,dx = \frac{X^2}{3} - \frac{1}{3} \rightarrow \quad X = \sqrt{3U+1}$$

We want to compute:

$$m := \mathbb{E}[X], \quad \sigma^2 := \mathbb{E}[\,(X-m)^2\,]$$

$$\begin{aligned}
m &= \frac{2}{3}\int_1^2 x^2\,dx = \frac{2}{3}\left.\frac{x^3}{3}\right|_1^2 = \frac{14}{9} \simeq 1.5555\ldots \\
\sigma^2 &= \mathbb{E}[\,X^2\,] - m^2 \\
&= \frac{2}{3}\int_1^2 x^3\,dx - m^2 \\
&= \frac{2}{3}\left.\frac{x^4}{4}\right|_1^2 - m^2 \\
&= \frac{5}{2} - \left(\frac{14}{9}\right)^2 \simeq 0.08024691358 \quad \sigma \simeq .28.
\end{aligned}$$

If we want to compute the average with an error of order 10^{-3} we need approximately $N = 10^4$.

1.3.2 Exponential Density

Let:

$$f(x) = Ze^{-x/\lambda}, \quad x \in [0, \infty].$$

Since:

$$\int_0^\infty e^{-x/\lambda} dx = \lambda e^{-x/\lambda} \Big|_0^\infty = \lambda,$$

we conclude that the correct normalisation is given by $Z = 1/\lambda$.
To generate such a RV we need to solve for X, where X is given by:

$$U = \frac{1}{\lambda} \int_0^X e^{-x/\lambda} dx = 1 - e^{-X/\lambda} \quad \to \quad X = -\lambda \log(1 - U).$$

Let's compute:

$$m := \mathbb{E}[X], \quad \text{and} \quad \sigma^2 := \mathbb{E}[\,(X - m)^2\,]$$

$$\begin{aligned}
m &= \frac{1}{\lambda} \int_0^\infty x e^{-x/\lambda} \\
&= -\frac{1}{\lambda} \frac{d}{dt} \int_0^\infty e^{-tx} \Big|_{t=1/\lambda} = -\frac{1}{\lambda} \frac{d}{dt} \frac{1}{t} \Big|_{t=1/\lambda} = \lambda \\
\sigma^2 &= \frac{1}{\lambda} \int_0^\infty x^2 e^{-x/\lambda} - m^2 \\
&= \frac{1}{\lambda} \frac{d^2}{dt^2} \int_0^\infty e^{-tx} \Big|_{t=1/\lambda} - m^2 = \lambda^2 \quad \to \quad \sigma = \lambda
\end{aligned}$$

If we want to compute the average with an error of order 10^{-3} we need approximately $N = \lambda^2 10^6$.

1.3.3 Pareto Density

$$f(x) = \frac{Z}{x^{1+\nu}}, \quad x \in [a, \infty], \quad \nu > 0$$

Since:

$$\int_a^\infty \frac{dx}{x^{1+\nu}} = -\frac{1}{\nu x^\nu}\bigg|_a^\infty = \frac{1}{\nu a^\nu}$$

we conclude that the correct normalisation is given by $Z = \nu a^\nu$.
To generate such a RV we need to solve for X, where X is given by:

$$\begin{aligned} U &= \nu a^\nu \int_a^X \frac{dx}{x^{1+\nu}} \\ &= -\nu a^\nu \frac{1}{\nu x^\nu}\bigg|_a^X \\ &= \nu a^\nu \frac{1}{\nu a^\nu} - \nu a^\nu \frac{1}{\nu X^\nu} \\ &= 1 - \left(\frac{a}{X}\right)^\nu \quad \to X = \frac{a}{(1-U)^{1/\nu}}. \end{aligned}$$

Let's compute:

$$m := \mathbb{E}[X], \quad \text{and} \quad \sigma^2 := \mathbb{E}[\,(X-m)^2\,]$$

$$\begin{aligned} m &= \nu a^\nu \int_a^\infty \frac{dx}{x^{1+\nu}}\, x \\ &= \nu a^\nu \int_a^\infty \frac{dx}{x^\nu} \\ &= \frac{\nu a^\nu}{1-\nu} \frac{1}{x^{\nu-1}}\bigg|_a^\infty \end{aligned}$$

and we have a problem:

$$m = \begin{cases} \infty & \nu \leq 1 \\ \frac{\nu a}{\nu - 1} & \nu > 1. \end{cases}$$

$$\begin{aligned} \sigma^2 &= \nu a^\nu \int_a^\infty \frac{dx}{x^{1+\nu}}\, x^2 - m^2 \\ &= \nu a^\nu \int_a^\infty dx\, x^{1-\nu} - m^2 \\ &= \frac{\nu a^\nu}{2-\nu} x^{2-\nu}\bigg|_a^\infty - m^2 \end{aligned}$$

and we have an even bigger problem unless $\nu > 2$. In this case we have:

$$\sigma^2 = \frac{\nu a^2}{\nu - 2} - \frac{\nu^2 a^2}{(\nu - 1)^2} \quad \rightarrow \quad \sigma = \sqrt{\frac{\nu a^2}{(\nu - 1)^2 (\nu - 2)}}$$

1.3.4 Homework

Problem 1.3.1. *Verify that things do not make sense for $\nu < 2$.*

1.4 Acceptance Rejection Method

1.4.1 Conditional Expectations

Let X a random variable with domain in $\mathcal{X}$ with law $f(x)$, that is, if $\mathcal{A} \subset \mathcal{X}$ we have:

$$\mathbb{P}(X \in \mathcal{A}) = \int_{\mathcal{A}} f(x)\, dx$$

If we want to compute

$$\mathbb{E}[G(X) \,|\, X \in \mathcal{A}] = \int_{\mathcal{A}} G(x) f(x)\, dx$$

we proceed in the following way:

1. generate the random variable Y with the law $f(x)$
2. if $Y \in \mathcal{A}$ use it, otherwise discard it and extract a new one

1.4.2 Acceptance Rejection Method

Let's assume that $f(x)$ is difficult to generate, but a different distribution $g(x)$ is easy. Furthermore we assume that there is a constant c such that

$$cg(x) \geq f(x) \quad \forall x.$$

We generate U, Y and accept Y with probability:

$$\frac{f(x)}{cg(x)}$$

We want to show that the random variable X generated this way has the law $f(x)$.

$$\begin{aligned}
\mathbb{P}(X \in \mathcal{A}) &= \mathbb{P}\left(Y \in \mathcal{A} \,|\, U < \frac{f(Y)}{cg(Y)}\right) \\
&= \frac{\mathbb{P}\left(Y \in \mathcal{A}, U < \frac{f(Y)}{cg(Y)}\right)}{\mathbb{P}\left(U < \frac{f(Y)}{cg(Y)}\right)}
\end{aligned}$$

$$\mathbb{P}\left(U < \frac{f(Y)}{cg(Y)}\right) = \int_{\mathcal{X}} \frac{f(x)}{cg(x)} g(x) dx = \frac{1}{c}$$

$$\mathbb{P}\left(Y \in \mathcal{A}, U < \frac{f(Y)}{cg(Y)}\right) = \int_{\mathcal{A}} \frac{f(y)}{cg(y)} g(y) dy$$

Finally:

$$\mathbb{P}(X \in \mathcal{A}) = \int_{\mathcal{A}} dx\, f(x).$$

1.5 The Normal Distribution

$$\mathbb{P}(X < \Lambda) = \int_{-\infty}^{\Lambda} \frac{dx}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Also for the normal we need a particular name:

$$N_{0,1}(x) := \int_{-\infty}^{x} \frac{dt}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)$$

1.5.1 The meaning of $(0,1)$

$$\begin{aligned}
\mathbb{E}[\,X\,] &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, t \, \exp\left(-\frac{t^2}{2}\right) \\
\mathbb{E}[\,(\,X - \mathbb{E}[X]\,)^2\,] &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, t^2 \, \exp\left(-\frac{t^2}{2}\right) = 1
\end{aligned}$$

More in general

$$N_{m,\sigma}(x) := \int_{-\infty}^{x} \frac{dt}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t-m)^2}{2\sigma^2}\right) = N_{0,1}\left(\frac{x-m}{\sigma}\right)$$

and:

$$\begin{aligned}
\mathbb{E}[\,X\,] &= \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi\sigma^2}}\, t \, \exp\left(-\frac{(t-m)^2}{2\sigma^2}\right) \\
& \quad s := t - m \\
&= \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{2\pi\sigma^2}}\, (s+m) \, \exp\left(-\frac{s^2}{2\sigma^2}\right) \\
& \quad v := \frac{s}{\sigma} \\
&= \int_{-\infty}^{+\infty} \frac{dv}{\sqrt{2\pi}}\, (\sigma v + m) \, \exp\left(-\frac{v^2}{2}\right) = m.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[\,(X - \mathbb{E}[X])^2\,] &= \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi\sigma^2}}\, (t-m)^2 \, \exp\left(-\frac{(t-m)^2}{2\sigma^2}\right) \\
& \quad s := t - m \\
&= \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{2\pi\sigma^2}}\, s^2 \, \exp\left(-\frac{s^2}{2\sigma^2}\right) \\
& \quad v := \frac{s}{\sigma} \\
&= \sigma^2 \int_{-\infty}^{+\infty} \frac{dv}{\sqrt{2\pi}}\, v^2 \, \exp\left(-\frac{v^2}{2}\right) = \sigma^2.
\end{aligned}$$

Chapter 2

Log Normal Processes

2.1 The Model

Let's study:

$$\frac{dS(t)}{S(t)} = \sigma dW_t \tag{2.1}$$

The standard solution goes like this:

$$X(t) =: \log\left(\frac{S(t)}{S(0)}\right), \tag{2.2}$$

application of Ito's theorem produces:

$$\begin{aligned} dX(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2}\left(\frac{dS(t)}{S(t)}\right)^2 \\ &= \left[\sigma dW_t - \frac{1}{2}\sigma^2 dt\right] \end{aligned}$$

togheter with the boundary condition: $X(0) = 0$.
The solution is given by:

$$X(t) = -\frac{\sigma^2 t}{2} + \sigma[\, W_t - W_0\,] \tag{2.3}$$

and from the transformation (2.2) we have:

$$S(t) = S(0)\exp\left(-\frac{\sigma^2}{2}t + \sigma[\, W_t - W_0]\right) \tag{2.4}$$

where

$$W_t - W_0 \stackrel{d}{=} N_{0,\sqrt{t}}$$

Let's suppose we are interested in pricing a title living up to a given T. We brak the time interval in N steps

$$0 = t_0 < t_1 < \cdots < t_N = T$$

then we can write:

$$X(T) = X(T) - X(0) = \sum_{n=0}^{N-1} \Delta X_n, \quad \Delta X_n = X(t_{n+1}) - X(t_n).$$

From eq.(2.3) it is easy to identify

$$\Delta X_n = -\frac{\sigma^2}{2}(t_{n+1} - t_n) + \sigma[\, W_{t_n+1} - W_{t_n} \,]$$

where

$$W_{t_{n+1}} - W_{t_n} \stackrel{d}{=} N_{0,\sqrt{t_{n+1}-t_n}}$$

2.2 The Martingale Property

We recall the defintion of eq. (2.5)

$$S(t_n) = S(t_0) + \sum_{i=0}^{n-1} S(t_i)(\, W_{t_{i+1}} - W_{t_i} \,) \tag{2.5}$$

It is immediate to see that:

$$\mathbb{E}[\, S(t_n) \,] = S(t_0) + \sum_{i=0}^{n-1} \mathbb{E}[\, S(t_i)(\, W_{t_{i+1}} - W_{t_i} \,) \,]$$

but $S(t_i)$ depends on

$$W_{t_1} - W_{t_0}, \ldots, W_{t_i} - W_{t_{i-1}},$$

so it is independent from $W_{t_{i+1}} - W_{t_i}$, therefore:

$$\mathbb{E}[\, S(t_i)(\, W_{t_{i+1}} - W_{t_i} \,) \,] \quad = \quad \mathbb{E}[\, S(t_i) \,]\mathbb{E}[\, (\, W_{t_{i+1}} - W_{t_i} \,) \,] = 0.$$

And finally:

$$\mathbb{E}[S(t)] = S(0) \quad \forall t.$$

This must be true even in the representation of eq.(2.4) that we must be able to show that:

$$\mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2} t + \sigma[\, W_t - W_0] \right) \right] = 1, \quad \forall t. \tag{2.6}$$

Remember that:

$$W_T - W_t \stackrel{d}{=} N_{0,\sqrt{T-t}},$$

therefore:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2}t + \sigma[W_t - W_0]\right)\right] \\
= \;& \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \exp\left(-\frac{\sigma^2}{2}t + \sigma x\right) \\
= \;& \exp\left(-\frac{\sigma^2}{2}t\right) \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t} + \sigma x\right)
\end{aligned}$$

completing the square we get

$$\begin{aligned}
= \;& \exp\left(-\frac{\sigma^2}{2}t\right) \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{(x-\sigma t)^2}{2t} + \frac{\sigma^2}{2}t\right) \\
= \;& \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{(x-\sigma t)^2}{2t}\right) = 1.
\end{aligned}$$

Back to equation:

$$X(t_n) = \sum_{i=0}^{n-1} \Delta X_i, \quad \Delta X_i = X(t_{i+1}) - X(t_i).$$

that we can rewrite as:

$$X(t_n) = X(t_{n-1}) + \Delta X_{n-1}$$

and for $S(t_n)$ we have:

$$S(t_n) = S(t_{n-1}) \exp\left(-\frac{\sigma^2}{2}(t_n - t_{n-1}) + \sigma(W_n - W_{n-1})\right)$$

In this way we can generate a trajectory, consistent with the stochastic differential equation (2.5), starting from an initial condition:

$$S(t_0), S(t_1), \ldots, S(t_n), \ldots, S(t_N)$$

In fact, when we do MC simulation we do generate e large number of such trajectories:

$$S(t_0), S^j(t_1), \ldots, S^j(t_n), \ldots, S^j(t_N) \quad 1 \le j \le J$$

and if we want the expectation value of some funcion G of $S(t_n)$, we estimate the value computing

$$\frac{1}{J}\sum_{j=1}^{J} G(S^j(t_n)).$$

Exercise 2.2.1. *We want to check the martingale property of the process $S(t)$ as defined before.*
Let $T = 1$ year, break it into monthly intervals. For each month estimate $\mathbb{E}[\,S(t_n)\,]$ togheter with the MC error.
For each time interval check the martingale property, and for a fixed number of trajectory try to determine how the MC error behaves as a function of t_n.

2.3 Introducing Interest Rates

If we introduce interestrates, the equation we have to deal with will be:

$$\frac{d\hat{S}}{\hat{S}} = rdt + \sigma dW \tag{2.7}$$

Let's define:

$$S(t) := \exp\left(-\int_0^t r(s)ds\right)\hat{S}(t),$$

from Ito's theorem we get:

$$\begin{aligned} dS(t) &= -r(t)S(t)dt + S(t)\frac{d\hat{S}}{\hat{S}} \\ &= \sigma S(t)dW_t. \end{aligned}$$

and we are back to out familiar martingale equation.
Since the interest rate curve does not depend on $S(t)$, once we have decided our time grid

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T,$$

we can build the array:

$$d_0 = 1,\ d_n = d_{n-1}\exp\left(-\int_{t_{n-1}}^{t_n} r(s)ds\right).$$

If we really need to compute trajectories for $\hat{S}_n^j$ we proceed as before: we generate several martingale tajectory for S_n^j and obtain the desired $\hat{S}_n^j$ trajectories as:

$$\hat{S}_n^j = d_n S_n^j.$$

2.4 Vanilla Options

2.4.1 The European Put

$$\Pi(\mathcal{P}) = \mathbb{E}\left[e^{-\int_0^T r(s)ds}(\,\kappa - \hat{S}(T)\,)^+\right]$$

where

$$(x - y)^+ := \max(x - y, 0).$$

The quantity:

$$\kappa(T) := e^{-\int_0^T r(s)ds}\kappa$$

is independent from the S process, therefore we can write:

$$\Pi(\mathcal{P}, T) \quad = \quad \mathbb{E}\left[(\kappa(T) - S(T))^+ \right]$$

Where we see clearly that we do not need to worry about the interest rate when we generate MC trajectories.
Once again, if we just need the RV $S(T)$ we can generate it in one step

$$S(T) = S(0) \exp\left(-\frac{\sigma^2}{2} T + \sigma(W_T - W_0) \right)$$

but since we are interested in trajectories we can as well compute

$$S_n = S_{n-1} \exp\left(-\frac{\sigma^2}{2} (t_n - t_{n-1}) + \sigma(W_{t_n} - W_{t_{n-1}}) \right)$$

Our option is estimated as:

$$\Pi(\mathcal{P}, t_n) \simeq \mathcal{P} := \frac{1}{J} \sum_{j=1}^{J} \max(\kappa(t_n) - S^j(t_n), 0)$$

while the MC error is estimated as:

$$\begin{aligned} \mathcal{V} &= \frac{1}{J} \sum_{j=1}^{J} \max(\kappa(t_n) - S^j(t_n), 0)^2 \\ \text{Err} &\simeq \sqrt{\frac{\mathcal{V} - \mathcal{P}^2}{J}} \end{aligned}$$

2.4.2 Sanity Check

We want to compute:

$$\Pi(\mathcal{P}, T) \quad = \quad \mathbb{E}\left[(\kappa(T) - S(T))^+ \right]$$

where:

$$\begin{aligned} \kappa(T) &:= e^{-\int_0^T r(s)ds}\kappa \\ S(T) &= S(0) \exp\left(-\frac{\sigma^2}{2} T + \sigma(W_T - W_0) \right) \end{aligned}$$

If wedefine:

$$\hat{\kappa}(T) := \frac{\kappa(T)}{S(0)},$$

we can write:

$$\Pi(\mathcal{P}, T) = S(0)\mathbb{E}\left[\left(\hat{\kappa}(T) - \exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right)\right)^+\right]$$

The argument of the expectation tells us that we get contributions only when:

$$\hat{\kappa}(T) - \exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right) > 0$$

that is:

$$W_T - W_0 < \frac{\log(\hat{\kappa}(T)) + \frac{\sigma^2}{2}T}{\sigma} =: d$$

It follows that:

$$\Pi(\mathcal{P}, T) \quad = \quad S(0)\int_{-\infty}^{d} \frac{dx}{\sqrt{2\pi T}} \left(\hat{\kappa}(T) - \exp\left(-\frac{\sigma^2}{2}T + \sigma x\right)\right) \exp\left(-\frac{x^2}{2T}\right)$$

The consider the two contributions to the integral:

$$\begin{aligned}
\mathcal{P}_{cn} &= \hat{\kappa}(T) \int_{-\infty}^{d} \frac{dx}{\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2T}\right) \\
\mathcal{P}_{an} &= \int_{-\infty}^{d} \frac{dx}{\sqrt{2\pi T}} \exp\left(-\frac{\sigma^2}{2}T + \sigma x - \frac{x^2}{2T}\right)
\end{aligned}$$

In the first integral we recognize the distribution function $N_{0,\sqrt{T}}(d)$, that we will compute as:

$$N_{0,\sqrt{T}}(d) = N_{0,1}\left(\frac{d}{\sqrt{T}}\right)$$

the second one can be written as:

$$\mathcal{P}_{an} \quad = \quad \int_{-\infty}^{d} \frac{dx}{\sqrt{2\pi T}} \exp\left(-\frac{(x - \sigma T)^2}{2T}\right) = N_{\sigma T,\sqrt{T}}(d)$$

that we will compute as:

$$N_{\sigma T,\sqrt{T}}(d) = N_{0,1}\left(\frac{d - \sigma T}{\sqrt{T}}\right)$$

Finally:

$$\Pi(\mathcal{P}, T) = S(0)\left[\hat{\kappa}(T)N_{0,1}\left(\frac{d}{\sqrt{T}}\right) - N_{0,1}\left(\frac{d - \sigma T}{\sqrt{T}}\right)\right]$$

2.4.3 The European Call

In this case the expectation value to compute is given by:

$$\Pi(\mathcal{C},T) \;=\; \mathbb{E}\left[e^{-\int_0^T r(s)ds}(\,\hat{S}(T)-\kappa\,)^+\right]$$

As before we procede to define

$$\kappa(T) := e^{-\int_0^T r(s)ds}\kappa$$

and all we are left with isthe expectation:

$$\Pi(\mathcal{C},T) \;=\; \mathbb{E}\left[(\,\hat{S}(T)-\kappa\,)^+\right]$$

Considerations concerning the generation of trajectories are fully unchanged, we merely haev to modify the computation of the payoff. Our option is estimated as:

$$\Pi(\mathcal{C},t_n) \simeq \mathcal{C} := \frac{1}{J}\sum_{j=1}^{J}\max(\,S_n^j - \kappa(t_n), 0\,)$$

while the MC error is estimated as:

$$\begin{aligned}
\mathcal{V} &= \frac{1}{J}\sum_{j=1}^{J}\max(\,\kappa(t_n) - S^j(t_n), 0\,)^2 \\
\mathrm{Err} &\simeq \sqrt{\frac{\mathcal{V}-\mathcal{C}^2}{J}}
\end{aligned}$$

This time, in performing the integral coming from the expectation we are bound to teh domain

$$\hat{\kappa}(T) - \exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right) < 0$$

that is:

$$W_T - W_0 > \frac{\log(\hat{\kappa}(T)) + \frac{\sigma^2}{2}T}{\sigma} =: d$$

It follows that:

$$\Pi(\mathcal{C},T) \;=\; S(0)\int_d^{+\infty}\frac{dx}{\sqrt{2\pi T}}\left(\exp\left(-\frac{\sigma^2}{2}T+\sigma x\right) - \hat{\kappa}(T)\right)\exp\left(-\frac{x^2}{2T}\right)$$

Again we consider one at a time the contribution to the integral:

$$\begin{aligned}
\mathcal{C}_{cn} &= \hat{\kappa}(T)\int_d^{+\infty}\frac{dx}{\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2T}\right) \\
\mathcal{C}_{an} &= \int_d^{+\infty}\frac{dx}{\sqrt{2\pi T}}\exp\left(-\frac{\sigma^2}{2}T+\sigma x\right)\exp\left(-\frac{x^2}{2T}\right)
\end{aligned}$$

Using properties of the normal distribution functions described in the appendix we can write the first integral as:

$$\mathcal{C}_{cn} = \hat{\kappa}(T)\left(1 - N_{0,\sqrt{T}}(d)\right) = \hat{\kappa}(T)\left(1 - N_{0,1}\left(\frac{d}{\sqrt{T}}\right)\right)$$

While for the second integral we get:

$$\begin{aligned}\mathcal{C}_{an} &= \int_d^{+\infty} \frac{dx}{\sqrt{2\pi T}} \exp\left(-\frac{(x-\sigma T)^2}{2T}\right) \\ &= 1 - N_{\sigma T,\sqrt{T}}(d) = 1 - N_{0,1}\left(\frac{d-\sigma T}{\sqrt{T}}\right)\end{aligned}$$

Finally:

$$\begin{aligned}\Pi(\mathcal{C},T) &= S(0)\left[1 - N_{0,1}\left(\frac{d-\sigma T}{\sqrt{T}}\right) - \hat{\kappa}(T)\left(1 - N_{0,1}\left(\frac{d}{\sqrt{T}}\right)\right)\right] \\ &= S(0)\,[\,1-\hat{\kappa}(T)\,] + S(0)\left[\hat{\kappa}(T)N_{0,1}\left(\frac{d}{\sqrt{T}}\right) - N_{0,1}\left(\frac{d-\sigma T}{\sqrt{T}}\right)\right] \qquad (2.8)\end{aligned}$$

2.4.4 The Call-Put Parity

If we look careful at eq.(2.8) we see that it can be rewritten as:

$$\Pi(\mathcal{C},T) = S(0) - \kappa(T) + \Pi(\mathcal{P},T).$$

This is not a mere accident, but a basic property. Remember the defintion:

$$(x-y)^+ := \max(x-y,0).$$

It is easy to check that:

$$x - y = (x-y)^+ - (y-x)^+,$$

therefore:

$$\mathbb{E}[\,(S(T)-\kappa(T))\,] = \mathbb{E}[\,(S(T)-\kappa(T))^+\,] - \mathbb{E}[\,(\kappa(T)-S(T))^+\,]$$

In the r.h.s. we recognize:

$$\Pi(\mathcal{C},T) - \Pi(\mathcal{P},T)$$

while the l.h.s is:

$$\mathbb{E}[\,(S(T)-\kappa(T))\,] = S(0)\mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right)\right] - \kappa(T).$$

the left over expectation has already been dealt with and its value is 1.
The situation can be described as follows: we have two random variables

$$X = (S(T) - \kappa(T))^+, \quad \text{and} \quad Y = (\kappa(T) - S(T))^+,$$

they are NOT the same, neither they have similar distributions. Actually we know nothing bout this concept. What we know is that the expectation of X and the expectation of Y differ by some constant. In fact

$$\mathbb{E}[\, X \,] = \mathbb{E}[\, Y \,] + S(0) - \kappa(T).$$

As fas as the extimate of the expectation is concerned, using the r.h.s. or the l.h.s. is immterial. They both provide an unbiased estimate of the correct value. Nonetheless, almost surely it could happen that one of the two has lower variance, leading to a beter error estimate and faster conergence. It may very well be that Y has a lower variance that X, so in this case we would be wise using Y.

Exercise 2.4.1. *Compute, using MC, the value of an option with the following pay off:*

$$\begin{cases} B_l & S(T) < B_l \\ S(T) & B_l \leq S(T) \leq B_h \\ B_h & S(T) > B_h. \end{cases}$$

If you are brave enough, you can compute the analytical value as well
(but it is NOT required).

Chapter 3

The Continuation Value

3.1 A Different View on Pricing

Definition 3.1.1. *A strategy is an adapted function to the filtration. In other words is a decision u take based solely on past and present information.*

In this context we talk about assuming financial decisions concerning our portfolio.
We know that the price of a claim is given by:

$$\Pi(\mathcal{C}) = \mathbb{E}[\,\mathcal{C}(T)\,]$$

where, let's say for a call, we have:

$$\mathcal{C}(T) = \max(S(T) - \kappa, 0).$$

A second though is enough to let us realize that we have defined a strategy:
when we reach maturity we check the market price of the assets $S(T)$. If $S(T) > \kappa$ we exercise the option otherwise we forgo it.
If we have a bermuda-like option (or an american one) things get a lot more complicated.

3.2 The America Put Option

Definition 3.2.1. *The right to sell, at any given time from now to maturity, for a price κ agreed is advance.*

Let's assume that our option has a one year maturity. A stregety could be something like:

1. do nothing for the first six months
2. after the first month, the first time time that the assets goes below κ, we are going to sell it.

Given this strategy, that we call π we can certainly compute its expectation value

$$\Pi_\pi(\mathcal{C}) = \mathbb{E}[\,\mathcal{C}_\pi(T)\,]$$

but this is by no means the usual 'non arbitrage price'. A proof is hard but intuition tells us that price will be something like:

$$\sup_\pi \Pi_\pi(\mathcal{C}) = \mathbb{E}[\,\mathcal{C}_\pi(T)\,]$$

We have contract that grants to us the right to sell an assets S at any time between now and its maturity T. Once a day, we check the price of the asset, and we decide what to do, hold on the asset or sell it. We call h the decision to hold, s the action corresponding to sell. As a consequence of our decision we will receive an amount:

$$\begin{cases} i(n,h) & = & 0 \\ i(n,s) & = & \kappa - S_n \end{cases}$$

Let's call a_n the decision we take in t_n, and ϕ the strategy emerging from the N choices we will make. The value of the ϕ strategy we decide to enact will be:

$$\mathcal{I}(\pi, S_0) = \mathbb{E}\left[\sum_{n=1}^{N} Z(t_n) i(S_n, a_n)\right]$$

Let's formalize the problem.

1. we break the time interval $[0 \le t \le T]$ in N intervals $0 = t_0 < t_1 < \ldots < t_N = T$.

2. evolution occurs in the interval $[t_{n-1}, t_n)$, and we apply some strategy in t_n having full knowledge of everything that happened up to that pont.

3. S is equally discretized, meaning that it can assume K values. The notation S_n will denote any one of the K possible values that S can assume at t_n, while S_n^k is one of these possible values.

4. after we have made our choice S_n will evolve to S_{n+1} with a probability described by the matrix $q_n(S^{k'} \mid S^k\,)$ or, if you prefer $q(S_{n+1}^{k'} \mid S_n^k)$

Let:

$$\begin{aligned} f_N(S) &= 0, \quad \forall S \\ f_{n-1}(S) &= \max_a \left(i(S, a_{n-1}) + \sum_k \frac{Z(t_n)}{Z(t_{n-1})} f_n(S^k) q_{n-1}(S^k \mid S) \right) \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[Z(t_n)f_n] &= \mathbb{E}\Big[\mathbb{E}[Z(t_n)f_n(S)\,|\,S_{n-1}]\Big] \\
&= \mathbb{E}\left[\sum_k Z(t_n)f_n(S^k)q_{n-1}(S^k|S_{n-1})\right] \\
&\leq \mathbb{E}[Z(t_{n-1})f_{n-1}(S_{n-1}) - Z(t_{n-1})i(S_{n-1},a)\,] \\
\mathbb{E}[Z(t_{n-1})f_{n-1}(S_{n-1})] - \mathbb{E}[Z(t_n)f_n(S_n)] &\geq \mathbb{E}[Z(t_{n-1})i(S_{n-1},a)\,] \\
f(S_0) &\geq \mathbb{E}\left[\sum_{n=1}^{N} i(Z(t_{n-1})S_{n-1},a)\right]
\end{aligned}
$$

Exercise 3.2.1. *We want to practice working with trajectories. Let τ the time that a trajectory hits a barrier for the first time. Compute the probability that*

$$\mathbb{P}(\tau < t)$$

If u discretize as usual your time interval

$$0 = t_0 < t_1 < \cdots < t_N = T$$

we can build the set of discrete probabilities:

$$p_0, p_1, \cdots, P_N, \quad p_n = \mathbb{P}(\tau < t_n).$$

more in class ...

Chapter 4

The LSM Algorithm in Practice

4.1 Going Backward

We know that at maturity the asset will assume a value $S(T)$, distributed according to the probability defined by the log normal process, therefore

$$f_N(S^j) = \max(\kappa - S^j, 0)$$

Let's recall the ideal strategy:

$$\begin{aligned} f_N(S) &= 0, \quad \forall S \\ f_{n-1}(S) &= \max_a \left(i(S, a_{n-1}) + \sum_k \frac{Z(t_n)}{Z(t_{n-1})} f_n(S^k) q_{n-1}(S^k \mid S) \right) \end{aligned}$$

That we approximate in the following way:

$$c_{n-1}(S^j_{n-1}) = \frac{e^{-rt_n}}{e^{-rt_{n-1}}} f_n(S^j_n)$$

$$f_{n-1}(S^j_{n-1}) = \max\left(\kappa - S^j_n,\ c_{n-1}(S^j_{n-1})\right)$$

This last step, is almost right. The notation might be somewhat confusing, but hat we want is something like this:

$$f_{n-1}(S),$$

that is a black box f_n , at every time step that, once we provide any value od S, the black box will return to us the continuation value.

4.1.1 Interpolation. A Quick Look

For each trajectory (at every time step) we have a pair

$$S_n, c_n(S_n)$$

Let's form a set of basis functions

$$\psi_k(S), \quad k = 0, K$$

for instance for $K = 3$ we could have:

$$\psi_0 = 1, \quad \psi_1(S) = S, \quad \psi_2(S) = S^2, \quad \psi_3 = S^3$$

or we could select, for $K - 5$

$$\psi_0 = 1, \quad \psi_1(S) = S, \quad \psi_2(S) = \Pi[S], \quad \psi_3 = S^2, \quad \psi_4 = S\,\Pi(S), \quad \psi_5 = \Pi^2(S)$$

where $\Pi(S)$ is the value of the put option in t_n with maturity in T.
The choiche of these functions is left to you and relies on your intuition on what you believe is a good way to approximate the continuation function.
Next, we want to find a set of constants $\pi_n^0, \pi_n^1, \dots, \pi_n^K$ such that

$$\hat{f}_n(S) := \sum_{k=0}^{K} \pi_n^k \psi_k(S)$$

is a good approximation to $f_n(S)$.
One way to do this is to solve the following minimisation problem:

$$\chi^2(\pi) := \frac{1}{2} \sum_j \left[f_n(S^j) - \sum_{k=0}^{K} \pi_n^k \psi_k(S^j) \right]^2 \tag{4.1}$$

then solve for:

$$\min_\pi \chi^2(\pi). \tag{4.2}$$

4.1.2 The Least Square Problem

If we expand the square of the r.h.s. of eq.(4.1) we get:

$$\frac{1}{2} \sum_j \left[f_n^2(S^j) + \left(\sum_{k=0}^{K} \pi_n^k \psi_k(S^j) \right)^2 - 2 f_n(S^j) \sum_{k=0}^{K} \pi_n^k \psi_k(S^j) \right]$$

Let's look at the three pieces separately:

$$A_1 = \frac{1}{2}\sum_j f_n^2(S^j) \tag{4.3}$$

$$\begin{aligned}
A_2 \quad &:= \quad \frac{1}{2}\sum_j \left(\sum_{k=0}^{K} \pi_n^k \psi_k(S^j)\right)^2 \\
&:= \quad \frac{1}{2}\sum_j \sum_{k,h=0}^{K} \pi_n^k \psi_k(S^j)\pi_n^h \psi_h(S^j)
\end{aligned}$$

exchanging the summations

$$:= \quad \frac{1}{2}\sum_{k,h=0}^{K} \pi_n^k \left(\sum_j (\psi_k(S^j)\psi_h(S^j) \right) \pi_n^h$$

that we can write, in matrix notation as:

$$A2 = \frac{1}{2}\sum_{k,h=0}^{K} \pi_n^k \, M_{kh}(S) \, \pi_n^h = \frac{1}{2}\boldsymbol{\pi}_n^\dagger \mathbf{M} \boldsymbol{\pi}_n$$

where:

$$M_{kh}(S) = \left(\sum_j (\psi_k(S^j)\psi_h(S^j) \right)$$

$$A_3 \quad = \quad \sum_j f_n(S^j) \sum_{k=0}^{K} \pi_n^k \psi_k(S^j)$$

exchanging the summations

$$\begin{aligned}
&= \quad \sum_{k=0}^{K} \pi_n^k \left(\sum_j f_n(S^j)\psi_k(S^j)\right) \\
&= \quad \sum_{k=0}^{K} \pi_n^k b_k = \boldsymbol{\pi}_n^\dagger \cdot \mathbf{b}.
\end{aligned}$$

where

$$b_k = \sum_j f_n(S^j)\psi_k(S^j)$$

Finally:

$$\chi^2(\pi) = A_1 + A_2 - A_3 = A_1 + \frac{1}{2}\boldsymbol{\pi}_n^\dagger \mathbf{M}\boldsymbol{\pi}_n - \boldsymbol{\pi}_n^\dagger \cdot \mathbf{b}.$$

The term A_1 was left unchanged given that it is constant that does not depend on π, so it is immaterial as far as teh minimisatio problem is concerned. Waht we are left is to minimize:

$$\frac{1}{2}\boldsymbol{\pi}_n^\dagger \mathbf{M}\boldsymbol{\pi}_n - \boldsymbol{\pi}_n^\dagger \cdot \mathbf{b}.$$

that is achieved solving the linear system of equations:

$$\mathbf{M}\boldsymbol{\pi}_n = \mathbf{b}.$$

Next we replace

$$f_n(S^j) \leftarrow \hat{f}_n(S^j) = \sum_{k=1}^{K} \pi_n^k \psi_k(S^j).$$

and we keep going all the way to $f_0(S^j)$
Clearly $S^j = S_o, \quad \forall j$ and we can compute the price, and the MC error, computing averge and standard deviation from this array.
The problem is that the error estimate, this time is not correct. The rule we did learn can be reliably applied to independent RV. In this case, although each trajectory is independent, the J different values of f_0 are not, in as much as the value of the parameters π we used to build the continuation value depends on the whole trajectory set.

4.2 The Forward MC

After we have performed our backward propagation, at each time step we have an efficient black box that we can use to compute the continuation value. Our black box is given by the coefficients π_n that togheter with the rules to build the basis functions provides us with a well defined 'strategy'. As we saw in earier lectures, any strategy (not forward looking) is sub-optimal, therefore will provide us with a value that is surely a lower bound to th actual price.

1. We start generating J independent trajectories S_n^j.
2. For each trajectory, we run upword from $t = 0$ to $t = T$,
3. at each time step t_n, given S_n^j we compute $\hat{f}_n(S_n^j)$, using the coefficients π_n obtained in the back propagation step.
4. if $\max(\kappa - S_n^j, 0)$ is larger that $\hat{f}_n$ we exercise the option, and the contribution of this trajectory to the price will be

$$p^j = e^{-rt_n}(\kappa - S_n^j)$$

5. if $\max(\kappa - S_n^j, 0)$ is less that $\hat{f}_n$ we hold on to teh option, and we proceed with the same analysis to step t_{n+1}.

This time we can rightfully apply our rule to estimate the MC error. Of course we should not forget that our result is also biaed by a systematic error. The wise pricer will look at the difference between the backward price and the forward price (the latter one should be less) and the MC error as an estimate for the statistical error.

Chapter 5

Examples

5.1 Examples

We are concerned with an amrican put option with strike κ and maturity T. The paramter definition is as follows:

$$\frac{dS(t)}{(t)S} = rdt + \sigma dW, \quad S(0) = S_o.$$

S_o	κ	T	r	σ	price
1.0	1.0	1.0	.05	.20	0.0609
1.0	1.2	1.0	.05	.20	0.2013
1.0	0.8	1.0	.05	.20	0.0072
1.0	1.0	1.0	.01	.20	0.0751
1.0	1.0	1.0	.00	.20	0.0797
1.0	1.0	1.0	.05	.40	0.1367
1.0	1.0	1.5	.05	.20	0.0702

Table 5.1: List of prices for the american put option

In all of the plots simulation was done with the following parameters:

$$\sigma = .2, \quad r = 0.05, \quad S_o = \kappa = 1, \quad T = 1 \text{ year}$$

The basis functions used to interpolate are:

$$\phi_0 = 1, \quad \phi_1 = S, \quad \phi_2 = S^2, \quad \phi_3 = S^3.$$

As far as the exam is concerned, it will be easier to chech the results if you run some selected set of parameters: keeping S_o fixed to 1, besides the results shown in the above table, you should produce results for:

$$\begin{array}{rcl} \kappa & : & 0.8,\ 0.9,\ 1.1,\ 1.2 \\ \sigma & : & .2,\ .4,\ .6 \\ r & : & -0.1,\ 0.0,\ 0.01,\ 0.05 \\ T & : & .5,\ \ 1.0,\ 1.5 \quad \text{years.} \end{array}$$

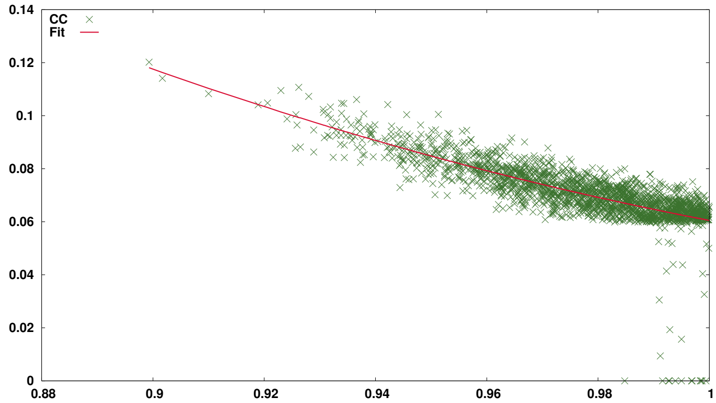

Figure 5.1: Interpolation of the continuation value one week after contract start. Trajectories have a δt of one day.

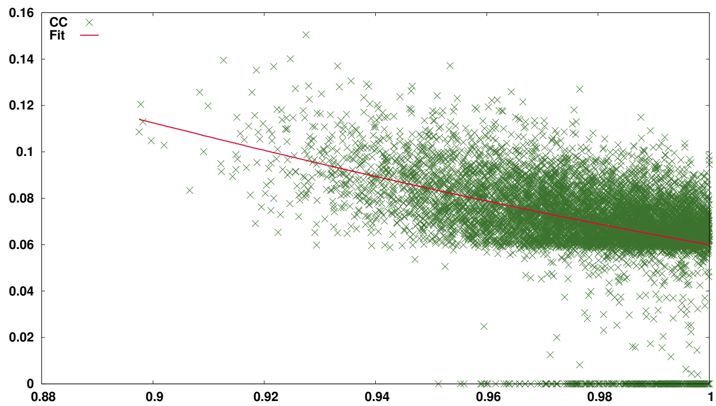

Figure 5.2: Interpolation of the continuation value one week after contract start. Trajectories have a δt of one week.

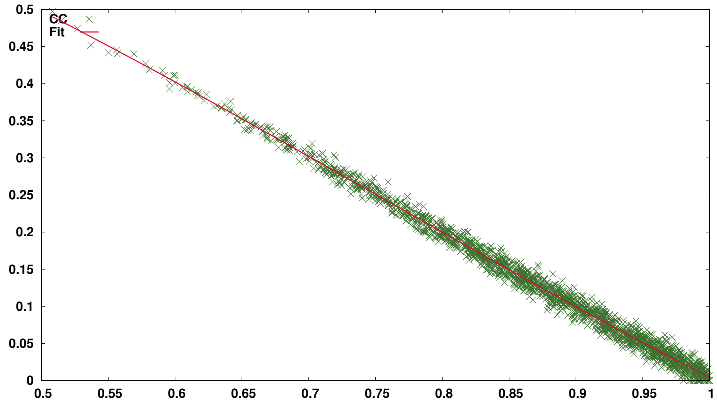

Figure 5.3: Interpolation of the continuation value one week before contract end. Trajectories have a δt of one day.

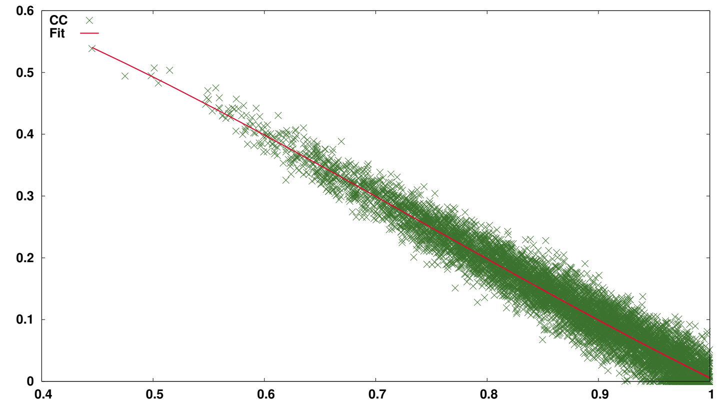

Figure 5.4: Interpolation of the continuation value one week before contract end. Trajectories have a δt of one week.

Appendix A

Gaussian integrals

A.1 Single variable

Our goal is to compute the integral:

$$I(\alpha) = \int_{-\infty}^{\infty} dx e^{-\alpha x^2} \quad \text{with} \quad \alpha > 0.$$

In order to proceed we notice that:

$$I^2(\alpha) = \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2} \int_{-\infty}^{+\infty} dy\, e^{-\alpha y^2} = \int_{-\infty}^{+\infty} dx dy\, e^{-\alpha(x^2+y^2)}$$

Performing the change of coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

we get:

$$I^2(\alpha) \quad = \quad \int_0^{2\pi} d\theta \int_0^{+\infty} r dr e^{-\alpha r^2} = \pi \int_0^{+\infty} dr^2 e^{-\alpha r^2} = \frac{\pi}{\alpha},$$

and finally:

$$I(\alpha) = \sqrt{\frac{\pi}{\alpha}}. \tag{A.1}$$

A.1.1 Normalized distribution

Let's consider the function:

$$f(x,\alpha) := \sqrt{\frac{\alpha}{\pi}} \exp\Big(- \alpha x^2 \Big), \tag{A.2}$$

then, for every α it is true that:

$$\int_{-\infty}^{+\infty} dx f(x,\alpha) = 1,$$

furthermore:

$$\begin{aligned}\int_{-\infty}^{+\infty} dx x^2 e^{-\alpha x^2} &= -\frac{d}{d\alpha}\int_{-\infty}^{+\infty} dx e^{-\alpha x^2}\\ &= -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\end{aligned}$$

and it follows that:

$$\int_{-\infty}^{+\infty} dx\, x^2\, f(x,\alpha) = \frac{1}{2\alpha}.$$

The choice $\alpha = 1/2$ is sort of natural and leads us to teh definition of the normal distribution density:

$$\phi_{0,1}(x) := \frac{1}{\sqrt{2\,\pi}} \exp\left(-\frac{x^2}{2}\right)$$

The meaning of the index $0,1$ will become apparent shortly.

A.1.2 Shift and σ

Let's consider the density function:

$$\phi_{m.\sigma}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

The following properties can be easily verified:

$$\begin{aligned}\int_{-\infty}^{+\infty} dx\,\phi_{m,\sigma}(x) &= 1\\ \int_{-\infty}^{+\infty} dx\,x\,\phi_{m,\sigma}(x) &= m\\ \int_{-\infty}^{+\infty} dx\,(x-m)^2\,\phi_{m,\sigma}(x) &= \sigma^2.\end{aligned}$$

The function:

$$\mathcal{N}_{m,\sigma}(x) := \int_{-\infty}^{x} dy\,\phi_{m,\sigma}(y),$$

is called the normal (m,σ) distribution function (df).

A.1.3 Some properties of the normal df

$$\begin{aligned}
\int_{\lambda}^{+\infty} dx\, \phi_{m,\sigma}(x) &= \left(\int_{-\infty}^{+\infty} - \int_{-\infty}^{\lambda} \right) dx\, \phi_{m,\sigma}(x) \\
&= 1 - \mathcal{N}_{m,\sigma}(\lambda).
\end{aligned}$$

Appendix B

The Martingale Process

Let $S(t)$ be the process:

$$S_{n+1} = S_n \exp\left(-\frac{\sigma^2}{2}(t_{n+1} - t_n) + \sigma(W_{n+1} - W_n)\right) \tag{B.1}$$

We want to compute:

$$\mathbb{E}[S_n] \quad \text{and} \quad \mathbb{E}[S_n^2].$$

The second expectation is needed to compute the 'expected' or theoretical MC error, when estimating the first one.
Iterating eq.(C.1) we get:

$$S_n = S_0 \exp\left(-\frac{\sigma^2}{2}t_n + \sigma(W_n - W_0)\right)$$

where

$$W_n - W_n \stackrel{d}{=} N_{0,\sqrt{t_n}}.$$

$$\begin{aligned}
\mathbb{E}[S_n] &= S_0 \mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2}t_n + \sigma(W_n - W_0)\right)\right] \\
&= S_0 \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{\sigma^2}{2}t_n + \sigma x - \frac{x^2}{2t_n}\right) \\
&= S_0 \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{(x - \sigma t_n)^2}{2t_n}\right) = S_0.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[S_n^2] &= S_0^2 \mathbb{E}\left[\exp\left(-\sigma^2 t_n + 2\sigma(W_n - W_0)\right)\right] \\
&= S_0^2 \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t_n}} \exp\left(-\sigma^2 t_n + 2\sigma x - \frac{x^2}{2t_n}\right) \\
&= S_0^2 \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi t_n}} \exp\left(-\frac{(x - 2\sigma t_n)^2}{2t_n} + \sigma^2 t_n\right) = S_0^2 e^{\sigma^2 t_n}.
\end{aligned}
$$

Our estimate for the MC error will be:

$$
\mathcal{E}rr = S_0 \sqrt{\frac{e^{\sigma^2 t_n} - 1}{J}} \simeq S_0 \frac{\sigma\sqrt{t_n}}{\sqrt{J}}
$$

Appendix C

The Cap And Floor Option

Let $S(t)$ be the process:

$$S_{n+1} = S_n \exp\left(-\frac{\sigma^2}{2}(t_{n+1} - t_n) + \sigma(W_{n+1} - W_n)\right) \tag{C.1}$$

where

$$S_n = e^{-rt_n}\hat{S}_n.$$

We want to compute price of a financial product define by the following payoff

$$\begin{cases} B_l & \hat{S}(T) < B_l \\ \hat{S}(T) & B_l < \hat{S}(T) < B_h \\ B_h & \hat{S}(T) > B_h \end{cases}$$

That is:

$$\Pi(\mathcal{O}) = \mathbb{E}\left[e^{-rT}\ \left(B_l \mathbf{1}_{[\hat{S}(T)<B_l]} + \hat{S}(T)\mathbf{1}_{[B_l<\hat{S}(T)<B_h]} + B_h \mathbf{1}_{[\hat{S}(T)>B_h]}\right)\right] \tag{C.2}$$

If we define:

$$B_l(T) := e^{-rT}B_l, \quad \text{and} \quad B_h(T) := e^{-rT}B_h$$

we can observe that

$$\begin{aligned} \mathbf{1}_{[\hat{S}(T)<B_l]} &= \mathbf{1}_{[S(T)<B_l(T)]} \\ \mathbf{1}_{[\hat{S}(T)>B_h]} &= \mathbf{1}_{[S(T)>B_g(T)]} \\ \mathbf{1}_{[B_l<\hat{S}(T)<B_h]} &= \mathbf{1}_{[B_l(T)<S(T)<B_g(T)]}, \end{aligned}$$

therefore:

$$\begin{aligned}
\Pi(\mathcal{O}) &= B_l(T)\mathbb{E}\left[\mathbf{1}_{[S(T)<B_l(T)]}\right] \\
&+ \mathbb{E}\left[S(T)\mathbf{1}_{[B_l(T)<S(T)<B_h(T)]}\right] \\
&+ B_h(T)\mathbb{E}\left[\mathbf{1}_{[S(T)>B_h(T)]}\right]
\end{aligned}$$

Let's look at the constraints:

$$\begin{aligned}
\mathbf{1}_{[S(T)<B_l(T)]} &\to S(0)\exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right) < B_l(T) \\
&\to (W_T - W_0) < \frac{\log\left(\frac{B_l(T)}{S(0)}\right) + \frac{\sigma^2}{2}T}{\sigma} := d_l \\
\mathbf{1}_{[S(T)>B_h(T)]} &\to S(0)\exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right) > B_h(T) \\
&\to (W_T - W_0) > \frac{\log\left(\frac{B_h(T)}{S(0)}\right) + \frac{\sigma^2}{2}T}{\sigma} := d_h \\
\mathbf{1}_{[B_l(T)<S(T)<B_h(T)]} &\to B_l(T) < S(0)\exp\left(-\frac{\sigma^2}{2}T + \sigma(W_T - W_0)\right) < B_h(T) \\
&\to \frac{\log\left(\frac{B_l(T)}{S(0)}\right) + \frac{\sigma^2}{2}T}{\sigma} < (W_T - W_0) < \frac{\log\left(\frac{B_h(T)}{S(0)}\right) + \frac{\sigma^2}{2}T}{\sigma} \\
&\to d_l < (W_T - W_0) < d_h.
\end{aligned}$$

Let's compute the three integrals separately:

$$\begin{aligned} I_1 &:= \mathbb{E}\left[\mathbf{1}_{[S(T)<B_l(T)]}\right] \\ &= \int_{-\infty}^{d_l} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2\sqrt{t}}\right) = N_{0,\sqrt{t}}(d_l) = N_{0,1}\left(\frac{d_l}{\sqrt{t}}\right) \end{aligned}$$

$$\begin{aligned} I_3 &:= \mathbb{E}\left[\mathbf{1}_{[S(T)>B_h(T)]}\right] \\ &= \int_{d_h}^{\infty} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2\sqrt{t}}\right) \\ &= 1 - \int_{-\infty}^{d_h} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2\sqrt{t}}\right) = 1 - N_{0,\sqrt{t}}(d_h) = 1 - N_{0,1}\left(\frac{d_h}{\sqrt{t}}\right) \end{aligned}$$

$$\begin{aligned} I_2 &:= \mathbb{E}\left[S(T)\mathbf{1}_{[B_l(T)<S(T)<B_h(T)]}\right] \\ &= S_0 \int_{d_l}^{d_h} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{\sigma^2}{2}t + \sigma x - \frac{x^2}{2\sqrt{t}}\right) \\ &= S_0 \int_{d_l}^{d_h} \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{(x-\sigma t)^2}{2\sqrt{t}}\right) \\ &= S_0 \left(\int_{-\infty}^{d_h} - \int_{-\infty}^{d_l}\right) \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{(x-\sigma t)^2}{2\sqrt{t}}\right) \\ &= S_0 \left(\int_{-\infty}^{d_h-\sigma t} - \int_{-\infty}^{d_l-\sigma t}\right) \frac{dx}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2\sqrt{t}}\right) \\ &= S_0 \left(N_{0,1}\left(\frac{d_h-\sigma t}{\sqrt{t}}\right) - N_{0,1}\left(\frac{d_l-\sigma t}{\sqrt{t}}\right)\right) \end{aligned}$$

and finally:

$$\Pi(\mathcal{O}) = B_l(T)N_{0,1}\left(\frac{d_l}{\sqrt{t}}\right) + B_h(T) - B_h(T)N_{0,1}\left(\frac{d_h}{\sqrt{t}}\right) + S_0N_{0,1}\left(\frac{d_h-\sigma t}{\sqrt{t}}\right) - S_0N_{0,1}\left(\frac{d_l-\sigma t}{\sqrt{t}}\right)$$