Chow Ring Classes of Varieties of Secant and Tangent Lines to Projective Varieties

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For simplicity, we work over an algebraically closed field of characteristic zero.

Main Theorem

Let X be a smooth projective variety of dimension $r \in \mathbb{Z}_{\geq 0}$, and let $\iota : X \hookrightarrow \mathbb{P}^n$ be a nondegenerate embedding for $n \in \mathbb{Z}_{\geq 1}$, so that $s := \operatorname{codim}_{\mathbb{P}^n}(X) = n - r \in \mathbb{Z}_{\geq 1}$. In $\mathbb{G}(1,n)$, let $\mathcal{S}X$ (resp. $\mathcal{T}X$) denote the subvariety of secant (resp. tangent) lines to X.

Theorem 1.1. (G. [2], 2023) Define the "higher degrees" $d_0, \ldots, d_r$ of X by

$$\iota_* c(\mathcal{N}_\iota) =: \sum_{j=0}^{r} d_j \zeta^{s+j} \in \mathbb{Z}[\zeta]/(\zeta^{n+1}) = \mathrm{CH}^*(\mathbb{P}^n).$$

Then

$$[\mathcal{S}X] = \frac{1}{2} d_0^2 \sigma_{s-1}^2 - \frac{1}{2} \sum_{i=0}^{s-1} \left[\sum_{j=0}^{s-1-i} (-1)^j \binom{i+j}{j} d_{s-1-i-j} \right] \sigma_{2s-2-i,i} \in \mathrm{CH}^{2s-2}\,\mathbb{G}(1,n), \text{ and}$$

$$[\mathcal{T}X] = \sum_{i=0}^{s-1} \left[\sum_{j=0}^{s-i} (-1)^i \binom{i+j}{j} d_{s-i-j} \right] \sigma_{2s-1-i,i} \in \mathrm{CH}^{2s-1}\,\mathbb{G}(1,n).$$

The higher degrees $(d_i)_{i=0}^r$ are stable under hyperplane sections and hence related to the coefficients of the Hilbert polynomial of (X, ι).

Corollary 1.2. Suppose further that X is not defective and $n \geq 2r+1$. Then

$$\deg \operatorname{Sec} X = \frac{1}{2\delta} \left[d^2 - \sum_{j=0}^{r} (-1)^{r-j} \binom{s-1-j}{r-j} d_j \right], \text{ and}$$

$$\deg \operatorname{Tan} X = \sum_{j=0}^{r} (-1)^{r-j} \binom{s-j}{r-j} d_j,$$

where $\delta := \deg J(X,X)/\operatorname{Sec}(X)$ is the number of secant lines to X on which a general point of $\operatorname{Sec}(X)$ lies.

Example 1.3. Let $C \subset \mathbb{P}^n$ be a nondegenerate curve of degree d and genus g. Then

$$d_1 = d(n+1) + 2g - 2$$

so that

$$[\mathcal{S}C] = \binom{d}{2} \sigma_{n-2,n-2} + \left[\binom{d-1}{2} - g \right] \sigma_{n-1,n-3} \quad \text{and} \quad [\mathcal{T}C] = (2d+2g-2)\sigma_{n-1,n-2}.$$

In particular,

$$\deg \operatorname{Sec} C = \binom{d-1}{2} - g \quad \text{and} \quad \deg \operatorname{Tan} C = 2d + 2g - 2$$

for $n \geq 4$ and $n \geq 3$ respectively.

Example 1.4. Similar formulae can be obtained when X is a Veronese variety, a Segre variety, a rational normal scroll, a Plücker-embedded Grassmannian, etc. For instance, for $r \in \mathbb{Z}_{\geq 1}$, we have

$$[\mathcal{S}(\mathbb{P}^{r-1} \times \mathbb{P}^1)] = \sum_{j=0}^{r-2} \binom{r-j}{2} \sigma_{r-2+j, r-2-j} \in \mathrm{CH}^{2r-4}\,\mathbb{G}(1, 2r-1)$$

and $[\mathcal{S}\mathbb{G}(1,4)] = \sigma_4 + 5\sigma_{3,1} + 10\sigma_{2,2} \in \mathrm{CH}^4\,\mathbb{G}(1,9)$.

Initial Motivation

For integers $r \in \mathbb{Z}_{\geq 0}$ and $d, n \in \mathbb{Z}_{\geq 1}$, study r-dimensional linear systems of degree d hypersurfaces in $\mathbb{P}^n$ up to equivalence, i.e., study

$$\mathbb{G}(r, |\mathcal{O}_{\mathbb{P}^n}(d)|) \,/\!/\, \mathrm{PGL}_{n+1}.$$

The case $r = 0$ is moduli of hypersurfaces. For $(r,d,n) = (1,2,2)$, Jordan showed in 1906 that there are 8 orbits $\mathcal{O}_1, \ldots, \mathcal{O}_8$ in $\mathbb{G}(1,5)$.

Theorem 2.1. (G. [1], 2022)

Orbit	Description	Base Locus Type	Codim	Class of Closure	Plücker Degree
$\mathcal{O}_1$	general	$(1,1,1,1)$	0	σ_0	14
$\mathcal{O}_2$	simply tangent	$(2,1,1)$	1	$6\sigma_1$	84
$\mathcal{O}_3$	bitangent	$(2,2)$	2	$4\sigma_2$	36
$\mathcal{O}_4$	osculating	$(3,1)$	2	$6\sigma_2 + 9\sigma_{1,1}$	99
$\mathcal{O}_5$	superosculating	(4)	3	$4\sigma_3 + 8\sigma_{2,1}$	56
$\mathcal{O}_6$	fixed point	$\{*\}$	4	$3\sigma_{3,1} + 6\sigma_{2,2}$	21
$\mathcal{O}_7$	fixed line	$L \cup \{*\} : * \notin L$	4	$6\sigma_{3,1} + 3\sigma_{2,2}$	24
$\mathcal{O}_8$	embedded point	$L \cup \{*\} : * \in L$	5	$6\sigma_{4,1} + 6\sigma_{3,2}$	18

There is some beautiful geometry here involving the Cayley cubic surface, the fibers of $j : \mathbb{P}^4 \dashrightarrow \mathbb{P}^1$, plane sextics with six cusps, the secant threefold to the rational normal quartic, generically non-reduced components of Fano schemes, etc., all discussed in [1].

Remark 2.2.

1. A lot (but not all) is known for $(r,d) = (1,2)$ (Segre-Weierstrass); this is related to the geometry of Fano schemes of spaces of symmetric matrices $\mathbf{F}_k(\mathrm{SD}_n^r)$ and compression spaces (Mokhtar [3]). Some results are known for $(r,d,n) = (2,2,2)$ (G.-Choudhary).
2. If $X \subset \mathbb{P}^5 = |\mathcal{O}_{\mathbb{P}^2}(2)|$ is the Veronese surface, then $\overline{\mathcal{O}}_6 = \mathcal{S}X$ and $\overline{\mathcal{O}}_8 = \mathcal{T}X$.

Main Proof Strategy

Proof 3.1. Consider the flag variety

$$\begin{tikzcd}[column sep=small] & \Phi := \mathbb{F}(0,1;n) = \mathbb{P}\mathcal{Q}_{/\mathbb{P}^n} = \mathbb{P}\mathcal{S}_{/\mathbb{G}} = \{(p,\ell) : p \in \ell \subset \mathbb{P}^n\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathbb{G} := \mathbb{G}(1,n). \end{tikzcd}$$

Let $Z := \Phi \times_{\mathbb{G}} \Phi = \{(p,q;\ell) : p, q \in \ell\}$ with projections p_1, p_2 to $\mathbb{P}^n$. Then

$$\mathrm{CH}(Z) = \mathrm{CH}(\mathbb{G})[\zeta_1, \zeta_2]/(\zeta_i^2 - \sigma_1 \zeta_i + \sigma_{1,1})_{i=1,2},$$

where $\zeta_i := p_i^* \zeta$ for $i = 1, 2$ is the pullback of the hyperplane class. The intersection $p_1^{-1}(X) \cap p_2^{-1}(X)$ is nontransverse with components $E \cong \mathbb{P}\mathcal{Q}_{/X}$ and $B \cong \mathrm{Bl}_\Delta(X \times X)$. Applying the Excess Intersection Formula yields

$$[p_1^{-1}(X)] \cap [p_2^{-1}(X)] = [B] + \left[\frac{\iota_* c(\mathcal{N}_\iota)}{1 + 2\zeta_1 - \sigma_1} (\zeta_1 + \zeta_2 - \sigma_1) \right]^{2s} \in \mathrm{CH}^{2s}(Z),$$

where $\iota_* c(\mathcal{N}_\iota) \in \mathbb{Z}[\zeta_1]/(\zeta_1^{n+1}) = \mathrm{CH}^*(\mathbb{P}^n)$. Then

$$[\mathcal{S}X] = \frac{1}{2} \pi_{2,*}[B] \quad \text{and} \quad [\mathcal{T}X] = \pi_{2,*}\left([B] \cap (\zeta_1 + \zeta_2 - \sigma_1)\right).$$

Proof 3.2. For $r < s$, by noting that a formula in terms of the higher degrees *exists*, reduce to the case of smooth complete intersection X, say of type $(a_1, \ldots, a_s)$. (In this case, $d_0 = d = \prod_{i=1}^s a_i$, and for $i = 1, \ldots, r$, we have $d_i = d \cdot e_i(a)$.) For $a \in \mathbb{Z}_{\geq 2}$, consider the exact sequence

$$0 \to \pi^* \operatorname{Sym}^{a-2} \mathcal{S}^\vee_{/\mathbb{G}} \otimes \mathcal{O}_{\mathbb{P}\operatorname{Sym}^2 \mathcal{S}^\vee_{/\mathbb{G}}}(-1) \to \pi^* \operatorname{Sym}^a \mathcal{S}^\vee_{/\mathbb{G}} \to Q_a \to 0$$

of vector bundles on $\mathbb{P}\operatorname{Sym}^2 \mathcal{S}^\vee_{/\mathbb{G}} = \mathcal{H}\mathrm{ilb}^2(\mathbb{P}\mathcal{S}_{/\mathbb{G}}) \xrightarrow{\pi} \mathbb{G}$. Then

$$[\mathcal{S}X] = \pi_* \prod_{i=1}^{s} c_2(Q_{a_i}).$$

Similarly, if $\mathcal{E} := \mathcal{P}^1_{\pi_2}\left(\pi_1^* \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^n}(a_i)\right)$ on Φ, then

$$[\mathcal{T}X] = \pi_{2,*}[c_{2s}(\mathcal{E})].$$

Further Corollaries

With some effort, the second proof strategy can be extended to multisecants and higher tangencies. For instance, we recover

Corollary 4.1. Let $C \subset \mathbb{P}^3$ be a nondegenerate curve of degree d and genus g.

1. (Berzolari-Cayley) The surface $S \subset \mathbb{P}^3$ swept out by trisecant lines to C has degree
$$2\binom{d-1}{3} - g(d-2).$$
2. (Cayley) The number (with multiplicity) of quadrisecant lines to C is
$$\frac{(d-2)(d-3)^2(d-4)}{12} - \frac{g(d^2 - 7d + 13 - g)}{2}.$$

Open Problems and Invitation to Collaborate

1. To generalise the main theorem to positive characteristic and singular varieties.
2. To work out more (all?) cases (r,d,n) of the problem in the "Initial Motivation" section, even fixing $d = 2$.
3. To systematize the case of multisecant and higher tangencies above to higher dimensional varieties, and see how far we can push this method.

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References

[1] Dhruv Goel. The Chow Ring Classes of PGL_3 Orbit Closures in $\mathbb{G}(1,5)$. `https://arxiv.org/abs/2310.18571`, 2022.

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[3] Ahmad Mokhtar. Fano schemes of symmetric matrices of bounded rank. `https://arxiv.org/abs/2310.07025`, 2023.

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