Algebraic Geometry

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2.1 Irreducible, Kolmogorov, and Sober Spaces

In this section, we introduce irreducible, Kolmogorov, and sober spaces. These concepts are absolutely fundamental to understanding the topology of schemes. To motivate the definition, we first recall the following facts about connected spaces.

Proposition/Definition 2.1.1 (Connected Spaces)**.**

- The following conditions on a topological space X are equivalent.
  (a) The space X cannot be expressed as a union of two proper clopen subspaces.
  (b) Any nonempty clopen subspace must be all of X.
  (c) The only subspaces of X with empty boundary are $\emptyset$ and X.
  (d) There does not exist a surjective continuous function $X \to \{0, 1\}$, where $\{0, 1\}$ is given the discrete topology.

  A topological space is said to be *connected* iff it satisfies these equivalent conditions.
- A *connected component* of a space is a maximal connected subspace.

Proof. Exercise. ■

Recall the following basic properies.

Lemma 2.1.2.

(a) If $f : X \to Y$ is a continuous map with X connected, then $f(X) \subset Y$ is connected.
(b) If $Y \subset X$ is a subspace and Y is connected, then so is $\overline{Y}$.[1] In particular, any connected component of X is closed in X.[2]
(c) A topological space that can be covered by a family $\{A_i\}$ of connected subspaces such that the total intersection $\bigcap_i A_i$ is nonempty is connected.
(d) Every connected subset of a topological space is contained in a unique connected component. In particular, distinct connected components of a space are disjoint, and every space is the (disjoint) union of its connected components.
(e) Any clopen subset of a topological space is a disjoint union of some of its connected components.

Proof. Exercise. We note only that (c) uses Zorn's Lemma for existence. ■

Often in algebraic geometry, we need a slightly stronger topological property than being connected. This is provided by

Proposition/Definition 2.1.3 (Irreducible Spaces)**.**

- The following conditions on a nonempty topological space are equivalent.
  (a) The space cannot be expressed as a union of two proper closed subspaces.
  (b) Any two nonempty open subspaces have nonempty intersection.
  (c) Every nonempty open subspace is dense.
  (d) A subspace is not dense iff it is nowhere dense.
  (e) Every nonempty open subspace is connected.

  A topological space is said to be *irreducible* iff it is nonempty and it satisfies these conditions.
- An *irreducible component* of a space is a maximal irreducible subspace.
- Given a space X, a point $\eta \in X$ is said to be a *generic point* of X iff $X = \overline{\{\eta\}}$.

[1] The converse is clearly not true.
[2] It is not true in general that a connected component is also necessarily open. A counterexample is given by taking the product topology on $2^{\mathbf{N}}$, or if you prefer $\operatorname{Spec} \mathbf{F}_2^{\mathbf{N}}$.

Proof. Exercise. ■

It is immediate that any irreducible space is connected.

Example 2.1.4. The space $\mathbf{C}$ with the Euclidean topology is connected but not irreducible, whereas the affine line $\mathbf{A}^1_{\mathbf{C}} \cong \operatorname{Spec} \mathbf{C}[x]$ with the Zariski topology is irreducible.

The analogous basic properties for irreducibility are given in

Lemma 2.1.5.

(a) If $f : X \to Y$ is a continuous map with X irreducible, then $f(X) \subset Y$ is irreducible.
(b) If X is any topological space and $Y \subset X$ a subspace, then Y is irreducible iff $\overline{Y}$ is. In particular, any irreducible component of X is closed in X.
(c) A topological space that can be covered by finitely many irreducible open subspaces with nonempty pairwise intersections is irreducible.
(d) Every irreducible subspace of a topological space is contained in an irreducible component. In particular, every topological space is the union of its irreducible components.[3]
(e) If $x \in X$ is a point in a space, then the closure $\overline{\{x\}} \subset X$ is an irreducible closed subspace of X. In particular, a space admitting a generic point is irreducible, and the set of generic points of such a space is precisely the intersection of all its nonempty open subsets.
(f) Every nonempty open subspace of an irreducible space is irreducible.
(g) If X is any topological space and $U \subset X$ a nonempty open subspace, then there is a bijection

$$\{\text{irreducible closed } Y \subset U\} \leftrightarrow \{\text{irreducible closed } Z \subset X \text{ with } Z \cap U \neq \emptyset\} \text{ given by}$$
$$Y \mapsto \overline{Y},$$
$$Z \cap U \leftarrow\!\shortmid Z.$$

Proof. Exercise; the proofs are very similar to those in 2.1.2. We prove only (b) to give a flavor of the proofs. Note that a subspace $Y \subset X$ is irreducible iff for any two open sets U, V of X meeting Y, the intersection $U \cap V$ meets Y. The result follows from noting that an open subspace of X meets Y iff it meets $\overline{Y}$. ■

A relationship between these two notions is provided in

Lemma 2.1.6.

(a) If X is a topological space, $Y \subset X$ a connected component and $Z \subset X$ an irreducible subspace, then $Y \cap Z \neq \emptyset$ implies $Z \subset Y$. In particular, every connected component of a topological space is the union of some of its irreducible components.
(b) A connected space is irreducible iff it is nonempty and it admits a cover by irreducible open subspaces.

Proof. Exercise. ■

Next, we recall some point-set topology before introducing sober spaces.

Proposition/Definition 2.1.7 (Kolmogorov Spaces)**.** The following conditions on a topological space X are equivalent.

[3] It is evident that not every topological space is the *disjoint* union of its irreducible components.

(a) If $x \neq y \in X$, then there is an open subspace $U \subset X$ containing exactly one of x and y.
(b) The map $X \to \mathrm{Irred}(X)$ given by $x \mapsto \overline{\{x\}}$ is an injection, where $\mathrm{Irred}(X)$ is the set of irreducible closed subsets of X.
(c) The specialization relation on X given by $x \rightsquigarrow y$ iff $y \in \overline{\{x\}}$ is a partial order.

A topological space is said to be *Kolmogorov* or T_0 if it satisfies these equivalent conditions.

Proof. Clear. ■

In a Kolmogorov space, the minimal points for the specialization preorder are the closed points, and the maximal points are the generic points of irreducible components (when they exist).

Definition 2.1.8 (Sober Spaces)**.** A topological space X is said to be *sober* if the map

$$X \to \mathrm{Irred}(X), \quad x \mapsto \overline{\{x\}}$$

is a bijection, i.e., every irreducible closed subspace of X has a unique generic point.

Remark 2.1.9.

(a) A sober space is, in particular, Kolmogorov. In [1], a Noetherian sober space is called a Zariski space.
(b) In an irreducible sober space, the intersection of all nonempty open subsets consists of a single point–*the* generic point.

Lemma 2.1.10.

(a) A topological space admitting an open cover by sober spaces is itself sober. In particular, the underlying topological space of any scheme is sober.
(b) A locally closed subspace[4] of a sober space is sober.

Proof.

(a) This is clear for Kolmogorov spaces, and generic points can be produced locally (using 2.1.5(g)). The second statement follows from the first because the underlying topological space of an affine scheme is clearly sober.
(b) Clear.

■

Remark 2.1.11. If X is a sober space and $Y \subset X$ an arbitrary subspace, it is not necessarily true that Y is also sober. For a simple counterexample, take $X = \mathbf{A}^1_{\mathbf{C}}$ with generic point $\eta \in X$ and let $Y := X \smallsetminus \{\eta\}$.

Here are a few of applications of these concepts that are often useful (and help provide clean solutions to the exercises in, say, [1, §II.3])

Proposition 2.1.12. Let $f : X \to Y$ be an open continuous map of topological spaces and suppose that Y is irreducible and has a generic point η. Consider the following conditions:

(a) There is a dense collection of points $y \in Y$ such that the fiber $X_y = f^{-1}(y)$ is irreducible.
(b) The generic fiber $X_\eta = f^{-1}(\eta)$ is irreducible.
(c) The space X is irreducible.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c).

[4] See 2.3.1 below if needed.

Proof.

(a) $\Rightarrow$ (c) Suppose $X = Z_1 \cup Z_2$ for proper closed $Z_1, Z_2 \subset X$. Since f is open, for $i = 1, 2$, the subset $f(X \smallsetminus Z_i) \subset Y$ is open; since Y is irreducible, then intersection $f(X \smallsetminus Z_1) \cap f(X \smallsetminus Z_2)$ is nonempty and open; by the density hypothesis, there is a $y \in f(X \smallsetminus Z_1) \cap f(X \smallsetminus Z_2)$ such that X_y is irreducible. Then $X_y \cap Z_i$ for $i = 1, 2$ are proper closed subspaces of X_y whose union is X_y, which is a contradiction.

(b) $\Leftrightarrow$ (c) For any subspace $Z \subset Y$, continuity of f tells us that $\overline{f^{-1}(Z)} \subset f^{-1}(\overline{Z})$ with equality when f is open. Applying this to $Z = \{\eta\}$ gives us that $\overline{X_\eta} = f^{-1}(\overline{\{\eta\}}) = f^{-1}(Y) = X$, and so the result follows from 2.1.5(b).

■

Proposition 2.1.13. Let $f : X \to Y$ be a continuous map of irreducible sober spaces, and let ξ (resp. η) be the generic point of X (resp. Y). The following are equivalent:

(a) $f(\xi) = \eta$.
(b) $\eta \in f(X)$.
(c) $\overline{f(X)} = Y$.
(d) If $\emptyset \neq U \subset Y$ is open, then $f^{-1}(U) \neq \emptyset$.

A map f satisfying these equivalent properties is said to be *dominant*, and we say that X *dominates* Y (via f).

When X and Y are sober but not necessarily irreducible, we say that a continuous map $f : X \to Y$ is *dominant* iff each irreducible component of Y is dominated by an irreducible component of X; in algebraic geometry, this is a slightly better definition than asking for $\overline{f(X)} = Y$, which is not as well-behaved.[5]

Proof. It is clear that the (c) $\Leftrightarrow$ (d) holds very generally (i.e., does not need the irreducible or sober hypotheses). The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear. For (d) $\Rightarrow$ (a), note that if $\emptyset \neq U \subset Y$ is open, then $f^{-1}(U) \ni \xi$, whence $f(\xi) \in U$, and so the result follows from 2.1.9(b). ■

Proposition 2.1.14. Let $f : X \to Y$ be a continuous map of sober spaces, and suppose that Y is irreducible. Let $\eta \in Y$ be the generic point and $X_\eta = f^{-1}(\eta)$ the generic fiber. There is an inclusion preserving bijection

$$\{\text{irreducible closed } Z \subset X_\eta\} \leftrightarrow \{\text{irreducible closed } W \subset X \text{ dominating } Y\} \text{ given by}$$
$$Z \mapsto \overline{Z},$$
$$W_\eta = W \cap X_\eta \leftarrow W.$$

Further, under this bijection, irreducible components of X_η correspond to irreducible components of X dominating Y.

Proof. Clear from 2.1.13. (Note that we are also using 2.1.10(b).) ■

[5]For an example where these definitions are not equivalent, consider the natural morphism $\coprod_{n\in\mathbf{Z}} \operatorname{Spec} \mathbf{C} \to \mathbf{A}^1_{\mathbf{C}}$ given by the inclusion of the integers.

2.2 Closure-Complete and Noetherian Spaces

In this section, we describe two fundamental finiteness conditions on topological spaces: closure-completeness and the Noetherian property.

Definition 2.2.1. A topological space is *closure-complete* if every point has a closed point in its closure.

Remark 2.2.2. A nonempty closed subspace of a closure-complete space contains a closed point, and in particular, every nonempty closure-complete space has a closed point. For a Kolmogorov space, closure-completeness is equivalent to saying that every point specializes to a minimal point for the specialization partial order.

Lemma 2.2.3. A Kolmogorov space admitting a finite open cover by closure-complete spaces is closure-complete. In particular, the underlying topological space of any quasicompact scheme is closure-complete and hence contains a closed point.

Proof. By induction on the size on the open cover, we are reduced to the case of two open sets, so say $X = U \cup V$ with U, V closure-complete open subspaces. It suffices to show that if $x \in U$, then x has a closed point in its closure. Since U is closure-complete, there is a $y \in \overline{\{x\}} \cap U$ which is closed in U. If $y \notin V$, then y is also closed in X. Else, $y \in V$ and so since V is closure-complete, there is a $z \in \overline{\{y\}} \cap V$ which is closed in V. Then z is closed in X. Indeed, if $w \in \overline{\{z\}} \smallsetminus \{z\}$, then $w \notin V$ and so $w \in \overline{\{z\}} \cap U \subset \overline{\{y\}} \cap U = \{y\}$, but $y \notin \overline{\{z\}}$ by the Kolmogorov hypothesis. The second result follows from the first, since the underyling space of an affine scheme is closure-complete (by Zorn's Lemma!). ■

For a scheme that is not closure-complet (and, in fact, has no closed points at all), see [2, Exercise 3.3.27].

Proposition/Definition 2.2.4 (Noetherian Spaces)**.** The following conditions on a topological space are equivalent.

(a) Every descending chain of closed subspaces is eventually stationary.
(b) Every non-empty collection of closed subspaces has a minimal element with respect to inclusion.
(c) Every ascending chain of open subspaces is eventually stationary.
(d) Every non-empty collection of open subspaces has a maximal element with respect to inclusion.

A topological space is *Noetherian* if it satisfies the above equivalent conditions.

Proof. Exercise. ■

Example 2.2.5. Let R be a Noetherian ring. Then the underlying topological space of $\operatorname{Spec} R$ is Noetherian. This follows from the inclusion-reversing bijection between closed subsets of $\operatorname{Spec} R$ and radical ideals of R.

Example 2.2.6. The space $\mathbf{C}$ equipped with the analytic topology is not Noetherian.

Here are some basic properties.

Lemma 2.2.7.

(a) A topological space that admits a finite cover by Noetherian spaces is Noetherian.

(b) Every subspace of a Noetherian space is Noetherian.
(c) A Noetherian space has only finitely many irreducible components, and hence only finitely many connected components.
(d) A connected component of a Noetherian space is open. In particular, the union of any collection of connected components is clopen.
(e) A Noetherian space is quasicompact.

Proof.

(a) Follows from 2.2.4(a).
(b) If X is Noetherian, $Y \subset X$ a subspace, and (Z_j) a descending chain of closed subspaces in Y, then the chain $(\overline{Z}_j)$ in X is eventually stationary, and hence we are done by $Z_j = Y \cap \overline{Z}_j$.
(c) Apply 2.2.4(b) to the collection of closed subspaces which cannot be written as a finite union of irreducible subspaces. The second result follows from the first, thanks to 2.1.6(a).
(d) By 2.1.6(a), a connected component of a Noetherian space is the complement of the union of some of its irreducible components (namely those disjoint from it); therefore, the first statement follows from (c). The second result follows from the first combined (c).
(e) Apply 2.2.4(d) to the collection of open subspaces consisting of finite unions of elements of a given open cover.

■

Remark 2.2.8.

(a) Note that if X is Noetherian scheme, then the underlying topological space of X is Noetherian, but the converse is false in general (e.g., take $X = \operatorname{Spec} k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$.) "Noetherian" (along with the corresponding "locally Noetherian" below) is one of the very few adjectives for which this remark holds.
(b) A scheme morphism out of a Noetherian scheme is automatically qcqs.

One of the key reasons we like Noetherian spaces is

Proposition 2.2.9 (Irreducible Decomposition of Noetherian Space)**.** Let X be a Noetherian space.

(a) There is an integer $r \geq 1$ and closed irreducible subspaces $Z_1, \dots, Z_r \subset X$ such that $X = Z_1 \cup \cdots \cup Z_r$ such that for all i, j with $1 \leq i \neq j \leq r$, we have $Z_i \not\subset Z_j$, and such that if $Z \subset X$ is any irreducible subspace, then there is an $i = 1, \dots, r$ such that $Z \subset Z_i$.
(b) The integer r and decomposition as in (a) are uniquely determibned (up to re-ordering).

The decomposition $X = Z_1 \cup \cdots \cup Z_r$ is called the *irreducible decomposition* of X.

Proof. Take Z_j to be the irreducible components of X, using 2.2.7(c). ■

Another reason we like Noetherian spaces is that we can do Noetherian induction on them, as in

Proposition 2.2.10 (Noetherian Induction)**.** Let X be a Noetherian topological space, and let $\mathscr{P}$ be a property satisfied by certain closed subsets of X. Suppose that for each closed subset $Z \subset X$, if $\mathscr{P}$ holds for each proper $Z' \subsetneq Z$, then it holds for Z. Then the property $\mathscr{P}$ holds for all closed subsets of X, and in particular for X itself.

Proof. This is just transfinite induction on the collection of closed subsets of X ordered by inclusion; as stated, this follows immediately from 2.2.4(b). ■

One kind of space that comes up often in algebraic geometry is a locally Noetherian space.

Definition 2.2.11. A topological space X is said to be *locally Noetherian* if for each $x \in X$, there is an open neighborhood U of x in X such that U is Noetherian as a topological space.

Note that if X is a locally Noetherian topological space and $U \subset X$ any open subspace, then U is also locally Noetherian. A few of the results from this section can be generalized to the locally Noetherian setting. Here's one example.

Proposition 2.2.12. Let X be a locally Noetherian space. An open subset $U \subset X$ is dense iff it meets all irreducible components of X.

Proof. If U meets all irreducible components of X, then it is dense in X by 2.1.3(c) and 2.1.5(d). To show the converse, suppose we are given a dense open subset $U \subset X$ and an irreducible component $Z \subset X$ such that $U \cap Z = \emptyset$. Since Z is irreducible, it is nonempty; pick any $z \in Z$ and a Noetherian open neighborhood V of z in X. Using 2.1.5(g), we may replace (X, U, Z) by $(V, U \cap V, Z \cap V)$ to assume that in fact X is Noetherian. Let $X = Z_1 \cup \cdots \cup Z_r$ for some $r \in \mathbf{Z}_{\geq 1}$ and (Z_i) be the irreducible decomposition of X as in 2.2.9(a), numbered so that $Z = Z_r$. Then check that $W = X \smallsetminus (Z_1 \cup \cdots \cup Z_{r-1})$ is a nonempty open subset of X contained in Z; this cannot happen, since then U must both meet W (since it is dense) and not (since $W \subset Z$ and $U \cap Z = \emptyset$). ■

It is an easy and instructive exercise, left to the reader, to come up with a space X (necessarily non-locally-Noetherian) and a dense open subset $U \subset X$ that does not meet some irreducible components of X.

2.3 Locally Closed, Constructible, and Very Dense Subspaces

Now we would like to review some point-set topology about locally closed, constructible, and very dense subspaces.

Proposition/Definition 2.3.1 (Locally Closed Subspaces)**.** Let X be a topological space. The following conditions on a subspace $Y \subset X$ are equivalent.

(a) For each $y \in Y$, there is an open subspace $U \subset X$ containing y such that $Y \cap U$ is closed in U.
(b) There is an open subspace $U \subset X$ such that $Y \subset U$ and Y is closed in U.
(c) There is a closed subspace $Z \subset X$ such that $Y \subset Z$ and Y is open in Z.
(d) There is an open subspace $U \subset X$ and a closed subspace $Z \subset X$ such that $Y = U \cap Z$.
(e) There are open subspaces $V \subset U \subset X$ such that $Y = U \smallsetminus V$.
(f) There are closed subspaces $W \subset Z \subset X$ such that $Y = Z \smallsetminus W$.
(g) Y is an open subspace of its closure $\overline{Y}$ in X.
(h) $\overline{Y} \smallsetminus Y$ is a closed subspace of X.

A subspace Y satisfying these equivalent conditions is said to be a *locally closed subspace* of X. In this case, the largest open subspace U of X as in (b), i.e., containing Y and in which Y is closed, is $U = X \smallsetminus (\overline{Y} \smallsetminus Y)$.

Proof. The implications (a) $\Leftarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) and (c) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) are clear. For (a) $\Rightarrow$ (b), for each $y \in Y$ pick a U_y, and let $U := \bigcup_{y \in Y} U_y$. The last claim is clear. ∎

Corollary 2.3.2.

(a) The preimage of a locally closed subspace is locally closed.
(b) If $Z \subset Y \subset X$ are such that Z is locally closed in Y and Y locally closed in X, then Z is locally closed in X.
(c) The intersection of two locally closed subspaces is locally closed.
(d) The complement of a locally closed subspace is the disjoint union of an open subspace and a closed subspace.

Proof. The statements (a) and (b) follow from 2.3.1(d), the statement (c) follows from (a) and (b), and (d) follows from either (e) or (f) of 2.3.1. ∎

Proposition/Definition 2.3.3 (Constructible Subspaces)**.** Let X be a topological space. The following conditions on a subset $Y \subset X$ are equivalent.

(a) Y belongs to the algebra of subsets of X generated by the closed sets, i.e., the smallest collection of subsets containing the closed subsets which is closed under taking finite intersections and complements (and hence also under finite unions).
(b) Y can be written as a finite disjoint union of locally closed subsets of X.
(c) There is an integer $n \in \mathbf{Z}_{\geq 1}$ and a nested sequence $G_1 \supset G_2 \supset \cdots \supset G_n$ of closed subsets of X such that $Y = G_1 \smallsetminus (G_2 \smallsetminus (G_3 \smallsetminus (\cdots \smallsetminus G_n)) \cdots)$.

If X is Noetherian, these properties are also equivalent to

(d) For every closed irreducible $Z \subset X$, the intersection $Y \cap Z$ either contains a nonempty open subset of Z or is nowhere dense in Z.

A subspace satisfying equivalent conditions (a)-(c) is said to be a *constructible* subspace of X.

Proof. Here we prove the equivalence of (a), (b), and (c). The proof of equivalence with (d) in the Noetherian case is deferred to 2.3.5.

(a) $\Leftrightarrow$ (b) It suffices to show that the collection of subsets described by (b) is closed under taking finite intersections and complements. The first of these is clear from 2.3.2(c), and the second follows from the first and 2.3.2(d).

(c) $\Rightarrow$ (b) Clear, since $G_1 \setminus (G_2 \setminus (G_3 \setminus (\cdots \setminus G_n))\cdots) = (G_1 \setminus G_2) \amalg (G_3 \setminus G_4) \amalg \cdots$.

(a) $\Rightarrow$ (c) This is somewhat trickier than it looks, and may be skipped on a first reading. One possible approach is to show directly that the collection of subsets described by (c) is closed under taking finite intersections and complements. The latter is clear, but the former involves some nontrivial combinatorics and Boolean algebra. Here's a different approach.[6]

Given a topological space X, an $n \in \mathbf{Z}_{\geq 1}$, and a collection $\mathscr{Z} = \{Z_1, \ldots, Z_n\}$ of closed subsets of X, consider the Boolean subalgebra $\mathcal{S}(\mathscr{Z})$ of subsets of X generated by the Z_i. This consists of finite unions of subsets of the form $\bigcap_{i=1}^n Z_i^*$ where each Z_i^* is either Z_i or $X \setminus Z_i$; in particular, $\mathcal{S}(\mathcal{Z})$ has size at most 2^{2^n}, and is therefore finite.[7] The subset $\mathcal{S}(\mathscr{Z}) \subset 2^X$ inherits a natural partial order by inclusion, and we call a element $A \in \mathcal{S}(\mathscr{Z})$ an *atom* iff $A \neq \emptyset$ and A is minimal with respect to this partial order. Each atom A is locally closed (since it must itself be of the form $\bigcap_{i=1}^n Z_i^*$ as above by minimality), distinct atoms are disjoint, and every element of $\mathcal{S}(\mathscr{Z})$ is a (disjoint) union of atoms.

Suppose we are given a Y as in (a). Then there is a finite set $\mathscr{Z}$ of closed subsets of X such that $Y \in \mathcal{S}(\mathscr{Z})$; fix such a $\mathscr{Z}$. Write $Y \in \mathcal{S}(\mathscr{Z})$ as a disjoint union of atoms, say $Y = \coprod_{k=1}^m A_k$ for some $m \in \mathbf{Z}_{\geq 1}$, with the A_k pairwise distinct. Further express each A_k as $A_k = S_k \setminus T_k$ for some closed sets $T_k \subset S_k \subset X$. We claim that it suffices to take $n = 2m$, with

$$G_{2k-1} = \bigcup_{j=k}^{r} S_j \text{ and } G_{2k} = T_k \cup \bigcup_{j=k+1}^{r} S_j$$

for $1 \leq k \leq m$. Since the inclusions $G_1 \supset G_2 \supset \cdots \supset G_n$ are clear, it suffices to show that for each $k = 1, \ldots, m$, we have $G_{2k-1} \setminus G_{2k} = A_k$. The inclusion $G_{2k-1} \setminus G_{2k} \subset A_k$ is clear. For the reverse inclusion, it suffices to show that if $1 \leq k < k' \leq m$, then $A_k \cap S_{k'} = \emptyset$, but this follows from the fact that distinct atoms are disjoint.[8]

■

Lemma 2.3.4.

(a) The preimage of a constructible set under a continuous map is constructible. If $Z \subset Y \subset X$ are such that Z is constructible in Y and Y is constructible in X, then Z is constructible in X.

(b) Let X be an irreducible topological space and $C \subset X$ a constructible subspace. Then the following are equivalent:
   (i) C contains a nonempty open subspace of X.
   (ii) C is dense.

   If, further, X has a generic point η, then (i) and (ii) are also equivalent to

   (iii) C contains η.

(c) In a Noetherian sober space, a subspace is closed (resp. open) iff it is constructible and stable under specialization (resp. generization).

Proof.

[6] Collaboration acknowledgment: this solution was inspired by some conversations with ChatGPT on 09/20/25.

[7] Another way to see this is to note that a subalgebra of $\mathbf{F}_2^X$ generated by n elements, say $\chi_1, \ldots, \chi_n$ has dimension at most 2^n over $\mathbf{F}_2$, since it is spanned as a vector space over $\mathbf{F}_2$ by the elements $\chi_I = \prod_{i \in I} \chi_i$ as $I \subset 2^{\{1,\ldots,n\}}$ ranges over all subsets of $\{1, \ldots, n\}$.

[8] If the reader finds an easier proof of this implication, please let me know!

(a) Follows from 2.3.2 and 2.3.3 (check!).
(b) The implication (i) $\Rightarrow$ (ii) follows from irreducibility. For (ii) $\Rightarrow$(i), use 2.3.3 to write C as a union of locally closed subspaces and use the irreducibility of X to reduce to the case where C is locally closed; then apply 2.3.1(g). When X has a generic point η, the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are clear.
(c) By taking complements for the assertion in parentheses, it suffices to show that if X is Noetherian sober, $C \subset X$ constructible, and $y \in \overline{C}$, then there is a $x \in C$ such that $x \rightsquigarrow y$. Use 2.2.7(b) and 2.2.9 to write an irreducible decomposition of C as $C = Z_1 \cup \cdots \cup Z_r$ for some $r \in \mathbf{Z}_{\geq 1}$ and closed irreducible $Z_1, \ldots, Z_r \subset C$. For each i, the closure $\overline{Z}_i$ of Z_i in X is irreducible (by 2.1.5(c)); since X is sober, $\overline{Z}_i$ has a generic point, say η_i. The intersection $C \cap \overline{Z}_i$ is a constructible set of $\overline{Z}_i$ by (a) and contains Z_i, so it is dense; therefore, by (b), we have $\eta_i \in C$. Now if $y \in \overline{C}$, then there is an i such that $y \in \overline{Z}_i$, and then taking $x = \eta_i$ suffices.

■

Corollary 2.3.5. Let X be a Noetherian topological space. A subset $Y \subset X$ is constructible iff for every closed irreducible $Z \subset X$, the intersection $Y \cap Z$ either contains a nonempty open subspace of Z or is nowhere dense in Z.

The following proof has been taken from [3, Proposition 10.14].

Proof. If Y is constructible, then for each such Z, so is $Y \cap Z \subset Z$ by 2.3.4(a). In this case, either $Y \cap Z$ is dense in Z, in which case it contains a nonempty open subspace of Z by 2.3.4(b), or it is not dense, in which case it is nowhere dense in Z by 2.1.3(d).

Conversely, by Noetherian Induction (2.2.10) applied to property of meeting Y in constructible subset of X, we are reduced to the case in which for every proper closed subset $W \subset X$, the intersection $W \cap Y$ is constructible in X (or equivalently in W, by 2.3.4(a)). If X is not irreducible, then for each irreducible component $X' \subset X$, the intersection $X' \cap W$ is constructible in X by hypothesis; this suffices, since constructible sets are closed under finite unions and X has only finitely many irreducible components (2.2.7(c)). Suppose now that X is irreducible. If Y contains a nonempty open subspace U of X, then the hypothesis applied to $W = X \smallsetminus U$ tells us that $W \cap Y = Y \smallsetminus U$ is constructible in X; then so is $Y = (Y \smallsetminus U) \cup U$. If Y is nowhere dense in X, then in particular the closure $\overline{Y} \subset X$ is a proper closed subset; then the hypothesis applied to $W = \overline{Y}$ tells us that $Y = W \cap Y$ is constructible in X. ■

The final notion we will need is that of *very dense subspaces.* To motivate this, note that if X is a topological space and $Y \subset X$ a subspace, then the map $W \mapsto W \cap Y$ taking open (resp. closed) subspaces $W \subset X$ to open (resp. closed) subspaces of Y is always a surjection.

Proposition/Definition 2.3.6 (Very Dense Subspaces)**.** Let X be a topological space. The following conditions on a subspace $Y \subset X$ are equivalent:

(a) The map $U \mapsto U \cap Y$ is a bijection from the set of open (resp. closed, resp. locally closed, resp. constructible) subspaces of X to those of Y. Equivalently, if $U, V \subset X$ are open (resp. closed, resp. locally closed, resp. constructible) subspaces such that $U \cap Y = V \cap Y$, then $U = V$.
(b) For every closed subspace $Z \subset X$, we have $Z = \overline{Z \cap Y}$.
(c) Every nonempty locally closed (resp. constructible) subset of X meets Y.

A subspace $Y \subset X$ satisfying these equivalent conditions is said to be *very dense* in X.

Proof.

(a) $\Rightarrow$ (b) The property (a) for constructible subsets implies it for locally closed subsets, which in turn implies it for open and closed subsets, and the properties for open and closed subsets are equivalent by taking complements. The property (a) for closed subsets then implies (b); indeed, if $Y \subset X$ is any subset and $Z \subset X$ closed, then $Z \cap Y = \overline{Z \cap Y} \cap Y$.

(b) $\Rightarrow$ (c) The two properties are equivalent by 2.3.3(b), and (b) implies (c) for locally closed subsets by 2.3.1(f).

(c) $\Rightarrow$ (a) The property (c) for constructible sets implies the property (a) for constructible sets: if $C, D \subset X$ are constructible subsets such that $C \cap Y = D \cap Y$, then $C \smallsetminus D$ and $D \smallsetminus C$ are constructible subsets of X not meeting Y, and hence are both empty.

■

Example 2.3.7. The subset $\mathbf{Q} \subset \mathbf{R}$ is dense but not very dense.

Corollary 2.3.8. Let X be a topological space and $Y \subset X$ a subspace.

(a) If $C \subset X$ is a constructible subspace and Y very dense in X, then $Y \cap C$ is very dense in C.
(b) If $X = \bigcup_\alpha X_\alpha$ for some $X_\alpha \subset X$ such that for each α, the intersection $Y \cap X_\alpha$ is very dense in X_α, then Y is very dense in X.

Proof.

(a) If $D \subset C$ is a nonempty constructible subspace, then D is constructible in X by 2.3.4(a) and hence meets Y (and so $Y \cap C$) by 2.3.6(c).
(b) Given a nonempty constructible $C \subset X$, pick a point $x \in C$ and an α such that $x \in X_\alpha$. Then $C \cap X_\alpha$ is a nonempty constructible subspace of X_α by 2.3.4(a), and so meets $Y \cap X_\alpha$ by hypothesis and 2.3.6(c). Therefore, C meeets Y, and we are done.

■

The exercise [2, Exercise 3.3.27] cited in the previous section gives an example of a pathological scheme which does not have *any* closed points. This, however, cannot happen for algebraic varieties. Indeed, the key example of very dense subspaces, and the reason we care, is

Theorem 2.3.9. Let k be a field and X be a scheme locally of finite type over k.

(a) The subspace of closed points of X is very dense in X. In particular, if X is nonempty, then X has a closed point.
(b) If Y is another scheme locally of finite type over k and $f, g : X \to Y$ two k-morphisms such that for some algebraic closure $\overline{k} \supset k$ we have $f(\overline{k}) = g(\overline{k}) : X(\overline{k}) \to Y(\overline{k})$ and X is geometrically reduced, then $f = g$.

Proof.

(a) Let $X^{\mathrm{cl}} \subset X$ denote the subset of closed points. If $W \subset X$ is any (locally closed) subscheme, then $W^{\mathrm{cl}} = W \cap X^{\mathrm{cl}}$, since closed points admit an intrinsic characterization (namely that their residue field is a finite extension of k). Therefore, by Corollary 2.3.8(b), it suffices to show the result when X is affine, so suppose X is an affine scheme of finite type over k and $Y \subset X$ a nonempty locally closed subset. We want to show $Y \cap X^{\mathrm{cl}} \neq \emptyset$. Replacing X by $(\overline{Y})^{\mathrm{red}}$, we may assume Y is open in X (using 2.3.1(g) and that every closed subscheme of affine scheme is affine), and by shrinking Y we may assume it is a nonempty principal open subset, say $Y = D_f$ for some $f \in \mathcal{O}(X)$. Since Y is nonempty, f lies in some prime ideal, and then f lies in some maximal ideal; the corresponding point of X then belongs to $Y \cap X^{\mathrm{cl}}$.

(b) Base change to $\overline{k}$ to assume k is algebraically closed, using that $f_{\overline{k}} = g_{\overline{k}}$ implies $f = g$. (Indeed, this is fppf descent. Directly, the map $X_{\overline{k}} \to X$ is faithfully flat, and hence an injection on local rings by the algebraic result that a flat local homomorphism of local rings is injective. All of this shows that $X_{\overline{k}} \to X$ is an epimorphism of schemes.) In this case, $X(k)$ can be identified with the set of closed points, which is very dense by (a). The (underlying subspace) of the equalizer $\mathrm{Eq}(f, g) \subset X$ is locally closed, since the diagonal morphism for schemes is always a locally closed immersion, and so its complement is constructible; if it were to be nonempty, then it would meet $X(k)$ by Proposition/Definition 2.3.6(e), which it does not. Therefore, $\mathrm{Eq}(f, g) = X$ as topological spaces, and then as schemes because X is reduced, i.e., $f = g$.

■

Informally, for geometrically reduced schemes locally of finite type over a field k, equality of morphisms can be checked at the level of $\overline{k}$-points for an(y choice of an) algebraic closure $\overline{k}$ of k. The hypothesis of geometric reduceness cannot be removed; for a counterexample, consider the ring $\mathbf{C}[\varepsilon] := \mathbf{C}[x]/(x^2)$, $X = Y = \operatorname{Spec} \mathbf{C}[\varepsilon]$, and $f, g : X \to Y$ the two morphisms corresponding to $\varepsilon \mapsto 0, \varepsilon$. It also does not suffice to work with k points when k is not algebraically closed, since a scheme locally of finite type over k need not have any k-points at all (e.g. when $k = \mathbf{R}$ and $X = \operatorname{Spec} \mathbf{C}$).

Remark 2.3.10. The result of the previous proposition holds more generally in other contexts, e.g., for schemes locally of finite type over $\mathbf{Z}$. A scheme is called a *Jacobson scheme* if its subset of closed points is very dense; then 2.3.9 is saying that a scheme locally of finite type over a field is Jacobson. The affine version of a Jacobson scheme is a Jacobson ring, which admits many different characterizations (see [3, Exercises 10.15-16] and [4, §6.2.2]). It is easy to see that $\mathbf{Z}$ is a Jacobson ring, and hence that $\operatorname{Spec} \mathbf{Z}$ is a Jacobson scheme; also, a field k is trivially seen to be a Jacobson ring. The final result needed is that if S is a Jacobson scheme and $f : X \to S$ a morphism locally of finite type, then X is a Jacobson scheme ([3, Exercise 10.16]).

2.4 Dimension

Next, we discuss *dimension*, a purely topological notion that is very important in algebraic geometry.

Definition 2.4.1 (Dimension)**.** Let X be a topological space.

(a) A *(finite) chain of irreducible closed subsets of* X consists of an integer $n \in \mathbf{Z}_{\geq 0}$ and irreducible closed subsets
$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X.$$
Here n is called the *length* of the chain.

(b) The *dimension* of X, denoted $\dim X$, is the supremum of all $n \in \mathbf{Z}_{\geq 0}$ such that there is a chain of irreducible closed subsets of X of length n. We say that X is *equidimensional* or *pure of dimension* r if every irreducible component of X has dimension r.

(c) For a point $x \in X$, we define the *local dimension of* X *at* x, denoted $\dim_x X$, to be the infimum of $\dim U$, where U ranges over all open neighborhoods of x in X.[9]

(d) For an irreducible subset $Y \subset X$, we define the *codimension of* Y *in* X, denoted $\operatorname{codim}_X Y$, to be the supremum of all $n \in \mathbf{Z}_{\geq 0}$ such that there is a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X$ of irreducible closed subsets of X of length n containing Y, i.e., such that $Y \subset X_0$.

(e) For any arbitrary closed subset $Y \subset X$, we define the *codimension of* Y *in* X to be $\operatorname{codim}_X Y := \inf_{Z \in \operatorname{Irred}(Y)} \operatorname{codim}_X Z$. We say that Y is *pure of codimension* r *in* X for some $r \in \mathbf{Z}_{\geq 0}$ if all irreducible subsets of Y have codimension r in X.

The dimension of a scheme is the defined to be the dimension of its underlying topological space; the codimension of a closed subscheme is the codimension of its underlying closed subspace.

Remark 2.4.2.

(a) For any space X, we have $\dim X \in \mathbf{Z}_{\geq 0} \cup \{\pm\infty\}$, with $\dim X = -\infty$ iff $X = \emptyset$.

(b) The local dimension is a local property in the sense that if X is a space, $x \in X$, and $U \subset X$ an open neighborhood of x in X, then $\dim_x X = \dim_x U$. Further, if X has only finitely many irreducible components and $x \in X$ is contained in all of them, then $\dim X = \dim_x X$.

(c) When X is a sober space, the above definition(s) can be made using point specializations instead.

(d)

(e) If $X = \operatorname{Spec} R$ is an affine scheme, then it is clear by definition that the dimension of X is the Krull dimension $\dim R$ of R.

(f) If $Y \subset X$ is an irreducible subspace of a topological space, then by definition we have that $\operatorname{codim}_X Y = \operatorname{codim}_X \overline{Y}$, and that $\operatorname{codim}_X Y + \dim Y \leq \dim X$. Unfortunately, equality does not always hold. A simple counterexample is given as follows: let R be a DVR with uniformizer π, and let $X = \mathbf{A}^1_R = \operatorname{Spec} R[x]$, so that $\dim X = 2$. If $Y := \mathbf{V}(\pi x - 1)$, then Y is a closed subscheme with $\operatorname{codim}_X Y = 1$ (by Krull's Hauptidealsatz) and $\dim Y = 0$. However, we will show below (2.4.6(a)) that equality holds in reasonable circumstances (e.g., for varieties).

Lemma 2.4.3.

(a) Let X be a space and $Y \subset X$ a subspace. Then $\dim Y \leq \dim X$. If X is irreducible, finite dimensional, and $Y \subsetneq X$ is a proper closed subset, then the inequality is strict.

[9] This definition disagrees with that given in [1, Prop. III.9.5], which is poorly behaved, and agrees with the definition in EGA or [5, Definition 3.3.6]. For the relationship between these definitions (at least for schemes locally of finite type over a field), see 2.4.6(b).

(b) For any space X, we have

$$\dim X = \sup_{Y \in \mathrm{Irred}(X)} \dim Y = \sup_{x \in X} \dim_x X.$$

(c) If $\{U_\alpha\}$ is an open cover of X, then $\dim X = \sup_\alpha \dim U_\alpha$.
(d) If X is a closure-complete space (e.g., the underlying space of a quasicompact scheme), then $\dim X = \sup_{x \in X^{\mathrm{cl}}} \dim_x X$, where X^{cl} is the subset of closed points.

Proof.

(a) Follows from 2.1.5(c) and the observation that if $Z \subset Y$ is any closed subset with closure $\overline{Z}$ in X, then $Z = \overline{Z} \cap Y$. The second statement follows from appending X to a chain of irreducible closed subsets of Y.
(b) By (a), these suprema are at most $\dim X$. The first result is clear. For the second, if $X_0 \subsetneq \cdots \subsetneq X_n \subset X$ is a chain of irreducible closed subsets, then there is an $x \in X_0$; the result follows from applying 2.1.5(d) to any open neighborhood U of x in X.
(c) Follows from (b).
(d) In any chain as above, the set X_0 contains a closed point.

■

Let's first see an algebraic interpretation of the (co)dimension for (possibly non-affine) schemes.

Lemma 2.4.4. Let X be a scheme.

(a) If $Y \subset X$ is an irreducible subset, then $\mathrm{codim}_X Y = \dim \mathcal{O}_{X,Y}$, where $\mathcal{O}_{X,Y}$ is the local ring of X at the generic point of $\overline{Y}$.
(b) In particular, we have $\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x}$.

Proof.

(a) By replacing Y by its generic point, we are immediately reduced to showing that if X is a scheme and $x \in X$, then $\dim \mathcal{O}_{X,x} = \mathrm{codim}_X\{x\}$. By 2.1.5(d), we are then reduced to the case of affine X, where the result is clear by the inverse correspondence between prime ideals and irreducible subsets.
(b) Clear from (a).

■

Now let us focus on the case of schemes locally of finite type over a field k; in this section, we follow [3, §5.6]. By 2.4.3(b), we may restrict to considering irreducible schemes.[10]

Theorem 2.4.5. Let X be an irreducible scheme locally of finite type over a field k, and let $\eta \in X$ be the generic point of X. Let $k(X) := \kappa(\eta)$. Then

(a) We have $\dim X = \mathrm{trdeg}_k\, k(X)$.
(b) If $U \subset X$ is any nonempty open subscheme, then $\dim U = \dim X$.
(c) More generally, let $f : Y \to X$ be a dominant finite-type morphism. Then $\dim Y \geq \dim X$.
(d) If $x \in X$ is any closed point, then $\dim X = \dim \mathcal{O}_{X,x}$.
(e) If $f : Y \to X$ is a quasi-finite finite-type morphism, then $\dim Y \leq \dim X$.

In particular, if $f : Y \to X$ is a finite-type morphism which is both dominant and quasi-finite, then $\dim Y = \dim X$.

[10] Since dimension is a topological property, we may pass to the underlying reduced subscheme to restrict ourselves to considering integral schemes. The added generality in considering irreducible and not just integral schemes is somewhat superficial.

Proof.

(a) Replacing X by X_{red}, we may assume that X is integral. Using 2.4.3(c), we are reduced to the case of affine X. This reduces us to a statement in commutative algebra: if R is a domain finitely generated over a field k, then $\dim R = \operatorname{trdeg}_k k(R)$, where $k(R) = \operatorname{Frac} R$ is the fraction field of R. This is an immediate consequence of the Noether Normalization lemma (see, e.g., [4, Theorem 6.2.7(a)]).
(b) Clear from (a).
(c) By 2.1.13, there is a $\theta \in Y$ such that $f(\theta) = \eta$. Then f induces a k-embedding $\kappa(\eta) \hookrightarrow \kappa(\theta)$, and so
$$\dim X = \operatorname{trdeg}_k \kappa(\eta) \leq \operatorname{trdeg}_k \kappa(\theta) = \dim \overline{\{\theta\}} \leq \dim Y,$$
where we are repeatedly using (a) and 2.4.3(a).
(d) Using (b), we may assume that X is integral and affine. This reduces us to: if R is a domain finitely generated over a field and $\mathfrak{m} \subset R$ a maximal ideal, then $\dim R = \operatorname{ht} \mathfrak{m}$; this is a standard consequence of Noether Normalization as well ([4, Theorem 6.2.7(d)]).
(e) By 2.4.3(b), it suffices to show the result assuming Y to be irreducible. By (b), we may further reduce to the case when X and Y are affine, and again we may replace X and Y be their reductions to assume that both are integral. This reduces us to: if R and S are domains finitely generated over a field k and $f : R \to S$ a k-algebra homomorphism such that for all primes $\mathfrak{p} \subset R$, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_R S$ has only finitely many primes, then $\dim S \leq \dim R$. To show this, take $\mathfrak{p} := \ker \varphi$. Then $\operatorname{Spec}(\kappa(\mathfrak{p}) \otimes_R S)$ is a nonempty finite-type $\kappa(\mathfrak{p})$-scheme whose underlying topological space is *finite*. The subset of non-closed points is then open, and hence by 2.3.9, must then be empty, i.e., every point is closed. In particular, applying this to the prime corresponding to $(0) \subset S$ shows us that $\operatorname{Frac} S$ is a *finite* extension of $\kappa(\mathfrak{p})$, and hence that $\dim S = \operatorname{trdeg}_k S = \operatorname{trdeg}_k \kappa(\mathfrak{p}) = \operatorname{coht} \mathfrak{p} \leq \dim R$ as needed.

■

Parts (b) and (d) of 2.4.5 combine to tell us that, in the above setting, if $x \in X$ is any closed point, then $\dim_x X = \dim \mathcal{O}_{X,x}$. When x is not necessarily closed, the correct generalization is

Theorem 2.4.6. Let X be a scheme that is locally of finite type over a field k.

(a) If X is of finite type over k and $Y \subset X$ a closed subset that is contained in every irreducible component of X (e.g., if X is irreducible), then $\dim X = \operatorname{codim}_X Y + \dim Y$.
(b) In general, for any $x \in X$, we have $\dim_x X = \dim \mathcal{O}_{X,x} + \operatorname{trdeg}_k \kappa(x)$.

Note that some sort of hypothesis as in (a) is clearly necessary, as easily constructed counterexamples otherwise show.

Proof.

(a) Note that X and Y are Noetherian and so have only finitely many irreducible components (2.2.7(c)). A little thought shows that it suffices to deal with the case of irreducible X and Y.[11] Using 2.4.5(b) to replace X by an affine open, we may assume that $X = \operatorname{Spec} R$ is affine; then if (the generic point of) Y corresponds to the prime ideal $\mathfrak{p} \subset R$, we have $\operatorname{codim}_X Y = \dim R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p}$ by 2.4.4, and $\dim Y = \dim R/\mathfrak{p} = \operatorname{coht} \mathfrak{p}$. This reduces us to the following standard result in commutative algebra: if R is a domain that is finitely

[11] Having done the case of irreducible X and Y, do the case of X irreducible but Y possibly reducible by noting that $\operatorname{codim}_X Y$ and $\dim Y$ only both depend on the component of Y of maximal dimension. Finally, do the general case, noting that both $\dim X$ and $\operatorname{codim}_X Y$ only depend on the component of X of maximal dimension–when all the components contain Y.

generated over a field k and $\mathfrak{p} \subset R$ a prime ideal, then $\dim R = \operatorname{ht} \mathfrak{p} + \operatorname{coht} \mathfrak{p}$ ([4, Theorem 6.2.7(d)]).

(b) We will reduce to (a). First, replacing X by an affine open containing x, we may assume that X is of finite type over k. In particular, X is Noetherian and so has only finitely many irreducible components. Removing all irreducible components of X which do not contain x, we may assume that x is contained in every irreducible component of X. In this case, we have by 2.4.2(b) that $\dim_x X = \dim X$. Applying (a) to $Y := \overline{\{x\}}$, and using 2.4.4(a) and 2.4.5(a) gives the result.

■

Finally, we check that the dimension of a scheme locally of finite type over a field behaves well under changing the base field.

Theorem 2.4.7. Let X be a scheme over a field k, equidimensional of dimension $n \in \mathbf{Z}_{\geq 0}$, and let $k \subset K$ be a field extension. If either

(a) X is affine over k and K/k algebraic, or
(b) X is locally of finite type over k (and K/k arbitrary), then

the base-change X_K of X to K is again equidimensional of dimension n.

In the course of the proof below, we will also show

Corollary 2.4.8. Let X be an affine integral scheme over a field k and K/k be a purely transcendental extension. Then X_K is also (affine and) integral.

This result is (I think) more subtle than is often given credit for, and the (only detailed) proof (I know of it) is somewhat tricky. A simple counterexample if we drop some hypotheses is given by taking $X = \operatorname{Spec} k(t)$ and $K = k(s)$, both purely transcendental; then X_K is a suitable localization of $\mathbf{A}^1_{k(s)}$ and can be easily shown to be one-dimensional. (I suspect similar counterexamples exist when we drop *any* of the hypotheses of the theorem.) The following proof strategy has been taken from [6, Chapter 12].

Proof. In both cases, we can reduce to the case of integral X. Indeed, we reduce to the case of irreducible X by noting that an irreducible component of X_K must be an irreducible component of Y_K for some irreducible component Y of X. Also, the reduction map $X^{\text{red}} \to X$ is a surjective closed immersion and hence a universal homeomorphism; in particular, the natural map $(X_K)^{\text{red}} \to (X^{\text{red}})_K$ is a homeomorphism, and so both of these spaces have the same dimension (which is a topological property).

(a) Since K/k is algebraic, every component of X_K maps to X via an integral morphism. Since X is affine and integral, it suffices to show that every component of X_K dominates X. This is a result in commutative algebra: if R is a domain over a field k and K/k an algebraic extension, then for any minimal prime $\mathfrak{p}$ of $R_K = K \otimes_k R$, the natural map $R \to R_K \to R_K/\mathfrak{p}$ is injective. This itself follows from two facts: (i) that R_K is a free R-module and R is a domain, and (ii) if S is any ring and $\mathfrak{p} \subset S$ a minimal prime, then any element of $\mathfrak{p}$ is a zero divisor.[12]

(b) By 2.4.5(b), it suffices to deal with the case of affine X. Since any K/k can be written as a composite of a purely transcendental extension followed by an algebraic extension, by part (a), it suffices to treat the case when K/k is purely transcendental. At this point, let us first note the result of 2.4.8: indeed, if $X = \operatorname{Spec} R$ for a k-domain R and $K = k(e_i)_{i\in I}$,

[12]Indeed, since $\mathfrak{p}$ is minimal, the ring $S_\mathfrak{p}$ has a unique prime ideal $\mathfrak{p}S_\mathfrak{p}$. Therefore, $\operatorname{Nil}(S_\mathfrak{p}) = \mathfrak{p}S_\mathfrak{p}$, so that any element of $\mathfrak{p}$ is nilpotent in $S_\mathfrak{p}$ and hence a zero-divisor in S.

then $X_K = \operatorname{Spec} R_K$ and

$$R_K = K \otimes_k R \cong k(e_i) \otimes_{k[e_i]} (k[e_i] \otimes_k R) \cong (k[e_i] \smallsetminus \{0\})^{-1} R[e_i]$$

is a localization of the domain $R[e_i]$ and hence also a domain. Finally, by 2.4.5, it remains only to justify that $\operatorname{trdeg}_k k(R) = \operatorname{trdeg}_K K(R_K)$, but indeed $K(R_K) = k(R)(e_i)$, so the result is clear by, say, [4, Excercise 5.10].

■

2.5 Separatedness and Properness from a Topological Perspective

In this final section, we provide a few topological results that might help motivate the algebro-geometric definitions of separated and proper morphisms. The first is the classic

Proposition/Definition 2.5.1. (Hausdorff Spaces) Let X be a topological space. The following conditions on X are equivalent:

(a) For any two distinct points $x, y \in X$, there are disjoint open subsets $U, V \subset X$ such that $x \in U$ and $y \in V$.
(b) Limits of nets in X, when they exist, are unique.
(c) Every point $x \in X$ is the intersection of all its closed neighborhoods (where by a closed neighborhood of x we meant a closed subset of X containing x in its interior).
(d) The diagonal $\Delta(X) \subset X \times X$ is a closed subset when $X \times X$ is given the product topology.

A topological space satisfying these equivalent conditions is called *Hausdorff.*

Proof. Exercise. ■

Schemes are usually not Hausdorff, and so the above definition needs to be modified to obtain the usual definition for separated morphisms. One can also define, just as in the scheme case, a continuous map $X \to Y$ to be separated if the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a closed topological embedding, and this notion can then be characterized in terms of the fibers being relatively Hausdorff ([7, Proposition 1.25]), but to me that seems a little artificial; besides, it's less useful to algebraic geometers because it does not translate over to the schemes setting, precisely because the Zariski topology on $X \times_Y X$ is not the usual (subspace-of-) product topology.

There is another related topological notion that is used in scheme theory, namely the notion of quasiseparated topological spaces. I believe this is best handled in a scheme-theoretic manner ([6, §11.1.4]), so I will content myself with giving the definition and one lemma which is sometimes useful (as in 3.4.2 below and in the solution to [6, Exercise 5.1.H]).

Definition 2.5.2. A topological space X is said to be *quasiseparated* if the intersection of any two quasicompact[13] open subsets of X is again quasicompact.

Lemma 2.5.3 ([8])**.** Let X be a topological space admitting a basis of quasicompact open subsets, and $\mathscr{U}$ a cover of X by quasiseparated open subsets, all of whose pairwise intersections are quasicompact. If $W \subset X$ is any quasicompact open subset, then for all $U \in \mathscr{U}$, the intersection $W \cap U$ is quasicompact.

Proof. Cover W by finitely many basic quasicompact open subsets V_i each contained in some element say U_i of the cover $\mathscr{U}$. It suffices to show that for each $U \in \mathscr{U}$, the intersection $U \cap V_i$ is quasicompact, but $U \cap V_i = V_i \cap (U \cap U_i)$ is an intersection of two quasicompact open subsets inside the quasiseparated U_i. ■

Au contraire, it is easier and more direct to relate the topological notion of compactness to the algebro-geometric notion of properness. The following exposition has been taken from [7, §1.5]. Recall the definition of quasicompact spaces.

[13] See 2.5.4 below if needed.

Proposition/Definition 2.5.4 (Quasicompact Spaces)**.** The following conditions on a topological space X are equivalent:

(a) Every open cover has a finite subcover.
(b) Every collection of closed subspaces with the finite intersection property (i.e., such that every finite intersection of which is nonempty) has nonempty intersection.
(c) Every net in X has a convergent subnet.
(d) The final morphism $X \to \{*\}$ is universally closed, i.e., for every topological space Z, the projection $\pi : Z \times X \to Z$ is a closed map.

A topological space satisfying these equivalent properties is said to be *quasicompact.* A topological space is said to be *compact* if it is both quasicompact and Hausdorff. Further:

(e) A closed subspace of a quasicompact space is quasicompact.
(f) The continuous image of a quasicompact space is quasicompact.
(g) A quasicompact subspace of a Hausdorff space is closed.

Proof. The implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) are standard and left as an exercise, as are the statements (e), (f), (g). Since a proof of the equivalence with (d) is trickier (and harder to locate), we reproduce it here.

(a) $\Rightarrow$ (d) Let Z be given, $C \subset Z \times X$ closed, and $z \in Z \smallsetminus \pi(C)$. The following lemma (2.5.5)[14] applied to $\pi^{-1}(z) = \{z\} \times X \subset (Z \times X) \smallsetminus C \subset Z \times X$ tells us that there is an open neighborhood V of z in Z such that $V \times X \subset (Z \times X) \smallsetminus C$. This shows that $V \subset Z \smallsetminus \pi(C)$. Since z was arbitrary, this proves that $Z \smallsetminus \pi(C)$ is open as needed.

(d) $\Rightarrow$ (c) An argument can be found in [7, Prop. 1.29], but I prefer the following argument adapted from [9, Lemma 4.3]. Suppose X is a topological space and $\mathscr{U}$ an open cover of X with no finite subcover. Replace $\mathscr{U}$ by the collection of open subspaces consisting of all finite unions of elements of $\mathscr{U}$ to assume without loss of generality that $\mathscr{U}$ is directed. We put a topology on $Z = \mathscr{T}(X) \subset 2^X$, the collection of open subsets of X, such that the projection map $\pi : Z \times X \to Z$ is not closed. Define a subset $V \subset Z$ to be open iff both (i) and (ii) below hold.

(i) The collection V is upward closed, i.e., for all $U \in V$, if $U' \subset X$ is open and $U \subset U'$, then $U' \in V$.

(ii) If $X \in V$, then there is a $U \in \mathscr{U}$ such that $U \in V$.

It follows from the fact that $\mathscr{U}$ is directed that the above defines a topology on Z, and further that if $U \subset X$ is an open subspace of some element of $\mathscr{U}$, then the upward closure $U^{\uparrow} := \{U' \in Z : U' \supset U\} \subset Z$ is open. It remains to show that the subset $C \subset Z \times X$ defined by $C := \{(U, x) : U \not\ni x\}$ is a closed subset such that $\pi(C) = Z \smallsetminus \{X\}$ is not closed. The latter part follows from the fact that $X \notin \mathscr{U}$. For the former part, we need to show that if $(U, x) \in Z \times X$ is such that $U \ni x$, then there is an open neighborhood of (U, x) in $Z \times X$ contained in $Z \times X \smallsetminus C$. But now since $\mathscr{U}$ covers X, there is a $U' \in \mathscr{U}$ such that $U' \ni x$, and then $(U, x) \in (U \cap U')^{\uparrow} \times (U \cap U') \subset Z \times X \smallsetminus C$ as needed.

■

Lemma 2.5.5. Let X, Y be topological spaces and $K \subset X$ and $L \subset Y$ quasicompact subspaces. Then for any open neighborhood U of $K \times L$ in $X \times Y$, there are open neighborhoods V of K in X and W of L in Y such that $V \times W \subset U$.

Proof. Exercise. ■

[14] In order to avoid circular arguments, the reader is encouraged to prove the Lemma 2.5.5 using definition (a) of compactness given above.

Proposition/Definition 2.5.6 (Proper Maps)**.** The following conditions on a continuous map $f : X \to S$ between topological spaces are equivalent:

(a) The map f is universally closed in the topological category, i.e., for every continuous map $T \to S$, the projection map $f_T : X_T := T \times_S X \to T$ after base-change to T is closed.
(b) For every topological space T, the map $f_{T\times S} : T \times X \to T \times S$ is closed.
(c) The map f is closed and for every quasicompact subspace $K \subset S$, the preimage $f^{-1}(K) \subset X$ is quasicompact.
(d) The map f is closed and for all $s \in S$, the fiber $X_s = f^{-1}(s) \subset X$ is quasicompact.

Further, when S is locally (quasi)compact Hausdorff, these are also equivalent to:

(e) For every (quasi)compact subset $K \subset S$, the preimage $f^{-1}(K) \subset X$ is quasicompact.

A continuous map $f : X \to S$ is said to be *proper* iff it is satisfies equivalent conditions (a)-(d).

Note that as a consequence, we have that a space X is quasicompact iff the final morphism $X \to \{*\}$ is proper.

Proof.

(a) ⇒ (b) Base-change along the projection $T \times S \to T$.
(b) ⇒ (c) Taking T to be a point shows f is closed. Let $K \subset S$ be a quasicompact subspace and $T = Z$ an arbitrary space. Since $f_Z : Z \times X \to Z \times S$ is closed, so is the restriction $f_Z|_{Z\times f^{-1}(K)} : Z \times f^{-1}(K) \to Z \times K$.[15] By 2.5.4(d), the projection $Z \times K \to Z$ is also closed. Therefore, the composition $Z \times f^{-1}(K) \to Z \to K \to Z$, which is just the projection onto Z, is also closed. Since this is true for all Z, another application of 2.5.4(d) tells us that $f^{-1}(K)$ is quasicompact.
(c) ⇒ (d) The one point subspace $\{s\} \subset S$ is quasicompact.
(d) ⇒ (a) Given $g : T \to S$, the image $f_T(X_T) = g^{-1}(f(X))$ is closed because f is. Given a closed $Z \subset X_T$, we have to show that $T \smallsetminus f_T(Z)$ is open, i.e., for all $t \in T \smallsetminus f_T(Z)$, there is an open neighborhood U of t in T contained in $T \smallsetminus f_T(Z)$. When $t \notin f_T(X_T)$, we can take $U := T \smallsetminus f_T(X_T)$. When $t \in f_T(X_T)$, note that $f_T^{-1}(t) = \{t\} \times X_{g(t)} \subset X_T \smallsetminus Z$. Since $X_T \subset T \times X$ is given the subspace topology, there is a closed subset $\overline{Z}$ of $T \times X$ such that $Z = \overline{Z} \cap X_T$; then applying 2.5.5 to $\{t\} \times X_{g(t)} \subset T \times X \smallsetminus \overline{Z}$ gives us a neighborhood V of t in T and a neighborhood W of $X_{g(t)}$ in X such that $V \times W \subset T \times X \smallsetminus \overline{Z}$. It then suffices to take $U := V \cap g^{-1}(S \smallsetminus f(X \smallsetminus W))$.

Suppose now that S is locally (quasi)compact Hausdorff. It only remains to prove

(e) ⇒ (c) We have to show that f is closed. Let $C \subset X$ be closed; we have to show that for all $s \in S \smallsetminus f(C)$, there is an open neighborhood U of s in S contained in $S \smallsetminus f(C)$. Pick an open neighborhood V of s in S whose closure $\overline{V} \subset S$ is (quasi)compact. Then $f^{-1}(\overline{V}) \subset X$ is quasicompact, and hence so is $C \cap f^{-1}(\overline{V})$ by 2.5.4(e). By 2.5.4(f), the subspace $f(C \cap f^{-1}(\overline{V})) = f(C) \cap \overline{V} \subset S$ is also quasicompact, and hence closed by 2.5.4(g). It suffices to take $U := V \smallsetminus (f(C) \cap \overline{V}) = V \smallsetminus f(C)$.

■

For an example where (e) is strictly weaker than (a)-(d), see [7, Problem 1.21]. We end this section by stating without proof the following wonderful result relating these modern and classical notions. For this recall that when X is scheme of finite type over $\mathbf{C}$, then we can associate to it a complex analytic space X^{an} whose underlying topological space is the set $X(\mathbf{C})$

[15] This uses the following elementary fact in point-set topology: if $p : A \to B$ is a continuous closed map and $C \subset B$ any subset, then the restriction $p|_{p^{-1}(C)} : p^{-1}(C) \to C$ is also closed. Indeed, any closed subspace of $p^{-1}(C)$ is of the form $p^{-1}(C) \cap D$ for some closed $D \subset A$; then it is easy to check that $p(p^{-1}(C) \cap D) = C \cap p(D)$.

of complex points of X equipped with the classical (i.e., analytic) topology (c.f. [1, Appendix B]).

Theorem 2.5.7. Let X be a scheme of finite type over $\mathbf{C}$.

(a) The scheme X is separated iff $X(\mathbf{C})$ with the classical topology is Hausdorff.
(b) The scheme X is complete (i.e., proper over $Y = \operatorname{Spec} \mathbf{C}$) iff $X(\mathbf{C})$ with the classical topology is compact (i.e., quasicompact Hausdorff).

TOCITE. ■

Chapter 3

Affine Communication Lemma and Properties of Morphisms

3.1 The Affine Communication Lemma

Let X be a locally ringed space. An open subset $D \subset X$ is said to be *distinguished* if there is an $f \in \mathcal{O}(X)$ such that $D = X \smallsetminus \mathbf{V}(f) = \{p \in X : f(p) \neq 0\}$; in general, there are many such f for which this holds–for instance, every power of f gives the same D.

A scheme X is affine iff the natural morphism $X \to \operatorname{Spec} \mathcal{O}(X)$ is an isomorphism. In this case, if $D \subset X$ is a distinguished open subset, then $(D, \mathcal{O}|_D)$ is also an affine scheme, almost by construction of the sheaf $\mathcal{O}$. The following results will be used repeatedly.

Lemma 3.1.1.

(a) If $\pi : X \to Y$ is a morphism of locally ringed spaces and $D \subset Y$ a distinguished open, then so is $\pi^{-1}(D) \subset X$.
(b) The intersection of two distinguished opens in a locally ringed space is again distinguished.
(c) Suppose X is a locally ringed space and $D' \subset D \subset X$ open subsets. If D' is distinguished in X, then it is also in D. Conversely, if X is an affine scheme, D distinguished in X, and D' distinguished in D, then D is distinguished in X.
(d) Distinguished open subsets are a basis for the topology on an affine scheme. Affine open subsets are a basis for the topology on any scheme.
(e) Suppose X is an affine scheme, $\{f_i\} \subset \mathcal{O}(X)$ a family of elements, and for each i the set $D_i := X \smallsetminus \mathbf{V}(f_i)$. Then $X = \bigcup_i D_i$ iff $(f_i) = (1)$ in $\mathcal{O}(X)$. In particular, any affine scheme is quasicompact.

Proof. Exercise. ■

Example 3.1.2. On a general locally ringed space, distinguised opens do not have to be a basis for the topology. A simple example is obtained by taking a non-affine scheme like $X = \mathbf{P}^1_{\mathbf{C}}$, in which the only distinguished opens as $\emptyset$ and X.

The first fundamental result we need is

Lemma 3.1.3. Let X be scheme and $U, V \subset X$ affine opens. Then $U \cap V$ can be covered by affine opens which are simultaneously distinguished in both U and V.

Proof. Let $x \in U \cap V$. Using 3.1.1(d), pick a distinguished open D in U such that $p \in D \subset U \cap V$. Next, again using Lemma 3.1.1(d), pick a distinguished open D' in V such that $p \in D' \subset D$. Then D' is distinguished in D and hence in U by 3.1.1(c). ■

Remark 3.1.4. The above results say that for any scheme X, the site $X^{\text{dist}}_{\text{Zar}}$ consisting of affine opens and distinguished inclusions is enough to capture all (Zariski-) sheafy business on X. Precisely, the continuous inclusion $X^{\text{dist}}_{\text{Zar}} \hookrightarrow X_{\text{Zar}}$ gives rise to an adjoint equivalent of categories $\mathsf{Shv}(X^{\text{dist}}_{\text{Zar}}, \mathcal{O}) \rightleftharpoons \mathsf{Shv}(X_{\text{Zar}}, \mathcal{O})$, where $\mathcal{O}$ denotes either a constant sheaf of rings or the structure sheaf ([6, Theorem 6.2.2]). This adjoint equivalence preserves various structures on these categories such as (quasi)coherence; concretely, we shall see below (3.3.1) that specifying a quasicoherent sheaf $\mathcal{F}$ on a scheme X is the same data as specifying for each affine open $U \subset X$ an $\mathcal{O}(U)$-module $\mathcal{F}(U)$, along with comptaible restriction morphisms $\mathcal{F}(U) \to \mathcal{F}(D)$ for every distinguished $D \subset U$.

The wonderful theory-building tool we will repeatedly use is

Lemma 3.1.5 (Affine Communication Lemma)**.** Let X be a scheme and $\mathscr{P}$ be a property satisfied by some affine open subsets of X such that for any affine open $U \subset X$, the following two conditions hold.

(a) For any distinguished open $D \subset U$, if U has $\mathscr{P}$ then so does D.
(b) Suppose $\{D_\alpha\}$ is a finite collection of distinguished opens in U that cover U (i.e., such that $U = \bigcup_\alpha D_\alpha$). If each D_α has $\mathscr{P}$, then so does U.

In this case, if X admits some affine open cover all of whose elements have $\mathscr{P}$, then every affine open $U \subset X$ has $\mathscr{P}$.

Proof. Suppose we are given an affine open cover $\{V_\beta\}$ of X such that each V_β has $\mathscr{P}$. Using 3.1.3, cover U by distinguished opens D_α that are simultaneously also distinguished in *some* $V_{\beta(\alpha)}$. Since each V_β has $\mathscr{P}$, property (a) tells us that each D_α has $\mathscr{P}$. To apply (b), it remains to only note that finitely many D_α cover U, and this is true because U is quasicompact. ■

Lemma 3.1.6. Let $\mathscr{P}$ be a property of morphisms of schemes that is affine-local on the target.[1] If $\mathscr{P}$ is stable under affine base change[2], then $\mathscr{P}$ is stable under base change.

Proof. Let $X \to S$ have $\mathscr{P}$ and let $T \to S$ be a morphism. Since $\mathscr{P}$ is affine-local on the target, to show that the pullback $X_T \to T$ has $\mathscr{P}$, it suffices to produce an affine open cover $\mathscr{U}$ of T such that for each $U \in \mathscr{U}$, the morphism $X_U \to U$ has $\mathscr{P}$. For this, let $\mathscr{U}$ be the cover of T consisting of the affine opens $U \subset T$ whose image in S lies in some affine open of S; we show this $\mathscr{U}$ works. Given a $U \in \mathscr{U}$, pick an affine open V of S such that U maps to V in S. Since $\mathscr{P}$ is affine-local on the target, the basechange $X_V \to V$ has $\mathscr{P}$. Then the morphism $X_U \to U$ is the affine base change of $X_V \to V$ along $U \to V$ and hence also has $\mathscr{P}$. ■

A slightly less useful result is

Lemma 3.1.7. Let $\mathscr{P}$ be a property of morphisms of schemes that is affine-local on the target. If $\mathscr{P}$ is further

(a) affine-local on the source,[3] and
(b) stable under affine composition,[4]

then it is stable under arbitrary composition.

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms having $\mathscr{P}$; we want to show that so does $gf : X \to Z$. Since $\mathscr{P}$ is affine-local on the target, we may assume without loss of generality that Z is affine. Since $\mathscr{P}$ is affine-local on the source, to show that gf has $\mathscr{P}$, it suffices to produce an affine open cover $\mathscr{U}$ of X such that for each $U \in \mathscr{U}$, the composite $U \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ has $\mathscr{P}$. For this, let $\mathscr{U}$ be the cover of X consisting of affine opens $U \subset X$ whose image in Y under f lies in some affine open of Y; we show that this $\mathscr{U}$ works. Given $U \in \mathscr{U}$, pick an affine open V of Y such that $U \subset f^{-1}(V)$. Since $\mathscr{P}$ is affine-local on the source, the restriction $g|_V : V \to Z$ has $\mathscr{P}$. Since $\mathscr{P}$ is affine-local on the target, $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ has $\mathscr{P}$. Since $\mathscr{P}$ is affine-local on the source, the further restriction $f|_U : U \to V$ has $\mathscr{P}$. Finally, since $\mathscr{P}$ is stable under affine composition, the composite $g|_V \circ f|_U : U \to Z$ has $\mathscr{P}$, as needed. ■

Lemma 3.1.8.

Lemma 3.1.9.

[1] In other words, if $\pi : X \to Y$ is a morphism of schemes, then π has the property $\mathscr{P}$ iff for each $y \in Y$, there is an affine open $V \subset Y$ such that the restriction $\pi^{-1}(V) \to V$ has $\mathscr{P}$.

[2] An *affine base change* is a base change in which the source and target bases are affine.

[3] In other words, if $\pi : X \to Y$ is a morphism of schemes, then π has the property $\mathscr{P}$ iff for each $x \in X$, there is an affine open $U \subset X$ such that the restriction $\pi|_U : U \to Y$ has $\mathscr{P}$.

[4] An *affine composition* is a composition of two morphisms, both of whose sources and targets are affine.

3.2 (Locally) Noetherian Schemes and (Locally) Finite Type and Presentation Morphisms

Proposition/Definition 3.2.1 ((Locally) Noetherian Schemes)**.** The following conditions on a scheme X are equivalent:

(a) For every affine open $U \subset X$, the ring $\mathcal{O}(U)$ is Noetherian.
(b) There is an affine open cover $X = \bigcup_i U_i$ such that for each i, the ring $\mathcal{O}(U)$ is Noetherian.

A scheme X satisfying these equivalent conditions is said to be *locally Noetherian.* A scheme X is said to be *Noetherian* iff it is locally Noetherian and quasicompact. Further:

(c) If X is a (locally) Noetherian scheme, then the underlying topological space of X is also (locally) Noetherian. Further, any open subscheme $U \subset X$ is also (locally) Noetherian.

Proof. By 3.1.5 it suffices to show the following.

(a) If A is a Noetherian ring, then for each $f \in A$, so is $A[f^{-1}]$. This follows, for instance, from the Hilbert Basis Theorem, since $A[f^{-1}] \cong A[X]/(fX - 1)$.
(b) Suppose A is a ring, $n \in \mathbf{Z}_{\geq 1}$, and $f_1, \dots, f_n \in A$ such that $(f_1, \dots, f_n) = 1$. If for each i the ring $A[f_i^{-1}]$ is Noetherian, then so is A. Let $\mathfrak{a} \subset A$ be an ideal. For each i, pick finitely many elements $a_{ij} \in \mathfrak{a}$ such that $\mathfrak{a}A[f_i^{-1}]$ is generated by the images of the a_{ij} in $A[f_i^{-1}]$. Then it remains to check that $\mathfrak{a} = (a_{ij})$ as an ideal of A, which we leave to the reader.

The first statement in (c) follows from 2.2.5 and 2.2.7(a). The second statement for the locally Noetherian case follows immediately from the equivalence of (a) and (b), while the in the Noetherian case follows from the locally Noetherian case coupled with the first statement and 2.2.7(b). ■

Let A be a ring. Recall that an A-algebra B is said to be of *finite type (over A)* if B is finitely generated as an A-algebra, i.e. there is an $n \in \mathbf{Z}_{\geq 1}$ and an A-algebra epimorphism $A[X_1, \dots, X_n] \to B$.

Proposition/Definition 3.2.2 (Locally Finite Type Schemes)**.** Let A be a ring and X be an A-scheme. Then X is said to be *locally of finite type* (abbreviated *lft*) *over* A if the following equivalent conditions hold:

(a) For every affine open $U \subset X$, the A-algebra $\mathcal{O}(U)$ is of finite type.
(b) There is an affine open cover $X = \bigcup_i U_i$ such that for each i, the A-algebra $\mathcal{O}(U_i)$ is of finite type.

Further:

(c) If X is lft over A, then so is any open subscheme $U \subset X$. Conversely, if X admits an open cover by lft A-schemes, then X is lft over A.

Proof. By 3.1.5, it suffices to show the following. Suppose B is an A-algebra.

(i) If B is of finite type over A, then for each $f \in B$, so is $B[f^{-1}]$; this is clear.
(ii) If there is an $n \in \mathbf{Z}_{\geq 1}$ and $f_1, \dots, f_n \in B$ such that $(f_1, \dots, f_n) = (1)$ and for each i, the localization $B[f_i^{-1}]$ is of finite type over A, then so if B. Indeed, there is an $N \gg 1$ such that for each i, there is a finite set of $g_{ij} \in B$ such that $B[f_i^{-1}]$ is generated as an A-algebra by the $g_{ij} \cdot f_i^{-N}$. If we pick $h_1, \dots, h_n \in B$ such that $\sum_{i=1}^n h_i f_i = 1$, then $B = A[f_i, g_{ij}, h_i]$.

The statement (c) follows immediately from the equivalence of (a) and (b). ■

The relative version of the above property is given in

Proposition/Definition 3.2.3 (Locally Finite Type Morphisms)**.** A morphism $\pi : X \to Y$ of schemes is said to be *locally of finite type* (abbreviated *lft*) if the following equivalent conditions hold:

(a) For every affine open $V \subset Y$, the $\mathcal{O}(V)$-scheme $\pi^{-1}(V)$ is lft over $\mathcal{O}(V)$.
(b) There is an affine open cover $Y = \bigcup_i V_i$ such that for each i, the $\mathcal{O}(V_i)$-scheme $\pi^{-1}(V_i)$ is lft over $\mathcal{O}(V_i)$.

Further:

(c) Lft morphisms are affine local on both the source and target. In particular, an open immersion is lft.
(d) Lft morphisms are stable under base-change and composition.
(e) If $\pi : X \to Y$ is lft and Y locally Noetherian, then so is X.

If A is a ring and X an A-scheme, then X is locally of finite type over A iff the structure morphism $X \to \operatorname{Spec} A$ is locally of finite type.

Proof. By 3.1.5, it suffices to show the following. Let A be a ring.

(i) If X is an A-scheme lft over A, then for each $f \in A$, the scheme $\varphi^{-1}(D_f)$ is lft over $\mathcal{O}(D_f) = A[f^{-1}]$. Indeed, for each affine open $U \subset \varphi^{-1}(D_f)$, we know that $\mathcal{O}(U)$ is a finite type A-algebra and hence also a finite type $A[f^{-1}]$-algebra.
(ii) If $n \in \mathbf{Z}_{\geq 1}$ and $f_1, \ldots, f_n \in A$ are such that $(f_1, \ldots, f_n) = 1$ and for each i, the preimage $\varphi^{-1}(D_{f_i})$ is lft over $A[f_i^{-1}]$, then X is lft over A. Indeed, for each i, there is an affine open cover $\varphi^{-1}(D_{f_i}) = \bigcup_j U_{ij}$ such that for each j, the $A[f_i^{-1}]$-algebra $\mathcal{O}(U_{ij})$ is of finite type. Then $X = \bigcup_i \bigcup_j U_{ij}$ is an affine open cover such that for each (i, j), the A-algebra $\mathcal{O}(U_{ij})$ is of finite type over $A[f_i^{-1}]$ and hence also over A, so we are done.

Now we show the remaining properties.

(c) Affine-locality on the target follows from the equivalence of (a) and (b). To prove affine-locality on the source, use affine-locality on the target to reduce the statement to 3.2.2(c).
(d) Stability under base-change follows from 3.1.6 and stability under composition follows from 3.1.7.
(e) Clear.

∎

3.3 Quasicoherent Sheaves

Let X be an affine scheme with coordinate ring $A := \mathcal{O}(X)$. Then we have adjunctions

$$x$$

Theorem/Definition 3.3.1 (Quasicoherent Sheaves)**.** Let X be a scheme and $\mathcal{F} \in \mathcal{O}_X\text{-}\mathsf{Mod}$. The following are equivalent:

(a) For every affine open $U \subset X$ and distinguished open $D \subset U$, the natural $\mathcal{O}(D)$-module morphism $\mathcal{O}(D) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \to \mathcal{F}(D)$ is an isomorphism.
(b) For every affine open $U \subset X$, we have $\mathcal{F}|_U \in \mathsf{QCoh}(U)$.
(c) There is an affine open cover $\mathscr{U}$ of X such that for all $U \in \mathscr{U}$, we have $\mathcal{F}|_U \in \mathsf{QCoh}(U)$.
(d) For each $x \in X$, there is a neighborhood U of x in X, sets I and J and an exact sequnece of $\mathcal{O}_U$-modules[5] of the form $\mathcal{O}_U^{\oplus J} \to \mathcal{O}_U^{\oplus I} \to \mathcal{F} \to 0$.

An $\mathcal{F} \in \mathcal{O}_X\text{-}\mathsf{Mod}$ is said to be *quasicoherent* if it satisfies these equivalent conditions. Further:

(e) The full subcategory $\mathsf{QCoh}(X)$ of $\mathcal{O}_X\text{-}\mathsf{Mod}$ consisting of quasicoherent $\mathcal{O}_X$-modules is closed under arbitrary direct sums and taking (co)kernels, and is hence an abelian subcategory of $\mathcal{O}_X\text{-}\mathsf{Mod}$.

Proof. ■

[5] Here $\mathcal{O}_U := \mathcal{O}_X|_U$.

3.4 Quasicompact and Quasiseparated Morphisms

The first order of business is to understand quasicompact morphisms.

Proposition/Definition 3.4.1 (Quasicompact Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes. The following conditions on π are equivalent.

(a) For every quasicompact open $V \subset Y$, the preimage $\pi^{-1}(V) \subset X$ is quasicompact.
(b) For every affine open $V \subset Y$, the preimage $\pi^{-1}(V)$ is quasicompact.
(c) There is an affine open cover $Y = \bigcup_i V_i$ such that for each i, the preimage $\pi^{-1}(V_i) \subset X$ is quasicompact.

A morphism π of schemes satisfying these equivalent conditions is said to be *quasicompact.* Further:

(d) Quasicompact morphisms are stable under composition and base change, and are affine local on the target.

Proof.

(a) $\Rightarrow$ (b) Every affine open subset is quasicompact.
(b) $\Rightarrow$ (a) Every quasicompact open subset is a finite union of affine opens.
(b) $\Leftrightarrow$ (c) By the Affine Communication Lemma (Lemma 3.1.5), it remains to check the following in the case Y is affine.

(i) If X is quasicompact, then for any distinguished $D \subset Y$ the preimage $\pi^{-1}(D)$ is quasicompact. Since X is quasicompact, it is a finite union of affines; this reduces us to the case where X is also affine, but then we can invoke 3.1.1(a).

(ii) If $\{D_\alpha\}$ is a finite cover of Y by distinguished opens and each preimage $\pi^{-1}(D_\alpha) \subset X$ is quasicompact, then so is X. This is clear because $X = \pi^{-1}(Y) = \bigcup_\alpha \pi^{-1}(D_\alpha)$.

(d) Stability under composition follows from (a), and the rest is clear from 3.1.6.

■

The next order of business is to understand quasiseparated morphisms, but for that we need to understand quasiseparated schemes first.

Proposition/Definition 3.4.2 (Quasiseparated Schemes)**.** The following conditions on a scheme X are equivalent:

(a) If $U, V \subset X$ are quasicompact opens, then so is $U \cap V$.
(b) If $U, V \subset X$ are affine opens, then $U \cap V$ is a finite union of affine opens in X.
(c) There is an affine open cover $\mathscr{U}$ of X all of whose pairwise intersections are quasicompact.

A scheme X satisfying these equivalent conditions is said to be *quasiseparated.* Further:

(d) An open subscheme of an quasiseparated scheme is quasiseparated.
(e) A locally Noetherian scheme is quasiseparated.

Proof.

Step 1. The implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) follow from the fact that an open subset of X is quasicompact iff it can be written as a finite union of affine opens of X.

Step 2. We show that an affine scheme X satisfies (b). Indeed, let X be affine and $U, V \subset X$ be affine opens. By 3.1.3, U (resp. V) can be written as the finite union of affine opens U_i (resp. V_j) which are simultaneously distinguished in U and X (resp. in V and X). Then $U \cap V = \bigcup_{i,j} U_i \cap V_j$, with each intersection $U_i \cap V_j$ being distinguished in X by 3.1.1(b).

Step 3. Now we show (c) $\Rightarrow$ (b). Let $\mathscr{U}$ and $U, V \subset X$ be as given. By 2.5.3 applied to $\mathscr{U}$ and $W = U$, we conclude that for each $U' \in \mathscr{U}$, the intersection $U \cap U'$ is quasicompact. Then 3.1.5 applied to $\mathscr{U} \cup \{U\}$ and $W = V$ shows that $U \cap V$ is quasicompact.

The statement (d) is clear from (a). The statement (e) is clear from (b) combined with (b) and (e) of 2.2.7. ■

Corollary 3.4.3 (Qcqs Schemes)**.** A scheme X is quasicompact and quasiseparated (abbreviated *qcqs*) iff it admits a finite affine open cover, all of whose pairwise intersections are also covered by finitely many affine opens.

We like qcqs schemes for various reasons; one is

Lemma 3.4.4 (Qcqs Lemma)**.** Let X be a qcqs scheme and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then for any $f \in \mathcal{O}(X)$, the natural morphism $\mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \to \mathcal{F}(X_f)$ is an isomorphism.

Proof. Note that if $U \subset X$ is any affine, then $U \cap X_f = D(f|_U)$ is a distinguished open in U. Now cover X by finitely many affines U_i such that each pairwise intersection $U_i \cap U_j$ is the finite union of affines U_{ijk}. Then for any sheaf $\mathcal{F}$ on X, we have an exact sequence

$$0 \to \mathcal{F}(X) \to \bigoplus_i \mathcal{F}(U_i) \to \bigoplus_{i,j,k} \mathcal{F}(U_{ijk}). \tag{3.4.5}$$

Tensoring with the flat $\mathcal{O}(X)$-algebra $\mathcal{O}(X)[f^{-1}]$ and using that tensor products commute with direct sums, along with some natural identifications, yields the analogous sequence

$$0 \to \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \to \bigoplus_i \mathcal{O}(U_i \cap X_f) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i) \to \bigoplus_{i,j,k} \mathcal{O}(U_{ijk} \cap X_f) \otimes_{\mathcal{O}(U_{ijk})} \mathcal{F}(U_{ijk}).$$

Map this to the sequence corresponding to 3.4.5 for the cover $U_i \cap X_f$ of X_f, and use that the middle and last maps are then isomorphisms (because $\mathcal{F} \in \mathsf{QCoh}(X)$) along with the five lemma to finish the proof. ■

Now we are ready to define quasiseparated morphisms.

Proposition/Definition 3.4.6 (Quasiseparated Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes. The following conditions on π are equivalent.

(a) For every quasiseparated open $V \subset Y$, the preimage $\pi^{-1}(V) \subset X$ is quasiseparated.
(b) For every affine open $V \subset Y$, the preimage $\pi^{-1}(V)$ is quasiseparated.
(c) There is an affine open cover $Y = \bigcup_i V_i$ such that for each i, the preimage $\pi^{-1}(V_i)$ is quasiseparated.
(d) The diagonal morphism $\Delta_\pi : X \to X \times_Y X$ is quasicompact.

A morphism π of schemes satisfying these equivalent conditions is said to be *quasiseparated*. Further:

(e) Quasiseparated morphisms are stable under composition and base change, and are affine local on the target.
(f) Any morphism out of a quasiseparated scheme is quasiseparated.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (b) By the Affine Communication Lemma (3.1.5), it remains to check the following in the case Y is affine:

(i) If X is quasiseparated, then for any distinguished $D \subset Y$, so is the preimage $\pi^{-1}(D)$; this follows from 3.4.2(d).

(ii) If $\{D_\alpha\}$ is a finite cover of Y by distinguished opens and each $\pi^{-1}(D_\alpha)$ is quasiseparated, then so is X. Indeed, suppose we are given affines $U, V \subset X$. Covering U by finitely many distinguished opens in U, each of which lies in $\pi^{-1}(D_\alpha)$ for some α, we reduce to the case in which $U \subset \pi^{-1}(D_\alpha)$ for some α. Then

$$U \cap V = U \cap \pi^{-1}(D_\alpha) \cap V = U \cap \pi|_V^{-1}(D_\alpha).$$

Here U is affine and hence quasicompact, and $\pi|_V^{-1}(D_\alpha)$ is distinguished in V (by 3.1.1(a) applied to $\pi|_V$) and hence quasicompact; since this intersection takes place in $\pi^{-1}(D_\alpha)$, which is quasiseparated, it follows that $U \cap V$ is quasicompact.

(c) $\Rightarrow$ (d) We produce an affine open cover of $X \times_Y X$ such that for each element of this cover, its preimage under Δ_π is quasicompact. For this, for each i pick an affine open cover $\pi^{-1}(V_i) = \bigcup_j U_{ij}$ of $\pi^{-1}(U_i)$. Then we claim that the affine open cover $\{U_{ij} \times_{V_i} U_{ij'}\}_{i,j,j'}$ of $X \times_Y X$ has this property; indeed, we have $\Delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij'}) = U_{ij} \cap U_{ij'}$, which is an intersection of affines in the quasiseparated $\pi^{-1}(V_i)$ and hence quasicompact.

(d) $\Rightarrow$ (b) Let $U, U' \subset \pi^{-1}(V)$ be affine opens; we want to show that $U \cap U'$ is quasicompact. Well, $U \times_V U' \xrightarrow{\sim} U \times_Y U' \subset X \times_Y X$ is an affine open and hence quasicompact, therefore its preimage under Δ_π, namely $U \cap U'$, is also quasicompact.

At this point, we have shown the equivalence of (b), (c), and (d), and it is clear that a scheme X is quasiseparated in the sense of 3.4.2 iff the final morphism $X \to \operatorname{Spec} \mathbf{Z}$ satisfies conditions (d). Next, we note that, with definition (d) of quasiseparatedness, it is clear from the abstract formalism (3.1.8) that quasiseparated morphisms are stable under composition. Given this, we are ready to show

(d) $\Rightarrow$ (a) Let $V \subset Y$ be any open subset and let $X_V = \pi^{-1}(V)$. Then the diagram

$$\begin{tikzcd} X_V \arrow[r, hook] \arrow[d, "\Delta"] & X \arrow[d, "\Delta"] \\ X_V \times_V X_V \arrow[r, hook] & X \times_Y X \end{tikzcd}$$

is a pullback, so stability of quasicompact morphisms under basechange (3.4.1(d)), we conclude that $X_V \to V$ is also quasiseparated in the sense of (d). Since V is quasiseparated, we know that the final morphism $V \to \operatorname{Spec} \mathbf{Z}$ is quasiseparated in the sense of (d). Since quasiseparated morphisms in the sense of (d) are stable under composition, we conclude that the composite $X_V \to V \to \operatorname{Spec} \mathbf{Z}$ is quasiseparated in the sense of (d), but this is just saying that X_V is a quasiseparated scheme.

That quasiseparated morphisms are preserved by basechange follows from 3.1.9, and (f) is clear from 3.4.2(d). ■

A morphism is said to be *qcqs* if it is both quasicompact and quasiseparated. We like such morphisms because of the corresponding analog of the qcqs lemma:

Lemma 3.4.7. Let $\pi : X \to Y$ be a qcqs morphism and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then $\pi_*\mathcal{F} \in \mathsf{QCoh}(Y)$.

Proof. Since qcqs morphisms are affine-local on the target and the question of quasicoherence is likewise local on Y, we may assume that Y is affine; then X is a qcqs scheme. We have to show that if $D \subset Y$ is distinguished, then the natural morphism $\mathcal{O}(D) \otimes_{\mathcal{O}(Y)} \pi_*\mathcal{F}(Y) \to \pi_*\mathcal{F}(D)$ is an isomorphism. If $g \in \mathcal{O}(Y)$ is such that $D = D(g)$ and $f := \mathcal{O}(\pi)(g) \in \mathcal{O}(X)$ the image of g under the ring homomorphism $\mathcal{O}(\pi) : \mathcal{O}(Y) \to \mathcal{O}(X)$, then $\pi^{-1}(D) = X_f$. Since g acts by f on $\mathcal{F}(X)$, in light of 3.4.4, it remains to note only that the natural morphism $\mathcal{O}(Y)[g^{-1}] \otimes_{\mathcal{O}(Y)} \mathcal{F}(X) \to \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X)$ is an isomorphism. ■

3.5 Affine, Finite Morphisms, and Integral Morphisms

Before we define affine morphisms, we use the following consequence of the qcqs lemma taken from [1, Exercise 2.17].

Lemma 3.5.1. A scheme X is affine iff there is an $n \in \mathbf{Z}_{\geq 1}$ and $f_1, \ldots, f_n \in \mathcal{O}(X)$ generating the unit ideal such that each X_{f_i} is affine.

Proof. If X is affine, it suffices to take $n = 1$ and $f_1 = 1$. Conversely, the hypothesis implies that X is qcqs (3.4.3), so for each i, 3.4.4 applied to $\mathcal{F} = \mathcal{O}_X$ and $f = f_i$ tells us that $\mathcal{O}(X)[f_i^{-1}] \xrightarrow{\sim} \mathcal{O}(X_{f_i})$. Since X_{f_i} is affine, we get a series of isomorphisms

$$X_{f_i} \xrightarrow{\sim} \operatorname{Spec} \mathcal{O}(X_{f_i}) \xrightarrow{\sim} \operatorname{Spec} \mathcal{O}(X)[f_i^{-1}] \cong D(f_i) \subset \operatorname{Spec} \mathcal{O}(X).$$

This says exactly that the natural morphism $X \to \operatorname{Spec} \mathcal{O}(X)$ is an isomorphism over the cover $\{D(f_i)\}_i$ of $\operatorname{Spec} \mathcal{O}(X)$, so by the affine locality of isomorphisms we conclude that $X \to \operatorname{Spec} \mathcal{O}(X)$ is also an isomorphism. ■

Proposition/Definition 3.5.2 (Affine Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes. The following conditions on π are equivalent.

(a) For every affine open $V \subset Y$, the preimage $\pi^{-1}(V) \subset X$ is affine.
(b) There is an open affine cover $Y = \bigcup_i V_i$ such that for each i, the preimage $\pi^{-1}(V_i)$ is affine.

A morphism satisfying these equivalent conditions is said to be *affine*. Further:

(c) Affine morphisms are stable under composition and base-change, and are affine-local on the target.

Proof. Using 3.1.5, reduce to 3.1.1(a) and 3.5.1. For (c), use 3.1.6. ■

Recall that an algebra $A \to B$ is said to be *finite* if B is a finitely generated A-module. THe globalization of this definition is

Proposition/Definition 3.5.3 (Finite Morphisms)**.** Let $\pi : X \to Y$ be a morphism of schemes. The following conditions on π are equivalent.

(a) For every affine open $V \subset Y$, the preimage $\pi^{-1}(V) \subset X$ is affine and $\mathcal{O}(\pi^{-1}(V))$ is a finite $\mathcal{O}(V)$-algebra.
(b) There is an open affine cover $Y = \bigcup_i V_i$ such that for each i, the preimage $\pi^{-1}(V_i)$ is affine and $\mathcal{O}(\pi^{-1}(V_i))$ is a finite $\mathcal{O}(V_i)$-algebra.

A morphism satisfying these equivalent conditions is said to be *finite*. Further:

(c) Finite morphisms are stable under composition and base-change, and are affine-local on the target.
(d) Finite morphisms have finite discrete fibers.

Proof. Using 3.1.5, reduce to showing the following.

(i) If $\phi : A \to B$ is a finite algebra, then for any $f \in A$, the localization $A[f^{-1}] \to B[\phi(f)^{-1}]$ is also finite. This is clear.
(ii) If $\phi : A \to B$ is an A-algebra, $n \in \mathbf{Z}_{\geq 1}$, and $f_1, \ldots, f_n \in A$ with $(f_1, \ldots, f_n) = 1$ such that for each i, the algebra $A[f_i^{-1}] \to B[\phi(f_i)^{-1}]$ is finite, then $A \to B$ is finite. To show this, for each i, pick finitely many elements $(b_{ij})_j$ in B such that the b_{ij} generate $B[\phi(f_i)^{-1}]$

as an $A[f_i^{-1}]$-module; then use a partition-of-unity argumen to check that the $\{b_{ij}\}_{i,j}$ generate B as an A-module.

For (c), use 3.1.6 as usual. For (d), we use (c) to reduce to the case of the target being $\operatorname{Spec} k$ for a field k; then X is an Artinian scheme over k. ■

3.6 Closed Embeddings

Let X be a locally ringed space and $\mathcal{I} \subset \mathcal{O}_X$ an ideal sheaf. We want to define a closed subscheme $Z = \mathbf{V}(\mathcal{I})$ of X corresponding to the ideal sheaf $\mathcal{I}$. For this, as a topological space we take $Z := \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}) \subset X$, which is a closed subset since $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings on X. Let $\iota : Z \hookrightarrow X$ denote the inclusion map. Define a sheaf of rings on Z by taking $\mathcal{O}_Z := \iota^{-1}(\mathcal{O}_X/\mathcal{I})$, so that at a point $z \in Z$, the stalk $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}/\mathcal{I}_z$ is a nonzero quotient of a local ring $\mathcal{O}_{X,z}$ and hence local. This construction therefore gives us a locally ringed space Z. Define a morphism $\iota : Z \to X$ of locally ringed spaces given on the underlying topological spaces by the ι previously defined and with the morphism of sheaves corresponding to the natural surjection $\iota^\sharp : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \cong \iota_*\mathcal{O}_Z$.[6] In particular, there is an exact sequence of abelian sheaves on X of the form

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \iota_*\mathcal{O}_Z \to 0 \tag{3.6.1}$$

called the *ideal sheaf exact sequence*.[7] With this construction, it is clear that the morphism $\iota : Z \to X$ is a monomorphism in the category of locally ringed spaces.

As with ideals in a ring, this construction satisfies the expected universal property.

Proposition 3.6.2. Let X be a locally ringed space and $\mathcal{I}$ an ideal sheaf on X. Let Z and ι be as above. If Y is any other locally ringed space and $f : Y \to X$ a morphism such that $\mathcal{I} \subset \ker(f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y)$, then f factors uniquely through Z, i.e., there is a unique morphism of locally ringed spaces $\widetilde{f} : Y \to Z$ such that $f = \iota \circ \widetilde{f}$.

We could take this universal property as a *definition* of $\mathbf{V}(\mathcal{I})$; then the above paragraph would amount to giving a *construction*. Note that conversely if a morphism $f : Y \to X$ admits a factorization of the form $f = \iota \circ \widetilde{f}$ for some $\widetilde{f} : Y \to Z$, then automatically we have that $\mathcal{I} \subset \ker(f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y)$.

Proof. The uniqueness of such an f follows from ι being a monomorphism. To show existence, first note that $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ admits a factorization as a morphism of sheaves of rings of the form

$$\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y. \tag{3.6.3}$$

This implies that set-theoretically $f(Y) \subset Z$; indeed, if $y \in Y$, then we have a factorization $\mathcal{O}_{X,f(y)} \to (\mathcal{O}_X/\mathcal{I})_{f(y)} \to \mathcal{O}_{Y,y}$ of $f^\sharp_y$; since the composite is a nonzero homomorphism of rings between nonzero rings, the middle ring cannot be zero. Therefore, by definition of the subspace topology, there is a unique continuous map $\tilde{f} : Y \to Z$ such that $f = \iota \circ \tilde{f}$ as continuous maps. To upgrade this to a morphism of locally ringed spaces, we need to give a morphism $\widetilde{f}^\sharp : \mathcal{O}_Z \to \widetilde{f}_*\mathcal{O}_Y$ of sheaves of rings on Z such that the resulting composition $f = \iota \circ \widetilde{f}$ holds as morphisms of locally ringed spaces. Since $\iota_* : \mathsf{Ab}(Z) \to \mathsf{Ab}(X)$ is fully faithful (and preserves ring sheaves), it suffices to produce a morphism $\iota_*\widetilde{f}^\sharp : \iota_*\mathcal{O}_Z \to \iota_*\widetilde{f}_*\mathcal{O}_Y$ of sheaves on X; take this to be the composition

$$\iota_*\mathcal{O}_Z = \iota_*\iota^{-1}(\mathcal{O}_X/\mathcal{I}) \xrightarrow{\eta^{-1}_{\mathcal{O}_X/\mathcal{I}}} \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y \cong \iota_*\widetilde{f}_*\mathcal{O}_Y,$$

[6] Here we are using the following fact: if $\pi : Z \to X$ is a continuous map, then we have an adjunction $\pi^{-1} \dashv \pi_*$ between $\mathsf{Ab}(Z)$ and $\mathsf{Ab}(X)$. Further, if $\pi = \iota$ is a topological closed embedding (i.e., up to identifications, the inclusion of a closed subset), then the counit $\epsilon : \iota^{-1}\iota_* \to 1$ is an isomorphism, which tells us that $\iota_* : \mathsf{Ab}(Z) \to \mathsf{Ab}(X)$ is fully faithful with essential image consisting of the abelian sheaves on X supported on Z. In particular, if $\mathcal{F}$ is an abelian sheaf on X supported on Z, then the unit $\eta_\mathcal{F} : \mathcal{F} \to \iota_*\iota^{-1}\mathcal{F}$ is an isomorphism. This follows also from the fact that if $j : U \hookrightarrow X$ denotes the open complement of Z, then we have a natural exact sequence $0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to \iota_*\iota^{-1}\mathcal{F} \to 0$ on $\mathcal{F}$.

[7] Since the functor ι_* is fully faithful, some texts drop the ι_* from 3.6.1, writing it instead as $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$, but this means exactly the same thing.

where the second-to-last morphism comes from the factorization (3.6.3). Then the identity $f = \iota \circ \widetilde{f}$ as morphisms of locally ringed spaces holds by construction using (3.6.3). ■

That this is the right construction in our context follows from

Proposition 3.6.4. Let X be a scheme and $\mathcal{I}$ a *quasicoherent* ideal sheaf on X. Then the locally ringed space Z constructed above is a scheme and $\iota : Z \to X$ is a morphism of schemes. Further, the quotient $\iota_*\mathcal{O}_Z$ is also quasicoherent on X, i.e., the ideal sheaf exact sequence (3.6.1) is an exact sequence in $\mathsf{QCoh}(X)$.

Proof. The question is local on X, so we may assume without loss of generality that X is affine. Let $A := \mathcal{O}(X)$, and let $I := \mathcal{I}(X) \subset A$. The quasicoherence of $\mathcal{I}$ tells us that there is an isomorphism $\mathcal{O}_X/\mathcal{I} \cong \widetilde{A/I}$ of sheaves of X, from which it follows that as a topological space $Z = \operatorname{Supp}(\widetilde{A/I}) = \operatorname{Supp}(A/I) = \mathbf{V}(I) \subset \operatorname{Spec} A$. Therefore, the underlying topological space of Z is homeomorphic to $\operatorname{Spec}(A/I)$, and so admits the sheaf $\mathcal{O}_{\operatorname{Spec}(A/I)}$ also. Now consider the adjunctions (and inclusion)

$$\begin{aligned}
\operatorname{Hom}_{A\text{-}\mathsf{Mod}}\left(A/I, \Gamma(Z, \mathcal{O}_{\operatorname{Spec}(A/I)})\right) &\cong \operatorname{Hom}_{\mathcal{O}_X\text{-}\mathsf{Mod}}\left(\widetilde{A/I}, \iota_*\mathcal{O}_{\operatorname{Spec}(A/I)}\right) \\
&\subset \operatorname{Hom}_{\mathsf{Ab}(X)}\left(\widetilde{A/I}, \iota_*\mathcal{O}_{\operatorname{Spec}(A/I)}\right) \\
&\cong \operatorname{Hom}_{\mathsf{Ab}(Z)}\left(\iota^{-1}(\widetilde{A/I}), \mathcal{O}_{\operatorname{Spec}(A/I)}\right).
\end{aligned}$$

The obvious isomorphism in the first term above therefore gives rise to an abelian sheaf morphism $\iota^{-1}\widetilde{A/I} \to \mathcal{O}_{\operatorname{Spec}(A/I)}$, which is easily seen to be an isomorphism on stalks and hence an isomorphism. In particular, Z is a(n) (affine) scheme. Further, considering global sections shows that the morphism $i : Z \to X$ constructed above corresponds exactly to the natural morphism $\operatorname{Spec}\pi$ arising from $\pi : A \twoheadrightarrow A/I$. ■

Remark 3.6.5.

(a) This statement is *not* true in general for a scheme X if we do not require $\mathcal{I}$ to be a quasicoherent. Here's a standard counterexample: let $X = \mathbf{A}^1_{\mathbf{C}}$ and $U = \mathbf{A}^1_{\mathbf{C}} \smallsetminus \{0\} \subset X$, with $j : U \hookrightarrow X$ the inclusion. Then $j_!\mathcal{O}_U \subset \mathcal{O}_X$ is a sheaf of ideals. Applying the above construction to this sheaf yields $Z = \{0\}$ equipped with the structure sheaf $\mathcal{O}_{X,0} \cong \mathbf{C}[t]_{(t)}$. This Z is certainly a locally ringed space, but not a scheme.

(b) Different quasicoherent ideal sheaves $\mathcal{I}$ can give rise to the same underlying topological space Z (e.g., the sheaves (t) and (t^2) on $X = \mathbf{A}^1_{\mathbf{C}} = \operatorname{Spec}\mathbf{C}[t]$), although the *scheme* Z along with the morphism $\iota : Z \to X$ determines $\mathcal{I}$ uniquely, namely as the kernel of the map of sheaves $\iota^\sharp : \mathcal{O}_X \to \iota_*\mathcal{O}_Z$.

Proposition/Definition 3.6.6 (Closed Embedding)**.** The following conditions on a morphism $\iota : Z \to X$ of schemes are equivalent:

(a) For each open affine $U \subset X$, the preimage $\iota^{-1}(U)$ is affine, and $\mathcal{O}(U) \twoheadrightarrow \mathcal{O}(\iota^{-1}(U))$.

(b) There is an affine open cover $X = \bigcup U_i$ of X such that for each i, the preimage $\iota^{-1}(U_i)$ is affine and $\mathcal{O}(U_i) \twoheadrightarrow \mathcal{O}(\iota^{-1}(U_i))$.

(c) The (underlying continuous) map (of) ι is a topological closed embedding (i.e., a homeomorphism onto a closed subset of X), and further the sheaf morphism $\iota^\sharp : \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is surjective.

(d) There is a quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that the map ι factors as an isomorphism $Z \xrightarrow{\sim} \mathbf{V}(\mathcal{I})$ followed by the inclusion morphism $\mathbf{V}(\mathcal{I}) \hookrightarrow Z$ constructed above.

A morphism ι satisfying these equivalent conditions is said to be a *closed embedding* or a *closed immersion*. In this case, the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ and the isomorphism $Z \to \mathbf{V}(\mathcal{I})$ in (d) are determined uniquely by ι. Further:

(e) Closed immersions are stable under composition and base change, and are affine-local on the target.

Proof.

(a) ⇒ (b) Clear.

(b) ⇒ (c) Topological closed embeddings are a "reasonable class" of continuous maps, and the surjectivity of $\iota^\sharp$ is a local question.

(c) ⇒ (d) The conditions of (c) imply that ι is qcqs; indeed, a topological closed embedding is a quasicompact, and (c) implies that ι is a monomorphism in the category of locally ringed spaces and hence *separated* (so that the property 3.4.6(d) for ι is obvious). By 3.4.7, the sheaf $\iota_*\mathcal{O}_Z$ on X is quasicoherent, and hence so is the kernel $\mathcal{I} := \ker(\iota^\sharp)$ by 3.3.1(e). We claim that this $\mathcal{I}$ works. By the universal property (3.6.2), there is a unique factorization of ι as $Z \xrightarrow{\widetilde{\iota}} \mathbf{V}(\mathcal{I}) \hookrightarrow X$ as a morphism of schemes, and we have to show that $\widetilde{\iota} : Z \to \mathbf{V}(\mathcal{I})$ is an isomorphism. Topologically, since ι and $\mathbf{V}(\mathcal{I}) \hookrightarrow X$ are closed embeddings, so is $\widetilde{\iota}$; it is then a homeomorphism because it is also surjective: the sheaf $\iota_*\mathcal{O}_Z$ is supported exactly on $\iota(Z)$. To check that $\widetilde{\iota}^\sharp : \mathcal{O}_{\mathbf{V}(\mathcal{I})} \to \widetilde{\iota}_*\mathcal{O}_Z$ is an isomorphism, we may check it after pushing forward via the inclusion to X, at which point we have to check that the natural morphism $\mathcal{O}_X/\mathcal{I} \to \iota_*\mathcal{O}_Z$ induced by $\iota^\sharp$ is an isomorphism, but indeed this is true by definition of $\mathcal{I}$ since we assume that $\iota^\sharp$ is surjective as a morphism of sheaves.

(d) ⇒ (a) Follows from (the proof of) 3.6.4.

Affine-locality on the target follows from the equivalence of (a) and (b), stability under composition is clear from (a), and stability under base change follows from 3.1.6. ■

We could also prove (b) ⇒ (a) above directly using 3.1.5; this proof is left to the reader as an exercise.

Definition 3.6.7 (Closed Subschemes)**.** Let X be a scheme and $\iota : Z \to X$ and $\iota' : Z' \to X$ denote two closed embeddings.

(a) We say that Z' *majorizes*[8] Z, written $Z \preceq Z'$, iff there is a morphism $\widetilde{\iota} : Z \to Z'$ such that $\iota = \iota' \circ \widetilde{\iota}$.

By 3.6.2, this happens iff the corresponding ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ satisfy $\mathcal{I}' \subset \mathcal{I}$, and in this case $\widetilde{\iota}$ is uniquely determined and also necessarily a closed embedding.

(b) We say that Z' and Z are *equivalent* if both $Z \preceq Z'$ and $Z' \preceq Z$.

This notion of equivalence is an equivalence relation on closed embeddings (check!).

(c) A *closed subscheme* of X is an equivalence class of closed embeddings mapping to X.

Therefore, closed subschemes of X correspond to quasicoherent ideal sheaves on X, with the correspondence given by taking a quasicoherent ideal sheaf $\mathcal{I}$ to the (equivalence class of) the morphism $\mathbf{V}(\mathcal{I}) \hookrightarrow X$ constructed above.

Remark 3.6.8. On a locally Noetherian scheme (e.g. a scheme locally of finite type over a field k), every quasicoherent ideal sheaf is automatically coherent; therefore, in this case, closed subschemes correspond to coherent ideal sheaves.

[8] Being more pedantic, one could insist on saying that ι' majorizes ι instead, since this notion does depend on the morphisms ι and ι' and not just the abstract isomorphism classes of Z and Z', but we leave this level of pedantry to the pedants.

Chapter 4

Intersection Theory

Following [10]. Throughout (except in the Appendices), we will fix an algebraically closed field k. By a *scheme*, we mean a separated finite-type scheme over k, and by a *variety*, we mean an integral scheme. We will denote the function field of a variety X by $k(X)$. In particular, all schemes are assumed to be Noetherian.

Chapter 5

Appendices

5.1 Generic Freeness

Theorem 5.1.1 (Geometric Nakayama)**.** Let X be a locally Noetherian scheme, $\mathscr{F} \in \mathsf{Coh}(X)$ and $x \in X$. Let $n \in \mathbf{Z}_{\geq 0}$.

(a) Given an epimorphism $\psi(x) : \kappa(x)^{\oplus n} \twoheadrightarrow \mathscr{F}(x)$ of $\kappa(x)$-vector spaces, there is an open subset $V \subset X$ containing x and an $\mathcal{O}_V$-module epimorphism $\psi_V : \mathcal{O}_V^{\oplus n} \twoheadrightarrow \mathscr{F}|_V$ lifting $\psi(x)$. Further, V can be chosen to be affine and such that $\psi(V) : \mathcal{O}_X(V)^{\oplus n} \to \mathscr{F}(V)$ is an $\mathcal{O}_X(V)$-module epimorphism.

(b) Given an isomorphism $\psi_x : \mathcal{O}_{X,x}^{\oplus n} \xrightarrow{\sim} \mathscr{F}_x$ of $\mathcal{O}_{X,x}$-modules, then there is an open subset $W \subset X$ containing x and an isomorphism $\psi_W : \mathcal{O}_W^{\oplus n} \to \mathscr{F}|_W$ of $\mathcal{O}_W$-modules lifting ψ_x.

In particular, $\mathscr{F}$ is flat iff it is locally free of finite rank, i.e. a vector bundle.

Proof.

(a) Pick an open subset $U \subset X$ containing x and elements $a_1, \ldots, a_n \in \mathscr{F}(U)$ such that for $i = 1, \ldots, n$, the map $\psi(x)$ sends the i^{th} basis vector to $a_i(x)$. Without loss of generality, we may assume $U = \operatorname{Spec} A$ for a Noetherian ring A and $\mathscr{F} = \widetilde{M}$ for a finite A-module M; then $\mathscr{F}(U) = M$. Let x correspond to the prime ideal $\mathfrak{p} \subset A$. The hypothesis when combined with Nakayama's Lemma tells us that (the classes of) $a_1, \ldots, a_n$ generate $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module. Let $M' \subset M$ be the submodule of M generated by $a_1, \ldots, a_n$; since M is finite over A, there is an $s \in A \setminus \mathfrak{p}$ such that $sM \subset M'$. It suffices to take $V := D(s)$.

(b) By (the proof of) (a), there is an affine open $V \subset U$ and a lifting $\psi_V : \mathcal{O}_V^{\oplus n} \twoheadrightarrow \mathscr{F}|_V$ of ψ_x to an $\mathcal{O}_V$-module epimorphism. Then $\ker \psi_V \in \mathsf{Coh}(V)$; taking $W := V \setminus \operatorname{supp} \ker \psi_V$ works.

The last statement follows from (b) and Lemma 5.2.1. ∎

Remark 5.1.2. For an abitrary scheme, the correct statement is that vector bundles, i.e. locally free sheaves of finite rank, are exactly the flat modules of finite presentation.

Corollary 5.1.3 (Generic Freeness)**.** Let X be a locally Noetherian integral scheme and $\mathscr{F} \in \mathsf{Coh}(X)$.

(a) Let $\eta \in X$ denote the generic point. If $\mathscr{F}_\eta = 0$, then there is a dense open $U \subset X$ with $\mathscr{F}|_U = 0$.

(b) There is a dense open $U \subset X$ such that $\mathscr{F}|_U$ is locally free of finite rank, i.e. a vector bundle.

5.2 Flatness

Lemma 5.2.1. . Let A be a ring, and M be a finitely presented flat module. (For instance, this holds if A is Noetherian and M finite.) Then M is locally free, i.e. projective.

Proof. Localizing, we may assume that $(A, \mathfrak{m}, k)$ is a local ring; then we have to show that a finitely presented flat A-module M is free. Since M is finitely generated, $k \otimes_A M$ is a finite-dimensional k-vector space, of dimension say $r \in \mathbf{Z}_{\geq 0}$. Pick $m_1, \ldots, m_r \in M$ such that the images of these in $k \otimes_A M$ constitute a k-basis; we show that the corresponding map $A^{\oplus r} \to M$ is an isomorphism. Indeed, it is surjective by another application of Nakayama's Lemma. For injectivity, let I denote the kernel, so that we have a short exact sequence $0 \to I \to A^{\oplus r} \to M \to 0$. Note that I is a finitely generated A-module.[1] Since B is flat, we have $\mathrm{Tor}_1^A(k, B) = 0$, and hence tensoring with k yields the exact sequence $0 \to k \otimes_A I \to k^{\oplus r} \to k \otimes_A B \to 0$. Since the corresponding map $k^{\oplus r} \to k \otimes_A B$ is an injection, we conclude that $k \otimes_A I = 0$, and so by a final application of Nakayama's Lemma we get $I = 0$. ■

Lemma 5.2.2. Let A be a Noetherian domain with fraction field $A \to K$, let B an irreducible Noetherian ring (i.e. a Noetherian ring such that $\gamma := \mathrm{Nil}(B)$ is a prime ideal), and $\varphi : A \hookrightarrow B$ be a finite flat injective morphism. The natural map $K \otimes_A B \cong (A \smallsetminus \{0\})^{-1} B \to B_\gamma$ is an isomorphism of K-vector spaces.

Proof. The idea is that in this setting, the nilpotent elements of $K \otimes_A B$ are precisely the zero-divisors. Injectivity then comes from the fact that the localization map obtained by inverting non-zero-divisors is injective. Surjectivity comes from checking that [TODO]. ■

[1]This is due to the "finitely presented implies always finitely presented" property.

5.3 Counterexamples in Algebraic Geometry

Example 5.3.1. A locally ringed space that is not a scheme. Take X to be a point with local ring that has more than one prime ideal.

Example 5.3.2. Open subscheme of an affine scheme that is not affine.

Example 5.3.3. Scheme with no closed point.

Example 5.3.4. An affine open subscheme of an affine scheme that is not a distinguished affine open. (Elliptic curve - non-torsion)

Example 5.3.5. A non-Noetherian scheme whose underlying space is Noetherian.

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