The Joys of the Atiyah-Singer Index Theorem

A Dissertation Presented by

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to the Department of Mathematics

In partial fulfillment of the requirements
for the degree of
Bachelor of Arts with Honors
in the subject of
Mathematics

Advised by

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Cambridge, Massachusetts

March 2024

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Contents

Abstract 5

Foreword 6

Acknowledgments 7

Conventions and Fundamentals 8

1 Statements **13**

1.1 The Atiyah-Singer Index Theorem 14

1.2 Chern-Gauss-Bonnet and Poincaré-Hopf 20

1.3 The Hirzebruch Signature Theorem 26

1.4 The Hirzebruch-Riemann-Roch Theorem 29

1.5 Integrality of the $\hat{A}$-Genus for Spin Manifolds 36

2 Examples **41**

2.1 Spheres 42

2.1.1 Almost Complex Structures on Spheres 48

2.2 Projective Spaces 58

2.2.1 Complex Projective Spaces 64

2.2.2 Quaternionic Projective Spaces 69

2.3 Grassmannians 80

2.3.1 Grassmannians of Lines 90

2.4 Smooth Complete Intersections 93

2.4.1 Hypersurfaces 100

3 Appendices **103**

3.1 Differential Operators 104

3.2 The Pfaffian and the Euler Class 110

3.3 Multiplicative Sequences 116

3.4 Vector Bundles and Characteristic Classes 122

3.5 Multiplicativity of Genera 133
3.6 Almost Complex Structures on Manifolds 137
3.7 Cohomology of Complex Grassmannians 143
References 147

Abstract

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.

Mathematics and History, Mathematical Intelligencer, v. 4, no. 4,
W. S. Anglin

The Atiyah-Singer Index Theorem, discovered by the geometers Sir Michael Atiyah and Isadore Singer in 1963, is one of the crown jewels of differential geometry of the 20$^{\text{th}}$ century. Roughly speaking, the theorem allows us to study solutions to partial differential equations on manifolds by relating analytic information about their solution spaces to the topology of the underlying manifolds; in other words, it builds a bridge between analysis and topology through geometry. More precisely, the theorem expresses the index of an elliptic differential complex on a closed orientable smooth manifold in terms of characteristic classes and other cohomological data on the manifold. This theorem unifies our understanding of a variety of apparently unrelated results that are central to several subdisciplines of mathematics, including, but not limited to, the Chern-Gauss-Bonnet Theorem, the Hirzebruch Signature Theorem, the Hirzebruch-Riemann-Roch Theorem, and the integrality of the $\hat{A}$-genus for spin manifolds. The Index Theorem and its several cousins, like the Atiyah-Segal-Singer Fixed Point Theorem and the Atiyah-Patodi-Singer Theorem, are some of the most powerful modern techniques used to study the geometry and topology of smooth manifolds. They open avenues connecting differential geometry and topology to several other mathematical disciplines such as algebraic topology, algebraic geometry, mathematical physics, and even combinatorics and number theory.

This thesis is structured roughly in sonata form. In Chapter 1: Statements, we introduce the main characters of the above story, namely the Atiyah-Singer Index Theorem and its various consequences. We do not give complete proofs of these results, but we give detailed sketches of, and references to, several complete proofs. We aim, rather, to narrate the (mathematical) story connecting these results to each other and to various other mathematical gems such as the Lefschetz Fixed Point Theorem, Milnor's exotic 7-sphere, the Riemann-Roch Theorem, and non-smoothable manifolds. Then, in Chapter 2: Examples, we carry out several computations–on (hyper)spheres, projective spaces of various flavors, complex Grassmannians, and complex smooth complete intersection varieties–verifying and illustrating the general theory of the previous chapter. Along the way, we stop to explore several related questions about these manifolds, such as those about the existence of almost complex structures and metrics of positive scalar curvature. Finally, in Chapter 3: Appendices, we return to the original characters and spell out the details of the technical tools used throughout the exposition, such as differential operators, characteristic classes of vector bundles, and the cohomology of Grassmannians. The primary goal of this thesis is to unify and present in modern language several classical results collected from a plethora of textbooks, journals, and websites, and to provide thus an introduction to the breadth of applications–the many joys–of the Atiyah-Singer Index Theorem.

Foreword

> So the whole book consists of using the Atiyah-Singer Index Theorem, one of the deepest and hardest results in mathematics, to prove a series of perfectly elementary identities which can be proved much more easily by direct means.
>
> *Foreword, The Atiyah-Singer Index Theorem and Elementary Number Theory,*
> FRIEDRICH HIRZEBRUCH AND DON ZAGIER

The idea behind this thesis first arose when I stumbled upon the survey article [1]. Titled simply "The Atiyah-Singer Index Theorem" and published in the Bulletin of the American Mathematical Society, this captivating survey was written by Dan Freed, one of the world's renowned experts in index theory. Upon reading it (and even understanding maybe less than half of it), I felt particularly attracted not only to the theorem itself–which connected several of my mathematical interests–but also to Freed's clear and effective narrative style. Imagine, then, my delight when I found out in July of 2023 that Dan Freed would be moving to Harvard as the director of the CMSA starting my senior year! Of course, I had to write my thesis on the Atiyah-Singer Index Theorem.

None of the results in this thesis are mine, or even new–indeed, most of them (particularly the computations in Chapter 2) are very classical results on the geometry of manifolds and varieties, obtained decades, if not centuries, before the Index Theorem. An attempt then, to understand them using the framework of the Index Theorem, may seem like "a rather pointless course requiring justification," to quote Hirzebruch and Zagier once more. Unlike them, however, I do not attempt to give any such justification, other than mentioning that this attempt serves very well the purpose of illustrating the beauty and unity of mathematics, which I find well worth writing about.

This thesis barely scratches the tip of the iceberg of the subject of index theory. Missing from it, in addition to a proof of the Index Theorem, is any treatment of operator theory, K-theory and Bott periodicity, equivariant index theory, Atiyah-Segal-Singer fixed point formulae, or the physical motivation behind index theory–all topics well worth including. The choice of manifolds on which I carry out computations also leaves something to be desired: for instance, the discussion of complex Grassmannians begs to be generalized to one of real, oriented real, Lagrangian, and isotropic Grassmannians, and, in a different direction, to one of flag manifolds, or more generally relative flag manifolds and their towers. The eclectic and uneven treatment–such as assuming more familiarity with complex algebraic geometry and combinatorics than with differential geometry–reflects my own somewhat untraditional background and, of course, biases.

The most interesting results in mathematics are often the ones that admit several different proofs and are connected to a wealth of different results. It is for this reason that I have tried to give multiple proofs of each result I mention, drawing on tools from, and highlighting connections between, differential topology, differential geometry, algebraic topology, algebraic geometry, and combinatorics. In order to keep this thesis at a reasonable size, I have not been able to give every detail in every proof of every result I include–I can at least pretend to deflect some of the blame in this regard towards the breadth of the math involved (while retaining other blame on the incompleteness of my understanding). More than anything, this thesis is an excuse for me to write about and share pieces of math that I find beautiful. I hope that the reader derives as much joy from reading as I did from writing it.

Acknowledgments

A poem is never finished, only abandoned.

PAUL VALERY

I would like to thank, first and foremost, my first and biggest mentor at Harvard, Professor Joe Harris. I was fortunate enough to experience first-hand during my first year at Harvard his infectious enthusiasm for, and love of, math; it was this enthusiasm and love that convinced me to study math in college. From teaching me what a group is, to enabling me to dive into the abysses of algebraic geometry, Harris has always been there for me during my mathematical journey at Harvard. He is somehow both incredibly smart and an epitome of humility–no topic for him is too simple to explain, no question too trivial to patiently answer. In my moments of mathematical and existential crises–I had several of them writing this thesis–he has been my pillar of support. When I spoke to him about the possibility of never finishing the project I had initially intended to write, he redirected to me to find solace in the above quote by Valery, gently reminding me that I have all of my life ahead of me to do math! Some version of the following sentence can surely be found in every undergraduate and graduate thesis advised by him (of which there are many!), and I want to say it too: if I could one day be but half the mathematician, a quarter the guide, an eighth the teacher, a sixteenth the human being, etc., that he is, then I would consider myself a wholly successful mathematician.

Of course, deep gratitude is due also to Professor Dan Freed, whom I deeply admire. Ever since our first meeting less than eight months ago, he has pushed me to grow as a thinker and mathematician. He has taught me to be skeptical, and to think calmly and deeply about the simplest and most difficult of mathematical concepts. In addition to possessing an encyclopaedic knowledge of both vast fields of mathematics and their history, Freed is also a very gifted lecturer–which is something I am both very grateful for, and very inspired by. Learning from Harris and Freed, I have grown more in this past year–as a student, thinker, and as a human being–than I have ever in my life.

I would also like to thank Profs. Denis Auroux, Laura DeMarco, Noam Elkies, Mike Hopkins, Curt McMullen, Mihnea Popa, and Lauren Williams, from whom, in addition to Harris and Freed, I have learned most of the (very little) mathematics I know. I am also very grateful to Joshua Wang–it was taking several courses taught by him that sparked in me a love for all things topological and geometric. I want to thank my postdoc friends–Nathan Chen, Chris Eur, Roderic Guigo Corominas and Aaron Landesman–and graduate student friends–Amanda Burcroff, Sanath Devalapurkar, Benjy Firester, Grant Barkley, Keeley Hoek, Ollie Thakar, and Natalie Stewart–for the several illuminating conversations about mathematics (and life) that we have had. Particular thanks are due also to my closest friends–Hari Iyer, Kaiying Hou, Hahn Lheem, and Dora Woodruff–for their companionship in my mathematical journey, and to my suitemates–Jesse Hernandez, Andy Kim, and Ike Park–for both tolerating and incessantly supporting my thesising.

I thank Mrs.Komal Ghadigaonkar–Komal Ma'am–for introducing me to abstract mathematics. Finally, and most importantly, I wish to thank my family–my Dadima, Mumma, Papa, Gunna, Varun Bhaiya, and Simba–to whom I owe my very being, and without whose undying love and support, I would not have made it this far. I love you all very much!

Conventions and Fundamentals

The following notations and conventions will be adopted throughout the article. We[1] also use this opportunity to review standard definitions and results that will be used throughout the article without further comment.

Rings and Ringed Spaces

- All rings are commutative and unitary. We do not disallow the zero ring.
- Given a ring R and a power series $Q(z) \in R[\![z]\!]$, we denote for each integer $n \geq 0$ the coefficient of z^n in $Q(z)$ by $[z^n]Q(z)$.
- A semiring S is an algebraic object that satisfies all axioms in the definition of a ring, except for the existence of additive inverses, so that $(S, +)$ and $(S, \times)$ are both monoids, with the two operations $+$ and $\times$ related by the distributive law.
- Let R be a ring. An R-ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X equipped with a sheaf of R-algebras. A $\mathbb{Z}$-ringed space is simply called a ringed space.
- To generalize the previous notion slightly, by an $\mathscr{R}$-ringed space, we mean a triple $(X, \mathscr{R}, \mathcal{O}_X)$, where X is a topological space, $\mathscr{R}$ a sheaf of rings on X, and $\mathcal{O}_X$ a sheaf of $\mathscr{R}$-algebras on X.[2] The sheaf $\mathcal{O}_X$ is called the structure sheaf of X. Often, we will supress $\mathscr{R}$ and $\mathcal{O}_X$, and call X the $\mathscr{R}$-ringed space.

Partitions

- A partition λ is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_k)$ of some size $k \geq 0$, i.e. of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. We call the integers λ_j the parts of λ. We call their sum $\lambda_1 + \cdots + \lambda_k$ the size of λ and denote it by $|\lambda|$. Finally, we call the number of nonzero λ_j the length of λ and denote it by $\ell(\Lambda)$. If $n = |\lambda|$ is the size of λ, then we also write $\lambda \vdash n$ and call λ a partition of n. It is often helpful to add or delete as many zero parts as needed.
- The lexicographic ordering on partitions is the total order on all partitions, given by saying $\lambda > \mu$ if $\lambda_j > \mu_j$ when $j \geq 1$ is the first index at which λ and μ differ.
- Given a partition $\lambda \vdash n$ of $n \geq 0$, we also denote λ as $\lambda = (n^{i_n}, (n-1)^{i_{n-1}}, \dots, 1^{i_1})$, where for $1 \leq j \leq n$, we set $i_j := \#\{s : \lambda_s = j\}$ to be the number of times j appears in λ. Therefore, i_j are nonnegative integers satisfying $\sum_{j=1}^n j i_j = n$.
- Given a partition λ, the partition obtained by flipping the Ferrers diagram across the principal diagonal is called the conjugate partition of λ, and denoted λ^*. So, for instance, for any $k \geq 0$, the partition (j) has Ferrers diagram a single row of j boxes, whereas $(j)^* = (1^j)$ denotes the partition $(1, \dots, 1)$, with 1 appearing j times, which has Ferrers diagram a single column of j boxes. Note that for any integers $a, b \geq 0$, a partition λ has at most a parts of size at most b iff the Ferrers diagram of λ fits in the rectangle with a rows and b columns; we denote this by writing $\lambda \subset a \times b$.

[1] One convention will be the usage of the formal academic "we". All the writing and figures are mine, unless explicitly noted.

[2] Therefore, an R-ringed space $(X, \mathcal{O}_X)$ is an R_X-ringed space $(X, R_X, \mathcal{O}_X)$, where R_X is the constant sheaf with values in R.

Symmetric Functions

- Given an integer $n \geq 1$ and variables $x := x_1, \ldots, x_n$, we define the ring of the symmetric polynomials in x to be the subring $\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ of $\mathbb{Z}[x_1, \ldots, x_n]$ fixed under the natural action of the symmetric group $\mathcal{S}_n$. This is naturally a graded ring $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$, where Λ_n^k consists of symmetric polynomials of degree k.
- Given a partition λ with $\ell(\lambda) \leq n$, we define the monomial symmetric function in the variables x of type λ, denoted $m_\lambda(x)$, to be the unique polynomial Λ_n with the smallest number of nonzero coefficients that contains the monomial $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.
- Given an integer $0 \leq j \leq n$, we define the elementary, resp. complete, symmetric polynomial of degree j in x, written $e_j(x)$, resp. $h_j(x)$, by

  $$e_j(x) := m_{(1^j)}(x), \text{ resp. } h_j(x) := \sum_{\lambda \vdash j} m_\lambda(x).$$

  Given a partition λ with parts at most n, we also define the elementary, resp. complete, symmetric function in the variables x of type λ, written $e_\lambda(x)$, resp. $h_\lambda(x)$, to be the product

  $$e_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(x), \text{ resp. } h_\lambda(x) := \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}(x).$$

  The Fundamental Theorem of Symmetric Polynomials says that the ring Λ_n is given by

  $$\Lambda_n = \mathbb{Z}[e_1(x), \ldots, e_n(x)] = \mathbb{Z}[h_1(x), \ldots, h_n(x)],$$

  i.e. is a polynomial ring generated by the $e_j(x)$ or $h_j(x)$.
- Given any integer $j \geq 0$, we define the power sum of degree j in x, written $p_j(x)$ by

  $$p_j(x) = m_{(j)}(x).$$

- For each $n \geq 1$, there is a natural graded epimorphism $\Lambda_{n+1} \to \Lambda_n$ given by setting $x_{n+1} = 0$. We define the group of symmetric polynomials of degree k in countably many variables as the projective limit

  $$\Lambda^k := \varprojlim \Lambda_n^k$$

  of the degree k part of this system, and define the ring of symmetric polynomials in countably many variables as the graded ring given by

  $$\Lambda := \bigoplus_{k \geq 0} \Lambda^k,$$

  where multiplication comes from the structures in Λ_n. This often also called simply the ring of all symmetric polynomials. For each partition λ, there is a unique element $m_\lambda \in \Lambda$ that projects under the natural map $\Lambda \to \Lambda_n$ to $m_\lambda(x_n)$ for all $n \geq \ell(\lambda)$; call this element m_λ. Define e_λ, h_λ, and p_j similarly. Then Λ is generated as a $\mathbb{Z}$-algebra by the sequence $e_1, e_2, \ldots$ (resp. $h_1, h_2, \ldots$), and as an abelian group by the sequence e_λ (resp. h_λ) as λ ranges over all partitions.

- For any ring R, let $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$ be the corresponding ring of symmetric polynomials with coefficients in R. Note that the Newton's Identities are the universal identities in Λ, which for each $n \geq 1$ relate the elementary symmetric and power sum polynomials via

$$p_n = (-1)^{n-1} n e_n + \sum_{i=1}^{n-1} (-1)^{n-1+i} e_{n-i} p_i. \tag{1}$$

These identities imply that $\Lambda_{\mathbb{Q}}$ is generated as a $\mathbb{Q}$-algebra also by $p_1, p_2, \ldots$.

Topology

- We say that a topological space X is of finite type if its total singular cohomology group with integral coefficients $\mathrm{H}^*(X;\mathbb{Z})$ is a finitely generated abelian group.[3] Note that all compact manifolds (possibly with boundary) and finite CW complexes are of finite type, as are all spaces homotopy equivalent to these, by the homotopy invariance of singular cohomology. In this case, we define for $i \geq 0$ the i^{th} Betti number of X to be

$$b_i(X) = \operatorname{rank}_{\mathbb{Z}} \mathrm{H}^i(X;\mathbb{Z}) = \dim_{\mathbb{Q}} \mathrm{H}^i(X;\mathbb{Q});$$

note that $b_i(X) = 0$ for all but finitely many i. More generally, given any field k, we define the i^{th} Betti number of X with coefficients in k by

$$b_i(X;k) = \dim_k \mathrm{H}^i(X;k);$$

note that $b_i(X;k) = b_i(X)$ whenever k has characteristic 0. We then call the generating function of the Betti numbers the k-Poincaré polynomial of X and denote it by $p_t(X;k)$, so that

$$p_t(X;k) := \sum_{i=0}^{\infty} b_i(X;k) t^i \in \mathbb{Z}[t].$$

The $\mathbb{Q}$-Poincaré polynomial is denoted by $p_t(X)$ and is called the Poincaré polynomial of X. The Universal Coefficient Theorem tells us that the quantity obtained by evaluating $p_t(X;k)$ at $t = -1$ is independent of the choice of k; this quantity is called the Euler characteristic of X. In other words,

$$\chi(X) = p_{-1}(X;k) = \sum_{i=0}^{\infty} (-1)^i b_i(X;k)$$

for any field k. By a standard theorem in algebraic topology, if X is given a finite CW structure with c_i cells of dimension i, then we have also

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i c_i.$$

[3] The Universal Coefficient Theorem tells us that we get the same definition if we ask this of the homology or cohomology groups.

- Since the quaternions $\mathbb{H}$ are not commutative, but form a skew-field, we need to be careful about our definition of right and left vector spaces over $\mathbb{H}$. We will take a vector space over $\mathbb{H}$ to mean a right $\mathbb{H}$-module, i.e. to say that scalar multiplication is done on the right. So, for instance, subspaces are defined to be sub-right-modules, and $\mathbb{H}^n$ is the space of column vectors of size n with right multiplication, on which $n \times n$-matrices with values in $\mathbb{H}$ act on the left as $\mathbb{H}$-linear maps. With this convention, we can define uniformly for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and integer $n \geq 0$ the projective space of dimension n over $\mathbb{F}$, denoted $\mathbb{FP}^n$, to be the quotient

$$\mathbb{FP}^n := \{v \in \mathbb{F}^{n+1} \setminus \{0\}\} / \left(v \sim v\lambda \text{ for all } v \in \mathbb{F}^{n+1} \setminus \{0\}, \lambda \in \mathbb{F}^\times = \mathbb{F} \setminus \{0\}\right).$$

- Let $n \geq 0$ and let X be a connected, closed, orientable topological n-manifold. Then standard algebraic topology of manifolds asserts that the top homology group $\mathrm{H}_n(X;\mathbb{Z})$ of X is isomorphic to $\mathbb{Z}$. A choice of orientation corresponds to a choice of isomorphism, i.e. if X is oriented, then we have a canonical generator $[X]$ of $\mathrm{H}_n(X;\mathbb{Z})$ called the fundamental class of X. Poincaré duality then asserts that, in this case, the map

$$\mathscr{D} : \mathrm{H}^n(X;\mathbb{Z}) \to \mathrm{H}_0(X;\mathbb{Z}) = \mathbb{Z}[*] \cong \mathbb{Z}, \quad \eta \mapsto [X] \frown \eta$$

is then an isomorphism, where $\frown$ denotes the cap product operation and $[*]$ is the class of a point. In particular, there is a unique distinguished generator in the top cohomology group, $\eta_X \in \mathrm{H}^n(X;\mathbb{Z})$, which is the Poincaré dual to a point and algebraic dual to the fundamental class, i.e. which satisfies

$$[X] \frown \eta_X = 1.$$

Of course, the same remarks hold if we replace $\mathbb{Z}$ by an arbitrary coefficient ring R and the word "orientable" by "R-orientable".[4] Note that when X is a smooth manifold, taking R to be $\mathbb{R}$ or $\mathbb{C}$ and identifying $\mathrm{H}^*(X;R) \cong \mathrm{H}^*_{\mathrm{dR}}(X;R)$ with the de Rham cohomology of X with R-coefficients using the de Rham theorem, the map $\mathscr{D}$ can be given by

$$\mathscr{D} : \mathrm{H}^n_{\mathrm{dR}}(X;R) \to R, \quad [\omega] \mapsto \int_X \omega,$$

where ω is an n-form on X, and $\int_X$ denotes integration on X, an operation that is well-defined on cohomology classes by Stokes' theorem. For this reason, for any coefficient ring R and even when X is simply a topological manifold, the map $\mathscr{D}$ is sometimes also written as

$$\mathscr{D}(\eta) = \int_X \eta,$$

even though integration doesn't really make sense for topological manifolds or when R is, say, a finite field. We will also follow this convention without further comment.

- In addition to $\mathcal{C}^r$-manifolds for $r = 0, 1, 2, \ldots, \infty$, we say a real analytic manifold is smooth of class $\mathcal{C}^\omega$, and a complex manifold is smooth of class $\mathcal{C}^{\mathrm{hol}}$. In this article, we will work only with topological (i.e. $\mathcal{C}^0$), smooth (i.e. $\mathcal{C}^\infty$), and complex (i.e. $\mathcal{C}^{\mathrm{hol}}$) manifolds.

[4] Note, however, that there are really only two kinds of orientability: a $\mathbb{Z}$-orientable manifold is R-orientable for all R, while a $\mathbb{Z}$-nonorientable manifold is R-orientable iff R is an $\mathbb{F}_2$-algebra.

- When X is a smooth manifold, we denote the tangent bundle to X by $\mathrm{T}X$, which is real vector bundle of rank $\dim_{\mathbb{R}} X$. When X is a complex manifold, we denote its holomorphic tangent bundle by $\mathscr{T}X$, which is a complex vector bundle of rank $\dim_{\mathbb{C}} X$. In this case, the underlying space of X is also a smooth (in fact real analytic) manifold of dimension $2n$, and the relationship between the real tangent bundle $\mathrm{T}X$ and the holomorphic tangent bundle $\mathscr{T}X$ is that, as real vector bundles of rank $2n$, we have
$$\mathrm{T}X \cong \mathscr{T}X_{\mathbb{R}},$$
where the subscript $\mathbb{R}$ denotes restriction of scalars to the reals, i.e. considering the underlying real vector bundle. Conversely, the complexification $\mathrm{T}X_{\mathbb{C}}$, a complex vector bundle of rank $2n$, decomposes as
$$\mathrm{T}X_{\mathbb{C}} = \mathscr{T}X \oplus \overline{\mathscr{T}X},$$
where $\overline{\mathscr{T}X}$ is the complex conjugate bundle to $\mathscr{T}X$.
- If X is a compact complex manifold and $E \to X$ a holomorphic vector bundle, the sheaf cohomology groups $\mathrm{H}^*(X, E)$ are finite dimensional (see Example 1.1.8 and Theorem 1.1.4), and we define for each $j \geq 0$ the quantity $h^j(X, E)$ as
$$h^j(X, E) := \dim_{\mathbb{C}} \mathrm{H}^j(X, E).$$
The holomorphic Euler characteristic $\chi(X, E)$ of E is then defined to be
$$\chi(X, E) = \sum_j (-1)^j h^j(X, E).$$
- Throughout the article, we will use non-calligraphic font (e.g. $E \to X$) to denote the total space of a vector bundle, and calligraphic font (e.g. $\mathcal{E} \to X$) to denote the corresponding locally free sheaf of sections, moving freely between these notions as needed. For instance, the trivial complex vector bundle on a complex manifold X will be denoted either by $\underline{\mathbb{C}}$ or $\mathcal{O}_X$, according to the conventions of the discipline (differential geometry or algebraic geometry) we are talking about. See Appendix 3.4 for a discussion of this choice.

Chapter 1

Statements

In this chapter, we precisely state the Atiyah-Singer Index Theorem and its several corollaries, namely the Chern-Gauss-Bonnet Theorem, the (generalized) Hirzebruch-Riemann-Roch Theorem, the Hirzebruch Signature Theorem, and the integrality of the $\hat{A}$-genus for spin manifolds. We give only sketches, and precise references to, a few different proofs of each result.

In addition to the topics mentioned in Conventions and Fundamentals, as well as those reviewed in the appendices, we will assume familiarity with the basics of K-theory, as can be found in, say, [2, Chapter I, §9] and the references therein.

1.1 The Atiyah-Singer Index Theorem

We can associate to each differential operator $D : E \to F$ of order $k \geq 0$ between vector bundles E, F on a manifold X its symbol $\sigma(D)$ (see Appendix 3.1). If $\pi : \mathrm{T}^\vee X \to X$ is the cotangent bundle of X, then the symbol $\sigma(D)$ of D is a vector bundle morphism

$$\sigma(D) : \pi^* E \to \pi^* F$$

such that the underlying map $(x, \xi, v) \mapsto \sigma(D)(x, \xi, v)$ is locally a homogenous polynomial of degree k in ξ, where $x \in X$ is a point, $\xi \in \mathrm{T}_x^\vee X$ is a cotangent vector, and $v \in E(x)$. One of the central definitions in the theory of differential operators is then

Definition 1.1.1. Let X be a smooth manifold.

(a) A differential complex $E^\bullet$ on X is a sequence

$$\cdots \to E^{j-1} \xrightarrow{\partial^{j-1}} E^j \xrightarrow{\partial^j} E^{j+1} \to \cdots$$

of smooth vector bundles $E^j \to X$ and differential operators $\partial^j : E^j \to E^{j+1}$ with $\partial^2 = 0$, i.e. $\partial^j \circ \partial^{j-1} = 0$ for all j, and $E^j = 0$ for all but finitely many j.

(b) We say that a differential complex $E^\bullet$ is an elliptic complex, or simply elliptic, if its symbol complex, i.e. the vector bundle complex

$$\cdots \to \pi^* E^{j-1} \xrightarrow{\sigma(\partial^{j-1})} \pi^* E^j \xrightarrow{\sigma(\partial^j)} \pi^* E^{j+1} \to \cdots$$

on the cotangent bundle $\mathrm{T}^\vee X$ is exact outside of the zero section $X \subset \mathrm{T}^\vee X$. In other words, $E^\bullet$ is elliptic if for each $x \in X$ and $0 \neq \xi \in \mathrm{T}_x^\vee X$, the sequence

$$\cdots \to E^{j-1}(x) \xrightarrow{\sigma(\partial^{j-1},\xi)} E^j(x) \xrightarrow{\sigma(\partial^j,x)} E^{j+1}(x) \to \cdots$$

of vector spaces and linear maps is exact.

Note that E^j could be real or complex vector bundles; it does not matter for the definition, although we will soon specialize to the case of complex vector bundles.

Example 1.1.2. The data of a complex $E^\bullet$ supported in two consecutive degrees, say $j = 0, 1$, is a single differential operator $D : E^0 \to E^1$, and then $E^\bullet$ (i.e. D) is elliptic iff for each $x \in X$ and $0 \neq \xi \in \mathrm{T}_x^\vee X$, the map $\sigma(D, \xi) : E^0(x) \to E^1(x)$ is an isomorphism.

Example 1.1.3. Let $X = U \subset \mathbb{R}^2$ be a domain with coordinates x, y and cotangent coordinates $\xi\, \mathrm{d}x + \eta\, \mathrm{d}y$. Let $E^0 = E^1 = \underline{\mathbb{R}}$ be the trivial vector bundles on X. A differential operator $D : \Gamma(U, \mathbb{R}) \to \Gamma(U, \mathbb{R})$ of order 2 can be given as

$$D = a\, \partial_x^2 + 2h\, \partial_x \partial_y + b\, \partial_y^2 + 2e\, \partial_x + 2f\, \partial_y + c$$

for some smooth functions a, b, c, e, f, h on U. Then the ellipticity of the complex $\underline{\mathbb{R}} \xrightarrow{D} \underline{\mathbb{R}}$ is equivalent to the inequality $h^2 < ab$ on U, which amounts to saying that the equation

$$a\xi^2 + 2h\xi\eta + b\eta^2 + 2e\xi + 2f\eta + c = 0$$

defines an ellipse in the (ξ, η)-plane $\mathrm{T}_{(x,y)}U$ at each $(x, y) \in U$. This is the origin of the nomenclature “elliptic”.

The fundamental result in the theory of elliptic partial differential equations is the finite dimensionality of the space of solutions. In the above terminology, this can be expressed as follows. Let $E^\bullet$ be a differential complex on a manifold X, so that in particular, the sequence of spaces of global sections

$$\cdots \to \Gamma(X, E^{j-1}) \xrightarrow{\partial^{j-1}} \Gamma(X, E^j) \xrightarrow{\partial^j} \Gamma(X, E^{j+1}) \to \cdots$$

is a complex of vector spaces. With this, we define the cohomology groups $\mathrm{H}^j(E^\bullet)$ of E as

$$\mathrm{H}^j(E^\bullet) := \frac{\ker \partial^j : \Gamma(X, E^j) \to \Gamma(X, E^{j+1})}{\operatorname{im} \partial^{j-1} : \Gamma(X, E^{j-1}) \to \Gamma(X, E^j)}.$$

Note that each $\mathrm{H}^j(E^\bullet)$ is a vector space, real or complex as $E^\bullet$ is. Then this fundamental result can be stated as

Theorem 1.1.4 (Finiteness)**.** Let $E^\bullet$ be an elliptic differential complex on a compact manifold X. Then for each j, the cohomology space $\mathrm{H}^j(E^\bullet)$ is finite dimensional.

Proof. This is usually proven using a fair bit of machinery from analysis, including elliptic regularity applied to harmonic sections in Sobolev spaces of sections of E^j–this is the same machinery that proves, for instance, the Hodge theorem as a special case. For complete proofs, see [3, Theorem 2.6] and [4, Chapter 0, §6]. ■

In light of Theorem 1.1.4, we may make

Definition 1.1.5. Let $E^\bullet$ be a differential complex on a compact manifold X. Then we define the index of $E^\bullet$, denoted $\chi(E^\bullet)$ or $\operatorname{ind}(E^\bullet)$, to be

$$\chi(E^\bullet) = \operatorname{ind}(E^\bullet) := \sum_j (-1)^j \dim \mathrm{H}^j(E^\bullet).$$

Example 1.1.6. In the simplest case of a single elliptic operator $D : E^0 \to E^1$ between vector bundles on a compact manifold X, the above theorem is saying that the kernel and cokernel of $D : \Gamma(X, E^0) \to \Gamma(X, E^1)$ are finite-dimensional, and the index of D is

$$\operatorname{ind} D = \dim \ker D - \dim \operatorname{coker} D.$$

Example 1.1.7. The de Rham complex on a smooth manifold is the differential complex $E^\bullet$ given by taking $E^j := \Lambda^j \mathrm{T}^\vee X$ and $\partial^j : \Lambda^j \mathrm{T}^\vee X \to \Lambda^{j+1} \mathrm{T}^\vee X$ to be the de Rham differential. This complex is often denoted simply as $\Lambda^\bullet \mathrm{T}^\vee X$. In this case, for $(x, \xi) \in \mathrm{T}^\vee X$, the symbol map

$$\sigma(\partial^j, \xi) : \Lambda^j \mathrm{T}^\vee_x X \to \Lambda^{j+1} \mathrm{T}^\vee_x X \text{ is given by } \omega \mapsto \xi \wedge \omega,$$

so that $\Lambda^\bullet \mathrm{T}^\vee X$ is, in particular, elliptic. In this case, the j^{th} cohomology group of this complex, $\mathrm{H}^j(\Lambda^\bullet \mathrm{T}^\vee X) =: \mathrm{H}^j_{\mathrm{dR}}(X)$, is by definition the j^{th} de Rham cohomology group of X. Theorem 1.1.4 therefore implies that the de Rham cohomology groups of a compact manifold are finite dimensional. In this case, the de Rham theorem then tells us that the index $\chi(\Lambda^\bullet \mathrm{T}^\vee X)$ of the de Rham complex is simply the Euler characteristic $\chi(X)$ of X, explaining the notation.

Example 1.1.8. Let X be a complex manifold and $E \to X$ be a holomorphic vector bundle. Then the Dolbeaux complex of E is the differential complex $E^\bullet = (\mathcal{A}^{0,\bullet} \otimes E, \overline{\partial}_E)$ given by taking $E^j := \mathcal{A}^{0,j} \otimes_{C^\infty(-,\mathbb{C})} E$, where $\mathcal{A}^{p,q}$ denotes the sheaf of complex differential forms of type (p,q) on X, and taking $\overline{\partial}_E : E^j \to E^{j+1}$ to be the $\overline{\partial}$-operator corresponding to E. In this case, for $(x,\xi) \in \mathrm{T}^\vee X \cong (\mathscr{T}^\vee X)_\mathbb{R}$, the symbol map

$$\sigma(\partial^j, \xi) : \Lambda_x^{0,j} X \to \Lambda_x^{0,j+1} X \text{ is given by } \omega \mapsto \mathrm{i}\xi^{0,1} \wedge \omega,$$

so that again, $E^\bullet$ is elliptic. In this case, the j^{th} cohomology group of this complex is called the j^{th} Dolbeaux cohomology group of E, and is denoted by $\mathrm{H}^j_{\overline{\partial}}(X,E)$. Theorem 1.1.4 says in this case that the Dolbeoux cohomology groups of a holomorphic vector bundle on a compact complex manifold are finite dimensional. In this case, the index of the Dolbeaux complex associated to E is called the holomorphic Euler characteristic of E and is denoted by $\chi(X,E)$.

Note that if we temporarily write E^{hol} to emphasize the holomorphic structure on E and denote by E^∞ the corresponding smooth complex vector bundle, then the $\overline{\partial}$-Poincaré Lemma says exactly that the sequence

$$0 \to E^{\text{hol}} \to E^\infty \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1} \otimes E^\infty \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{0,n} \otimes E^\infty \to 0,$$

where $n := \dim_\mathbb{C} X$, is a soft resolution of E^{hol}. Therefore, from the general machinery of soft resolutions, we conclude that there is an isomorphism

$$\mathrm{H}^*(X, E^{\text{hol}}) \xrightarrow{\sim} \mathrm{H}^*_{\overline{\partial}}(X,E)$$

between the sheaf cohomology of X with coefficients in E to the Dolbeaux cohomology of E–this is the Dolbeaux Theorem, the complex analog of the de Rham Theorem. Theorem 1.1.4 then tells us that the sheaf cohomology groups of a vector bundle on a compact complex manifold are finite dimensional. This is the differential geometric analog of the theorem in algebraic geometry which ensures the coherence of higher pushforwards of coherent sheaves along proper morphisms of Noetherian schemes; see [5, Theorem 8.8(b)] and the references there.

Roughly speaking, the Atiyah-Singer Index Theorem says that that the index of an elliptic complex on a compact manifold can be computed with the help of cohomological data on it. To state this we first recall the definition of the compactly supported K-theory of a locally compact paracompact Hausdorff space. Recall that if $F^\bullet = (F^j, \partial^j)_j$ is a complex of vector bundles on a space Y[1], then we define its support to be

$$\operatorname{supp}(F^\bullet) := \{y \in Y : \cdots \to F^j(y) \xrightarrow{\partial^j} F^{j+1}(y) \to \cdots \text{ is exact}\}.$$

By the upper semicontinuity of the kernel dimension of a vector bundle morphism, this is a closed subset of Y. We say that the complex $F^\bullet$ is compactly supported if $\operatorname{supp}(F^\bullet)$ is compact. Now given a locally compact paracompact Hausdorff space Y, we consider the set $L_c(Y)$ of isomorphism classes of compactly supported complexes of vector bundles on Y. In $L_c(Y)$, a complex $F^\bullet$ is said to be elementary iff it is supported in only two consecutive degrees, say i and $i+1$, and the map $\partial^i : F^i \to F^{i+1}$ is an isomorphism of

[1] This is, as opposed to a differential complex, so that the maps $\partial^j : F^j \to F^{j+1}$ are vector bundle homomorphisms (i.e. $\partial^j : \mathcal{F}^j \to \mathcal{F}^{j+1}$ are $\mathcal{O}_X$-module homomorphisms).

vector bundles. There is an operation of direction sum on $L_c(Y)$, and we now define an equivalence relation on $L_c(Y)$ by saying that two complexes $F^\bullet$ and $G^\bullet$ are equivalent, written $F^\bullet \sim G^\bullet$, if there are elementary complexes $S_1, \ldots, S_r, T_1, \ldots, T_s \in L_c(Y)$ such that

$$F \oplus S_1 \oplus \cdots \oplus S_r \cong G \oplus T_1 \oplus \cdots \oplus T_s. \tag{1.1}$$

Definition 1.1.9. Given a locally compact, paracompact Hausdorff space Y, we define the compactly supported K-theory of Y to be

$$K_c(Y) := L_c(Y)/\sim$$

the set of equivalence classes of compactly supported complexes of complex vector bundles under the equivalence relation (1.1).

On the equivalence of this definition to the more traditional definition of $K_c(Y) = \tilde{K}(Y^+)$ of the compactly supported K-theory as the reduced K-theory of the one point compactification Y^+, see [6]. We remark only that if Y is already compact, then any complex on Y has compact support, and the isomorphism with the usual definition of K-theory (as the Grothedieck ring of the semiring $\mathrm{Vect}(Y)$ of complex vector bundles on Y; see Appendix 3.4) is given simply by taking the class in $K_c(Y) = L_c(Y)/\sim$ of a complex $E^\bullet \in L_c(Y)$ to the element

$$\sum_j (-1)^j [E^j] \in K(Y).$$

In particular, this class, at least for compact Y, does not depend on the morphisms $\partial^j : E^j \to E^{j+1}$ in the complex at all. Finally, just as in ordinary K-theory, for nice spaces Y there is a Chern character homomorphism

$$\mathrm{ch} : K_c(Y) \to \mathrm{H}^*_c(Y; \mathbb{Q}),$$

where $\mathrm{H}^*_c(Y;\mathbb{Q})$ denotes the compactly supported (say) singular cohomology of Y with rational coefficients. Again, this Chern character is natural with respect to pullbacks along proper morphisms.

Given the above set-up, let's return to the case of a differential complex $E^\bullet$ on a smooth manifold X. In this case, if we take $Y := \mathrm{T}^\vee X$ to be (the total space of) thecotangent bundle of X, then the support of the elliptic complex $(\pi^* E^\bullet, \sigma(\partial))$ on Y contains at least the zero section $X \subset \mathrm{T}^\vee X$ (at least when all differential operators have positive order), and saying that $E^\bullet$ is elliptic is equivalent to saying that the symbol complex is supported exactly on the zero section. In particular, if $E^\bullet$ is elliptic and X is compact, then the symbol complex is compactly supported, and hence defines a class in the compactly supported K-theory of $\mathrm{T}^\vee X$, which we will call the symbol class of $E^\bullet$ and denote by

$$\sigma(E^\bullet) \in K_c(\mathrm{T}^\vee X).$$

In particular, taking its Chern character then defines an element

$$\mathrm{ch}\,\sigma(E^\bullet) \in \mathrm{H}^*_c(\mathrm{T}^\vee X; \mathbb{Q}).$$

With this notation, we can state the eponymous

Theorem 1.1.10 (Atiyah-Singer)**.** Let X be a closed oriented smooth n-manifold, where $n \geq 1$, and let $E^\bullet$ be an elliptic complex on X. Then the index of $E^\bullet$ is given by

$$\chi(E^\bullet) = (-1)^{n(n+1)/2} \int_X \Theta_{\mathrm{T}^\vee X}^{-1} \operatorname{ch} \sigma(E^\bullet) \cdot \mathsf{Td}(\mathrm{T}X_{\mathbb{C}}),$$

where $\Theta_{\mathrm{T}^\vee X} : \mathrm{H}^*(X;\mathbb{Q}) \to \mathrm{H}^*_c(\mathrm{T}^\vee X;\mathbb{Q})$ is the Thom isomorphism in cohomology and the class $\mathsf{Td}(\mathrm{T}X_{\mathbb{C}}) \in \mathrm{H}^*(X;\mathbb{Q})$ is the total Todd class of the complexified tangent bundle $\mathrm{T}X_{\mathbb{C}}$ of X.

The Thom Isomorphism Theorem is reviewed in Appendix 3.2.1 and the definition of the total Todd class of a complex vector bundle is reviewed in Appendix 3.4. It is not clear at all that the quantity on the right, a priori only a rational number, is even an integer; the theorem implies that it is, and indeed this gives us a way to prove a series of integrality results for genera on manifolds (see, for instance, §1.5). Note how remarkable this theorem is–it expresses deep analytic information about the spaces of solutions to partial differential equations, namely $\chi(E^\bullet)$, in terms of the topology of the underlying space on which we have formulated these differential equations.

Proof. We refer the reader to the survey article [1] for a history of the theorem and ideas behind several proofs of it–using cobordism, K-theory and pseudodifferential operators, and the heat equation–and to the textbooks [2] and [7] for complete expositions. ■

Instead of giving the proof, we now emphasize two qualitative results that can be derived already from Theorem 1.1.10. The first observation is

Proposition 1.1.11. Let X be a closed oriented smooth n-manifold with n odd. If $E^\bullet$ is any elliptic complex on X, then the index

$$\chi(E^\bullet) = 0.$$

Proof Sketch. If X is an odd-dimensional manifold, then there is an orientation-reversing involution on $\mathrm{T}X \cong \mathrm{T}^\vee X$ given by $(x, \xi) \mapsto (x, -\xi)$. Using this involution, one can show that the formula on the right side of Theorem 1.1.10 gives you something equal to its own negative, so the result follows from Theorem 1.1.10. See [2, Ch. III, Thm. 13.12]. ■

This proposition, combined with Example 1.1.7, implies that the Euler characteristic $\chi(X)$ of a closed, odd-dimensional manifold is zero, something which is often proven as a consequence of Poincaré duality. The second observation is that the quantity on the right side in Theorem 1.1.10 can in a lot of cases of interest be simplified via

Lemma 1.1.12. Let $F \to X$ be a vector bundle over a compact space X, and let $\iota : X \hookrightarrow F$ denote the inclusion of the zero section. Then for any $\gamma \in \mathrm{H}^*_c(F;\mathbb{Q})$, we have that

$$\iota^*\gamma = \mathrm{e}(F) \smile \Theta_F^{-1}\gamma,$$

where $\Theta_E : \mathrm{H}^*(X;\mathbb{Q}) \to \mathrm{H}^*_c(F;\mathbb{Q})$ is the Thom isomorphism, and $\iota^* : \mathrm{H}^*_c(F;\mathbb{Q}) \to \mathrm{H}^*(X;\mathbb{Q})$ is the restriction to the zero section.

Proof. Write $\gamma = \Theta_F(\eta)$ for some $\eta \in \mathrm{H}^*(X;\mathbb{Q})$ using the Thom Isomorphism Theorem, and then use the formula that

$$\Theta_F(\eta) = \Theta_F \smile \pi^*\eta,$$

where Θ_F on the right side is the Thom class (see Appendix 3.2). Pulling back both sides via ι^* and using the push-pull formula, we then conclude as needed that

$$\iota^*\gamma = \iota^*\Theta_F(\eta) = \iota^*(\Theta_F \smile \pi^*\eta) = \iota^*(\Theta_F) \smile \eta = \mathrm{e}(F) \smile \eta = \mathrm{e}(F) \smile \Theta_F^{-1}\gamma,$$

where in the second to step we have used the definition $\mathrm{e}(F) := \iota^*\Theta_F$. ■

Let's return to the case where X is a closed oriented smooth manifold, and $E^\bullet$ is an elliptic complex on X. Applying the above lemma to $F := \mathrm{T}^\vee X$ and $\gamma = \operatorname{ch}\sigma(E^\bullet)$, we get that

$$\iota^* \operatorname{ch}\sigma(E^\bullet) = \mathrm{e}(\mathrm{T}^\vee X) \smile \Theta^{-1}_{\mathrm{T}^\vee X} \operatorname{ch}\sigma(E^\bullet) = \mathrm{e}(\mathrm{T}X) \smile \Theta^{-1}_{\mathrm{T}^\vee X} \operatorname{ch}\sigma(E^\bullet),$$

where in the last step we have used that $\mathrm{T}^\vee X \cong \mathrm{T}X$ as bundles. On the other hand, the naturality of the Chern character homomorphism, along with the fact that the symbol complex $\sigma(E^\bullet)$ pulls back via ι^* to the complex on X which has the same vector bundles $E^\bullet$ but zero differentials, we conclude that

$$\iota^* \operatorname{ch}\sigma(E^\bullet) = \operatorname{ch}\iota^*\sigma(E^\bullet) = \operatorname{ch}\left(\sum_j (-1)^j E^j\right) = \sum_j (-1)^j \operatorname{ch} E^j.$$

Therefore, if $\mathrm{e}(\mathrm{T}X) \in \mathrm{H}^*(X;\mathbb{Q})$ were invertible, then this would allow us to write

$$\Theta^{-1}_{\mathrm{T}^\vee X} \operatorname{ch}\sigma(E^\bullet) = \frac{\sum_j (-1)^j \operatorname{ch} E^j}{\mathrm{e}(\mathrm{T}X)},$$

where by this division we mean multiplication on the left by $\mathrm{e}(\mathrm{T}X)^{-1}$. However, in most cases of interest, this Euler class will not be invertible. For instance, if X is odd dimensional, then this class is simply zero. Even if X is even-dimensional, nothing guarantees in general that for an arbitrary manifold X, this class will be invertible. There is, nonetheless, a way to get around this problem for some interesting choices of $E^\bullet$ by appealing to the "universal case" of $X = \mathrm{BSO}_n$. Of course, this doesn't make sense literally, since BSO_n is not even close to being a manifold. Nonetheless, Atiyah and Singer explain already in the third [8] of their series of papers in which they lay out the proof of Theorem 1.1.10 how to deal with this issue–they introduce, for Lie groups H, the notion of H-structures on manifolds and elliptic complexes, and then show that if $\rho : H \to \mathrm{SO}_n$ is a homomorphism (with n even!) such that the maximal torus of H has no fixed nonzero vector in $\mathbb{R}^n$, then the pullback $\rho^*(\mathrm{e}) \in \mathrm{H}^*(\mathrm{B}H;\mathbb{Q})$ of the universal Euler class $\mathrm{e} \in \mathrm{H}^*(\mathrm{BSO}_n;\mathbb{Q})$ is nonzero. Since the cohomology $\mathrm{H}^*(\mathrm{B}H;\mathbb{Q})$ is polynomial, we are now allowed to divide by $\rho^*(\mathrm{e})$. In particular, if the elliptic complex $E^\bullet$ comes from complex H-modules M^j (in the sense that E^j is obtained from the principal H-bundle lift of $\mathrm{T}X$ and M^j via the mixing construction), then in fact there is a well-defined element

$$\frac{\sum_j (-1)^j \operatorname{ch} M^j}{\rho^*(\mathrm{e})} \in \mathrm{H}^*(\mathrm{B}H;\mathbb{Q}),$$

which we are pulling back to X via a classifying map, so that we may pretend that we are allowed to divide by the Euler class. See [8, Proposition 2.17] and [2, Ch. III, Thm. 13.13] on precise formulations of this heuristic argument; in the following sections, we will show rather how to use this argument in practice.

1.2 Chern-Gauss-Bonnet and Poincaré-Hopf

One of the simplest local-to-global theorems in differential geometry is the Gauss-Bonnet Theorem, which relates the curvature of a closed surface (a local, geometric invariant) to its Euler characteristic (a global topological invariant). Chern later extended this result to manifolds of higher dimension, giving a formula relating the curvature of a Riemannian manifold–specifically the Pfaffian of the curvature matrix–to its topological Euler characteristic; this is the content of the famed Chern-Gauss-Bonnet Theorem. This theorem can be proven immediately as a consequence of the machinery of the Atiyah-Singer Index Theorem, and we present (detailed sketches of) two proofs of this result below. Atiyah and Singer even call this application "not very exciting" (see [8, §6])–I disagree.

Theorem 1.2.1 (Chern-Gauss-Bonnet)**.** Let X be an oriented closed smooth manifold. Then the Euler characteristic $\chi(X)$ of X can be computed as

$$\chi(X) = \int_X \mathrm{e}(\mathrm{T}X),$$

where $\mathrm{e}(\mathrm{T}X)$ is the Euler class of the tangent bundle $\mathrm{T}X$ of X.

Since the proof is not explained in detail in [8], we give a slightly more detailed explanation of this consequence of Theorem 1.1.10 here.

Sketch of Proof 1 of Theorem 1.2.1. We use Theorem 1.1.10. Of course, the result is only nontrivial when X is even dimensional (see Proposition 1.1.11 and the following discussion), so we assume hence that X has dimension $2n$ for some $n \geq 1$. Consider the complexified de Rham complex $\Lambda^j \mathrm{T}^\vee X_{\mathbb{C}}$ on X. Combining the observation in Example 1.1.7 along with the Universal Coefficient Theorem tells us that that the index of the complexified de Rham complex

$$\chi(\Lambda^\bullet \mathrm{T}^\vee X_{\mathbb{C}}) = \chi(X)$$

is just the Euler characteristic of X. On the other hand, we can compute the right hand side of Theorem 1.1.10 as follows. By taking $H = \mathrm{SO}_{2n}$ in the argument at the end of §1.1, we may pretend that we are allowed to divide by the Euler class, and use this to write the right side of Theorem 1.1.10 as

$$(-1)^{n(2n+1)} \int_X \frac{\sum_j (-1)^j \operatorname{ch} \Lambda^j \mathrm{T}^\vee X_{\mathbb{C}}}{\mathrm{e}(\mathrm{T}X)} \cdot \mathsf{Td}(\mathrm{T}X_{\mathbb{C}}).$$

To compute this quantity, we note that by a version of the splitting principle (Theorem 3.4.11), proven identically to the usual one, we may assume that the tangent bundle $\mathrm{T}X$ splits as

$$\mathrm{T}X = E_1 \oplus E_2 \oplus \cdots \oplus E_n \tag{1.2}$$

for some collection of oriented real 2-plane bundles $E_1, \ldots, E_n$. Now any oriented real 2-plane bundle is the underlying real vector bundle of a complex line bundle (by $\mathrm{U}_1 = \mathrm{SO}_2$), and hence there are complex line bundles $L_1, \dots, L_n$ such that $E_j = (L_j)_\mathbb{R}$ for $1 \le j \le n$. If we write $\gamma_i := c_1(L_j) = \mathrm{e}(E_j)$ for $1 \le j \le n$, then we get from (1.2) that

$$\mathrm{e}(\mathrm{T}X) = \prod_{j=1}^{n} \mathrm{e}(E_j) = \prod_{j=1}^{n} \gamma_j.$$

On the one hand, we have

$$\mathrm{T}X_\mathbb{C} \cong \bigoplus_{j=1}^{n} L_j \oplus \overline{L_j},$$

so that

$$\mathsf{Td}(\mathrm{T}X_\mathbb{C}) = \prod_{j=1}^{n} \frac{\gamma_j}{1-\mathrm{e}^{-\gamma_j}} \cdot \frac{(-\gamma_j)}{1-\mathrm{e}^{\gamma_j}}.$$

On the other hand, the Chern roots of $\mathrm{T}^\vee X_\mathbb{C}$ are also those of $\mathrm{T}X_\mathbb{C}$, namely

$$\{\beta_1, \dots, \beta_{2n}\} = \{\gamma_1, \dots, \gamma_n, -\gamma_1, \dots, -\gamma_n\},$$

and hence for each $1 \le j \le n$, we have

$$\mathrm{ch}\, \Lambda^j \mathrm{T}^\vee X_\mathbb{C} = \sum_{\substack{I \subset \{1,\dots,2n\} \\ |I|=j}} \mathrm{e}^{\beta_I},$$

where $I = \{i_1, \dots, i_j\}$ is a subset of $\{1, \dots, 2n\}$ of size j and $\beta_I := \beta_{i_1} + \beta_{i_2} + \dots + \beta_{i_j}$. It follows from this that

$$\sum_{j=0}^{2n} (-1)^j \,\mathrm{ch}\, \Lambda^j \mathrm{T}^\vee X_\mathbb{C} = \prod_{i=1}^{2n} (1-\mathrm{e}^{\beta_i}) = \prod_{j=1}^{n} (1-\mathrm{e}^{\gamma_j})(1-\mathrm{e}^{-\gamma_j}).$$

Putting these all together, we conclude from Theorem 1.1.10 that

$$\begin{aligned}
\chi(X) &= (-1)^n \int_X \frac{\prod_{j=1}^{n}(1-\mathrm{e}^{\gamma_j})(1-\mathrm{e}^{-\gamma_j})}{\prod_{j=1}^{n}\gamma_j} \cdot \prod_{j=1}^{n} \frac{\gamma_j}{1-\mathrm{e}^{-\gamma_j}} \cdot \frac{(-\gamma_j)}{1-\mathrm{e}^{\gamma_j}} \\
&= \int_X \prod_{j=1}^{n} \gamma_j = \int_X \mathrm{e}(\mathrm{T}X),
\end{aligned}$$

as needed. ■

Sketch of Proof 2 of Theorem 1.2.1. The more classical proof of this result, using differential topology, proceeds as follows. As noted in the second definition of the Euler class in Appendix 3.2, the integer

$$\int_X \mathrm{e}(\mathrm{T}X)$$

is the oriented self intersection number of the zero section $X \subset \mathrm{T}X$ in the tangent bundle of X. To compute this, note that the normal bundle of the zero section $X \subset \mathrm{T}X$ is simply the bundle $\mathrm{T}X$ itself. Further, if we denote by $\iota : X \to X \times X$ the embedding as the diagonal $\Delta \subset X \times X$, then the pullback of the normal bundle $\mathcal{N}_{\Delta/X\times X}$ of Δ in $X \times X$ under the diagonal embedding ι is also $\mathrm{T}X$, i.e. we have an isomorphism

$$\iota^* \mathcal{N}_{\Delta/X\times X} \cong \mathrm{T}X.$$

Since this oriented self intersection number is a local quantity depending only on the normal bundle of X in $\mathrm{T}X$, we may compute it also as the self intersection of Δ in $X \times X$. The advantage of this formulation is that, since $X \times X$ is compact, we can use the Künneth formula to explicitly write down the cohomology ring $\mathrm{H}^*(X \times X; \mathbb{Q})$ of the product and compute the class $\eta_\Delta \in \mathrm{H}^n(X \times X; \mathbb{Q})$ of the diagonal, where $\dim X = N$. Then it is not hard to check directly that this oriented self-intersection number

$$I(\Delta, \Delta) = \int_{X \times X} \eta_\Delta \smile \eta_\Delta = \chi(X)$$

is the Euler characteristic of X. For details, see [4, Chapter 3, §4] or [9, Prop. 11.24]. ■

When X has dimension 2, we recover from Theorem 1.2.1 the classical Gauss-Bonnet Theorem. Indeed, the topological classification of oriented closed surfaces tells us that any such surface X is diffeomorphic to the connect sum of g tori for some unique $g \geq 0$ called the genus of X. In this case, standard algebraic topology gives us

$$\mathrm{H}^*(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0, 2, \\ \mathbb{Z}^{2g}, & * = 1, \text{ and} \\ 0 & \text{else.} \end{cases}$$

Therefore, the Euler characteristic of X is simply

$$\chi(X) = 2 - 2g.$$

We may also easily compute the Euler class $\mathrm{e}(\mathrm{T}X)$ using Chern-Weil Theory (see Definition 3 of the Euler class in Appendix 3.2.1)–namely, if we fix a Riemannian metric on X, then

$$\mathrm{e}(\mathrm{T}X) = \left[\frac{1}{2\pi} \kappa \, \mathrm{dvol}\right],$$

where $\kappa : X \to \mathbb{R}$ is the scalar curvature of X and dvol is the volume form on X with respect to the chosen Riemannian metric. Indeed, this is because the Pfaffian of a skew-symmetric 2×2 matrix is given by

$$\mathrm{Pf} \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} = x.$$

With this notation, applying Theorem 1.2 to this case gives us

Theorem 1.2.2 (Gauss-Bonnet)**.** Let X be an oriented closed surface of genus g. If κ denotes the scalar curvature of X with respect to a given Riemannian metric, then we have

$$2 - 2g = \frac{1}{2\pi} \int_X \kappa \, \mathrm{dvol}.$$

This theorem implies, for instance, that the volume of the round sphere S^2, which has constant curvature $\kappa = 1$ and genus $g = 0$ is exactly 4π. Also, if X is a surface of genus $g \geq 2$ equipped with a hyperbolic metric of constant curvature $\kappa = -1$, then its hyperbolic volume is simply $\mathrm{Vol}(X) = 4\pi(g - 1)$.

The Chern-Gauss-Bonnet theorem gives us a way to relate $\chi(X)$ with the Euler class $\mathrm{e}(\mathrm{T}X)$, but, in practice, is not the easiest way to compute $\chi(X)$, because determination of the Euler class $\mathrm{e}(\mathrm{T}X)$ for higher dimensional X is fairly nontrivial in general. There are, however, two approaches that can often be made to work.

The first approach applies when X is a(n) (almost) complex manifold. If X is complex of dimension n (so real dimension $N = 2n$) and $\mathscr{T}X$ denotes the holomorphic tangent bundle of X, then by Remark 3.4.23, the Euler class of $\mathrm{T}X \cong (\mathscr{T}X)_{\mathbb{R}}$ is given as

$$\mathrm{e}(\mathrm{T}X) = c_n(\mathscr{T}X),$$

i.e. top Chern class of $\mathrm{T}X$, often also denoted simply by $c_n(X)$. More generally, if X is almost complex, so that there is a complex vector bundle E such that $\mathrm{T}X \cong E_{\mathbb{R}}$, then we have

$$\mathrm{e}(\mathrm{T}X) = c_n(E).$$

If we can use characteristic class techniques to compute this Chern class, then we may use this computation to figure out the Euler characteristic of X. This is done in a few special cases in an ad hoc manner in sections §2.2, 2.3 and 2.4 below. Carrying out this computation systematically (using the definition of Chern classes as degeneracy loci and the Thom-Porteous formula) can also be used to give another proof of Theorem 1.2.1 in this special case of complex manifolds; see [4, §3.3] and [10, Formula 3.5.10].

The second approach applies more generally to any oriented closed manifold. To explain this, note that Proof 2 of Theorem 1.2.1 allows us to write the Euler characteristic of X as the self-intersection number

$$\chi(X) = I(\Delta, \Delta)$$

of the diagonal $\Delta \subset X \times X$. (This is sometimes also taken as the **definition** of the Euler characteristic, as in [11, §3.3].) The homotopy invariance of the intersection number then allows us to say that if $f : X \to X$ is any smooth map homotopic to the identity map such that its graph $\Gamma_f := \{(x, f(x)) : x \in X\}$ intersects the diagonal Δ transversally, written $\Gamma_f \pitchfork \Delta$, then we can compute the Euler characteristic of X as

$$\chi(X) = I(\Gamma_f, \Delta).$$

This motivates the following definition.

Definition 1.2.3. Let X be any manifold and $f : X \to X$ be a smooth map.

(a) Suppose that $x \in X$ is a fixed point of f so $f(x) = x$. Then x is called a **nondegenerate, or Lefschetz, fixed point** if the graph Γ_f of $X \times X$ intersects the diagonal Δ transversally at (x, x). Equivalently, x is a Lefschetz fixed point of f is the differential

$$\mathrm{d}f_x : \mathrm{T}_x \to \mathrm{T}_x$$

does not have $+1$ as an eigenvalue, i.e. $\det(\mathrm{d}f_x - \mathrm{id}_{\mathrm{T}_x X}) \neq 0$.

(b) We call f a **Lefschetz map** if all its fixed points are Lefschetz fixed points.

The equivalence of the two definitions of x being a Lefschetz fixed point is seen easily in local coordinates (see [11, §3.4]); the advantage of the second definition is that it gives us a convenient criterion to check explicitly. Since both a graph and Δ are half-dimensional in $X \times X$, any Lefschetz fixed point is isolated. In particular, if X is compact, then any Lefschetz map $f : X \to X$ has only finitely many fixed points. Note also that any smooth map f without fixed points is tautologically Lefschetz.

Now suppose that X is a closed oriented manifold. Then to any smooth map $f : X \to X$ and Lefschetz fixed point $x \in X$, we can associate a number, called the **local Lefschetz number** of f at x and denoted $L_x(f)$, which is simply the local oriented intersection number of Γ_f and Δ at (x, x), given via the formula

$$L_x(f) = I_x(\Gamma_f, \Delta) = \operatorname{sign}\det(\mathrm{d}f_x - \mathrm{id}_{\mathrm{T}_x X}).$$

If f is Lefschetz, we may define its **Lefschetz number**, denoted $L(f)$, to be the total oriented intersection number of Γ_f and Δ, so that, by definition, we have

$$L(f) = I(\Gamma_f, \Delta) = \sum_{x \in X : f(x) = x} L_x(f).$$

Then the main result here is a cohomological interpretation of $L(f)$ as in

Theorem 1.2.4 (Lefschetz Fixed Point Theorem)**.** Let X be a closed oriented n-manifold and $f : X \to X$ a Lefschetz map. Then the Lefschetz number of f is

$$L(f) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}\big(f^* : \mathrm{H}^i(X; \mathbb{R}) \to \mathrm{H}^i(X; \mathbb{R})\big).$$

Proof. The proof is very similar to Proof 2 of Theorem 1.2.1; we refer the reader to [4, §3.4] or [9, Exercise 11.26]. ∎

We mention two immediate consequences of this theorem.

Corollary 1.2.5. If X is a closed oriented manifold and $f : X \to X$ a Lefschetz map homotopic to the identity, then

$$\chi(X) = L(f).$$

Proof. By homotopy invariance of cohomology, we know for each i that the pullback map $f^* : \mathrm{H}^i(X; \mathbb{R}) \to \mathrm{H}^i(X; \mathbb{R})$ is the identity map, and so has trace equal to

$$\operatorname{Tr}\big(f^* : \mathrm{H}^i(X; \mathbb{R}) \to \mathrm{H}^i(X; \mathbb{R})\big) = \dim_{\mathbb{R}} \mathrm{H}^i(X; \mathbb{R}) = b_i(X).$$

Therefore, by Theorem 1.2.4, we have

$$L(f) = \sum_i (-1)^i b_i(X) =: \chi(X).$$

Alternatively, as noted above, this follows from the homotopy invariance of the oriented intersection number, which gives

$$L(f) = I(\Gamma_f, \Delta) = I(\Delta, \Delta) = \chi(X).$$

∎

Corollary 1.2.6. If X is a closed oriented manifold and $f : X \to X$ is any smooth map such that the alternating sum of traces

$$\sum_i (-1)^i \operatorname{Tr}\big(f^* : \mathrm{H}^i(X;\mathbb{R}) \to \mathrm{H}^i(X;\mathbb{R})\big)$$

does not vanish, then any smooth map $f' : X \to X$ homotopic to f has a fixed point. In particular, if X has nonzero Euler characteristic, then any smooth map $f : X \to X$ homotopic to the identity has a fixed point.

Proof. If f did not have any fixed points, then f would tautologically be Lefschetz with $L(f) = 0$, contradicting Theorem 1.2.4. ■

Note that this corollary does not imply that if the alternating sum of traces is nonzero, then f is a Lefschetz map or even has isolated fixed points, and indeed that is false in general (take $f = \mathrm{id}_X$ when $\chi(X) \neq 0$). Corollary 1.2.6 shows the double-edged sword that Theorem 1.2.4 is; it can be used both to compute the Euler characteristic, and then prove fun results about the existence of fixed points of maps on manifolds. For instance, it can be used to show the Fundamental Theorem of Algebra (Corollary 2.2.4).

One way to find a map $f : X \to X$ that is homotopic to the identity is to consider the flow of a vector field on X. Note that if V is a vector field on X, then since X is closed, we have for all time $t \in \mathbb{R}$ a flow map $f_t : X \to X$ given by flowing along V for time t (i.e. given by integrating V). In this case, for small $t \neq 0$, it is easy to show that the fixed points of f_t correspond exactly to the zeroes of V. We say that a zero of V is nondegenerate if for all small time t, it is a Lefschetz fixed point of the flow map f_t. In this case, we define the index of this zero $x \in X$, denoted $\operatorname{ind}_x V$, to be the local Lefschetz number of f_t at x, which can also be computed as the degree of the map given in local coordinates as $V/|V|$ from a small sphere around x to S^{N-1}, where $N = \dim X$. This last interpretation allows us to generalize the definition of the index to isolated but possible degenerate zeroes; for points $x \in X$ where $V(x) \neq 0$, this quantity is also simply zero. In these terms, we have

Theorem 1.2.7 (Poincaré-Hopf Index Theorem)**.** Let X be a closed oriented manifold. If V is any global vector field on X with isolated zeroes, then we have

$$\chi(X) = \sum_{x \in X} \operatorname{ind}_x(V).$$

Proof. In the case of nondegenerate zeroes, this follows from the above discussion: apply Corollary 1.2.5 to the flow f_t of V. For the general case of isolated zeroes, see [9, Theorem 11.25]. ■

This sequence of results (Theorem 1.2.4, Corollary 1.2.5, and Theorem 1.2.7) is what is often used to compute Euler characteristics of various spaces. We give several examples of such computations in Chapter 2; see §2.1, 2.2, and 2.3.

1.3 The Hirzebruch Signature Theorem

Let X be a closed oriented topological manifold of dimension $4k$ for some $k \geq 0$. Poincaré duality tells us that the intersection pairing in the middle cohomology

$$\mathrm{H}^{2k}(X;\mathbb{Z})^{\text{free}} \times \mathrm{H}^{2k}(X;\mathbb{Z})^{\text{free}} \to \mathbb{Z}, \quad (\alpha,\beta) \mapsto \int_X \alpha \smile \beta$$

is a perfect symmetric bilinear pairing of abelian groups, where the superscript "free" denotes the torsion-free part of this group (see [12, Proposition 3.8]). This result, then, holds also with $\mathbb{Z}$ replaced by any field of characteristic zero. Over $\mathbb{R}$, this pairing induces a nondegenerate symmetric bilinear form on the middle cohomology

$$\mathrm{H}^{2k}(X;\mathbb{R}) \cong \mathbb{R}^{b_{2k}(X)},$$

called the intersection pairing. Now Sylvester's Law of Inertia says that a symmetric bilinear form $Q = \langle\cdot,\cdot\rangle$ on a finite dimensional real vector space V of a fixed dimension $n \geq 0$ is determined upto isomorphism by its signature $\mathrm{Sign}(Q)$, which is defined as

$$\mathrm{Sign}(Q) := r - s,$$

where r (resp. s) is the maximal dimension of a subspace of V, the restriction of Q to which is positive definite (resp. negative definite), so that $r + s = n$. In particular, to each X as above, we may associate the signature of its intersection form on $\mathrm{H}^{2k}(X;\mathbb{R})$, a quantity we will denoted by $\mathrm{Sign}(X)$. It was Hirzebruch's observation that if X is given a smooth structure, then this topological quantity $\mathrm{Sign}(X)$ can be expressed in terms of characteristic classes on X. Specifically, if we let $L(X)$ denote the L-genus of the manifold X defined as

$$L(X) = \int_X \mathsf{L}(\mathrm{T}X),$$

where $\mathsf{L}(\mathrm{T}X)$ denotes the total L-class of the tangent bundle $\mathrm{T}X$ (see Appendix 3.4), then we have

Theorem 1.3.1 (Hirzebruch Signature Theorem)**.** For a closed oriented smooth manifold X of dimension divisible by 4, we have

$$\mathrm{Sign}(X) = L(X).$$

In particular, $L(X)$ is a topological invariant.

Sketch of Proof 1 of Theorem 1.3.1. This is obtained by applying Theorem 1.1.10 to the Hodge-Dirac operator D. To say more, if we pick a Riemannian metric on X and let d^* be the formal adjoint of the de Rham differential d on X, then the operator $D := d + d^*$, called the Hodge-Dirac operator, is a square root of the Laplacian operator, i.e. satisfies $\Delta = D^2$. The complexified cotangent bundle $\Lambda^*\mathrm{T}^\vee X_{\mathbb{C}}$ admits an involution τ given by

$$\tau\omega = \mathrm{i}^{q(q-1)+n} * \omega,$$

if $\omega \in \Gamma(\Lambda^q\mathrm{T}^\vee X_{\mathbb{C}})$ is a complex q-form, where $\dim X = 2n$ for $n \geq 1$ and $*$ denotes the Hodge star operator. Therefore, $\Lambda^*\mathrm{T}^\vee X_{\mathbb{C}}$ ecomposes as the sum of ± 1 eigenbundles, denoted $\Lambda^{\pm}$, of the involution τ. It is easy to see then that $D\tau = -\tau D$, so that $D^+ := D|_{\Lambda^+}$ maps Λ^+ to Λ^-, and that, by Hodge theory, the index of D^+ is exactly

$$\operatorname{ind} D^+ = \operatorname{Sign}(X).$$

Since the Laplacian Δ is elliptic, the equation $\Delta = D^2$ can be used to show that $D^+ : \Lambda^+ \to \Lambda^-$ is elliptic as well, and we may apply the Atiyah-Singer Index Theorem (Theorem 1.1.10) to compute this quantity. To simplify the right hand side of the theorem, we may use the discussion at the end of Section 1.1 to pretend that $\mathrm{e}(\mathrm{T}X)$ is invertible; it then remains only to compute $\operatorname{ch}\Lambda^+ - \operatorname{ch}\Lambda^-$–and this is where the L-polynomials enter into the proof. For details, see [8, Theorem 6.6]. A slightly different way to phrase this argument is to consider the complexified Clifford bundles $\mathrm{Cl}^{\pm}_{\mathbb{C}}(X)$ instead and to apply the Index Theorem to the signature operator $D^+ : \mathrm{Cl}^+_{\mathbb{C}}(X) \to \mathrm{Cl}^-_{\mathbb{C}}(X)$; this proof can be found in [2, Ch. III, Thm. 13.9]. ■

Sketch of Proof 2 of Theorem 1.3.1. The first proof of this result, due to Hirzebruch, uses cobordism. The idea is to consider the oriented cobordism ring Ω_*, and show that both sides of Theorem 1.3.1 give ring homomorphisms $\Omega_* \to \mathbb{Z}$. This reduces the proof to showing the result for generators of the ring Ω_*, which by work of Thom can be taken to be the various complex projective spaces $\mathbb{CP}^n$, for which the result can be shown by direct computation. For details, see Hirzebruch's exposition in [13, Theorem 8.2.2]. ■

Remark 1.3.2. In low dimensions, the content of Theorem 1.3.1 can be written down rather explicitly using the computations in Example 3.3.9. For instance, if X is a closed oriented 4-manifold, then

$$\operatorname{Sign}(X) = \frac{1}{3} p_1(X),$$

explaining the otherwise somewhat mysterious result that the (unique) Pontryagin number of any closed 4-manifold is a multiple of 3. Similarly, if X is a closed oriented 8-manifold, then

$$\operatorname{Sign}(X) = \frac{1}{45}(7p_2(X) - p_1(X)^2). \tag{1.3}$$

We mention in closing that the formula (1.3) can be used to show the existence of exotic 7-spheres; a version of this argument was carried out first by Milnor and was the first example of sphere shown to be exotic. We will now sketch a proof of this result.

Proposition 1.3.3 (Milnor)**.** There is a closed oriented 7-manifold X that is homeomorphic but not diffeomorphic to the standard seven-sphere S^7.

Proof Sketch. We closely follow [14, §2.1], to which we refer the reader for details. The exceptional isomorphism (see [2, Ch. I, Thm. 8.1]) given by

$$\mathrm{Spin}_4 \cong \mathrm{SU}_2 \times \mathrm{SU}_2 \cong S^3 \times S^3$$

tells us that

$$\pi_3 \,\mathrm{SO}_4 \cong \pi_3 \,\mathrm{Spin}_4 \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

In fact, for each pair (i, j) of integers and $x \in S^3$ thought of as a unit quaternion, the map $y \mapsto x^i y x^j$ preserves the usual inner product on $\mathbb{H} \cong \mathbb{R}^4$, and so we get a map

$$f_{i,j} : S^3 \to \mathrm{SO}_4, \quad f(x)y = x^i y x^j$$

which can be shown to have class (i, j) in $\pi_3 \operatorname{SO}_4$. Given a fixed (i, j), we can apply the clutching construction to the map $f_{i,j}$ to obtain a rank 4 real vector bundle on S^4; we let $D_{i,j}$ be the corresponding disk bundle with respect to some choice of metric. It can then be shown that $X_{i,j} := \partial D_{i,j}$, which is an S^3 bundle over S^4, has the homotopy type of a sphere iff $i + j = \pm 1$. Let's stick to the case $j = 1 - i$. We will show that $X = X_{i,1-i}$ is an exotic sphere if $i \not\equiv 0, 1 \pmod 7$.

First, using the fact that $X_{i,j}$ is an S^3-bundle over S^4, one can explicitly write down a Morse function on $X_{i,1-i}$ with only two critical points, because of which it follows from Reeb's Theorem that $X_{i,1-i}$ is homeomorphic S^7. Suppose now that $X_{i,1-i}$ is diffeomorphic to S^7. Then gluing back an 8-disk $\mathbb{D}^8$ to $D_{i,1-i}$ along $X_{i,1-i}$ results in a closed smooth 8-manifold, say Y_i. It is clear by construction that Y_i is homotopy equivalent to a CW complex with one cell each in dimensions $1, 4, 8$, so that the cohomology of Y_i is

$$\mathrm{H}^*(Y_i; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0, 4, 8, \text{ and} \\ 0 & \text{else.} \end{cases}$$

Since Y_i is a manifold, Poincaré duality then tells us that the generator of the middle cohomology must square to a generator of the top cohomology, determining the cohomology ring as

$$\mathrm{H}^*(Y_i; \mathbb{Z}) = \mathbb{Z}[\lambda]/(\lambda^3) \text{ where } |\lambda| = 4.$$

In particular, we must have $\operatorname{Sign}(Y_i) = \pm 1$. We may orient Y_i so this signature is 1. In particular, Theorem 1.3.1 in the form of (1.3) tells us that the Pontryagin numbers $A_i := p_1(Y_i)^2$ and $B_i := p_2(Y_i)$ are related as

$$7B_i - A_i = 45. \tag{1.4}$$

Now, since the restriction map $\mathrm{H}^4(Y_i; \mathbb{Z}) \to \mathrm{H}^4(S^4)$ is an isomorphism, the naturality of characteristic classes tells us that B_i is determined by $p_1(\mathrm{T}Y_i|_{S^4})$, from which it can be shown that

$$p_1(Y_i) = 2(i - j)\lambda = 2(2i - 1)\lambda \text{ so } A_i = 4(2i - 1)^2. \tag{1.5}$$

It follows from (1.4) and (1.5) that

$$B_i = \frac{4(2i - 1)^2 + 45}{7}.$$

In particular, since Pontryagin numbers of smooth manifolds are integers, we conclude from this that if $X_{i,1-i}$ is diffeomorphic to S^7, then we must have

$$4(2i - 1)^2 + 45 \equiv 0 \pmod 7,$$

which is equivalent to saying that i is 0 or 1 mod 7. ■

One can combine this result with a theorem of Novikov saying that rational Pontryagin classes are topological invariants to conclude that for $i \not\equiv 0, 1 \pmod 7$, the Y_i constructed above is a topological 8-manifold that does not admit a smooth structure; see the references in [14, §2.1] and compare this result with that in Remark 1.5.8. Note that this argument does not show that the spheres $X_{i,1-i}$ are not all pairwise non-diffeomorphic; indeed, there are only 28 distinct diffeomorphism classes of smooth structures on S^7, which follows from work of Kervaire and Milnor [15].

1.4 The Hirzebruch-Riemann-Roch Theorem

In this section, we present the Hirzebruch-Riemann-Roch Theorem (Theorem 1.4.1), and its generalization to the Hirzebruch χ_y-characteristic for vector bundles (Theorem 1.4.3), as a consequence of the Atiyah-Singer Index Theorem (Theorem 1.1.10). The first proof of Theorem 1.4.1 was obtained by Hirzebruch in 1954, which was published in [13], and it served as central impetus to the establishment of generalizations of this result by Grothendieck, and ultimately also the Atiyah-Singer Index Theorem. After sketching a proof of Theorem 1.4.1, we then discuss its applications including the classical Riemann-Roch Theorem for Riemann surfaces and its consequences, as well as Noether's formula for complex surfaces. Finally, we briefly discuss how the proof via the Atiyah-Singer Index Theorem can be used to generalize these theorems to almost complex manifolds.

Recall (see Example 1.1.8) that to each complex manifold X and holomorphic vector bundle $E \to X$, we can associate an elliptic differential complex $(E^\bullet, \overline{\partial}_E)$ on X called the Dolbeaux complex of E. The Dolbeaux Theorem says that the cohomology groups of this complex are exactly the sheaf cohomology groups of E, i.e. there is an isomorphism

$$\mathrm{H}^*(X,E) \to \mathrm{H}^*_{\overline{\partial}}(X,E).$$

In particular, if X is compact, then Theorem 1.1.4 tells us that these cohomology groups are finite dimensional, and hence the holomorphic Euler characteristic

$$\chi(X,E) = \sum_j (-1)^j h^j(X,E)$$

of E is well-defined. The Hirzebruch-Riemann-Roch Theorem allows us to compute this holomorphic Euler characteristic in terms of cohomological information.

Theorem 1.4.1 (Hirzebruch-Riemann-Roch)**.** If $E \to X$ is a holomorphic vector bundle over a compact complex manifold X, then the holomorphic Euler characteristic of E is given by

$$\chi(X,E) = \int_X \operatorname{ch} E \cdot \mathsf{Td}(X),$$

where $\mathsf{Td}(X) = \mathsf{Td}(\mathscr{T}X)$ is the total Todd class of X.

Sketch of Proof of Theorem 1.4.1. The key idea is to apply the Atiyah-Singer Index Theorem 1.1.10 to the Dolbeaux complex $(E^\bullet, \overline{\partial}_E)$. By the above observations, the index of the Dolbeux complex is exactly

$$\chi(E^\bullet, \overline{\partial}_E) = \chi(X,E),$$

the holomorphic Euler characteristic of E. To compute the quantity on the right side of the formula, we can use the remarks at the end of §1.1 to pretend that we may divide by $\mathrm{e}(\mathrm{T}X) = c_n(\mathscr{T}X)$ as before. More specifically, if we let $\gamma_1, \ldots, \gamma_n$ be the Chern roots of $\mathscr{T}_X$, then we can show that

$$\Theta^{-1}_{\mathrm{T}^\vee X} \operatorname{ch} \sigma(E^\bullet) = \operatorname{ch} E \cdot \prod_{i=1}^n \frac{1 - \mathrm{e}^{\gamma_i}}{\gamma_i}.$$

Also, we have that
$$\mathrm{T}X_{\mathbb{C}} \cong \mathscr{T}X \oplus \overline{\mathscr{T}X},$$
so that
$$\mathsf{Td}(\mathrm{T}X_{\mathbb{C}}) = \prod_{i=1}^{n} \frac{\gamma_i}{1-\mathrm{e}^{-\gamma_i}} \cdot \frac{(-\gamma_i)}{1-\mathrm{e}^{\gamma_i}}.$$
Therefore, remembering that the real dimension of X is $2n$, Theorem 1.1.10 implies that
$$\begin{aligned}\chi(X,E) &= (-1)^{n(2n+1)} \int_X \operatorname{ch} E \cdot \prod_{i=1}^{n} \frac{1-\mathrm{e}^{\gamma_i}}{\gamma_i} \prod_{i=1}^{n} \frac{\gamma_i}{1-\mathrm{e}^{-\gamma_i}} \cdot \frac{(-\gamma_i)}{1-\mathrm{e}^{\gamma_i}} \\ &= \int_X \operatorname{ch} E \cdot \prod_{i=1}^{n} \frac{\gamma_i}{1-\mathrm{e}^{-\gamma_i}} \\ &= \int_X \operatorname{ch} E \cdot \mathsf{Td}(X)\end{aligned}$$
as needed. For details, see [8, Thm. 4.3] or [2, Ch. III, Thm. 13.15]. ■

Remark 1.4.2. Note how the right hand side of this equality does not depend on the holomorphic structure of E at all, but only the complex structure. It is possible to write down a compact complex manifold X and holomorphic vector bundles $L_1, L_2 \to X$ such that the $h^0(X, L_1) \neq h^0(X, L_2)$, so that L_1 and L_2 are not isomorphic as holomorphic vector bundles, but such that $L_1^\infty \cong L_2^\infty$, i.e. the underlying smooth complex bundles are isomorphic. In this case, the above theorem tells us that the higher cohomology groups $h^j(X, L_i)$ compensate for this discrepancy in such a way that we get
$$\chi(X, L_1) = \chi(X, L_2).$$
Here's a simple example of this phenomenon. Take X to be any smooth curve, $L_1 = \mathcal{O}_X$ and $L_2 = \mathcal{O}_X(D)$ for some divisor D with $\deg D = 0$ but $D \not\sim 0$, i.e. such that D is not linearly equivalent to zero. For instance, we may take X to be an elliptic curve with $D = p - q$ for $p \neq q \in X$; the fact that $D \not\sim 0$ is saying that there is no degree 1 map $f : X \to \mathbb{CP}^1$, which is a topological observation. In this case, we have $h^0(X, L_1) = 1$ whereas $h^0(X, L_2) = 0$, since if a meromorphic f satisfies $\operatorname{div} f \geq -D$, then by equality of degree we would conclude that $\operatorname{div} f = -D$, so $D \sim 0$. On the other hand, the only invariant of a smooth complex line bundle on a curve is its degree, and so we conclude from this that if X is any curve and D on X a divisor with $\deg D = 0$ but $D \not\sim 0$, then
$$0 - h^1(X, \mathcal{O}_X(D)) = 1 - h^1(X, \mathcal{O}_X) = 1 - h^{0,1}(X)$$
so that
$$h^1(X, \mathcal{O}_X(D)) = h^{0,1}(X) - 1 = g - 1,$$
where g is the genus of X; see Proposition 1.4.8.

We may generalize Theorem 1.4.1 slightly. To see this, note that we can encode a lot of information about a vector bundle $E \to X$ in its Hirzebruch χ_y-characteristic, defined as
$$\chi_y(X, E) := \sum_{i=0}^{n} \chi(X, \Omega^i E) y^i,$$

where n is the complex dimension of X and $\Omega^i := \Lambda^i \mathscr{T}_X^\vee$, so that $\Omega^i E := \Omega^i \otimes E$ is the bundle of E-valued holomorphic i-forms. Note that taking $y = 0$ simply returns

$$\chi_0(X, E) = \chi(X, E).$$

Here's where things get interesting. Taking $E = \mathcal{O}_X$ to be the trivial bundle, we get that

$$\chi_y(X) := \chi_y(X, \mathcal{O}_X) = \sum_{i=0}^{n} \chi(X, \Omega^i) y^i.$$

Since

$$\chi(X, \Omega^i) = \sum_{j=0}^{n} (-1)^j h^j(X, \Omega^i) = \sum_{j=0}^{n} (-1)^j h^{i,j}(X),$$

where $h^{i,j}(X)$ is (either by definition or by the Dolbeaux Theorem) the $(i,j)^{\text{th}}$ Hodge number of X, we conclude that

$$\chi_y(X) = \sum_{i,j=0}^{n} (-1)^j h^{i,j}(X) y^i.$$

This has the following specializations:

(a) Taking $y = 0$ gives us

$$\chi_0(X) = \sum_{j=0}^{n} (-1)^j h^{0,j}(X) = \chi(X, \mathcal{O}_X),$$

which is the holomorphic Euler characteristic of X.

(b) Taking $y = -1$ gives us

$$\chi_{-1}(X) = \sum_{i,j=0}^{n} (-1)^{i+j} h^{i,j}(X) = \sum_{k=0}^{2n} (-1)^k b_k(X) = \chi(X),$$

which is the topological Euler characteristic of X, also written $\chi_{\text{top}}(X)$.

(c) Finally, taking $y = 1$ gives us

$$\chi_1(X) = \sum_{j=0}^{n} (-1)^j h^{i,j}(X).$$

If X is a compact Kähler manifold of even complex dimension, then one can show directly (see [16, Corollary 3.3.18]) that this quantity is the signature $\operatorname{Sign}(X)$.

Now suppose $E \to X$ is a complex vector bundle. If E has Chern roots μ_i, then we define the generalized Chern character $\operatorname{ch}_y E$ of E to be

$$\operatorname{ch}_y E := \sum_i \mathrm{e}^{\mu_i(1+y)} = \operatorname{rank} E + \sum_{j=1}^{\infty} (1+y)^j \frac{p_j(\mu)}{j!} \in \mathrm{H}^*(X; \mathbb{Q}[y]),$$

where p_j as before is the j^{th} power sum in the μ_i. If $\mathsf{Td}_y(X) := \mathsf{Td}_y(\mathscr{T}X)$ denotes the total generalized Todd class of X in the sense of Appendix 3.4, i.e. the total characteristic class corresponding to the series

$$Q_y(z) = \frac{z(1 + y\mathrm{e}^{-z(1+y)})}{1 - \mathrm{e}^{-z(1+y)}},$$

then the generalization of Theorem 1.4.1 is

Theorem 1.4.3 (Generalized Hirzebruch-Riemann-Roch Theorem). If $E \to X$ is a holomorphic vector bundle on a compact complex manifold X, then the χ_y-characteristic of E is given by

$$\chi_y(X,E) = \int_X \mathrm{ch}_y E \cdot \mathsf{Td}_y(X).$$

Since it is tricky to extract a proof of this result from [13], we present a complete proof.

Proof. We have

$$\chi_y(X,E) = \sum_i \chi(X, \Omega^i E) y^i = \sum_i y^i \int_X \mathrm{ch}(\Omega^i E) \cdot \mathsf{Td}(X) = \int_X \mathrm{ch}\, E \cdot \mathsf{Td}(X) \cdot \sum_i y^i \,\mathrm{ch}\, \Omega^i,$$

where the first step is the definition, the second step uses Theorem 1.4.1, and the third step uses the multiplicativity of the Chern character to conclude that $\mathrm{ch}(\Omega^i E) = \mathrm{ch}(\Omega^i)\,\mathrm{ch}(E)$. If $\gamma_1, \dots, \gamma_n$ are the Chern roots of $\mathscr{T}X$, then those of Ω^i are $-\gamma_J$, where J ranges over subsets $J \subset \{1, \dots, n\}$ of size i and $\gamma_J := \sum_{j \in J} \gamma_j$. It follows from this that

$$\sum_i y^i \,\mathrm{ch}\, \Omega^i = \prod_{j=1}^n (1 + y\mathrm{e}^{-\gamma_j}),$$

and hence that

$$\chi_y(X,E) = \int_X \mathrm{ch}\, E \cdot \prod_{j=1}^n \frac{\gamma_j}{1 - \mathrm{e}^{-\gamma_j}} \prod_{j=1}^n (1 + y\mathrm{e}^{-\gamma_j}).$$

In computing this integral, we are extracting only the degree $2n$ component of the integrand. Therefore, scaling all degree 2 components (i.e. Chern roots) involved by $1+y$ and dividing throughout by $(1+y)^n$ does not change this quantity. In other words, this formula gives us

$$\begin{aligned}\chi_y(X,E) &= \frac{1}{(1+y)^n} \int_X \mathrm{ch}_y E \cdot \prod_{j=1}^n \frac{\gamma_j(1+y)}{1 - \mathrm{e}^{-\gamma_j(1+y)}} \prod_{j=1}^n (1 + y\mathrm{e}^{-\gamma_j(1+y)}) \\ &= \int_X \mathrm{ch}_y E \cdot \prod_{j=1}^n Q_y(\gamma_j) \\ &= \int_X \mathrm{ch}_y E \cdot \mathsf{Td}_y(X)\end{aligned}$$

as needed. ∎

Taking $y = 0$ in Theorem 1.4.3 recovers Theorem 1.4.1. Specializing to the trivial line bundle $E = \mathscr{O}_X$ then gives us

Corollary 1.4.4. If X is a compact complex manifold, then

$$\chi_y(X) = \int_X \mathsf{Td}_y(X).$$

Further specializing to $y \in \{0, \pm 1\}$, and using the specializations of Q_y mentioned in Example 3.3.10, we then obtain the following three corollaries.

Corollary 1.4.5. If X is a compact complex manifold, then the holomorphic Euler characteristic of X is the Todd genus of X, i.e.

$$\chi(X, \mathcal{O}_X) = \mathrm{Td}(X).$$

Corollary 1.4.6. If X is a compact complex manifold, then the topological Euler characteristic of X is the Chern genus of X, i.e.

$$\chi_{\mathrm{top}}(X) = c(X).$$

Corollary 1.4.7. If X is a compact Kähler manifold of even complex dimension, then the signature of X the L-genus of X, i.e.

$$\mathrm{Sign}(X) = L(X).$$

Note that since $c_n(X) = \mathrm{e}(\mathrm{T}X)$ by Remark 3.4.23, Corollary 1.4.6 is just the Chern-Gauss-Bonnet Theorem (Theorem 1.2.1) for compact complex manifolds. Similarly, Corollary 1.4.7 is just the Hirzebruch Signature Theorem (Theorem 1.3.1) for compact Kähler manifolds.

Let's now look at special cases of the Hirzebruch-Riemann-Roch Theorem for complex curves (i.e. Riemann surfaces) and complex surfaces. First suppose that X is a compact Riemann surface, i.e. a closed connected complex manifold of dimension 1.[2] Then every holomorphic line bundle $L \to X$ is of the form $\mathcal{O}_X(D)$ for some divisor D on X, where in this case D is simply a formal sum of the form $\sum_{x\in X} a_x x$ where the $a_x \in \mathbb{Z}$ are zero for all but finitely many x.[3] It is then not hard to show (see [4, Ch. 1, Prop. 1]) that the first Chern class of $\mathcal{O}_X(D)$ is given by

$$c_1(\mathcal{O}_X(D)) = \deg D \cdot \eta_X,$$

where $\deg D := \sum_{x\in X} a_x$ and $\eta_X \in \mathrm{H}^2(X;\mathbb{Z})$ is the generator of the top cohomology, i.e. the Poincaré dual to a point. Finally, it is a consequence of Serre Duality that for any divisor D if we let $\Omega^1(D) := \Omega^1(\mathcal{O}_X(D))$, then

$$\chi(X, \Omega^1(D)) = -\chi(X, -D).$$

If we let $c_1(X)$ denote the (unique) Chern number of X, then the Generalized Hirzebruch-Riemann-Roch Theorem (Theorem 1.4.3) says in this case that for any divisor D on X, we have

$$\chi(X, D) - y \cdot \chi(X, -D) = (1+y)\deg D + \frac{(1-y)}{2} c_1(X). \tag{1.6}$$

Plugging in $D = 0$ and $y = -1$ in this equation gives us

$$2 - 2g = \chi_{\mathrm{top}}(X) = c_1(X),$$

[2] See, for instance, [4, Ch. 2] or [17] for the basic theory of Riemann surfaces.

[3] Because of this fact, we will use the notation of divisors and line bundles interchangably, so, for instance, we use the notation $\chi(X, D) := \chi(X, \mathcal{O}_X(D))$.

where g is the (topological) genus of X–but we knew this already, since this is the content of the Gauss-Bonnet Theorem (Theorem 1.2.2). Now plugging in $D = 0$ in (1.6) gives us that

$$1 - h^{0,1}(X) = \chi(X, \mathcal{O}_X) = 1 - g,$$

so that $h^{0,1}(X) = g$. Finally, Serre Duality gives us that $h^{1,0}(X) = g$ as well, where $h^{1,0}(X) = h^0(X, \Omega^1)$ is the dimension of the space $\Omega(X)$ of global holomorphic 1-forms on X. Therefore, we have shown

Proposition 1.4.8 (Equality of the Three Genera)**.** Let X be a compact Riemann surface. Then the following three quantities assocaited to X are all equal:

- its topological genus g,
- its arithmetic genus $h^1(X, \mathcal{O}_X)$, and
- its analytic genus $\dim_{\mathbb{C}} \Omega(X)$.

Serre Duality tells us that for any divisor D we have $h^1(X, D) = h^0(X, \Omega^1(-D))$. Using this observation and taking $y = 0$ in (1.6) gives us the awaited

Theorem 1.4.9 (Riemann-Roch)**.** Let X be a compact Riemann surface of genus g. Then for any divisor D on X, we have

$$h^0(X, D) - h^0(X, \Omega^1(-D)) = \deg D + 1 - g.$$

In particular, we have

$$h^0(X, D) \geq \deg D + 1 - g,$$

with equality if $\deg D \geq 2g - 1$.

The last statement follows from the observation that

$$\deg \Omega^1(-D) = -c_1(X) - \deg D = 2g - 2 - \deg D,$$

so that if $\deg D \geq 2g - 1$, there are no global sections of $\Omega^1(-D)$.

Theorem 1.4.9 is really the starting point of the theory of curves; see [17, Ch. VII] for several applications, some of which we now mention. Firstly, Theorem 1.4.9 implies that any divisor D on X of degree $\deg D \geq 2g + 1$ is very ample. In particular, every compact Riemann surface is projective, and hence by Chow's Theorem, a smooth projective variety. Therefore, the analytic machinery that goes into the Atiyah-Singer Index Theorem subsumes the tools needed to show the existence of meromorphic functions on Riemann surfaces, as is done in any text on the analytic theory of Riemann surfaces, e.g. [18, Ch. 2]. Other delicious consequences of Theorem 1.4.9 include the following:

(a) any curve of genus 0 is biholomorphic to $\mathbb{CP}^1$,
(b) any curve of genus 1 is biholomorphic to a smooth plane cubic and a complex torus,
(c) any curve of genus 2 is hyperelliptic,
(d) any nonhyperelliptic curve of genus 3 is a smooth plane quartic curve,
(e) any nonhyperelliptic curve of genus 4 is the smooth complete intersection variety $X^1_{2,3} \subset \mathbb{CP}^3$, i.e. is the intersection of a quadric and cubic hypersurface in $\mathbb{CP}^3$, etc.

See [17, Ch. VII] for details.

Now suppose that X is a connected complex surface, i.e. complex 2-manifold. Then the Generalized Hirzebruch-Riemann-Roch Theorem (Theorem 1.4.3) for X can be made quite explicit. Namely, we have

Theorem 1.4.10. Let X be a compact complex surface. If $E \to X$ is a holomorphic vector bundle of rank $r \geq 0$, then the χ_y-characteristic of E is given by

$$\chi_y(X, E) = r\chi_y(X) + \int_X \left(\frac{(1-y)^2}{2} c_1(\mathscr{T}X)c_1(E) + \frac{(1+y)^2}{2} \left(c_1(E)^2 - 2c_2(E)\right) \right)$$

where

$$\chi_y(X) = \frac{1 - 10y + y^2}{12} c_2(X) + \frac{(1+y)^2}{12} c_1(X)^2 \in \mathbb{Q}[y].$$

Proof. It follows from Example 3.3.10 that the total generalized Todd class of X is

$$\begin{aligned} \mathsf{Td}_y(X) &= 1 + \frac{1-y}{2} c_1(\mathscr{T}X) + \frac{1 - 10y + y^2}{12} c_2(\mathscr{T}X) + \frac{(1+y)^2}{12} c_1(\mathscr{T}X)^2 \\ &= 1 + \frac{1-y}{2} c_1(\mathscr{T}X) + \chi_y(X) \cdot \eta_X \in \mathrm{H}^*(X; \mathbb{Q}[y]), \end{aligned}$$

where $\chi_y(X)$ has the required form from Corollary 1.4.4. The generalized Chern character of E is given by

$$\mathrm{ch}_y(E) = r + (1+y)c_1(E) + \frac{(1+y)^2}{2} \left[c_1(E)^2 - 2c_2(E)\right].$$

Therefore, the required result follows from Theorem 1.4.3. ■

To put this result in familiar terms, consider now the case where $E = L$ is a line bundle, so we denote $c_1(L) \in \mathrm{H}^2(X; \mathbb{Z})$ simply by L and $c_2(L) = 0$. Note also that $c_1(\mathscr{T}X) = -K_X \in \mathrm{H}^2(X; \mathbb{Z})$ is the negative of the canonical class of X. In this notation, specializing Theorem 1.4.10 to $y = 0$ gives us

Corollary 1.4.11 (Noether)**.** If X is a compact complex surface then

$$\chi(X, \mathcal{O}_X) = \frac{1}{12} \left(c_2(X) + c_1(X)^2\right).$$

Further, if $L \to X$ is a holomorphic line bundle, then

$$\chi(X, L) = \chi(X, \mathcal{O}_X) + \int_X \frac{L \cdot (L - K_X)}{2}.$$

Remark 1.4.12. The machinery of the Atiyah-Singer Index Theorem also allows us to generalize these results to almost complex manifolds. However, we now have to be careful in how we define the holomorphic Euler characteristic or the Dolbeaux complex of a vector bundle, since if (X, J) is only almost complex, then we might not have $\overline{\partial}_J^2 = 0$ (indeed, this is one of the equivalent conditions for the almost complex structure J to be integrable–see Theorem 3.6.3). For an explanation of how to generalize this machinery to the almost complex case, we refer the reader to [19].

1.5 Integrality of the $\hat{A}$-Genus for Spin Manifolds

Recall (see Appendices 3.3 and 3.4) that the $\hat{A}$-genus of a closed oriented manifold X is given as

$$\hat{A}(X) = \int_X \hat{\mathsf{A}}(\mathrm{T}X),$$

where $\hat{\mathsf{A}}(\mathrm{T}X)$ is the total $\hat{A}$-class of the tangent bundle $\mathrm{T}X$, coming from the (reduced) characteristic series

$$\tilde{Q}_{\hat{A}}(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}.$$

If X is a complex manifold, then $\mathrm{Td}(\mathscr{T}X) = \mathrm{e}^{c_1(X)/2}\hat{A}(X)$; it is probably this observation (along with the fact that $c_1(X) \equiv w_2(X) \pmod 2$; see Corollary 1.5.3 below) which first led Hirzebruch to ask: if a real manifold X of dimension $4k$ has second Stiefel-Whitney class $w_2(X) = 0$, a condition called being spin, then is $\hat{A}(X)$ an integer? As the story goes (recounted from [1]), the first proof of the integrality of $\hat{A}(X)$ for a spin manifold was obtained by Atiyah and Hirzebruch via a version of the Riemann-Roch Theorem for smooth manifolds, but this explanation was not considered satisfactory. Hirzebruch had realized that the signature is the difference in dimensions of spaces of harmonic forms, and asked for a similar analytic interpretation of $\hat{A}(X)$. When Singer came to Oxford in 1962 for a sabbatical, Atiyah's first question to him was "Why is the A-roof genus an integer for a spin manifold?" Singer responded, "Michael, why are you asking me that question? You know the answer to that." Singer knew, however, that Atiyah was looking for something deeper, and within months, the two had discovered the Dirac operator and the index formula. Therefore, this integrality result for the $\hat{A}$-genus served as key impetus that led to the discovery of Theorem 1.1.10; it is this result we discuss now.

The story starts with the definition of a spin structure. Recall that for $n \geq 0$, the special orthogonal group SO_n is connected and has fundamental group

$$\pi_1 \mathrm{SO}_n = \begin{cases} 0, & n = 1, \\ \mathbb{Z}, & n = 2, \text{ and} \\ \mathbb{Z}/2, & n \geq 3. \end{cases}$$

In particular, for each $n \geq 0$, there is a unique degree two cover of SO_n, which is called the spin group in dimension n and is denoted $\mathrm{Spin}_n \to \mathrm{SO}_n$. Note that $\mathrm{Spin}_1 = \mathbb{Z}/2$, and $\mathrm{Spin}_2 \cong \mathrm{SO}_2 \cong S^1$ with the map $\pi : \mathrm{Spin}_2 \to \mathrm{SO}_2$ given by the unique two-fold covering map, written $z \mapsto z^2$ when we think of SO_2 as S^1. For $n \geq 3$, the group Spin_n is a connected, simply connected Lie group which is the universal cover of SO_n, and can be constructed explicitly as a matrix group using Clifford algebras (see [2, Chapter I]).

Definition 1.5.1. Given an oriented Riemannian vector bundle E of rank $n \geq 1$ over a space X, a spin structure on E is a lift of the bundle $\mathrm{SO}(E) \to X$ of orthonormal frames of E to a principal Spin_n-bundle.

A spin structure on an oriented Riemannian manifold X is defined to be a spin structure on its tangent bundle $\mathrm{T}X$. If X admits a spin structure, then it is said to be a spin manifold, or simply spin.

In other words, a spin structure on a bundle E is a two sheeted cover $\mathrm{Spin}(E) \to \mathrm{SO}(E)$ that is nontrivial on the fibers of $\mathrm{SO}(E) \to X$. When $n = 1$ this is saying simply that a spin structure on E is a double cover of $\mathrm{SO}(E) \cong X$. The first result in the theory simply characterizes when such structures exist.

Theorem 1.5.2 (Borel-Hirzebruch)**.** Let E be an oriented vector bundle over a manifold X. Then there exists a spin structure on E iff the second Stiefel-Whitney class $w_2(E) \in \mathrm{H}^2(X;\mathbb{Z}/2)$ vanishes. Further, if this is case, then the distinct spin structures on E are in bijective correspondence with the elements of $\mathrm{H}^1(X;\mathbb{Z}/2)$.

Proof. This is standard obstruction theory; here's one way to phrase this. Consider the Leray-Serre Spectral Sequence with $\mathbb{Z}/2$ coefficients of the fibre bundle $\mathrm{SO}_n \to \mathrm{SO}(E) \to X$, where $n = \operatorname{rank} E$, which is given by

$$E_2^{p,q} = \mathrm{H}^p(X;\boldsymbol{\mathcal{H}}^q(\mathrm{SO}_n;\mathbb{Z}/2)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{SO}(E);\mathbb{Z}/2).$$

Note that for $q = 0, 1$, the local system $\boldsymbol{\mathcal{H}}^q(\mathrm{SO}_n;\mathbb{Z}/2)$ is necessarily trivial because $\mathrm{H}^q(\mathrm{SO}_n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for these q. Assuming without loss of generality that X is path connected (else we may work on each component individually), the five term exact sequence in low degrees for this first quadrant spectral sequence looks like

$$0 \to \mathrm{H}^1(X;\mathbb{Z}/2) \to \mathrm{H}^1(\mathrm{SO}(E);\mathbb{Z}/2) \to \mathrm{H}^1(\mathrm{SO}_n;\mathbb{Z}/2) \xrightarrow{\partial_2^{0,1}} \mathrm{H}^2(X;\mathbb{Z}/2) \to \mathrm{H}^2(\mathrm{SO}(E);\mathbb{Z}/2).$$

Note that double coverings of $\mathrm{SO}(E)$ are in bijection with index two subgroups of $\pi_1 \mathrm{SO}(E)$, and hence with $\mathrm{Hom}(\pi_1(\mathrm{SO}(E));\mathbb{Z}/2) \cong \mathrm{H}^1(\mathrm{SO}(E);\mathbb{Z}/2)$, with the map

$$\rho : \mathrm{H}^1(\mathrm{SO}(E);\mathbb{Z}/2) \to \mathrm{H}^1(\mathrm{SO}_n;\mathbb{Z}/2)$$

in the above sequence given simply by restriction to fibers of the double covering; in particular, E admits a spin structure iff this map is surjective, which happens iff $\partial_2^{0,1} = 0$. The image of the nontrivial element $1 \in \mathrm{H}^1(\mathrm{SO}_n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ under $\partial_2^{0,1}$ is, however, the second Stiefel-Whitney class

$$\partial_2^{0,1}(1) = w_2(E),$$

because this is clearly true in the "universal" case of $X = \mathrm{BSO}_n$, for which the map $\partial_2^{0,1} : \mathrm{H}^1(\mathrm{SO}_n;\mathbb{Z}/2) \to \mathrm{H}^2(\mathrm{BSO}_n;\mathbb{Z}/2)$ is an isomorphism. Combining these observations tells us that E is spin iff $\partial_2^{0,1}(1) = w_2(E)$ vanishes, in which case the set of distinct spin structures is the coset $\rho^{-1}(1)$, which is in bijection with $\ker \rho = \mathrm{H}^1(X;\mathbb{Z}/2)$ as needed. ■

This theorem gives us many different ways to interpret the condition of being spin; see, for instance, Proposition 1.12 and Theorem 2.10 of [2, Ch. II] and their corollaries. For complex manifolds, we also have

Corollary 1.5.3. If X is a compact complex manifold, then X admits a spin structure iff the first Chern class $c_1(X) \in \mathrm{H}^2(X;\mathbb{Z})$ is even in the sense that its mod two reduction vanishes, i.e.

$$[c_1(X)] = 0 \in \mathrm{H}^2(X;\mathbb{Z}/2).$$

Proof. This follows from Theorem 1.5.2 along with the fact that $w_2(X)$ is the reduction of $c_1(X)$ mod 2, which follows from the isomorphism $\mathrm{T}X \cong (\mathscr{T}X)_{\mathbb{R}}$ and Theorem 3.4.19. ■

Corollary 1.5.4. Every Riemann surface is spin. If X is a Riemann surface of genus g, then there are exactly 2^{2g} distinct spin structures on X.

Proof. The Gauss-Bonnet Theorem (Theorem 1.2.1) tells us that if X is a Riemann surface of genus g, then $c_1(X) = (2-2g)\eta_X \in \mathrm{H}^2(X;\mathbb{Z})$ is always even. The distinct spin structures on X are then in bijection with elements of $\mathrm{H}^1(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$. ■

In fact, this is a special case of the more general result that the set of spin structures on a compact, spin Kähler manifold X are in bijection with the holomorphic square roots of the canonical bundle X; see [2, Ch. II, Example 2.6] and the references mentioned there.

Recall that given a topological (or Lie) group G, a finite-dimensional representation (V,ρ) of G, so that $\rho : G \to \mathrm{GL}(V)$ is a continuous (resp. smooth) homomorphism, and a principal G-bundle $G \to P \to X$, the balanced product $P \times_G V \to X$ is a vector bundle over X, called the vector bundle **associated** to the representation V. In our case, if X is a spin manifold with corresponding principal Spin_n-bundle $\mathrm{Spin}(X)$, then associated to any representation $\mathrm{Spin}_n \to \mathrm{GL}(V)$ of Spin_n, we get a vector bundle on X. Two cases of particular interest are the **Clifford bundles**, given by taking $V = \mathrm{Cl}(\mathbb{R}^n)$ or $\mathrm{Cl}(\mathbb{R}^n)_{\mathbb{C}}$ to be the real or complexified Clifford algebras (which really come from representations of SO_n), and the **spinor bundles** given by taking the spin representation of Spin_n (which does not). To say more, the group Spin_n has a complex representation $\mathcal{S}$ of dimension 2^n called the **spin representation**. If n is odd, this representation is irreducible; if n is even, then this representation splits into two irreducible representations of dimension $2^{n/2}$ each, denoted $\mathcal{S}^{\pm}$ and called the **half spin representations**. If X is a spin manifold of even dimension $n \geq 2$, the associated bundle construction mentioned above gives rise to two complex vector bundles $\mathcal{S}^{\pm}(X)$ on X called the **spinor bundles**, sections of which are called **spinor fields** or simply **spinors** on X. In this case, we can construct using the Levi-Civita connection of X and Clifford multiplication a first order elliptic differential operator

$$\not{D}^+ : \mathcal{S}^+(X) \to \mathcal{S}^-(X)$$

called the **Dirac** operator. In analogy with the Hodge-Dirac operator introduced in §1.3, spinors annihilated by $\not{D}^+$ are called **harmonic spinors**. More generally, if $E \to X$ is a complex vector bundle, then the there is a first order elliptic differential operator

$$\not{D}^+_E : \mathcal{S}^+(X) \otimes E \to \mathcal{S}^-(X) \otimes E$$

called the **twisted Dirac operator**, also called the **twisted Atiyah-Singer operator**. In this terminology, we have

Theorem 1.5.5. Let X be a closed spin manifold of even dimension $n \geq 2$. If $E \to X$ is any complex vector bundle over X, then the index of the twisted Dirac operator $\not{D}^+_E$ is given by

$$\mathrm{ind}(\not{D}^+_E) = \int_X \mathrm{ch}(E) \cdot \hat{\mathsf{A}}(X).$$

In particular, the quantity on the right is an integer.

Proof. The strategy is to use Theorem 1.1.10 for the twisted Dirac operator $\not{D}_E^+$ and to massage the right side of Theorem 1.1.10 to obtain the required form. For details, we refer the reader to [8, Thm. 5.3] or [2, Ch. III, Thm. 13.10]. ■

From this result, we have now obtained the awaited

Corollary 1.5.6. Let X be a closed spin manifold of even dimension $n \geq 2$. Then the $\hat{A}$-genus $\hat{A}(X)$ of X is an integer. If $n \equiv 4 \pmod 8$, then $\hat{A}(X)$ is an even integer.

Proof. Take E to be the trivial bundle in Theorem 1.5.5. In the case $n \equiv 4 \pmod 8$, the additional factor of two comes from observing that in these dimensions, the spin representations are actually quaternionic. In particular, the kernel and cokernel of the Dirac operator $\not{D}^+$ are (finite dimensional) quaternionic vector spaces, and hence their complex dimensions are even. Therefore, the equality $\hat{A}(X) = \operatorname{ind}(\not{D}^+) = \dim\ker \not{D}^+ - \dim\operatorname{coker} \not{D}^+$ tells us that $\hat{A}(X)$ must be even as well. See [2, Ch. IV, Thm. 1.1]. ■

Corollary 1.5.7 (Rochlin)**.** The signature of a closed orientable smooth spin 4-manifold is a multiple of 16.

Proof. Recall (see Examples 3.3.9 and 3.3.11) that the first L and $\hat{A}$ polynomials are given by

$$L_1 = \frac{1}{3}p_1 \text{ and } \hat{A}_1 = -\frac{1}{24}p_1.$$

These formulae, combined with Theorem 1.3.1, imply that if X is a closed, oriented 4-manifold, then the signature and $\hat{A}$-genus of X are related as

$$\operatorname{Sign}(X) = -8\hat{A}(X).$$

In particular, irrespective of whether X is spin, we have that $\hat{A}(X) \in \frac{1}{8}\mathbb{Z}$. In case X *is* spin, the result then follows from Corollary 1.5.6. ■

Remark 1.5.8. This theorem of Rochlin (which was known before the whole machinery of Index Theory was developed) also answers the question of the existence of smooth structures on topological manifolds. Specifically, a very natural first question to ask in the theory of manifolds is whether every topological manifold admits a smooth structure. In 1982, Michael Freedman showed by construction that if Q is any unimodular symmetric bilinear form, then there is a simply connected topological closed oriented 4-manifold X with intersection form Q. In particular, if we choose a unimodular symmetric bilinear form with signature 8 (one such example being the E_8 lattice–the unique positive definite, even, unimodular lattice of rank 8), then we may construct a topological 4-manifold $X(E_8)$ with intersection form E_8. One can then show that if $X(E_8)$ admitted a smooth structure, then it would be spin, but then by Corollary 1.5.7, its signature would be divisible by 16, which is a contradiction. This argument proves that the topological manifold $X(E_8)$ does not admit any smooth structure, and $X(E_8)$ was one of the first manifolds shown to have this property. For details, see the discussion following [2, Ch. IV, Cor. 1.2], and the references mentioned there.

One other result that we will mention, a proof of which requires even more machinery than we have mentioned so far, is

Theorem 1.5.9 (Lichnerowicz)**.** If X is a closed spin manifold that admits a Riemannian metric with positive scalar curvature, then $\hat{A}(X) = 0$.

Proof. This uses the vanishing of the ring homomorphism

$$\hat{\mathscr{A}} : \Omega_*^{\mathrm{Spin}} \to KO^{-*}(\mathrm{pt})$$

on a compact spin manifold admitting a Riemannian metric with positive scalar curvature; see Theorem [2, Ch. IV, Thm. 4.1] for details. ■

We do not use Theorem 1.5.9 in what follows in any serious way. Rather, we only point out verifications of it via direct computation in specific examples.

Remark 1.5.10. Classical examples of spaces with nonnegative scalar curvatures are homogenous spaces with normal metrics: given a compact real Lie group G with Lie algebra $\mathfrak{g}$, an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ (e.g. the Killing form when $\mathfrak{g}$ is semisimple), and a closed (hence Lie) subgroup $H \subset G$, the homogenous space $X = G/H$ admits a G-invariant metric making $G \to X$ a Riemannian submersion. Except for the one case of the flat torus, this metric then always has positive scalar curvature. Examples of this sort are spheres and real and complex flag varieties (so in particular, real and complex Grassmannians and hence projective spaces). This is a special case of a more general result by Lawson and Yau that a compact manifold admitting a smooth, effective action by a connected non-abelian Lie group admits a metric with positive scalar curvature. The above theorem then says that whenever such a manifold is spin, its $\hat{A}$ genus must necessarily vanish. See the discussion following [2, Ch. IV, Cor. 4.2] and the article by Lawson and Yau [20] in which they prove this result.

In closing, we remark only that this chapter barely scratches the surface of index theory. Besides the theorems and applications already mentioned, there are

(a) equivariant versions of Theorem 1.1.10, generalizing, for instance, the Lefschetz Fixed Point Formula to the Atiyah-Segal-Singer Fixed Point Theorem,
(b) generalizations of Theorem 1.1.10 to manifolds with boundary, such as the Atiyah-Patodi-Singer Index Theorem,

and many more. These techniques then have a wide variety of applications in geometry and topology, with ramifications even in theoretical physics; sample applications include, in addition to the contents of Chapter 2, applications to problems of vector fields on spheres, immersions of manifolds into Euclidean space, twistor geometry, reduced holonomy on Calabi-Yau manifolds, and even to the positive mass conjecture in general relativity. See the original article [8] and the textbooks [2] and [7] for an introduction to the wide variety of things index theory can do.

Chapter 2

Examples

The only way to learn mathematics is to do mathematics.

PAUL HALMOS

In this chapter, we pursue an eclectic list of computations verifying, illustrating, or giving sample applications of, the general theorems discussed in the previous chapter, hopefully illuminating how far-reaching the consequences–how expansive the joy–of the famed Atiyah-Singer Index Theorem really is. This chapter is the heart of the thesis.

2.1 Spheres

I think our lives are surely but the dreams
Of spirits, dwelling in the distant spheres,
Who as we die, do one by one awake.

Poppies and Mandragora
Edgar Saltus

We start with the simplest examples of closed oriented manifolds, namely the spheres. For $n \geq 0$, let S^n denote the n-sphere, i.e.

$$S^n := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}. \tag{2.1}$$

The first, warm-up, computation is

Proposition 2.1.1. For $n \geq 0$, we have

$$\chi(S^n) = 1 + (-1)^n.$$

Proof 1. The space S^n admits a CW decomposition with one 0-cell and one n-cell[1], and so we are done by the definition of the Euler characteristic of a finite CW complex as

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i c_i,$$

where c_i is the number of i-cells. Alternatively, we could give S^n a CW decomposition with exactly two cells in each dimension j for $0 \leq j \leq n$, with the property that the j-skeleton of S^n is exactly S^j, and the two attaching maps of the $(j+1)$-cells are both the identity map $S^j \to S^j$.[2] With respect to this structure, the Euler characteristic is

$$\chi(S^n) = \sum_{i=0}^{n} (-1)^i \cdot 2 = 1 + (-1)^n.$$

Another slightly different way to phrase the same argument is to use that the cohomology ring of S^n is given by

$$\mathrm{H}^*(S^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2) = \mathbb{Z} \oplus \mathbb{Z}\alpha \text{ with } |\alpha| = n.$$[3]

It follows that the Betti numbers $b_i(S^n)$ of the sphere are all zero except for $i \in \{0, n\}$, with $b_0(S^0) = 2$ and $b_0(S^n) = b_n(S^n) = 1$ for $n \geq 1$. In both cases, the Poincaré polynomial of S^n is

$$p_t(S^n) = 1 + t^n,$$

and plugging in $t = -1$ recovers the result. ■

[1] There is only one possible attaching map! Note that for $n = 0$, this is to be interpreted as saying that S^0 admits a CW decomposition with two 0-cells.

[2] The advantage of this decomposition is that it is $\mathbb{Z}/2$-invariant under the identifications of the new cells as the "upper" and "lower" hemispheres, and hence descends to a CW structure on $\mathbb{RP}^n$ (see Proposition 2.2.2).

[3] This really is saying something nontrivial, because, for instance, there is a way to compute this cohomology ring inductively without invoking any CW structure on the sphere.

Proof 2. This argument is adapted from [11, Ch. 3]. We compute the Lefschetz number $L(f)$ of a Lefschetz map f homotopic to the identity; the result then follows Corollary 1.2.5. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ denote the projection map $v \mapsto v/|v|$, and for $t \in [0, 1)$ the map $f_t : S^n \to S^n$ given by

$$f_t(x_1, x_2, \ldots, x_n, x_{n+1}) = \pi(x_1, x_2, \ldots, x_n, x_{n+1} - t);$$

from this formula, it is clear that each f_t is homotopic to $f_0 = \mathrm{id}_{S^n}$. For a fixed $t \in (0, 1)$, the map f_t has exactly two fixed points, namely at the "poles" $p_\pm := (0, \ldots, 0, \pm 1)$. A straightforward computation shows that

$$(\mathrm{d}f_t)(p_\pm) = \frac{1}{1 \mp t + t^2}\, \mathrm{id}_{\mathrm{T}_{p_\pm} S^n}\,.$$

In particular, for $t \in (0, 1)$, this matrix does not have eigenvalue $+1$, so that f_t is a Lefschetz map. See Figure 2.1.

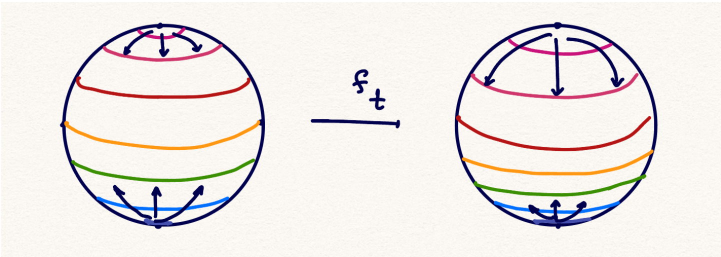

Figure 2.1: A pictorial representation of the map f_t on the sphere S^n for $t \in (0, 1)$. The map f_t is expanding near the "north pole" p_+ by a factor of $(1-t+t^2)^{-1}$, and contracting near the "south pole" by a factor of $1 + t + t^2$.

Finally, the local Lefschetz numbers of f_t at $p_\pm$ are given by

$$\begin{aligned} L_{p_\pm}(f) &= \operatorname{sign}\det\left[(\mathrm{d}f_t)(p_\pm) - \mathrm{id}_{\mathrm{T}_{p_\pm} S^n}\right] \\ &= \operatorname{sign}\det\left[\left(\frac{\pm t - t^2}{1 \mp t + t^2}\right)\mathrm{id}_{\mathrm{T}_{p_\pm} S^n}\right] \\ &= \operatorname{sign}\left(\frac{\pm t - t^2}{1 \mp t + t^2}\right)^n, \end{aligned}$$

and this gives us

$$L_{p_+}(f_t) = 1 \text{ and } L_{p_-}(f_t) = (-1)^n.$$

Therefore, it follows from Corollary 1.2.5 that for any $t \in (0, 1)$, we have

$$\chi(X) = L(f_t) = \sum_{x \in X} L_x(f_t) = L_{p_+}(f_t) + L_{p_-}(f_t) = 1 + (-1)^n$$

as needed. ∎

Proof 3. We explicitly compute $\int_{S^n} \mathrm{e}(\mathrm{T}S^n)$ and use the Chern-Gauss-Bonnet Theorem. Recall (see Appendix 3.2.1) that for any smooth, closed, oriented n-manifold X, the Euler class $\mathrm{e}(\mathrm{T}X)$ of $\mathrm{T}X$ can be written as

$$\mathrm{e}(\mathrm{T}X) = \left[\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)\right],$$

where Ω is locally the curvature matrix with respect to any connection and any orthonormal frame of $\mathrm{T}X$ (with respect to any Riemannian metric on X), and Pf denotes the Pfaffian of this $n \times n$ skew-symmetric matrix. We carry out the computation of this Pfaffian explicitly for the Levi-Civita connection on S^n equipped with the round metric.

To do this, we give an oriented parametrization of the sphere S^n using spherical coordinates $\theta_1, \ldots, \theta_n$, where $\theta_1 \in [0, 2\pi)$ and $\theta_j \in [0, \pi)$ for $j = 2, \ldots, n$. To simplify the notation, let

$$c_j := \cos\theta_j \text{ and } s_j := \sin\theta_j \text{ for } j = 1, \ldots, n.$$

With this notation, the parametrization is given by

$$x_j = c_{j-1} \prod_{k=j}^{n} s_k \text{ for } 1 \leq j \leq n+1,$$

where we adopt the convention that $c_0 = 1$. As a column vector,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} s_1 s_2 \cdots s_{n-1} s_n \\ c_1 s_2 \cdots s_{n-1} s_n \\ c_2 s_3 \cdots s_{n-1} s_n \\ \vdots \\ c_{n-2} s_{n-1} s_n \\ c_{n-1} s_n \\ c_n \end{bmatrix}$$

In these coordinates, the round metric on S^n is given by

$$g = \sum_{j=1}^{n} \left(\prod_{k=j+1}^{n} s_k^2 \right) \mathrm{d}\theta_j^2.$$

Let $U \subset S^n$ be the open subset defined by requiring $\theta_1 \in (0, 2\pi)$ and $\theta_j \in (0, \pi)$ for $j = 2, \ldots, n$; then $(\theta_1, \theta_2, \ldots, \theta_n) : (0, 2\pi) \times (0, \pi)^{n-1} \to U$ is a diffeomorphism that is orientation-preserving for even n, and orientation reversing for odd n, if $S^n = \partial\mathbb{D}^{n+1}$ is given its orientation as the boundary of the $n+1$ ball $\mathbb{D}^{n+1}$.[4] Since the complement $S^n \setminus U$ of U in S^n has measure zero, to compute $\int_{S^n} [\mathrm{Pf}(\Omega/2\pi)]$, it suffices to compute this Pfaffian explicitly on U and integrate over U. To do this, we use the oriented orthonormal frame $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ of $\mathrm{T}\mathbb{R}^{n+1}|_{S^n}$ defined by

$$\begin{aligned} \mathrm{e} &= [\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_{n-1}, \mathrm{e}_n, \mathrm{e}_{n+1}] \\ &:= \left[\frac{1}{s_2 \cdots s_n} \frac{\partial}{\partial\theta_1}, \frac{1}{s_3 \cdots s_n} \frac{\partial}{\partial\theta_2}, \ldots, \frac{1}{s_n} \frac{\partial}{\partial\theta_{n-1}}, \frac{\partial}{\partial\theta_n}, \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} \right]. \end{aligned}$$

This frame has the property that $\mathrm{e}_1, \ldots, \mathrm{e}_n$ is an oriented orthonormal frame for $\mathrm{T}S^n|_U$. The key step here is the explicit calculation of Ω as in

[4] This will not concern us, since we will stick to this orientation throughout, and the result about $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ being an oriented frame of $\mathbb{R}^{n+1}$ (see below) still remains correct. We will also only be computing the Pfaffian for even n anyway.

Lemma 2.1.2. Let $\Omega = [\Omega_{ij}]_{i,j=1}^{n}$ be the curvature matrix of the Levi-Civita connection on S^n with respect to the oriented orthonormal frame $\mathrm{e}_1, \dots, \mathrm{e}_n$ of $\mathrm{T}S^n|_U$. Then

$$\Omega_{ij} = \left(\prod_{p=i+1}^{j} s_p \prod_{q=j+1}^{n} s_q^2 \right) \mathrm{d}\theta_i \wedge \mathrm{d}\theta_j \text{ for } 1 \leq i < j \leq n,$$

with the rest of the entries determined by skew-symmetry.

For instance, for $n = 4$, we have

$$\Omega = \begin{bmatrix} 0 & s_2 s_3^2 s_4^2\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 & s_2 s_3 s_4^2\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_3 & s_2 s_3 s_4\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_4 \\ -s_2 s_3^2 s_4^2\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 & 0 & s_3 s_4^2\, \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 & s_3 s_4\, \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_4 \\ -s_2 s_3 s_4^2\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_3 & -s_3 s_4^2\, \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3 & 0 & s_4\, \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_4 \\ -s_2 s_3 s_4\, \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_4 & -s_3 s_4\, \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_4 & -s_4\, \mathrm{d}\theta_3 \wedge \mathrm{d}\theta_4 & 0 \end{bmatrix}.$$

Postponing the proof of this lemma for a moment, we proceed as follows. The result for odd n is clear, since the Pfaffian of a skew-symmetric matrix of odd order is zero. Suppose, therefore, that we are working with S^{2n} for $n \geq 1$. We can now use Lemma 2.1.2 with the explicit formula from Lemma 3.2.5 to compute the form representing $\mathrm{Pf}({}^1\!/\!{}_{2\pi}\Omega)$. To do this, let's recall the notation used in the statement of Lemma 3.2.5. Let Σ_n denote the set of unordered partitions of $\{1, 2, \dots, 2n\}$ into pairs, and write any element $\sigma \in \Sigma_n$ as $\sigma = \{(i_1, j_1), \dots, (i_n, j_n)\}$, where for $1 \leq k \leq n$ the $i_k = i_k(\sigma)$ and $j_k = j_k(\sigma)$ are integers between 1 and $2n$ satisfying $1 \leq i_1 < \cdots < i_n \leq 2n$ and $i_k < j_k$. Given a $\sigma \in \Sigma_n$ we also let $\pi_\sigma \in \mathcal{S}_{2n}$ be the permutation of $\{1, \dots, 2n\}$ such that for $1 \leq k \leq n$ we have

$$\pi_\sigma(2k-1) = i_k \text{ and } \pi_\sigma(2k) = j_k.$$

In this notation, Lemmas 2.1.2 and 3.2.5 tell us that

$$\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right) = \frac{1}{(2\pi)^n} \sum_{\sigma \in \Sigma_n} (-1)^{\pi_\sigma} \cdot \prod_{k=1}^{n} \left(\prod_{p=i_k+1}^{j_k} s_p \prod_{q=j_k+1}^{2n} s_q^2 \right) \mathrm{d}\theta_{i_k} \wedge \mathrm{d}\theta_{j_k}. \tag{2.2}$$

Here's the awaited magical simplification: for any $\sigma \in \Sigma_n$, we claim that

$$\prod_{k=1}^{n} \left(\prod_{p=i_k+1}^{j_k} s_p \prod_{q=j_k+1}^{2n} s_q^2 \right) = s_2 s_3^2 \cdots s_{2n-1}^{2n-2} s_{2n}^{2n-1} = \prod_{\ell=1}^{2n} s_\ell^{\ell-1}.$$

Indeed, this is an immediate consequence of the following combinatorial lemma.

Lemma 2.1.3. For each ℓ such that $1 \leq \ell \leq 2n$ and $\sigma \in \Sigma_n$, let a_ℓ and b_ℓ be integers defined by

$$\begin{aligned} a_\ell &= \#\{k : 1 \leq k \leq n \text{ such that } i_k < \ell \leq j_k\}, \text{ and} \\ b_\ell &= \#\{k : 1 \leq k \leq n \text{ such that } j_k < \ell\}, \end{aligned}$$

where the # denotes the cardinality of the specified set. Then

$$a_\ell + 2b_\ell = \ell - 1.$$

Proof. Let

$$c_\ell := a_\ell + b_\ell = \#\{k : 1 \le k \le n \text{ such that } i_k < \ell\}.$$

It follows from $i_1 < i_2 < \cdots < i_n$ that c_ℓ is the largest integer between 1 and n such that $i_{c_\ell} < \ell$. In particular, $i_1, \ldots, i_{c_\ell}$ constitutes a set of c_ℓ integers, each less than ℓ, and these are the only i_k's which are less than ℓ. The remaining $\ell - 1 - c_\ell$ integers less than ℓ must therefore be j's, and there are b_ℓ of these. Therefore,

$$\ell - 1 - c_\ell = b_\ell,$$

which is equivalent to the claim in the lemma. ■

Returning to the main proof, note that the definition of the permutation π_σ implies that

$$(-1)^{\pi_\sigma} \prod_{k=1}^{n} \mathrm{d}\theta_{i_k} \wedge \mathrm{d}\theta_{j_k} = \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \cdots \mathrm{d}\theta_{2n-1} \wedge \mathrm{d}\theta_{2n}.$$

Therefore, all the terms occuring in the sum in (2.2) are the same, this common value being

$$s_2 s_3^2 \cdots s_{2n-1}^{2n-2} s_{2n}^{2n-1} \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \cdots \mathrm{d}\theta_{2n-1} \wedge \mathrm{d}\theta_{2n}.$$

Since there are

$$\#\Sigma_n = (2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$$

of these, it follows that

$$\int_{S^n} \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right) = \frac{(2n-1)!!}{(2\pi)^n} \cdot \int_0^{2\pi} \mathrm{d}\theta_1 \cdot \prod_{\ell=2}^{2n} \int_0^{\pi} \sin^{\ell-1}\theta_\ell \, \mathrm{d}\theta_\ell.$$

Temporarily denoting the quantity on the right by t_n, the result then follows from noting that

$$t_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{\pi} \sin\theta_2 \, \mathrm{d}\theta_2 = \frac{1}{2\pi} \cdot 2\pi \cdot 2 = 2,$$

and that for $n \ge 2$, we have

$$\begin{aligned} \frac{t_n}{t_{n-1}} &= \frac{2n-1}{2\pi} \int_0^{\pi} \sin^{2n-2}\theta_{2n-1} \, \mathrm{d}\theta_{2n-1} \int_0^{\pi} \sin^{2n-1}\theta_{2n} \, \mathrm{d}\theta_{2n} \\ &= \frac{2n-1}{2\pi} \mathrm{B}\left(n - \frac{1}{2}, \frac{1}{2}\right) \mathrm{B}\left(n, \frac{1}{2}\right) \\ &= \frac{2n-1}{2\pi} \cdot \frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} \cdot \frac{\Gamma(n-1/2)\Gamma(1/2)}{\Gamma(n)} = 1, \end{aligned}$$

where B and Γ denote the beta and gamma functions respectively. ■

To finish the proof, it remains only to prove Lemma 2.1.2, which we do now.

Proof of Lemma 2.1.2. If ∂_x denotes the orthonormal basis for $\mathrm{T}\mathbb{R}^{n+1}$ given by

$$\partial_x := \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n+1}},\right]$$

then the change-of-basis matrix $a \in \mathrm{SO}_{n+1}$ defined by

$$\mathrm{e} = \partial_x \cdot a$$

is given explicitly by

$$a_{ij} = \begin{cases} c_{i-1}\left(\prod_{k=i}^{j-1} s_k\right) c_j, & 1 \leq i \leq j \leq n, \\ -s_j, & 1 \leq j \leq n \text{ and } i = j+1, \\ x_i, & 1 \leq i \leq n+1 \text{ and } j = n+1, \\ 0 & \text{else}, \end{cases},$$

i.e. the matrix a looks like

$$a = \begin{bmatrix} c_1 & s_1 c_2 & \cdots & s_1 \dots s_{n-1} c_n & s_1 \dots s_n \\ -s_1 & c_1 c_2 & \cdots & c_1 s_2 \cdots s_{n-1} c_n & c_1 s_2 \dots s_n \\ & -s_2 & \ddots & \vdots & \vdots \\ & & \ddots & c_{n-1} c_n & c_{n-1} s_n \\ & & & -s_n & c_n \end{bmatrix},$$

where the unfilled spots represent entries that are zero. Now the connection matrix $\theta^{\mathrm{Euc}}_{\mathrm{e}}$ for the Euclidean connection on $\mathrm{T}\mathbb{R}^{n+1}|_{S^n}$ with respect to e is related to the connection matrix $\theta^{\mathrm{Euc}}_{\partial}$ with respect to ∂_x via

$$\theta^{\mathrm{Euc}}_{\mathrm{e}} = a^{-1}\theta^{\mathrm{Euc}}_{\partial} a + a^{-1}\mathrm{d}a = a^t\,\mathrm{d}a, \tag{2.3}$$

where we have used both that $\theta^{\mathrm{Euc}}_{\partial} = 0$ and that $a \in \mathrm{SO}_{n+1}$. The connection matrix θ for the Levi-Civita connection on S^n is the matrix obtained from $\theta^{\mathrm{Euc}}_{\mathrm{e}}$ by deleting its last row and column. Using the explicit form of a and (2.3), we can then conclude from a straightforward, if lengthy, computation that $\theta = [\theta_{ij}]_{i,j=1}^n$ is given by

$$\theta_{ij} = -\left(\prod_{k=i+1}^{j-1} s_k\right) c_j\,\mathrm{d}\theta_i$$

for $1 \leq i < j \leq n$, where the rest of the entries are determined by skew-symmetry. For instance, for $n = 4$, this matrix is given by

$$\theta = \begin{bmatrix} 0 & -c_2\,\mathrm{d}\theta_1 & -c_3 s_2\,\mathrm{d}\theta_1 & -c_4 s_3 s_2\,\mathrm{d}\theta_1 \\ c_2\,\mathrm{d}\theta_1 & 0 & -c_3\,\mathrm{d}\theta_2 & -c_4 s_3\,\mathrm{d}\theta_2 \\ c_3 s_2\,\mathrm{d}\theta_1 & c_3\,\mathrm{d}\theta_2 & 0 & -c_4\,\mathrm{d}\theta_3 \\ c_4 s_3 s_2\,\mathrm{d}\theta_1 & c_4 s_3\,\mathrm{d}\theta_2 & c_4\,\mathrm{d}\theta_3 & 0 \end{bmatrix}.$$

The result then follows from the identity

$$\Omega = \mathrm{d}\theta + \theta \wedge \theta,$$

and again a straightforward computation. ■

2.1.1 Almost Complex Structures on Spheres

Next, we pursue the question of the existence of almost complex structures on spheres.[5] The key result here is:

Theorem 2.1.4. [Borel-Serre, 1953] Let $n \geq 1$ be an integer. If $n \notin \{1, 3\}$, then the sphere S^{2n} does not admit an almost complex structure compatible with any smooth structure on it.

TO DO: Borel-Serre. We will return to various proofs of this result shortly; let us discuss the case $n \in \{1, 3\}$ now. The two remaining spheres S^2 and S^6 *do* indeed admit almost complex structures. One the one hand, the situation with S^2 is as simple as it gets: S^2 has a unique smooth structure and complex structure.[6] The uniquess of the smooth structure follows from the classification of smooth oriented surfaces up to diffeomorphism. The uniqueness of the complex structure amounts to to saying that any Riemann surface of (topological) genus 0 is biholomorphic to $\mathbb{CP}^1$, which is a standard result often proved also as a consequence of the Riemann-Roch Theorem (see [17, Prop. VII.1.7]).

On the other hand, the six sphere S^6 also admits a unique smooth structure. This is a harder result and follows from the Smale's resolution of the generalized Poincaré conjecture (in the topological category) in dimensions at least 5, along with a computation by Kervaire and Milnor that $\Theta_6 = 0$, where Θ_n is the group of h-cobordism classes of oriented n-spheres; see [15] and [22].

The sphere S^6 also admits an almost complex structure: in 1947, A. Kirchhoff explicitly constructed an almost complex structure on S^6 using the octonions (see [23]). Subsequently, another proof of this result was obtained using obstruction theory. We isolate this result below, giving these two proofs. However, the question of the existence of an integrable complex structure on S^6, referred to as the Hopf problem, is much harder; see this expository article [24] for a history of this problem. Firstly, it was shown independently by Ehresmann and Libermann [25] and Eckmann and Frölicher [26] in 1951 that the particular almost complex structure constructed by Kirchhoff above is not integrable. Since these papers are hard to locate (and in French), we reproduce their argument below as well, following this excellent survey article [27]. For a slightly different argument that applies more generally to any closed orientable hypersurface $X \subset \mathrm{Im}(\mathbb{O})$, see [28]. This still leaves open the question of the existence of *some other* almost complex structure on S^6 that is integrable. This question has a long history riddled with controversy[7], but this problem is still open as of this writing in March 2024.

[5] We review the relevant definitions and fundamental results in Appendix 3.6.

[6] For surfaces, every almost complex structure is integrable. This follows simply from the Newlander-Nirenberg Theorem 3.6.3 and the fact that if (X, J) is a closed real surface X with an almost complex structure J, then $\overline{\partial}_J^2 = 0$ for dimensional reasons, because $\Omega^{p,q}(X) = 0$ for $q \geq 2$. It can also be proven quite directly; see for instance [21, Homework 10, Ex. 1(c)].

[7] See, for instance, the discussion on MathOverflow at `https://mathoverflow.net/questions/1973/is-there-a-complex-structure-on-the-6-sphere`, or the several others linked to in [24].

Let us now return to the case of S^6.

Proposition 2.1.5. The six sphere S^6 admits an almost complex structure.

Proof 1 of Proposition 2.1.5. We follow Kirchhoff's classical argument from [23], following [27]. Think of

$$S^6 = \{p \in \mathrm{Im}(\mathbb{O}) : p^*p = 1\} \subset \mathrm{Im}(\mathbb{O})$$

as the set of purely imaginary octonions of octonionic norm 1.[8] Recall the identity

$$pq = \langle p, q\rangle + p \times q \tag{2.4}$$

valid for all $p, q \in \mathrm{Im}(\mathbb{O})$, where $\times$ denotes the cross product of octonions defined by

$$p \times q = \frac{1}{2}\left(pq - qp\right),$$

and $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product on $\mathbb{O}$. This identity, along with the fact that $p \times q \perp p$, implies immediately that left multiplication by $p \in S^6$ preserves the subspace $p^\perp \cap \mathrm{Im}(\mathbb{O}) \subset \mathrm{Im}(\mathbb{O})$ of $\mathrm{Im}(\mathbb{O})$ parallel to T_pS^6, where $p^\perp$ denotes the orthogonal complement of the span of p with respect to $\langle\cdot,\cdot\rangle$. Further, this operation squares to -1: indeed, we have for any $p \in S^6$ and $q \in \mathbb{O}$ that

$$p(pq) = (pp)q = (-p^*p)q = -\left\|p\right\|^2 q = -q,$$

where in the first step we have used that the octonions, although not associative, form an alternating algebra. It follows that if N is the normal vector field on $S^6 \subset \mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$, then left multiplication by N defines an endomorphism $J = N\cdot$ of $\mathrm{T}S^6$ that satisfies $J^2 = -\mathrm{id}$, i.e. is an almost complex structure on S^6. ∎

Proof 2 of Proposition 2.1.5. We use obstruction theory as reviewed in Appendix 3.6; specifically, we use Theorem 3.6.4. This theorem, coupled with the Hurewicz Theorem, says that if X^{2n} is a $(2n-1)$-connected manifold, then the unique obstruction to the existence of an almost complex structure on X lies in $\mathrm{H}^{2n}(X; \pi_{2n-1}(\mathrm{SO}_{2n} / \mathrm{U}_n))$. In our case of $n = 3$ and $X = S^6$, the sphere S^6 is 5-connected, so that this unique obstruction lies in $\mathrm{H}^6(S^6; \pi_5\mathbb{CP}^3)$, where we have also used Proposition 3.6.5 to identify $\mathrm{SO}_6 / \mathrm{U}_3 \cong \mathbb{CP}^3$. But now the long exact sequence in homotopy associated to the fiber bundle $S^1 \to S^7 \to \mathbb{CP}^3$ shows that $\pi_5\mathbb{CP}^3 = 0$, and hence this obstruction vanishes. ∎

Remark 2.1.6. In fact, since also $\pi_6\mathbb{CP}^3 = 0$ from the same sequence, it follows again from Theorem 3.6.4 that the space of almost complex structures on S^6 is connected. In fact, it is a conjecture of Sullivan from 1977 (see [30]) that the space of all almost complex structures on S^6 is homotopy equivalent to $\mathbb{RP}^7$. As far as I can tell, this conjecture is still unresolved, although some progress has been made towards it. For instance, Milivojevic has shown (see [31]) that this space is rationally homotopy equivalent to S^7.

Next, we show that almost complex structure on S^6 constructed in Proof 1 is not integrable, again following [27].

[8] For a review of the basic properties of the octonions, see the excellent expository article [29].

Proposition 2.1.7. The almost complex structure on S^6 constructed in Proof 1 of Proposition 2.1.5 using the octonions is not integrable.

Proof. We compute the Nijenhuis tensor explicitly, and show that it does not vanish, concluding the argument using the Newlander-Nirenberg Theorem (Theorem 3.6.3). Recall that the Nijenhuis tensor on an almost complex manifold (X, J) is defined by the property that if ξ, η are local vector fields on X, then

$$\mathcal{N}(\xi, \eta) := [J\xi, J\eta] - J[\xi, J\eta] - J[J\xi, \eta] - [\xi, \eta]. \tag{2.5}$$

To compute the Nijenhuis tensor for the almost complex structure $(S^6, N\cdot)$ constructed above, note that we are in Euclidean space, so computation of Lie brackets is easy: if λ is a vector field on S^6, then we can think of it is a smooth map $\lambda : S^6 \to \mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ such that at any $p \in S^6$, we have $\lambda(p) \perp p$, and so its derivative $\mathrm{d}\lambda$ can be thought of as an $\mathrm{Im}(\mathbb{O})$-valued 1-form. In particular, given two vector fields λ, μ on S^6 it makes sense to apply $\mathrm{d}\lambda$ to μ and use the product structure of $\mathbb{O}$ to obtain a vector-valued function $\mathrm{d}\lambda(\mu) : S^6 \to \mathbb{O}$. In these terms, the Lie bracket $[\lambda, \mu]$ of two vector fields λ, μ on S^6 can then be written as

$$[\lambda, \mu] = \mathrm{d}\mu(\lambda) - \mathrm{d}\lambda(\mu). \tag{2.6}$$

Also, differentiating the identity

$$(J\lambda)_p = p \cdot \lambda_p$$

for $p \in S^6$ and using that the product structure on $\mathbb{O}$ is bilinear, we conclude from the following "product rule" that

$$\mathrm{d}(J\lambda)(\mu) = J(\mathrm{d}\lambda(\mu)) + \mu \cdot \lambda, \tag{2.7}$$

where the second term is the pointwise product of the vector fields μ and λ. It follows from (2.5), (2.6) and (2.7) after some straightforward simplification that the Nijenhuis tensor of two vector fields ξ, η on S^6 can be expressed as

$$\mathcal{N}(\xi, \eta) = J\xi \cdot \eta - J\eta \cdot \xi - J(\xi \cdot \eta) + J(\eta \cdot \xi);$$

in other words, we have for $p \in S^6$ and $q, r \in \mathrm{T}_p S^6$ that

$$\begin{aligned} \mathcal{N}_p(q, r) &= (pq)r - (pr)q - p(qr) + p(rq) \\ &= [p, q, r] - [p, r, q] \\ &= 2[p, q, r], \end{aligned}$$

where $[p, q, r] = (pq)r - p(qr)$ is the associator in $\mathbb{O}$ and in the last step we have used that the associator in $\mathbb{O}$ is alternating. In particular, the result follows from the fact that this associator is not identically zero. ■

Remark 2.1.8. The non-integrability of Kirchhoff's almost complex structure on S^6, therefore, comes really from the non-associativity of the octonions.

Let us now return to Theorem 2.1.4, which we restate here for convenience.

Theorem 2.1.4. [Borel-Serre, 1953] Let $n \geq 1$ be an integer. If $n \notin \{1, 3\}$, then the sphere S^{2n} does not admit an almost complex structure compatible with any smooth structure on it.

We will first give a proof using the Atiyah-Singer Index Theorem, which is quite similar in outline to the original proof by Borel and Serre, although the latter used K-Theory and Bott Periodicity directly to obtain Proposition 2.1.9 (see the original here [32], or the review paper [33] if you prefer English). After that, we will give two other proofs of this result that illustrate the breadth of the mathematics involved.

Proof 1 of Theorem 2.1.4. The key observation here is:

Proposition 2.1.9. If $E \to S^{2n}$ is a complex vector bundle of rank n, then the top Chern class

$$\int_{S^{2n}} c_n(E) \in \mathbb{Z}$$

of E is divisible by $(n-1)!$.

The result follows immediately from this observation: if for some $n \geq 1$ there is a complex rank n bundle $E \to S^{2n}$ such that $E_{\mathbb{R}} \cong \mathrm{T}S^{2n}$, then Proposition 2.1.9 combined with Remark 3.4.23 and the Chern-Gauss-Bonnet Theorem (Theorem 1.2.1) tells us that

$$(n-1)! \text{ divides } \int_{S^{2n}} c_n(E) = \int_{S^{2n}} \mathrm{e}(\mathrm{T}S^{2n}) = \chi(S^{2n}) = 2,$$

proving $n \leq 3$.[9] This leaves only the case of S^4, which we handle separately below in Proposition 2.1.16. ■

It then remains to prove this proposition.

Proof of Proposition 2.1.9. Note that for $n \geq 1$, the manifold S^{2n}, equipped with any smooth structure, admits a unique spin structure; indeed, this follows immediately for $n \geq 2$ from Theorem 1.5.2 and the sparsity of the cohomology ring $\mathrm{H}^*(S^{2n}; \mathbb{Z}/2)$, and it follows for $n = 1$ from the fact that S^2 admits a unique smooth structure which is diffeomorphic to the Riemann surface $\mathbb{CP}^1$, and the fact that every Riemann surface is spin (Corollary 1.5.4). It follows then from Theorem 1.5.5 that the integral

$$\int_{S^{2n}} \mathrm{ch}(E) \cdot \hat{\mathsf{A}}(S^{2n}) \tag{2.8}$$

can be expressed as the index of an elliptic operator, and is in particular an integer. Next, we observe that S^{2n} is stably parallelizable. Precisely stated, we have

[9] Note also that we did not assume that the smooth structure on S^{2n} is the "usual" one: this argument only uses the Chern-Gauss-Bonnet Theorem and $\chi(S^{2n}) = 2$, which is a topological statement, and is therefore vanish for any smooth structure on S^{2n}.

Lemma 2.1.10. If $k \geq 1$ is an integer and $X \subset \mathbb{R}^{k+1}$ is any closed orientable hypersurface, then X is stably parallelizable, i.e. the tangent bundle $\mathrm{T}X$ of X is stably trivial. In particular, the Pontryagin classes $p_j(X)$ vanish for $j \geq 1$.

In particular, it follows that in the above case, the total $\hat{A}$-class of X is trivial, i.e. is given by $\hat{\mathsf{A}}(X) = 1 \in \mathrm{H}^*(X; \mathbb{Q})$.

Proof of Lemma 2.1.10. This follows from considering the short exact sequence of vector bundles

$$0 \to \mathrm{T}X \to \mathrm{T}\mathbb{R}^{k+1}|_X \to \mathcal{N}_{X/\mathbb{R}^{k+1}} \to 0$$

on X, where the last bundle is the normal line bundle of the embedding. Since $\mathrm{T}X$ and $\mathrm{T}\mathbb{R}^{k+1}|_X$ are orientable, it follows from this sequence that $\mathcal{N}_{X/\mathbb{R}^{k+1}}$ is orientable; but an orientable real line bundle is trivial. Therefore,

$$\mathrm{T}X \oplus \underline{\mathbb{R}}_X \cong \mathrm{T}X \oplus \mathcal{N}_{X/\mathbb{R}^{k+1}} \cong \mathrm{T}\mathbb{R}^{k+1}|_X \cong \underline{\mathbb{R}}_X^{k+1},$$

telling us that $\mathrm{T}X$ is stably trivial. The vanishing of the higher Pontryagin classes then follows from their multiplicativity. ∎

Applying this lemma to $X = S^{2n}$ tells us that the quantity in (2.8) can be expressed as

$$\int_{S^{2n}} \mathrm{ch}_n(E).$$

Now suppose $\gamma_1, \ldots, \gamma_n$ are the Chern roots of E. Then

$$\mathrm{ch}_n(E) = \frac{1}{n!} p_n(\gamma) = \frac{1}{n!}(-1)^{n-1} n c_n(E) = (-1)^{n-1} \frac{c_n(E)}{(n-1)!},$$

where $p_n(\gamma) = \sum_{i=1}^n \gamma_i^n$ is the n^{th} power sum in the γ_i, and in the second step we have used the Newton's Identity (1) and the fact that $c_{n-i}(E) \in \mathrm{H}^{2n-2i}(S^{2n}; \mathbb{Z}) = 0$ for $i = 1, \ldots, n-1$. Therefore, the result follows from the fact that the integer in (2.8) can be written as

$$(-1)^{n-1} \int_{S^{2n}} \frac{c_n(E)}{(n-1)!}.$$

∎

Proof 2 of Theorem 2.1.4. A second proof carries out an outline initiated by Kirchhoff that was completed by many crowning achievements of algebraic topology in the 20th century. Kirchhoff showed in [23] that if S^{2n} admits an almost complex structure, then S^{2n+1} is parallelizable; again, this is easy to reproduce here and so we do it below. The second part of the proof, then, is finding all parallelizable spheres. This is a very classical problem in algebraic topology, closely related to the Hopf invariant one problem, and was resolved in 1958 by the Bott and Milnor and independently by Adams (with the answer being that the only parallelizable spheres are S^1, S^3 and S^7). The proof of this result uses either secondary operations in ordinary cohomology or primary operations (the Adams operations) in K-theory. Since the literature on this latter topic is quite abundant and standard (see the references below), we will not comment further on it. We now give Kirchhoff's argument, following again [27].

Lemma 2.1.11. Let $n \geq 0$. If S^n admits an almost complex structure, then S^{n+1} is parallelizable.

Proof. Using the coordinates (2.1), embed $S^n \hookrightarrow S^{n+1}$ as the hyperplane section defined by $x_{n+2} = 0$, or equivalently as the set of vectors perpendicular to the last basis vector e_{n+2}. Given a point $p \in S^n$, let $T_p \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$ be the n-dimensional subspace parallel to $\mathrm{T}_p S^n$, and $J_p : T_p \to T_p$ the endomorphism satisfying $J_p^2 = -\operatorname{id}_{T_p}$. By definition, we have $T_p \perp p$. Extend J_p to a linear endomorphism $\tilde{J}_p : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ by setting

$$\tilde{J}_p|_{T_p} = J_p, \quad \tilde{J}_p(\mathrm{e}_{n+2}) = p, \text{ and } \tilde{J}_p(p) = -\mathrm{e}_{n+2}.$$

Then it follows immediately that $\tilde{J}_p^2 = -\operatorname{id}_{\mathbb{R}^{n+2}}$ as well. Now any $q \in S^{n+1}$ other than $q = \pm\mathrm{e}_{n+2}$ can be written as $q = \sin\theta \cdot p + \cos\theta \cdot \mathrm{e}_{n+2}$ for some unique $\theta \in (0, \pi)$ and $p \in S^n$, and we define for such a q the linear endomorphism $\sigma_q : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ by

$$\sigma_q := \sin\theta \cdot \tilde{J}_p + \cos\theta \cdot \operatorname{id}_{\mathbb{R}^{n+2}} .$$

We also define

$$\sigma_{\pm\mathrm{e}_{n+2}} := \pm \operatorname{id}_{\mathbb{R}^{n+2}} .$$

Then each σ_q for $q \in S^{n+1}$ is a linear isomorphism, where for $q \neq \pm\mathrm{e}_{n+2}$, the inverse of σ_q is given by

$$\sigma_q^{-1} = -\sin\theta \cdot \tilde{J}_p + \cos\theta \cdot \operatorname{id}_{\mathbb{R}^{n+2}} .$$

The claim is that for any $q \in S^{n+1}$, the restriction $\sigma_q|_{\mathbb{R}^{n+1}}$ of σ_q to the hyperplane $\mathbb{R}^{n+1} = \{x_{n+2} = 0\} \subset \mathbb{R}^{n+2}$ is an isomorphism onto $q^\perp$, the orthogonal complement of q in $\mathbb{R}^{n+2}$; putting all of the $\sigma_q|_{\mathbb{R}^{n+1}}$ together then yields an isomorphism of vector bundles $\sigma : \underline{\mathbb{R}}^{n+1} \to \mathrm{T}S^{n+1}$ over S^{n+1}. This claim is trivial for $q = \pm\mathrm{e}_{n+2}$, so suppose that $q \neq \pm\mathrm{e}_{n+2}$. Then, since σ_q is injective and $\dim q^\perp = n+1$, it suffices to show that $\sigma_q(p) \in q^\perp$ and $\sigma_q(T_p) \subset q^\perp$, where $q = \sin\theta \cdot p + \cos\theta \cdot \mathrm{e}_{n+2}$ as before. The first of these follows from

$$\sigma_q(p) = \cos\theta \cdot p - \sin\theta \cdot \mathrm{e}_{n+2},$$

and the second of these follows from the fact that if $r \in T_p$, then

$$\sigma_q(r) = \sin\theta \cdot J_p(r) + \cos\theta \cdot r,$$

and both r and $J_p(r)$ are orthogonal to q. ■

For instance, for $n = 0$ and 2, the standard (almost) complex structures on S^0 and S^2 yield, via this construction, familiar parallelizations of the spheres S^1 and S^3. See Figure 2.2 for an illustration of the case $n = 0$.

As mentioned above, the rest of the proof is finished by the following hammer.

Theorem 2.1.12 (Adams)**.** Let $n \geq 1$. If S^n is parallelizable, then $n \in \{1, 3, 7\}$.

Proof. See [34, Chapter 24, §6] or [35, §2.3]. ■

■

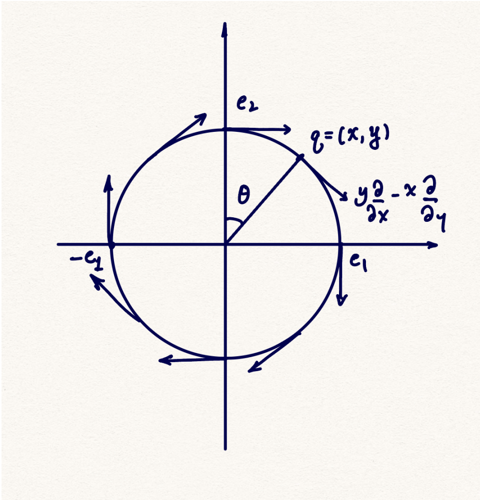

Figure 2.2: When $n = 0$, we have $S^0 = \{\pm \mathrm{e}_1\}$. Then $\tilde{J}_{\pm \mathrm{e}_1} = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and the map σ_q for a point $q = (x, y) \in S^1$ is given by $\sigma_{(x,y)} = \begin{bmatrix} y & x \\ -x & y \end{bmatrix}$. Thus, the trivialization of $\mathrm{T}S^1$ produced by this procedure is given by the nonvanishing vector field $y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ on S^1.

Proof 3 of Theorem 2.1.4. This third proof using Steenrod operations modulo an odd prime p is due to Borel and Serre [36]; the following presentation has been adapted from this MathOverflow answer by Sen, [37]. The key ingredient here is a weaker version of Proposition 2.1.9 that can be proved with the mod p Steenrod operations.

Proposition 2.1.13. If $E \to S^{2n}$ is a complex vector bundle of rank n, then the top Chern class

$$\int_{S^{2n}} c_n(E) \in \mathbb{Z}$$

of E is divisible by every odd prime $p < n$ not dividing n.

As before, the result follows immediately from this: for $n \geq 4$, there *is* an odd prime $p < n$ not dividing n, so if $\mathrm{T}S^{2n}$ were a complex vector bundle we would conclude as in Proof 1 that p divides 2, which is absurd. This shows that $n \leq 3$; the case $n = 2$ is again handled separately, as in Proposition 2.1.16. ■

Let's now proceed to the proof of Proposition 2.1.13.

Proof of Proposition 2.1.13. If $\mathrm{U}_1^n \subset \mathrm{U}_n$ is the maximal torus, then for any coefficient ring R, we know that

$$\mathrm{H}^*(\mathrm{BU}_n; R) \cong \mathrm{H}^*(\mathrm{B}\,\mathrm{U}_1^n; R)^{S_n} \cong \mathrm{H}^*((\mathbb{CP}^\infty)^n; R)^{S_n} \cong R[\gamma_1, \ldots, \gamma_n]^{S_n} \cong R[c_1, \ldots, c_n],$$

where the c_i are the elementary symmetric polynomials in the γ_i (see Appendix 3.4). Now, if $R = \mathbb{F}_p$ is a finite field of order p, then all of these cohomology algebras are modules over the Steenrod algebra $\mathcal{A}_p$. When $p > 2$, this algebra $\mathcal{A}_p$ is generated by the Bockstein homomorphism β and the Steenrod reduced power operations P^i. We recall some fundamental properties of these operations now; see [12, §4.L] for the construction and more details.

(a) For any space X and $i, j \geq 0$, the Steenrod powers are $\mathbb{F}_p$-linear maps

$$\mathrm{P}^i : \mathrm{H}^j(X;\mathbb{F}_p) \to \mathrm{H}^{i+2j(p-1)}(X;\mathbb{F}_p)$$

where $\mathrm{H}^*(X;\mathbb{F}_p)$ denotes the (say singular) cohomology of X with $\mathbb{F}_p$ coefficients. These maps are natural in X.[10]

(b) (Cartan Relation) We have $\mathrm{P}^0 = \mathrm{id}$, and

$$\mathrm{P}^i(xy) = \sum_{j+k=i} \mathrm{P}^j(x)\mathrm{P}^k(y)$$

for all $i \geq 0$ and $x, y \in \mathrm{H}^*(X;\mathbb{F}_p)$.

(c) If $x \in \mathrm{H}^*(X;\mathbb{F}_p)$ is homogenous of degree $|x|$, then for any $k \geq 1$, we have

$$\mathrm{P}^k(x) = \begin{cases} x^p, & \text{if } |x| = 2k, \text{ and} \\ 0, & \text{if } |x| < 2k. \end{cases}$$

Using these relations, we can now deduce

Lemma 2.1.14. If $p > 2$, then in the ring $\mathrm{H}^*(\mathrm{BU}_n;\mathbb{F}_p)$, we have for each k with $1 \leq k \leq n$ that

$$\mathrm{P}^1(c_k) = m_{(p,1^{k-1})}(\gamma) = \sum_{\mu \vdash k+p-1} b^{\mu}_{(p,1^{k-1})} c_\mu,$$

where $m_{(p,1^{k-1})}(\gamma)$ denotes the monomial symmetric function in γ of type given by the partition $(p, 1^{k-1})$ of $k+p-1$, and the notation in the final expression is as in Appendix 3.3, i.e. $b^\mu_\lambda \in \mathbb{Z}$ are the coefficients of the transition matrices from m_λ to e_μ and satisfy the identity

$$m_\lambda = \sum_\mu b^\mu_\lambda e_\mu$$

in the ring Λ of symmetric polynomials in countably many variables.

Proof. Since the γ_j have degree 2, it follows from observation (c) above that $\mathrm{P}^1(\gamma_j) = \gamma_j^p$ for each j. It then follows from the Cartan relation (b) that

$$\mathrm{P}^1(c_k) = \mathrm{P}^1(e_k(\gamma)) = m_{(p,1^{k-1})}(\gamma)$$

as needed. ∎

Now we use the following combinatorial observation:

[10] For this to be true as written, we must interpret $\mathrm{H}^0(X;\mathbb{F}_p)$ as the *reduced* singular cohomology of X (which is now required to be based); then naturality holds for based maps. This will not be a problem for us, since we will only deal with elements in positive degree.

Lemma 2.1.15. For any integers $s \geq 3$ and $t \geq 1$, we have

$$b^{(s+t-1)}_{(s,1^{t-1})} = (-1)^{s+1}(s+t-1).$$

Let us postpone the proof of this lemma for a moment, and show how this implies the result. Applying the lemma to $s = p$ and $t = k$, and recalling that p is an odd prime, yields in $\mathrm{H}^*(\mathrm{BU}_n; \mathbb{F}_p)$ the identity

$$(k+p-1)c_{k+p-1} = \mathrm{P}^1(c_k) - \sum_{\substack{\mu \vdash k+p-1 \\ \mu \neq (k+p-1)}} b^{\mu}_{(p,1^{k-1})} c_\mu. \tag{2.9}$$

Now suppose that $E \to S^{2n}$ is a complex vector bundle of rank n, and that $p < n$ is an odd prime not dividing n. Let $k := n - p + 1$. Then applying the universal identity (2.9) to the complex bundle $E \to S^{2n}$, we conclude that

$$nc_n(E) = \mathrm{P}^1(c_{n-p+1}(E)) - \sum_{\substack{\mu \vdash n \\ \mu \neq (n)}} b^{\mu}_{(p,1^{n-p})} c_\mu(E) \in \mathrm{H}^{2n}(S^{2n}; \mathbb{F}_p) \tag{2.10}$$

where by $c_j(E)$ here we mean the j^{th} mod p Chern class $c_j(E; \mathbb{F}_p) \in \mathrm{H}^{2j}(S^{2n}; \mathbb{F}_p)$. Now, however, we note that $c_{n-p+1}(E) = 0$ and $c_\mu(E) = 0$ for all partitions $\mu \vdash n, \mu \neq (n)$ simply because $\mathrm{H}^{2j}(S^{2n}; \mathbb{F}_p) = 0$ for $1 \leq j \leq n-1$. Therefore, recalling that n is invertible mod p, we conclude from the identity (2.10) that

$$c_n(E; \mathbb{F}_p) = 0.$$

Since the mod p top Chern class $c_n(E; \mathbb{F}_p)$ is simply the mod p reduction of the integral top Chern class $c_n(E) = c_n(E; \mathbb{Z})$, this finishes the proof of 2.1.13. ■

It remains only to prove Lemma 2.1.15.

Proof of Lemma 2.1.15. In the ring of symmetric polynomials Λ, we have for integers $s \geq 3$ and $t \geq 1$ that

$$m_{(s,1^{t-1})} = p_{s-1}e_t - m_{(s-1,1^t)},$$

and

$$m_{(2,1^{t-1})} = p_1 e_t - (t+1)e_{t+1}.$$

These two identities combined imply inductively that for $s \geq 3$ and $t \geq 1$,

$$m_{(s,1^{t-1})} = p_{s-1}e_t - p_{s-2}e_{t+1} + \cdots + (-1)^s p_1 e_{s+t-2} + (-1)^{s+1}(s+t-1)e_{s+t-1}. \tag{2.11}$$

We can now express the p_j for $1 \leq j \leq s-1$ in terms of the e_j using Newton's Identities (1), but the resulting expressions cannot involve the term e_{s+t-1} for degree reasons, since $t \geq 1$. It follows that the resulting coefficient $b^{(s+t-1)}_{(s,1^{t-1})}$ in the expansion

$$m_{(s,1^{t-1})} = \sum_{\mu \vdash s+t-1} b^{\mu}_{(s,1^{t-1})} e_\mu$$

is the same as in expression (2.11). ■

To end this discussion, it remains to handle the case of S^4.[11] This is the content of the following the result, attributed to Ehresmann and Hopf, which is regarded as the "beginning of a series of investigations into the question of existence of almost complex structures on smooth manifolds" (quote from Hirzebruch's [38], my translation).

Proposition 2.1.16. The four-sphere S^4 does not admit an almost complex structure compatible with any smooth structure on it.

The question of the existence of exotic 4-spheres is still open as of this writing, so it is possible that the only smooth structure (up to diffeomorphism) on S^4 is the standard one and the above result is not as general as it seems at first glance. We give two proofs of this result, one using characteristic classes, and another using obstruction theory and the fundamentals of twistor theory.

Proof 1 of Proposition 2.1.16. We use Theorem 3.6.8, which says that if X is an oriented closed 4-manifold and $E \to X$ a complex vector bundle such that $\mathrm{T}X \cong E_{\mathbb{R}}$, then

$$c_1(E)^2 = 2 \cdot \chi(X) + 3 \cdot \mathrm{Sign}(X).$$

In our case of $X = S^4$, we have that $c_1(E) \in \mathrm{H}^2(S^4; \mathbb{Z}) = 0$, whereas $\chi(S^4) = 2$ and $\mathrm{Sign}(S^4) = 0$. Putting these together gives us that if S^4 were almost complex, then

$$0 = 2 \cdot 2 + 3 \cdot 0,$$

which is absurd. ∎

Proof 2 of Proposition 2.1.16. Fix a Riemannian metric on S^4. The proof of Proposition 3.6.5 shows that the bundle $\mathrm{SOComp}(S^4) \to S^4$, the fiber of which at a given $x \in S^4$ is the set of orthogonal complex structures on $\mathrm{T}_x S^4$, and whose sections we are looking for, can be identified explicitly as the sphere bundle (with respect to the induced metric) of the vector bundle $\Lambda^2_- \mathrm{T}^\vee S^2 \to S^2$ of anti self-dual 2-forms on S^4. The resulting fibre bundle

$$S^2 \to S(\Lambda^2_- \mathrm{T}^\vee S^4) \to S^4$$

has zero Euler class, but is nontrivial; indeed, it is the Penrose construction of the twistor space of S^4, and by a standard argument (see [39, Ch. 13] or [40, §2]) is isomorphic to the bundle

$$\mathbb{CP}^1 \to \mathbb{P}\boldsymbol{\mathcal{S}}_{\mathbb{C}} = \mathbb{CP}\boldsymbol{\mathcal{O}}_{\mathbb{HP}^1}(-1) \to \mathbb{HP}^1$$

which we explain the notation for, and discuss, at some length in a more general context in the next section, §2.2.2. In particular, the total space of this fibration is a $\mathbb{CP}^3$. If this bundle admitted a smooth section (or even a topological section up to homotopy), then for each $k \geq 1$, the map $\pi_k(\mathbb{CP}^3) \to \pi_k(S^4)$ would be surjective, but $\pi_4(S^4) = \mathbb{Z}$, whereas $\pi_4(\mathbb{CP}^3) = 0$, this latter fact being an immediate consequence of the long exact homotopy sequence associated to the different fibration $S^1 \to S^7 \to \mathbb{CP}^3$. ∎

[11] Proposition 2.1.16 was the content of a homework exercise assigned for a class I took with Dan Freed in the fall of 2023. It is to this problem I owe my interest in the subject of almost complex structures on spheres. Thank you, Prof. Freed!

2.2 Projective Spaces

Die Bemerkung, dass irgend dreien Puncten einer Ebene immer solche Gewichte beigelegt werden können, dass ein gegebener vierter Punct der Ebene als Schwerpunct derselben betrachtet werden kann, und dass diese drei Gewichte in Verhältnissen zu einander stehen, die aus der gegenseitigen Lage der vier Puncte nur auf eine Weise bestimmbar sind, führte mich weiter zu einer neuen Methode, die Lage von Puncten in einer Ebene zu bestimmen.

The observation that to any three points of a plane we can always assign weights in such a way that a given fourth point of the plane can be regarded as their barycenter, and that these three weights stand in a relationships to one another that can be uniquely determined by the mutual positions of the four points, led me on to a new method of determining the positions of points in a plane.

Vorrede, Der Barycentrische Calcül
August Ferdinand Möbius,
describing his discovery of homogenous barycentric coordinates, which were arguably the beginnings of homogenous coordinates on projective space.

The next class of examples we treat are the projective spaces, which for geometers come in three flavors: the real, complex, and quaternionic projective spaces. The discussion for real projective spaces is simple: for each $n \geq 0$, the quotient by the antipodal involution gives a degree 2 covering map $\pi : S^n \to \mathbb{RP}^n$, which for $n \geq 3$ realizes S^n as the universal cover of $\mathbb{RP}^n$. This map tells us that most of the theory for $\mathbb{RP}^n$ is a minor extension of the theory for S^n. Consequently, we have little more to say about the real projective spaces. The case of complex projective spaces $\mathbb{CP}^n$ is certainly more interesting, but it is subsumed by our discussion of complex Grassmannians in §2.3 or smooth complete intersections in §2.4 below. We will, nonetheless, briefly treat complex projective spaces in this section (see §2.2.1) to give a preview of these more complicated results, and because the results for $\mathbb{CP}^n$ often serve as base cases when obtaining corresponding results for Grassmannians by induction. The spaces that we will be most interested in here are therefore the quaternionic projective spaces $\mathbb{HP}^n$; see §2.2.2.

We recall the definition of projective spaces here.

Definition 2.2.1. Given any field $\mathbb{F}$ and integer $n \geq 0$ the projective space of dimension n over $\mathbb{F}$, denoted $\mathbb{FP}^n$, is defined to be the quotient

$$\mathbb{FP}^n := \{v \in \mathbb{F}^{n+1} \smallsetminus \{0\}\}/\left(v \sim v\lambda \text{ for all } v \in \mathbb{F}^{n+1} \smallsetminus \{0\}, \lambda \in \mathbb{F}^\times = \mathbb{F} \smallsetminus \{0\}\right).$$

The same definition applies also when $\mathbb{F}$ is the skew field $\mathbb{H}$, as long as we follow the convention that we work with right $\mathbb{H}$-submodules (see Conventions and Fundamentals). In what follows, let $\mathbb{F}$ denote $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and let $d := \dim_{\mathbb{R}} \mathbb{F}$ be $1, 2$ or 4 correspondingly. We will try to give a uniform exposition of the basic theory for these three flavors of projective space for as long as possible. Recall that the points of $\mathbb{FP}^n$ can be given homogenous coordinates $[\xi_0 : \xi_1 : \cdots : \xi_n]$ with $\xi_i \in \mathbb{F}$ for $i = 0, \ldots, n$, not all zero, such that for each $\lambda \in \mathbb{F}^\times$ we have

$$[\xi_0 : \xi_1 : \cdots : \xi_n] = [\xi_0\lambda : \xi_1\lambda : \cdots : \xi_n\lambda]$$

As a set, $\mathbb{FP}^n$ is in bijection with the set of $\mathbb{F}$-lines (i.e. one dimensional vector subspaces over $\mathbb{F}$) through the origin in $\mathbb{F}^{n+1}$, and comes equipped with a unique smooth structure of dimension dn which makes the natural map $\pi : \mathbb{F}^{n+1} \smallsetminus \{0\} \to \mathbb{FP}^n$ a smooth quotient map. To verify that $\mathbb{FP}^n$ is a manifold, we can check that if U_j for $0 \leq j \leq n$ is the open subset of $\mathbb{FP}^n$ defined by the nonvanishing of the j^{th} coordinate ξ_j, then there is a natural diffeomorphism

$$\varphi_j : U_i \to \mathbb{F}^n, \quad [\xi_0 : \cdots : \xi_n] \to (\xi_0\xi_j^{-1}, \xi_1\xi_j^{-1}, \ldots, \xi_{j-1}\xi_j^{-1}, \xi_{j+1}\xi_j^{-1}, \ldots, \xi_n\xi_j^{-1}),$$

and the collection $\{(U_j, \varphi_j)\}_{j=0}^n$ then forms a smooth atlas on $\mathbb{FP}^n$. For future convenience, we define for $0 \leq j \leq n$, the **coordinate points** $[\mathrm{e}_j]$ in $\mathbb{FP}^n$ by setting

$$[\mathrm{e}_j] := \varphi_j^{-1}(0) = \varphi_j^{-1}(0, \ldots, 0);$$

these are, of course, the projectivizations (i.e. images under π) of the standard basis vectors $\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_n$ of $\mathbb{F}^{n+1}$. The group $\mathrm{GL}_{n+1}\,\mathbb{F}$ of invertible $(n+1)\times(n+1)$ matrices with entries in $\mathbb{F}$ then acts transitively on $\mathbb{FP}^n$ on the left. For $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, this map factors through the quotient $\mathrm{PGL}_{n+1}\,\mathbb{F}$ of $\mathrm{GL}_{n+1}\,\mathbb{F}$ by the scalar matrices, although the noncommutativity of $\mathbb{H}$ makes it difficult to define $\mathrm{PGL}_{n+1}\,\mathbb{H}$ in a way that is intuitive and consistent with the above definition.

The set of all unit vectors $v \in \mathbb{F}^{n+1}$ forms the sphere $S^{d(n+1)-1} \subset \mathbb{R}^{d(n+1)}$, and the restriction of π to $S^{d(n+1)-1}$ is a surjective map $\pi : S^{d(n+1)-1} \to \mathbb{FP}^n$ yielding the fibre bundle

$$S^{d-1} \to S^{d(n+1)-1} \to \mathbb{FP}^n.$$

This tells us that $\mathbb{FP}^n$ is a closed connected manifold. Here the groups

$$\begin{aligned} S^0 &= \{\pm 1\} \subset \mathbb{R}^\times, \\ S^1 &= \mathrm{U}_1 \subset \mathbb{C}^*, \text{ and} \\ S^3 &\cong \mathrm{SU}_2 \cong \mathrm{Sp}_1 \end{aligned}$$

are Lie groups, and these bundles are, in fact, (the universal) principal S^{d-1}-bundles. For $d = 1$ and $n \geq 1$, this action of $S^{d-1} \cong \mathbb{Z}/2$ on S^n is orientation-preserving iff n is odd, so that $\mathbb{RP}^n$ is orientable iff n is odd. For $d \in \{2, 4\}$ and $n \geq 0$, the corresponding action of S^{d-1} on $S^{d(n+1)-1}$ is orientation preserving, and hence $\mathbb{CP}^n$ and $\mathbb{HP}^n$ are orientable closed manifolds. The above natural quotient maps $\pi : S^{d(n+1)-1} \to \mathbb{FP}^n$ factor through each other in the sense that for each $n \geq 0$ there are maps

$$\begin{aligned} S^n &\to \mathbb{RP}^n, \\ S^{2n+1} \to \mathbb{RP}^{2n+1} &\to \mathbb{CP}^n, \text{ and} \\ S^{4n+3} \to \mathbb{RP}^{4n+3} \to \mathbb{CP}^{2n+1} &\to \mathbb{HP}^n, \end{aligned}$$

where at each stage all maps but the last one come from the previous stage, and the composite is the map $\pi : S^{d(n+1)-1} \to \mathbb{FP}^n$ described above. These maps already give a sizeable family of fibre bundles with interesting geometry, which we will return to momentarily.

Now we can give a uniform exposition of certain computations for projective spaces. For instance, we have:

Proposition 2.2.2. Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then

$$\chi(\mathbb{FP}^n) = \begin{cases} \frac{1+(-1)^n}{2}, & \text{if } \mathbb{F} = \mathbb{R}, \text{ and} \\ n+1 & \text{if } \mathbb{F} = \mathbb{C} \text{ or } \mathbb{F} = \mathbb{H}. \end{cases}$$

Proof 1 of Proposition 2.2.2. We can give $\mathbb{FP}^n$ a CW structure with one cell in each dimension dj for $0 \leq j \leq n$, such that for $j \leq n$, the dj^{th} skeleton is exactly $\mathbb{FP}^j$, and for $j < n$, the attaching map of the $d(j+1)$-cell is the map $\pi : S^{d(j+1)-1} \to \mathbb{FP}^j$ described above.[12] It follows then that

$$\chi(\mathbb{FP}^n) = \sum_{j=0}^{n} (-1)^{dj},$$

which is equivalent to the result claimed. Another CW structure on $\mathbb{FP}^n$ for with the same cells can be obtained by considering the Morse theory associated with the function

$$f_c([\xi]) = \frac{\sum_{j=0}^{n} c_j |\xi_j|^2}{\sum_{j=0}^{n} |\xi_j|^2}$$

for any collection $c = (c_j)_j$ of distinct real constants c_j (see [41, Chapter 1, §4]). Finally, this computation can also be done using our knowledge of the cohomology rings (see 2.12 below)[13], which allows us the write the k-Poincaré polynomial of $\mathbb{FP}^n$ as

$$p_t(\mathbb{FP}^n; k) = \sum_{j=0}^{n} t^{dj},$$

where $k = \mathbb{F}_2$ if $\mathbb{F} = \mathbb{R}$ and $k = \mathbb{Q}$ if $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}\}$. Evaluating at $t = -1$ then gives us the required formulae for the Euler characteristic. ■

Proof 2 of Proposition 2.2.2. We compute the Lefschetz number $L(f)$ of a Lefschetz map f homotopic to the identity and use Corollary 1.2.5. For $t \in [0, \infty)$, consider the element $f_t \in \mathrm{GL}_{n+1} \mathbb{F}$ given by the diagonal matrix

$$f_t = \mathrm{diag}(1, \mathrm{e}^t, \mathrm{e}^{2t}, \ldots, \mathrm{e}^{nt}),$$

and think of f_t as a map $f_t : \mathbb{FP}^n \to \mathbb{FP}^n$. Again, from the formula, it is clear the each f_t is homotopic to $f_0 = \mathrm{id}_{\mathbb{FP}^n}$. For a fixed $t \in (0, \infty)$, the map f_t has exactly $n+1$ fixed points, namely at the coordinate points $[\mathrm{e}_j]$.[14] To see that these are Lefschetz fixed points, we note that each f_t preserves each coordinate open subset U_j, and the action of f_t restricted to $U_j \xrightarrow{\varphi_j} \mathbb{F}^n$ is linear in the coordinates afforded by φ_j on $\mathbb{F}^n$, with the corresponding matrix representing f_t being given by

$$f_t|_{U_j} = \mathrm{diag}(\mathrm{e}^{-jt}, \mathrm{e}^{-(j-1)t}, \ldots, \mathrm{e}^{-t}, \mathrm{e}^{t}, \ldots, \mathrm{e}^{(n-j)t}).$$

[12] For $\mathbb{F} = \mathbb{R}$, this is the CW structure descended from the $\mathbb{Z}/2$-invariant CW structure on S^n mentioned above (see Chapter 1, Footnote 2).

[13] Howevever, this is not saying much unless we first compute the cohomology rings without using the CW structure. This is possible, for instance, using the Gysin sequence.

[14] I apologize for using the upright e for both Euler's constant and basis vectors. Hopefully, this will not cause any confusion.

Since $f_t|_{U_j}$ is linear, its derivative is correspondingly represented by the same matrix in $\mathrm{GL}_n\,\mathbb{F}$. To compute its Lefschetz number, however, we need to think of it as a matrix in $\mathrm{GL}_{dn}\,\mathbb{R}$. For $\mathbb{F} = \mathbb{R}$ itself, we get that

$$\begin{aligned}\det\left(\mathrm{d}f_t|_{[\mathrm{e}_j]} - \mathrm{id}_{\mathrm{T}_{[\mathrm{e}_j]}\mathbb{RP}^n}\right) &= \left(\mathrm{e}^{-jt}-1\right)\left(\mathrm{e}^{-(j-1)}t-1\right)\cdots\left(\mathrm{e}^{-t}-1\right)\left(\mathrm{e}^{t}-1\right)\cdots\left(\mathrm{e}^{(n-j)t}-1\right)\\ &= (-1)^j\prod_{k=1}^{j}\left(1-\mathrm{e}^{-kt}\right)\prod_{\ell=1}^{n-j}\left(\mathrm{e}^{\ell t}-1\right).\end{aligned}$$

In general, to interpret $\mathrm{d}f_t|_{[\mathrm{e}_j]}$ as an element not of $\mathrm{GL}_n\,\mathbb{F}$ but of of $\mathrm{GL}_{dn}\,\mathbb{R}$, we simply replace each diagonal entry with an identity matrix of size d. The same computation then tells us that for all $\mathbb{F}$, we have

$$\det\left(\mathrm{d}f_t|_{[\mathrm{e}_j]} - \mathrm{id}_{\mathrm{T}_{[\mathrm{e}_j]}\mathbb{FP}^n}\right) = \left[(-1)^j\prod_{k=1}^{j}\left(1-\mathrm{e}^{-kt}\right)\prod_{\ell=1}^{n-j}\left(\mathrm{e}^{\ell t}-1\right)\right]^d.$$

The nonvanishing of this determinant tells us that each $[\mathrm{e}_j]$ is a Lefschetz fixed point of f_t, and since each term in the product is positive, we conclude that the local Lefschetz number of the map f_t at the point $[\mathrm{e}_j]$ is

$$L_{[\mathrm{e}_j]}(f_t) = \operatorname{sign}\det\left(\mathrm{d}f_t|_{[\mathrm{e}_j]} - \mathrm{id}_{\mathrm{T}_{[\mathrm{e}_j]}\mathbb{FP}^n}\right) = (-1)^{dj}.$$

Therefore, for any $t \in (0,\infty)$, we get from Corollary 1.2.5 that

$$\chi(\mathbb{FP}^n) = L(f_t) = \sum_{j=0}^{n} L_{[\mathrm{e}_j]}(f_t) = \sum_{j=0}^{n}(-1)^{dj}.$$

■

Remark 2.2.3. There are certainly other proofs of this result. For instance, the result for $\mathbb{RP}^n$ can be obtained *from* Proposition 2.1.1 combined with Corollary 3.5.3. One could also do the same computation as in Proof 3 of Proposition 2.1.1 for the round metric (i.e. metric of constant curvature +1 descended from that of S^n), noting that the only modification necessary is to integrate over $\theta_1 \in (0,\pi)$ in stead of $\theta_1 \in (0,2\pi)$. For $\mathbb{CP}^n$, we may also the techniques of Chern classes (see Corollary 2.2.7). Note that Proof 2 amounts to constructing an $\mathbb{R}_+$ action on $\mathbb{FP}^n$ and counting the fixed points; in the complex case, this can also be rephrased as counting fixed points, or applying the holomorphic Lefschetz Theorem or equivariant Atiyah-Singer-Index Theorem, for a suitable $\mathbb{T}^{n+1} \subset \mathrm{GL}_{n+1}\,\mathbb{C}$ action on $\mathbb{CP}^n$. Finally, we can also obtain the result for $\mathbb{HP}^n$ from that for $\mathbb{CP}^n$ via the fiber bundle $S^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ discussed above along with the multiplicativity of the Euler characteristic (Theorem 3.5.1)–see Remark 2.2.16.

One consequence of this computation (and the Lefschetz Fixed Point Theorem machinery developed in §1.2) is the Fundamental Theorem of Algebra. This is a little bit of a digression from the main topic of this thesis, but it's a fun result, so we include it anyway.

Corollary 2.2.4.

(a) Any complex linear map between finite dimensional complex vector spaces has an eigenvector.
(b) (Fundamental Theorem of Algebra) The field $\mathbb{C}$ is algebraically closed.

Proof. For (a), by choosing a basis, we may reduce the problem to showing that if $n \geq 0$ is any integer and $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is any linear map, then f has an eigenvector. If f is not invertible, then any vector in the kernel $\ker f$ is an eigenvector of f with eigenvalue 0. If f is invertible, then it descends to a map on projective spaces $[f] : \mathbb{CP}^n \to \mathbb{CP}^n$; then saying that f has eigenvector is equivalent to saying that $[f]$ has a fixed point. Since by Proposition 2.2.2), the Euler characteristic of $\mathbb{CP}^n$ is $\chi(\mathbb{CP}^n) = n+1$, by Corollary 1.2.6, it suffices to show that f is homotopic to the identity, which follows from the fact that $\mathrm{GL}_{n+1}\mathbb{C}$ is path connected. Precisely, the polynomial $P(t) := \det((1-t)\,\mathrm{id} + tf) \in \mathbb{C}[t]$ satisfies $P(0) = 1$ and $P(1) \neq 0$, and is hence nonconstant. Since $\mathbb{C} \smallsetminus P^{-1}(0)$ is path connected (this is where the proof fails over $\mathbb{R}$!), there is a path $\gamma : [0,1] \to \mathbb{C}$ such that γ avoids $P^{-1}(0)$ and satisfies $\gamma(0) = 0$ and $\gamma(1) = 1$. Then $t \mapsto [(1-\gamma(t))\,\mathrm{id} + \gamma(t)f]$ gives a homotopy between $\mathrm{id}_{\mathbb{CP}^n}$ and $[f]$, finishing the proof.

For (b), note that if $K/\mathbb{C}$ is a finite algebraic extension and $\alpha \in K$, then multiplication by α denotes a complex linear map $T_\alpha : K \to K$. By (a), this has an eigenvector, so that there is a $\lambda \in \mathbb{C}$ such that $T_\alpha - \lambda\,\mathrm{id}_K = T_{\alpha-\lambda}$ is not invertible. Since K is a field, this is only possible if $\alpha - \lambda = 0$, i.e. $\alpha \in \mathbb{C}$. ■

One final general observation: for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we can use the CW structure described in Proof 1 of Proposition 2.2.2 above, along with Poincaré duality, to compute the cohomology rings of projective spaces as

$$\mathrm{H}^*(\mathbb{FP}^n; R) = R[\zeta]/(\zeta^{n+1}), \text{ where } |\zeta| = d, \tag{2.12}$$

where R is any $\mathbb{F}_2$-algebra when $d = 1$ and any ring when $d \in \{2, 4\}$. Here, ζ is the Poincaré dual to the fundamental class of the hyperplane $[\mathbb{FP}^{n-1}] \in \mathrm{H}_{d(n-1)}(\mathbb{FP}^n; R)$. Over each $\mathbb{FP}^n$, we also have the tautological line subbundle , denoted $\mathcal{S}_{\mathbb{FP}^n}$ or $\mathcal{O}_{\mathbb{FP}^n}(-1)$, of the trivial bundle $\mathbb{F}^{n+1} \times \mathbb{FP}^n$, defined by

$$\mathcal{S}_{\mathbb{FP}^n} = \mathcal{O}_{\mathbb{FP}^n}(-1) := \{(v, \ell) \in \mathbb{F}^{n+1} \times \mathbb{FP}^n : v \in \ell\} \subset \mathbb{F}^{n+1} \times \mathbb{FP}^n, \tag{2.13}$$

where we are thinking of $\mathbb{FP}^n$ as the set of lines through the origin in $\mathbb{F}^{n+1}$. Each of these three tautological bundles are nontrivial, as is evidence by the Stiefel-Whitney, Chern, and Pontryagin classes respectively. Explicitly, we have

$$\begin{aligned} w_1(\mathcal{O}_{\mathbb{RP}^n}(-1)) &= -\zeta \in \mathrm{H}^1(\mathbb{RP}^n; \mathbb{Z}/2), \\ c_1(\mathcal{O}_{\mathbb{CP}^n}(-1)) &= -\zeta \in \mathrm{H}^2(\mathbb{CP}^n; \mathbb{Z}), \text{ and} \\ p_1(\mathcal{O}_{\mathbb{HP}^n}(-1)) &= 2\zeta \in \mathrm{H}^4(\mathbb{HP}^n; \mathbb{Z}). \end{aligned}$$

The first two of these are almost tautological (depending on your definition of these characteristic classes!), whereas the third one is saying something nontrivial; we will prove it in Proposition 2.21 below.

Let's focus now on the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. Then on $\mathbb{FP}^n$, we define for each integer $k \in \mathbb{Z}$, the line bundle $\mathcal{O}_{\mathbb{FP}^n}(k)$ by

$$\mathcal{O}_{\mathbb{FP}^n}(k) := \mathcal{S}_{\mathbb{FP}^n}^{\otimes(-k)} := \begin{cases} \mathcal{S}_{\mathbb{FP}^n}^{\otimes(-k)}, & \text{when } k \leq 0, \text{ and} \\ (\mathcal{S}_{\mathbb{FP}^n}^{\vee})^{\otimes k}, & \text{when } k \geq 0. \end{cases}$$

Note that, $\mathcal{O}_{\mathbb{FP}^n} = \mathcal{O}_{\mathbb{FP}^n}(0)$ is just the structure sheaf (i.e. the trivial line bundle) on $\mathbb{FP}^n$, and, again almost by definition, we have for any $k \in \mathbb{Z}$ that

$$w(\mathcal{O}_{\mathbb{RP}^n}(k)) = 1 + k\zeta \in \mathrm{H}^*(\mathbb{RP}^n; \mathbb{Z}/2) \text{ and } c(\mathcal{O}_{\mathbb{CP}^n}(k)) = 1 + k\zeta \in \mathrm{H}^*(\mathbb{CP}^n; \mathbb{Z}).$$

Other than $\mathcal{S}_{\mathbb{FP}^n}$, the most important vector bundle on $\mathbb{FP}^n$ is the tautological quotient bundle $\mathcal{Q}_{\mathbb{FP}^n}$ defined by the short exact sequence

$$0 \to \mathcal{S}_{\mathbb{FP}^n} \to \mathcal{O}_{\mathbb{FP}^n}^{\oplus(n+1)} \to \mathcal{Q}_{\mathbb{FP}^n} \to 0, \tag{2.14}$$

where the first nonzero map is the inclusion in (2.13). It is a standard fact in the theory of projective spaces (and indeed Grassmannians, see 2.3), which we take as well-known (see [42, Lemma 4.4] or [43, §3.2.4]), that the tangent bundle $\mathrm{T}\mathbb{RP}^n$ of $\mathbb{RP}^n$ (resp. holomorphic tangent bundle $\mathscr{T}\mathbb{CP}^n$ of $\mathbb{CP}^n$) is given by

$$\mathrm{T}\mathbb{RP}^n \cong \mathcal{S}_{\mathbb{RP}^n}^{\vee} \otimes \mathcal{Q}_{\mathbb{RP}^n} \text{ (resp. } \mathscr{T}\mathbb{CP}^n \cong \mathcal{S}_{\mathbb{CP}^n}^{\vee} \otimes \mathcal{Q}_{\mathbb{CP}^n}). \tag{2.15}$$

In what follows, we will often drop the subscripts on $\mathcal{S}$ and $\mathcal{Q}$ for convenience. Now–and this is special to projective spaces–since $\mathcal{S}$ is a line bundle, we may twist the sequence (2.14) by $\mathcal{S}^{\vee} = \mathcal{O}_{\mathbb{FP}^n}(1)$ to obtain the Euler sequence, which for $\mathbb{F} = \mathbb{R}$ looks like

$$0 \to \mathcal{O}_{\mathbb{RP}^n} \to \mathcal{O}_{\mathbb{RP}^n}(1)^{\oplus(n+1)} \to \mathrm{T}\mathbb{RP}^n \to 0, \tag{2.16}$$

and for $\mathbb{F} = \mathbb{C}$ looks like

$$0 \to \mathcal{O}_{\mathbb{CP}^n} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus(n+1)} \to \mathscr{T}\mathbb{CP}^n \to 0. \tag{2.17}$$

The Euler sequence really lies at the core of all characteristic class computations on projective spaces, as we shall see below.

2.2.1 Complex Projective Spaces

Let's now study complex projective spaces in some detail. The bundles $\mathcal{O}_{\mathbb{CP}^n}(k)$ are of fundamental importance in the study of complex projective spaces, and indeed of complex projective varieties more generally. For instance, the line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a positive line bundle, where recall that a line bundle $L \to X$ over a compact complex manifold X is said to be positive if the first Chern class $c_1(L) \in \mathrm{H}^2(X;\mathbb{Z})$ can be represented in de Rham cohomology by a closed $(1,1)$-form ω with positive definite associated Hermitian form. In our case of $\mathcal{O}_{\mathbb{CP}^n}(1)$, this positive form can be taken to be, for instance, the Fubini-Study form ω_{FS} defined via

$$\omega_{\mathrm{FS}}(z) = \frac{\mathrm{i}}{2}\partial\overline{\partial}\log|z|^2.$$

The Kodaira Embedding Theorem says exactly that a compact complex manifold is projective, i.e. admits an embedding into projective space (and hence by Chow's Theorem is an algebraic variety), iff it admits a positive line bundle, and the necessity of this criterion really comes from the existence of this line bundle on $\mathbb{CP}^n$. The first fundamental computation here, therefore, is of the χ_y-characteristic of the bundles $\mathcal{O}_{\mathbb{CP}^n}(k)$.

Theorem 2.2.5. For any $k \in \mathbb{Z}$, we have the generating function

$$\sum_{n=0}^{\infty} \chi_y\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) t^n = \frac{(1+yt)^{k-1}}{(1-t)^{k+1}}.$$

Proof. The Euler sequence (2.17) tells us that for any $n \geq 0$, the total generalized Todd class of $\mathbb{CP}^n$ is given by

$$\mathsf{Td}_y(\mathbb{CP}^n) = Q_y(\zeta)^{n+1} \in \mathrm{H}^*(\mathbb{CP}^n;\mathbb{Q}[y]) = \mathbb{Q}[y,\zeta]/(\zeta^{n+1}),$$

where we recall that

$$Q_y(z) = \frac{z}{R(z)}, \text{ with } R(z) := \frac{1-\mathrm{e}^{-z(1+y)}}{1+y\mathrm{e}^{-z(1+y)}}.$$

From the Generalized Hirzebruch-Riemann-Roch Theorem (Theorem 1.4.3), we conclude that for any $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\chi_y\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) = \int_{\mathbb{CP}^n} \mathrm{ch}_y\, \mathcal{O}_{\mathbb{CP}^n}(k) \cdot \mathsf{Td}_y(\mathbb{CP}^n) = \int_{\mathbb{CP}^n} \mathrm{e}^{k\zeta(1+y)}\zeta^{n+1}R(z)^{-n-1}. \quad (2.18)$$

The evaluation (2.18) amounts to obtaining the coefficient of ζ^n in the series

$$\mathrm{e}^{k\zeta(1+y)}\frac{\zeta^{n+1}}{R(\zeta)^{n+1}},$$

which we can then express as a residue calculation as

$$[\zeta^n]\mathrm{e}^{k\zeta(1+y)}\frac{\zeta^{n+1}}{R(\zeta)^{n+1}} = \operatorname*{Res}_{\zeta=0}\frac{\mathrm{e}^{k\zeta(1+y)}}{R(\zeta)^{n+1}}\mathrm{d}\zeta.$$

To compute this residue, we make the change of variables $t = R(\zeta)$, which for all y is a holomorphic change of coordinates in a neighborhood of $\zeta = 0$. With this substitution, we have

$$\mathrm{e}^{\zeta(1+y)} = \frac{1+yt}{1-t},$$

so we conclude that the desired quantity is

$$\operatorname*{Res}_{t=0} \left(\frac{1+yt}{1-y}\right)^k \cdot \frac{1}{t^{n+1}} \cdot \frac{1}{(1+yt)(1-y)} \mathrm{d}t = [t^n]\frac{(1+yt)^{k-1}}{(1-y)^{k+1}},$$

as needed. ■

Now we relish in the consequences of this result.

Corollary 2.2.6. The generating function for the χ_y-characteristic of complex projective spaces is

$$\sum_{n=0}^{\infty} \chi_y(\mathbb{CP}^n)t^n = \frac{1}{(1+yt)(1-t)}.$$

In other words, we have for $n \geq 0$ that

$$\chi_y(\mathbb{CP}^n) = 1 - y + y^2 + \cdots + (-1)^n y^n = \frac{1-(-y)^{n+1}}{1+y}.$$

Proof. Set $k = 0$ in Theorem 2.2.5. ■

This result seems more natural to state using the "$-y$"-convention (see Remark 2.3.2) as

$$\chi_{-y}(\mathbb{CP}^n) = \frac{1-y^{n+1}}{1-y}.$$

Corollary 2.2.7. We have for $n \geq 0$ that $\chi(\mathbb{CP}^n) = n + 1$.

Proof. Set $y = -1$ in Corollary 2.2.6. ■

Corollary 2.2.8. We have for $n \geq 0$ that

$$\operatorname{Sign}(\mathbb{CP}^n) = \frac{1+(-1)^n}{2}.$$

Proof 1 of Corollary 2.2.8. Set $y = 1$ in Corollary 2.2.6. ■

Proof 2 of Corollary 2.2.8. For odd n, the signature $\operatorname{Sign}(\mathbb{CP}^n) = 0$ by definition, whereas for even $n = 2k$, the middle cohomology $\mathrm{H}^{2k}(\mathbb{CP}^{2k};\mathbb{Z})$ is generated as a $\mathbb{Z}$-module by ζ^k, with the intersection matrix given simply by $[1]$. ■

Corollary 2.2.9. For any $k \in \mathbb{Z}$, we have

$$\sum_{n=0}^{\infty} \chi\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) t^n = \frac{1}{(1-t)^{k+1}}.$$

In other words, we have for any $n \geq 0$ and $k \in \mathbb{Z}$ that

$$\chi\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) = \begin{cases} \binom{n+k}{k}, & \text{if } k \geq 0, \\ (-1)^n \binom{-k-1}{n}, & \text{if } k \leq -n-1, \\ 0 & \text{else.} \end{cases} .$$

Proof 1 of Corollary 2.2.9. Set $y = 0$ in Theorem 2.2.5. ∎

Proof 2 of Corollary 2.2.9. This result is often presented as a standard consequence of the following cohomology computation: for any $i \geq 0$, we have

$$h^i\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) = \begin{cases} \binom{n+k}{k} & \text{if } i = 0 \text{ and } k \geq 0 \\ \binom{-k-1}{n} & \text{if } i = n \text{ and } k \leq -n-1, \text{ and} \\ 0 & \text{else.} \end{cases} \tag{2.19}$$

Indeed, the top line expresses that the space of homogenous polynomials in degree $k \geq 0$ in $n+1$ variables has dimension $\binom{n+k}{k}$, whereas the second line is forced by Serre Duality and the computation that $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ (see [16, §2.4]). The vanishing of middle cohomology can be then proved in the holomorphic category[15] as a consequence of the Kodaira Vanishing Theorem (see [16, Example 5.2.5]) or of the ideal sheaf sequence corresponding to the hyperplane divisor $\mathbb{CP}^{n-1} = \mathbb{V}(\xi_n)$, namely

$$0 \to \mathcal{O}_{\mathbb{CP}^n}(-1) \to \mathcal{O}_{\mathbb{CP}^n} \to \mathcal{O}_{\mathbb{CP}^{n-1}} \to 0,$$

and its twists

$$0 \to \mathcal{O}_{\mathbb{CP}^n}(k-1) \to \mathcal{O}_{\mathbb{CP}^n}(k) \to \mathcal{O}_{\mathbb{CP}^{n-1}}(k) \to 0$$

for $k \in \mathbb{Z}$. Indeed, we can use these to sequences to perform double induction on (n, k), where $n \geq 1$ and we use forward and backward induction on k, starting with $k = 0$. The base case of $n = 1$ can be performed "by hand", whereas the base case of $k = 0$ uses the Borel-Weil-Bott Theorem, which says that $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ and complete homogenous space $X = G/P$, where G is a semisimple complex Lie group and $P \subset G$ a parabolic subgroup (the classic reference being [44]). A key step in the induction on k is the observation that the restriction map

$$\mathrm{H}^0\left(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)\right) \to \mathrm{H}^0\left(\mathbb{CP}^{n-1}, \mathcal{O}_{\mathbb{CP}^{n-1}}(k)\right)$$

is surjective for all $k \geq 0$ and an isomorphism for $k = 0$. The details are both standard and straightforward, and therefore omitted. ∎

It is very interesting to see by comparing these proofs how, once the hammer of the Atiyah-Singer Index Theorem is proven, these computations become trivial. We end this section with a couple of other fun results.

[15]That is, as opposed to performing it in the algebraic category as in [5, Ch. III] and invoking Serre's GAGA Theorems.

Proposition 2.2.10. For any $n \geq 0$, we have

$$\chi\left(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n\right) = n(n+2).$$

Proof. The Euler sequence (2.17) along with the additivity of the holomorphic Euler characteristic in short exact sequences yields

$$\chi(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = (n+1)\chi\left(\mathbb{CP}^n, \boldsymbol{\mathcal{O}}_{\mathbb{CP}^n}(1)\right) - \chi(\mathbb{CP}^n, \boldsymbol{\mathcal{O}}_{\mathbb{CP}^n}).$$

Using Corollary 2.2.9, we can write this as quantity as $(n+1)^2 - 1 = n(n+2)$ as needed. Another way to rephrase this argument is to say that the long exact sequence in cohomology arising from the Euler sequence (2.17), along with the cohomology computation (2.19), tells us that

$$h^i(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = 0 \text{ for } i > 0,$$

so that

$$\chi(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = h^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n),$$

which is the dimension of the space of global holomorphic vector fields on $\mathbb{CP}^n$.[16] Tracing through the isomorphism (2.15) tells us that in the short exact sequence

$$0 \to \mathrm{H}^0\left(\mathbb{CP}^n, \boldsymbol{\mathcal{O}}_{\mathbb{CP}^n}\right) \to \mathrm{H}^0\left(\mathbb{CP}^n, \boldsymbol{\mathcal{O}}_{\mathbb{CP}^n}(1)\right)^{\oplus(n+1)} \to \mathrm{H}^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) \to 0,$$

the surjection is given explicitly by taking $(\ell_0, \ldots, \ell_n) \mapsto \sum_{i=0}^n \ell_i \frac{\partial}{\partial \xi_i}$, where the $\ell_i \in \mathrm{H}^0(\mathbb{CP}^n, \boldsymbol{\mathcal{O}}_{\mathbb{CP}^n}(1))$ are linear polynomials in the coordinates ξ_i, subject only to the relation that the Euler vector field $\sum_{i=0}^n \xi_i \cdot \frac{\partial}{\partial \xi_i}$ vanishes on $\mathbb{CP}^n$.[17] Of course, this shows again that

$$\chi(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = h^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = (n+1)^2 - 1.$$

As a final remark, this result can also be obtained from differentiating the isomorphism

$$\mathrm{PSL}_{n+1}\mathbb{C} \to \mathrm{Aut}\,\mathbb{CP}^n$$

at the identity to obtain a Lie algebra isomorphism (up to a negative sign[18])

$$\mathfrak{sl}_{n+1}\mathbb{C} \to \mathrm{H}^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n),$$

so that

$$h^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^{n+1}) = \dim_{\mathbb{C}} \mathfrak{sl}_{n+1}\mathbb{C} = (n+1)^2 - 1.$$

■

[16] Note, by the way, that $h^1(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n) = 0$ tells us that the complex structure on $\mathbb{CP}^n$ is rigid–that there is no deformation theory of $\mathbb{CP}^n$.

[17] Here we are using that if the ℓ_i are linear, then the vector field $\sum_{i=0}^n \ell_i \frac{\partial}{\partial \xi_i}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is invariant under the action of $\mathbb{C}^\times$ and hence descends via π to a vector field on $\mathbb{CP}^n$.

[18] This negative shown shows up for an important and subtle reason. The map $\mathfrak{sl}_{n+1}\mathbb{C} \to \mathrm{H}^0(\mathbb{CP}^n, \mathscr{T}\mathbb{CP}^n)$ is given as follows: given a $X \in \mathfrak{sl}_{n+1}\mathbb{C}$, we can take a curve $\gamma : \Delta(\varepsilon) \to \mathrm{PSL}_{n+1}\mathbb{C}$ representing it, and then consider the vector field X on $\mathbb{CP}^{n+1}$ defined by taking a local holomorphic f to Xf defined by $Xf(z) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\gamma(t)z)$. The exponential map $\mathrm{Vect}(\mathbb{CP}^n) \to \mathrm{Aut}\,\mathbb{CP}^{n+1}$ is given by considering the flow, and hence for $X, Y \in \mathfrak{sl}_{n+1}\mathbb{C}$, we have

$$([X,Y]f)(z) = \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} f(\mathrm{e}^{tX}\mathrm{e}^{sY}\mathrm{e}^{-tX}z) = \left.\frac{\partial}{\partial t}\right|_{t=0} Y_{\mathrm{e}^{-tX}z}((\mathrm{e}^{tX})^* f) = -(\boldsymbol{\mathcal{L}}_X Y)f.$$

This negative sign shows up, by the same computation, whenever we have a Lie group G acting on a manifold X, and we consider similarly that induced map $\mathfrak{g} \to \mathrm{Vect}(X)$, which is a Lie algebra homomorphism up to this negative sign.

Finally, we compute the $\hat{A}$-genus of $\mathbb{CP}^n$.

Proposition 2.2.11. The generating function for the $\hat{A}$-genus of complex projective spaces is

$$\sum_{n=0}^{\infty} \hat{A}(\mathbb{CP}^n) t^n = \left(1+\frac{t^2}{4}\right)^{-1/2}.$$

In other words, we have for $k \geq 0$ that $\hat{A}(\mathbb{CP}^{2k+1}) = 0$, whereas

$$\hat{A}(\mathbb{CP}^{2k}) = (-1)^k \frac{1}{2^{4k}} \binom{2k}{k}.$$

Proof. Proceeding identically to the proof of Theorem 2.2.5, but this time using the series $Q_{\hat{A}}(z) = (z/2)\operatorname{csch}(z/2)$ instead, we conclude that the total $\hat{A}$-class of $\mathbb{CP}^n$ is exactly

$$\hat{\mathsf{A}}(\mathbb{CP}^n) = Q_{\hat{A}}(\zeta)^{n+1} = \left(\frac{\zeta}{2\sinh(\zeta/2)}\right)^{n+1} \in \mathrm{H}^*(\mathbb{CP}^n; \mathbb{Q}).$$

Therefore, we conclude that

$$\hat{A}(\mathbb{CP}^n) = \int_{\mathbb{CP}^n} \left(\frac{\zeta}{2\sinh(\zeta/2)}\right)^{n+1} = \operatorname*{Res}_{\zeta=0} \frac{1}{(2\sinh(\zeta/2))^{n+1}} \mathrm{d}\zeta,$$

where we have expressed the computation as a residue calculation as before. To carry out this computation, use the change of variables

$$t = 2\sinh(\zeta/2)$$

to get

$$\hat{A}(\mathbb{CP}^n) = \operatorname*{Res}_{t=0} \frac{1}{t^{n+1}} \left(1+\frac{t^2}{4}\right)^{-1/2} \mathrm{d}t = [t^n]\left(1+\frac{t^2}{4}\right)^{-1/2}$$

as needed. The explicit formula then follows from plugging $x = -t^2/16$ in the the generating function of the central binomial coefficients given by

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = (1-4x)^{-1/2}.$$

■

Remark 2.2.12. Note that for any $k \geq 0$, we have

$$\left|\hat{A}(\mathbb{CP}^{2k})\right| \leq \frac{1}{2^{2k}}.$$

In particular, for any $k \geq 1$, this has no hope of being an integer. This does not violate Corollary 1.5.6 because $\mathbb{CP}^{2k}$ is never spin; indeed, the Euler sequence (2.17) tells us that the total Chern class of $\mathbb{CP}^n$ is given by $c(\mathbb{CP}^n) = (1+\zeta)^{n+1}$, so that $c_1(\mathbb{CP}^n) = (n+1)\zeta$. In particular, we conclude from Corollary 1.5.3 that $\mathbb{CP}^n$ is spin iff n is odd. This gives us a family of (necessarily non-spin) manifolds with non-integral $\hat{A}$-genus. Note that $\mathbb{CP}^n$ admits a metric of positive Ricci curvature (the Fubini-Study metric!), so this example tells us that the condition that the manifold be spin is necessary for the conclusion of Theorem 1.5.9 to hold.

2.2.2 Quaternionic Projective Spaces

Let us investigate the existence of almost complex structures on projective spaces. Note first that for $n \geq 1$, the space $\mathbb{RP}^{2n}$ is non-orientable and hence cannot be almost complex, while complex projective spaces $\mathbb{CP}^n$ are complex (almost tautologically!). Therefore, the question of existence of almost complex structures on projective spaces is only really interesting for *quaternionic* projective spaces.

Hirzebruch showed in 1953 in [38] that the quaternionic projective space $\mathbb{HP}^n$ for $n = 1$ and $n \geq 4$ does not admit an almost complex structure compatible with its usual smooth structure. He also showed, using Wu's result that the the mod 3 Pontryagin classes are topological invariants of smooth manifolds, that the same statement is true with respect to *any* smooth structure on $\mathbb{HP}^n$ when $n \equiv 1 \pmod 3$ (so for $n = 1$, this result gives another proof of Proposition 2.1.16), as well as the fact that if $\mathbb{HP}^n$ for $n \in \{2, 3\}$ admits an almost complex structure, then $\mathbb{CP}^{2n+1}$ admits an "exotic" almost complex structure with Chern classes different from the usual complex structure. As the story goes, Hirzebruch announced at his 1958 lecture at the International Congress of Mathematicians that Milnor had been able to use K-theory to prove the result also for $n = 2, 3$, but before this result was published, William Massey also published a uniform proof of this result in the 1962 article [45]. In what follows, we will give Hirzebruch's and Massey's proofs (except for some standard results from K-theory which we will delegate to references, since we do not develop the machinery of K-theory in any detail). We will then also show how to use these characteristic class techniques to compute invariants such as $\chi(\mathbb{HP}^n)$, $\mathrm{Sign}(\mathbb{HP}^n)$, and $\hat{A}(\mathbb{HP}^n)$. The key player in this discussion will be the fibre bundle $\mathbb{CP}^1 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ mentioned in the previous section, which we analyze in more detail below.

To begin this discussion, recall that the the tautological line subbundle on $\mathbb{HP}^n$, denoted $\mathcal{S} := \mathcal{O}_{\mathbb{HP}^n}(-1) \to \mathbb{HP}^n$, is a quaternionic vector bundle, and hence by restriction of scalars can be thought of as both a complex vector bundle $\mathcal{S}_{\mathbb{C}}$ of rank 2 and a real vector bundle $\mathcal{S}_{\mathbb{R}}$ of rank 4. It then follows from the Gysin sequence that the cohomology of $\mathbb{HP}^n$ is given by

$$\mathrm{H}^*(\mathbb{HP}^n; \mathbb{Z}) = \mathbb{Z}[\xi]/(\xi^{n+1})$$

where we choose our cohomology generator to be

$$\xi := -\mathrm{e}(\mathcal{S}_{\mathbb{R}}) = -c_2(\mathcal{S}_{\mathbb{C}}).$$

Since $c_1(\mathcal{S}_{\mathbb{C}}) \in \mathrm{H}^2(\mathbb{HP}^n; \mathbb{Z}) = 0$, we conclude from the relation between Chern and Pontryagin classes (see Appendix 3.4) that the Pontryagin classes of $\mathcal{S}_{\mathbb{R}}$ are given by

$$p_1(\mathcal{S}_{\mathbb{R}}) = c_1(\mathcal{S}_{\mathbb{C}})^2 - 2c_2(\mathcal{S}_{\mathbb{C}}) = 2\xi \text{ and } p_2(\mathcal{S}_{\mathbb{R}}) = c_2(\mathcal{S}_{\mathbb{C}})^2 = \xi^2.$$

In other words, we have shown:

Lemma 2.2.13. With the above choice of generator $\xi \in \mathrm{H}^4(\mathbb{HP}^n; \mathbb{Z})$, we have

$$\mathsf{c}(\mathcal{S}_{\mathbb{C}}) = 1 - \xi \text{ and } \mathsf{p}(\mathcal{S}_{\mathbb{R}}) = 1 + 2\xi + \xi^2.$$

Note, in particular, how this result for $n = 1$ implies the existence of real 4-plane bundles on S^4 with first Pontryagin class an arbitrary even multiple of the generator of top cohomology (see [42, Lemma 20.9] and the following discussion).

The key observation now is that the complex projectivization $\mathbb{P}\mathcal{S}_{\mathbb{C}}$ of $\mathcal{S}$ can be naturally identified with the fiber bundle $\mathbb{CP}^1 \to \mathbb{CP}^{2n+1} \xrightarrow{\pi} \mathbb{HP}^n$ in way that identifies $\mathcal{O}_{\mathbb{P}\mathcal{S}_{\mathbb{C}}}(1)$ with $\mathcal{O}_{\mathbb{CP}^{2n+1}}(1)$. If we let $\zeta = c_1(\mathcal{O}_{\mathbb{CP}^{2n+1}}(1)) \in \mathrm{H}^2(\mathbb{CP}^{2n+1};\mathbb{Z})$ be the generator of the cohomology ring of $\mathrm{H}^*(\mathbb{CP}^{2n+1};\mathbb{Z})$, then we can describe the pullback map π with respect to these generators via

Lemma 2.2.14. If $\zeta \in \mathrm{H}^2(\mathbb{CP}^{2n+1};\mathbb{Z})$ and $\xi \in \mathrm{H}^4(\mathbb{HP}^n;\mathbb{Z})$ are the generators chosen above, then under the pullback map $\pi^* : \mathrm{H}^*(\mathbb{HP}^n;\mathbb{Z}) \to \mathrm{H}^*(\mathbb{CP}^{2n+1};\mathbb{Z})$, we have

$$\pi^*\xi = \zeta^2.$$

In particular, π^* is injective.

Proof 1 of Lemma 2.2.14. It it follows from the definition of the Chern classes and using the identification $\mathbb{CP}^{2n+1} = \mathbb{P}\mathcal{S}_{\mathbb{C}}$ that the relation satisfied by ζ in $\mathrm{H}^*(\mathbb{CP}^{2n+1};\mathbb{Z})$ is

$$0 = \zeta^2 + \pi^*\left(c_1(\mathcal{S}_{\mathbb{C}})\right)\zeta + \pi^*\left(c_2(\mathcal{S}_{\mathbb{C}})\right) = \zeta^2 - \pi^*\xi,$$

where in the last step we have used Lemma 2.2.13. ∎

Proof 2 of Lemma 2.2.14. If we make the identification

$$\mathbb{C}^{2n+2} \ni (z_0, \dots, z_{2n+1}) = (z_0 + \mathrm{j}z_1, \dots, z_{2n} + \mathrm{j}z_{2n+1}) \in \mathbb{H}^{n+1},$$

thought of as right $\mathbb{C}$-modules, then the fiber of the pullback bundle $\pi^*\mathcal{S}_{\mathbb{C}}$ over a point $[z_0 : z_1 : \dots : z_{2n+1}] \in \mathbb{CP}^{2n+1}$ consists exactly of the complex 2-plane spanned in $\mathbb{C}^{2n+2}$ by the two vectors of the form

$$(z_0, z_1, \dots, z_{2n+1}) \text{ and } (z_0, z_1, \dots, z_{2n+1}) \cdot \mathrm{j} = (-\overline{z}_1, \overline{z}_0, -\overline{z}_2, \overline{z}_1, \dots, -\overline{z}_{2n+1}, \overline{z}_{2n}),$$

for any choice of lift $(z_0, \dots, z_{2n+1})$ of $[z_0 : \dots : z_{2n+1}]$. This follows from the fact that if $u, v \in \mathbb{C}$, then in $\mathbb{H}$ we have the relation

$$(u + \mathrm{j}v)\mathrm{j} = -\overline{v} + \mathrm{j}\overline{u}.$$

The upshot of this discussion is that the complex 2-plane bundle $\pi^*\mathcal{S}_{\mathbb{C}}$ on $\mathbb{CP}^{2n+1}$ can be identified with the bundle

$$\pi^*\mathcal{S}_{\mathbb{C}} \cong \mathcal{O}_{\mathbb{CP}^{2n+1}}(-1) \oplus \overline{\mathcal{O}_{\mathbb{CP}^{2n+1}}(-1)} \cong \mathcal{O}_{\mathbb{CP}^{2n+1}}(-1) \oplus \mathcal{O}_{\mathbb{CP}^{2n+1}}(1),$$

where the first isomorphism is canonical, but the second one uses that if $E \to X$ is a complex vector bundle, then a choice of Hermitian metric on E induces an isomorphism $\overline{E} \to E^\vee$, where $\overline{E}$ denotes the conjugate bundle and $E^\vee$ the dual bundle to E. It follows from this isomorphism that

$$-\pi^*\xi = \pi^*(c_2(\mathcal{S}_{\mathbb{C}})) = c_2(\pi^*\mathcal{S}_{\mathbb{C}}) = c_2\left(\mathcal{O}_{\mathbb{CP}^n}(-1) \oplus \mathcal{O}_{\mathbb{CP}^n}(1)\right) = -\zeta^2,$$

where in the last step we have used

$$c_2\left(\mathcal{O}_{\mathbb{CP}^N}(a) \oplus \mathcal{O}_{\mathbb{CP}^N}(b)\right) = ab \cdot \zeta^2$$

for any $a, b \in \mathbb{Z}$ and $N \geq 0$. ∎

In what follows, we will give $\mathrm{H}^*(\mathbb{CP}^{2n+1};\mathbb{Z})$ the structure of a module over $\mathrm{H}^*(\mathbb{HP}^n;\mathbb{Z})$ via π^*, and in particular stop writing π^* explicitly.

The next step now is to study the relative tangent bundle T_π of the projection map $\pi : \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$, which by definition fits into the short exact sequence

$$0 \to \mathrm{T}_\pi \to \mathrm{T}\mathbb{CP}^{2n+1} \xrightarrow{\mathrm{d}\pi} \pi^*\mathrm{T}\mathbb{HP}^n \to 0 \tag{2.20}$$

of vector bundles on $\mathbb{CP}^{2n+1}$ (see the discussion preceding Proof 2 of Theorem 3.5.1 for a reminder on the basics). Since T_π is a real oriented bundle of rank 2, by the isomorphism $\mathrm{U}_1 \xrightarrow{\sim} \mathrm{SO}_2$, it can be thought of as a complex line bundle as well. In particular, we may speak of its first Chern class. Specifically, we have:

Lemma 2.2.15. The total Chern and Pontryagin classes of T_π are

$$\mathsf{c}(\mathrm{T}_\pi) = 1 + 2\zeta \text{ and } \mathsf{p}(\mathrm{T}_\pi) = 1 + 4\zeta^2.$$

Proof. Note that for any $q \in \mathbb{HP}^n$, the fiber of π over q, namely $\mathbb{CP}^1 \cong \pi^{-1}(q) \hookrightarrow \mathbb{CP}^{2n+1}$ is simply an embedded line. In particular, if $\iota : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^{2n+1}$ denotes the inclusion map, then the pullback ι^* via ι, i.e. the restriction map on the cohomology rings of $\mathbb{CP}^1$ and $\mathbb{CP}^{2n+1}$, is given simply by obvious projection

$$\iota : \mathbb{Z}[\zeta]/(\zeta^{2n+2}) \twoheadrightarrow \mathbb{Z}[\zeta]/(\zeta^2).$$

Next, we know that $\iota^*\mathrm{T}_\pi \cong \mathrm{T}\mathbb{CP}^1$, and, in fact, the structure of the complex vector bundle obtained from the above discussion on T_π, and hence $\iota^*\mathrm{T}_\pi$, is exactly that of the holomorphic tangent bundle $\mathscr{T}\mathbb{CP}^1$ of $\mathbb{CP}^1$. In particular, since $\chi(\mathbb{CP}^1) = 2$, we conclude from the Gauss-Bonnet Theorem that

$$\iota^* c_1(\mathrm{T}_\pi) = c_1(\iota^*\mathrm{T}_\pi) = c_1(\mathscr{T}\mathbb{CP}^1) = 2\zeta.$$

The first then follows from the above observation that $\iota^* : \mathrm{H}^2(\mathbb{CP}^{2n+1};\mathbb{Z}) \to \mathrm{H}^2(\mathbb{CP}^1;\mathbb{Z})$ is an isomorphism. The second follows as before from the relation between Chern and Pontryagin classes. ■

Remark 2.2.16. Note how this lemma allows us to give a different proof of the computation of $\chi(\mathbb{HP}^n)$, if we know it for $\mathbb{CP}^n$ (using characteristic classes, say). Indeed, (2.20), along with the multiplicativity and naturality of the Euler class, gives us that

$$\mathrm{e}(\mathrm{T}\mathbb{CP}^{2n+1}) = \mathrm{e}(\mathrm{T}_\pi) \cdot \pi^*\mathrm{e}(\mathrm{T}\mathbb{HP}^n).$$

By Lemma 2.2.15, we have $\mathrm{e}(\mathrm{T}_\pi) = 2\zeta$, so using the Chern-Gauss-Bonnet Theorem (Theorem 1.2.1), the calculation $\chi(\mathbb{CP}^{2n+1}) = 2n + 2$, and Lemma 2.2.14, we have

$$(2n + 2)\zeta^{2n+1} = 2\zeta \cdot \chi(\mathbb{HP}^n)\zeta^{2n}, \text{ so } \chi(\mathbb{HP}^n) = n + 1.$$

Of course, this is the same argument as in Proof 2 of Theorem 3.5.1.

Since it is clear from the computations in the previous section §2.2.1 that the total Pontryagin class of $\mathbb{CP}^{2n+1}$ is

$$\mathsf{p}(\mathrm{T}\mathbb{CP}^{2n+1}) = (1 + \zeta^2)^{2n+2},$$

the sequence (2.20), combined with Lemma 2.2.15 and the multiplicativity of the total Pontryagin class, gives us that

$$\mathsf{p}(\pi^*\mathrm{T}\mathbb{HP}^n) = (1 + \zeta^2)^{2n+2}(1 + 4\zeta^2)^{-1}.$$

The naturality of the total Pontryagin class, along with Lemma 2.2.14 then immediately yields the following result.

Lemma 2.2.17. The total Pontryagin class of $\mathbb{HP}^n$ is given by

$$\mathsf{p}(\mathbb{HP}^n) = (1+\xi)^{2n+2}(1+4\xi)^{-1}.$$

Note how for $n=1$, this says $\mathsf{p}(\mathbb{HP}^1)=1$, which is something we had observed already in Proof 1 of Proposition 2.1.16 to be a consequence of the stable parallelizability of $\mathbb{HP}^1 \cong S^4$. This computation shows us also that $\mathbb{HP}^n$ is not stably parallelizable if $n \geq 2$. We now quickly sketch a second proof of this result.

Sketch of Proof 2 of Lemma 2.2.17. Following [42, Exercise 20-A], we can use a version of the Euler sequence, or equivalently a direct computation of the tangent bundle for $\mathbb{HP}^n$ similarly to those for $\mathbb{RP}^n$ and $\mathbb{CP}^n$, to show that

$$\mathrm{T}\mathbb{HP}^n = \mathrm{Hom}_{\mathbb{H}}(\mathcal{S}, \mathcal{S}^{\perp}),$$

where $\mathcal{S}^{\perp} \subset \underline{\mathbb{H}}^{n+1}$ is the orthogonal complement to the tautological line bundle $\mathcal{S}$. It follows from this that

$$\mathrm{T}\mathbb{HP}^n \oplus \mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S}) \cong \mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S}^{\perp}\oplus\mathcal{S}) \cong \mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\underline{\mathbb{H}}^{n+1}) \cong \mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathbb{H})^{\oplus(n+1)}.$$

Next, we can argue from Lemma 2.2.13 that

$$\mathsf{p}\left(\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathbb{H})\right) = (1+\xi)^2 = 1+2\xi+\xi^2.$$

To show the result, therefore, it suffices to show that

$$\mathsf{p}\left(\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})\right) = 1+4\xi.$$

Note that $\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})$ is not a quaternionic vector bundle, but only a vector bundle over the center of $\mathbb{H}$, namely $\mathbb{R}$, i.e. a real bundle of rank 4. In fact, however, the identity map $\mathrm{id}_{\mathcal{S}}$ splits off a trivial summand $\underline{\mathbb{R}} \cong \mathbb{R}\,\mathrm{id}_{\mathcal{S}} \subset \mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})$, so the only nontrivial Pointryagin class of $\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})$ is the first one, p_1. To compute this, note that there is an isomorphism of complex vector bundles

$$\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})\otimes_{\mathbb{R}}\mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{S}_{\mathbb{C}},\mathcal{S}_{\mathbb{C}})$$

given locally by taking $\varphi\otimes(x+y\mathrm{i}) \mapsto x\cdot\varphi + y\cdot\varphi\circ\mathrm{i}$, where $x,y\in\mathbb{R}$, and $\varphi\circ\mathrm{i}$ represents the composition of the morphisms $\mathcal{S}_{\mathbb{C}}\to\mathcal{S}_{\mathbb{C}}$ given by multipling by i and then taking φ. It follows from this that

$$p_1\left(\mathrm{Hom}_{\mathbb{H}}(\mathcal{S},\mathcal{S})\right) = -c_2\left(\mathrm{Hom}_{\mathbb{C}}(\mathcal{S}_{\mathbb{C}},\mathcal{S}_{\mathbb{C}})\right) = -c_2(\mathcal{S}_{\mathbb{C}}^{\vee}\otimes_{\mathbb{C}}\mathcal{S}_{\mathbb{C}}).$$

To compute this last quanitity, we use the splitting principle. By Lemma 2.2.13, the Chern roots of both $\mathcal{S}_{\mathbb{C}}$ and $\mathcal{S}_{\mathbb{C}}^{\vee}$ are $\pm\sqrt{\xi}$, and so the splitting principle tells us that the total Chern class of $\mathcal{S}_{\mathbb{C}}^{\vee}\otimes_{\mathbb{C}}\mathcal{S}_{\mathbb{C}}$ is given by

$$\mathsf{c}(\mathcal{S}_{\mathbb{C}}^{\vee}\otimes_{\mathbb{C}}\mathcal{S}_{\mathbb{C}}) = (1+2\sqrt{\xi})\cdot(1-2\sqrt{\xi})\cdot(1+\sqrt{\xi}-\sqrt{\xi})\cdot(1-\sqrt{\xi}+\sqrt{\xi}) = 1-4\xi,$$

finishing the proof. ■

We are now ready to give Hirzebruch's proof from [38].

Theorem 2.2.18 (Hirzebruch)**.** For all integers $n \geq 1$, except possibly $n \in \{2, 3\}$, the quaternionic projective space $\mathbb{HP}^n$ does not admit an almost complex structure compatible with its usual smooth structure.

Proof. Suppose there were a complex vector bundle $E \to \mathbb{HP}^n$ of rank $2n$ such that $E_{\mathbb{R}} \cong T\mathbb{HP}^n$. Then we would conclude from Lemma 2.2.17 and the connection between Chern and Pontryagin classes of a complex vector bundle (Remark 3.4.22) that

$$(1-\xi)^{2n+2}(1-4\xi)^{-1} = \sum_{i=0}^{\infty}(-1)^i p_i(\mathbb{HP}^n) = \mathsf{c}(E)\mathsf{c}(\overline{E}). \tag{2.21}$$

Since $c_i(\overline{E}) = (-1)^i c_i(E)$ and $\mathrm{H}^{2i}(\mathbb{HP}^n;\mathbb{Z}) = 0$ for odd i, it follows that

$$\mathsf{c}(E) = \mathsf{c}(\overline{E})$$

and hence from (2.21) that the total Chern class of E is completely determined as

$$\mathsf{c}(E) = (1-\xi)^{n+1}(1-4\xi)^{-1/2},$$

so that recalling the identity $(1-4x)^{-1/2} = \sum_{i=0}^{\infty}\binom{2i}{i}x^i$, we get

$$c_{2n}(E) = \left(\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\binom{2j-2}{j-1}\right)\xi^n.$$

On the other hand, the Chern-Gauss-Bonnet Theorem (Theorem 1.2.1) tells us that

$$c_{2n}(E) = \chi(\mathbb{HP}^n)\xi^n = (n+1)\xi^n,$$

where we are using the computation in Proposition 2.2.2. It therefore suffices to show that if we let a_n be the sequence of integers defined by

$$a_n = \frac{1}{n+1}\left(\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\binom{2j-2}{j-1}\right), \tag{2.22}$$

then $a_n \neq 1$ for all integers $n \geq 1$ other than $n = 2, 3$. To do this, it suffices to show the recurrence relation

$$a_{n+1} = \sum_{j=0}^{n} a_j a_{n-j} + (-1)^{n+1} \tag{2.23}$$

valid for $n \geq 0$ along with $a_0 = 0$, since from these it follows that the a_n are nonnegative integers satisfying $a_{n+1} \geq 2a_n - 1$ for all $n \geq 0$, and hence, in particular, $a_n > 1$ for $n \geq 4$. To show this recurrence relation, note that from (2.22), it is clear that for any $n \geq 0$ we have $|a_n| \leq 2^{3n+1}$, so that if $f(t) = \sum_{n=0}^{\infty} a_n t^n$ is the generating function of the sequence a_n, then $f(t)$ defines a convergent power series and holomorphic function at least when $|t| < 1/8$. On here, if we let $g(t)$ be the function defined by

$$g(t) := f(t) + tf'(t) = \sum_{n=0}^{\infty}(n+1)a_n t^n,$$

then we can write $g(t)$ using the residue theorem as

$$g(t) = \sum_{n=0}^{\infty} t^n [\xi^n](1-\xi)^{n+1}(1-4\xi)^{-1/2} = \sum_{n=0}^{\infty} t^n \left(\frac{1}{2\pi \mathrm{i}} \oint_{|\xi|=\varepsilon} \left(\frac{1-\xi}{\xi} \right)^{n+1} (1-4\xi)^{-1/2} \mathrm{d}\xi \right).$$

for any $\varepsilon < 1/4$. For a fixed choice of $\varepsilon < 1/4$, we have for all ξ such that $|\xi| = \varepsilon$ that

$$|1-\xi^{-1}| \le 1 + \varepsilon^{-1} \text{ and } |(1-4\xi)^{-1/2}| \le (1-4\varepsilon)^{-1/2}.$$

Therefore, for $0 \ne t$ with $|t| \ll 1$ (specifically $t < \min\{1/8, \varepsilon(1+\varepsilon)^{-1}\}$), we are justified in switching the order of summation and integration to write this as

$$\begin{aligned} g(t) &= \frac{1}{2\pi \mathrm{i}} \oint_{|\xi|=\varepsilon} \frac{1}{t} \left[\sum_{n=0}^{\infty} \left(\frac{t(1-\xi)}{\xi} \right)^{n+1} \right] (1-4\xi)^{-1/2} \mathrm{d}\xi \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{|\xi|=\varepsilon} \frac{(1-\xi)(1-4\xi)^{-1/2}}{\xi(1+t)-t} \mathrm{d}\xi. \end{aligned}$$

To evaluate this integral, we note that for $|t| \ll 1$, the only pole of the integrand in the disc bound by $|\xi| = \varepsilon$ is $\xi = t/(1+t)$, so we may write this computation via the Residue Theorem as

$$g(t) = \operatorname*{Res}_{\xi = t/(1+t)} \frac{(1-\xi)(1-4\xi)^{-1/2}}{\xi(1+t)-t} \mathrm{d}\xi = (1-3t)^{-1/2}(1+t)^{-3/2}.$$

From this formula, we can recover $f(t)$ by solving the differential equation

$$f(t) + tf'(t) = (1-3t)^{-1/2}(1+t)^{-3/2}$$

as

$$f(t) = \frac{1}{t} \int_0^t (1-3s)^{-1/2}(1+s)^{-3/2} \mathrm{d}s = \frac{1}{2t} \left(1 - \sqrt{\frac{1-3t}{1+t}} \right).$$

Finally, from this last explicit formula, we can conclude that $f(t)$ satisfies the functional equation

$$f(t) = tf(t)^2 + (1+t)^{-1},$$

which is equivalent to the desired recurrence relation (2.23). ■

Remark 2.2.19. From the proof, it is clear that the reason this argument fails for $n = 2, 3$ is the numerical coincidence $a_2 = a_3 = 1$.

Now, we give Massey's argument, which works also for $n = 2, 3$. The idea is to study the image of the Chern character homomorphism

$$\mathrm{ch} : K(\mathbb{HP}^n) \to \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q}),$$

and to show that the image of the complexified tangent bundle $\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}$ of $\mathbb{C}$ under this homomorphism is "indivisible by two" in the image, which is what it would need to be if $\mathbb{HP}^n$ were almost complex. Let's now carry this program out.

Theorem 2.2.20 (Massey)**.** For any $n \geq 0$, the Chern character homomorphism is

$$\mathrm{ch} : K(\mathbb{HP}^n) \to \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q})$$

is an isomorphism onto the subring $\mathbb{Z}[\theta] \subset \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q})$ generated by

$$\theta := 2\cosh\sqrt{\xi} = 2\sum_{j=0}^{n} \frac{1}{(2j)!}\xi^j.$$

Proof. Recall that we showed in Lemma 2.21 that the Chern class of $\mathcal{S}_\mathbb{C}$ is $\mathsf{c}(\mathcal{S}_\mathbb{C}) = 1 - \xi$, so that the Chern roots of $\mathcal{S}_\mathbb{C}$ are $\pm\sqrt{\xi}$. It follows immediately from this that

$$\mathrm{ch}\,\mathcal{S}_\mathbb{C} = \mathrm{e}^{\sqrt{\xi}} + \mathrm{e}^{-\sqrt{\xi}} = \theta,$$

showing us that the image certainly contains $\mathbb{Z}[\theta]$.

To show that this is all of the image, we will first need the fact that $\mathrm{ch} : K(\mathbb{HP}^n) \to \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q})$ is a monomorphism, which follows from the fact that $\mathrm{H}^*(\mathbb{HP}^n; \mathbb{Z})$ has no torsion. More precisely, we have

Proposition 2.2.21 (Peterson)**.** Suppose that X is a manifold of dimension at most $2N$ for some integer $N \geq 1$. If for $1 \leq j \leq N$, the only torsion in $\mathrm{H}^{2j}(X; \mathbb{Z})$ is coprime to $(j-1)!$, then a complex vector bundle $E \to X$ is trivial iff its Chern classes vanish, i.e. $c_j(E) = 0$ for $1 \leq j \leq N$.

Note that, by the Newton Identities (1), saying that a complex bundle $E \to X$ is trivial iff $c_j(E) = 0$ for $1 \leq j \leq N$ is equivalent to saying that the Chern character homomorphism $\mathrm{ch} : K(X) \to \mathrm{H}^*(X; \mathbb{Q})$ is injective. In particular, if X has torsion-free integral cohomology, then this result always applies, telling us in particular that $\mathrm{ch} : K(\mathbb{HP}^n) \to \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q})$ is injective.

Sketch of Proof of Proposition 2.2.21. The idea is to look at the classifying space $\mathrm{BU}_n = \mathrm{Gr}_\mathbb{C}^{n,\infty}$ and its Postnikov tower. Recall that given a path connected space X, its Postnikov tower is an inverse system of spaces

$$\cdots \to X^{(k)} \to X^{(k-1)} \to \cdots \to X^{(1)} \to 0$$

with inverse limit X satisfying in particular that each $X^{(k)} \to X^{(k-1)}$ is a fibration with homotopy fiber the Eilenberg-MacLane space $K(\pi_k(X), k)$. When $X = \mathrm{BU}_n$, Bott Periodicity tells us that in the stable range $0 \leq j \leq n$, we have $\pi_j(\mathrm{BU}_n)$ is 0 if j is odd and $\mathbb{Z}$ if j is even, and also that

$$\pi_{2n+1}(\mathrm{BU}_n) = \mathbb{Z}/(n!). \tag{2.24}$$

This fact implies that in the Postnikov tower of BU_n, the maps $\mathrm{BU}_n^{(2k+1)} \to \mathrm{BU}_n^{(2k)}$ can be taken to be the identity whenever $0 \leq 2k+1 \leq n$. Now suppose $E \to X$ is a vector bundle of rank n and $f_E : X \to \mathrm{BU}_n$ its classifying map. If $c(E) = 1$, then we can solve the obstruction-theoretic problem of the homotopy-triviality of f_E by considering the maps $f_E^{(k)} : X \to \mathrm{BU}_n^{(k)}$ and lifting them succesively, using at the last stage the computation (2.24) and that $\mathrm{H}^{2n}(X; \mathbb{Z})$ has no $(n-1)!$-torsion. See [46, Theorem 3.2] for details. ■

Returning now to the proof of this injectivity, we can show that $\mathbb{Z}[\theta]$ is all of $\operatorname{ch} K(\mathbb{HP}^n)$ via induction on $n \geq 0$. This is where the following proof in this article becomes a little less self-contained, since we do not discuss K-theory exact sequences or Bott periodicity here, but we present the argument anyway. The case $n = 0$ is clear, and suppose $n \geq 1$ and that we have shown the result for $n-1$. Then we have a commutative diagram

$$\begin{tikzcd} & K(\mathbb{HP}^n, \mathbb{HP}^{n-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & K(\mathbb{HP}^n) \arrow[r] \arrow[d, "\mathrm{ch}"] & K(\mathbb{HP}^{n-1}) \arrow[d, "\mathrm{ch}"] \\ 0 \arrow[r] & \mathrm{H}^*(\mathbb{HP}^n, \mathbb{HP}^{n-1}; \mathbb{Q}) \arrow[r] & \mathrm{H}^*(\mathbb{HP}^n; \mathbb{Q}) \arrow[r] & \mathrm{H}^*(\mathbb{HP}^{n-1}; \mathbb{Q}), \end{tikzcd}$$

which in light of the injectivity of ch gives rise to an exact sequence of the images

$$0 \to \operatorname{ch} K(\mathbb{HP}^n, \mathbb{HP}^{n-1}) \to \operatorname{ch} K(\mathbb{HP}^n) \to \operatorname{ch} K(\mathbb{HP}^{n-1}).$$

Now $\mathbb{HP}^n/\mathbb{HP}^{n-1} \cong S^{4n}$, and Bott Periodicity tells us also that for any even $m \geq 0$, the map

$$K(S^m) \to \mathrm{H}^*(S^m; \mathbb{Q})$$

maps to the generator of $\mathrm{H}^m(S^m; \mathbb{Z})$. Applying this to $\mathbb{HP}^n/\mathbb{HP}^{n-1} \cong S^{4n}$, we conclude that

$$\operatorname{ch} K(\mathbb{HP}^n, \mathbb{HP}^{n-1}) = \mathbb{Z}\langle \xi^n \rangle = \mathbb{Z}\langle (\theta - 2)^n \rangle,$$

On the other hand, by our inductive hypothesis, we have

$$\operatorname{ch} K(\mathbb{HP}^{n-1}) = \mathbb{Z}[\theta] \subset \mathrm{H}^*(\mathbb{HP}^{n-1}; \mathbb{Q}).$$

These two results combined then tell us that $\operatorname{ch} K(\mathbb{HP}^n)$ cannot be any larger than $\mathbb{Z}[\theta]$, as needed. ∎

Now, we compute the Chern character of the complexified tangent bundle $\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}$.

Lemma 2.2.22. For any $n \geq 0$, we have

$$\operatorname{ch} \mathrm{T}\mathbb{HP}^n_{\mathbb{C}} = (2n+2)\theta - \theta^2.$$

Proof. From Lemma 2.2.17, we know that

$$\mathsf{c}(\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}) = \sum_{i=0}^{\infty} (-1)^i p_i(\mathbb{HP}^n) = (1-\xi)^{2n+2}(1-4\xi)^{-1}.$$

Suppose the Chern roots of $\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}$ are $\gamma_1, \ldots, \gamma_{4n}$. Then taking the logarithm on both sides of

$$\prod_{i=1}^{4n}(1+\gamma_i) = (1-\xi)^{2n+2}(1-4\xi)^{-1}$$

and using the Taylor expansion of $\log(1 \pm z)$ around $z = 0$, we can write this as

$$\sum_{i=1}^{\infty} \left(\frac{(-1)^{i+1}}{i} \right) p_i(\gamma) = -\sum_{i=1}^{\infty} \left(\frac{2n+2-4^i}{i} \right) \xi^i,$$

where $p_i(\gamma)$ is the power sum of degree i in γ. It follows that

$$p_i(\gamma) = \begin{cases} 0 & i \text{ odd}, \\ 2(2n+2-4^{i/2})\xi^{i/2}, & i \text{ even}, i \geq 2. \end{cases}$$

It follows that

$$\begin{aligned} \operatorname{ch}(\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}) &= 4n + \sum_{i=1}^{\infty} \frac{p_i(\gamma)}{i!} \\ &= 4n + \sum_{i=1}^{\infty} \frac{2(2n+2-4^i)\xi^i}{(2i)!} \\ &= (2n+2)(2\cosh\sqrt{\xi}) - 2\cosh\sqrt{4\xi} - 2 \\ &= (2n+2)\theta - \theta^2 \end{aligned}$$

as needed. ■

We are now ready to give Massey's proof.

Theorem 2.2.23 (Massey). For any $n \geq 1$, the space $\mathbb{HP}^n$ does not admit an almost complex structure compatible with its usual smooth structure.

Proof. Suppose there is some complex vector bundle $E \to \mathbb{HP}^n$ of rank $2n$ such that $\mathrm{T}\mathbb{HP}^n \cong E_{\mathbb{R}}$. Then $\mathrm{T}\mathbb{HP}^n_{\mathbb{C}} \cong E \oplus \overline{E}$, so that

$$\operatorname{ch}(\mathrm{T}\mathbb{HP}^n_{\mathbb{C}}) = \operatorname{ch}(E) + \operatorname{ch}(\overline{E}).$$

For any integer $i \geq 0$, the i^{th}-graded component of $\operatorname{ch}(\overline{E})$ is

$$\operatorname{ch}_i(\overline{E}) = (-1)^i \operatorname{ch}(E).$$

Since $\mathrm{H}^{2i}(\mathbb{HP}^n; \mathbb{Z}) = 0$ for odd i, it follows from this as before that

$$\operatorname{ch}(\overline{E}) = \operatorname{ch}(E).$$

In particular, Lemma 2.2.22 gives us

$$\operatorname{ch}(E) = \frac{1}{2} \operatorname{ch} \mathrm{T}\mathbb{HP}^n_{\mathbb{C}} = (n+1)\theta - \frac{1}{2}\theta^2,$$

which does not lie in $\mathbb{Z}[\theta]$, contradicting the result of Theorem 2.2.20. ■

In both approaches, the proof involves computing certain characteristic classes of the tangent bundle $\mathrm{T}\mathbb{HP}^n$ and showing that these characteristic classes cannot be "halved", which is what would be needed for $\mathbb{HP}^n$ to have a complex structure.

We end this section by computing using characteristic class techniques the signature and $\hat{A}$-genus of quaternionic projective spaces.

Proposition 2.2.24. For any integer $n \geq 0$, we have

$$\mathrm{Sign}(\mathbb{HP}^n) = \frac{1+(-1)^n}{2}.$$

Proof 1 of Proposition 2.2.24. When n is odd, the middle cohomology $\mathrm{H}^{2n}(\mathbb{HP}^n;\mathbb{Z})$ has rank zero, so that so is the signature $\mathrm{Sign}(\mathbb{HP}^n)$. On the other hand, when n is even, the same middle cohomology group has rank 1, and is in fact generated by $\xi^{n/2}$ with the intersection matrix given by $[1]$. ■

Proof 2 of Proposition 2.2.24. We use the Hirzebruch Signature Theorem (Theorem 1.3.1). The exact sequence (2.20) of vector bundles tells us that the total L-classes of these vector bundles on $\mathbb{CP}^{2n+1}$ are related as

$$\mathsf{L}(\mathbb{CP}^{2n+1}) = \mathsf{L}(\mathrm{T}_\pi)\cdot \pi^*\mathsf{L}(\mathbb{HP}^n).$$

We know from Subsection 2.2.1 and Lemma 2.2.15 respectively that

$$\mathsf{L}(\mathbb{CP}^{2n+1}) = \left(\frac{\zeta}{\tanh\zeta}\right)^{2n+2} \text{ and } \mathsf{L}(\mathrm{T}_\pi) = \frac{2\zeta}{\tanh 2\zeta}.$$

Combining these three results with Lemma 2.2.14 then gives us

$$\mathsf{L}(\mathbb{HP}^n) = \left(\frac{\sqrt{\xi}}{\tanh\sqrt{\xi}}\right)^{2n+2}\cdot\frac{\tanh 2\sqrt{\xi}}{2\sqrt{\xi}},$$

where we have by definition (see Appendix 3.3) that

$$\frac{\sqrt{z}}{\tanh\sqrt{z}} := \sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}z^n.$$

Therefore, Theorem 1.3.1 tells us that

$$\mathrm{Sign}(\mathbb{HP}^n) = L(\mathbb{HP}^n) = \int_{\mathbb{HP}^n}\left(\frac{\sqrt{\xi}}{\tanh\sqrt{\xi}}\right)^{2n+2}\cdot\frac{\tanh 2\sqrt{\xi}}{2\sqrt{\xi}}.$$

Evaluating this integral amounts to computing the coefficient of ξ^n in the power series given by the integrand, which can express as the residue calculation

$$[\xi^n]\left(\frac{\sqrt{\xi}}{\tanh\sqrt{\xi}}\right)^{2n+2}\cdot\frac{\tanh 2\sqrt{\xi}}{2\sqrt{\xi}} = \operatorname*{Res}_{\xi=0}\frac{1}{(\tanh\sqrt{\xi})^{2n+2}}\cdot\frac{\tanh 2\sqrt{\xi}}{2\sqrt{\xi}}\mathrm{d}\xi.$$

To evaluate this residue, make the change of variables $t = \tanh^2\sqrt{\xi}$.[19] Via this change of variables, and using the identity $\tanh 2x = 2\tanh x(1+\tanh^2 x)^{-1}$, we can express this residue calculation as

$$\operatorname*{Res}_{t=0}\frac{1}{t^{n+1}}\cdot\frac{2\sqrt{t}}{(1+t)\tanh^{-1}\sqrt{t}}\cdot\frac{\tanh^{-1}\sqrt{t}}{(1-t)\sqrt{t}}\mathrm{d}t = [t^n]\frac{1}{1-t^2},$$

which is equivalent to the claim. ■

[19] In other words, we let $t(\xi)$ be the unique holomorphic function $t(\xi)$ defined and biholomorphic in a neighborhood of 0 and satisfying $t(0) = 1$ and $t(\xi^2) = \tanh^2\xi$.

Proposition 2.2.25. For any integer $n \geq 1$, we have
$$\hat{A}(\mathbb{HP}^n) = 0.$$

Clearly, $\hat{A}(\mathbb{HP}^0) = 1$.

Proof 1 of Proposition 2.2.25. It follows from Theorem 1.5.2 that $\mathbb{HP}^n$ always has a unique spin structure compatible with any given smooth structure on it. Since $\mathbb{HP}^n$ is a homogenous space for the compact symplectic group $\mathrm{Sp}(n+1)$, it follows from Remark 1.5.10 that for $n \geq 1$, the space $\mathbb{HP}^n$ dmits a metric of positive scalar curvature, and hence from Theorem 1.5.9 that $\hat{A}(\mathbb{HP}^n) = 0$. ■

Proof 2 of Proposition 2.2.25. Repeating the argument from Proof 2 of Proposition 2.2.24, we can write the total $\hat{A}$ class of $\mathbb{HP}^n$ as
$$\hat{\mathsf{A}}(\mathbb{HP}^n) = \left(\frac{\sqrt{\xi}/2}{\sinh\big(\sqrt{\xi}/2\big)}\right)^{2n+2} \cdot \frac{\sinh\sqrt{\xi}}{\sqrt{\xi}},$$
where, as before (see Appendix 3.3),
$$\tilde{Q}_{\hat{A}}(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$
is the unique function defined, holomorphic, and nonzero in a neighborhood of $z = 0$ satisfying
$$\tilde{Q}_{\hat{A}}(z^2)\sinh\left(\frac{z}{2}\right) = \frac{z}{2}.$$
Therefore,
$$\hat{A}(\mathbb{HP}^n) = [\xi^n]\left(\frac{\sqrt{\xi}/2}{\sinh\big(\sqrt{\xi}/2\big)}\right)^{2n+2} \cdot \frac{\sinh\sqrt{\xi}}{\sqrt{\xi}} = \operatorname*{Res}_{\xi=0} \frac{1}{\big(2\sinh\big(\sqrt{\xi}/2\big)\big)^{2n+2}} \cdot \frac{\sinh\sqrt{\xi}}{\sqrt{\xi}} \mathrm{d}\xi.$$
To evaluate this residue, we make the change of variables $t = 4\sinh^2(\sqrt{\xi}/2)$ to write this as
$$\operatorname*{Res}_{t=0} \frac{1}{t^{n+1}} \cdot \frac{\sinh\big(2\sinh^{-1}(\sqrt{t}/2)\big)}{2\sinh^{-1}(\sqrt{t}/2)} \cdot \frac{2\sinh^{-1}(\sqrt{t}/2)}{\sqrt{t}\sqrt{1+(t/4)}}\mathrm{d}t.$$
Noting now that
$$\begin{aligned}\sinh\left(2\sinh^{-1}\left(\frac{\sqrt{t}}{2}\right)\right) &= \frac{1}{2}\left[\left(\sqrt{1+\frac{t}{4}} + \frac{\sqrt{t}}{2}\right)^2 - \left(\sqrt{1+\frac{t}{4}} - \frac{\sqrt{t}}{2}\right)^2\right] \\ &= \sqrt{t}\sqrt{1+\frac{t}{4}},\end{aligned}$$
we find that this residue is
$$\hat{A}(\mathbb{HP}^n) = \operatorname*{Res}_{t=0} \frac{1}{t^{n+1}}\mathrm{d}t = [t^n](1),$$
which is equivalent to the desired result. ■

2.3 Grassmannians

Denn die Wahrheit ist ewig, ist göttlich; und keine Entwicklungsphase der Wahrheit, wie geringe auch das Gebiet sei, was sie umfasst, kann spurlos vorübergehen; sie bleibt bestehen, wenn auch das Gewand, in welches schwache Menschen sie kleiden, in Staub zerfällt.

[This is] because truth is eternal, is godly; and no period of discovery of the truth, no matter how minor the subject that it encompasses be, can pass over without a trace; it endures, even when the mantle in which mortal men clothe it crumbles to dust.

Vorrede, Die Ausdehnungslehre: Vollständing und in strenger Form bearbeitet von
Hermann Grassmann,
on mathematical truth and the rejection of his work by his contemporaries.

As our next collection of examples, we generalize the results of the previous section by considering the Grassmannian manifolds, often simply called Grassmannians. Generally speaking, Grassmannians, named after the prolific German mathematician Hermann Grassmann, are manifolds that parametrize linear subspaces of a given vector space. They come in various flavors: of course, we have real, complex, and quaternionic Grassmannians, but we also have oriented real Grassmannians, as well as Lagrangian and isotropic variants. Grassmannians, and their generalizations, flag manifolds, are the simplest examples of complete homogenous spaces and have a very rich geometric structure. In this section, we briefly recall the definition some basic properties of Grassmannians and flag bundles. We then specialize to the case of complex Grassmannians, for which we compute the basic invariants such as the χ_y-characteristic, Euler characteristic, signature, etc. using "direct methods". Due to the complicated nature of their holomorphic tangent bundles and the resulting combinatorics, it is not clear (to me) how to carry out these computations at this level of generality purely using characteristic classes, as we have done in previous sections. Of course, we've handled one special case in §2.2; in the next subsection, §2.3.1, we show how to do these computations for the simplest new case of the Grassmannian parametrizing 2-dimensional linear subspaces (or equivalently projective lines).

We begin by recalling the basic definition.

Definition 2.3.1. For any field $\mathbb{F}$ and integers $m, n \geq 0$, we define the Grassmannian manifold of m-planes in $\mathbb{F}^{m+n}$ as

$$\mathrm{Gr}_{\mathbb{F}}^{m,n} := \{V^m \subset \mathbb{F}^{m+n}\},$$

i.e. as the set of m-dimensional $\mathbb{F}$-linear subspaces of $\mathbb{F}^{m+n}$. More generally, suppose that $r \geq 1$ is an integer and $m = (m_1, \ldots, m_r)$ is sequence of nonnegative integers m_j. Let $|m| := m_1 + \cdots + m_r$. We define the flag manifold of type m over $\mathbb{F}$, denoted $\mathrm{Fl}_{\mathbb{F}}^{m}$ or $\mathrm{Fl}_{\mathbb{F}}^{m_1,\ldots,m_r}$, to be the set of all flags of type m in $\mathbb{F}^{|m|}$, i.e. all sequences $(V_1, \ldots, V_{r-1})$ of vector subspaces of $\mathbb{F}^{m+n}$ such that

$$0 =: V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V_r = \mathbb{F}^{|m|},$$

and that for $j = 1, \ldots, r$, we have $\dim_{\mathbb{F}} V_j/V_{j-1} = m_j$.

We may also speak of the flag manifold of a given type m consisting of subspaces of an abstract $\mathbb{F}$-vector space W of dimension $|m|$; this is usually denoted by $\mathrm{Fl}^m_{\mathbb{F}}(W)$. For any $n \geq 0$, we have $\mathbb{FP}^n = \mathrm{Gr}^{1,n}_{\mathbb{F}}$ and for any $m, n \geq 0$ that $\mathrm{Gr}^{m,n}_{\mathbb{F}} = \mathrm{Fl}^{m,n}_{\mathbb{F}}$. It is clear by definition that the Grassmannian $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ also parametrizes projective $(m-1)$-planes in $\mathbb{FP}^{m+n-1}$, and is hence sometimes also written as $\mathbb{FG}(m-1, m+n-1)$, although we will not use this notation. In particular, for $m = 2$ and any $n \geq 0$, the Grassmannian $\mathrm{Gr}^{2,n}_{\mathbb{F}}$ parametrizes projective lines $\ell \cong \mathbb{FP}^1$ in $\mathbb{FP}^{n+1}$, and is hence often called the Grassmannian of lines. Note finally that $\mathrm{Gr}^{m,n}_{\mathbb{F}} \cong \mathrm{Gr}^{n,m}_{\mathbb{F}}$ for any $m, n \geq 0$, given by taking a $V^m \subset \mathbb{F}^{m+n}$ to its annihilator $\mathrm{Ann}(V)^n \subset (\mathbb{F}^{m+n})^\vee$. Note also that we call a flag of type $(1, 1, \ldots, 1)$ a complete flag, and the corresponding flag manifold $\mathrm{Fl}^{1^r}_{\mathbb{F}}$ the complete flag manifold.

An element of $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ is given by (the image of) a full rank $(m+n)\times m$ matrix with values in $\mathbb{F}$, where we identify two such matrices iff they differ by an action of a matrix in $\mathrm{GL}_m \mathbb{F}$ on the right. This gives us a surjection $\pi : \mathrm{Mat}^{\mathrm{fr}}_{(m+n)\times m} \mathbb{F} \to \mathrm{Gr}^{m,n}_{\mathbb{F}}$ from the space of full rank $(m+n) \times m$ matrices onto $\mathrm{Gr}^{m,n}_{\mathbb{F}}$, which allows to give $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ the quotient topology if $\mathbb{F}$ is a topological field. When $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the space $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ comes equipped with a unique smooth structure of real dimension dmn, where $d = \dim_{\mathbb{R}} \mathbb{F} \in \{1, 2, 4\}$ as before, which makes π a smooth quotient map. To verify that $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ is a manifold, we can give charts generalizing those for projective spaces as follows. Given a collection $I = \{i_1, \ldots, i_m\}$ of integers satisfying $1 \leq i_1 < i_2 < \cdots < i_m \leq m+n$, consider the locus $U_I \subset \mathrm{Gr}^{m,n}_{\mathbb{F}}$ given by those matrices with non-vanishing $m \times m$ minor coming from taking rows in I. Then $U_I \subset \mathrm{Gr}^{m,n}_{\mathbb{F}}$ is open and, similarly to projective space, we can write down a diffeomorphism $\varphi_I : U_I \to \mathbb{F}^{mn}$: each element in U_I can be represented by a unique $(m+n) \times m$ matrix A such that the $m \times m$ minor of A with rows in I is just the identity matrix, and the remaining mn entries give the coordinates of $\varphi_I : U_I \to \mathbb{F}^{mn}$. More coordinate-invariantly, U_I is the set of subspaces that may be written as the graph of a map $\mathbb{F}^I \to \mathbb{F}^{(m+n)-I}$, where $\mathbb{F}^I$ is spanned by the basis vectors $\mathrm{e}_{i_1}, \ldots, \mathrm{e}_{i_m}$, and $\mathbb{F}^{(m+n)-I}$ is spanned by the remaining basis vectors and may be identified by $\mathbb{F}^{m+n}/\mathbb{F}^I$. For $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{C}$, $\mathbb{H}$), the group SO_{m+n} (resp. $\mathrm{SU}_{m+n}, \mathrm{Sp}_{m+n}$) acts transitively on $\mathrm{Gr}^{m+n}_{\mathbb{F}}$ on the left, showing us that $\mathrm{Gr}^{m+n}_{\mathbb{F}}$ is in fact a connected closed manifold. When $\mathbb{F}$ is a field, there is a natural map $\Lambda^m : \mathrm{Gr}^{m,n}_{\mathbb{F}} \to \mathbb{FP}^{\binom{m+n}{m}-1}$ taking a vector subspace $V^m \subset \mathbb{F}^{m+n}$ to its determinant line $\Lambda^m V \subset \Lambda^m \mathbb{F}^{m+n}$ in the m^{th} exterior power of $\mathbb{F}^{m+n}$. This map is injective with closed image (in the say Zariski topology or classical topology when available), the image being the set of totally decomposable vectors in $\Lambda^m \mathbb{F}^{m+n}$, and in the case of $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is easily seen to give us a smooth embedding, called the Plücker embedding, of $\mathrm{Gr}^{m,n}_{\mathbb{F}}$ into projective space as a closed submanifold.[20] In particular, in the case $\mathbb{F} = \mathbb{C}$, the Grassmannian $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ is a compact Kähler manifold.

Similarly, it is a standard result that when $\mathbb{F} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the space $\mathrm{Fl}^m_{\mathbb{F}}$ is a smooth manifold of real dimension $d \cdot e_2(m)$, where $d \in \{1, 2, 4\}$ as before and $e_2(m)$ is the second elementary symmetric polynomial in m_j. The obvious map $\mathrm{Fl}^m_{\mathbb{F}} \to \prod_{j=1}^r \mathrm{Gr}^{m_j, |m|-m_j}_{\mathbb{F}}$ embeds $\mathrm{Fl}^m_{\mathbb{F}}$ as a closed subvariety of a product of Grassmannians, and hence expresses it as a smooth projective variety when $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In particular, $\mathrm{Fl}^m_{\mathbb{C}}$ is a compact Kähler manifold of complex dimension $e_2(m)$.

[20] The Plücker embedding, does not, in fact, exist when $\mathbb{F} = \mathbb{H}$. Not only is it not clear how to define the usual notions of linear algebra, such as exterior powers, when working with quaternionic vector spaces, but there is also a topological obstruction: there is no continuous map $\mathrm{Gr}^{m,n}_{\mathbb{H}} \to \mathbb{HP}^N$ for any m, n, N such that the pullback of the first Pontryagin class of the tautological bundle $\mathcal{O}_{\mathbb{HP}^N}(-1)$ over $\mathbb{HP}^N$ is the first Pontryagin class of the tautological line bundle $\mathcal{S}$ over $\mathrm{Gr}^{m,n}_{\mathbb{H}}$; see [47].

As a final general remark, we note that the construction of flag bundles is natural, so, for instance, when $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$, and we have a sequence m and an $\mathbb{F}$-vector bundle $E \to X$ on a space X, then we can form an associated bundle

$$\mathrm{Fl}^m(E) \to X,$$

whose fiber at a point $x \in X$ is the flag manifold $\mathrm{Fl}^m_{\mathbb{F}}(E(x))$ consisting of tuples of subspaces of the fiber $E(x) \cong \mathbb{F}^{|m|}$. In particular, we get this way the projectivization $\mathbb{P}E \to X$ of E, as well as the Grassmannian bundles $\mathrm{Gr}^m(E) \to X$ for $0 \leq m \leq \operatorname{rank} E$.

We now compute some numerical invariants of complex Grassmannians. In the following discussion, we will assume familiarity with their cohomology rings. For the convenience of the reader, we summarize this beautiful bit of theory in Appendix 3.7.

Remark 2.3.2. In stating the following formulae relating to the χ_y-characteristic of Grassmannians, it is convenient to adopt the convention where we flip the sign of y, talking about the "χ_{-y}-characteristic" instead.

Theorem 2.3.3. For any $m, n \geq 0$, we have

$$\chi_{-y}\left(\mathrm{Gr}^{m,n}_{\mathbb{C}}\right) = \prod_{j=1}^{m} \frac{1 - y^{n+j}}{1 - y^j}.$$

Proof. The result is clear for $n = 0$; suppose then that $n \geq 1$. The key to proving this result is the multiplicativity of the χ_y-characteristic in fibre bundles (Theorem 3.5.6), along with the observation that for any $m \geq 0$, the flag manifold $\mathrm{Fl}^{m,1,n-1}_{\mathbb{C}}$ fibers over both $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ and $\mathrm{Gr}^{m+1,n-1}_{\mathbb{C}}$ with fibers projective spaces. Specifically, we can look at the diagram

$$\begin{tikzcd}
& \mathrm{Fl}^{m,1,n-1}_{\mathbb{C}} \arrow[dl, "\mathbb{P}\mathcal{Q}"'] \arrow[dr, "\mathrm{Gr}^m \mathcal{S}"] & \\
\mathrm{Gr}^{m,n}_{\mathbb{C}} & & \mathrm{Gr}^{m+1,n-1}_{\mathbb{C}},
\end{tikzcd}$$

where the first projection map

$$\mathrm{Fl}^{m,1,n-1}_{\mathbb{C}} = \{V^m \subset W^{m+1} \subset \mathbb{C}^{m+n}\} \to \mathrm{Gr}^{m,n}_{\mathbb{C}} = \{V^m \subset \mathbb{C}^{m+n}\}$$

expresses the flag manifold as the projectivized tautological quotient bundle $\mathbb{P}\mathcal{Q}$ of $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ with fibers $\mathbb{CP}^{m-1}$, whereas the the second projection map

$$\mathrm{Fl}^{m,1,n-1}_{\mathbb{C}} = \{V^m \subset W^{m+1} \subset \mathbb{C}^{m+n}\} \to \mathrm{Gr}^{m+1,n-1}_{\mathbb{C}} = \{W^{m+1} \subset \mathbb{C}^{m+n}\}$$

expresses the flag manifold as the m^{th} Grassmannian bundle $\mathrm{Gr}^m \mathcal{S}$ of the tautological subbundle of $\mathrm{Gr}^{m+1,n-1}_{\mathbb{C}}$, with fibers $\mathrm{Gr}^{m,1}_{\mathbb{C}} = (\mathbb{CP}^m)^\vee \cong \mathbb{CP}^m$. Using Corollary 3.5.7 to write the χ_y-characteristic of $\mathrm{Fl}^{m,1,n-1}_{\mathbb{C}}$ in two ways, we get

$$\chi_y\left(\mathrm{Fl}^{m,1,n-1}_{\mathbb{C}}\right) = \chi_y\left(\mathrm{Gr}^{m,n}_{\mathbb{C}}\right) \chi_y(\mathbb{CP}^{m-1}) = \chi_y\left(\mathrm{Gr}^{m+1,n-1}_{\mathbb{C}}\right) \chi_y(\mathbb{CP}^m), \tag{2.25}$$

so the result follows from (2.25) by induction, with the base case being Corollary 2.2.6. ∎

Remark 2.3.4. We mention three other proofs of Theorem 2.3.3.

(a) For complex algebraic homogeous manifolds $X = G/H$, we have

$$\chi_{-y}(X) = p_{\sqrt{y}}(X),$$

i.e. the χ_y-characteristic and the Poincaré polynomial agree up to a sign and up to replacing y by $\sqrt{y}$. Then, by the Leray-Hirsch Theorem, the Poincaré polynomial of a homogenous manifold G/H, where G and H have the same rank, can be computed from those of G and H. See the classic references [48] and [49] for details.

(b) Alternatively, once this observation is made, we can also reduce the computation of the Poincaré polynomial to a calculation of the size $\#\mathrm{Gr}^{m,n}_{\mathbb{F}_q}$ of the Grassmannian over a finite field $\mathbb{F}_q$, which can be done explicitly using either the Orbit-Stabilizer Theorem or the Schubert cell decomposition. See [10, Corollary 3.2.5], where it is explained why these approaches are the same, and why the equivalence of these approaches is a special case of the Weil conjectures noticed by Weil himself.

(c) Finally, we can construct action of the torus $\mathbb{T}^{m+n}$ on $\mathrm{Gr}^{m+n}_{\mathbb{C}}$ and apply the holomorphic Lefschetz formula, or equivalently the equivariant Atiyah-Singer Index Theorem, to the resulting action. A version of this argument is carried out in [50], and we will do a version of this argument in Proof 3 of Corollary 2.3.5.

Corollary 2.3.5. For any $m, n \geq 0$, the Euler characteristic of $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ is

$$\chi\left(\mathrm{Gr}^{m,n}_{\mathbb{C}}\right) = \binom{m+n}{m}.$$

Proof 1 of Corollary 2.3.5. From the previous result, we get

$$\chi\left(\mathrm{Gr}^{m,n}_{\mathbb{C}}\right) = \lim_{y\to 1} \chi_{-y}\left(\mathrm{Gr}^{m,n}_{\mathbb{C}}\right) = \prod_{j=1}^{m} \frac{n+j}{j} = \binom{m+n}{m}.$$

■

Proof 2 of Corollary 2.3.5. The Schubert cell decomposition of $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ gives us a CW structure on $\mathrm{Gr}^{m,n}_{\mathbb{C}}$ with only even dimensional cells in one-to-one correspondence with partitions $\lambda \subset m \times n$ or equivalently, m-element subsets of $\{1, \ldots, m+n\}$ (see Lemma 3.7.1 if needed). In particular, there are exactly $\binom{m+n}{m}$ of these. ■

Proof 3 of Corollary 2.3.5. Similarly to Proof 2 of Proposition 2.2.2, we calculate the Lefschetz number $L(f)$ of a map f homotopic to the identity. In fact, a version of the same map works; for $t \in [0, \infty)$, consider the map $f_t : \mathbb{F}^{m+n} \to \mathbb{F}^{m+n}$ given by the diagonal matrix

$$f_t = \mathrm{diag}(\mathrm{e}^t, \mathrm{e}^{2t}, \ldots, \ldots, \mathrm{e}^{(m+n)t}) \in \mathrm{GL}_{m+n}\,\mathbb{C}$$

and the induced map $f_t : \mathrm{Gr}^{m,n}_{\mathbb{C}} \to \mathrm{Gr}^{m,n}_{\mathbb{C}}$ taking V to $f_t(V)$. Then for $t \in (0, \infty)$, it is easy to check that f_t has exactly $\binom{m+n}{n}$ fixed points, namely the $\binom{m+n}{n}$ subspaces $V_I := \langle \mathrm{e}_{i_1}, \ldots, \mathrm{e}_{i_m} \rangle$ spanned by some m of the $n+m$ standard basis vectors $\mathrm{e}_1, \ldots, \mathrm{e}_{m+n}$, indexed by subsets $I \subset \{1, \ldots, m+n\}$ of size m. Indeed, let $V^m \subset \mathbb{C}^{m+n}$ be fixed under f_t for some $t \in (0, \infty)$. Then $f|_V : V \to V$ must have a nonzero eigenvector v, and so by the Proof 2 of Proposition 2.2.2, this v must be one of the e_j's. Then we can replace V by $V/\langle \mathrm{e}_j\rangle \subset \mathbb{C}^{m+n}/\langle \mathrm{e}_j\rangle$ and proceed by induction. We claim that each V_I is a Lefschetz fixed point of f with Lefschetz number $L_{V_I}(f) = 1$; this would finish the proof.

Given an $I = \{i_1, \ldots, i_m\}$, let $J = \{1, \ldots, m+n\} \smallsetminus I = \{j_1, \ldots, j_n\}$ be its complement. To compute the Lefschetz number, we can look at the neighborhood U_I of V_I in $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ and the chart $\varphi_I : U_I \to \mathbb{F}^{nm}$ defined above; recall that U_I is defined as the subset of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ consisting of $V \in \mathrm{Gr}_{\mathbb{C}}^{m,n}$ described by the non-vanishing of the row-I minor of size $m \times m$ of any $(m+n) \times m$ matrix representing V, and the chart $\varphi_I : U_I \to \mathbb{F}^{nm}$ is given by setting this minor be the identity, and taking the coefficients to be the entries of the remaining $n \times m$ matrix. It is easy to see that f_t preserves U_I, and the map $\varphi_I \circ f_t \circ \varphi_I^{-1} : \mathbb{F}^{nm} \to \mathbb{F}^{nm}$ given by taking an $n \times m$ matrix A to the matrix

$$\mathrm{diag}(\mathrm{e}^{j_1 t}, \mathrm{e}^{j_2 t}, \ldots, \mathrm{e}^{j_n t}) \cdot A \cdot \mathrm{diag}(\mathrm{e}^{-i_1 t}, \mathrm{e}^{-i_2 t}, \ldots, \mathrm{e}^{-i_m t}),$$

i.e. by $[a_{\ell k}] \mapsto [\mathrm{e}^{(j_\ell - i_k)t} a_{\ell k}]$ for $1 \le \ell \le n$ and $1 \le k \le m$. In particular, $\varphi_I \circ f_t \circ \varphi_I^{-1}$ is linear, and hence so equals its own derivative at any point. The eigenvalues of this map are clearly $\mathrm{e}^{(j_\ell - i_k)t}$ for $1 \le \ell \le n$ and $1 \le k \le m$, and this quantity never equals 1, since I and J are disjoint and $t \in (0, \infty)$. Therefore, V_I is a Lefschetz fixed point. Arguing similarly to Proof 2 of Proposition 2.2.2, we compute that

$$\det\left(\mathrm{d}f_t|_{V_I} - \mathrm{id}_{\mathrm{T}_{V_I}\mathrm{Gr}_{\mathbb{C}}^{m,n}}\right) = \prod_{\substack{1 \le \ell \le n,\\ 1 \le k \le m}} \left(\mathrm{e}^{(j_\ell - i_k)t} - 1\right)^2.$$

In particular, this determinant is the square of some nonzero real number, and is hence positive, finishing the proof. ■

Remark 2.3.6. We chose to do the case $\mathbb{F} = \mathbb{C}$ above because it is the simplest and most illustrative, but the same arguments as in Proofs 2 and 3 above show also that

$$\chi\left(\mathrm{Gr}_{\mathbb{H}}^{m,n}\right) = \binom{m+n}{n}.$$

A similar argument can be made for $\mathrm{Gr}_{\mathbb{R}}^{m,n}$, but now we have to account for signs. Either using Proof 2 or 3, we can then conclude that

$$\chi\left(\mathrm{Gr}_{\mathbb{R}}^{m,n}\right) = \sum_{\lambda \subset m \times n} (-1)^{|\lambda|} = \begin{cases} \binom{\lfloor (m+n)/2 \rfloor}{\lfloor m/2 \rfloor} & \text{if } 2 \mid mn, \text{ and} \\ 0, & \text{else,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function, and the second equality is a simple combinatorial check, a version of which will be done in Proof 2 of Corollary 2.3.8 below. Note that trying to adapt Proof 1 of Corollary 2.3.5 using the multiplicativity of the Euler characteristic (Theorem 3.5.1) doesn't work on the nose because $\chi(\mathbb{RP}^n) = 0$ for odd n, but its extension to Poincaré polynomials (Theorem 3.5.4) works and shows in exactly the same way that the $\mathbb{F}_2$-Poincaré polynomial of $\mathrm{Gr}_{\mathbb{R}}^{m,n}$ is given by

$$p_t\left(\mathrm{Gr}_{\mathbb{R}}^{m,n}; \mathbb{F}_2\right) = \prod_{i=1}^{m} \frac{1 - t^{n+j}}{1 - t^j} \in \mathbb{Z}[t],$$

and so the result follows from Lemma 2.3.7 below.

Lemma 2.3.7. For any integers $m, n \geq 0$, we have

$$\lim_{t \to -1} \prod_{i=1}^{m} \frac{1 - t^{n+j}}{1 - t^j} = \begin{cases} \binom{\lfloor (m+n)/2 \rfloor}{\lfloor m/2 \rfloor} & \text{if } 2 \mid mn, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Proof. This is a simple case-by-case check. When m and n are both odd, the exact power of $1+t$ dividing the numerator is $(1+t)^{(m+1)/2}$, while that dividing the denominator is $(1+t)^{(m-1)/2}$, so the quotient has limit 0 as $t \to -1$. When either one of m or n is even, the numerator and denominator are divisible by the same power of $1+t$, namely $(1+t)^{\lfloor m/2 \rfloor}$, and the limit is nonzero. The evaluation of this limit follows from the observation that

$$\lim_{t \to -1} \frac{1 - t^a}{1 - t^b}$$

is 1 if $a \equiv b \equiv 1 \pmod 2$ and a/b if $a \equiv b \equiv 0 \pmod 2$. ■

The same calculation gives also the signature of the complex Grassmannian.

Corollary 2.3.8. For any $m, n \geq 0$, the signature of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is given by

$$\operatorname{Sign}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = \begin{cases} \binom{\lfloor (m+n)/2 \rfloor}{\lfloor m/2 \rfloor} & \text{if } 2 \mid mn, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Proof 1 of Corollary 2.3.8. Take $y \to -1$ in Theorem 2.3.3 and use Lemma 2.3.7. ■

Proof 2 of Corollary 2.3.8. By Proposition 3.7.4, we have a complete description of the intersection pairing on the middle cohomology

$$\mathrm{H}^{mn}(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z}) = \bigoplus_{\substack{\lambda \subset m \times n \\ |\lambda| = mn/2}} \mathbb{Z}\sigma_\lambda$$

as follows. We say that a partition $\lambda \subset m \times n$ is **self-complementary** if $\lambda = \hat{\lambda}$, i.e. if the complement of the Ferrers diagram of λ in the $m \times n$ rectangle is an inverted copy of itself. The non-self-complementary partitions λ satisfying $|\lambda| = mn/2$ come in pairs $(\lambda, \hat{\lambda})$ and the corresponding Schubert cycles $\sigma_\lambda, \sigma_{\hat{\lambda}}$ span a hyperbolic plane $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the middle cohomology, whereas σ_λ for self-complementary λ gives rise to an orthonormal summand. In other words, if $P_{m,n}$ is the number of pairs of non-self-complementary λ with $|\lambda| = mn/2$, and $Q_{m,n}$ is the number of self-complementary λ so that the middle Betti number of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is $b_{mn}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = 2P_{m,n} + Q_{m,n}$, then the matrix representing the intersection pairing on $\mathrm{H}^{mn}(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z})$ can be written as a block direct sum

$$U^{P_{m,n}} \oplus [1]^{Q_{m,n}}.$$

Since this is an orthogonal decomposition and $\operatorname{Sign}(U) = 0$, we conclude that

$$\operatorname{Sign}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = P_{m,n}\operatorname{Sign}(U) + Q_{m,n}\operatorname{Sign}([1]) = Q_{m,n}.$$

Therefore, the signature of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is the number of self-complementary partitions $\lambda \subset m \times n$. If mn is odd, this number is clearly zero (in fact, in this case $P_{m,n} = 0$ as well and $b_{mn}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = 0$). When m is odd but n is even, a self-complementary λ determines and is determined by the sequence $n \geq \lambda_1 \geq \cdots \lambda_{(m-1)/2} \geq n/2$ with $\lambda_{(m+1)/2} = n/2$, and when m is even, with n arbitrary, a self-complementary λ determines and is determined the sequence $n \geq \lambda_1 \geq \cdots \lambda_{m/2} \geq n/2$. In either case, it is not hard to see that there are exactly

$$Q_{m,n} = \binom{\lfloor (m+n)/2 \rfloor}{\lfloor m/2 \rfloor}$$

such sequences.

■

Remark 2.3.9. The signature of real and quaternionic Grassmannians can be described as follows. Firstly, in the quaternionic case, the signature agrees with the complex case, namely

$$\mathrm{Sign}\left(\mathrm{Gr}_{\mathbb{H}}^{m,n}\right) = \mathrm{Sign}\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}\right) = \begin{cases} \binom{\lfloor (m+n)/2 \rfloor}{\lfloor m/2 \rfloor} & \text{if } 2 \mid mn, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Finally, the signature of the real Grassmannian is giving by pushing further in the 2-divisibility, namely

$$\mathrm{Sign}\left(\mathrm{Gr}_{\mathbb{R}}^{m,n}\right) = \begin{cases} \binom{\lfloor (m+n)/4 \rfloor}{\lfloor m/4 \rfloor} & \text{if } 8 \mid mn, \text{ and} \\ 0, & \text{else.} \end{cases}$$

To prove these results, one could try to mimic Proof 2 of Corollary 2.3.8 above, but the computation seems to be much more involved. A better proof can be given by constructing suitable torus actions on these Grassmannians and using the equivariant Atiyah-Singer Index Theorem; since we do not build the technology of the equivariant case, we refer the reader to [50] and [51] for details.

Let's talk briefly about the two other invariants we have been interested in–holomorphic Euler characteristics of vector bundles, and the $\hat{A}$-genus. Firstly, we have:

Proposition 2.3.10. We have for any sequence $m = (m_1, \ldots, m_r)$ that

$$\chi\left(\mathrm{Fl}_{\mathbb{C}}^{m}, \mathcal{O}_{\mathrm{Fl}_{\mathbb{C}}^{m}}\right) = h^0\left(\mathrm{Fl}_{\mathbb{C}}^{m}, \mathcal{O}_{\mathrm{Fl}_{\mathbb{C}}^{m}}\right) = 1.$$

Proof. This is an immediate consequence of the Borel-Weil-Bott Theorem (see [44])–since $\mathrm{Fl}_{\mathbb{C}}^{m}$ is a complete homogenous space, we have $h^i\left(\mathrm{Fl}_{\mathbb{C}}^{m}, \mathcal{O}_{\mathrm{Fl}_{\mathbb{C}}^{m}}\right) = 0$ for all $i > 0$. ■

It is possible to calculate also the cohomology and hence Euler characteristic of other vector bundles, such as those arising from linear algebraic operations on the tautological bundels $\mathcal{S}_j$ on $\mathrm{Fl}_{\mathbb{C}}^{m}$ using techniques in Geometric Representation Theory. The key idea is to reduce the computation to that of line bundles via the following two observations:

- If $f : Y \to X$ is a holomorphic fibre bundle of closed complex manifolds and $\mathcal{F} \to X$ is a coherent sheaf, then

$$\chi(Y, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \chi(X, \mathrm{R}^i f_* \mathcal{F}),$$

where $\mathrm{R}^i f_*$ is the i^{th} derived functor of the pushforward f_*. This is a consequence of the Leray-Serre Spectral sequence $\mathrm{H}^p(Y, \mathrm{R}^q f_* \mathcal{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F})$, along with the fact that the Euler characteristic of the pages of a spectral sequence is constant.
- If $E \to X$ is a holomorphic vector bundle of rank $n + 1$ and $\pi : \mathbb{P}E \to X$ its projectivization, then we have for each $k \in \mathbb{Z}$ the line bundle $\mathcal{O}_{\mathbb{P}E}(k) \to \mathbb{P}E$ whose restriction to a given fiber $\mathbb{P}E(x) \cong \mathbb{CP}^n$ is exactly $\mathcal{O}_{\mathbb{P}E(x)}(k) \cong \mathcal{O}_{\mathbb{CP}^n}(k)$. Then we can concretely describe the higher pushforwards of this vector bundle as

$$\mathrm{R}^i \pi_* \mathcal{O}_{\mathbb{P}E}(k) = \begin{cases} \mathrm{Sym}^k E^\vee, & \text{if } i = 0, k \geq 0, \\ \mathrm{Sym}^{-n-k-1} E, & \text{if } i = n, k \leq -n-1, \text{ and} \\ 0 & \text{else.} \end{cases}$$

This is simply the relative version of the cohomology computation we carried out in Proof 2 of Corollary 2.2.9, and essentially follows from that computation.

Therefore, if $E \to X$ is a holomorphic vector bundle of rank $n + 1$, then

$$\chi\left(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(k)\right) = \begin{cases} \chi(X, \mathrm{Sym}^k E^\vee), & k \geq 0, \\ (-1)^n \chi\left(X, \mathrm{Sym}^{-n-k-1} E\right), & k \leq -n-1, \\ 0 & \text{else.} \end{cases}$$

For instance, this allows us to conclude, for Grassmannians $\mathrm{Gr}_{\mathbb{C}}^{2,n}$ of lines, that

$$\chi\left(\mathrm{Gr}_{\mathbb{C}}^{2,n}, \mathcal{S}_{\mathrm{Gr}_{\mathbb{C}}^{2,n}}\right) = -\chi\left(\mathrm{Fl}_{\mathbb{C}}^{1,1,n}, \mathcal{S}_0^{\otimes 3}\right),$$

reducing the computation to line bundles on the flag manifold $\mathrm{Fl}_{\mathbb{C}}^{1,1,n}$. Once we have performed this reduction, we can then appeal to Geometric Representation Theory to express this calculation using weights of certain representations. A detailed discussion will take us too far afield, and we refer the reader to [52, Chapter 4] for details.

Finally, for the $\hat{A}$-genus, we have:

Proposition 2.3.11. If for $m = (m_1, \ldots, m_r)$, the flag manifold $\mathrm{Fl}_{\mathbb{C}}^m$ is spin, then

$$\hat{A}\left(\mathrm{Fl}_{\mathbb{C}}^m\right) = 0.$$

Proof. As observed in Remark 1.5.10, this follows from Theorem 1.5.9. ∎

We will show below (see Proposition 2.3.13) that a Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is spin iff $m \equiv n \pmod 2$. In particular, we conclude that $\hat{A}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = 0$ whenever $m \equiv n \pmod 2$. When m and n have different parity, there is no guarantee that $\hat{A}$ will be an integer. We saw this for $m = 1$ and $n = 2k$ in Remark 2.2.12. One can check also that $\hat{A}(\mathrm{Gr}_{\mathbb{C}}^{2,3}) = -1/1024$.

We now explain why obtaining the above results using the techniques of the Atiyah-Singer Index Theorem or Hirzebruch-Riemann-Roch Theorem is very difficult. The key difficulty lies in computing the Todd class

$$\mathsf{Td}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) \in \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z})$$

at this level of generality. To do this, we need to talk about characteristic class of $\mathscr{T}\mathrm{Gr}_{\mathbb{C}}^{m,n}$ more generally, which we now do.

On the flag bundle $\mathrm{Fl}_{\mathbb{F}}^{m}$, we have a sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ of r vector bundles, called the **tautological subquotient bundles**, where for $j = 1, \ldots, r$, the fiber of $\mathcal{S}_j$ at a flag $(V_1, \ldots, V_{r-1})$ is exactly V_j/V_{j-1}. By definition, the bundle $\mathcal{S}_j$ is an $\mathbb{F}$-vector bundle of rank m_j; this is the reason for our choice of notational convention for Grassmannians and flag bundles.[21] When the flag manifold is a one-step flag bundle, i.e. $r = 2$ and the manifold is a Grassmannian $\mathrm{Gr}_{\mathbb{F}}^{m,n}$, the bundle $\mathcal{S}_1$ is simply denoted by $\mathcal{S}$ and is called the **tautological subbundle**, whereas the bundle $\mathcal{S}_2$ is denoted by $\mathcal{Q}$ and is called the **tautological quotient bundle**. By definition, for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we have a short exact sequence of vector bundles

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathrm{Gr}_{\mathbb{F}}^{m,n}}^{\oplus(m+n)} \to \mathcal{Q} \to 0 \tag{2.26}$$

on the Grassmannian $\mathrm{Gr}_{\mathbb{F}}^{m,n}$. Similarly to projective spaces, then the tangent bundle $\mathrm{TGr}_{\mathbb{R}}^{m,n}$ (resp. holomorphic tangent bundle $\mathscr{T}\mathrm{Gr}_{\mathbb{C}}^{m,n}$) can be described as

$$\mathrm{TGr}_{\mathbb{R}}^{m,n} \cong \mathcal{S}_{\mathbb{R}}^{\vee} \otimes \mathcal{Q}_{\mathbb{R}} \text{ (resp. } \mathscr{T}\mathrm{Gr}_{\mathbb{C}}^{m,n} \cong \mathcal{S}_{\mathbb{C}}^{\vee} \otimes \mathcal{Q}_{\mathbb{C}}), \tag{2.27}$$

where as before we have included the base field for emphasis and to distinguish the two cases. However, since $\mathcal{S}$ is not a line bundle in general, this makes it obtaining the total Chern class, or other characteristic classes and numbers of $\mathrm{Gr}_{\mathbb{F}}^{m,n}$ more difficult by direct means.

The first computation, in any case, is that of the Chern classes of $\mathcal{S}^{\vee}$ and $\mathcal{Q}$:

Theorem 2.3.12. For any $m, n \geq 0$, if $\mathcal{S}$ and $\mathcal{Q}$ denote the tautological sub- and quotient bundles on $\mathrm{Gr}_{\mathbb{C}}^{m,n}$, then

$$\begin{aligned} \mathsf{c}(\mathcal{S}^{\vee}) &= 1 + \sigma_1 + \sigma_{1^2} + \cdots + \sigma_{1^m}, \text{ and} \\ \mathsf{c}(\mathcal{Q}) &= 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n. \end{aligned}$$

This we take as a standard theorem, so we only indicate sketches of two proofs.

Sketch of Proof 1 of Theorem 2.3.12. Any $\ell \in (\mathbb{F}^{m+n})^{\vee}$ gives rise to a global section of $\mathcal{S}^{\vee}$, whereas any $v \in \mathbb{F}^{m+n}$ gives rise to a global section of $\mathcal{Q}$. We can then finish by using these sections and the description of Chern classes as degeneracy loci; see [43, §5.6.2]. ■

Sketch of Proof 2 of Theorem 2.3.12. To do the case of $\mathcal{Q}$, using the nondegeneracy of the intersection product and Proposition 3.7.4, it suffices to show that $c_k(\mathcal{Q}) \cdot \sigma_\lambda$ is 1 iff $\lambda = \hat{k}$ is the complementary partition to (k) (see [10, Prop. 3.5.5]). Then to deduce the class of $\mathcal{S}^{\vee}$, it suffices to consider the sequence (2.26) and to use the algebraic identity

$$(1 - \sigma_1 + \sigma_{1^2} + \cdots + (-1)^m \sigma_{1^m})(1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n) = 1,$$

which can be proven using Pieri's formula (see [43, Corollary 4.10]). See also [4, §3.3]. ■

[21] This is also how, for instance, Macaulay2 encodes flag varieties.

Theorem 2.3.12, combined with the decomposition 2.27 and the splitting principle, allows us to calculate the total K-class of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ for any multiplicative sequence K. Namely, suppose that $\gamma_1, \ldots, \gamma_m$ are the Chern roots of $\mathcal{S}^\vee$ so that for $1 \leq i \leq m$ we have

$$e_i(\gamma) = \sigma_{1^i},$$

and suppose that $\delta_1, \ldots, \delta_n$ are the Chern roots of $\mathcal{Q}$, so that for $1 \leq j \leq n$ we have

$$e_j(\delta) = \sigma_j.$$

Then if the multiplicative sequence K corresponds to the characteristic series $Q(z)$, we can write the total K-class of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ as

$$\mathsf{K}\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}\right) = \prod_{i=1}^{m} \prod_{j=1}^{n} Q\left(\gamma_i + \delta_j\right).$$

For instance, we have

$$\mathsf{Td}_y(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{(\gamma_i + \delta_j)\left(1 + y\mathrm{e}^{-(\gamma_i+\delta_j)(1+y)}\right)}{1 - \mathrm{e}^{-(\gamma_i+\delta_j)(1+y)}} \in \mathrm{H}^*\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Q}[y]\right)$$

and now it is clear how difficult of a problem it is to express this in terms of Schubert cycles in general. Already for $K = \mathrm{id}$ and $Q(z) = 1 + z$, this gives us a nontrivial expression for the total Chern class of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ as

$$\mathsf{c}\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}\right) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + \gamma_i + \delta_j), \tag{2.28}$$

and it is not clear how to extract individual c_j from this for general j, although it is not to hard do so for small j. For $j = 1$, we get:

Proposition 2.3.13. The first Chern class of the Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is given by

$$c_1\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}\right) = (m + n)\sigma_1.$$

In particular, $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ is spin iff $m \equiv n \pmod 2$.

Proof. From (2.28), we get that

$$c_1\left(\mathrm{Gr}_{\mathbb{C}}^{m,n}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} (\gamma_i + \delta_j) = \sum_{i=1}^{m} (n\gamma_i + \sigma_1) = n\sigma_1 + m\sigma_1 = (m + n)\sigma_1.$$

The second part follows from this computation and Corollary 1.5.3. ∎

Similarly, it is possible, although more painful, to derive a formula for $c_2(\mathrm{Gr}_{\mathbb{C}}^{m,n})$. The top Chern class, on the other hand, is

$$c_{mn}(\mathrm{Gr}_{\mathbb{C}}^{m,n}) = \prod_{i=1}^{m} \prod_{j=1}^{n} (\gamma_i + \delta_j) = \mathrm{Res}(t^m - \sigma_1 t^{m-1} + \cdots + (-1)^m \sigma_{1^m}, t^n + \sigma_1 t^{n-1} + \cdots + \sigma_n),$$

where Res denotes the resultant of the two polynomials with coefficients in $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n})$. It is harder to go any further and to calculate this resultant using only combinatorics and Schubert calculus and get the answer $\binom{m+n}{m}\sigma_{n^m}$ predicted by Chern-Gauss-Bonnet. In the next section, we carry out this computation for $m = 2$.

2.3.1 Grassmannians of Lines

In this section, we carry out the computation of the top Chern class of the Grassmannian of lines $\mathrm{Gr}_{\mathbb{C}}^{2,n}$. In this case, we let for simplicity $\alpha := \gamma_1$ and $\beta := \gamma_2$. Then, following the notation of the previous section, the top Chern class of $\mathrm{Gr}_{\mathbb{C}}^{2,n}$ is given by

$$\prod_{j=1}^{n}(\alpha+\delta_j)(\beta+\delta_j) = (\alpha^n+\sigma_1\alpha^{n-1}+\cdots+\sigma_n)(\beta^n+\sigma_1\beta^{n-1}+\cdots+\sigma_n),$$

where this computation is to be carried out in the ring

$$R^n := \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,n};\mathbb{Z})[\alpha,\beta]/(\alpha+\beta-\sigma_1, \alpha\beta-\sigma_{1,1}). \tag{2.29}$$

To aid in this computation, we can actually "let $n\to\infty$", i.e. work instead with the infinite Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{2,\infty}$ (see Appendix 3.7). Doing this allows us to use induction to do this computation, having done which, we may extract the desired quantity in the cohomology $\mathrm{H}^{4n}(\mathrm{Gr}_{\mathbb{C}}^{2,n};\mathbb{Z})$ of the finite Grassmannian as the term corresponding to $\sigma_{n,n}$. Therefore, the required verification boils down to

Theorem 2.3.14. In the ring

$$R := \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,\infty};\mathbb{Z})[\alpha,\beta]/(\alpha+\beta-\sigma_1, \alpha\beta-\sigma_{1,1}),$$

we have for each $n \geq 0$ the identity

$$(\alpha^n+\sigma_1\alpha^{n-1}+\cdots+\sigma_n)(\beta^n+\sigma_1\beta^{n-1}+\cdots+\sigma_n) = \sum_{j=0}^{n}\left[\binom{n+2}{2}-\binom{j+1}{2}\right]\sigma_{n+j,n-j}.$$

To prove this result, we first make the following combinatorial observation. In what follows, we adopt the convention that $\sigma_{i,j}=0$ if $0\leq i<j$.

Lemma 2.3.15. In the ring R, we have for each $k\geq 1$ that

$$\alpha^k+\beta^k = \sigma_k - \sigma_{k-1,1}.$$

Proof. We induct on k. The result is clear for $k=1$, and for $k=2$ we have

$$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \sigma_1^2-2\sigma_{1,1} = (\sigma_2+\sigma_{1,1})-2\sigma_{1,1} = \sigma_2-\sigma_{1,1}.$$

Suppose now that $k\geq 2$. Then we have

$$\begin{aligned}\alpha^{k+1}+\beta^{k+1} &= (\alpha+\beta)(\alpha^k+\beta^k)-\alpha\beta(\alpha^{k-1}+\beta^{k-1})\\ &= \sigma_1(\sigma_k-\sigma_{k-1,1})-\sigma_{1,1}(\sigma_{k-1}-\sigma_{k-2,1}).\end{aligned}$$

Using the product formula for the Grassmannian of lines, Proposition 3.7.5, we know that for all $a\geq b\geq 0$, we have

$$\sigma_1\sigma_{a,b} = \sigma_{a+1,b}+\sigma_{a,b+1} \text{ and } \sigma_{1,1}\sigma_{a,b} = \sigma_{a+1,b+1}.$$

Therefore, we conclude that

$$\alpha^{k+1}+\beta^{k+1} = (\sigma_{k+1}+\sigma_{k,1})-(\sigma_{k,1}+\sigma_{k-1,2})-\sigma_{k,1}+\sigma_{k-1,2} = \sigma_{k+1}-\sigma_{k,1}$$

as needed. ∎

Remark 2.3.16. This lemma also implies, by the way, a formula for the Chern character of $\mathcal{S}$ or $\mathcal{S}^\vee$ on the Grassmannian of lines. Namely, we have for any $n \geq 1$, we have in $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,n}; \mathbb{Z})$ that

$$\operatorname{ch} \mathcal{S}^\vee = 2 + \sum_{k=1}^{n} \frac{\sigma_k - \sigma_{k-1,1}}{k!} - \frac{\sigma_{n,1}}{(n+1)!}.$$

We can now proceed to the proof of Theorem 2.3.14.

Proof of Theorem 2.3.14. We proceed by induction, with the case $n = 0$ being clear. For any $n \geq 0$, let

$$\xi_n := (\alpha^n + \sigma_1 \alpha^{n-1} + \cdots + \sigma_n)(\beta^n + \sigma_1 \beta^{n-1} + \cdots + \sigma_n).$$

The key step in the induction is the observation that for any $n \geq 0$, we have the recursion

$$\xi_{n+1} = \alpha\beta \cdot \xi_n + \sigma_{n+1} \sum_{j=0}^{n} \sigma_j (\alpha^{n+1-j} + \beta^{n+1-j}) + \sigma_{n+1}^2.$$

Using that $\alpha\beta = \sigma_{1,1}$, and using the previous lemma, this can then be written as

$$\xi_{n+1} = \sigma_{1,1} \xi_n + \sigma_{n+1} \sum_{j=0}^{n} \sigma_j \left(\sigma_{n+1-j} - \sigma_{n-j,1}\right) + \sigma_{n+1}^2.$$

It is easy to see from Proposition 3.7.5 that

$$\sigma_j(\sigma_{n+1-j} - \sigma_{n-j,1}) = \begin{cases} \sigma_{n+1} - \sigma_{n-j,j+1}, & \text{if } 2j \leq n-1, \\ \sigma_{n+1}, & \text{if } 2j = n, \text{ and} \\ \sigma_{n+1} + \sigma_{j,n+1-j}, & \text{if } 2j \geq n+1. \end{cases}$$

From this calculation, it follows that

$$\sum_{j=0}^{n} \sigma_j (\sigma_{n+1-j} - \sigma_{n-j,1}) = (n+1)\sigma_{n+1}$$

and so

$$\xi_{n+1} = \sigma_{1,1} \xi_n + (n+2)\sigma_{n+1}^2.$$

But now, by induction,

$$\sigma_{1,1}\xi_n = \sum_{j=0}^{n} \left[\binom{n+2}{2} - \binom{j+1}{2} \right] \sigma_{n+1+j,n+1-j} = \sum_{j=0}^{n+1} \left[\binom{n+2}{2} - \binom{j+1}{2} \right] \sigma_{n+1+j,n+1-j}$$

and by Pieri's formula (Proposition 3.7.5) we have

$$\sigma_{n+1}^2 = \sum_{j=0}^{n+1} \sigma_{n+1+j,n+1-j}.$$

The result then follows from combining these identities and using the simple observation that

$$\binom{n+2}{2} + (n+2) = \binom{n+3}{2}.$$

■

Remark 2.3.17. In fact, the ring R^n (resp. R) can be identified itself as the cohomology ring a different space, namely $\mathrm{Fl}_{\mathbb{C}}^{1,1,n}$ (resp. the direct limit $\varinjlim_n \mathrm{Fl}_{\mathbb{C}}^{1,1,\infty}$). We will explain this for finite n; the result for $n = \infty$ simply follows by taking the direct limit over all n. Note that by eliminating one of α or β, we can write R^n as

$$R^n = \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,n}; \mathbb{Z})[\zeta]/(\zeta^2 - \sigma_1\zeta + \sigma_{1,1}),$$

and by the Leray-Hirsch Theorem for the cohomology of projective bundles (or by the definition of Chern classes–depending on your definition!), this is the cohomology ring of some projective bundle $\mathbb{P}E \to \mathrm{Gr}_{\mathbb{C}}^{2,n}$, where $E \to \mathrm{Gr}_{\mathbb{C}}^{2,n}$ is a complex vector bundle of rank 2 with Chern classes $c_1(E) = \sigma_1$ and $c_2(E) = \sigma_{1,1}$. Indeed, $\mathcal{S}$ is one such vector bundle, and we conclude that pulling back via the projection map

$$\pi : \mathrm{Fl}_{\mathbb{C}}^{1,1,n} = \mathbb{P}\mathcal{S} \to \mathrm{Gr}_{\mathbb{C}}^{2,n}$$

gives us an isomorphism of $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,n}; \mathbb{Z})$-algebras between R^n and $\mathrm{H}^*(\mathrm{Fl}_{\mathbb{C}}^{1,1,n}; \mathbb{Z})$, where $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{S}}(1)) = c_1(\mathcal{S}_1^\vee)$. In this algebra, we have the relation

$$\zeta^k = \sigma_{k-1}\zeta - \sigma_{k-1,1}$$

for any $k \geq 1$, proven easily via induction on k, which gives another way of seeing why

$$\alpha^k + \beta^k = \zeta^k + (\sigma_1 - \zeta)^k = \sigma_{k-1}\sigma_1 - 2\sigma_{k-1,1} = \sigma_k - \sigma_{k-1,1}.$$

More generally, for any complex vector bundle $E \to X$ of rank r on any space X, we may describe the cohomology of the complete relative flag bundle $\pi : \mathrm{Fl}^{1^r}E \to X$, given an $\mathrm{H}^*(X; \mathbb{Z})$-module via π^*, as

$$\mathrm{H}^*(\mathrm{Fl}^{1^r}E; \mathbb{Z}) = \mathrm{H}^*(X; \mathbb{Z})[\zeta_1, \ldots, \zeta_r]/(e_i(\zeta) - c_i(E))_{i=1}^r,$$

where $e_k(\zeta)$ is the k^{th} elementary symmetric polynomial in the ζ_j's. Here, $\zeta_j = c_1(\mathcal{S}_j)$ for $j = 1, \ldots, r$, where $\mathcal{S}_j$ as before denotes the j^{th} tautological subquotient bundle on $\mathrm{Fl}^{1^r}E$. For a proof of this result, see [10, Proposition 3.8.1]. We may now apply this formula to the tautological subbundle $\mathcal{S} \to \mathrm{Gr}_{\mathbb{C}}^{m,n}$ for arbitrary m, n, and noting that $\mathrm{Fl}^{1^m}\mathcal{S} = \mathrm{Fl}_{\mathbb{C}}^{1^m,n}$, we conclude that

$$\mathrm{H}^*(\mathrm{Fl}_{\mathbb{C}}^{1^m,n}; \mathbb{Z}) = \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z})[\gamma_1, \ldots, \gamma_m]/(e_i(\gamma) - \sigma_{1^i})_{i=1}^m,$$

where we have used γ_j to be consistent with the notation above. Then by the same argument as before, we can write the computation of the top Chern class of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ as a computation in this ring, namely that of

$$\prod_{i=1}^m (\gamma_i^n + \sigma_1\gamma_i^{n-1} + \cdots + \sigma_n) = \sum_{\lambda \subset m \times n} \sigma_{\hat{\lambda}} m_\lambda(\gamma) = \sum_{\lambda \subset m \times n} \sum_\mu \sigma_{\hat{\lambda}} b_\mu^\lambda e_\mu(\gamma),$$

where we have argued as in, and used the notation of, Appendix 3.3, and where for any λ we define $\sigma_\lambda := \prod_{i=1}^{\ell(\lambda)} \sigma_{\lambda_i}$. (Recall that $\hat{\lambda}$ is the complementary partition to λ.) Note that $e_\mu(\gamma) = \prod_{i=1}^{\ell(\mu)} \sigma_{1^{\mu_i}}$. But now it is not clear to me at all how we can massage this last expression for general m, n into looking like $\binom{m+n}{n}\sigma_{n^m}$, which is what we expect it to be.

2.4 Smooth Complete Intersections

As our final collection of examples, we treat smooth complete intersection varieties, following [13, Appendix One]. We start with some general remarks about smooth projective varieties, i.e. complex submanifolds of $\mathbb{CP}^N$. Recall that if $X \subset \mathbb{CP}^N$ is any smooth complex submanifold of dimension say n with $1 \leq n \leq N$, then we denote for each $k \in \mathbb{Z}$ by $\mathcal{O}_X(k)$ the restriction of $\mathcal{O}_{\mathbb{CP}^N}(k)$ to X (i.e. the pullback under the inclusion map). For $k \geq 1$, the bundle $\mathcal{O}_X(k)$ is again a positive line bundle; indeed, naturality of the Chern class implies that we have for each $k \in \mathbb{Z}$ that

$$c(\mathcal{O}_X(k)) = 1 + k\zeta_X,$$

where $\zeta_X \in \mathrm{H}^2(X;\mathbb{Z})$ is the hyperplane class in X, i.e. the restriction of the hyperplane class $\zeta \in \mathrm{H}^2(\mathbb{CP}^N;\mathbb{Z})$ to X. The Hard Lefschetz Theorem says that for each i with $0 \leq i \leq n$, multiplication by ζ_X^i induces an isomorphism $\mathrm{H}^{n-i}(X;\mathbb{Q}) \to \mathrm{H}^{n+i}(X;\mathbb{Q})$, making Poincaré duality for X explicit. With integer coefficients, this map is not quite an isomorphism; we have for instance that

$$\zeta_X^n = (\deg X)\eta_X,$$

where $\eta_X \in \mathrm{H}^{2n}(X;\mathbb{Z})$ is the Poincaré dual to a point (i.e. algebraic dual to the fundamental class of X) and $\deg X$ is the degree of X. Indeed, this can be taken to be a definition of the degree of a smooth projective variety.

The simplest class of such varieties is the class of smooth complete intersections. Given an integer $r \geq 0$ and a tuple of positive integers $d = (d_1, \ldots, d_r)$ of length r, let $X_d^n \subset \mathbb{CP}^{n+r}$ be a complete intersection of type d, i.e. a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_r$. This has degree $\deg X_d^n = |d| := \prod_{i=1}^r d_i$. In what follows, we let $e_j(d)$ be the elementary symmetric polynomial in $d_1, \ldots, d_r$ of degree j for each $j \geq 1$, so we have that $|d| = e_r(d)$. The principal idea employed in this section is that the numerical invariants of X_d^n such as $\chi(X_d^n), \mathrm{Sign}(X_d^n), \chi(X_d^n, \mathcal{O}_X)$, $\hat{A}(X_d^n)$, or any other invariants computed purely using characteristic classes, do not depend on the specific isomorphism type of X_d^n, but are functions of n and d (or more precisely $e_j(d)$).

The key to these computations is the fact that we have a much better understanding of the cohomology groups $\mathrm{H}^*(X_d^n;\mathbb{Z})$ and the tangent bundle $\mathscr{T}X_d^n$ of X_d^n than we do for general X. This is what we explain now.

- Since the line bundles $\mathcal{O}_X(k)$ for $k \geq 1$ are all positive, an inductive application of the Weak Lefschetz Theorem tells us that the restriction (resp. inclusion) map

  $$\mathrm{H}^i(\mathbb{CP}^{n+r};\mathbb{Z}) \to \mathrm{H}^i(X_d^n;\mathbb{Z}) \text{ (resp. } \mathrm{H}_i(X_d^n;\mathbb{Z}) \to \mathrm{H}_i(\mathbb{CP}^{n+r};\mathbb{Z}))$$

  is an isomorphism for all $i \leq n-1$ and injective for $i = n$. Combined with Poincaré duality, this gives us all the cohomology groups $\mathrm{H}^i(X_d^n;\mathbb{Z})$ for $0 \leq i \leq 2n$, except possibly for the middle one, i.e. where $i = n$. However, we can completely determine this as well. Indeed, the Universal Coefficient Theorem tells us that we have a short exact sequence

  $$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_{n-1}(X_d^n;\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^n(X_d^n;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X_d^n;\mathbb{Z}),\mathbb{Z}) \to 0.$$

  Since $\mathrm{H}_{n-1}(X_d^n;\mathbb{Z}) = 0$, it follows that the natural map

  $$\mathrm{H}^n(X_d^n;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(X_d^n;\mathbb{Z}),\mathbb{Z})$$

is an isomorphism; in particular, $\mathrm{H}^n(X_d^n;\mathbb{Z})$ is torsion-free, and hence, since it is of finite rank, it is free abelian of rank $b_n(X_d^n)$. This determines the cohomology groups completely, namely

$$\mathrm{H}^i(X_d^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } 0 \leq i \leq 2n, \text{with even } i \neq n, \\ \mathbb{Z}^{\oplus b_n(X_d^n)}, & \text{if } i = n, \text{ and} \\ 0 & \text{else.} \end{cases}$$

In particular, the Euler characteristic of X_d^n is given by

$$\chi(X_d^n) = \begin{cases} n + b_n(X_d^n) & \text{for } n \text{ even, and} \\ n + 1 - b_n(X_d^n) & \text{for } n \text{ odd.} \end{cases} \tag{2.30}$$

Therefore, the calculation of the middle Betti number $b_n(X_d^n)$ of X_d^n is equivalent to the calculation of this Euler characteristic. This can be achieved using the characteristic class computations which we carry out below; see Corollary 2.4.4.

- Recall that if Y is a compact complex manifold and $X \subset Y$ a smooth complex hypersurface, then the normal bundle $\mathcal{N}_{X/Y}$ is simply the restriction of the bundle $\mathcal{O}_Y(X)$ to X, what is often denoted by $\mathcal{O}_X(X)$; this comes from the identification of the conormal bundle $\mathcal{N}_{X/Y}^\vee$ as $\mathcal{I}_X/\mathcal{I}_{X^2}$, where $\mathcal{I}_X$ is the ideal sheaf of X in Y. The upshot of this is that repeated application of this formula then tells us that the normal bundle $\mathcal{N}_{X_d^n/\mathbb{CP}^{n+r}}$ of X_d^n can be identified with $\bigoplus_{i=1}^r \mathcal{O}_{X_d^n}(d_i)$[22], i.e. there is a short exact sequence of vector bundles on X_d^n given by

$$0 \to \mathscr{T} X_d^n \to \mathscr{T}\mathbb{CP}^{n+r}|_{X_d^n} \to \bigoplus_{i=1}^r \mathcal{O}_{X_d^n}(d_i) \to 0. \tag{2.31}$$

Combining this with the Euler sequence (2.17) for $\mathbb{CP}^{n+r}$ restricted to X_d^n, namely

$$0 \to \mathcal{O}_{X_d^n} \to \mathcal{O}_{X_d^n}(1)^{\oplus(n+r+1)} \to \mathscr{T}\mathbb{CP}^{n+r}|_{X_d^n} \to 0, \tag{2.32}$$

allows us to easily carry out the characteristic class computations.

For instance, we have the two following results:

Lemma 2.4.1. Let X_d^n be a complete intersection of dimension n and type d. Then the canonical bundle of X_d^n is given by

$$\omega_{X_d^n} = \mathcal{O}_{X_d^n}\left(e_1(d) - n - r - 1\right).$$

Proof. From the multiplicativity of the determinant in short exact sequences, we conclude from (2.31) that the anticanonical bundles of X_d^n and $\mathbb{CP}^{n+r}$ are related via

$$\omega_{\mathbb{CP}^{n+r}}^{-1}|_{X_d^n} = \omega_{X_d^n}^{-1} \cdot \mathcal{O}_{X_d^n}(e_1(d)).$$

From this, we can finish using that $\omega_{\mathbb{CP}^N}^{-1} = \mathcal{O}_{\mathbb{CP}^N}(N+1)$ for any $N \geq 0$. ■

[22] More precisely, this only tells us that the normal bundle admits a filtration with successive quotients the $\mathcal{O}_{X_d^n}(d_i)$. But, in fact, this normal bundle splits as a direct sum of these line bundles. Of course, in the topological category, every exact sequence is split; since we are only dealing with characteristic class computations, this will not concern us.

Lemma 2.4.2. The complete intersection variety $X = X_d^n$ has first Chern class

$$c_1(X) = (n + r + 1 - e_1(d))\zeta_X,$$

where, as above, ζ_X is the hyperplane class on X. In particular, X is always spin when $n = 1$ and for $n \geq 2$, it is spin iff

$$n + r + 1 \equiv e_1(d) \pmod 2.$$

Proof. The sequences (2.31) and (2.32) imply that the total Chern class of X is

$$\mathsf{c}(X) = (1 + \zeta_X)^{n+r+1} \prod_{i=1}^{r} (1 + d_i\zeta_X)^{-1}, \tag{2.33}$$

giving us $c_1(X)$. Note that for $n = 1$, we have $\zeta_X = |d|\eta_X$, and hence

$$c_1(X_d^1) = |d|(r + 2 - e_1(d))\eta_X,$$

the coefficient of which is always even (if $|d|$ is odd, then $e_1(d) \equiv r \pmod 2$), as it must be, being the Euler characteristic of a Riemann surface; Riemann surfaces are always spin (Corollary 1.5.4). For $n \geq 2$, the Weak Lefschetz Theorem tells us that ζ_X generates $\mathrm{H}^2(X; \mathbb{Z})$, so that Corollary 1.5.3 tells us that for $n \geq 2$, the manifold X is spin iff this coefficient of ζ_X in $c_1(X)$ is even, which is equivalent to the above condition. ■

The first important computation here is the content of

Theorem 2.4.3. For any d as above and $k \in \mathbb{Z}$, the generating function of the series of χ_y-characteristics of the line bundles $\mathcal{O}_{X_d^n}(k)$ on X_d^n for $n \geq 0$ is given by

$$\sum_{n=0}^{\infty} \chi_y\left(X_d^n, \mathcal{O}_{X_d^n}(k)\right) t^{n+r} = \frac{(1+yt)^{k-1}}{(1-t)^{k+1}} \prod_{i=1}^{r} \frac{(1+yt)^{d_i} - (1-t)^{d_i}}{(1+yt)^{d_i} + y(1-t)^{d_i}}.$$

Proof. Given any k, n and d, he sequences (2.31) and (2.32) allow us to express the total generalized Todd class of $X = X_d^n$ as

$$\mathsf{Td}_y(X) = Q_y(\zeta_X)^{n+r+1} \prod_{i=1}^{r} Q_y(d_i\zeta_X)^{-1} \in \mathrm{H}^*(X; \mathbb{Q}[y]),$$

where as before

$$Q_y(z) = \frac{z}{R(z)}, \text{ with } R(z) := \frac{1 - \mathrm{e}^{-z(1+y)}}{1 + y\mathrm{e}^{-z(1+y)}}.$$

Therefore, the Generalized Hirzebruch-Riemann-Roch Theorem 1.4.3 tells us that we have

$$\chi_y(X, \mathcal{O}_X(k)) = \int_X \mathrm{ch}_y\, \mathcal{O}_X(k) \cdot \mathsf{Td}_y(X) = \int_X \mathrm{e}^{k\zeta_X(1+y)} \cdot \frac{\zeta_X^{n+r+1}}{R(\zeta_X)^{n+r+1}} \prod_{i=1}^{r} \frac{R(d_i\zeta_X)}{d_i\zeta_X}.$$

Since $\int_X \zeta_X^j$ is $|d| = \prod_{i=1}^r d_i$ for $j = n$ and zero otherwise, this integral can be obtained as the coefficient of ζ^n in the power series $\mathrm{e}^{k\zeta(1+y)}\zeta^{n+1}R(\zeta)^{-n-r-1}\prod_{i=1}^r R(d_i\zeta)$, which can in turn be expressed as the residue calculation

$$[\zeta^n]\mathrm{e}^{k\zeta(1+y)}\frac{\zeta^{n+1}}{R(\zeta)^{n+r+1}}\prod_{i=1}^r R(d_i\zeta) = \operatorname*{Res}_{\zeta=0}\frac{\mathrm{e}^{k\zeta(1+y)}}{R(\zeta)^{n+r+1}}\prod_{i=1}^r R(d_i\zeta)\,\mathrm{d}\zeta.$$

To compute this residue, we make the same substitution as in the proof of Theorem 2.2.5, namely $t = R(\zeta)$, and note that this implies

$$R(d_i\zeta) = \frac{(1+yt)^{d_i} - (1-t)^{d_i}}{(1+yt)^{d_i} + y(1-t)^{d_i}}.$$

It follows then that this residue is given by

$$\begin{aligned}&\operatorname*{Res}_{t=0}\left(\frac{1+yt}{1-t}\right)^k\cdot\frac{1}{t^{n+r+1}}\cdot\prod_{i=1}^r\frac{(1+yt)^{d_i}-(1-t)^{d_i}}{(1+yt)^{d_i}+y(1-t)^{d_i}}\cdot\frac{1}{(1+yt)(1-t)}\mathrm{d}t\\ =&[t^{n+r}]\frac{(1+yt)^{k-1}}{(1-t)^{k+1}}\prod_{i=1}^r\frac{(1+yt)^{d_i}-(1-t)^{d_i}}{(1+yt)^{d_i}+y(1-t)^{d_i}},\end{aligned}$$

which is equivalent to the claimed result. ∎

As previously, we obtain from this computation a delicious sequence of corollaries.

Corollary 2.4.4. For any d as above, we have

$$\sum_{n=0}^\infty \chi(X_d^n)t^n = \frac{|d|}{(1-t)^2}\prod_{i=1}^r\frac{1}{1-(1-d_i)t}.$$

In other words, we have for any $n \geq 0$ and d that

$$\chi(X_d^n) = |d|\sum_{i=0}^n (n-i+1)h_i(1-d) = |d|\sum_{i=0}^n(-1)^i\binom{n+r+1}{n-i}h_i(d),$$

where $h_i(1-d)$ (resp. $h_i(d)$) denotes the i^{th} complete symmetric polynomial in $1-d_1,\dots,1-d_r$ (resp. $d_1,\dots,d_r$).

Proof. Plug in $k = 0$ and then take the limit as $y \to -1$ in Theorem 2.4.3, noting that

$$\lim_{y\to -1}\frac{(1+yt)^{d_i}-(1-t)^{d_i}}{(1+yt)^{d_i}+y(1-t)^{d_i}} = \frac{d_i t}{1-(1-d_i)t}.$$

The first explicit formula then follows from $(1-t)^{-2} = \sum_{i=0}^\infty (i+1)t^i$ and the fact that for any variables $x = (x_1,\dots,x_r)$, the generating function for the complete symmetric polynomials $h_j(x)$ in x is

$$\sum_{j=0}^\infty h_j(x)t^j = \prod_{i=1}^r \frac{1}{1-x_i t}.$$

The second explicit formula can then be obtained by relating $h_i(1-d)$ and $h_i(d)$, but we can also obtain it as an immediate consequence of (2.33). ∎

Remark 2.4.5. The computation of this Euler characteristic along with (2.30) finishes the computation of $b_n(X_d^n)$ and hence of the cohomology groups (or equivalently Betti numbers, by the freeness of cohomology) of X_d^n .

Corollary 2.4.6. For any d as above, we have

$$\sum_{n=0}^{\infty} \mathrm{Sign}(X_d^n) t^{n+r} = \frac{1}{1-t^2} \prod_{i=1}^{r} \frac{(1+t)^{d_i} - (1-t)^{d_i}}{(1+t)^{d_i} + (1-t)^{d_i}}.$$

Proof. Plug in $k = 0$ and $y = 1$ in Theorem 2.4.3. ■

Corollary 2.4.7. For any d as above and $k \in \mathbb{Z}$, we have

$$\sum_{n=0}^{\infty} \chi\left(X_d^n, \mathcal{O}_{X_d^n}(k)\right) t^n = \frac{1}{(1-t)^{k+1}} \prod_{i=1}^{r} \left[\frac{1-(1-t)^{d_i}}{t}\right].$$

Proof. Plug in $y = 0$ in Theorem 2.4.3. ■

Finally, we compute that $\hat{A}$-genera of the spaces X_d^n.

Theorem 2.4.8. For any d as above, the generating function of the $\hat{A}$-genus of X_d^n for $n \geq 0$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \hat{A}\left(X_d^n\right) t^n &= \left(1+\frac{t^2}{4}\right)^{-1/2} \prod_{i=1}^{r} \frac{\sinh\left(d_i \sinh^{-1}(t/2)\right)}{t/2} \\
&= \left(1+\frac{t^2}{4}\right)^{-1/2} \prod_{i=1}^{r} \frac{1}{t}\left[\left(\sqrt{1+\frac{t^2}{4}}+\frac{t}{2}\right)^{d_i} - \left(\sqrt{1+\frac{t^2}{4}}-\frac{t}{2}\right)^{d_i}\right].
\end{aligned}$$

Proof. Applying the same technique as in the proof of Theorem 2.4.3, this time to the characteristic series $Q_{\hat{A}}(z)$, we conclude that

$$\begin{aligned}
\hat{A}(X_d^n) &= \int_X \left[\frac{\zeta_X/2}{\sinh(\zeta_X/2)}\right]^{n+r+1} \prod_{i=1}^{r} \frac{\sinh(d_i\zeta_X/2)}{d_i\zeta_X/2} \\
&= \operatorname*{Res}_{\zeta=0} \frac{1}{[2\sinh(\zeta/2)]^{n+r+1}} \prod_{i=1}^{r} 2\sinh\left(\frac{d_i\zeta}{2}\right) \mathrm{d}\zeta,
\end{aligned}$$

where again we have used that $\int_X \zeta_X^j$ is $|d| = \prod_{i=1}^r d_i$ for $j = n$ and zero otherwise. This time, we use the change of variables $t = 2\sinh(\zeta/2)$ to get

$$\hat{A}(X_d^n) = \operatorname*{Res}_{t=0} \frac{1}{t^{n+r+1}} \prod_{i=1}^{r} \left[2\sinh\left(d_i \sinh^{-1}\left(\frac{t}{2}\right)\right)\right] \cdot \left(1+\frac{t^2}{4}\right)^{-1/2} \mathrm{d}t,$$

which is equivalent to the first result. The second formula can then be obtained by using the identities $\sinh^{-1}(x) = \ln\left(\sqrt{1+x^2}+x\right)$ and $(\sqrt{1+x^2}+x)(\sqrt{1+x^2}-x) = 1$. ■

Remark 2.4.9. Plugging in $r = 0$ (so $d = \emptyset$) in the above results recovers the results of §2.2.1. Note also how, in all the above expressions, adding additional 1's at the end of the d, i.e. replacing $d = (d_1, \dots, d_r)$ by $d = (d_1, \dots, d_r, 1, 1, \dots, 1)$ for any number of 1's, does not change the value of the invariants $\chi(X_d^n), \mathrm{Sign}(X_d^n), \chi(X_d^n, \mathcal{O}_{X_d^n}(k))$ and $\hat{A}(X_d^n)$. It is clear why this must be the case; indeed, intersecting with additional hyperplanes amounts to just working in a smaller dimensional projective space.

Although these closed-form expressions for the generating functions are quite neat, extracting an individual coefficient seems quite untractable in general. However, for small n and r, this is not too difficult, and we do this now for $n \leq 2$ and arbitrary r (complete intersection curves and surfaces) here, and $r = 1$ and arbitrary n (hypersurfaces) in the next subsection, §2.4.1.

Corollary 2.4.10. For a given d, consider the complete intersection curve X_d^1. Then:

(a) For each $k \in \mathbb{Z}$, we have

$$\chi_y(X_d^1, \mathcal{O}_{X_d^1}(k)) = |d| \left(\frac{r + 2 - e_1(d)}{2}(1 - y) + k(1 + y) \right),$$

where $e_1(d) = \sum_{i=1}^r d_i$.

(b) The Euler characteristic of X_d^1 is

$$\chi(X_d^1) = |d| \left(r + 2 - e_1(d)\right).$$

Therefore, the genus of X_d^1 is

$$g(X_d^1) = 1 + \frac{1}{2}|d| \left(e_1(d) - r - 2\right).$$

Proof. For (a), we divide both sides of the result in Theorem 2.4.3 by t^r and extract the coefficient of t. For (b), this follows from Lemma 2.4.1 and noting that for a Riemann surface X of genus g, we have $\deg \omega_X = 2g - 2$; we can also specalize the result in (a) to $k = 0$ and $y = -1$, or note using Corollary 2.4.4 that

$$\chi(X_d^1) = |d| \left[h_1(1 - d) + 2h_0(1 - d)\right] = |d| \left[r - e_1(d) + 2\right].$$

■

Remark 2.4.11. Note that this formula tells us that the genus increases with the d_i, whereas (as we observed above) stays put when extending the tuple d by 1's. Therefore, we may easily enumerate all possible numbers that occur as genera of complete intersection curves as

$$0, 1, 3, 4, 5, 6, 9, 10, 13 \dots.$$

In particular, $g = 2$ is not possible–indeed, any curve of genus 2 is hyperelliptic (see [17, Prop. VII.1.10], and hyperelliptic curves of genus $g \geq 2$ cannot be complete intersections, since the canonical sheaf of a complete intersection curve is very ample. Indeed, for X_d^1 of genus $g \geq 2$, Lemma 2.4.1 gives us $\omega_{X_d^1} = \mathcal{O}_{X_d^1}(e_1(d) - r - 2)$ with $e_1(d) - r - 2 = 2g - 2 \geq 1$, so that since already $\mathcal{O}_{X_d^1}$ is very ample, so is this multiple $\omega_{X_d^1}$. On the other hand, the canonical sheaf of hyperelliptic curve is not very ample ([17, Prop. VII.2.2]).

Similarly, we have:

Corollary 2.4.12. For a given d, consider the complete intersection surface X_d^2. Then:

(a) The Euler characteristic of X_d^2 is

$$\chi(X_d^2) = |d| \left[\binom{r+3}{2} - (r+3)e_1(d) + (e_1(d)^2 - e_2(d)) \right].$$

(b) The signature of X_d^2 is

$$\mathrm{Sign}(X_d^2) = \frac{|d|}{3} \left(r + 3 - e_1(d)^2 + 2e_2(d) \right).$$

(c) The Todd genus of X_d^2 is

$$\chi(X_d^2, \mathcal{O}_{X_d^2}) = \frac{|d|}{12} \left[2\left(e_1(d)^2 + e_2(d)\right) - 6(r+1)e_1(d) + (r^2 + 9r + 12) \right].$$

(d) The $\hat{A}$ genus of X_d^2 is

$$\hat{A}(X_d^2) = -\frac{|d|}{24} \left(r + 3 - e_1(d)^2 + 2e_2(d) \right).$$

Proof. This amounts to extracting the coefficient of t^2 (or t^{r+2} as the case may be) from the above formulae, and is a straightforward computation. ∎

Remark 2.4.13. This last computation nontrivially verifies Corollary 1.5.6, which says in our case that, when X_d^2 is spin, $\hat{A}(X_d^2)$ is an even integer. To show this from the formula above, note first that for any d we have $\hat{A}(X_d^2) \in \frac{1}{8}\mathbb{Z}$ (this was also observed in the proof of Corollary 1.5.7). Indeed, if $3 \nmid |d|$, then $d_i^2 \equiv 1 \pmod 3$ for $i = 1, \ldots, r$, so

$$r + 3 - e_1(d)^2 + 2e_2(d) = r + 3 - \sum_{i=1}^{r} d_i^2 \equiv 0 \pmod 3.$$

Therefore, it suffices to consider powers of 2. By Lemma 2.4.2, X_d^2 is spin iff

$$r + 1 \equiv e_1(d) \pmod 2,$$

which happens iff there is an odd number, say k, of i's such that d_i is even. It suffices to show that in this case, $|d|(r + 3 - e_1(d)^2 + 2e_2(d))$ is divisible by 16. Since $|d|$ is divisible by 2^k, the only cases of interest are $k \in \{1, 3\}$. Relabel the d_i's if needed to assume that $d_1, \ldots, d_k$ are even and that $d_{k+1}, \ldots, d_r$ are odd. When $k = 3$, we have

$$r + 3 - e_1(d)^2 + 2e_2(d) = r + 3 - \sum_{i=1}^{3} 4\left(\frac{d_i}{2}\right)^2 - \sum_{i=4}^{r} d_i^2 \equiv r + 3 - (r-3) \equiv 2 \pmod 4,$$

and we are done. When $k = 1$, so d_1 is even, we have

$$r + 3 - \sum_{i=1}^{r} d_i^2 \equiv r + 3 - 4\left(\frac{d_1}{2}\right)^2 - (r-1) \equiv 4\left(1 - \left(\frac{d_1}{2}\right)^2\right) \pmod 8.$$

Therefore, if d_1 is divisible by 4, then $|d|$ contributes the required additional power of 2, whereas if $d_1 \equiv 2 \pmod 4$, this power comes from the second factor.

2.4.1 Hypersurfaces

An interesting special case of the computations in the previous section is when $r = 1$, so that d is simply a positive integer and $X_d^n \subset \mathbb{CP}^{n+1}$ is a smooth hypersurface of degree d and dimension n. For these hypersurfaces, we can obtain surprisingly explicit formulae for the above numerical invariants. Firstly, from Corollary 2.4.4, we immediately obtain

Corollary 2.4.14. For integers $n \geq 0$ and $d \geq 1$, the Euler characteristic of X_d^n is given by

$$\chi(X_d^n) = \sum_{j=0}^{n} (-1)^j \binom{n+2}{j+2} d^{j+1} = \frac{1}{d}\left((1-d)^{n+2} + (n+2)d - 1\right).$$

Similarly, Corollary 2.4.6 gives us

Corollary 2.4.15. For a given integer $d \geq 1$, the generating function of the signatures of X_d^n for $n \geq 0$ is given by

$$\sum_{n=0}^{\infty} \mathrm{Sign}(X_d^n) t^n = \frac{1}{t(1-t^2)} \cdot \frac{(1+t)^d - (1-t)^d}{(1+t)^d + (1-t)^d}.$$

It is harder to extract an explicit formula for $\mathrm{Sign}(X_d^n)$ from this; however, not all is lost. Note that $\mathrm{Sign}(X_d^n)$ is zero for odd n, and for even $n = 2k$, is expressible as an odd polynomial of degree $2k+1$ in d. Since this polynomial takes integer values at integer d, and it is expressible as an integral linear combination of the binomial coefficients $\binom{d}{j}$ for $1 \leq j \leq 2k+1$. For $0 \leq k \leq 3$ these polynomials are

$$\begin{aligned}
\mathrm{Sign}(X_d^0) &= d \\
\mathrm{Sign}(X_d^2) &= d - 2\binom{d}{2} - 2\binom{d}{3} \\
\mathrm{Sign}(X_d^4) &= d + 16\binom{d}{3} + 32\binom{d}{4} + 16\binom{d}{5}, \text{ and} \\
\mathrm{Sign}(X_d^6) &= d - 2\binom{d}{2} - 50\binom{d}{3} - 368\binom{d}{4} - 864\binom{d}{5} - 816\binom{d}{6} - 272\binom{d}{7}.
\end{aligned}$$

Next, from Corollary 2.4.7, we obtain

Corollary 2.4.16. For integers n, d, k with $n \geq 0$ and $d \geq 1$, the holomorphic Euler characteristic of the bundle $\mathcal{O}_{X_d^n}(k)$ on X_d^n is

$$\chi(X_d^n, \mathcal{O}_{X_d^n}(k)) = \sum_{j=0}^{n} (-1)^j \binom{n-j+k}{n-j} \binom{d}{j+1}.$$

Of course, this expression is the Hilbert polynomial of X_d^n in k; as a sanity check, this clearly has degree n and leading coefficient $d/n!$ as a polynomial in k.

Finally, from Theorem 2.4.8, we can compute the $\hat{A}$-genera of X_d^n as in

Corollary 2.4.17. For any $k \geq 0$, we have $\hat{A}(X_d^{2k+1}) = 0$, whereas

$$\hat{A}(X_d^{2k}) = \frac{1}{2^{2k}(2k+1)!} \prod_{j=-k}^{k} (d-2j) = 2\binom{(d/2)+k}{2k+1}.$$

Proof. From Theorem 2.4.8, we conclude that

$$\sum_{n=0}^{\infty} \hat{A}(X_d^n) t^n = \frac{1}{t}\left(1+\frac{t^2}{4}\right)^{-1/2} \left[\left(\sqrt{1+\frac{t^2}{4}}+\frac{t}{2}\right)^d - \left(\sqrt{1+\frac{t^2}{4}}-\frac{t}{2}\right)^d\right] \tag{2.34}$$

Using that d is a positive integer, we may use the Binomial Theorem to expand this as

$$\sum_{i=0}^{\infty} \binom{d}{2i+1}\left(1+\frac{t^2}{4}\right)^{(d/2)-i-1}\left(\frac{t}{2}\right)^{2i} = \sum_{i,j=0}^{\infty} \binom{d}{2i+1}\binom{(d/2)-i-1}{j}\left(\frac{t}{2}\right)^{2(i+j)},$$

from which we conclude that

$$\hat{A}(X_d^{2k}) = \frac{1}{2^{2k}} \sum_{i=0}^{k} \binom{d}{2i+1}\binom{(d/2)-i-1}{k-i}.$$

To bring this into the desired form, it suffices to show the corresponding identity of polynomials. Observe that for $0 \leq i \leq k$, the i^{th} term in the sum is a polynomial of degree $k+i+1$ in d, so the sum is a polynomial of degree $2k+1$ in d, with leading coefficient the same as that of $2^{-2k}\binom{d}{2k+1}$, namely

$$\frac{1}{2^{2k}(2k+1)!}.$$

Since it is clear from (2.34) that $\hat{A}(X_d^{2k})$ is an odd polynomial in d, to show the result, it suffices to show that $\hat{A}(X_d^{2k}) = 0$ whenever $d = 2j$ for some $1 \leq j \leq k$. This follows from the fact that when $1 \leq j \leq k$ and $0 \leq i \leq k$, the product

$$\binom{2j}{2i+1}\binom{j-i-1}{k-i}$$

is always zero, since the first factor is zero for $i \geq j$, whereas the second factor is zero for $i < j$, which is equivalent to $0 \leq j-i-1 < k-i$. ∎

Remark 2.4.18. Again, Corollary 2.4.17 verifies Corollary 1.5.6. By Lemma 2.4.2, the hypersurface X_d^n for $n \geq 2$ is spin iff $n \equiv d \pmod 2$. In particular, when $n = 2k$ for $k \geq 1$, this tells us that X_d^{2k} is spin iff d is even, in which case $\hat{A}(X_d^{2k}) = 2\binom{(d/2)+k}{2k+1}$ is clearly an even integer for all k (and not just odd k, as guaranteed by Corollary 1.5.6). The evenness of $\hat{A}(X_d^{2k})$ for even k seems to come from some additional symmetry of these hypersurfaces, which is not shared by all manifolds[23]; an independent proof of this result would be illuminating. In contrast, when d is odd, it is clear again from the first formula in Corollary 2.4.17 that $\hat{A}(X_d^{2k})$ is not an integer, giving us another huge class of (necessarily non-spin) manifolds with non-integral $\hat{A}$-genus.

[23]For instance, there is a smooth spin manifold of dimension 8, which can be constructed by plumbing, which has $\hat{A}$-genus 1; see [53]

Remark 2.4.19. We can also show that $\hat{A}(X_{2j}^{2k}) = 0$ for $1 \leq j \leq k$ using the Calabi-Yau Theorem combined with Theorem 1.5.9. Indeed, as observed in (2.33) above, the first Chern class of $X = X_{2j}^{2k}$ is

$$c_1(X) = 2(k+1-j)\zeta_X.$$

In particular, for $1 \leq j \leq k$, the first Chern class of X can be represented by the positive form $2(k+1-j)\omega_{\mathrm{FS}}|_X$, where ω_{FS} is the Fubini-Study form on the ambient $\mathbb{CP}^{2k+1}$. It follows from the Calabi-Yau Theorem that X admits a Kähler metric with Ricci form given by $2(k+1-j)\omega_{\mathrm{FS}}|_X$, and then the underlying Riemannian metric must consequently have positive Ricci, and hence scalar, curvature. It follows then from Theorem 1.5.9 that $\hat{A}(X_{2j}^{2k}) = 0$.

The next case, namely $j = k+1$, is then the critical Calabi-Yau case, i.e. the case of vanishing Ricci curvature. In this case we have $\hat{A}(X_{2k+2}^{2k}) = 2$. The nonvanishing of the $\hat{A}$-genus in this case then implies the existence of harmonic spinors, and hence globally parallel spinor fields, since the scalar curvature κ is 0 (see [2, Cor. II.8.10]). This then reduces the holonomy group of the Calabi-Yau manifolds X_{2k+2}^{2k}; see [2, §IV.10].

Example 2.4.20. The simplest case of the above Calabi-Yau case is when $k = 1$; then $X_4^2 \subset \mathbb{CP}^3$ is a smooth quartic surface, which is an example of K3 surface. Corollary 2.4.14 tells us that $\chi(X_4^2) = 24$ so that $b_2(X_4^2) = 22$, and we can determine the cohomology groups of X_4^2 as

$$\mathrm{H}^*(X_4^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0, 4, \\ \mathbb{Z}^{22}, & * = 2, \text{ and} \\ 0, & \text{else.} \end{cases}$$

Now Corollary 2.4.15 tells us that the signature of X_4^2 is $\mathrm{Sign}(X_4^2) = -16$. One can then use both of these facts–true of any K3 surface–along with unimodularity of this pairing and the observation by Wu that the intersection pairing of any spin complex manifold is even, to pin down the intersection matrix of X_4^2 uniquely as

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic plane given by

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and $E_8(-1)$ is the E_8 lattice mentioned in Remark 1.5.8 twisted by -1. This observation is the starting point for the moduli theory of complex K3 surfaces; see [54, Ch. 1].

Chapter 3

Appendices

In these appendices, we collect the details of several technical tools used in the above exposition, which we find to be worth reviewing in more detail than is achieved in the section on Conventions and Fundamentals.

3.1 Differential Operators

One important character in the story narrated in this thesis is the notion of a differential operator between vector bundles and its symbol. There are many settings and many equivalent ways to define this notion. Some of these, ordered roughly in increasing order of abstractness and generality, include

(I) using local coordinates,
(II) using jet bundles,
(III) proceeding inductively and using the Leibniz rule,
(IV) using the "universal" sheaf of relative differential operators.

Here we briefly present all four. In what follows, we will use the letters E, F to denote total spaces of vector bundles, and $\mathcal{E}, \mathcal{F}$ for their locally free sheaves of sections; we will then use the notation $\mathcal{E}_x$ to denote the stalk of the sheaf $\mathcal{E}$ at a point x in the base, and $E(x)$ or $\mathcal{E}(x)$ to denote the fiber of E over x; see Appendix 3.4 for a discussion of this relationship. We will also use $\Gamma(X, E)$ or $\Gamma(X, \mathcal{E})$ to denote the vector space of smooth sections on X, and use the Einstein summation convention as needed.

Definition (I), using local coordinates, is perhaps the easiest to explain.

Definition 3.1.1 (I). Let X be a smooth manifold, $E, F \to X$ be smooth complex vector bundles, and $k \geq 0$ be an integer. Then a differential operator $D : E \to F$ of order at most k is a complex linear map $D : \Gamma(X, E) \to \Gamma(X, F)$ that has the following local form. Around each point $x \in X$, pick coordinates $x = (x^1, \ldots, x^n)$ on X and local frames $\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_r)$ of E and $\mathrm{f} = (\mathrm{f}_1, \ldots, \mathrm{f}_s)$ of F. If $u \in \Gamma(X, E)$, then in these coordinates, u can be written as $u = u^\mu \mathrm{e}_\mu$ for some local smooth functions u^μ. In this case, $Du \in \Gamma(X, F)$ is required to locally have the form

$$Du = a^\nu_{\lambda\mu}(\partial^\lambda u^\mu)\mathrm{f}_\nu,$$

where $a^\nu_{\lambda\mu}$ are local complex-valued smooth functions on X that depend only on x, e, and f, but not u, and where $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a sequence of nonnegative integers with $|\lambda| := \lambda_1 + \cdots + \lambda_r \leq k$ and

$$\partial^\lambda := (-\mathrm{i})^{|\lambda|} \left(\frac{\partial}{\partial x^1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x^r}\right)^{\lambda_r}.$$

In this case, we also write

$$D = a^\nu_{\lambda\mu}(\partial^\lambda \mathrm{e}^\mu \cdot) f_\nu.$$

Remark 3.1.2. A similar definition to the one above can also be made for smooth real vector bundles on smooth manifolds (without the $(-\mathrm{i})^{|\lambda|}$), as well as holomorphic vector bundles on complex manifolds. The factor of $(-\mathrm{i})^{|\lambda|}$ is usually included to make the formulae involving Fourier inversion look nicer, and is a matter of convention.

Remark 3.1.3. Despite the notation $D : E \to F$, a differential operator is not a vector bundle (i.e. $\mathcal{O}_X$-module) morphism; in fact, it follows easily from the definition that differential operators satisfy a higher analog of the Leibniz rule. For this (good) reason, some authors choose to use a different notation for a differential operator; we will not. Hopefully, it is clear from context what is meant by a map $D : E \to F$.

We note here also that a differential operator D is said to have order k if it has order at most k but not at most $k-1$; this amounts to saying that at least one $a^{\nu}_{\lambda\mu}$ is nonzero somewhere for some λ with $|\lambda| = k$. The same definition will apply everywhere below, and we will not repeat it.

To give the second definition, we first recall the notion of a jet bundle. Given a smooth vector bundle $E \to X$, for each $k \geq 0$, there is a vector bundle $J^k E \to X$ called the k^{th} jet bundle over E whose fiber $(J^k E)(x)$ over a point x is the set of equivalence classes of local sections of E around x under the equivalence relation that their Taylor series in x (with respect to some local chart on X and frame on E) agree to order k. Equivalently, the fiber of $J^k E$ over x is exactly $J^k(E)(x) = \mathcal{E}_x/\mathfrak{m}^{k+1}_{X,x}\mathcal{E}_x$, where $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$ is the maximal ideal of functions vanishing at x and $\mathcal{E}_x$ is the stalk of of $\mathcal{E}$ at x. This formulation makes it clear that if E has rank r and $\dim X = n$, then the bundle $J^k E$ has rank $r\binom{n+k}{n}$. Another construction of this bundle in the complex or holomorphic category is given by taking

$$J^k(\mathcal{E}) = \pi_{2,*}(\pi_1^* \mathcal{E} \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{k+1}_{\Delta}),$$

where $\pi_i : X \times X \to X$ are the projection maps, and $\mathcal{I}_\Delta \subset \mathcal{O}_{X\times X}$ is the ideal sheaf of the diagonal Δ; in other words, $\mathcal{O}_{X\times X}/\mathcal{I}^{k+1}_{\Delta}$ is the structure sheaf of an order k-thickening of the diagonal $\Delta \subset X \times X$; then this defintion can be applied not just to vector bundles but arbitrary quasicoherent sheaves $\mathcal{E}$ (see [4, §7.2]).

Returning to the case of a smooth vector bundle $E \to X$ there is universal prolongation differential operator $j^k_E : \Gamma(X,E) \to \Gamma(X, J^k E)$ given simply by taking a section $\sigma \in \Gamma(X,E)$ its equivalence class at each fiber in $\Gamma(X, J^k E)$. This is, by definition, a "differential operator of degree at most k" in the above sense, and we may use this universal case to define all other differential operators, as in

Definition 3.1.4 (II). Let X be a smooth manifold, $E, F \to X$ smooth complex vector bundles, and $k \geq 0$ be an integer. Then a differential operator $D : E \to F$ of order at most $k \geq 0$ is a vector bundle homomorphism $D : J^k(E) \to F$.

From this definition, we may recover the map $\Gamma(X,E) \to \Gamma(X,F)$ on global sections as the composition $\Gamma(X,D) \circ j^k_E$.

The first definition leaves something to be desired, implicit already in the second definition. Namely, differential operators are local operators, and the definition using spaces $\Gamma(X,E)$ and $\Gamma(X,F)$ of global sections obscures this fact. A better definition, therefore, involves working with the sheaves $\mathcal{E}$ and $\mathcal{F}$ directly. The second definition, along with the observation that

$$\mathcal{E}_x/\mathfrak{m}^{k+1}_{X,x}\mathcal{E}_x \cong \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}^{k+1}_{X,x}$$

also makes it clear that it suffices to work with the trivial bundle $E = \underline{\mathbb{C}}$; all the differential information is captured in the jet bundle $J^k\underline{\mathbb{C}}$, elements of which can be thought of as differential operators $D : \underline{\mathbb{C}} \to \underline{\mathbb{C}}$ of order at most k. For instance, it is clear from either definition that a differential operator $D : \underline{\mathbb{C}} \to \underline{\mathbb{C}}$ of order 1 is given by a map $D : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ of the form $f \mapsto \xi(f) + gf$ for some vector field $\xi \in \Gamma(X, \mathrm{T}X)$ and smooth function $g \in \Gamma(X, \mathcal{O}_X)$.

Finally, it is clear also from either of the first two definitions that if $D: E \to F$ is a differential operator of order k and $D' : F \to G$ is a differential operator of order k', then we may compose D and D' to obtain a differential operator $D' \circ D : E \to G$ of order $k + k'$. Now if $E = F = G = \underline{\mathbb{C}}$, then we may also consider the composition $D \circ D'$ and hence the bracket

$$[D, D'] := D \circ D' - D' \circ D.$$

A priori, this is only a differential operator of degree $k + k'$ on $\underline{\mathbb{C}}$, but, in fact, it is easy to see using the Leibniz rule that the "top" differential terms cancel out and that

$$[D, D'] \text{ is an operator of degree } k + k' - 1. \tag{3.1}$$

When $k = k' = 1$, the observation (3.1) is saying that the Lie bracket of two vector fields is another vector field. The observation (3.1) then enables us to make an indutive definition of differential operators as in

Definition 3.1.5 (III). Let X be a smooth manifold, $\mathcal{O}_X$ be the sheaf of smooth complex valued functions on X, and and $k \geq 0$ be an integer. The sheaf Diff^k of k^{th} order differential operators on X is the left $\mathcal{O}_X$-submodule

$$\mathsf{Diff}^k \subset \mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)$$

of the complex endomorphism sheaf $\mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)$ of $\mathcal{O}_X$ defined inductively as follows. We define $\mathsf{Diff}^{-1} = 0$ and for $k \geq 0$, and $U \subset X$ open, we say that an operator

$$\mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)(U) \ni D : \mathcal{O}_X|_U \to \mathcal{O}_X|_U$$

belongs to $\mathsf{Diff}^k(U)$ if for each open $V \subset U$ and $f \in \mathcal{O}_X(V)$, the map

$$[D, m_f] : \mathcal{O}_X|_V \to \mathcal{O}_X|_V$$

lies in $\mathsf{Diff}^{k-1}(V)$, where $m_f \in \mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)(U)$ is the map given by multiplication by f and the bracket is taken in $\mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)(U)$.

Left multiplication then allows us to think of each Diff^k as a left $\mathcal{O}_X$-module.[1] Using this definition as model, given any vector bundles $\mathcal{E}, \mathcal{F}$ on X, we can then define the sheaf $\mathsf{Diff}^k(\mathcal{E}, \mathcal{F})$ of differential operators from $\mathcal{E}$ to $\mathcal{F}$ of order at most $k \geq 0$ to be a subsheaf

$$\mathsf{Diff}^k(\mathcal{E}, \mathcal{F}) \subset \mathsf{Hom}_{\mathbb{C}}(\mathcal{E}, \mathcal{F})$$

either similarly to, or using, Definition III, with the details being straightforward, if tedious. Admittedly, this definition is hard to parse, but it nonetheless makes it clear that a differential operator is, in fact, a local operator. The Diff^k form an increasing sequence of sheaves that exhausts $\mathsf{End}_{\mathbb{C}}(\mathcal{O}_X)$, in the sense that if we allow differential operators to have orders that are locally finite but globally arbitrarily large, then, in fact, any complex sheaf endomorphism, $D : \mathcal{O}_X \to \mathcal{O}_X$ is a differential operator. This is a theorem of Jaak Peetre; see [55] and the Wikipedia page linked there. Finally, this definition makes it clear that there was nothing special to smooth manifolds that we used

[1] In fact, Diff^k clearly has a structure of an $\mathcal{O}_X$-bimodule, but the right and left module structures are very different.

here, and we could have also worked with arbitrary $\mathscr{R}$-ringed spaces (see Conventions and Fundamentals for a reminder on the definition), the above special case corresponding to when X is a smooth manifold, $\mathscr{R} = \mathbb{C}_X$ is constant sheaf with values in $\mathbb{C}$, and $\mathcal{O}_X$ is the sheaf of smooth complex valued functions on X. This motivates

Definition 3.1.6 (IV)**.** Let X be an $\mathscr{R}$-ringed space, let X be a topological space equipped with a sheaf $\mathscr{R}$ of rings and another sheaf $\mathcal{O}_X$ of $\mathscr{R}$-algebras.

(a) The **tangent sheaf** of X is the left $\mathcal{O}_X$-submodule

$$\mathscr{T}_X \subset \mathsf{End}_{\mathscr{R}\text{-}\mathsf{Mod}}(\mathcal{O}_X)$$

of $\mathscr{R}$-module endomorphisms of $\mathcal{O}_X$ that satisfy the Leibniz rule, i.e. such that if $U \subset X$ is any open set and $\mathscr{T}_X(U) \ni D : \mathcal{O}_X|_U \to \mathcal{O}_X|_U$, then for any open $V \subset U$ and $f, g \in \mathcal{O}_X(V)$, we have

$$D(fg) = Df \cdot g + f \cdot Dg,$$

where in this formula, D denotes $D(V) : \mathcal{O}_X(V) \to \mathcal{O}_X(V)$.

(b) We define the sheaf Diff^1 of first order differential operators to be the subsheaf

$$\mathsf{Diff}^1 := \mathscr{T}_X \oplus \mathcal{O}_X \subset \mathrm{End}_{\mathscr{R}\text{-}\mathsf{Mod}}(\mathcal{O}_X).$$

Note that this is a sub-$\mathcal{O}_X$-bimodule.

(c) We define the **connection algebra** of X to be an $\mathscr{R}$-algebra $\mathscr{C}_X$ with an $\mathscr{R}$-module homomorphism $\rho : \mathcal{O}_X \to \mathscr{C}_X$, turning $\mathscr{C}_X$ into an $\mathcal{O}_X$-bimodule, and an $\mathcal{O}_X$-bimodule homomorphism

$$\rho^1 : \mathsf{Diff}^1 \to \mathscr{C}_X$$

that restricts to ρ on $\mathcal{O}_X$, that is universal with respect to these properties. The class in $\mathscr{C}_X$ of any local section η of $\mathscr{T}_X$ is denoted by $\nabla_\eta = \rho^1(\eta)$.

(d) We define the **algebra of differential operators** to be the quotient of $\mathscr{C}_X$ by the curvature relations, i.e.

$$\mathscr{D}_X := \mathscr{C}_X / ([\nabla_\eta, \nabla_\xi] - \nabla_{[\eta,\xi]})_{\eta,\xi}.$$

Both $\mathscr{C}_X$ and $\mathscr{D}_X$ are filtered via the obvious epimorphisms from the tensor algebra $\mathbb{T}^*_{\mathcal{O}_X}(\mathsf{Diff}^1) \to \mathscr{C}_X$, and this filtration is nothing but that by order. We denote these filtrations by $\mathscr{C}_X^{(\bullet)}$ and $\mathscr{D}_X^{(\bullet)}$ respectively. Finally,

(e) Finally, we define a differential operator of order at most k between $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ to be a right $\mathcal{O}_X$-module homomorphism $\mathcal{E} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(k)}$, or equivalently a global section of the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(k)} \otimes_{\mathcal{O}_X} \mathcal{E}$, where we have used the bimodule structure of $\mathscr{D}_X$.

For more details about this definition, see [3]. Note that in the above terminology, a **connection** on an $\mathcal{O}_X$-module $\mathcal{E}$ is nothing but a lift of its $\mathcal{O}_X$-module structure to one of a $\mathscr{C}_X$-module; this justifies the name. Similarly, a $\mathscr{D}_X$-module is nothing but an $\mathcal{O}_X$-module with a flat connection. Definition (IV), being the most abstract, is also the most general. For instance, if we take $\pi : X \to S$ to be a relative scheme and $\mathscr{R} := \pi^{-1}\mathcal{O}_S$, then the above definition of $\mathscr{T}_X$ recovers the the relative tangent sheaf $\mathrm{T}_{X/S}$ of X over S, and $\mathscr{C}_X$ recovers the sheaf of relative differential operators. This level of generality is the beginning of the theory of $\mathscr{D}$-modules, describing even the basics of which would take us way too far afield. To quote Michael Ende, das ist aber eine andere Geschichte, und soll ein andermal erzählt werden.

We end this appendix with some brief remarks on the symbol of a differential operator, interpreting it from these various perspectives. If $D : E \to F$ is a differential operator between smooth vector bundles on a smooth manifold X, then the symbol of D, denoted $\sigma(D)$, gives us for each $(x, \xi) \in \mathrm{T}^\vee X$ a linear map $\sigma(D, \xi) : E(x) \to F(x)$ that for fixed x is polynomial of degree k in ξ if D has order k. The symbol can be thought of as

- a vector bundle morphism $\pi^*E \to \pi^*F$ on the total space of the cotangent bundle $\mathrm{T}^\vee X$ between the pullbacks of E and F via $\pi : \mathrm{T}^\vee X \to X$, or
- a vector bundle homomorphism on X given as

$$\sigma(D) : \mathrm{Sym}^k \mathrm{T}^\vee X \to \mathsf{Hom}(E, F).$$

Roughly speaking, the symbol captures exactly the k^{th} order part of D. This symbol has the following property. If $f : X \to \mathbb{C}$ is any smooth function, then the operator

$$\Gamma(X, E) \to \Gamma(X, F), \quad u \mapsto \mathrm{e}^{-\mathrm{i}tf} D\, \mathrm{e}^{\mathrm{i}tf} u$$

can be written as $\sum_{j=0}^{k} t^{k-j} D_j^f u$, where $D_j^f : E \to F$ is a differential operator of order j that does depends only D and f but not u. Then the vector bundle homomorphism $D_0^f : E \to F$ depends only on the 1-form $\mathrm{d}f$ in the sense we have

$$\sigma(D, \mathrm{d}f) = D_0^f.$$

This result is often expressed as

$$\sigma(D, \mathrm{d}f) = \lim_{t \to \infty} \frac{1}{t^k} \mathrm{e}^{-\mathrm{i}tf} D\, \mathrm{e}^{\mathrm{i}tf}.$$

Since this property is local, this equality can be used to define the symbol $\sigma(D)$, but we can also connect this to our four definitions above as follows.

(I) If D is given in local coordinates as $D = \sum_{|\lambda| \leq k|} a^\nu_{\lambda\mu} (\partial^\lambda \mathrm{e}^\mu \cdot) \mathrm{f}_\nu$, then the symbol in the same local coordinates, along with cotangent coordinates $\xi_i \mathrm{d}x^i$, is given

$$\sigma(D, \xi) = \sum_{|\lambda| = k} a^\nu_{\lambda\mu} \xi^\lambda \mathrm{e}^\mu \mathrm{f}_\nu,$$

where $\xi^\lambda := \xi_1^{\lambda_1} \cdots \xi_n^{\lambda_n}$. That this is polynomial in ξ of degree k is clear; one has to then check that this definition glues together to give a well-defined global vector bundle morphism $\sigma(D) : \pi^*E \to \pi^*F$.

(II) If $x^1, \ldots, x^n$ are local coordinates on X and $\mathrm{e}_1, \ldots, \mathrm{e}_r$ a local frame of E, then the jet bundle $J^k E$ has a local frame $\mathrm{e}_\mu \delta_\lambda$ for $\mu = 1, \ldots, r$ and $|\lambda| \leq k$, where δ_λ is the "universal dual" to ∂^λ in the sense that the prolongation map j_E^k is given by taking a section $u = u^\mu \mathrm{e}_\mu$ to $(\partial^\lambda u^\mu)\mathrm{e}_\mu \delta_\lambda$. In these terms, the symbol of the universal prolongation operator $j^k_E : E \to J^k E$ is simply given as the map

$$\sigma(j^k_E, \xi) : E(x) \to J^k E(x), \quad \gamma^\mu \mathrm{e}_\mu(x) \mapsto \sum_{|\lambda|=k} \xi^\lambda \gamma^\mu \mathrm{e}_\mu \delta_\lambda(x).$$

For an abritrary differential operator $D : E \to F$ thought of as a vector bunder homomorphism $D : J^k(E) \to F$ the symbol $\sigma(D, \xi)$ is then given as the composition

$$\sigma(D, \xi) = D(x) \circ \sigma(j^k_E, \xi) : E(x) \to F(x).$$

That this is polynomial in ξ of degree k is again clear; one has to then check again that this gives us a global section $\sigma(D) : \pi^* E \to \pi^* F$.

(III) If $f_1, \dots, f_k$ are any local smooth functions, then the inductive definition of Diff^k tells us that the the operator $[\dots[[D, m_{f_1}], m_{f_2}], \dots, m_{f_k}]$ is an order zero operator, i.e. vector bundle homomorphism, $E \to F$. This can be now used as follows: given D and given cotangent vectors $\xi_1, \dots, \xi_k$ at a point $x \in X$, we may pick local functions $f_1, \dots, f_n$ at x with $\mathrm{d}f_j = \xi_j$ for $1 \le j \le k$. Then the map

$$[\dots[[D, m_{f_1}], m_{f_2}], \dots, m_{f_k}](x) : E(x) \to F(x)$$

is easily seen to be independent of the choice of f_j's, giving us a multilinear map $(\mathrm{T}^\vee X)^k \to \mathrm{Hom}_{\mathbb{R}}(E(x), F(x))$. Finally, using the Jacobi identity, this is easily seen to be symmetric in the ξ_j and hence gives us a morphism

$$\sigma(D)(x) : \mathrm{Sym}^k \mathrm{T}^\vee_x X \to \mathrm{Hom}_{\mathbb{R}}(E(x), F(x)).$$

One, then has to check again, that these homomorphisms patch together to give us a smooth vector bundle homomorphism $\sigma(D) : \mathrm{Sym}^k \mathrm{T}^\vee X \to \mathsf{Hom}(E, F)$.

(IV) Finally, we can show that for each $k \ge 0$, there is a symbol sequence

$$0 \to \mathscr{D}_X^{(k-1)} \to \mathscr{D}_X^{(k)} \xrightarrow{\sigma} \mathrm{Sym}^k \mathscr{T}_X \to 0,$$

which is to say that the associated graded algebra of the filtered algebra of differential operators $\mathscr{D}_X^{(\bullet)}$ is nothing but the symmetric algebra $\mathrm{Sym}^\bullet \mathscr{T}_X$ of the tangent sheaf $\mathscr{T}_X$. This follows essentially from the observation (3.1). If we think of an order k differential operator D between $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ as a right $\boldsymbol{\mathcal{O}}_X$-module homomorphism $\boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathscr{D}_X^{(k)}$, then σ gives rise to an $\boldsymbol{\mathcal{O}}_X$-module homomorphism

$$\boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{F}} \otimes_{\mathcal{O}_X} \mathrm{Sym}^k \mathscr{T}_X,$$

which can then be tought of as an $\boldsymbol{\mathcal{O}}_X$-module homomorphism

$$\sigma : \mathrm{Sym}^k \mathscr{T}_X^\vee \to \boldsymbol{\mathcal{E}}^\vee \otimes_{\mathcal{O}_X} \boldsymbol{\mathcal{F}}$$

as needed.

It is then a straightforward exercise–important for the reader encountering these ideas for the first time–to check that these five (!) definitions of a symbol all agree.

3.2 The Pfaffian and the Euler Class

The determinant of a skew-symmetric matrix is, somewhat surprisingly, the square of a polynomial in the coefficients of that matrix with integer coefficients. This polynomial, suitably normalized, is called the Pfaffian of the matrix. Pfaffians, named by Arthur Cayley after the German mathematician Johann Friedrich Pfaff, show up in a variety of mathematical contexts, from the combinatorics of braids, to Chern-Weil Theory of the the Euler class, and to the geometry of the Grassmannian of lines. Here we will give a quick overview of the proof of their existence. We will then mention an explicit formula for the Pfaffian and include a small digression on the Grassmannian on lines. Finally, we will review definitions of the Euler class of an oriented vector bundle and connect Pfaffians and the Euler class via Chern-Weil Theory. The starting point here is

Theorem 3.2.1. Let $n \geq 1$ be a positive integer, and let $R := \mathbb{Z}[x_{ij}]_{1\leq i<j\leq 2n}$ be the polynomial ring generated by the $n(2n-1)$ variables x_{ij}. Let X_{2n} be the unique skew-symmetric square matrix of size $2n$ with entries in R such that for $1 \leq i < j \leq 2n$, the (i, j) entry of X_{2n} is x_{ij}. Then there is an element $\mathrm{Pf}_n \in R$ such that

$$\det X_{2n} = (\mathrm{Pf}_n)^2.$$

Since R is an integral domain, the element Pf_n is unique up to sign, and we pick a normalization to pin it down uniquely, as in

Definition 3.2.2. For any $n \geq 1$,the Pfaffian polynomial of degree n is defined to be the polynomial $\mathrm{Pf}_n \in R$ given by Theorem 3.2.1 and normalized so that upon setting $x_{12} = x_{34} = \cdots = x_{2n-1,2n} = 1$ and the remaining variables to 0, we get $\mathrm{Pf}_n = 1$.

The first couple of these are given by

$$\mathrm{Pf}_1 = x_{12} \text{ and } \mathrm{Pf}_2 = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}.$$

Note that Pf_n has degree exactly n in the x_{ij}. Clearly, the above universal set-up tells us that the same result holds when we specialize to an arbitrary skew-symmetric matrix over any ring. Evaluating the Pfaffian polynomial at a specific skew-symmetric matrix X yields *its* Pfaffian, denoted $\mathrm{Pf}(X)$, so that

$$\mathrm{Pf}(X) = \mathrm{Pf}_n|_X$$

when X has order $2n$, and by convention $\mathrm{Pf}(X) = 0$ when X has odd order.

We now proceed to a proof of Theorem 3.2.1, following [56, §25.3]. The main ingredient in the proof is the following linear algebraic result. To state this result, note first that any $N \times N$ matrix X with coefficients in a field k, where $N \geq 1$, can be interpreted as a bilinear form ω_X on the vector space k^N with Gram matrix X. Then ω_X is symmetric (resp. alternating), iff X is, where recall that a matrix is called alternating if it is skew-symmetric and the entries on its principal diagonal are zero. When the characteristic of k is different from 2, alternating is the same as skew-symmetric, but in characteristic 2, a matrix is symmetric iff it is skew-symmetric and alternating matrices form a subclass of symmetric matrices. It is in this subclass that we are interested, but eventually, to do the "universal" case, we will restrict to characteristic 0 anyway. We now have

Lemma 3.2.3. Let k be a field and $N \geq 1$ an integer. Given an alternating $N \times N$-matrix $X \in \mathrm{Mat}_N(k)$, there is a $T \in \mathrm{GL}_N(k)$ such that the matrix $T^t XT$ can be expressed in block diagonal form as

$$T^t XT = S^{\oplus r} \oplus [0]^{\oplus s}$$

where

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

appears r times and the zero matrix $[0]$ appears s times for some integers r, s satisfying $2r + s = N$. The integers r and s are uniquely determined by X.

Proof. As in the discussion above, think of X as the Gram matrix of an alternating form ω on the vector space $V := k^N$, and proceed by induction on N. When $N = 1$, the result is clear; suppose $N \geq 2$. If $\omega = 0$, we are done; else, there are $e_1, f_1 \in V$ such that $\omega(e_1, f_1) \neq 0$, and so, by rescaling f_1 as needed, we may assume that $\omega(e_1, f_1) = 1$. Consider the subspace $W := ke_1 + kf_1 \subset V$ and its ω-orthogonal complement subspace $W^\perp := \{v \in V : \omega(v, W) = 0\}$. The claim, then, is that the natural map $W \oplus W^\perp \to V$ is an isomorphism; translating this result back into the language of matrices then finishes the proof by induction. To show injectivity, note that if $w = \lambda e_1 + \mu f_1 \in W \cap W^\perp$, then

$$\mu = \omega(e_1, w) = 0 \text{ and } \lambda = \omega(-f_1, w) = 0$$

so that $w = 0$. To show surjectivity, note simply that any $v \in V$ can be written as

$$v = (-\omega(f_1, v)e_1 + \omega(e_1, v)f_1) + (v + \omega(f_1, v)e_1 - \omega(e_1, v)f_1),$$

where the term in the first pair of parentheses is in W, and that in the second pair of parentheses is in $W^\perp$. The integer s is determined as the dimension of the kernel of ω. ■

This result shows also that there are no nondegenerate alternating forms on a vector space of odd dimension in any characteristic.

Proof of Theorem 3.2.1. Apply Lemma 3.2.3 to the fraction field $k = \mathbb{Q}(x_{ij})$ of R, taking $N = 2n$ and $X = X_{2n}$ to find a matrix $T \in \mathrm{GL}_{2n}(k)$ as in the conclusion of the lemma. Then $r = n$ and $s = 0$; indeed, otherwise, it would follow that $\det X_{2n} = 0$, which is incorrect. (This we know, for instance, by specalizing to $x_{12} = x_{34} = \cdots = x_{2n-1,2n} = 1$ and setting the other variables to 0.) From the resulting equation, we conclude then that

$$(\det T)^2 \det X_{2n} = \det\bigl(T^t X_{2n} T\bigr) = \det\bigl(S^{\oplus n}\bigr) = (\det S)^n = 1,$$

so taking $\mathrm{Pf}_n := 1/\det T \in K$ shows the existence of such an element in K. It remains to show that Pf_n is in R. But now R is a unique factorization domain, so we may write $\mathrm{Pf}_n = p/q$ for some coprime $p, q \in R$. The defining equation then yields $q^2 \det X_{2n} = p^2$, but since p and q are coprime, it follows from this that $q \in R$ is a unit, which is to say that $q = \pm 1$, proving the claim.

■

Next, we will need the following "change of variables" property of the Pfaffian.

Lemma 3.2.4. Let $n \geq 1$ be a positive integer, and let $R = \mathbb{Z}[x_{ij}]_{1 \leq i < j \leq 2n}$ and X_{2n} be as before. Consider further the ring $R[a] := R[a_{ij}]_{1 \leq i,j \leq 2n}$. Let $A = [a_{ij}] \in \mathrm{Mat}_{2n}(R[a])$ be another matrix of indeterminates. Then we have

$$\mathrm{Pf}(A^t X_{2n} A) = \det(A) \cdot \mathrm{Pf}_n \in R[a] = \mathbb{Z}[x_{ij}, a_{ij}].$$

Proof. Note that $A^t X_{2n} A$ is also skew-symmetric and that

$$\mathrm{Pf}(A^t X_{2n} A)^2 = \det\bigl(A^t X_{2n} A\bigr) = \det(A)^2 \det X_{2n} = \det(A)^2 \mathrm{Pf}_n^2,$$

so, we must have

$$\mathrm{Pf}(A^t X A) = \pm \det(A) \cdot \mathrm{Pf}_n$$

for some universally defined choice of sign. Specializing to the identity matrix $A = \mathrm{id}_{2n}$ tells us that the positive sign must hold. ∎

Note again that this universal set-up implies the same result holds when we specialize to an arbitrary ring. We end by mentioning a combinatorial formula for the Pfaffian, namely

Lemma 3.2.5. Let Σ_n be the set of all unordered partitions of $\{1, 2, \dots, 2n\}$ into pairs. Given an element $\sigma \in \Sigma_n$, write $\sigma = \{(i_1, j_1), \dots, (i_n, j_n)\}$, where $1 \leq i_k, j_k \leq 2n$ for $k = 1, \dots, n$ satisfy $1 \leq i_1 < \cdots < i_n \leq 2n$ and $i_k < j_k$ for $k = 1, \dots, n$.[a] For each $\sigma \in \Sigma$, let $\pi_\sigma \in S_{2n}$ be the permutation of $\{1, \dots, 2n\}$ given by

$$\pi_\sigma(2k-1) = i_k \text{ and } \pi_\sigma(2k) = j_k \text{ for } 1 \leq k \leq n.$$

Then

$$\mathrm{Pf}_n = \sum_{\sigma \in \Sigma_n} (-1)^{\pi_\sigma} \prod_{k=1}^{n} x_{i_k, j_k}.$$

[a] Note that the integers $i_k = i_k(\sigma)$ and $j_k = j_k(\sigma)$ depend on, and uniquely determine, σ, but we suppress them in the notation.

Proof. A proof can be found in the classic reference [57, Chapter 7]. ∎

Remark 3.2.6. Pfaffians are closely related to the Plücker relations describing Grassmannians of lines. When V is a vector space over a field $\mathbb{F}$, the Grassmannian of lines in $\mathbb{P}V$, i.e. the Grassmannian $\mathrm{Gr}^2_{\mathbb{F}}(V)$ embeds via the Plücker embedding

$$\mathrm{Gr}^2_{\mathbb{F}}(V) \hookrightarrow \mathbb{P}\Lambda^2 V$$

as (the image of) the set of totally decomposable vectors in $\Lambda^2 V$. The space $\Lambda^2 V$ can be then identified with the space of all alternating forms on V. It then turns out that, when $\mathrm{ch}\,\mathbb{F} \neq 2$, the set of all totally decomposable alternating forms, i.e. $\mathrm{Gr}^2_{\mathbb{F}}(V)$, can be described as set of all $[\omega] \in \mathbb{P}\Lambda^2 V$ satisfying the equation

$$\omega \wedge \omega = 0.$$

Indeed, to show the nontrivial direction, supoose that this is true and take any nonzero $\lambda \in V^*$. Then the interior product $\lambda \lrcorner \omega \in V$ has the property that

$$0 = \lambda \lrcorner (\omega \wedge \omega) = 2\omega \wedge (\lambda \lrcorner \omega),$$

so that if $\operatorname{ch} \mathbb{F} \neq 2$, we can conclude from this that $\lambda \lrcorner \omega$ divides ω, so that, in particular, ω is decomposable. If we pick a basis $\mathrm{e}_1, \ldots, \mathrm{e}_n$ of V (where $n := \dim V$) and write $\omega = \sum_{1 \leq i < j \leq n} x_{ij} \mathrm{e}_i \wedge \mathrm{e}_j$ for some x_{ij}, then we have

$$\begin{aligned} \omega \wedge \omega &= \sum_{1 \leq k_1 < k_2 < k_3 < k_4 \leq n} (x_{k_1k_2}x_{k_3k_4} - x_{k_1k_3}x_{k_2k_4} + x_{k_1k_4}x_{k_1k_3})\, \mathrm{e}_{k_1} \wedge \mathrm{e}_{k_2} \wedge \mathrm{e}_{k_3} \wedge \mathrm{e}_{k_4} \\ &= \sum_{1 \leq k_1 < k_2 < k_3 < k_4 \leq n} \mathrm{Pf}_{k_1k_2k_3k_4}(X) \cdot \mathrm{e}_{k_1} \wedge \mathrm{e}_{k_2} \wedge \mathrm{e}_{k_3} \wedge \mathrm{e}_{k_4}, \end{aligned}$$

where $\mathrm{Pf}_{k_1k_2k_3k_4}(X)$ is the Pfaffian of the 4×4 minor of the skew-symmetric matrix $X = [x_{ij}]_{i,j=1}^n$ representing ω given by taking rows and columns in k_1, k_2, k_3, k_4. It follows that $\mathrm{Gr}^2_{\mathbb{F}}(V)$ is described as the subvariety of $\mathbb{P}\Lambda^2 V \cong \mathbb{FP}^{\binom{n}{2}-1}$ cut out by the vanishing of these $\binom{n}{4}$ Pfaffians of all 4×4 "symmetric" minors of X. In particular, $\mathrm{Gr}^2_{\mathbb{F}}(V)$ is the intersection of quadrics. For instance, when $n = 4$, this expresses $\mathrm{Gr}^{2,2}_{\mathbb{F}} \subset \mathbb{FP}^5$ as a quadric with very interesting geometry; see [58, Lecture 6].

Finally, we explain how the Pfaffian relates to the Euler class of oriented real bundles. Suppose that X is a sufficiently nice (e.g. paracompact Hausdorff) topological space, and $\pi : E \to X$ an **oriented** real vector bundle of rank r. To each such bundle E, we can associate a Characteristic class

$$\mathrm{e}(E) \in \mathrm{H}^r(X; \mathbb{Z})$$

called the **Euler class** of E.[2] There are quite a few ways to define this class, and the equivalence of these definitions is a nontrivial result that unifies different disciplines in math. We mention three perspectives–those of algebraic topology, differential topology, and from differential geometry. In what follows, we also assume for simplicity that X is compact; for noncompact X, one must either use cohomology relative to the complement of the zero section or cohomology with compact vertical supports.We do not give proofs or pretend to give a comprehensive treatment of this very broad subject; missing details can be found in [9], [42] and [56].

(a) (Algebraic Topology) The **Thom Isomorphism Theorem** in cohomology says that there is a uniquely defined class

$$\Theta_E \in \mathrm{H}^r_c(E; \mathbb{Z})$$

called the **Thom class** of E, with the property that for each point $x \in X$, under the restriction map

$$\mathrm{H}^r_c(E; \mathbb{Z}) \to \mathrm{H}^r_c(E(x); \mathbb{Z})$$

[2]Here we could work with singular cohomology or sheaf cohomology when X is nice (e.g. a manifold, or a locally contractible space). In sheaf cohomology, the Thom isomorphism can be proven from the Leray-Serre spectral sequence associated to the map $E \to X$. In singular cohomology, if X is a topological manifold then we can also use Poincaré duality to prove it.

to the fiber $E(x) \subset E$ of E over x, the Thom class Θ_E maps to the specified orientation class of E. This class has the property that the map Θ_E defined by

$$\Theta_E : \mathrm{H}^*(X;\mathbb{Z}) \to \mathrm{H}_c^{*+r}(E;\mathbb{Z}), \quad \eta \mapsto \Theta_E \smile \pi^*\eta$$

is an isomorphism, called the Thom isomorphism, where $\pi^* : \mathrm{H}^*(X,\mathbb{Z}) \to \mathrm{H}^*(E;\mathbb{Z})$ is the pullback map, and

$$\smile : \mathrm{H}^i(E;\mathbb{Z}) \times \mathrm{H}_c^j(E;\mathbb{Z}) \to \mathrm{H}_c^{i+j}(E;\mathbb{Z})$$

is the cup product in cohomology with compact supports. If $j : X \to E$ is the inclusion of the zero section then, since j is proper, we get a pullback map in compactly supported cohomology

$$j^* : \mathrm{H}_c^*(E;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z}),$$

and the Euler class of E is defined as the pullback

$$\mathrm{e}(E) := j^*\Theta_E$$

of the Thom class by j. Of course, the same thing can be done by replacing $\mathbb{Z}$ with any ring R (and slightly more generally if we replace "orientable" with "R-orientable").[3] Note that the resulting characteristic class is natural, and so has one further important property. If E is a vector bundle of odd rank r, then the map

$$E \to E, \quad (x,v) \mapsto (x,-v)$$

given by fiberwise negation is an orientation-reversing isomorphism of E. Therefore, the naturality of the Euler class says that

$$\mathrm{e}(E) = -\mathrm{e}(E) \text{ so } 2\mathrm{e}(E) = 0 \in \mathrm{H}^r(X;\mathbb{Z}).$$

In particular, if $\mathrm{H}^r(X;\mathbb{Z})$ has no two-torsion, or if we are working over a $\mathbb{Z}[1/2]$-algebra R, then we must have in this case also that $\mathrm{e}(E) = 0$.[4]

(b) (Differential Topology) Note that we can also perform the same computation as above replacing $\mathbb{Z}$ with say $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Suppose now that X is a smooth manifold and $E \to X$ is a smooth real vector bundle; for simplicity, we assume also that X is closed and oriented. Then translating between singular cohomology with $\mathbb{R}$ or $\mathbb{C}$ coefficients and de Rham cohomology via the de Rham theorem then tells us that

$$\Theta_E \in \mathrm{H}_c^r(E;\mathbb{R})$$

can be thought of as the Poincaré dual to the zero section. In particular, the Euler class

$$\mathrm{e}(E) = j^*\Theta_E \in \mathrm{H}_{\mathrm{dR}}^r(X;\mathbb{R})$$

is the self intersection class of the zero section of E. This gives us a way to compute this Euler class in practice: since any two sections of $E \to X$ are homotopic, if we

[3]See, however, Footnote 4 in Conventions and Fundamentals.

[4]Note, although, that it is possible for the Euler class to be torsion in general. For instance, if we consider a lens space $L_p = S^5/(\mathbb{Z}/p)$ for some prime p and if $\mathbb{Z}/p \to \mathrm{SO}_2$ is any homomorphism, then we can extend the principal $\mathbb{Z}/p$ bundle $\pi : S^5 \to L_p$ via π to a principle SO_2-bundle $S^5 \times_{\mathbb{Z}/p} \mathrm{SO}_2 \to L_p$, (the vector bundle corresponding to) which is easily seen to have nontrivial Euler class in $\mathrm{H}^2(L_p;\mathbb{Z}) \cong \mathbb{Z}/p$.

can find a section $\sigma : X \to E$ that is transverse to the zero section E_0, written $\sigma \pitchfork E_0$, then the preimage of E_0 under σ,

$$\sigma^{-1}(E_0) = \mathbb{V}(\sigma),$$

or equivalently the vanishing locus of σ, is a smooth embedded submanifold of X of codimension r in X, and hence by Poincaré duality represents a cohomology class $\eta_{\mathbb{V}(\sigma)} \in \mathrm{H}^r_{\mathrm{dR}}(X;\mathbb{R})$. It follows then that this class is independent of the choice of σ, and is none other other than the Euler class $\mathrm{e}(E) \in \mathrm{H}^r_{\mathrm{dR}}(X;\mathbb{R})$. In particular, if the rank r of E coincides with the dimension n of X, then $\mathrm{e}(E)$ is a multiple of the generator η_X of the top cohomology of X by the number of zeroes, counted with signed multiplicity, of a generic section $\sigma : X \to E$.

(c) (Differential Geometry) Finally, suppose that we are working with a vector bundle of even rank r, and we reduce the structure group of E from $\mathrm{GL}^+_r\,\mathbb{R}$ to $\mathrm{SO}_r \subset \mathrm{GL}^+_r\,\mathbb{R}$, i.e. we equip E with a Riemannian metric. Suppose further that we have a metric connection ∇ on E. Then given any local oriented orthonormal frame

$$\mathrm{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_r)$$

of E, the curvature matrix Ω of ∇ with respect to e is a skew-symmetric $r \times r$ matrix of 2-forms on X. In particular, since r is even, we may speak of the Pfaffian

$$\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right),$$

which is a locally defined r-form on X. In fact, if

$$\mathrm{e}' = \mathrm{e}a$$

is a different oriented orthonormal frame, so a is a local function on X with values in SO_r, then the curvature matrix Ω' of ∇ with respect to e' is given by

$$\Omega' = a^{-1}\Omega a.$$

In particular, we get from Lemma 3.2.4 that

$$\mathrm{Pf}\left(\frac{1}{2\pi}\Omega'\right) = \mathrm{Pf}\left(\frac{1}{2\pi}a^{-1}\Omega a\right) = \mathrm{Pf}\left(\frac{1}{2\pi}a^{t}\Omega a\right) = \det a \cdot \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right) = \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right),$$

where in the second and last equalities we have used that a is SO_r-valued. In particular, this tells us that this r-form is well-defined irrespective of the choice of oriented orthornormal frame e, and hence glues together to give a global r-form, also denoted by $\mathrm{Pf}(1/2\pi\Omega)$ on X. The general machinery of Chern-Weil Theory then says that this form is closed, and the corresponding cohomology class

$$\left[\mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)\right] \in \mathrm{H}^r_{\mathrm{dR}}(X;\mathbb{R})$$

is well-defined and independent of the choice of metric and connection on E; and indeed, this class is none other than the Euler class $\mathrm{e}(E)$ of E.

3.3 Multiplicative Sequences

The theory of multiplicative sequences was formalized by Hirzebruch in his seminal book *Neue topologische Methoden in der algebraischen Geometrie* published in 1956. It serves as algebraic preparation for the theory of characteristic classes. We will now do this preparation, following closely the revised English translation of this classic, [13].

Definition 3.3.1. Given a ring R, let $R[c]$ be the graded R-algebra defined by

$$R[c] := R[c_1, c_2, \dots], \quad \text{where for } n \geq 1, \text{we have } \deg c_n = 2n.^a$$

It is often convenient to let $c_0 := 1$. For each $n \geq 0$, the degree $2n$ component of $R[c]$ is

$$R[c]_{2n} = R[c_1, \dots, c_n]_{2n},$$

i.e. consists of polynomials with R-coefficients of total degree $2n$ in the variables $c_1, \dots, c_n$. Let $\mathrm{U}_1(R)$ be the subgroup of the group of units of the power series ring $R[\![z]\!]$ consisting of the 1-units, i.e. let $\mathrm{U}_1(R)$ be defined by

$$\mathrm{U}_1(R) := 1 + zR[\![z]\!] := \left\{ \sum_{n=0}^{\infty} q_n z^n : q_n \in R, q_0 = 1 \right\} \subset R[\![z]\!]^{\times}.$$

(a) A sequence $K = (K_n)_{n \geq 0}$ of elements of $R[c]$ is said to be multiplicative if $K_0 = 1$ and $K_n \in R[c]_{2n}$, and the map

$$K : \sum_{n=0}^{\infty} q_n z^n \mapsto \sum_{n=0}^{\infty} K_n(q_1, \dots, q_n) z^n$$

is an endomorphism of the group $\mathrm{U}_1(R)$.

(b) Given a multiplicative sequence K, we define its characteristic series Q_K by

$$Q_K(z) := K(1+z) = \sum_{n=0}^{\infty} K_n(1, 0, \dots, 0) z^n \in \mathrm{U}_1(R).$$

[a] This grading is not the same as the one chosen by Hirzebruch in [13, §1]. The reason for our choice is that for any complex vector bundle $E \to X$, the evaluation at the Chern classes of E gives us a ring homomorphism $\mathrm{eval}_E : R[c] \to \mathrm{H}^*(X; R)$, which with our convention becomes a morphism of graded R-algebras. See §3.4.

Example 3.3.2. The sequence $K = c$ defined by $K_n = c_n$ is called the identity sequence, since the corresponding endomorphism K is the identity map on $\mathrm{U}_1(R)$. Its characteristic series is $Q(z) = 1 + z$.

The first important result here is that a multiplicative sequence is completely characterized by its characteristic series:

Lemma 3.3.3. The map $K \mapsto Q_K(z)$ is a bijection from the set of all multiplicative sequences to $\mathrm{U}_1(R)$.

Proof. We define an inverse map. For each $N \geq 1$, let

$$R[c]^{(N)} := R[c][\gamma_1, \ldots, \gamma_N]/(c_j - e_j(\gamma_1, \ldots, \gamma_N))_{j=1}^{N},$$

where $e_j(\gamma_1, \ldots, \gamma_N)$ is the j^{th} elementary symmetric polynomial in $\gamma_1, \ldots, \gamma_N$. Let $Q(z) \in \mathrm{U}_1(R)$ be given. Consider the product

$$\prod_{i=1}^{N} Q(\gamma_i z) \in \mathrm{U}_1\left(R[c]^{(N)}\right).$$

Since the coefficients of the powers of z in the expansion of this product are symmetric in the γ_j, by the Fundamental Theorem of Symmetric Polynomials, they can be written as polynomials in the c_j with coefficients in R. Therefore, for $n \geq 0$, there are $K_n^{(N)} \in R[c]_{2n}$ such that

$$\prod_{i=1}^{N} Q(\gamma_i z) = \sum_{n=0}^{\infty} K_n^{(N)}(c_1, \ldots, c_n) z^n.$$

For $N \geq n$, the polynomial $K_n^{(N)}$ is independent of N, and we define K_n to be this common value. It is easy to see that the resulting sequence $K = (K_n)_{n\geq 0}$ is multiplicative, and that these operations give us inverse bijections. For more details, see [13, §1]. ■

The previous proof is constructive and allows us to give an explicit formula for the sequence K_n in terms of the series $Q(z)$. To state this formula, we introduce some terminology. First suppose that

$$Q(z) = \sum_{i=0}^{\infty} q_i z^i = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots.$$

For each partition $\lambda \vdash n$, we define

$$q^\lambda := \prod_{j=1}^{\ell(\lambda)} q_{\lambda_j}, \text{ and } c_\lambda := e_\lambda(\gamma) = \prod_{j=1}^{\ell(\lambda)} c_{\lambda_j}.$$

Finally, we let $b_\lambda^\mu \in \mathbb{Z}$ be the entries of the transition matrices that allow us to express the monomial symmetric functions in terms of the elementary symmetric functions, i.e. satisfy the universal identity

$$m_\lambda = \sum_{\mu \vdash n} b_\lambda^\mu e_\mu$$

in the ring Λ of symmetric polynomials in countably many variables (see Conventions and Fundamentals). Then the proof of Lemma 3.3.3 also establishes

Proposition 3.3.4. If K is the multiplicative sequence corresponding to the characteristic series $Q(z) = \sum_{i=0}^{\infty} q_i z^i$, then we have for each $n \geq 0$ that

$$K_n = \sum_{\lambda, \mu \vdash n} q^\lambda b_\lambda^\mu c_\mu.$$

The matrix $B_n := [b^\mu_\lambda]_{\lambda,\mu\vdash n}$, where we order partitions reverse lexicographically from left to right and top to bottom, has the property that entries of its inverse matrix $B_n^{-1} = [a^\lambda_\mu]_{\lambda,\mu\vdash n}$ admit a simple combinatorial description, namely that a^λ_μ is the number of matrices with entries 0 or 1 such that the successive row (resp. column) sums are given by the parts of λ (resp. μ). In particular,

(a) the coefficients of B_n^{-1} are nonnegative integers,
(b) the matrix B_n is symmetric, i.e. $b^\mu_\lambda = b^\lambda_\mu$,
(c) and the matrix B_n has 1's along its nonprincipal diagonal, and is zero below it, i.e. $b^\mu_\lambda \neq 0$ implies $\mu \geq \lambda^*$, with $b^{\lambda^*}_\lambda = 1$.[5]

For instance, the matrices B_n for $1 \leq n \leq 4$ are

$$\begin{aligned}
B_1 &= \begin{bmatrix} 1 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} 3 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and} \\
B_4 &= \begin{bmatrix} -4 & 4 & 2 & -4 & 1 \\ 4 & -1 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

which correspond to the formulae

$$\begin{aligned}
K_1 &= q_1 c_1, \\
K_2 &= (-2q_2 + q_1^2)c_2 + q_2 c_1^2, \\
K_3 &= (3q_3 - 3q_2 q_1 + q_1^3)c_3 + (-3q_3 + q_2 q_1)c_2 c_1 + q_3 c_1^3, \text{ and} \\
K_4 &= (-4q_4 + 4q_3 q_1 + 2q_2^2 - 4q_2 q_1^2 + q_1^4)c_4 + (4q_4 - q_3 q_1 - 2q_2^2 + q_2 q_1^2)c_3 c_1 \\
&\quad + (2q_4 - 2q_3 q_1 + q_2^2)c_2^2 + (-4q_4 + q_3 q_1)c_2^2 c_1 + q_4 c_1^4.
\end{aligned}$$

Before moving to a few examples, we develop one more crucial bit of theory.

Definition 3.3.5. An element $Q(z) \in \mathrm{U}_1(R)$ is said to be even if $Q(z)$ only consists of even powers of z, i.e. for each $n \geq 0$, the coefficient $[z^{2n+1}]Q(z)$ of z^{2n+1} in $Q(z)$ is zero, or equivalently when there is a (necessarily unique) $\widetilde{Q}(z) \in \mathrm{U}_1(R)$ such that $Q(z) = \widetilde{Q}(z^2)$.

If $Q(z)$ is even, then in the corresponding multiplicative sequence (K_n), we also have for each $n \geq 0$ that $K_{2n+1} = 0$. In this case, the series $\tilde{Q}(z)$ is called the reduced series and the sequence of polynomials $(\widetilde{K}_n)_{n\geq 0}$ defined by $\widetilde{K}_n = K_{2n}$ is called the corresponding reduced (multiplicative) sequence.[6]

[5] Here λ^* denotes the conjugate partition of λ, and $\geq$ refers to the lexicographic order.
[6] Note that this is not multiplicative in the above sense, and indeed $\deg \widetilde{K}_n = 4n$.

Example 3.3.6. The reduced sequence corresponding to $Q(z) = 1 + z^2$ is called the Pontryagin sequence and is denoted by (p_n). Explicitly, we have for $n \geq 0$ that

$$p_n = c_n^2 - 2\sum_{k=1}^{n}(-1)^k c_{n-k}c_{n+k},$$

or, in other words, the formal identity

$$\sum_{j=0}^{\infty}(-1)^j p_j z^{2j} = \left(\sum_{i=0}^{\infty} c_i z^i\right)\left(\sum_{i=0}^{\infty}(-1)^i c_i z^i\right). \tag{3.2}$$

Again by the Fundamental Theorem of Symmetric Polynomials, this time applied to the γ_j^2, we conclude that if $Q(z)$ is any even series and $(\widetilde{K}_n)$ the corresponding reduced sequence, then for each $n \geq 0$, we have

$$\widetilde{K}_n \in R[p_1, \dots, p_n]_{4n},$$

i.e. $\widetilde{K}_n$ can be written as a polynomial of total degree $4n$ in the variables $p_1, \dots, p_n$.

Remark 3.3.7. Note that the whole formalism of multiplicative sequences and characteristic series can also be done on the R-algebra

$$R[p] := R[p_1, p_2, \dots,],$$

where, this time, for $n \geq 1$, we have $\deg p_n = 4n$. In this case, one defines a multiplicative sequence $\tilde{K}$ to be have elements $\tilde{K}_n \in R[p]_{4n}$ instead. This contrast amounts to a choice of working with Chern or Pontryagin classes, and is largely a matter of personal taste. When working with characteristic series of complex vector bundles, Example 3.3.6 then furnishes the required relationship between the two series. This will become clearer when we discuss the relationship between Chern and Pontryagin classes of complex vector bundles in Remark 3.4.22 below.

Let's now work out a few examples.

Example 3.3.8. Let R be any $\mathbb{Q}$-algebra, and consider

$$Q_{\mathrm{Td}}(z) := \frac{z}{1-\mathrm{e}^{-z}} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n = 1 + \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \frac{1}{30240}z^6 - \frac{1}{1209600}z^8 + \cdots,$$

the exponential generating function of the Bernoulli numbers B_n.[7] The corresponding multiplicative sequence is called the sequence of Todd polynomials Td_n, and the first few of these are given by

$$\begin{aligned}
\mathrm{Td}_0 &= 1,\\
\mathrm{Td}_1 &= \frac{1}{2}c_1,\\
\mathrm{Td}_2 &= \frac{1}{12}(c_2 + c_1^2),\\
\mathrm{Td}_3 &= \frac{1}{24}c_2c_1,\\
\mathrm{Td}_4 &= \frac{1}{720}(-c_4 + c_3c_1 + 3c_2^2 + 4c_2c_1^2 - c_1^4).
\end{aligned}$$

[7] Here we are choosing the "positive" convention $B_1 = 1/2$.

Example 3.3.9. Let R be any $\mathbb{Q}$-algebra, and consider

$$Q_L(z) := \frac{z}{\tanh z} = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n} = 1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \frac{2}{945}z^6 - \frac{1}{4725}z^8 + \cdots .$$

The corresponding reduced series is

$$\tilde{Q}_L(z) = \frac{\sqrt{z}}{\tanh \sqrt{z}} := \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^n,$$

and the corresponding reduced sequence is called the sequence of L-polynomials and is denoted by (L_n). The first few of these are given by

$$\begin{aligned}
L_0 &= 1, \\
L_1 &= \frac{1}{3}p_1, \\
L_2 &= \frac{1}{45}(7p_2 - p_1^2), \\
L_3 &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3), \\
L_4 &= \frac{1}{14175}(831p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4).
\end{aligned}$$

Example 3.3.10. To generalize the previous two examples, let R be any algebra over the polynomial ring $\mathbb{Q}[y]$, and consider the series

$$\begin{aligned}
Q_y(z) &:= \frac{z(1+y)}{1 - \mathrm{e}^{-z(1+y)}} - yz \\
&= \frac{z\left(1 + y\mathrm{e}^{-z(1+y)}\right)}{1 - \mathrm{e}^{-z(1+y)}} \\
&= \sum_{n=0}^{\infty} \frac{B_n}{n!}(1+y)^n z^n - yz \\
&= 1 + \frac{1-y}{2}z + \frac{(1+y)^2}{12}z^2 - \frac{(1+y)^4}{120}z^4 + \frac{(1+y)^6}{30240}z^6 - \frac{(1+y)^8}{1209600}z^8 \cdots ,
\end{aligned}$$

the exponential generating function of the Bernoulli polynomials $B_n^-(y)$.[8] Note that we have for $y \in \{0, \pm 1\}$ the specializations

$$Q_0(z) = Q_{\mathrm{Td}}(z), \quad Q_1(z) = Q_L(z), \text{ and } Q_{-1}(z) = 1 + z.$$

The corresponding multiplicative sequence is called the sequence of generalized Todd

[8] There is a good argument to choose instead the definition of $Q_y(z)$ to be what we would, in our convention, denote by $Q_{-y}(z)$; see Remark 2.3.2. However, the convention used here, established already Hirzebruch's [13], is too firmly rooted to easily change.

polynomials and is denoted by $(\mathrm{Td}_n(y))$. The first few of these are given by

$$\begin{aligned}
\mathrm{Td}_0(y) &= 1,\\
\mathrm{Td}_1(y) &= \frac{1-y}{2}c_1,\\
\mathrm{Td}_2(y) &= \frac{1-10y+y^2}{12}c_2 + \frac{(1+y)^2}{12}c_1^2,\\
\mathrm{Td}_3(y) &= -\frac{y(1-y)}{2}c_3 + \frac{(1+y)^2(1-y)}{24}c_2c_1,\\
\mathrm{Td}_4(y) &= -\frac{1+124y-474y^2+124y^3+y^4}{720}c_4 + \frac{(1-58y+y^2)(1+y)^2}{720}c_3c_1\\
&\quad + \frac{(1+y)^4}{240}c_2^2 + \frac{(1+y)^4}{180}c_2c_1^2 - \frac{(1+y)^4}{720}c_1^4.
\end{aligned}$$

Example 3.3.11. Again, let R be any $\mathbb{Q}$-algebra and consider

$$Q_{\hat{A}}(z) := \frac{z/2}{\sinh(z/2)} = 1 - \frac{1}{24}z^2 + \frac{7}{5760}z^4 - \frac{31}{967680}z^6 + \frac{127}{154828800}z^8 + \cdots .$$

The corresponding reduced series is denoted

$$\tilde{Q}_{\hat{A}}(z) = \frac{\sqrt{z}/2}{\sinh(\sqrt{z}/2)}$$

and the corresponding reduced sequence is called the sequence of $\hat{A}$-polynomials and is denoted by $(\hat{A}_n)$. The first few of these are given by

$$\begin{aligned}
\hat{A}_0 &= 1\\
\hat{A}_1 &= -\frac{1}{24}p_1,\\
\hat{A}_2 &= \frac{1}{5760}(-4p_2+7p_1^2),\\
\hat{A}_3 &= \frac{1}{967680}(-16p_3+44p_2p_1-31p_1^3),\\
\hat{A}_4 &= \frac{1}{464486400}(-192p_4+512p_3p_1+208p_2^2-904p_2p_1^2+381p_1^4).
\end{aligned}$$

Example 3.3.12. Although we won't work with this series at all, it is still a fun construction: let's take $R = \mathbb{Q}[\![q]\!]$ to be the ring of rational power series in the variable q, and consider the series

$$Q_{\mathrm{W}}(z) := \frac{z/2}{\sinh(z/2)} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n\mathrm{e}^z)(1-q^n\mathrm{e}^{-z})}.$$

This is called the Witten series or the universal elliptic series, and is a modular form with respect to q. Note that for $q=0$, the Witten series specializes to $Q_{\hat{A}}$. The corresponding reduced sequence is called the sequence of Witten polynomials and is denoted by (W_n). We have, for instance, that $W_0 = 1$ and

$$W_1 = \left(-\frac{1}{24} + \sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}\right)p_1 = \left(-\frac{1}{24} + \frac{\psi_q^1(1)}{\log^2 q}\right)p_1,$$

where $\psi_q^n(z)$ denotes the q-polygamma function.

3.4 Vector Bundles and Characteristic Classes

In this section, we review the fundamentals of vector bundles and characteristic classes. Since the classical literature on this subject is abundant, too vast and ingrained in the "general culture" to even cite properly, we will pursue a somewhat eccentric, modern approach to the subject.

It has become increasingly evident in the past century that the "correct" setting for the definition of a vector bundle is the context of ringed spaces, with the "correct"[9] definition being

Definition 3.4.1.

(a) Let X be a ringed space and $n \geq 0$ be an integer. A vector bundle of rank n on X, written $\mathcal{E} \to X$, is a locally free $\mathcal{O}_X$-module of rank n.[a]

(b) Given a vector bundle $\mathcal{E} \to X$ and a morphism $f : Y \to X$ of ringed spaces, we define the pullback bundle to be

$$f^*\mathcal{E} := f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y.$$

If $\mathcal{E}$ has rank n, then so does $f^*\mathcal{E}$.

[a] It is possible to define a vector bundle $\mathcal{E}$ as a locally free $\mathcal{O}_X$-module of finite rank, without requiring the rank to be constant. There is something to be said about the resulting locally constant function called the rank; we will, however, have no occasion to deal with mixed-rank vector bundles in this article.

The set of isomorphism classes of vector bundles (of arbitrary, but globally constant, rank) on a ringed space X, denoted $\mathrm{Vect}(X)$, naturally forms a commutative graded semiring under the operations of direct sum and tensor product, with the grading being by rank, and the pullback construction associates to each morphism $f : Y \to X$ of ringed spaces a graded semiring homomorphism

$$f^* : \mathrm{Vect}(X) \to \mathrm{Vect}(Y).$$

This construction therefore gives rise to a contravariant functor

$$\mathrm{Vect} : \mathsf{RingSp}^{\mathrm{op}} \to \mathsf{GrSemiRing}$$

from the category of ringed spaces to the category of commutative graded semirings, called the vector bundle functor.

Recall that the forgetful functor $F : \mathsf{Ring} \to \mathsf{SemiRing}$ admits a left-adjoint functor, called the Grothendieck ring construction. In other words, there is a functor

$$\mathrm{GR} : \mathsf{SemiRing} \to \mathsf{Ring},$$

such that for any ring R and semiring S, we have the natural isomorphisms

$$\mathrm{Hom}_{\mathsf{SemiRing}}(S, FR) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Ring}}(\mathrm{GR}(S), R).$$

[9] Of course, the notion of "correctness" of a setting or definition is subjective. What we mean here is that this seems to be the most general (in the sense of widely applicable) and conceptually simple definition of this concept.

The exact construction of $\mathrm{GR}(S)$ does not matter at all[10], what matters is the universality of this construction, which makes this functor GR unique upto unique natural isomorphism making the relevant diagrams commute. Of course, there is a version of this functor that applies to graded semirings as well, which yields graded rings, but we'll forget that for a moment and use the faithful embedding $\mathsf{GrSemiRing} \hookrightarrow \mathsf{SemiRing}$ in making the following definition.

Definition 3.4.2. The K-theory functor is the contravariant functor

$$K := \mathrm{GR} \circ \mathrm{Vect} : \mathsf{Ring}\,\mathrm{Sp}^{\mathrm{op}} \to \mathsf{Ring}$$

from the category of ringed spaces to the category of rings that takes a ringed space X to the Grothendieck ring of $\mathrm{Vect}(X)$.

The definition of this functor is the starting point of a beautiful extraordinary cohomology theory, which is intimately connected to our discussions, but about which we will, unfortunately, not say much in this article. To see why Definition 3.4.1 might be a very general one, consider the following specializations:

Example 3.4.3. Let $(X, \mathcal{O}_X)$ be a topological space equipped with the sheaf $\mathcal{O}_X$ of continuous real (resp. complex) valued functions on X. Then a vector bundle on $(X, \mathcal{O}_X)$ is just a continuous real (resp. complex) vector bundle on X.

Example 3.4.4. Let $(X, \mathcal{O}_X)$ be a $\mathcal{C}^r$-manifold, where $r \in \{0, 1, \ldots, \infty, \omega, \mathrm{hol}\}$.[11] equipped with the sheaf $\mathcal{O}_X$ of $\mathcal{C}^r$ real (resp. complex) valued functions on X.[12] Then a vector bundle on $(X, \mathcal{O}_X)$ is exactly a $\mathcal{C}^r$ real (resp. complex) vector bundle on X.

Example 3.4.5. Let $(X, \mathcal{O}_X)$ be a scheme (for example, a variety over a field). Then a vector bundle on $\mathcal{E} \to X$ is called an algebraic vector bundle.

Definition 3.4.1 therefore simultaneously encompasses all these different notions of vector bundles. It also makes very evident what is happening when a single topological space is equipped with different sheaves, i.e. we are "shifting structures". For instance, let X be a topological space and $\mathcal{C}^0_{\mathbb{R}}$ (resp. $\mathcal{C}^0_{\mathbb{C}}$) denote the sheaves of continuous real (resp.complex) valued functions on X. Then the natural injection of sheaves $\mathcal{C}^0_{\mathbb{R}} \hookrightarrow \mathcal{C}^0_{\mathbb{C}}$ induces a morphism of ringed spaces $\iota : (X, \mathcal{C}^0_{\mathbb{R}}) \to (X, \mathcal{C}^0_{\mathbb{C}})$ that is the identity map on the underlying space, such that the corresponding pullback

$$\iota^* : \mathrm{Vect}(X, \mathcal{C}^0_{\mathbb{C}}) \to \mathrm{Vect}(X, \mathcal{C}^0_{\mathbb{R}})$$

takes a complex vector bundle $\mathcal{E} \to X$ and forgets its complex structure, i.e. $\mathcal{E}_{\mathbb{R}} := \iota^*\mathcal{E} \to X$ remembers only the real vector bundle structure on X. This gives us a systematic way to treat complex vector bundles as real vector bundles, smooth vector bundles as continuous vector bundles, algebraic vector bundles on smooth complex varieties as holomorphic vector bundles, etc.

[10] One can take as an example the free ring $\mathbb{Z}[S]$ generated by the elements of S, written e_s for $s \in S$ in this ring, subject to relations of the form $\mathrm{e}_1 = 1, \mathrm{e}_{s+t} = \mathrm{e}_s + \mathrm{e}_t$ and $\mathrm{e}_{st} = \mathrm{e}_s\mathrm{e}_t$ for all s, t. Alternatively, you could put an equivalence relation on the set of pairs (s, t) by saying $(s, t) = (s', t')$ iff $s + t' = s' + t$, and define a multiplicative structure on the set of equivalence classes.

[11] Here, as always, C^∞ means "smooth", C^ω means "analytic", and C^{hol} means "holomorphic".

[12] Of course, only complex-valued functions are allowed when $r = \mathrm{hol}$.

In various categories, there is a way to think of the vector bundle $\mathcal{E}$ as an object $|\mathcal{E}|$ in that category equipped with a map $\pi : |\mathcal{E}| \to X$, with the corresponding locally free sheaf $\mathcal{E}$ recovered from $|\mathcal{E}|$ as the sheaf of local sections of π. In this case, this "geometric realization" $|\mathcal{E}|$ of $\mathcal{E}$ is referred to as the total space of the vector bundle $\mathcal{E}$. This is done by writing the gluing data that defines $\mathcal{E}$ by comparing transition functions across different trivializations and using this to glue together local "models" of the vector bundle (although sometimes this can be done globally in one sweep, for instance for a vector bundle $\mathcal{E} \to X$ over a scheme we have $|\mathcal{E}| := \mathrm{Spec}_X \, \mathsf{Sym}_{\mathscr{O}_X} \mathcal{E}^\vee$ via the relative Spec construction). What we have used as a definition of a vector bundle here is often referred to as the sheaf of sections of this total space. It is sometimes important to keep this distinction between locally free sheaves and their total spaces in mind; for instance, when dealing with ranks of morphisms between vector bundles.

Having said this, we will work exclusively in the topological or smooth category in this article, where working with total spaces is just as easy. Therefore, we will conflate the vector bundle $\mathcal{E}$ with its total space $|\mathcal{E}|$, and calling either one the "vector bundle". We will further restrict ourselves to working with complex vector bundles on paracompact Hausdorff spaces, although much is there to be said about real vector bundles (the framework of KO-theory) too .

Let us now develop briefly review theory of characteristic classes. There are many ways to do this–at least four of them being

(a) via representable functors and classifying spaces (which we do below),
(b) using the Leray-Hirsch Theorem (see [35, Chapter 3]),
(c) as degeneracy loci of generic sections (which is how they were historically conceived; see [10, Proposition 3.5.9] and [43, §5.2]), and
(d) via Chern-Weil Theory (see, for instance, [56, §23]).

For certain special cases, we may also use other techniques; for instance, we can construct the Chern classes via the Euler class (see Remark 3.4.23) and the Stiefel-Whitney classes via the Steenrod square operations (see [42, §4]). Perhaps the cleanest and pedagogically most effective approach to defining characteristic classes and showing the equivalence of these definitions is to first characterize them axiomatically, à la Grothendieck (see [35, Chapter 3]).

We will follow a slightly different route–namely that of (a). Let Top denote the category of paracompact Hausdorff spaces with continuous maps between them, and HoTop the corresponding naive homotopy category, i.e. the category whose objects are the same as in Top but morphisms are homotopy classes of continuous maps, so that there is a "forgetful" functor $F : \mathsf{Top} \to \mathsf{HoTop}$ remembering only the homotopy type of a space. Then isomorphisms in HoTop are, by definition, homotopy equivalences of spaces.

Now consider the functor $\mathsf{Top} \to \mathsf{RingSp}$ given by equipping a space X with the sheaf of continuous complex-valued functions on X, and hence a (complex) vector bundle functor

$$\mathrm{Vect} : \mathsf{Top}^{\mathrm{op}} \to \mathsf{GrSemiRing}. \tag{3.3}$$

The first fundamental result in the theory is

Proposition 3.4.6 (Homotopy Invariance of Vector Bundles)**.** The functor (3.3) factors through HoTop, i.e. there is a functor

$$\mathrm{Vect}^{\circ} : \mathsf{HoTop}^{\mathrm{op}} \to \mathsf{GrSemiRing}$$

and a natural isomorphism $\mathrm{Vect}^{\circ} \circ F \Rightarrow \mathrm{Vect}$.

Proof. See [35, Theorem 1.6]. ∎

In what follows, we will drop the little ring on Vect° and confound the two functors without a qualm. In English, Proposition 3.4.6 is saying that if $f_0, f_1 : Y \to X$ are homotopic maps and $E \to Y$ a vector bundle, then the pullbacks f_0^*E and f_1^*E of E via f_0 and f_1 on Y are isomorphic. In particular, every vector bundle over a contractible space is trivial. This can be shown by reducing to the "universal" case of when $X = Y \times [0,1]$, and f_t denotes the inclusion $Y \cong Y \times \{t\} \hookrightarrow Y \times [0,1]$, and then using the compactness of $[0,1]$ to construct a sequence of isomorphisms, a suitable limit of which provides the required isomorphism. When Y and E are smooth, this can then also be shown using parallel transport with respect to a connection on E, and then the result amounts to the continuous dependence of the solutions to differential equations on initial conditions.

The second fundamental result is

Proposition 3.4.7 (Representability of Vector Bundle Functor)**.** For each integer $n \geq 0$, the functor $\mathrm{Vect}^n_{\mathbb{C}} : \mathsf{HoTop}^{\mathrm{op}} \to \mathsf{Set}$ taking a space X to the set of isomorphism classes of rank n complex vector bundles on it is **representable**, i.e. there is space BU_n, necessarily unique up to homotopy equivalence, and a natural isomorphism of functors

$$\mathsf{HoTop}(-, \mathrm{BU}_n) \Rightarrow \mathrm{Vect}^n_{\mathbb{C}}$$

Proof. See [35, Theorem 1.16]. ∎

In fact, an explicit model for the homotopy space BU_n representing rank n-vector bundles is the infinite Grassmannian $\mathrm{Gr}^{n,\infty}_{\mathbb{C}}$ of n-planes in $\mathbb{C}^{\infty}$; see also Appendix 3.7. In this case, the natural isomorphism above is not hard to describe; indeed, if $\gamma^n := \mathcal{O}_{\mathrm{Gr}^{n,\infty}_{\mathbb{C}}}(-1) \to \mathrm{Gr}^{n,\infty}_{\mathbb{C}}$ is the tautological n-plane bundle over $\mathrm{Gr}^{n,\infty}_{\mathbb{C}}$, then for any space X, the above natural isomorphism is given by

$$\mathsf{HoTop}(X, \mathrm{Gr}^{n,\infty}_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Vect}^n_{\mathbb{C}}(X), \quad [f] \mapsto f^*(\gamma^n),$$

with the fact that this is well-defined being a consequence of Proposition 3.4.6. Note that when $n = 1$, this is saying that complex line bundles are classified by the space $\mathrm{BU}_1 \simeq \mathrm{Gr}^{1,\infty}_{\mathbb{C}} = \mathbb{CP}^{\infty}$.

Remark 3.4.8. The representability of Vect_n falls into the more general framework of representability of principal G-bundles, which we now briefly explain. Given any topological group G, we can consider the functor

$$\mathrm{Bun}_G : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Set}$$

which takes a space X to the set of isomorphism classes of principal G-bundles on X, i.e. fibre bundles $G \to P \to X$ such that G acts on the fibre via right multiplication as the structure group. This functor again factors through

$$\mathrm{Bun}_G : \mathsf{HoTop}^{\mathrm{op}} \to \mathsf{Set},$$

and the resulting functor is again representable, via the general framework of Brown representability. This space representing it, which is unique up to homotopy equivalence, is then called the **classifying space** for the group G. Often, when working with Lie groups G, it suffices to restrict to maximal compact subgroups $H \subset G$, since the inclusion $H \hookrightarrow G$ is then a homotopy equivalence; applying this to $G = \mathrm{GL}_n \mathbb{C}$ with $\mathrm{Bun}_{\mathrm{GL}_n \mathbb{C}} \cong \mathrm{Vect}_n$ and $\mathrm{U}_n \subset \mathrm{GL}_n \mathbb{C}$ explains our choice of notation for the space representing Vect_n. The following remarks about characteristic classes of vector bundles can be made very similarly for principal bundles; for Lie groups G, this can be achieved on smooth manifolds also via Chern-Weil Theory of principal G-bundles.

With this terminology, we can now define a characteristic class.

Definition 3.4.9. A **characteristic class** of complex vector bundles of rank n with coefficients in a ring R is a natural transformation

$$\mathrm{Vect}^n_{\mathbb{C}} \Rightarrow \mathrm{H}^*(-; R),$$

where $\mathrm{H}^*(-; R)$ denotes (say) singular cohomology with coefficients in R.

In light of Proposition 3.4.7 and the Yoneda Lemma, characteristic classes of complex vector bundles of rank n are in bijection with the cohomology ring

$$\mathrm{H}^*(\mathrm{BU}_n; R),$$

reducing the study of characteristic classes to that of cohomology rings of Grassmannians; see Appendix 3.7. One way to carry out this computation is to consider the central torus $\mathbb{T}^n = \mathrm{U}_1^n \subset \mathrm{U}_n$, and the corresponding morphism $(\mathbb{CP}^\infty)^n \cong \mathrm{BU}_1^n \to \mathrm{BU}_n$. Note that principal $\mathbb{T}^n$-bundles are simply the **split** vector bundles, i.e. vector bundles that are sums of line bundles. Associated to this inclusion $\mathrm{BU}_1^n \to \mathrm{BU}_n$, we have a pullback map

$$\mathrm{H}^*(\mathrm{BU}_n; R) \to \mathrm{H}^*(\mathrm{BU}_1^n; R) = \mathrm{H}^*((\mathbb{CP}^\infty)^n; R)] \cong R[\gamma_1, \dots, \gamma_n],$$

where in the last step we have used the computation $\mathrm{H}^*(\mathbb{CP}^\infty; R) \cong R[\gamma]$ with $|\gamma| = 2$ (done using say the Gysin sequence, the Leray-Serre Spectral Sequence, or using cellular cohomology), alongside the Künneth Formula. It is then a standard result that this pullback map is injective, with its image correspond to the $\mathcal{S}_n$-invariant polynomials in $\gamma_1, \dots, \gamma_n$. Precisely, we get

$$\mathrm{H}^*(\mathrm{BU}_n; R) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{BU}_1^n; R)^{\mathcal{S}_n} \cong R[\gamma_1, \dots, \gamma_n]^{\mathcal{S}_n} \cong R[c_1, \dots, c_n],$$

where the c_j are elementary symmetric polynomials in the γ_j's. The classes c_j so defined are called the (universal) **Chern classes**, and γ_j the (universal) **Chern roots**. For a specific vector bundle $E \to X$, we can use Proposition 3.4.7 to find a map classifying it $f_E : X \to \mathrm{BU}_n$, and define for $0 \leq j \leq n$ its j^{th} Chern class with coefficients in R to be

$c_j(E;R) := f_E^* c_j$. The classifying map f_E is only well-defined up to homotopy, but by the homotopy invariance of cohomology, this Chern class is well-defined. Using this definition, one can then prove various properties about Chern classes as consequences of appropriate results in the cohomology of Grassmannians. Here we summarize the story in

Theorem 3.4.10 (Chern Classes)**.** To each complex vector bundle $E \to X$ over a paracompact Hausdorff space X and ring R, we can associate for each $j \geq 0$ a characteristic class $c_j(E;R) \in \mathrm{H}^{2j}(X;R)$ for $j \geq 0$ called the j^{th} Chern class. The Chern classes classes are uniquely characterized by the following properties:

(a) The zeroth class $c_0(E;R) = 1 \in \mathrm{H}^0(X;R)$ is the cohomology unit, and we have $c_j(E;R) = 0$ if $j > \operatorname{rank} E$.

(b) (Naturality) If $f : Y \to X$ is a continuous map, then under the induced pullback map $f^* : \mathrm{H}^*(X;R) \to \mathrm{H}^*(Y;R)$, we get for each $j \geq 0$ that

$$c_j(f^*(E)) = f^*(c_j(E)).$$

(c) (Whitney Product Formula) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles on X, then the Chern classes of E, E' and E'' are related via the product formula, which says that for each $k \geq 0$ that

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'').$$

(d) (Normalization) For the tautological bundle $\mathcal{O}_{\mathbb{CP}^1}(1) \to \mathbb{CP}^1$, we have

$$c_1(\mathcal{O}_{\mathbb{CP}^1}(1)) = \zeta \in \mathrm{H}^2(\mathbb{CP}^1;R),$$

where ζ is the hyperplane class (in this case, the Poincaré dual to a point).

These properties also give the axiomatic characterization of Chern classes mentioned above; in practice, the actual construction of Chern classes does not matter at all, and only these properties do. We will often suppress the coefficient ring in the notation and define the total Chern class of E to be

$$\mathsf{c}(E) := \sum_{j=0}^{\infty} c_j(E).$$

With this notation, the Whitney product formula says simply that if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles, then we have

$$\mathsf{c}(E) = \mathsf{c}(E') \cdot \mathsf{c}(E'') \in \mathrm{H}^*(X;R).$$

One another important remark: to work with vector bundles obtained from linear algebraic operations on existing vector bundles, we may "pretend" that all our vector bundles actually split as direct sums of line bundles. This is justified via

Theorem 3.4.11 (Splitting Principle)**.** An identity among Chern classes of bundles that is true for bundles that are direct sums of lines bundles is true in general. More precisely, if X is a paracompact Hausdorff space, and $E \to X$ is a (real or complex) vector bundle on X, then there is a space Y and a morphism $f : Y \to X$ such that

(a) for any coefficient ring R, the pullback map $f^* : \mathrm{H}^*(X;R) \to \mathrm{H}^*(Y;R)$ is injective, and
(b) the pullback bundle f^*E admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = f^*E,$$

where $r = \operatorname{rank} E$, by vector subbundles $E_j \subset f^*E$ such that for each $j = 1, \ldots, r$, the successive subquotient bundle E_j/E_{j-1} is a line bundle.

Proof. Take Y to be the complete flag bundle $\mathrm{Fl}^{1^r}(E)$ of E, and $f : \mathrm{Fl}^{1^r}(E) \to X$ the projection map. The bundles E_j are then the obvious bundles: for $0 \le j \le r$, the fiber of E_j over a complete flag $(V_1, \ldots, V_{r-1})$ is simply V_j. The only thing that remains to be shown is that $f^* : \mathrm{H}^*(X;R) \to \mathrm{H}^*(Y;R)$ is injective, but that is an easy inductive consequence of the Leray-Hirsch Theorem. ■

The proof above shows us also that if X is a smooth (resp. complex) manifold and $E \to X$ is a smooth (resp. holomorphic) vector bundle, then the splitting space Y, the morphism $f : Y \to X$, and the splitting on Y of f^*E can also be taken to be in the same category.

Having developed the theory of Chern classes, we may now also speak of other characteristic classes coming from multiplicative sequences as follows. Note that the machinery of Appendix 3.3 is developed in such a way that for any vector bundle $E \to X$, evaluation at E yields a graded ring homomorphism

$$\mathrm{eval}_E : R[c] \to \mathrm{H}^*(X;R),$$

and by functoriality of 1-units of a ring also a homomorphism

$$\mathrm{eval}_E : \mathrm{U}_1(R[c]) \to \mathrm{U}_1(\mathrm{H}^*(X;R)). \tag{3.4}$$

Definition 3.4.12.

(a) Given a multiplicative sequence (K_n) and a complex vector bundle $E \to X$, the **total K-series of E**, denoted

$$\mathsf{K}(E,z) \in \mathrm{U}_1(\mathrm{H}^*(X;R))$$

is the image of the series $\sum_{i=0}^{\infty} K_i z^i \in \mathrm{U}_1(R[c])$ under the evaluation map (3.4).
(b) If further X is of finite type (i.e. $\operatorname{rank}_{\mathbb{Z}} \mathrm{H}^*(X;\mathbb{Z}) < \infty$), then evaluating the series $\mathsf{K}(E,z)$ element $z_0 \in \mathrm{H}^*(X;R)$ gives an element in $\mathrm{H}^*(X;R)$. In particular, taking $z_0 = 1$ yields the **total K-sequence** of E, denoted

$$\mathsf{K}(E) := \mathsf{K}(E,1) \in \mathrm{H}^*(X;R).$$

Example 3.4.13. When K is the identity sequence (see Example 3.3.2), then the corresponding total K-series is denoted

$$\mathsf{c}(E,z)=\sum_{i=0}^{\infty}c_j(E)z^j$$

and is called the Chern polynomial of E. Note that, irrespective of the cohomology of the base X, this is indeed a polynomial because $c_j(E)=0$ for $j>\operatorname{rank}E$. Of course, the total K-sequence in this case is simply the total Chern clas as defined above.

Remark 3.4.14. If X is not of finite type, then the total K-sequence of E lives rather in total product $\mathrm{H}^{**}(X;R):=\prod_{i=0}^{\infty}\mathrm{H}^i(X;R)$.

Remark 3.4.15. We will often use sans serif fonts applied to the same letter as in the series to denote total K-series and classes of vector bundles.

Now, the Whitney Product Formula for Chern classes implies that for any short exact sequence

$$0\to E'\to E\to E''\to 0$$

of vector bundles on a space X and multiplicative sequence K, we have the product formula

$$\mathsf{K}(E,z)=\mathsf{K}(E',z)\cdot\mathsf{K}(E'',z)\in\mathrm{U}_1(\mathrm{H}^*(X;R));$$

indeed, the definition of multiplicative sequences is set-up to make this true. This formula, along with the Splitting Principle (Theorem 3.4.11), gives us a recipe to compute these series: if E has rank $r\geq 0$, and we formally factor the Chern polynomial as

$$\mathsf{c}(E,z)=\prod_{i=1}^{r}(1+\gamma_i z),$$

so that the γ_i are the Chern roots of E, then the total K-series of E is given by

$$\mathsf{K}(E,z)=\prod_{i=1}^{r}Q(\gamma_i z),$$

where Q is the characteristic series corresponding to K.

Example 3.4.16. If $E\to X$ is a complex vector bundle with Chern roots γ_i, then, respectively, the total Pontryagin, Todd, L-, generalized Todd, and $\hat{A}$-series of E are

$$\begin{aligned}
\mathsf{p}(E,z)&=\prod_i(1+\gamma_i^2z^2),\\
\mathsf{Td}(E,z)&=\prod_i\gamma_i z/(1-\mathrm{e}^{-\gamma_i z}),\\
\mathsf{L}(E,z)&=\prod_i\gamma_i z/\tanh(\gamma_i z),\\
\mathsf{Td}_y(E,z)&=\prod_i\gamma_i z(1+y\mathrm{e}^{-\gamma_i z(1+y)})/(1-\mathrm{e}^{-\gamma_i z(1+y)}),\ \text{and}\\
\hat{\mathsf{A}}(E,z)&=\prod_i(\gamma_i z/2)/\sinh(\gamma_i z/2).
\end{aligned}$$

Evaluating these at $z=1$ yields, respectively, the total Pontryagin, Todd, L-, generalized Todd, and $\hat{A}$-classes of E.

Suppose now that X is a closed, R-orientable manifold of dimension $2n \geq 0$, so that the top cohomology $\mathrm{H}^{2n}(X;R)$ is cyclic, generated by a specified generator η_X coming from the orientation. Then, as noted in Conventions and Fundamentals, there is an evaluation map

$$\int_X : \mathrm{H}^*(X;R) \to R$$

which sends a cohomology class to the coefficient of η_X in its degree $2n$ component.

Definition 3.4.17. In the above set-up, given a complex vector bundle $E \to X$ and multiplicative sequence K, we define the K-characteristic of E to be

$$K(E) := \int_X \mathsf{K}(E) \in R.$$

The K-characteristic of E is sometimes also referred to as the "K-genus" of E. In particular, in light of the previous examples, we have now defined the Chern genus, Pontryagin genus, Todd genus, L-genus, generalized Todd genus and $\hat{A}$-genus of a complex vector bundle on a closed orientable R-manifold, where the R are as in the definitions of these multiplicative sequences.

Remark 3.4.18. We also adopt the following convention: if X is a closed complex manifold and K a multiplicative sequence, then by the total K-series, total K-class, or K-genus of X, we mean the corresponding quantity for its holomorphic tangent bundle $\mathscr{T}X$. These are denoted by $\mathsf{K}(X,z), \mathsf{K}(X)$, and $K(X)$ respectively.[13]

So far, we have only spoken about characteristic classes of complex vector bundles; let's now speak about characteristic classes of real vector bundles. Of these, two of primary importance are the Stiefel-Whitney classes (w_j) and the Pontryagin classes (p_j).

The discussion for Stiefel-Whitney classes is very similar to the one for Chern classes above, where we work instead with real vector bundles and $\mathbb{F}_2$-coefficients. Again, we have a natural isomorphism $\mathsf{HoTop}(-,\mathrm{BO}_n) \Rightarrow \mathrm{Vect}^n_{\mathbb{R}}$, where $\mathrm{BO}_n \cong \mathrm{Gr}^{n,\infty}_{\mathbb{R}}$ is the infinite real Grassmannian. As before, we may define characteristic classes of real vector bundles of rank $n \geq 0$ with coefficients in an $\mathbb{F}_2$-algebra R as a natural transformation $\mathrm{Vect}^n_{\mathbb{R}} \Rightarrow \mathrm{H}^*(-;R)$, and these are again in bijection with elements of the cohomology ring $\mathrm{H}^*(\mathrm{BO}_n;R)$. The key difference, here, now is that this last cohomology ring is given by

$$\mathrm{H}^*(\mathrm{BO}_n;R) = R[w_1,\ldots,w_n],$$

where the $|w_j| = j$ for $1 \leq j \leq n$ and the w_j are called the universal Stiefel-Whitney classes. A theorem analogous to Theorem 3.4.10 still holds, and the Splitting Principle (Theorem 3.4.11) applies verbatim. We will not develop this theory again, but rather refer the reader to [42, §4] or [35, Chapter 3] for more on Stiefel-Whitney classes. The one result that we will need and use will be the relationship between Chern and Stiefel-Whitney classes, namely

[13] This notation $K(X)$ has nothing to do with the K-theory of X. Usually, we will work with specific multiplicative sequences, so no confusion should arise from this overloaded notation.

Theorem 3.4.19. Let $E \to X$ be a complex vector bundle, and $E_{\mathbb{R}} \to X$ its underlying real vector bundle. Then the reduction mod 2 of coefficients takes the total Chern class of E to the total Stiefel-Whitney class of $E_{\mathbb{R}}$, i.e.

$$\mathrm{H}^{**}(X; \mathbb{Z}) \ni \mathsf{c}(E) \mapsto \mathsf{w}(E_{\mathbb{R}}) \in \mathrm{H}^{**}(X; \mathbb{Z}/2).$$

In other words, we have for each $j \geq 0$ that

$$w_{2j+1}(E) = 0 \text{ and } w_{2j}(E) \equiv c_j(E) \pmod 2.$$

Proof. See [35, Proposition 3.8]. ■

The Pontryagin classes, on the other hand, can be described in terms of the Chern classes, as in

Definition 3.4.20. Let $E \to X$ be a (topological) real vector bundle of rank $n \geq 0$. Then for each $i \geq 0$ and coefficient ring R, we define the i^{th} Pontryagin class of E with coefficients in R via

$$p_i(E; R) := (-1)^i c_{2i}(E_{\mathbb{C}}; R) \in \mathrm{H}^{4i}(X; R),$$

where $E_{\mathbb{C}} = E \otimes \mathbb{C}$ is the complexification of E.

In what follows, as before, we will often suppress the coefficient ring R, although it is not unimportant. Note that we have intentionally left it ambiguous in the above definition what we mean by $i \geq 0$. Certainly, $p_i(E; R) = 0$ for all $i > (\operatorname{rank} E)/2$. The definition certainly works for integer i, but it can also be somewhat made sense of (without the factor of $(-1)^i$ say) when i is a half-integer. Note that since $E_{\mathbb{C}} \cong \overline{E}_{\mathbb{C}}$, we have for each $j \geq 0$ that

$$c_j(E_{\mathbb{C}}) = c_j(\overline{E_{\mathbb{C}}}) = (-1)^j c_j(E_{\mathbb{C}}),$$

so that if j is odd, then

$$2c_j(E_{\mathbb{C}}) = 0.$$

This tells us that when j is odd, $c_j(E_{\mathbb{C}}; R)$ is 2-torsion, and hence there is a notion of half-integer Pontryagin classes, which are 2-torsion classes. This small technicality about half-integer Pontryagin classes implies also that if

$$0 \to E' \to E \to E'' \to 0$$

is a short exact sequence of real vector bundles, then for each $k \geq 0$, the product formula

$$p_k(E) = \sum_{i+j=k} p_i(E') \cdot p_j(E'') \tag{3.5}$$

does not hold on the nose, but only modulo 2 torsion.

Therefore, when working with Pontryagin classes, it is often convenient to take the coefficient ring R to be a $\mathbb{Z}[1/2]$-algebra (e.g. a field of characteristic other than 2), in which case it is true that $c_j(E_\mathbb{C}) = 0$ for odd j and the product formula (3.5) does hold on the nose. In this case, we define the **total Pontryagin class** of the bundle E to be

$$\mathsf{p}(E) = \sum_{j=0}^{\infty} p_j(E).$$

Now, if we had worked with the reduced convention (see Remark 3.3.7), then evaluation at a vector bundle E would yield again a graded ring homomorphism

$$\mathrm{eval}_E : R[p] \to \mathrm{H}^*(X; R).$$

Working with reduced sequences $\tilde{K}$ then allows us to define, as before, then total $\tilde{K}$-series, class, and genus of a vector bundle E.

Remark 3.4.21. Similarly to the complex case (see Remark 3.4.18), whenever we speak of the Stiefel-Whitney classes or $\tilde{K}$-series, classes, or genera of real smooth manifolds X, we always mean the corresponding quantity for its smooth tangent bundle $\mathrm{T}X$. In particular, we have now defined the Pontryagin, L- and $\hat{A}$-genera of a manifold X. Note that these all vanish unless $\dim X \equiv 0 \pmod 4$.

Remark 3.4.22. Note that if $E \to X$ is a complex vector bundle, then we now have two apparently distinct definitions of its total Pontryagin series (resp. class, resp. genus): namely as its total K-class corresponding to the series $Q(z) = 1 + z^2$, and as the total Pontryagin class $\mathsf{p}(E_\mathbb{R})$ of its underyling real vector bundle defined above. We end this section by noting that these two definitions coincide. Indeed, if E is a complex vector bundle, then we have

$$(E_\mathbb{R})_\mathbb{C} \cong E \oplus \overline{E}$$

as complex vector bundles, so that

$$\sum_{j=0}^{\infty} (-1)^j p_j(E_\mathbb{R}) = \left(\sum_{i=0}^{\infty} c_i(E) \right) \left(\sum_{i=0}^{\infty} (-1)^i c_i(E) \right),$$

where on the left side we have used the definition of the Pontryagin classes, and on the right side we have used the Whitney product formula along with the result that $c_i(\overline{E}) = (-1)^i c_i(E)$ for all $i \geq 0$, which is an immediate consequence of the splitting principle and the fact that for any complex line bundle L we have $c_1(\overline{L}) = c_1(L^\vee) = -c_1(L)$. The equivalence of the two definitions $\mathsf{p}(E_\mathbb{R})$ and $\mathsf{p}(E)$ then follows from the formula (3.2).

Remark 3.4.23. As a final remark, we note that if $E \to X$ is a complex vector bundle of rank n, then the underlying real vector bundle $E_\mathbb{R}$ is canonically oriented, and we then have

$$c_n(E) = \mathrm{e}(E_\mathbb{R}).$$

This result is clear when E is a line bundle (from $\mathrm{U}_1 = \mathrm{SO}_2$, or from considering the "universal case" of $\mathcal{O}_{\mathbb{CP}^1}(-1) \to \mathbb{CP}^1$), and follows from the line bundle case by the splitting principle: if $E \cong \bigoplus_i L_i$, with L_i line bundles, then

$$c_n(E) = \prod_i c_1(L_i) = \prod_i \mathrm{e}((L_i)_\mathbb{R}) = \mathrm{e}(E_\mathbb{R}).$$

This equality can also be taken as an alternative definition of the top Chern class, with the lower Chern classes defined inductively to satisfy Theorem 3.4.10. This is, for instance, the approach adopted in [42, §14].

3.5 Multiplicativity of Genera

It is often the case that, given a fibre bundle $F \to X \to B$ of spaces, numerical invariants (such as genera) of F, X, and B are related via a multiplicative formula of the form

$$I(X) = I(B) \cdot I(F),$$

where I stands for a generic numerical invariant. We will now state precise formulations of several instances of this general principle, proving some of them, and delegating proofs of others to references. The core of these proofs usually relies on some version of the Leray-Serre spectral sequence associated to this fibre bundle. Firstly, we have

Theorem 3.5.1. Let $F \to X \xrightarrow{\pi} B$ be a fibre bundle of closed manifolds. Then

$$\chi(X) = \chi(F) \cdot \chi(B).$$

Proof 1 of Theorem 3.5.1. If $X = B \times F$ is the product bundle, then the result follows from the Künneth Formula or equivalently Leray-Hirsch Theorem in (say) rational cohomology. In general, the result follows from the product case (or even simply the homotopy invariance of the Euler characterstic) by using the Mayer-Vietoris sequence to induct on the cardinality of a finite good cover (in the sense of [9, §5]) of B that trivializes the bundle $F \to X \to B$.[14] ■

To give the second proof, we quickly review the theory of relative tangent bundles and integration along fibers, following say [9, §6]. If $\pi : X \to B$ is any submersion of smooth manifolds, then the total differential $\mathrm{d}\pi : \mathrm{T}X \to \pi^*\mathrm{T}B$ of π is a surjective morphism of vector bundles on X, the kernel of which, denoted T_π, is called the relative tangent bundle of π. In other words, we have by definition a short exact sequence of vector bundles on X given by

$$0 \to \mathrm{T}_\pi \to \mathrm{T}X \xrightarrow{\mathrm{d}\pi} \pi^*\mathrm{T}B \to 0. \tag{3.6}$$

Further, for any $b \in B$, the fiber $X_b := \pi^{-1}(b) \subset X$ is an embedded submanifold of codimension equal to $\dim B$, and the restriction of T_π to the fiber recovers the tangent bundle of X_b, i.e. for all $b \in B$ we have

$$\mathrm{T}_\pi|_{X_b} \cong \mathrm{T}X_b.$$

Finally, if X and B are closed, oriented manifolds, then integration along fibers gives rise to the Gysin homomorphism

$$\int_{X/B} : \mathrm{H}^*_{\mathrm{dR}}(X;\mathbb{R}) \to \mathrm{H}^{*-r}_{\mathrm{dR}}(B;\mathbb{R}),$$

where $r := \dim X - \dim B$, which satisfies the push-pull formula

$$\int_X \alpha \wedge \pi^*\beta = \int_B \left(\int_{X/B} \alpha \right) \wedge \beta \tag{3.7}$$

for any $\alpha \in \mathrm{H}^s_{\mathrm{dR}}(X;\mathbb{R})$ and $\beta \in \mathrm{H}^t_{\mathrm{dR}}(B;\mathbb{R})$ with $s, t \geq 0$ such that $s + t = \dim X$. Now we can give

[14] This last statement is, of course, a statement about Čech cohomology of finite good covers of B and X, and the equivalence of singular and Čech cohomology for manifolds.

Proof 2 of Theorem 3.5.1, when X, B and F are oriented and $F \to X \to B$ is an oriented fibre bundle. By Poincaré duality, the formula is only saying something nontrivial if $\dim B \equiv \dim F \equiv 0 \pmod 2$, which we assume hence. In this case, the sequence (3.6) along with the multiplicativity of the Euler class implies the formula

$$\mathrm{e}(\mathrm{T}X) = \mathrm{e}(\mathrm{T}_\pi) \cdot \pi^* \mathrm{e}(\mathrm{T}B),$$

where the wedge product in this case is symmetric because of the dimension hypothesis, so we need not worry about the order of multiplication. Integrating over X and using the push-pull formula (3.7) then yields

$$\int_X \mathrm{e}(\mathrm{T}X) = \int_B \left(\int_{X/B} \mathrm{e}(\mathrm{T}_\pi) \right) \mathrm{e}(\mathrm{T}B).$$

Note that $\int_{X/B} \mathrm{e}(T_\pi) \in \mathrm{H}^0(B; \mathbb{R}) \cong \mathbb{R}$ is just a number[15], and indeed by definition of integration along the fiber and the Chern-Gauss-Bonnet Theorem (Theorem 1.2.1) applied to each fiber $F = X_b$, this number is nothing but $\chi(F)$. Applying Chern-Gauss-Bonnet two more times–to X and B–then yields

$$\chi(X) = \int_X \mathrm{e}(\mathrm{T}X) = \int_B \chi(F) \cdot \mathrm{e}(\mathrm{T}B) = \chi(F) \int_B \mathrm{e}(\mathrm{T}B) = \chi(F) \cdot \chi(B).$$

■

Proof 3 of Theorem 3.5.1. This third proof works more generally for topological fiber bundles, but under the assumption that local system $\boldsymbol{\mathcal{H}}^*(F)$ of coefficients on the base B coming from this fibration is trivial, i.e. that the fundamental group $\pi_1(B)$ acts trivially on the cohomology $\mathrm{H}^*(F)$. Then we may form the Leray-Serre Spectral Sequence coming from this fibration, say with rational coefficients, which has E_2 page given by

$$E_2^{p,q} = \mathrm{H}^p(B; \boldsymbol{\mathcal{H}}^*(F; \mathbb{Q})) \cong \mathrm{H}^p(B; \mathrm{H}^q(F; \mathbb{Q})) \cong \mathrm{H}^p(B; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^q(F; \mathbb{Q}),$$

and which abuts to $\mathrm{H}^*(X; \mathbb{Q})$. It follows from this that the $E_2^{p,q}$ page is finitely supported. In general for a bigraded spectral sequence $E = (E_r^{p,q})$ of rational vector spaces which is eventually finitely supported (i.e. there is an $r \geq 1$, such that $E_r^{p,q} = 0$ for all but finitely many p, q), then we define its Euler characteristic to be

$$\chi_r(E) = \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} E_r^{p,q}.$$

By standard linear algebra (the rank-nullity theorem!), we have then that

$$\chi_r(E) = \chi_{r+1}(E) = \cdots = \chi_\infty(E).$$

In our case, the Leray-Serre spectral sequence E is finitely supported for $r \geq 2$ and that

$$\begin{aligned}
\chi(F) \cdot \chi(B) &= \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{Q}} \left(\mathrm{H}^p(B; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^q(F; \mathbb{Q}) \right) \\
&= \chi_2(E) \\
&= \chi_\infty(E) \\
&= \sum_n (-1)^n \dim_{\mathbb{Q}} \mathrm{H}^*(X; \mathbb{Q}) = \chi(X),
\end{aligned}$$

as needed. ■

[15] Here we are imlicitly assuming that B is connected. If it is not, we may work over each component separately to get the same result.

Remark 3.5.2. There are certainly other ways to prove Theorem 3.5.1 as well that apply to various other contexts (e.g. finite CW complexes, or more generally spaces of finite type). For instance, for fiber bundles over finite CW complexes, one could proceed by induction on the dimension of the base and analyze what happens when attaching exactly one cell at a time, noting that the fibration over an open cell (or more generally any contractible base) must necessarily be homotopically trivial. We can also use a variation of the argument in Proof 3 to get rid of the assumption that the local system $\boldsymbol{\mathcal{H}}^*(\boldsymbol{\mathcal{F}})$ be trivial; namely, using $\mathbb{Z}/2$ coefficients, since the image of $\pi_1(B) \to \mathrm{Aut}(\mathrm{H}^*(F;\mathbb{Z}/2))$ is finite (note that F is a space of finite type), we may pull the fibration back along the corresponding finite cover of B, on which the required triviality of the local system holds by construction. We may then finish by using Corollary 3.5.3, which can also be proven more easily by direct means (e.g. by lifting a CW structure on the base to one on the cover). Note also the proofs obtained by specializing Theorem 3.5.4 (resp. Corollary 3.5.7) to $t = -1$ (resp. $y = -1$) in the special cases of the specified additional hypotheses on the bundle.

Corollary 3.5.3. If $X \to B$ is a covering map of closed manifolds of degree d, then

$$\chi(X) = d \cdot \chi(B).$$

Proof. Apply the previous theorem to the fibre bundle $\mathbb{Z}/d \to E \to B$. ■

Next, we have

Theorem 3.5.4. Let $F \to X \xrightarrow{\pi} B$ be a fibre bundle of spaces of finite type and k a field such that the conditions of the Leray-Hirsch Theorem are satisfied, i.e. the cohomology $\mathrm{H}^*(F;k)$ is free, and there are global classes $\alpha_1, \ldots, \alpha_N \in \mathrm{H}^*(X;k)$ such that for each $b \in B$, the restriction of α_j to the fiber $F_b = \pi^{-1}(b)$ freely generate $\mathrm{H}^*(F_b;k)$. Then we have the multiplicativity of k-Poincaré polynomials in this bundle, i.e.

$$p_t(X;k) = p_t(F;k) \cdot p_t(B;k) \in \mathbb{Z}[t].$$

Proof. This follows immediately from the conclusion of the Leray-Hirsch Theorem, which says that $\mathrm{H}^*(X;k)$, thought of as a $\mathrm{H}^*(B;k)$-module via the pullback π^*, is freely generated as a module by the classes $\alpha_1, \ldots, \alpha_N$, along with the fact that

$$p_t(F;k) = \sum_{i=1}^{N} t^{|\alpha_i|}.$$

■

One important special case when the hypotheses of the Leray-Hirsch Theorem are satisfied is when $X = \mathrm{Fl}^m E \to B$ is the flag bundle of some type m for some vector bundle $E \to B$. Note how plugging in $t = -1$ in Theorem 3.5.4 recovers the result of Theorem 3.5.1 in this special case, but says more in general.

Remark 3.5.5. We cannot expect the multiplicativity of the Poincaré polynomial for arbitrary smooth fibre bundles of closed manifolds. For instance, for the Hopf fibration $S^1 \to S^3 \to S^2$, we have

$$p_t(S^3) = 1 + t^3 \neq (1+t^2)(1+t) = p_t(S^2) \cdot p_t(S^1).$$

The problem here is that the Leray-Serre spectral sequence has nontrivial differentials.

A similar spectral sequence argument in $\overline{\partial}$-cohomology also works for relating the χ_y-characteristics of vector bundles holomorphic on spaces in holomorphic fibre bundles (see §1.4 for the relevant definitions). Here we state without proof

Theorem 3.5.6. Let $F \to X \xrightarrow{\pi} B$ be a holomorphic fibre bundle of connected closed complex manifolds with connected structure group and F Kählerian. Then for any vector bundle $E \to B$, we have

$$\chi_y(X, \pi^* E) = \chi_y(F) \cdot \chi_y(B, E).$$

Proof. See [13, Appendix Two]. ■

Corollary 3.5.7. Let $F \to X \xrightarrow{\pi} B$ be a holomorphic fibre bundle of connected closed complex manifolds with connected structure group and F Kählerian. Then we have

$$\chi_y(X) = \chi_y(F) \cdot \chi_y(B).$$

Proof. Take $E = \underline{\mathbb{C}} \to B$ to be the trivial bundle in Theorem 3.5.6. ■

Plugging in $y = -1$ in Corollary 3.5.7 recovers Theorem 3.5.1 in the special case of the hypothesis. We end by stating, again without proof, a couple of other multiplicativity results along these lines. We will not use these results in this text.

Theorem 3.5.8 (Chern-Hirzebruch-Serre)**.** Let $F \to X \to B$ be an oriented fibre bundle of connected closed oriented manifolds. Suppose that the fundamental group $\pi_1(B)$ of the base acts trivially on the cohomology $\mathrm{H}^*(F)$ of F. Then the signatures of F, X, and B are related as

$$\mathrm{Sign}(X) = \mathrm{Sign}(F) \cdot \mathrm{Sign}(B).$$

Proof. Again a spectral sequence argument, although more involved; see [59]. ■

Theorem 3.5.9 (Bott-Taubes)**.** Let $F \to X \to B$ be a fibre bundle of closed manifolds with compact, connected Lie structure group and F spin. Then

$$\hat{A}(X) = \hat{A}(F) \cdot \hat{A}(B).$$

Proof. Indeed, this is true more generally of the Witten genus (i.e. the universal elliptic genus), and is a manifestation of the rigidity of the elliptic genus; see [60]. ■

3.6 Almost Complex Structures on Manifolds

Note that a complex manifold of dimension $n \geq 0$ is, in particular, an orientable real smooth manifold of dimension $2n$. A fundamental question we can ask in the theory of manifolds is–given an orientable real smooth manifold of even dimension, when is it the underlying real manifold of a complex manifold, i.e. when can it be given a complex structure? One necessary condition is easy to give. If X is a complex manifold of dimension n, then the real tangent bundle $\mathrm{T}X$ of X is the underlying real vector bundle of a complex vector bundle, namely the holomorphic tangent bundle $\mathscr{T}X$ on X. Therefore, we may ask more generally: given a smooth real $2k$-plane bundle $E \to X$, when is E the underlying real vector bundle of a complex vector bundle, i.e. when is there a complex vector bundle $F \to X$ such that $E \cong F_{\mathbb{R}}$? Suppose that this is the case. Then multiplication by $\mathrm{i} = \sqrt{-1}$ defines a real-linear endomorphism of F that squares to $-\operatorname{id}_F$. Therefore, under the isomorphism $E \cong F_{\mathbb{R}}$, the real vector bundle E also admits a real linear endomorphism $J : E \to E$ such that $J^2 = -\operatorname{id}_E$. This motivates the following definition.

Definition 3.6.1. Given a real vector bundle $E \to X$ on a space X, a complex structure on E is a vector bundle endomorphism $J : E \to E$ such that $J^2 = -\operatorname{id}_E$.

Therefore, a necessary condition for a real vector bundle E to be the underlying vector bundle of a complex vector bundle is that it must admit an almost complex structure. This condition is also sufficient, and indeed we can define multiplication by i on the fiber E_x of E at $x \in X$ by J_x, and check that this turns E into a complex vector bundle. In this language, a necessary condition for a real manifold X to be the underlying real manifold of a complex manifold is that its tangent bundle $\mathrm{T}X$ must admit a complex structure. This condition has a name.

Definition 3.6.2. An almost complex structure on a real smooth manifold X is a complex structure on its tangent bundle $\mathrm{T}X$, i.e. a real vector bundle endomorphism $J : \mathrm{T}X \to \mathrm{T}X$ such that $J^2 = -\operatorname{id}_{\mathrm{T}X}$.

A smooth manifold X is said to be almost complex if it admits an almost complex structure. We sometimes say that this almost complex structure J is compatible with the smooth structure of X. Note that the existence of almost complex structures already has many consequences. We list them briefly; for a more detailed discussion, see [21, Part V]. An almost complex structure implies a decomposition of the complexification $\mathrm{T}X_{\mathbb{C}}$ of the tangent bundle into the J-holomorphic and J-antiholomorphic parts $\mathrm{T}^{1,0}X$ and $\mathrm{T}^{0,1}X$ respectively, and then a similarly decomposition on the cotangent bundle and the corresponding exterior powers as well. From this, we get for each pair (p, q) a bundle $\Omega^{p,q}(X)$ of (p, q)-forms with respect to J, nonzero only for $p, q \leq n$, and we have corresponding ∂_J and $\overline{\partial}_J$ operators, with $\partial_J : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\overline{\partial}_J : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$. Already, this is a nontrivial necessary condition, and there are manifolds that do not admit almost complex structures: examples include $\mathbb{CP}^2 \# \mathbb{CP}^2$ (Corollary 3.6.9) or more generally $\#^{2m}\mathbb{CP}^{2n}$ for any $m, n \geq 1$ (Example 3.6.7), the spheres S^{2n} for $n = 2$ and $n \geq 4$ (Theorem 2.1.4), and quaternionic projective spaces $\mathbb{HP}^n$ for $n \geq 1$ (Theorem 2.2.23).

However, this is still not quite a sufficient condition, because it still does not give us a way to construct holomorphic coordinates, or equivalently holomorphic functions, on X–indeed, in general, there are many holomorphic maps into X (this is the key principle underlying the theory of pseudoholomorphic or J-holomorphic curves), but no holomorphic functions out of X. When an almost complex structure J on a manifold X comes from an actual complex structure, so we can give holomorphic coordinates on X compatible with J, we say that J is integrable. Not all almost complex structures are integrable; for instance, we show in Proposition 2.1.7 that the almost complex structure constructed on S^6 using the octonions is a nonintegrable due to the failure of associativity of the octonions. There also exist manifolds which admit almost complex structures, but no complex structures, the simplest example being perhaps $\#^3\mathbb{CP}^2$, i.e. the connect sum of three complex projective planes with their usual orientation, as first shown by Van de Ven in [61].

The definitive theorem on the integrability of almost complex structures is the Newlander-Nirenberg Theorem. To state this theorem, we first introduce the Nijenhuis tensor $\mathcal{N}$ on an almost complex manifold (X, J), which is a tensor of type $(1, 2)$ on X and is defined by

$$\mathcal{N}(\xi, \eta) := [J\xi, J\eta] - J[\xi, J\eta] - J[J\xi, \eta] - [\xi, \eta] \tag{3.8}$$

for local vector fields ξ, η on X. It is easy to see that this is a well-defined tensor, i.e. for any $x \in X$, the value $\mathcal{N}(\xi, \eta)_x \in \mathrm{T}_x X$ of it at x depends only on ξ_x and η_x. Now we are ready to state

Theorem 3.6.3 (Newlander-Nirenberg)**.** Let (X, J) be an almost complex manifold. Then the following are equivalent:

(a) (X, J) is a complex manifold, i.e. J is integrable.
(b) We have $[\mathrm{T}^{1,0}X, \mathrm{T}^{1,0}X] \subset \mathrm{T}^{1,0}X$, i.e. the Lie bracket of two J-holomorphic vector fields is J-holomorphic.
(c) The Nijenhuis tensor $\mathcal{N}$ of (X, J) vanishes identically.
(d) The de Rham differential d is the sum $\mathrm{d} = \partial_J + \overline{\partial}_J$.
(e) We have $\overline{\partial}_J^2 = 0$.

Proof. This is a hard theorem and its proof uses deep analytic techniques. For a discussion of this result and references to several proofs, see [21, Theorem 15.4]. ■

In the rest of this appendix, we present two ways to study the existence of almost complex structures on manifolds: using obstruction theory, and using a criterion of Hirzebruch. Firstly, we have

Theorem 3.6.4. Let X be a smooth orientable manifold of real dimension $2n$. Then the obstructions to the existence of an almost complex structure on X lie in the groups

$$\mathrm{H}^*(X; \pi_{*-1}(\mathrm{SO}_{2n} / \mathrm{U}_n)).$$

In particular, if for $1 \leq j \leq 2n$, we have $\mathrm{H}^j(X; \pi_{j-1}(\mathrm{SO}_{2n} / \mathrm{U}_n)) = 0$, then X admits an almost complex structure. In this case, the obstruction to the uniqueness (up to homotopy) of this almost complex structure lies in

$$\mathrm{H}^{2n}(X;\pi_{2n}(\mathrm{SO}_{2n}/\mathrm{U}_n))\cong\pi_{2n}(\mathrm{SO}_{2n}/\mathrm{U}_n),$$

so that, if this group is trivial well, then there is a unique almost complex structure on X up to homotopy, i.e. the space of almost complex structures on X is (path) connected.

Proof. Given a real vector space V of dimension $2n$, the set

$$\mathrm{Comp}(V) := \{J\in\mathrm{GL}_{\mathbb{R}}(V) : J^2 = -\mathrm{id}_V\}\subset\mathrm{GL}_{\mathbb{R}}(V)$$

of complex structures on V is a closed submanifold of dimension $2n^2$; indeed, it is diffeomorphic to the homogenous space $\mathrm{GL}_{\mathbb{R}}(V)/\mathrm{GL}_{\mathbb{C}}(V,J)$ for any given $J\in\mathrm{Comp}(V)$. This manifold has two connected components $\mathrm{Comp}^{\pm}(V)$ corresponding to the two orientations on V, since a complex structure naturally determines an orientation.[16] If V is further equipped with an inner product, then we may also consider the manifold

$$\mathrm{SOComp}(V) := \{J\in\mathrm{SO}(V) : J^2 = -\mathrm{id}_V\}\subset\mathrm{Comp}^+(V)$$

of orthogonal complex structures on V, which is diffeomorphic to the homogenous space $\mathrm{SO}(V)/\mathrm{U}(V,J)$ for any $J\in\mathrm{SOComp}(V)$, i.e. diffeomorphic to the homogenous space $\mathrm{SO}_{2n}/\mathrm{U}_n$, and has in particular dimension $2\binom{n}{2}$. The addition of this metric amounts to restricting to maximal compact subgroups.

Given a smooth manifold X of dimension $2n$, the manifolds $\mathrm{Comp}^+(\mathrm{T}_xX)$ for $x\in X$ glue together to form a fibre bundle $\mathrm{Comp}^+(X)\to X$ over X, where by definition a smooth section of this fibre bundle is an almost complex structure on X. Similarly, if we equip X with a Riemannian metric, then we get a fibre bundle $\mathrm{SOComp}(X)\to X$ with fibres diffeomorphic to $\mathrm{SO}_{2n}/\mathrm{U}_n$, sections of which correspond to almost complex structures on X orthogonal with respect to this Riemannian metric. Indeed, $\mathrm{Comp}^+(X)$ deformation retracts onto $\mathrm{SOComp}(X)$. The result then follows from obstruction theory[17] applied to this bundle $\mathrm{SOComp}(X)\to X$.

Said another way, an almost complex structure amounts to a lift of the principal SO_n-bundle $\mathrm{SO}_n(X)$ of orthonormal tangent frames on X (with respect to any specified Riemannian metric) to a principal U_n bundle, which amounts to a lift

$$\begin{tikzcd} & \mathrm{BU}_n \arrow[d] & \mathrm{SO}_{2n}/\mathrm{U}_n \arrow[l] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{BSO}_{2n}, & \end{tikzcd}$$

of the classifying map $X\to\mathrm{BSO}_{2n}$ of the tangent bundle $\mathrm{T}X$ to a map $X\to\mathrm{BU}_n$. Since the homotopy fiber of the map $\mathrm{B}H\to\mathrm{B}G$ induced by a inclusion $H\hookrightarrow G$ of a compact Lie subgroup is again the quotient G/H; applying this to $H=\mathrm{U}_n\hookrightarrow G=\mathrm{SO}_{2n}\mathbb{R}$ gives us again that the obstructions lie in $\mathrm{H}^*(X;\pi_{*-1}(\mathrm{SO}_{2n}/\mathrm{U}_n))$. ■

[16] Note that $\mathrm{Comp}(V)\subset\mathrm{GL}^+_{\mathbb{R}}(V)$, so the existence of these two components is not, in fact, due to $\mathrm{GL}_{\mathbb{R}}(V)$ being disconnected. For instance, when $n=1$ and $V=\mathbb{R}^2$, we have the explicit description $\mathrm{Comp}(\mathbb{R}^2)=\left\{\begin{bmatrix}a & b\\ c & -a\end{bmatrix} : a^2+bc=-1\right\}\subset\mathrm{GL}^+_2\mathbb{R}$, and the two components of $\mathrm{Comp}(\mathbb{R}^2)$ come from the two sheets of the hyperboloid $a^2+bc=-1$.

[17] See, for instance, [62, Lecture 18] for the fundamentals of obstruction theory.

The manifolds $\mathrm{SO}_{2n} / \mathrm{U}_n$ are examples of Hermitian symmetric spaces and are in particular complex projective varieties. They also admit a few alternate descriptions. For instance, $\mathrm{SO}_{2n} / \mathrm{U}_n$ can be thought of as the isotropic Grassmannian $\mathrm{IsoGr}^{2n}_{\mathbb{C}}$ consisting of maximal oriented isotropic subspaces of $\mathbb{C}^{2n}$, which can itself be described as one component of the Fano variety of projective $(n-1)$-planes on a smooth quadric $X_2^{2n-2} \subset \mathbb{CP}^{2n-1}$. Alternatively, if $\pi : \mathrm{Spin}_{2n} \to \mathrm{SO}_{2n}$ denotes the universal cover of SO_{2n}, then if $\tilde{\mathrm{U}}_n := \pi^{-1}\,\mathrm{U}_n \subset \mathrm{Spin}_{2n}$ is the preimage of U_n under π, then $\tilde{\mathrm{U}}_n \to \mathrm{U}_n$ is a nontrivial degree two cover of U_n. This cover can be described as follows: we have the determinant homomorphism $\det : \mathrm{U}_n \to \mathrm{U}_1 \cong S^1$, and $\tilde{\mathrm{U}}_n$ is the pullback via det of the degree two cover $z \mapsto z^2$ of S^1. Elements of $\tilde{\mathrm{U}}_n$ then can be written as pairs $(U, s) \in \mathrm{U}_n \times S^1$, where $U \in \mathrm{U}_n$ and $s^2 = \det U$. In particular, $\tilde{\mathrm{U}}_n$ is still diffeomorphic to U_n. For small values of n, the exceptional isomorphisms of spin groups then allow us to give concrete descriptions of $\mathrm{SO}_{2n} / \mathrm{U}_n \cong \mathrm{Spin}_{2n} / \tilde{\mathrm{U}}_n$. For instance, we have

Proposition 3.6.5. For $n \in \{1, 2, 3\}$, we have diffeomorphisms $\mathrm{SO}_{2n} / \mathrm{U}_n \cong \mathbb{CP}^{n(n-1)/2}$.

Proof. The case $n = 1$ is trivial. For $n = 2$, note that a smooth quadric $X_2^2 \subset \mathbb{CP}^3$ is the Segre product $\mathbb{CP}^1 \times \mathbb{CP}^1$, and hence the Fano variety of projective lines on it has two components, each isomorphic to $\mathbb{CP}^1$. We could also argue as follows: if $\mathbb{R}^4$ with is usual inner product and standard orientation is the defining representation of $\mathrm{SO}_4 \mathbb{R}$, then the action of $\mathrm{SO}_4 \mathbb{R}$ on $\Lambda^2(\mathbb{R}^4)^\vee$ preserves the ± 1-eigenspaces $\Lambda^2_\pm(\mathbb{R}^4)^\vee$ of the Hodge star involution on $\Lambda^2(\mathbb{R}^4)^\vee$, as well as the induced norm on them. The $(+1)$-eigenspace is called the space of self dual alternating forms, whereas the (-1)-eigenspace is called the space of anti self dual (ASD) alternating forms. In particular, this $\mathrm{SO}_4 \mathbb{R}$ action preserves the unit sphere $S(\Lambda^2_-(\mathbb{R}^4)^\vee) \cong S^2$ in the space $\Lambda^2_-(\mathbb{R}^4)^\vee$ of ASD alternating bilinear forms on $\mathbb{R}^4$. It is then a standard fact that this action is transitive, and the stabilizer of any unit-norm ASD alternating form is U_2; indeed, if $\mathrm{e}_1, \ldots, \mathrm{e}_4$ denotes an oriented orthonormal basis of $\mathbb{R}^4$, then the stabilizer of the unit-norm ASD alternating form

$$\frac{1}{\sqrt{2}}\left(\mathrm{e}_1 \wedge \mathrm{e}_3 + \mathrm{e}_2 \wedge \mathrm{e}_4\right),$$

is exactly

$$\mathrm{SO}_4 \mathbb{R} \cap \mathrm{Sp}_4 \mathbb{R} = \mathrm{U}_2 .^{18}$$

Finally, for $n = 3$, we use that $\mathrm{SO}_6 / \mathrm{U}_3 \cong \mathrm{Spin}_6 / \tilde{\mathrm{U}}_3$. Note that we have an exceptional isomorphism $\mathrm{Spin}_6 \cong \mathrm{SU}_4$ which comes from observing that the complex Clifford algebra $\mathrm{Cl}_5^{\mathbb{C}} \cong \mathrm{Mat}_{4\times 4}(\mathbb{C})$ is the algebra of 4×4 matrices over $\mathbb{C}$, giving us an injection $\mathrm{Spin}_6 \hookrightarrow \mathrm{U}_4$, along with the fact that the Lie algebra $\mathfrak{so}_6$ is simple. Now SU_4 acts transitively on $\mathbb{CP}^3$, with the stabilizer of a point being diffeomorphic to U_3. It is not hard to see then that under these identifications, we have $\mathrm{SO}_6 / \mathrm{U}_3 \cong \mathrm{Spin}_6 / \tilde{\mathrm{U}}_3 \cong \mathrm{SU}_4 / \mathrm{U}_3 \cong \mathbb{CP}^3$. ■

In the next case of $n = 4$, the space $\mathrm{SO}_8 / \mathrm{U}_4 \cong X_2^6 \subset \mathbb{CP}^7$ is a smooth quadric hypersurface in $\mathbb{CP}^7$. A different exposition of these results can also be found in [40].

[18] In the last step we have used that for any $n \geq 1$, we have $\mathrm{Sp}_{2n} \mathbb{R} \subset \mathrm{SL}_{2n} \mathbb{R}$, as well as the "Kähler trichotomy" which says that

$$\mathrm{GL}_n \mathbb{C} \cap \mathrm{O}_{2n} \mathbb{R} = \mathrm{GL}_n \mathbb{C} \cap \mathrm{Sp}_{2n} \mathbb{R} = \mathrm{Sp}_{2n} \mathbb{R} \cap \mathrm{O}_{2n} \mathbb{R} = \mathrm{U}_n,$$

where the intersection is carried out in $\mathrm{GL}_{2n} \mathbb{R}$.

We end by mentioning the promised criterion by Hirzebruch.

Theorem 3.6.6 (Hirzebruch)**.** Let X be a smooth manifold of real dimension $4k$ for some $k \geq 0$. If X admits an almost complex structure, then the Euler characteristic $\chi(X)$ and the signature $\mathrm{Sign}(X)$ of X are related by

$$\chi(X) \equiv (-1)^k \mathrm{Sign}(X) \pmod 4.$$

We will not give a complete proof of this result, although is it not too hard, once we assume the Atiyah-Singer Index Theorem.[19]

Proof Sketch. The key idea here to use the Hirzebruch χ_y-genus $\chi_y(X)$ and the generalization of the Hirzebruch-Riemann-Roch Theorem (or more specifically Corollaries 1.4.6 and 1.4.7) to the setting of almost complex manifolds via the machinery of the Atiyah-Singer Index Theorem (see Remark 1.4.12). For full details, see [19]. ■

Example 3.6.7. As an application, also taken from [19], we can look at $m, n \geq 1$, let $\#^m \mathbb{CP}^n$ denote the m-fold connect sum of complex projective spaces $\mathbb{CP}^n$. When n is odd, complex conjugation gives us an orientation-reversing diffeomorphism of $\mathbb{CP}^n$, so that $\mathbb{CP}^n$ and $\overline{\mathbb{CP}^n}$ are diffeomorphic as oriented manifolds. (This is not true when n is even, because $\mathrm{Sign}(\mathbb{CP}^{2k}) = 1$ whereas $\mathrm{Sign}(\overline{\mathbb{CP}^{2k}}) = -1$.) In particular, we have

$$\#^m \mathbb{CP}^n \cong \mathbb{CP}^n \# \left(\#^{m-1}\overline{\mathbb{CP}^n}\right) \cong \mathrm{Bl}_{p_1,\ldots,p_{m-1}} \mathbb{CP}^n$$

is diffeomorphic to the blow-up of $\mathbb{CP}^n$ at $m-1$ general points $p_1, \ldots, p_{m-1}$, and hence, in fact, admits an integrable almost complex structure.

When n is even, we can argue as follows. Note that for any $k \geq 2$ and k-manifolds X, Y, the connect sum $X \# Y$ has Euler characteristic given by

$$\chi(X \# Y) = \chi(X) + \chi(Y) - \chi(S^k).$$

In particular, we have $\chi(\#^m X) = m\chi(X) - (m-1)\chi(S^k)$. By contrast, the signature Sign is simply additive: we have

$$\mathrm{Sign}(X \# Y) = \mathrm{Sign}(X) + \mathrm{Sign}(Y)$$

so that $\mathrm{Sign}(\#^m X) = m \cdot \mathrm{Sign}(X)$. It follows that for any $n, m \geq 1$ we have

$$\chi(\#^m \mathbb{CP}^n) = mn - m + 2,$$

whereas for even n only we have $\mathrm{Sign}(\#^m \mathbb{CP}^n) = m$. Therefore, if m and n are both even, then the criterion of Theorem 3.6.6 cannot be satisfied, and $\#^m \mathbb{CP}^n$ is not almost complex. When n is even but m is odd, in fact the manifolds $\#^m \mathbb{CP}^n$ do all admit almost complex structures, but not all of these are integrable. For instance, $\#^{2k+1}\mathbb{CP}^2$ for $k \geq 0$ admits a complex structure iff $k = 0$, as is shown in [61]. I am not aware of a complete answer to the question of when the manifolds $\#^{2k+1}\mathbb{CP}^{2\ell}$ for $\ell, k \geq 0$ admit complex structures.

[19] To be honest, this is because I ran out of time.

Since we do not prove Theorem 3.6.6, let us at least work out the case of 4-manifolds “by hand”. Here one criterion is

Theorem 3.6.8. Let X be a closed oriented manifold of dimension 4. If X admits an almost complex structure, i.e. if E is a complex vector budle such that $\mathrm{T}X \cong E_{\mathbb{R}}$, then we have must have

$$\int_X c_1(E)^2 = 2 \cdot \chi(X) + 3 \cdot \mathrm{Sign}(X).$$

Proof. Since by Example 3.3.9 we have

$$L_1 = \frac{1}{3} p_1,$$

it follows from the Hirzebruch Signature Theorem (Theorem 1.3.1) that

$$p_1(\mathrm{T}X) = 3 \cdot \mathrm{Sign}(X) \cdot \eta_X,$$

where $\eta_X \in \mathrm{H}^4(X;\mathbb{Z})$ is the chosen generator of the top cohomology. If $\mathrm{T}X \cong E_{\mathbb{R}}$, then Remark 3.4.22 tells us that

$$p_1(\mathrm{T}X) = c_1(E)^2 - 2c_2(E).$$

Next, Remark 3.4.23 tells us that

$$c_2(E) = \mathrm{e}(\mathrm{T}X),$$

so using the Chern-Gauss-Bonnet Theorem (Theorem 1.2) we get

$$3 \cdot \mathrm{Sign}(X) = \int_X p_1(\mathrm{T}X) = \int_X c_1(E)^2 - 2\int_X \mathrm{e}(\mathrm{T}X) = \int_X c_1(E)^2 - 2 \cdot \chi(X)$$

as needed. ∎

As an application of this criterion, we show a special case of the computation in Example 3.6.7, namely that $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit an almost complex structure.

Corollary 3.6.9. The 4-fold $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit an almost complex structure.

Proof. If $X := \mathbb{CP}^2 \# \mathbb{CP}^2$, then the cohomology ring of X is easily seen to be

$$\mathrm{H}^*(X;\mathbb{Z}) = \mathbb{Z}[\alpha,\beta]/(\alpha^3, \beta^3, \alpha\beta, \alpha^2 - \beta^2) = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\eta_X,$$

where the generator η_X of the top cohomology is $\eta_X = \alpha^2 = \beta^2$; in particular,we have $\chi(X) = 4$ and $\mathrm{Sign}(X) = 2$. If X admitted an almost complex structure and E were as in Theorem 3.6.8, then writing $c_1(E) = r\alpha + s\beta$ for some $r, s \in \mathbb{Z}$, Theorem 3.6.8 would give us that

$$r^2 + s^2 = \int_X c_1(E)^2 = 2 \cdot 4 + 3 \cdot 2 = 14.$$

This is a contradiction, since 14 is not the sum of two squares. ∎

3.7 Cohomology of Complex Grassmannians

In this section, we will review the Schubert cell decomposition of Grassmannians, and the description of their cohomology rings using the algebra of symmetric polynomials. The basic references for this material are [4], [10] and [43].In what follows, we will stick to the case of complex Grassmannians, although similar remarks can be made over any field.

The Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ can be given a CW structure, called the Schubert cell decomposition of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$, with cells in bijection with partitions $\lambda \subset m \times n$, i.e. partitions of at most m parts with each part at most n. Let's now motivate this cell decomposition. To start, we fix a complete flag in $\mathbb{F}^{m+n}$, i.e. a sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{m+n-1} \subset V_{m+n} = \mathbb{C}^{m+n}$$

satisfying $\dim_{\mathbb{C}} V_j/V_{j-1} = 1$ for $j = 1, \ldots, m+n$. Then for any $V \in \mathrm{Gr}_{\mathbb{C}}^{m,n}$, as i goes from 0 to $m+n$, the dimension $\dim(V \cap V_i)$ increases from 0 to m in steps of size 0 or 1. Let's say these jumps happen at $i_1, i_2, \ldots, i_m$, so that the i_k for $1 \leq k \leq m$ are integers such that $1 \leq i_1 < \cdots < i_m \leq m+n$ and such that for all $k = 1, \ldots, m$, we have

$$\dim(V \cap V_{i_k}) = \dim(V \cap V_{i_k - 1}) + 1.$$

In this way, every element $V \in \mathrm{Gr}_{\mathbb{C}}^{m,n}$ gives rise to a sequence $I(V) = \{i_1, \ldots, i_m\}$ of m integers between 1 and $m+n$.[20] Conversely, given any such sequence I, we may consider the set $\Omega_I = \{V \in \mathrm{Gr}_{\mathbb{C}}^{m,n} : I(V) = V\}$ of elements of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ that give rise to this sequence. This clearly gives us a set-theoretic decomposition

$$\mathrm{Gr}_{\mathbb{C}}^{m,n} = \coprod_{\substack{I \subset \{1,\ldots,m+n\} \\ |I| = m}} \Omega_I,$$

of the Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ as a disjoint union of the Ω_I, where the decomposition is over all m-elements subsets I of $\{1, 2, \ldots, m+n\}$. This, as it turns out, is the sought-after Schubert cell decomposition of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$. To state this result more precisely, and to relate the notation to that of partitions, we use the following simple observation, the proof of which is clear.

Lemma 3.7.1. Given a sequence $I = \{i_1, \ldots, i_m\}$ of integers with the property $1 \leq i_1 < \cdots < i_m \leq m+n$, we can associate to it a partition $\lambda \subset m \times n$ by defining $\lambda_k := n + i_k - k$ for $k = 1, \ldots, m$. This map is a one-to-one correspondence between the set of m-element subsets of $1, \ldots, m+n$, and the set of partitions $\lambda \subset m \times n$.

It is fun exercise to think about how to recover the set I from the Ferrers diagram of λ geometrically. In our case, given a partition $\lambda \subset m \times n$, we can form the corresponding sequence $I = I(\lambda)$, and then look at the subset

$$\Omega_\lambda := \Omega_{I(\lambda)} = \{V \in \mathrm{Gr}_{\mathbb{C}}^{m,n} : \dim(V \cap V_j) = i \text{ for } n + i - \lambda_i \leq j \leq n + i - \lambda_{i+1}\} \subset \mathrm{Gr}_{\mathbb{C}}^{m,n}.$$

This is called the Schubert cell corresponding to the partition λ, and its closure $X_\lambda = \overline{\Omega_\lambda}$ (in the classical or Zariski topology–it does not matter) is then called the corresponding Schubert variety. The first fundamental result in the theory is

[20] If we take $V_j = \langle \mathrm{e}_1, \ldots, \mathrm{e}_j \rangle$ for $0 \leq j \leq m+n$, then I can also be obtained by putting the $(m+n) \times m$ matrix representing a $V \in \mathrm{Gr}_{\mathbb{C}}^{m,n}$ in "column echelon" form, but we will not pursue this argument further.

Theorem 3.7.2. For all $\lambda \subset m \times n$, we have:

(a) The closed subset $X_\lambda \subset \mathrm{Gr}_{\mathbb{C}}^{m,n}$ is an algebraic subvariety, i.e. is defined by vanishing of homogenous polynomials in Plücker coordinates.

(b) There is a diffeomorphism $\Omega_\lambda \cong \mathbb{C}^{mn-|\lambda|}$, so that Ω_λ is an open cell of (complex) codimension $|\lambda|$.

(c) We have $X_\lambda = \coprod_{\mu \supset \lambda} \Omega_\mu$, where $\mu \supset \lambda$ means that the Ferrers diagram of λ is contained in that of μ.

(d) The decomposition $\mathrm{Gr}_{\mathbb{C}}^{m,n} = \coprod_\lambda \Omega_\lambda$ is a CW decomposition of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$. In particular, the Grassmannian $\mathrm{Gr}_{\mathbb{C}}^{m,n}$ can be given a CW decomposition with cells in bijection with partitions $\lambda \subset m \times n$, such that a partition λ corresponds to a cell of complex codimension $|\lambda|$.

Proof. See [4, Ch. 1, §5] or [10, §3.2] ■

In particular, this CW decomposition determines the (co)homology groups of $\mathrm{Gr}_{\mathbb{C}}^{m,n}$. Namely, the fundamental classes $[X_\lambda] \in \mathrm{H}_{2(mn-|\lambda|)}(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$ for $\lambda \subset m \times n$ form an additive basis for $\mathrm{H}_*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$. Similarly, if we denote the Poincaré dual to $[X_\lambda]$ by σ_λ, then the $\sigma_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$ form an additive basis for the cohomology groups $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$, i.e. we have

$$\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z}) = \bigoplus_{\lambda \subset m \times n} \mathbb{Z}\sigma_\lambda.$$

As a matter of convention, we also denote σ_λ by $\sigma_{\lambda_1,\ldots,\lambda_{\ell(\lambda)}}$. Note in particular that the generator of top cohomology is simply

$$\eta_{\mathrm{Gr}_{\mathbb{C}}^{m,n}} = \sigma_{n^m}.$$

The multiplicative structure in cohomology is trickier to describe as explicitly, although certain ad hoc techniques can be applied in special cases. For instance, we have formulae due Pieri and Giambelli for special Schubert cycles– see [43, Chapter 4] or [10, Chapter 3]– and the intersection for Grassmannians of lines $\mathrm{Gr}_{\mathbb{C}}^{2,n}$ can also be made similarly explicit. Here we record two special cases which we will need. To state the first one, we will need the notion of the complementary partition.

Definition 3.7.3. Given integers $m, n \geq 0$ and a partition $\lambda \subset m \times n$, we define the complementary partition $\hat{\lambda} \subset m \times n$ to λ by

$$\hat{\lambda}_k := n - \lambda_{m+1-k} \text{ for } 1 \leq k \leq m.$$

By definition, the Ferrers diagram of $\hat{\lambda}$ is the (inverted) complement of the Ferrers diagram of λ in the box $m \times n$; in particular, $|\lambda| + |\hat{\lambda}| = mn$. This terminology is helpful in stating many results about the geometry of finite Grassmannians; for instance, the degree of the Schubert variety X_λ under the Plücker embeddings is the number of standard Young tableu of shape $\hat{\lambda}$, and hence can be obtained via the Hook-Length Formula (see [10, Corollary 3.2.14])–this is essentially a consequence of Pieri's rule for multiplication and the observation that this degree is nothing but

$$\int_{\mathrm{Gr}_{\mathbb{C}}^{m,n}} \sigma_\lambda \cdot \sigma_1^{|\hat{\lambda}|}.$$

The first result about the multiplicative structure says that for any λ, the Schubert cycles σ_λ and $\sigma_{\hat{\lambda}}$ in the Grassmannian are algebraic duals to each other under the cup-product pairing. Precisely stated, we have

Proposition 3.7.4 (Intersection in Complementary Dimension)**.** For integers $m, n \geq 0$ and for any partitions $\lambda, \mu \subset m \times n$ such that $|\lambda| + |\mu| = mn$, we have in $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z})$ that

$$\sigma_\lambda \sigma_\mu = \delta_{\hat{\lambda},\mu} \sigma_{n^m},$$

where $\delta_{\hat{\lambda},\mu}$ is the Kronecker delta. In other words, we have

$$\int_{\mathrm{Gr}_{\mathbb{C}}^{m,n}} \sigma_\lambda \sigma_\mu = \begin{cases} 1, & \mu = \hat{\lambda}, \\ 0 & \text{else.} \end{cases}$$

Proof. See [43, Prop. 4.6]. ■

Next, note that for Grassmannians of lines $\mathrm{Gr}_{\mathbb{C}}^{2,n}$ a partition $\lambda \subset 2 \times n$ is simply a pair of integers (a_1, a_2) such that $n \geq a_1 \geq a_2 \geq 0$. We can then completely describe the product structure on $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{2,n}; \mathbb{Z})$ as follows.

Proposition 3.7.5 (Grassmannian of Lines)**.** Given an integer $n \geq 0$, and partitions $(a_1, a_2), (b_1, b_2) \subset 2 \times n$ such that $a_1 - a_2 \geq b_1 - b_2$, we have

$$\sigma_{a_1,a_2} \sigma_{b_1,b_2} = \sum_{j=0}^{b_2 - b_1} \sigma_{a_1+b_1-j, a_2+b_2+j} = \sum_{\substack{|c|=|a|+|b| \\ a_1+b_1 \geq c_1 \geq a_1+b_2}} \sigma_{c_1,c_2}.$$

Proof. See [43, Prop. 4.11]. ■

Note how Propositions 3.7.4 and 3.7.5 agree in the case of complementary dimension on the Grassmannian of lines.

These ad-hoc techniques are often useful for direct computations in low dimensions. However, a better, and more general, way to understand the multiplicative structure of $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n}; \mathbb{Z})$, is to relate it to the algebra of symmetric polynomials as follows. The inclusion $\mathbb{F}^{m+n} \subset \mathbb{F}^{m+n+1}$ as the hyperplane defined by the vanishing of the last coordinate induces an injection $\mathrm{Gr}_{\mathbb{C}}^{m,n} \hookrightarrow \mathrm{Gr}_{\mathbb{C}}^{m,n+1}$, and we may consider the infinite Grassmannian of m-planes in $\mathbb{C}^\infty$ defined to be the inductive limit $\mathrm{Gr}_{\mathbb{C}}^{m,\infty} := \varinjlim_n \mathrm{Gr}_{\mathbb{C}}^{m,n}$. Then we have

$$\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,\infty}; \mathbb{Z}) = \bigoplus_{\lambda \subset m \times \infty} \mathbb{Z}\sigma_\lambda,$$

where $\lambda \subset m \times \infty$ simply denotes the set of partitions with at most m parts, and the pullback map on cohomology induced by the inclusion $\mathrm{Gr}_{\mathbb{C}}^{m,n} \hookrightarrow \mathrm{Gr}_{\mathbb{C}}^{m,\infty}$ is then defined simply by the projection map

$$\bigoplus_{\lambda \subset m \times \infty} \mathbb{Z}\sigma_\lambda \to \bigoplus_{\lambda \subset m \times n} \mathbb{Z}\sigma_\lambda$$

that sends the class σ_λ of a partition with some part greater than n to 0, i.e. "forgets" all such classes. Since this map is also a ring homomorphism, to describe the ring structure on $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$, it suffices to do that on $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,\infty};\mathbb{Z})$. Now the key point is that, as observed in the remarks after Proposition 3.4.7, $\mathrm{Gr}_{\mathbb{C}}^{m,\infty}$ is a model for the classifying space BU_m for the unitary group U_m, and hence by the general theory of classifying spaces and characteristic classes (see Appendix 3.4), we can describe this ring as

$$\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,\infty};\mathbb{Z}) = \mathrm{H}^*(\mathrm{BU}_m;\mathbb{Z}) = \mathbb{Z}[\gamma_1,\dots,\gamma_m]^{S_m} = \mathbb{Z}[c_1,\dots,c_m],$$

where the c_j are the universal Chern classes. In these terms, for any $\lambda \subset m \times \infty$, the Schubert class σ_λ corresponds to the symmetric polynomial in $\gamma_1,\dots,\gamma_m$ known as the Schur polynomial of type λ, written $s_\lambda(\gamma)$, which can be defined in many (non-obviously) equivalent ways (see [10, Chapter 1]). This gives us a way, in theory to, carry out any computation in the cohomology ring of the Grassmannian. This argument can then be taken a little further to give explicit generators and relations for the $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z})$ as a ring; see, for instance, [10, Exercise 3.2.13] or [43, §5.8].

Even more fascinatingly, for any $m \geq 0$ there is an inclusion $\mathrm{Gr}_{\mathbb{C}}^{m,\infty} \hookrightarrow \mathrm{Gr}_{\mathbb{C}}^{m+1,\infty}$ (given simply by "including a new basis vector"), such that the resulting pullback map

$$\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m+1,\infty};\mathbb{Z}) \to \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,\infty};\mathbb{Z})$$

in the above terms is given by setting γ_{m+1} to 0, i.e. takes $c_j \mapsto c_j$ for $1 \leq j \leq m$ and $c_m \mapsto 0$. Then inductive limit $\mathrm{Gr}_{\mathbb{C}}^{\infty,\infty} = \varinjlim_m \mathrm{Gr}_{\mathbb{C}}^{m,\infty}$, the doubly infinite Grassmannian has the property that the its cohomology ring $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{\infty,\infty};\mathbb{Z})$ has as an additive basis the σ_λ over all λ, and is isomorphic to the ring Λ of symmetric polynomials in countably many variables.

It is in this universal ring that we will do our computations. For instance, the multiplication of Schur polynomials s_λ in Λ is expressed as

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu,$$

where the $c_{\lambda\mu}^\nu \in \mathbb{Z}$ are called the Littlewood-Richardson coefficients. Note that $c_{\lambda\mu}^\nu = 0$ unless $|\nu| = |\lambda| + |\mu|$. The above discussion then tells us that for any $\lambda, \mu \subset m \times n$, we have

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subset m \times n} c_{\lambda\mu}^\nu \sigma_\nu \in \mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}^{m,n};\mathbb{Z}).$$

See Subsection 2.3.1 for another example.

Remark 3.7.6. Similarly to Grassmannians, the flag manifold $\mathrm{Fl}_{\mathbb{F}}^m$ admits a decomposition into Schubert cells, this time indexed by $W(G,P)$, where $G = \mathrm{GL}_{|m|}\,\mathbb{F}$ and $P \subset G$ is the parabolic subgroup preserving any given flag of type m, and $W(G,P)$ is the set of left cosets of the Weyl group $W(P)$ of P in the Weyl group $W(G)$ of G. This is one starting point of Geometric Representation Theory, and a great ending point for this thesis.

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