

# Algebraic Geometry

Dhruv Goel

# Contents

Preface . . . . .	2
<b>1 Fundamentals</b>	<b>3</b>
1.1 Definitions and Examples . . . . .	4
1.2 The Zariski Base . . . . .	5
1.3 Quasicoherent Sheaves . . . . .	7
1.4 Exercises . . . . .	10
<b>2 Topology for Algebraic Geometers</b>	<b>11</b>
2.1 Irreducible, Kolmogorov, and Sober Spaces . . . . .	12
2.2 Closure-Complete and Noetherian Spaces . . . . .	16
2.3 Locally Closed, Constructible, and Very Dense Subspaces . . . . .	19
2.4 Separatedness and Properness from a Topological Perspective . . . . .	23
<b>3 Affine Communication Lemma and Properties of Morphisms</b>	<b>26</b>
3.1 The Affine Communication Lemma and Reasonable Classes of Morphisms . . . . .	27
3.2 (Locally) Noetherian Schemes and (Locally) Finite Type and Presentation Morphisms . . . . .	29
3.3 Quasically compact and Quasiseparated Morphisms . . . . .	32
3.4 Affine, Integral, and Finite Morphisms . . . . .	36
3.5 Closed Embeddings . . . . .	37
3.6 Separated and Proper Morphisms . . . . .	40
3.7 Exercises . . . . .	43
<b>4 Dimension</b>	<b>44</b>
4.1 Basics . . . . .	45
4.2 Krull's Theorems . . . . .	50
4.3 Smoothness . . . . .	51
4.4 Exercises . . . . .	53
<b>5 Divisors and Line Bundles</b>	<b>55</b>
5.1 Weil Divisors . . . . .	56
5.2 Meromorphic Functions and Cartier Divisors . . . . .	59
<b>6 Intersection Theory</b>	<b>62</b>
<b>7 Appendices</b>	<b>63</b>

7.1	Generic Freeness . . . . .	64
7.2	Flatness . . . . .	65
7.3	Counterexamples in Algebraic Geometry . . . . .	66
7.4	Possible Hints to Selected Exercises . . . . .	67
<b>Bibliography</b>		<b>68</b>

## Preface

## Chapter 1

# Fundamentals

## 1.1 Definitions and Examples

## 1.2 The Zariski Base

Let  $X$  be a locally ringed space. An open subset  $D \subset X$  is said to be *distinguished* if there is an  $f \in \mathcal{O}(X)$  such that  $D = X \setminus \mathbb{V}(f) = \{p \in X : f(p) \neq 0\}$ ; in general, there are many such  $f$  for which this holds—for instance, every power of  $f$  gives the same  $D$ .

A scheme  $X$  is affine iff the natural morphism  $X \rightarrow \operatorname{Spec} \mathcal{O}(X)$  is an isomorphism. In this case, if  $D \subset X$  is a distinguished open subset, then  $(D, \mathcal{O}|_D)$  is also an affine scheme, almost by construction of the sheaf  $\mathcal{O}$ . The following results will be used repeatedly.

**Lemma 1.2.1.**

- (a) If  $\pi : X \rightarrow Y$  is a morphism of locally ringed spaces and  $D \subset Y$  a distinguished open, then so is  $\pi^{-1}(D) \subset X$ .
- (b) The intersection of two distinguished opens in a locally ringed space is again distinguished.
- (c) Suppose  $X$  is a locally ringed space and  $D' \subset D \subset X$  open subsets. If  $D'$  is distinguished in  $X$ , then it is also in  $D$ . Conversely, if  $X$  is an affine scheme,  $D$  distinguished in  $X$ , and  $D'$  distinguished in  $D$ , then  $D'$  is distinguished in  $X$ .
- (d) Distinguished open subsets are a basis for the topology on an affine scheme. Affine open subsets are a basis for the topology on any scheme.
- (e) Suppose  $X$  is an affine scheme,  $\{f_i\} \subset \mathcal{O}(X)$  a family of elements, and for each  $i$  the set  $D_i := X \setminus \mathbb{V}(f_i)$ . Then  $X = \bigcup_i D_i$  iff  $(f_i) = (1)$  in  $\mathcal{O}(X)$ . In particular, any affine scheme is quasicompact.

*Proof.* Exercise. ■

**Example 1.2.2.** On a general locally ringed space, distinguished opens do not have to be a basis for the topology. A simple example is obtained by taking a non-affine scheme like  $X = \mathbb{P}_{\mathbb{C}}^1$ , in which the only distinguished opens are  $\emptyset$  and  $X$ .

The first fundamental result we need is

**Lemma 1.2.3.** Let  $X$  be a scheme and  $U, V \subset X$  affine opens. Then  $U \cap V$  can be covered by affine opens which are simultaneously distinguished in both  $U$  and  $V$ .

*Proof.* Let  $p \in U \cap V$ . Using 1.2.1(d), pick a distinguished open  $D$  in  $U$  such that  $p \in D \subset U \cap V$ . Next, again using Lemma 1.2.1(d), pick a distinguished open  $D'$  in  $V$  such that  $p \in D' \subset D$ . Then  $D'$  is distinguished in  $D$  and hence in  $U$  by 1.2.1(c). ■

The above observations allow us to construct sheaves very explicitly on a scheme.

**Theorem 1.2.4.** Let  $X$  be a scheme. Suppose  $\mathcal{F}$  is a sheaf on  $X_{\text{Zar}}^{\text{dist}}$ , i.e., an assignment to each affine open  $U \subset X$  a set<sup>1</sup>  $\mathcal{F}(U)$ , and to each distinguished inclusion  $D \subset U$  a restriction morphism  $\mathcal{F}(U) \rightarrow \mathcal{F}(D)$ , satisfying the usual presheaf axioms, such that for every affine open  $U \subset X$  and finite cover  $U = \bigcup_{\alpha} D_{\alpha}$  of  $U$  by distinguished affine opens, the associated sequence

$$* \rightarrow \mathcal{F}(U) \rightarrow \prod_{\alpha} \mathcal{F}(D_{\alpha}) \rightrightarrows \prod_{\alpha, \beta} \mathcal{F}(D_{\alpha\beta})$$

is an equalizer diagram, where  $D_{\alpha\beta} = D_{\alpha} \cap D_{\beta}$ . Then there is a sheaf  $\mathcal{F}^+$  on  $X$  along with an isomorphism  $\mathcal{F} \rightarrow \mathcal{F}^+_{\text{Zar}} := \mathcal{F}^+|_{X_{\text{Zar}}^{\text{dist}}}$ , and this pair is unique up to unique isomorphism.

*Proof.*

Existence. For  $x \in X$ , define  $\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$ , where the colimit is taken over affine opens  $U$  containing  $x$  and distinguished inclusions. For any open  $V \subset X$ , define  $\mathcal{F}^+(V) \subset \prod_{x \in V} \mathcal{F}_x$  via compatible germs, namely those  $(\sigma_x)_{x \in V}$  such that for each  $x \in V$  there is an affine open  $U$  with  $x \in U \subset V$  and an element  $\sigma_U \in \mathcal{F}(U)$  such that for all  $y \in U$ , we have  $\sigma_y = \sigma_U|_y$ . Because of 1.2.1(d), the natural projections make  $\mathcal{F}^+$  into a presheaf on  $X$ , which is automatically a sheaf by construction. There is clearly a natural morphism  $\mathcal{F} \rightarrow \mathcal{F}^+|_{X_{\text{Zar}}^{\text{dist}}}$ . To check that it is an isomorphism, first note

<sup>1</sup>Or a group, or ring, or a module over a fixed ring, etc.

that injectivity follows from 1.2.1(de) and the separatedness axiom for  $\mathcal{F}$ : if  $U \subset X$  is an affine open and  $\sigma, \tau \in \mathcal{F}(U)$  map to the same element of  $\mathcal{F}^+(U)$ , then there is a finite cover  $U = \bigcup_{\alpha} D_{\alpha}$  of  $U$  by distinguished affine opens such that for all  $\alpha$  we have  $\sigma|_{D_{\alpha}} = \tau|_{D_{\alpha}}$ , and so  $\sigma = \tau$  by the injectivity of  $\mathcal{F}(U) \rightarrow \prod_{\alpha} \mathcal{F}(D_{\alpha})$ . Similarly, we show surjectivity: given  $(\sigma_x) \in \mathcal{F}^+(U)$ , pick a finite cover  $U = \bigcup_{\alpha} D_{\alpha}$  of  $U$  by distinguished affine opens and elements  $\sigma_{\alpha} \in \mathcal{F}(D_{\alpha})$  such that for all  $\alpha$  and all  $x \in D_{\alpha}$ , we have  $\sigma_x = \sigma_{\alpha}|_x$ . Then on each intersection  $D_{\alpha\beta}$ , the restrictions  $\sigma_{\alpha}|_{D_{\alpha\beta}}$  and  $\sigma_{\beta}|_{D_{\alpha\beta}}$  map to the same element of  $\mathcal{F}^+(D_{\alpha\beta})$  and hence agree by the injectivity proven above. Therefore, by the gluing axiom for  $\mathcal{F}$ , there is a  $\sigma \in \mathcal{F}(U)$  such that for all  $\alpha$ , we have that  $\sigma_{\alpha} = \sigma|_{D_{\alpha}}$ , and then  $\sigma$  maps to  $(\sigma_x) \in \mathcal{F}^+(U)$ .

**Uniqueness.** Given any sheaf  $\mathcal{G}$  on  $X$ , it is easy to see that given any morphism  $\varphi : \mathcal{F} \rightarrow \mathcal{G}_{\text{Zar}}$  gives rise to a unique morphism  $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$  whose restriction to  $X_{\text{Zar}}^{\text{dist}}$  precomposed with the isomorphism  $\mathcal{F} \rightarrow \mathcal{F}^+|_{\text{Zar}}$  is  $\varphi$ ; this combined with the fact that if  $\varphi$  is an isomorphism, then so is  $\varphi_x : \mathcal{F}_x \rightarrow (\mathcal{G}_{\text{Zar}})_x \xrightarrow{\sim} \mathcal{G}_x$  finishes the proof. Indeed, one shows existence by the following construction: given an open  $V \subset X$  and  $\sigma = (\sigma_x) \in \mathcal{F}^+(V)$ , pick an affine open cover  $V = \bigcup U_{\alpha}$  and  $\sigma_{\alpha} \in \mathcal{F}(U_{\alpha})$  such that for all  $\alpha$  and all  $x \in U_{\alpha}$  we have  $\sigma_x = \sigma_{\alpha}|_x$ . Consider the elements  $\varphi(\sigma_{\alpha}) \in \mathcal{G}(U_{\alpha})$ . We claim that for all  $\alpha, \beta$  we have  $\varphi(\sigma_{\alpha})|_{U_{\alpha\beta}} = \varphi(\sigma_{\beta})|_{U_{\alpha\beta}}$ , and this follows from the sheaf axiom for  $\mathcal{G}$  and using 1.2.3 to produce an open cover  $U_{\alpha\beta} = \bigcup_{\gamma} U_{\alpha\beta\gamma}$  where  $\sigma_{\alpha}|_{U_{\alpha\beta\gamma}} = \sigma_{\beta}|_{U_{\alpha\beta\gamma}}$  and  $U_{\alpha\beta\gamma}$  is simultaneously distinguished in both  $U_{\alpha}$  and  $U_{\beta}$ , for all  $\alpha, \beta, \gamma$ . Therefore, the  $\varphi(\sigma_{\alpha})$  glue to produce a unique element of  $\mathcal{G}(V)$  which we call  $\varphi^+(\sigma)$ . The verification that this is well-defined, restricts to  $\varphi$ , and is uniquely determined by this condition is left to the reader. ■

**Remark 1.2.5.** In fancy language, the above results say that for any scheme  $X$ , the site  $X_{\text{Zar}}^{\text{dist}}$  consisting of affine opens and distinguished inclusions is enough to capture all (Zariski-) sheafy business on  $X$ . Precisely, the continuous inclusion  $X_{\text{Zar}}^{\text{dist}} \hookrightarrow X_{\text{Zar}}$  gives rise to an adjoint equivalent of categories  $\text{Shv}(X_{\text{Zar}}^{\text{dist}}, \mathcal{O}) \rightleftarrows \text{Shv}(X_{\text{Zar}}, \mathcal{O})$ , where  $\mathcal{O}$  denotes either a constant sheaf of rings or the structure sheaf ([1, Theorem 6.2.2]). This adjoint equivalence preserves various structures on these categories such as (quasi)coherence (1.3.2).

### 1.3 Quasicoherent Sheaves

**Theorem/Definition 1.3.1** (QCoh(Affine)). Let  $X$  be an affine scheme with  $A := \mathcal{O}(X)$ .

- (a) The global sections functor  $\Gamma : \mathcal{O}_X\text{-Mod} \rightarrow A\text{-Mod}$  admits a left adjoint  $\widetilde{\phantom{x}}$ .
- (b) This left adjoint  $\widetilde{\phantom{x}}$  is a fully faithful and faithfully exact additive monoidal functor.

The full subcategory on the essential image of  $\widetilde{\phantom{x}}$  is called the subcategory of *quasicoherent sheaves* on  $X$ , and denoted  $\text{QCoh}(X)$ .

- (c) The subcategory  $\text{QCoh}(X) \subset \mathcal{O}_X\text{-Mod}$  is closed under arbitrary direct sums, taking (co)kernels, and taking tensor products, and is hence a full abelian symmetric monoidal subcategory of  $\mathcal{O}_X\text{-Mod}$ .
- (d) An  $\mathcal{O}_X$ -module  $\mathcal{F}$  lies in  $\text{QCoh}(X)$  iff the co-unit  $\varepsilon_{\mathcal{F}} : \widetilde{\mathcal{F}}(X) \rightarrow \mathcal{F}$  is an isomorphism.
- (e) The functors  $\widetilde{\phantom{x}} : A\text{-Mod} \rightarrow \text{QCoh}(X) : \Gamma$  are inverse equivalences of abelian symmetric monoidal categories.

*Proof.*

- (a) We give two constructions of  $\widetilde{\phantom{x}}$ .<sup>2</sup>

1. If  $\tau : (X, \mathcal{O}_X) \rightarrow (X, \underline{A}_X)$  is the change of structure morphism, then we have the adjunctions

$$\begin{array}{ccccc}
 A\text{-Mod} & \xrightarrow{[\cdot]} & \underline{A}_X\text{-PMod} & \xrightarrow{\text{sh}} & \underline{A}_X\text{-Mod} & \xrightarrow{\tau^*} & \mathcal{O}_X\text{-Mod} \\
 \uparrow \Gamma & \perp & \uparrow \text{incl} & \perp & \uparrow \text{Res}_{\underline{A}_X}^{\mathcal{O}_X} = \tau_* & \perp & \\
 & & & & & & 
 \end{array}$$

Therefore, the adjoint is given by the composite  $M \mapsto \widetilde{M} = \mathcal{O}_X \otimes_{\underline{A}_X} \underline{M}_X$ .

2. We factor via the adjunctions

$$\begin{array}{ccccc}
 A\text{-Mod} & \xrightarrow{[\cdot]_{\text{Zar}}} & \mathcal{O}_{X_{\text{Zar}}}^{\text{dist}}\text{-Mod} & \xrightarrow{+} & \mathcal{O}_X\text{-Mod} \\
 \uparrow \Gamma & \perp & \uparrow [\cdot]_{\text{Zar}} = [\cdot]_{X_{\text{Zar}}}^{\text{dist}} & \perp & \\
 & & & & 
 \end{array}$$

Explicitly, given  $M \in A\text{-Mod}$ , consider the  $\mathcal{O}_{X_{\text{Zar}}}^{\text{dist}}$ -premodule  $\widetilde{M}_{\text{Zar}}$  given by  $U \mapsto \mathcal{O}(U) \otimes_A M$ . Check, as in the definition of  $\mathcal{O}$  [TODO], that this is actually an  $\mathcal{O}_{X_{\text{Zar}}}^{\text{dist}}$ -module: if  $U = \bigcup D_{\alpha}$  is a finite cover of  $U$  by distinguished affine opens, then the sequence

$$0 \rightarrow \mathcal{O}(U) \otimes_A M \rightarrow \bigoplus_{\alpha} \mathcal{O}(D_{\alpha}) \otimes_A M \rightarrow \bigoplus_{\alpha, \beta} \mathcal{O}(D_{\alpha\beta}) \otimes_A M$$

of  $A$ -modules is exact. The proof 1.2.4 (or its slightly strengthened version 1.2.5), produces an  $\mathcal{O}_X$ -module  $\widetilde{M}$  such that for every affine  $U \subset X$ , the map  $\mathcal{O}(U) \otimes_A M \rightarrow \widetilde{M}(U)$  is an isomorphism, i.e.,  $\widetilde{M}_{\text{Zar}} \xrightarrow{\sim} \widetilde{M}|_{X_{\text{Zar}}^{\text{dist}}}$ . Then for any  $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ , we have the natural bijections

$$\text{Hom}_{\mathcal{O}_X\text{-Mod}}(\widetilde{M}, \mathcal{F}) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_{X_{\text{Zar}}}^{\text{dist}}}(\widetilde{M}_{\text{Zar}}, \mathcal{F}_{\text{Zar}}) \xrightarrow{\sim} \text{Hom}_{A\text{-Mod}}(M, \mathcal{F}(X)),$$

where the first map is given by restriction (1.2.5), and the second by taking global sections (check!).

- (b) To show that  $\widetilde{\phantom{x}}$  is fully faithful, it suffices to show that for each  $M \in A\text{-Mod}$ , the unit  $\eta_M : M \rightarrow \widetilde{M}(X)$  is an  $A$ -module isomorphism, but this is clear from the second construction above. For  $M \in A\text{-Mod}$  and  $p \in X$ , there is a natural isomorphism  $M_p \rightarrow \widetilde{M}_p$  of modules over  $\mathcal{O}_{X,p} = A_p$ , so faithful exactness follows from the corresponding fact in commutative algebra about the localizations of a module at primes. This functor is clearly additive and preserves tensor products, for instance, because all the functors it is a composite of in the first construction do.

The statements in (c), (d), and (e) follow immediately from (a) and (b). ■

<sup>2</sup>These two constructions are not *obviously* the same, but they are a posteriori by the uniqueness of adjoints.

**Theorem/Definition 1.3.2** ( $\mathbf{QCoh}(\text{Scheme})$ ). Let  $X$  be a scheme and  $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ . The following conditions on  $\mathcal{F}$  are equivalent.

- (a) For every affine open  $U \subset X$  and distinguished open  $D \subset U$ , the natural  $\mathcal{O}(D)$ -module morphism  $\mathcal{O}(D) \otimes_{\mathcal{O}(U)} \mathcal{F}(U) \rightarrow \mathcal{F}(D)$  is an isomorphism.
- (b) For every affine open  $U \subset X$ , we have  $\mathcal{F}|_U \in \mathbf{QCoh}(U)$ .
- (c) There is an affine open cover  $\mathcal{U}$  of  $X$  such that for all  $U \in \mathcal{U}$ , we have  $\mathcal{F}|_U \in \mathbf{QCoh}(U)$ .
- (d) For each  $x \in X$ , there is a neighborhood  $U$  of  $x$  in  $X$ , sets  $I$  and  $J$  and an exact sequence of  $\mathcal{O}_U$ -modules<sup>3</sup> of the form  $\mathcal{O}_U^{\oplus J} \rightarrow \mathcal{O}_U^{\oplus I} \rightarrow \mathcal{F} \rightarrow 0$ .

An  $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$  is said to be *quasicoherent* if it satisfies these equivalent conditions. Further:

- (e) The full subcategory  $\mathbf{QCoh}(X)$  of  $\mathcal{O}_X\text{-Mod}$  consisting of quasicoherent  $\mathcal{O}_X$ -modules is closed under arbitrary direct sums, taking (co)kernels, and taking tensor products, and is hence a full abelian symmetric monoidal subcategory of  $\mathcal{O}_X\text{-Mod}$ .

*Proof.*

- (a)  $\Leftrightarrow$  (b) Follows from 1.3.1(d).
- (b)  $\Rightarrow$  (c) Clear from 1.2.1(d).
- (c)  $\Rightarrow$  (d) Modules admit presentations, and so the result follows from 1.3.1(b).
- (d)  $\Rightarrow$  (b) Without loss of generality,  $X$  is affine. If there were a global such exact sequence, then  $\mathcal{F}$  would be quasicoherent because  $\mathbf{QCoh}(X)$  is closed under taking cokernels and free sheaves are obviously quasicoherent. Therefore, the result follows immediately from the following lemma (1.3.3).

The statement in (e) then follows from 1.3.1(c). ■

**Lemma 1.3.3.** Let  $X$  be an affine scheme and  $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ . Suppose there is a finite cover  $X = \bigcup_{\alpha} D_{\alpha}$  by distinguished open affines such that for all  $\alpha$ , we have  $\mathcal{F}|_{D_{\alpha}} \in \mathbf{QCoh}(D_{\alpha})$ . Then  $\mathcal{F} \in \mathbf{QCoh}(X)$ .

*Proof.* We want to show that for all distinguished opens  $D \subset X$ , the natural morphism  $\mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \rightarrow \mathcal{F}(D)$  is an isomorphism. For this, consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(X) & \longrightarrow & \bigoplus_{\alpha} \mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(D_{\alpha}) & \longrightarrow & \bigoplus_{\alpha, \beta} \mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(D_{\alpha\beta}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{F}(D) & \longrightarrow & \bigoplus_{\alpha} \mathcal{F}(D \cap D_{\alpha}) & \longrightarrow & \bigoplus_{\alpha, \beta} \mathcal{F}(D \cap D_{\alpha\beta}),
 \end{array}$$

where the top row comes from the sheaf axiom for  $\mathcal{F}$ , the finiteness of the cover, and the fact that the  $\mathcal{O}(X)$ -algebra  $\mathcal{O}(D)$  is flat. Consider the middle column: for each  $\alpha$ , we have

$$\mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(D_{\alpha}) \cong \mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{O}(D_{\alpha}) \otimes_{\mathcal{O}(D_{\alpha})} \mathcal{F}(D_{\alpha}) \cong \mathcal{O}(D \cap D_{\alpha}) \otimes_{\mathcal{O}(D_{\alpha})} \mathcal{F}(D_{\alpha}),$$

and under this isomorphism, the middle map is the obvious one; a similar observation holds for the last column. Since for all  $\alpha$ , we have  $\mathcal{F}|_{D_{\alpha}} \in \mathbf{QCoh}(D_{\alpha})$ , the middle map is an isomorphism; since for all  $\alpha$  and  $\beta$ , we have  $\mathcal{F}|_{D_{\alpha\beta}} \in \mathbf{QCoh}(D_{\alpha\beta})$  (which follows because the restriction of a quasicoherent sheaf on affine scheme to a distinguished open is clearly quasicoherent, and  $D_{\alpha\beta}$  is distinguished in both  $D_{\alpha}$  and  $D_{\beta}$ ), the last column is an isomorphism. Then the five-lemma tells us that  $\mathcal{O}(D) \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \rightarrow \mathcal{F}(D)$  is also an isomorphism, as needed. ■

**Lemma 1.3.4.** Let  $X$  be an affine scheme and  $\mathcal{F} \in \mathbf{QCoh}(X)$ .

- (a) Let  $f \in \mathcal{O}(X)$  and  $D = D(f) \subset X$  be the corresponding distinguished open.
  - (i) If  $\sigma \in \mathcal{F}(X)$  is such that  $\sigma|_D = 0$ , then there is an  $n \gg 1$  such that  $f^n \sigma = 0 \in \mathcal{F}(X)$ .
  - (ii) Given a section  $\sigma \in \mathcal{F}(D)$ , there is an  $n \gg 1$  such that  $f^n \sigma$  comes from an element of  $\mathcal{F}(X)$ .
- (b) Suppose we are given a finite cover  $X = \bigcup_{\alpha} D_{\alpha}$  of  $X$  by distinguished affine opens, and suppose for all  $\alpha, \beta$ , we are given  $\sigma_{\alpha\beta} \in \mathcal{F}(D_{\alpha\beta})$  such that for all  $\alpha, \beta, \gamma$  we have  $\sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} = 0$  in  $\mathcal{F}(D_{\alpha\beta\gamma})$ .<sup>4</sup> Then there are elements  $\tau_{\alpha} \in \mathcal{F}(D_{\alpha})$  for all  $\alpha$  such that for all  $\alpha, \beta$ , we have  $\tau_{\beta} - \tau_{\alpha} = \sigma_{\alpha\beta} \in \mathcal{F}(D_{\alpha\beta})$ .

<sup>3</sup>Here  $\mathcal{O}_U := \mathcal{O}_X|_U$ .

<sup>4</sup>Here, and in what follows, restrictions are left implicit.

*Proof.* The statement in (a) is clear. For (b), first pick  $f_\alpha \in \mathcal{O}(X)$  such that for all  $\alpha$  we have  $D_\alpha = D(f_\alpha)$ . Next, fix a  $\gamma$ . Using (a)(ii), produce an  $n \gg 1$  and for all  $\alpha$  a section  $\tau_\alpha^\gamma \in \mathcal{F}(D_\alpha)$  that restricts to  $f_\gamma^n \cdot \sigma_{\gamma\alpha} \in \mathcal{F}(D_{\alpha\gamma})$ . Use (a)(i) and the cocycle condition to increase  $n$  if needed to assume further that for all  $\alpha, \beta$  we have  $\tau_\beta^\gamma - \tau_\alpha^\gamma = f_\gamma^n \sigma_{\alpha\beta} \in \mathcal{F}(D_{\alpha\beta})$ . Since  $(f_\gamma^n) = (1)$  (1.2.1(e)), there are  $c_\gamma \in \mathcal{O}(X)$  such that  $\sum_\gamma c_\gamma f_\gamma^n = 1$ . For each  $\alpha$ , let  $\tau_\alpha := \sum_\gamma c_\gamma \tau_\alpha^\gamma$ . This works. ■

**Theorem 1.3.5.** Let  $X$  be a scheme and  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  be an exact sequence of  $\mathcal{O}_X$ -modules.

- (a) If  $\mathcal{F}' \in \text{QCoh}(X)$ , then for any affine open  $U \subset X$ , the sequence  $0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U) \rightarrow 0$  of sections on  $U$  is exact.
- (b) If  $\mathcal{F}', \mathcal{F}'' \in \text{QCoh}(X)$ , then also  $\mathcal{F} \in \text{QCoh}(X)$ . In other words,  $\text{QCoh}(X)$  is stable under extensions.

*Proof.*

- (a) This is a formal consequence of 1.3.4(b).
- (b) The result follows from 1.3.2(a), our (a), and the five lemma. ■

**Remark 1.3.6.** 1.3.4(b) is saying, of course, that for any affine scheme  $X$ , cover  $\mathcal{D}$  by finitely many distinguished affines, and  $\mathcal{F} \in \text{QCoh}(X)$ , the Čech cohomology  $\check{H}^1(\mathcal{D}, \mathcal{F})$  vanishes. This is an important idea that will come back repeatedly.

## 1.4 Exercises

**Exercise 1.1.** Let  $X$  be a scheme and  $p \in X$ .

- (a) Produce a natural bijection between the set of irreducible components of  $X$  containing  $p$  and the minimal primes of the ring  $\mathcal{O}_{X,p}$ . (The morphism  $\mathrm{Spec} \mathcal{O}_{X,p} \rightarrow X$  “sees” all the irreducible components of  $p$  in  $X$ .)
- (b) Show that if  $\mathcal{O}_{X,p}$  is a domain, then there is a unique irreducible component of  $X$  passing through  $p$ . In particular, if this is true of all  $p \in X$ , then the irreducible components of  $X$  are disjoint.

**Exercise 1.2.** Let  $X \rightarrow Z \leftarrow Y$  be morphisms of schemes. For each  $p \in Z$  with residue field  $\kappa(p)$ , produce a natural isomorphism between the fiber  $(X \times_Z Y)_p$  and  $X_p \times_{\kappa(p)} Y_p$ . (In this sense, the fiber product of schemes is the “fiber-wise product”.)

**Exercise 1.3.** Let  $R$  be a Dedekind domain with  $K := \mathrm{Frac} R$ , and let  $X := \mathrm{Spec} R$  have generic point  $\eta$  so that  $\mathcal{O}_{X,\eta} \xrightarrow{\sim} K$ .

- (a) Produce a natural bijection between the set of nonzero fractional ideals of  $R$  and the set of pairs  $(\mathcal{L}, \varphi)$ , where  $\mathcal{L} \rightarrow X$  is a line bundle and  $\varphi : \mathcal{L}_\eta \xrightarrow{\sim} K$  is an  $R^\times$ -equivalence class of  $K$ -vector-space trivializations of  $\mathcal{L}$  at the generic point.
- (b) Use (a) to give a group isomorphism  $\mathrm{Pic}(R) \xrightarrow{\sim} \mathrm{Pic}(X)$ .

## Chapter 2

# Topology for Algebraic Geometers

## 2.1 Irreducible, Kolmogorov, and Sober Spaces

In this section, we introduce irreducible, Kolmogorov, and sober spaces. These concepts are absolutely fundamental to understanding the topology of schemes. To motivate the definition, we first recall the following facts about connected spaces.

**Proposition/Definition 2.1.1** (Connected Spaces).

- The following conditions on a topological space  $X$  are equivalent.
  - (a) The space  $X$  cannot be expressed as a union of two proper clopen subspaces.
  - (b) Any nonempty clopen subspace must be all of  $X$ .
  - (c) The only subspaces of  $X$  with empty boundary are  $\emptyset$  and  $X$ .
  - (d) There does not exist a surjective continuous function  $X \rightarrow \{0, 1\}$ , where  $\{0, 1\}$  is given the discrete topology.
- A topological space is said to be *connected* iff it satisfies these equivalent conditions.
- A *connected component* of a space is a maximal connected subspace.

*Proof.* Exercise. ■

Recall the following basic properties.

**Lemma 2.1.2.**

- (a) If  $f : X \rightarrow Y$  is a continuous map with  $X$  connected, then  $f(X) \subset Y$  is connected.
- (b) If  $Y \subset X$  is a subspace and  $Y$  is connected, then so is  $\bar{Y}$ .<sup>1</sup> In particular, any connected component of  $X$  is closed in  $X$ .<sup>2</sup>
- (c) A topological space that can be covered by a family  $\{A_i\}$  of connected subspaces such that the total intersection  $\bigcap_i A_i$  is nonempty is connected.
- (d) Every connected subset of a topological space is contained in a unique connected component. In particular, distinct connected components of a space are disjoint, and every space is the (disjoint) union of its connected components.
- (e) Any clopen subset of a topological space is a disjoint union of some of its connected components.

*Proof.* Exercise. We note only that (c) uses Zorn's Lemma for existence. ■

Often in algebraic geometry, we need a slightly stronger topological property than being connected. This is provided by

**Proposition/Definition 2.1.3** (Irreducible Spaces).

- The following conditions on a nonempty topological space are equivalent.
  - (a) The space cannot be expressed as a union of two proper closed subspaces.
  - (b) Any two nonempty open subspaces have nonempty intersection.
  - (c) Every nonempty open subspace is dense.
  - (d) A subspace is not dense iff it is nowhere dense.
  - (e) Every nonempty open subspace is connected.
- A topological space is said to be *irreducible* iff it is nonempty and it satisfies these conditions.
- An *irreducible component* of a space is a maximal irreducible subspace.
- Given a space  $X$ , a point  $\eta \in X$  is said to be a *generic point* of  $X$  iff  $X = \overline{\{\eta\}}$ .

*Proof.* Exercise. ■

It is immediate that any irreducible space is connected.

**Example 2.1.4.** The space  $\mathbb{C}$  with the Euclidean topology is connected but not irreducible, whereas the affine line  $\mathbb{A}_{\mathbb{C}}^1 \cong \text{Spec } \mathbb{C}[x]$  with the Zariski topology is irreducible.

<sup>1</sup>The converse is clearly not true.

<sup>2</sup>It is not true in general that a connected component is also necessarily open. A counterexample is given by taking the product topology on  $2^{\mathbb{N}}$ , or if you prefer  $\text{Spec } \mathbb{F}_2^{\mathbb{N}}$ .

The analogous basic properties for irreducibility are given in

**Lemma 2.1.5.**

- (a) If  $f : X \rightarrow Y$  is a continuous map with  $X$  irreducible, then  $f(X) \subset Y$  is irreducible.
- (b) If  $X$  is any topological space and  $Y \subset X$  a subspace, then  $Y$  is irreducible iff  $\overline{Y}$  is. In particular, any irreducible component of  $X$  is closed in  $X$ , and a dense subspace of an irreducible space is irreducible.
- (c) Every nonempty open subspace of an irreducible space is irreducible.
- (d) A topological space that admits an open cover by irreducible open subspaces with nonempty pairwise intersections is irreducible.
- (e) Every irreducible subspace of a topological space is contained in an irreducible component. In particular, every topological space is the union of its irreducible components.<sup>3</sup>
- (f) If  $x \in X$  is a point in a space, then the closure  $\overline{\{x\}} \subset X$  is an irreducible closed subspace of  $X$ . In particular, a space admitting a generic point is irreducible, and the set of generic points of such a space is precisely the intersection of all its nonempty open subsets.
- (g) If  $X$  is any topological space and  $U \subset X$  a nonempty open subspace, then there is an inclusion-preserving bijection

$$\begin{aligned} \{\text{irreducible closed } Y \subset U\} &\leftrightarrow \{\text{irreducible closed } Z \subset X \text{ with } Z \cap U \neq \emptyset\} \text{ given by} \\ Y &\mapsto \overline{Y}, \\ Z \cap U &\mapsto Z. \end{aligned}$$

This bijection restricts to one between irreducible components of  $U$  and those of  $X$  meeting  $U$  (or equivalently those irreducible subsets  $Z \subset X$  such that the intersection  $Z \cap U \subset Z$  is dense).

*Proof.* Exercise; the proofs are very similar to those in 2.1.2. We prove only (b) to give a flavor of the proofs. Note that a subspace  $Y \subset X$  is irreducible iff for any two open sets  $U, V$  of  $X$  meeting  $Y$ , the intersection  $U \cap V$  meets  $Y$ . The result follows from noting that an open subspace of  $X$  meets  $Y$  iff it meets  $\overline{Y}$ . ■

A relationship between these two notions is provided in

**Lemma 2.1.6.**

- (a) If  $X$  is a topological space,  $Y \subset X$  a connected component and  $Z \subset X$  an irreducible subspace, then  $Y \cap Z \neq \emptyset$  implies  $Z \subset Y$ . In particular, every connected component of a topological space is the union of some of its irreducible components.
- (b) A connected space is irreducible iff it is nonempty and it admits a cover by irreducible open subspaces.

*Proof.* Exercise. ■

Next, we recall some point-set topology before introducing sober spaces.

**Proposition/Definition 2.1.7** (Kolmogorov Spaces). The following conditions on a topological space  $X$  are equivalent.

- (a) If  $x \neq y \in X$ , then there is an open subspace  $U \subset X$  containing exactly one of  $x$  and  $y$ .
- (b) The map  $X \rightarrow \text{Irred}(X)$  given by  $x \mapsto \overline{\{x\}}$  is an injection, where  $\text{Irred}(X)$  is the set of irreducible closed subsets of  $X$ .
- (c) The specialization relation on  $X$  given by  $x \rightsquigarrow y$  iff  $y \in \overline{\{x\}}$  is a partial order.

A topological space is said to be *Kolmogorov* or  $T_0$  if it satisfies these equivalent conditions.

*Proof.* Clear. ■

<sup>3</sup>It is evident that not every topological space is the *disjoint* union of its irreducible components.

In a Kolmogorov space, the minimal points for the specialization preorder are the closed points, and the maximal points are the generic points of irreducible components (when they exist).

**Definition 2.1.8** (Sober Spaces). A topological space  $X$  is said to be *sober* if the map

$$X \rightarrow \text{Irred}(X), \quad x \mapsto \overline{\{x\}}$$

is a bijection, i.e., every irreducible closed subspace of  $X$  has a unique generic point.

**Remark 2.1.9.**

- (a) A sober space is, in particular, Kolmogorov. In [2], a Noetherian sober space is called a Zariski space.
- (b) In an irreducible sober space, the intersection of all nonempty open subsets consists of a single point—the generic point.

**Lemma 2.1.10.**

- (a) A topological space admitting an open cover by sober spaces is itself sober. In particular, the underlying topological space of any scheme is sober.
- (b) A constructible subspace<sup>4</sup> of a sober space is sober.

*Proof.*

- (a) This is clear for Kolmogorov spaces, and generic points can be produced locally (using 2.1.5(g)). The second statement follows from the first because the underlying topological space of an affine scheme is clearly sober.
- (b) The statement is clearly true for locally closed subspaces; for constructible subspaces it follows from 2.3.3(b). ■

**Remark 2.1.11.** If  $X$  is a sober space and  $Y \subset X$  an arbitrary subspace, it is not necessarily true that  $Y$  is also sober. For a simple counterexample, take  $X = \mathbb{A}_{\mathbb{C}}^1$  with generic point  $\eta \in X$  and let  $Y := X \setminus \{\eta\}$ .

Here are a few of applications of these concepts that are often useful (and help provide clean solutions to the exercises in, say, [2, §II.3])

**Proposition 2.1.12.** Let  $f : X \rightarrow Y$  be an open continuous map of topological spaces and suppose that  $Y$  is irreducible and has a generic point  $\eta$ . The following conditions are equivalent.

- (a) The generic fiber  $X_{\eta} = f^{-1}(\eta)$  is irreducible.
- (b) There is a dense collection of points  $y \in Y$  such that the fiber  $X_y = f^{-1}(y)$  is irreducible.
- (c) The space  $X$  is irreducible.

*Proof.*

- (a)  $\Rightarrow$  (b) Consider the collection  $\{\eta\}$ .
- (b)  $\Rightarrow$  (c) Suppose  $X = Z_1 \cup Z_2$  for proper closed  $Z_1, Z_2 \subset X$ . Since  $f$  is open, for  $i = 1, 2$ , the subset  $f(X \setminus Z_i) \subset Y$  is open; since  $Y$  is irreducible, then intersection  $f(X \setminus Z_1) \cap f(X \setminus Z_2)$  is nonempty and open; by the density hypothesis, there is a  $y \in f(X \setminus Z_1) \cap f(X \setminus Z_2)$  such that  $X_y$  is irreducible. Then  $X_y \cap Z_i$  for  $i = 1, 2$  are proper closed subspaces of  $X_y$  whose union is  $X_y$ , which is a contradiction.
- (c)  $\Leftrightarrow$  (a) For any subspace  $Z \subset Y$ , continuity of  $f$  tells us that  $\overline{f^{-1}(Z)} \subset f^{-1}(\overline{Z})$  with equality when  $f$  is open. Applying this to  $Z = \{\eta\}$  gives us that  $\overline{X_{\eta}} = f^{-1}(\overline{\{\eta\}}) = f^{-1}(Y) = X$ , and so the result follows from 2.1.5(b). ■

**Proposition/Definition 2.1.13.** Let  $f : X \rightarrow Y$  be a continuous map of irreducible sober spaces, and let  $\xi$  (resp.  $\eta$ ) be the generic point of  $X$  (resp.  $Y$ ). The following are equivalent:

- (a)  $f(\xi) = \eta$ .

---

<sup>4</sup>See 2.3.3 below if needed.

- (b)  $\eta \in f(X)$ .
- (c)  $\overline{f(X)} = Y$ .
- (d) If  $\emptyset \neq U \subset Y$  is open, then  $f^{-1}(U) \neq \emptyset$ .

A map  $f$  satisfying these equivalent properties is said to be *dominant*, and we say that  $X$  *dominates*  $Y$  (via  $f$ ).

*Proof.* It is clear that the (c)  $\Leftrightarrow$  (d) holds very generally (i.e., does not need the irreducible or sober hypotheses). The implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d) are clear. For (d)  $\Rightarrow$  (a), note that if  $\emptyset \neq U \subset Y$  is open, then  $f^{-1}(U) \ni \xi$ , whence  $f(\xi) \in U$ , and so the result follows from 2.1.9(b). ■

**Definition 2.1.14.** When  $X$  and  $Y$  are sober (but not necessarily irreducible) topological spaces, we say that a continuous map  $f : X \rightarrow Y$  is *dominant* iff each irreducible component of  $Y$  is dominated by an irreducible component of  $X$ .

**Remark 2.1.15.** In algebraic geometry, this is a slightly better definition than “dense image” (conditions (c) and (d) above), which is not as well-behaved. In particular, our definition of dominant implies dense image, but is in general stronger.<sup>5</sup> Besides, in most cases of interest (e.g., when the morphism  $f$  is quasicompact and  $X, Y$  are schemes; see 3.3.3(a)), the two definitions are equivalent.

**Proposition 2.1.16.** Let  $f : X \rightarrow Y$  be a continuous map of sober spaces, and suppose that  $Y$  is irreducible. Let  $\eta \in Y$  be the generic point and  $X_\eta = f^{-1}(\eta)$  the generic fiber. There is an inclusion preserving bijection

$$\begin{aligned} \{\text{irreducible closed } Z \subset X_\eta\} &\leftrightarrow \{\text{irreducible closed } W \subset X \text{ dominating } Y\} \text{ given by} \\ Z &\mapsto \overline{Z}, \\ W_\eta = W \cap X_\eta &\leftarrow W. \end{aligned}$$

Further, under this bijection, irreducible components of  $X_\eta$  correspond to irreducible components of  $X$  dominating  $Y$ .

*Proof.* Clear from 2.1.13. (Note that we are also using 2.1.10(b).) ■

<sup>5</sup>Note that this convention differs from the one in [3]. For an example where these definitions are not equivalent, consider the natural morphism  $\coprod_{n \in \mathbb{Z}} \text{Spec } \mathbb{C} \rightarrow \mathbb{A}_{\mathbb{C}}^1$  given by the inclusion of the integers.

## 2.2 Closure-Complete and Noetherian Spaces

In this section, we describe two fundamental finiteness conditions on topological spaces: closure-completeness and the Noetherian property.

**Definition 2.2.1.** A topological space is *closure-complete* if every point has a closed point in its closure.

**Remark 2.2.2.** A nonempty closed subspace of a closure-complete space contains a closed point, and in particular, every nonempty closure-complete space has a closed point. For a Kolmogorov space, closure-completeness is equivalent to saying that every point specializes to a minimal point for the specialization partial order.

**Lemma 2.2.3.** A Kolmogorov space admitting a finite open cover by closure-complete spaces is closure-complete. In particular, the underlying topological space of any quasicompact scheme is closure-complete and hence contains a closed point.

*Proof.* By induction on the size of the open cover, we are reduced to the case of two open sets, so say  $X = U \cup V$  with  $U, V$  closure-complete open subspaces. It suffices to show that if  $x \in U$ , then  $x$  has a closed point in its closure. Since  $U$  is closure-complete, there is a  $y \in \overline{\{x\}} \cap U$  which is closed in  $U$ . If  $y \notin V$ , then  $y$  is also closed in  $X$ . Else,  $y \in V$  and so since  $V$  is closure-complete, there is a  $z \in \overline{\{y\}} \cap V$  which is closed in  $V$ . Then  $z$  is closed in  $X$ . Indeed, if  $w \in \overline{\{z\}} \setminus \{z\}$ , then  $w \notin V$  and so  $w \in \overline{\{z\}} \cap U \subset \overline{\{y\}} \cap U = \{y\}$ , but  $y \notin \overline{\{z\}}$  by the Kolmogorov hypothesis. The second result follows from the first, since the underlying space of an affine scheme is closure-complete (by Zorn's Lemma!). ■

For a scheme that is not closure-complete (and, in fact, has no closed points at all), see [4, Exercise 3.3.27].

---

**Proposition/Definition 2.2.4** (Noetherian Spaces). The following conditions on a topological space are equivalent.

- (a) Every descending chain of closed subspaces is eventually stationary.
- (b) Every non-empty collection of closed subspaces has a minimal element with respect to inclusion.
- (c) Every ascending chain of open subspaces is eventually stationary.
- (d) Every non-empty collection of open subspaces has a maximal element with respect to inclusion.

A topological space is *Noetherian* if it satisfies the above equivalent conditions.

*Proof.* Exercise. ■

**Example 2.2.5.** Let  $R$  be a Noetherian ring. Then the underlying topological space of  $\text{Spec } R$  is Noetherian. This follows from the inclusion-reversing bijection between closed subsets of  $\text{Spec } R$  and radical ideals of  $R$ .

**Example 2.2.6.** The space  $\mathbb{C}$  equipped with the analytic topology is not Noetherian.

Here are some basic properties.

**Lemma 2.2.7.**

- (a) A topological space that admits a finite cover by Noetherian spaces is Noetherian.
- (b) Every subspace of a Noetherian space is Noetherian.
- (c) A Noetherian space has only finitely many irreducible components, and hence only finitely many connected components.
- (d) A connected component of a Noetherian space is open. In particular, the union of any collection of connected components is clopen.
- (e) A Noetherian space is quasicompact.

*Proof.*

- (a) Follows from 2.2.4(a).

- (b) If  $X$  is Noetherian,  $Y \subset X$  a subspace, and  $(Z_j)$  a descending chain of closed subspaces in  $Y$ , then the chain  $(\overline{Z_j})$  in  $X$  is eventually stationary, and hence we are done by  $Z_j = Y \cap \overline{Z_j}$ .
- (c) Apply 2.2.4(b) to the collection of closed subspaces which cannot be written as a finite union of irreducible subspaces. The second result follows from the first, thanks to 2.1.6(a).
- (d) By 2.1.6(a), a connected component of a Noetherian space is the complement of the union of some of its irreducible components (namely those disjoint from it); therefore, the first statement follows from (c). The second result follows from the first combined (c).
- (e) Apply 2.2.4(d) to the collection of open subspaces consisting of finite unions of elements of a given open cover.

■

**Remark 2.2.8.**

- (a) Note that if  $X$  is Noetherian scheme, then the underlying topological space of  $X$  is Noetherian, but the converse is false in general (e.g., take  $X = \operatorname{Spec} k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ .) “Noetherian” (along with the corresponding “locally Noetherian” below) is one of the very few adjectives for which this remark holds.
- (b) A scheme morphism out of a Noetherian scheme is automatically qcqs.

One of the key reasons we like Noetherian spaces is

**Proposition 2.2.9** (Irreducible Decomposition of Noetherian Space). Let  $X$  be a Noetherian space.

- (a) There is an integer  $r \geq 1$  and closed irreducible subspaces  $Z_1, \dots, Z_r \subset X$  such that  $X = Z_1 \cup \dots \cup Z_r$  such that for all  $i, j$  with  $1 \leq i \neq j \leq r$ , we have  $Z_i \not\subset Z_j$ , and such that if  $Z \subset X$  is any irreducible subspace, then there is an  $i = 1, \dots, r$  such that  $Z \subset Z_i$ .
- (b) The integer  $r$  and decomposition as in (a) are uniquely determined (up to re-ordering).

The decomposition  $X = Z_1 \cup \dots \cup Z_r$  is called the *irreducible decomposition* of  $X$ .

*Proof.* Take  $Z_j$  to be the irreducible components of  $X$ , using 2.2.7(c). ■

Another reason we like Noetherian spaces is that we can do Noetherian induction on them, as in

**Proposition 2.2.10** (Noetherian Induction). Let  $X$  be a Noetherian topological space, and let  $\mathcal{P}$  be a property satisfied by certain closed subsets of  $X$ . Suppose that for each closed subset  $Z \subset X$ , if  $\mathcal{P}$  holds for each proper  $Z' \subsetneq Z$ , then it holds for  $Z$ . Then the property  $\mathcal{P}$  holds for all closed subsets of  $X$ , and in particular for  $X$  itself.

*Proof.* This is just transfinite induction on the collection of closed subsets of  $X$  ordered by inclusion; as stated, this follows immediately from 2.2.4(b). ■

One kind of space that comes up often in algebraic geometry is a locally Noetherian space.

**Definition 2.2.11.** A topological space  $X$  is said to be *locally Noetherian* if for each  $x \in X$ , there is an open neighborhood  $U$  of  $x$  in  $X$  such that  $U$  is Noetherian as a topological space.

Note that if  $X$  is a locally Noetherian topological space and  $U \subset X$  any open subspace, then  $U$  is also locally Noetherian. Further, a topological space is Noetherian iff it is locally Noetherian and quasicompact. A few of the results from this section can be generalized to the locally Noetherian setting. Here’s one example.

**Proposition 2.2.12.** Let  $X$  be a locally Noetherian space. An open subset  $U \subset X$  is dense iff it meets all irreducible components of  $X$ .

*Proof.* If  $U$  meets all irreducible components of  $X$ , then it is dense in  $X$  by 2.1.3(c) and 2.1.5(d). To show the converse, suppose we are given a dense open subset  $U \subset X$  and an irreducible component  $Z \subset X$  such that  $U \cap Z = \emptyset$ . Since  $Z$  is irreducible, it is nonempty; pick any  $z \in Z$  and a Noetherian open neighborhood  $V$  of  $z$  in  $X$ . Using 2.1.5(g), we may replace  $(X, U, Z)$  by  $(V, U \cap V, Z \cap V)$  to

assume that in fact  $X$  is Noetherian. Let  $X = Z_1 \cup \cdots \cup Z_r$  for some  $r \in \mathbb{Z}_{\geq 1}$  and  $(Z_i)$  be the irreducible decomposition of  $X$  as in 2.2.9(a), numbered so that  $Z = Z_r$ . Then check that  $W = X \setminus (Z_1 \cup \cdots \cup Z_{r-1})$  is a nonempty open subset of  $X$  contained in  $Z$ ; this cannot happen, since then  $U$  must both meet  $W$  (since it is dense) and not (since  $W \subset Z$  and  $U \cap Z = \emptyset$ ). ■

It is an easy and instructive exercise, left to the reader, to come up with a space  $X$  (necessarily non-locally-Noetherian) and a dense open subset  $U \subset X$  that does not meet some irreducible components of  $X$ .

## 2.3 Locally Closed, Constructible, and Very Dense Subspaces

Now we would like to review some point-set topology about locally closed, constructible, and very dense subspaces.

**Proposition/Definition 2.3.1** (Locally Closed Subspaces). Let  $X$  be a topological space. The following conditions on a subspace  $Y \subset X$  are equivalent.

- (a) For each  $y \in Y$ , there is an open subspace  $U \subset X$  containing  $y$  such that  $Y \cap U$  is closed in  $U$ .
- (b) There is an open subspace  $U \subset X$  such that  $Y \subset U$  and  $Y$  is closed in  $U$ .
- (c) There is a closed subspace  $Z \subset X$  such that  $Y \subset Z$  and  $Y$  is open in  $Z$ .
- (d) There is an open subspace  $U \subset X$  and a closed subspace  $Z \subset X$  such that  $Y = U \cap Z$ .
- (e) There are open subspaces  $V \subset U \subset X$  such that  $Y = U \setminus V$ .
- (f) There are closed subspaces  $W \subset Z \subset X$  such that  $Y = Z \setminus W$ .
- (g)  $Y$  is an open subspace of its closure  $\overline{Y}$  in  $X$ .
- (h)  $\overline{Y} \setminus Y$  is a closed subspace of  $X$ .

A subspace  $Y$  satisfying these equivalent conditions is said to be a *locally closed subspace* of  $X$ . In this case, the largest open subspace  $U$  of  $X$  as in (b), i.e., containing  $Y$  and in which  $Y$  is closed, is  $U = X \setminus (\overline{Y} \setminus Y)$ .

*Proof.* The implications (a)  $\Leftarrow$  (b)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e)  $\Leftrightarrow$  (f) and (c)  $\Leftrightarrow$  (g)  $\Leftrightarrow$  (h) are clear. For (a)  $\Rightarrow$  (b), for each  $y \in Y$  pick a  $U_y$ , and let  $U := \bigcup_{y \in Y} U_y$ . The last claim is clear. ■

**Corollary 2.3.2.**

- (a) The preimage of a locally closed subspace is locally closed.
- (b) If  $Z \subset Y \subset X$  are such that  $Z$  is locally closed in  $Y$  and  $Y$  locally closed in  $X$ , then  $Z$  is locally closed in  $X$ .
- (c) The intersection of two locally closed subspaces is locally closed.
- (d) The complement of a locally closed subspace is the disjoint union of an open subspace and a closed subspace.

*Proof.* The statements (a) and (b) follow from 2.3.1(d), the statement (c) follows from (a) and (b), and (d) follows from either (e) or (f) of 2.3.1. ■

**Proposition/Definition 2.3.3** (Constructible Subspaces). Let  $X$  be a topological space. The following conditions on a subset  $Y \subset X$  are equivalent.

- (a)  $Y$  belongs to the algebra of subsets of  $X$  generated by the closed sets, i.e., the smallest collection of subsets containing the closed subsets which is closed under taking finite intersections and complements (and hence also under finite unions).
- (b)  $Y$  can be written as a finite disjoint union of locally closed subsets of  $X$ .
- (c) There is an integer  $n \in \mathbb{Z}_{\geq 1}$  and a nested sequence  $G_1 \supset G_2 \supset \cdots \supset G_n$  of closed subsets of  $X$  such that  $Y = G_1 \setminus (G_2 \setminus (G_3 \setminus (\cdots \setminus G_n))) \cdots$ .

If  $X$  is Noetherian, these properties are also equivalent to

- (d) For every closed irreducible  $Z \subset X$ , the intersection  $Y \cap Z$  either contains a nonempty open subset of  $Z$  or is nowhere dense in  $Z$ .

A subspace satisfying equivalent conditions (a)-(c) is said to be a *constructible* subspace of  $X$ .

*Proof.* Here we prove the equivalence of (a), (b), and (c). The proof of equivalence with (d) in the Noetherian case is deferred to 2.3.5.

- (a)  $\Leftrightarrow$  (b) It suffices to show that the collection of subsets described by (b) is closed under taking finite intersections and complements. The first of these is clear from 2.3.2(c), and the second follows from the first and 2.3.2(d).
- (c)  $\Rightarrow$  (b) Clear, since  $G_1 \setminus (G_2 \setminus (G_3 \setminus (\cdots \setminus G_n))) \cdots = (G_1 \setminus G_2) \amalg (G_3 \setminus G_4) \amalg \cdots$ .
- (a)  $\Rightarrow$  (c) This is somewhat trickier than it looks, and may be skipped on a first reading. One possible approach is to show directly that the collection of subsets described by (c) is closed under taking

finite intersections and complements. The latter is clear, but the former involves some nontrivial combinatorics and Boolean algebra. Here's a different approach.<sup>6</sup>

Given a topological space  $X$ , an  $n \in \mathbb{Z}_{\geq 1}$ , and a collection  $\mathcal{Z} = \{Z_1, \dots, Z_n\}$  of closed subsets of  $X$ , consider the Boolean subalgebra  $\mathcal{S}(\mathcal{Z})$  of subsets of  $X$  generated by the  $Z_i$ . This consists of finite unions of subsets of the form  $\bigcap_{i=1}^n Z_i^*$  where each  $Z_i^*$  is either  $Z_i$  or  $X \setminus Z_i$ ; in particular,  $\mathcal{S}(\mathcal{Z})$  has size at most  $2^{2^n}$ , and is therefore finite.<sup>7</sup> The subset  $\mathcal{S}(\mathcal{Z}) \subset 2^X$  inherits a natural partial order by inclusion, and we call an element  $A \in \mathcal{S}(\mathcal{Z})$  an *atom* iff  $A \neq \emptyset$  and  $A$  is minimal with respect to this partial order. Each atom  $A$  is locally closed (since it must itself be of the form  $\bigcap_{i=1}^n Z_i^*$  as above by minimality), distinct atoms are disjoint, and every element of  $\mathcal{S}(\mathcal{Z})$  is a (disjoint) union of atoms.

Suppose we are given a  $Y$  as in (a). Then there is a finite set  $\mathcal{Z}$  of closed subsets of  $X$  such that  $Y \in \mathcal{S}(\mathcal{Z})$ ; fix such a  $\mathcal{Z}$ . Write  $Y \in \mathcal{S}(\mathcal{Z})$  as a disjoint union of atoms, say  $Y = \coprod_{k=1}^m A_k$  for some  $m \in \mathbb{Z}_{\geq 1}$ , with the  $A_k$  pairwise distinct. Further express each  $A_k$  as  $A_k = S_k \setminus T_k$  for some closed sets  $T_k \subset S_k \subset X$ . We claim that it suffices to take  $n = 2m$ , with

$$G_{2k-1} = \bigcup_{j=k}^r S_j \text{ and } G_{2k} = T_k \cup \bigcup_{j=k+1}^r S_j$$

for  $1 \leq k \leq m$ . Since the inclusions  $G_1 \supset G_2 \supset \dots \supset G_n$  are clear, it suffices to show that for each  $k = 1, \dots, m$ , we have  $G_{2k-1} \setminus G_{2k} = A_k$ . The inclusion  $G_{2k-1} \setminus G_{2k} \subset A_k$  is clear. For the reverse inclusion, it suffices to show that if  $1 \leq k < k' \leq m$ , then  $A_k \cap S_{k'} = \emptyset$ , but this follows from the fact that distinct atoms are disjoint.<sup>8</sup>

■

#### Lemma 2.3.4.

- (a) The preimage of a constructible set under a continuous map is constructible. If  $Z \subset Y \subset X$  are such that  $Z$  is constructible in  $Y$  and  $Y$  is constructible in  $X$ , then  $Z$  is constructible in  $X$ .
- (b) Let  $X$  be an irreducible topological space and  $C \subset X$  a constructible subspace. Then the following are equivalent:
  - (i)  $C$  contains a nonempty open subspace of  $X$ .
  - (ii)  $C$  is dense.
 If, further,  $X$  has a generic point  $\eta$ , then (i) and (ii) are also equivalent to
  - (iii)  $C$  contains  $\eta$ .
- (c) In a locally Noetherian sober space, a subspace is closed (resp. open) iff it is constructible and stable under specialization (resp. generization).

*Proof.*

- (a) Follows from 2.3.2 and 2.3.3 (check!).
- (b) The implication (i)  $\Rightarrow$  (ii) follows from irreducibility. For (ii)  $\Rightarrow$  (i), use 2.3.3 to write  $C$  as a union of locally closed subspaces and use the irreducibility of  $X$  to reduce to the case where  $C$  is locally closed; then apply 2.3.1(g). When  $X$  has a generic point  $\eta$ , the implications (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii) are clear.
- (c) Noting that the question is local, and by taking complements for the assertion in parentheses, it suffices to show that if  $X$  is Noetherian sober,  $C \subset X$  constructible, and  $y \in \overline{C}$ , then there is a  $x \in C$  such that  $x \rightsquigarrow y$ . Use 2.2.7(b) and 2.2.9 to write an irreducible decomposition of  $C$  as  $C = Z_1 \cup \dots \cup Z_r$  for some  $r \in \mathbb{Z}_{\geq 1}$  and closed irreducible  $Z_1, \dots, Z_r \subset C$ . For each  $i$ , the closure  $\overline{Z_i}$  of  $Z_i$  in  $X$  is irreducible (by 2.1.5(c)); since  $X$  is sober,  $\overline{Z_i}$  has a generic point, say  $\eta_i$ . The intersection  $C \cap \overline{Z_i}$  is a constructible set of  $\overline{Z_i}$  by (a) and contains  $Z_i$ , so it is dense; therefore, by (b), we have  $\eta_i \in C$ . Now if  $y \in \overline{C}$ , then there is an  $i$  such that  $y \in \overline{Z_i}$ , and then taking  $x = \eta_i$  suffices.

■

<sup>6</sup>Collaboration acknowledgment: this solution was inspired by some conversations with ChatGPT on 09/20/25.

<sup>7</sup>Another way to see this is to note that a subalgebra of  $\mathbb{F}_2^X$  generated by  $n$  elements, say  $\chi_1, \dots, \chi_n$  has dimension at most  $2^n$  over  $\mathbb{F}_2$ , since it is spanned as a vector space over  $\mathbb{F}_2$  by the elements  $\chi_I = \prod_{i \in I} \chi_i$  as  $I \subset 2^{\{1, \dots, n\}}$  ranges over all subsets of  $\{1, \dots, n\}$ .

<sup>8</sup>If the reader finds an easier proof of this implication, please let me know!

**Corollary 2.3.5.** Let  $X$  be a Noetherian topological space. A subset  $Y \subset X$  is constructible iff for every closed irreducible  $Z \subset X$ , the intersection  $Y \cap Z$  either contains a nonempty open subspace of  $Z$  or is nowhere dense in  $Z$ .

The following proof has been taken from [5, Proposition 10.14].

*Proof.* If  $Y$  is constructible, then for each such  $Z$ , so is  $Y \cap Z \subset Z$  by 2.3.4(a). In this case, either  $Y \cap Z$  is dense in  $Z$ , in which case it contains a nonempty open subspace of  $Z$  by 2.3.4(b), or it is not dense, in which case it is nowhere dense in  $Z$  by 2.1.3(d).

Conversely, by Noetherian Induction (2.2.10) applied to property of meeting  $Y$  in constructible subset of  $X$ , we are reduced to the case in which for every proper closed subset  $W \subset X$ , the intersection  $W \cap Y$  is constructible in  $X$  (or equivalently in  $W$ , by 2.3.4(a)). If  $X$  is not irreducible, then for each irreducible component  $X' \subset X$ , the intersection  $X' \cap W$  is constructible in  $X$  by hypothesis; this suffices, since constructible sets are closed under finite unions and  $X$  has only finitely many irreducible components (2.2.7(c)). Suppose now that  $X$  is irreducible. If  $Y$  contains a nonempty open subspace  $U$  of  $X$ , then the hypothesis applied to  $W = X \setminus U$  tells us that  $W \cap Y = Y \setminus U$  is constructible in  $X$ ; then so is  $Y = (Y \setminus U) \cup U$ . If  $Y$  is nowhere dense in  $X$ , then in particular the closure  $\bar{Y} \subset X$  is a proper closed subset; then the hypothesis applied to  $W = \bar{Y}$  tells us that  $Y = W \cap Y$  is constructible in  $X$ . ■

The final notion we will need is that of *very dense subspaces*. To motivate this, note that if  $X$  is a topological space and  $Y \subset X$  a subspace, then the map  $W \mapsto W \cap Y$  taking open (resp. closed) subspaces  $W \subset X$  to open (resp. closed) subspaces of  $Y$  is always a surjection.

**Proposition/Definition 2.3.6** (Very Dense Subspaces). Let  $X$  be a topological space. The following conditions on a subspace  $Y \subset X$  are equivalent:

- (a) The map  $U \mapsto U \cap Y$  is a bijection from the set of open (resp. closed, resp. locally closed, resp. constructible) subspaces of  $X$  to those of  $Y$ . Equivalently, if  $U, V \subset X$  are open (resp. closed, resp. locally closed, resp. constructible) subspaces such that  $U \cap Y = V \cap Y$ , then  $U = V$ .
- (b) For every closed subspace  $Z \subset X$ , we have  $Z = \bar{Z} \cap \bar{Y}$ .
- (c) Every nonempty locally closed (resp. constructible) subset of  $X$  meets  $Y$ .

A subspace  $Y \subset X$  satisfying these equivalent conditions is said to be *very dense* in  $X$ .

*Proof.*

- (a)  $\Rightarrow$  (b) The property (a) for constructible subsets implies it for locally closed subsets, which in turn implies it for open and closed subsets, and the properties for open and closed subsets are equivalent by taking complements. The property (a) for closed subsets then implies (b); indeed, if  $Y \subset X$  is any subset and  $Z \subset X$  closed, then  $Z \cap Y = \bar{Z} \cap \bar{Y} \cap Y$ .
- (b)  $\Rightarrow$  (c) The two properties are equivalent by 2.3.3(b), and (b) implies (c) for locally closed subsets by 2.3.1(f).
- (c)  $\Rightarrow$  (a) The property (c) for constructible sets implies the property (a) for constructible sets: if  $C, D \subset X$  are constructible subsets such that  $C \cap Y = D \cap Y$ , then  $C \setminus D$  and  $D \setminus C$  are constructible subsets of  $X$  not meeting  $Y$ , and hence are both empty. ■

**Example 2.3.7.** The subset  $\mathbb{Q} \subset \mathbb{R}$  is dense but not very dense.

**Corollary 2.3.8.** Let  $X$  be a topological space and  $Y \subset X$  a subspace.

- (a) If  $C \subset X$  is a constructible subspace and  $Y$  very dense in  $X$ , then  $Y \cap C$  is very dense in  $C$ .
- (b) If  $X = \bigcup_{\alpha} X_{\alpha}$  for some  $X_{\alpha} \subset X$  such that for each  $\alpha$ , the intersection  $Y \cap X_{\alpha}$  is very dense in  $X_{\alpha}$ , then  $Y$  is very dense in  $X$ .

*Proof.*

- (a) If  $D \subset C$  is a nonempty constructible subspace, then  $D$  is constructible in  $X$  by 2.3.4(a) and hence meets  $Y$  (and so  $Y \cap C$ ) by 2.3.6(c).

- (b) Given a nonempty constructible  $C \subset X$ , pick a point  $x \in C$  and an  $\alpha$  such that  $x \in X_\alpha$ . Then  $C \cap X_\alpha$  is a nonempty constructible subspace of  $X_\alpha$  by 2.3.4(a), and so meets  $Y \cap X_\alpha$  by hypothesis and 2.3.6(c). Therefore,  $C$  meets  $Y$ , and we are done. ■

The exercise [4, Exercise 3.3.27] cited in the previous section gives an example of a pathological scheme which does not have *any* closed points. This, however, cannot happen for algebraic varieties. Indeed, the key example of very dense subspaces, and the reason we care, is

**Theorem 2.3.9.** Let  $k$  be a field and  $X$  be a scheme locally of finite type over  $k$ .

- (a) The subspace of closed points of  $X$  is very dense in  $X$ . In particular,  $X$  is closure-complete, so that if it is nonempty, then it has a closed point.
- (b) Suppose  $Y$  is another scheme locally of finite type over  $k$ , and  $f, g : X \rightarrow Y$  two  $k$ -morphisms such that for some algebraic closure  $\bar{k} \supset k$  we have  $f(\bar{k}) = g(\bar{k}) : X(\bar{k}) \rightarrow Y(\bar{k})$ . If  $X$  is geometrically reduced, then  $f = g$ .

*Proof.*

- (a) Let  $X^{\text{cl}} \subset X$  denote the subset of closed points. If  $W \subset X$  is any (locally closed) subset, then  $W^{\text{cl}} = W \cap X^{\text{cl}}$ , since closed points admit an intrinsic characterization (namely that their residue field is a finite extension of  $k$ ). Therefore, by 2.3.8(b), it suffices to show the result when  $X$  is affine, so suppose  $X$  is an affine scheme of finite type over  $k$  and  $Y \subset X$  a nonempty locally closed subset. We want to show  $Y \cap X^{\text{cl}} \neq \emptyset$ . Replacing  $X$  by  $(\bar{Y})^{\text{red}}$ , we may assume  $Y$  is open in  $X$  (using 2.3.1(g) and that every closed subscheme of affine scheme is affine), and by shrinking  $Y$  we may assume it is a nonempty principal open subset, say  $Y = D(f)$  for some  $f \in \mathcal{O}(X)$ . Since  $Y$  is nonempty,  $f$  lies in some prime ideal, and then  $f$  lies in some maximal ideal; the corresponding point of  $X$  then belongs to  $Y \cap X^{\text{cl}}$ .
- (b) Base change to  $\bar{k}$  to assume  $k$  is algebraically closed, using that  $f_{\bar{k}} = g_{\bar{k}}$  implies  $f = g$  (exercise; see [1, 10.2.J]) In this case,  $X(k)$  can be identified with the set of closed points, which is very dense by (a). The (underlying subspace) of the equalizer  $\text{Eq}(f, g) \subset X$  is locally closed, since the diagonal morphism for schemes is always a locally closed immersion, and so its complement is constructible; if it were to be nonempty, then it would meet  $X(k)$  by 2.3.6(e), which it does not. Therefore,  $\text{Eq}(f, g) = X$  as topological spaces, and then as schemes because  $X$  is reduced, i.e.,  $f = g$ . ■

Informally, for geometrically reduced schemes locally of finite type over a field  $k$ , equality of morphisms can be checked at the level of  $\bar{k}$ -points for an(y choice of an) algebraic closure  $\bar{k}$  of  $k$ . The hypothesis of geometric reducedness cannot be removed; for a counterexample, consider the ring  $\mathbb{C}[\varepsilon] := \mathbb{C}[x]/(x^2)$ ,  $X = Y = \text{Spec } \mathbb{C}[\varepsilon]$ , and  $f, g : X \rightarrow Y$  the two morphisms corresponding to  $\varepsilon \mapsto 0, \varepsilon$ . It also does not suffice to work with  $k$  points when  $k$  is not algebraically closed, since a scheme locally of finite type over  $k$  need not have any  $k$ -points at all (e.g. when  $k = \mathbb{R}$  and  $X = \text{Spec } \mathbb{C}$ ).

**Remark 2.3.10.** The result of the previous proposition holds more generally in other contexts, e.g., for schemes locally of finite type over  $\mathbb{Z}$ . A scheme is called a *Jacobson scheme* if its subset of closed points is very dense; then 2.3.9 is saying that a scheme locally of finite type over a field is Jacobson. The affine version of a Jacobson scheme is a Jacobson ring, which admits many different characterizations (see [5, Exercises 10.15-16] and [6, §6.2.2]). It is easy to see that  $\mathbb{Z}$  is a Jacobson ring, and hence that  $\text{Spec } \mathbb{Z}$  is a Jacobson scheme; also, a field  $k$  is trivially seen to be a Jacobson ring. The final result needed is that if  $S$  is a Jacobson scheme and  $f : X \rightarrow S$  a morphism locally of finite type, then  $X$  is a Jacobson scheme ([5, Exercise 10.16]).

## 2.4 Separatedness and Properness from a Topological Perspective

In this final section, we provide a few topological results that might help motivate the algebro-geometric definitions of separated and proper morphisms. The first is the classic

**Proposition/Definition 2.4.1.** (Hausdorff Spaces) Let  $X$  be a topological space. The following conditions on  $X$  are equivalent:

- (a) For any two distinct points  $x, y \in X$ , there are disjoint open subsets  $U, V \subset X$  such that  $x \in U$  and  $y \in V$ .
- (b) Limits of nets in  $X$ , when they exist, are unique.
- (c) Every point  $x \in X$  is the intersection of all its closed neighborhoods (where by a closed neighborhood of  $x$  we meant a closed subset of  $X$  containing  $x$  in its interior).
- (d) The diagonal  $\Delta(X) \subset X \times X$  is a closed subset when  $X \times X$  is given the product topology.

A topological space satisfying these equivalent conditions is called *Hausdorff*.

*Proof.* Exercise. ■

Schemes are usually not Hausdorff, and so the above definition needs to be modified to obtain the usual definition for separated morphisms. One can also define, just as in the scheme case, a continuous map  $X \rightarrow Y$  to be separated if the diagonal  $\Delta_{X/Y} : X \rightarrow X \times_Y X$  is a closed topological embedding, and this notion can then be characterized in terms of the fibers being relatively Hausdorff ([7, Proposition 1.25]), but to me that seems a little artificial; besides, it's less useful to algebraic geometers because it does not translate over to the schemes setting, precisely because the Zariski topology on  $X \times_Y X$  is not the usual (subspace-of-) product topology.

There is another related topological notion that is used in scheme theory, namely the notion of quasiseparated topological spaces. I believe this is best handled in a scheme-theoretic manner ([1, §11.1.4]), so I will content myself with giving the definition and one lemma which is sometimes useful (as in 3.3.5 below and in the solution to [1, Exercise 5.1.H]).

**Definition 2.4.2.** A topological space  $X$  is said to be *quasiseparated* if the intersection of any two quasicompact<sup>9</sup> open subsets of  $X$  is again quasicompact.

**Lemma 2.4.3** ([8]). Let  $X$  be a topological space admitting a basis of quasicompact open subsets, and  $\mathcal{U}$  a cover of  $X$  by quasiseparated open subsets, all of whose pairwise intersections are quasicompact. If  $W \subset X$  is any quasicompact open subset, then for all  $U \in \mathcal{U}$ , the intersection  $W \cap U$  is quasicompact.

*Proof.* Cover  $W$  by finitely many basic quasicompact open subsets  $V_i$  each contained in some element say  $U_i$  of the cover  $\mathcal{U}$ . It suffices to show that for each  $U \in \mathcal{U}$ , the intersection  $U \cap V_i$  is quasicompact, but  $U \cap V_i = V_i \cap (U \cap U_i)$  is an intersection of two quasicompact open subsets inside the quasiseparated  $U_i$ . ■

---

Au contraire, it is easier and more direct to relate the topological notion of compactness to the algebro-geometric notion of properness. The following exposition has been taken from [7, §1.5]. Recall the definition of quasicompact spaces.

**Proposition/Definition 2.4.4** (Quasicompact Spaces). The following conditions on a topological space  $X$  are equivalent:

- (a) Every open cover has a finite subcover.
- (b) Every collection of closed subspaces with the finite intersection property (i.e., such that every finite intersection of which is nonempty) has nonempty intersection.
- (c) Every net in  $X$  has a convergent subnet.

---

<sup>9</sup>See 2.4.4 below if needed.

- (d) The final morphism  $X \rightarrow \{*\}$  is universally closed, i.e., for every topological space  $Z$ , the projection  $\pi : Z \times X \rightarrow Z$  is a closed map.

A topological space satisfying these equivalent properties is said to be *quasicompact*. A topological space is said to be *compact* if it is both quasicompact and Hausdorff. Further:

- (e) A closed subspace of a quasicompact space is quasicompact.
- (f) The continuous image of a quasicompact space is quasicompact.
- (g) A quasicompact subspace of a Hausdorff space is closed.

*Proof.* The implications (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c) are standard and left as an exercise, as are the statements (e), (f), (g). Since a proof of the equivalence with (d) is trickier (and harder to locate), we reproduce it here.

- (a)  $\Rightarrow$  (d) Let  $Z$  be given,  $C \subset Z \times X$  closed, and  $z \in Z \setminus \pi(C)$ . The following lemma (2.4.5)<sup>10</sup> applied to  $\pi^{-1}(z) = \{z\} \times X \subset (Z \times X) \setminus C \subset Z \times X$  tells us that there is an open neighborhood  $V$  of  $z$  in  $Z$  such that  $V \times X \subset (Z \times X) \setminus C$ . This shows that  $V \subset Z \setminus \pi(C)$ . Since  $z$  was arbitrary, this proves that  $Z \setminus \pi(C)$  is open as needed.
- (d)  $\Rightarrow$  (c) An argument can be found in [7, Prop. 1.29], but I prefer the following argument adapted from [9, Lemma 4.3]. Suppose  $X$  is a topological space and  $\mathcal{U}$  an open cover of  $X$  with no finite subcover. Replace  $\mathcal{U}$  by the collection of open subspaces consisting of all finite unions of elements of  $\mathcal{U}$  to assume without loss of generality that  $\mathcal{U}$  is directed. We put a topology on  $Z = \mathcal{T}(X) \subset 2^X$ , the collection of open subsets of  $X$ , such that the projection map  $\pi : Z \times X \rightarrow Z$  is not closed. Define a subset  $V \subset Z$  to be open iff both (i) and (ii) below hold.
- (i) The collection  $V$  is upward closed, i.e., for all  $U \in V$ , if  $U' \subset X$  is open and  $U \subset U'$ , then  $U' \in V$ .
  - (ii) If  $X \in V$ , then there is a  $U \in \mathcal{U}$  such that  $U \in V$ .

It follows from the fact that  $\mathcal{U}$  is directed that the above defines a topology on  $Z$ , and further that if  $U \subset X$  is an open subspace of some element of  $\mathcal{U}$ , then the upward closure  $U^\uparrow := \{U' \in Z : U' \supset U\} \subset Z$  is open. It remains to show that the subset  $C \subset Z \times X$  defined by  $C := \{(U, x) : U \not\ni x\}$  is a closed subset such that  $\pi(C) = Z \setminus \{X\}$  is not closed. The latter part follows from the fact that  $X \notin \mathcal{U}$ . For the former part, we need to show that if  $(U, x) \in Z \times X$  is such that  $U \ni x$ , then there is an open neighborhood of  $(U, x)$  in  $Z \times X$  contained in  $Z \times X \setminus C$ . But now since  $\mathcal{U}$  covers  $X$ , there is a  $U' \in \mathcal{U}$  such that  $U' \ni x$ , and then  $(U, x) \in (U \cap U')^\uparrow \times (U \cap U') \subset Z \times X \setminus C$  as needed. ■

**Lemma 2.4.5.** Let  $X, Y$  be topological spaces and  $K \subset X$  and  $L \subset Y$  quasicompact subspaces. Then for any open neighborhood  $U$  of  $K \times L$  in  $X \times Y$ , there are open neighborhoods  $V$  of  $K$  in  $X$  and  $W$  of  $L$  in  $Y$  such that  $V \times W \subset U$ .

*Proof.* Exercise. ■

**Proposition/Definition 2.4.6** (Proper Maps). The following conditions on a continuous map  $f : X \rightarrow S$  between topological spaces are equivalent:

- (a) The map  $f$  is universally closed in the topological category, i.e., for every continuous map  $T \rightarrow S$ , the projection map  $f_T : X_T := T \times_S X \rightarrow T$  after base-change to  $T$  is closed.
- (b) For every topological space  $T$ , the map  $f_{T \times S} : T \times X \rightarrow T \times S$  is closed.
- (c) The map  $f$  is closed and for every quasicompact subspace  $K \subset S$ , the preimage  $f^{-1}(K) \subset X$  is quasicompact.
- (d) The map  $f$  is closed and for all  $s \in S$ , the fiber  $X_s = f^{-1}(s) \subset X$  is quasicompact.

Further, when  $S$  is locally (quasi)compact Hausdorff, these are also equivalent to:

- (e) For every (quasi)compact subset  $K \subset S$ , the preimage  $f^{-1}(K) \subset X$  is quasicompact.

A continuous map  $f : X \rightarrow S$  is said to be *proper* iff it satisfies equivalent conditions (a)-(d).

<sup>10</sup>In order to avoid circular arguments, the reader is encouraged to prove the Lemma 2.4.5 using definition (a) of compactness given above.

Note that as a consequence, we have that a space  $X$  is quasicompact iff the final morphism  $X \rightarrow \{*\}$  is proper.

*Proof.*

- (a)  $\Rightarrow$  (b) Base-change along the projection  $T \times S \rightarrow T$ .
- (b)  $\Rightarrow$  (c) Taking  $T$  to be a point shows  $f$  is closed. Let  $K \subset S$  be a quasicompact subspace and  $T = Z$  an arbitrary space. Since  $f_Z : Z \times X \rightarrow Z \times S$  is closed, so is the restriction  $f_Z|_{Z \times f^{-1}(K)} : Z \times f^{-1}(K) \rightarrow Z \times K$ .<sup>11</sup> By 2.4.4(d), the projection  $Z \times K \rightarrow Z$  is also closed. Therefore, the composition  $Z \times f^{-1}(K) \rightarrow Z \rightarrow K \rightarrow Z$ , which is just the projection onto  $Z$ , is also closed. Since this is true for all  $Z$ , another application of 2.4.4(d) tells us that  $f^{-1}(K)$  is quasicompact.
- (c)  $\Rightarrow$  (d) The one point subspace  $\{s\} \subset S$  is quasicompact.
- (d)  $\Rightarrow$  (a) Given  $g : T \rightarrow S$ , the image  $f_T(X_T) = g^{-1}(f(X))$  is closed because  $f$  is. Given a closed  $Z \subset X_T$ , we have to show that  $T \setminus f_T(Z)$  is open, i.e., for all  $t \in T \setminus f_T(Z)$ , there is an open neighborhood  $U$  of  $t$  in  $T$  contained in  $T \setminus f_T(Z)$ . When  $t \notin f_T(X_T)$ , we can take  $U := T \setminus f_T(X_T)$ . When  $t \in f_T(X_T)$ , note that  $f_T^{-1}(t) = \{t\} \times X_{g(t)} \subset X_T \setminus Z$ . Since  $X_T \subset T \times X$  is given the subspace topology, there is a closed subset  $\bar{Z}$  of  $T \times X$  such that  $Z = \bar{Z} \cap X_T$ ; then applying 2.4.5 to  $\{t\} \times X_{g(t)} \subset T \times X \setminus \bar{Z}$  gives us a neighborhood  $V$  of  $t$  in  $T$  and a neighborhood  $W$  of  $X_{g(t)}$  in  $X$  such that  $V \times W \subset T \times X \setminus \bar{Z}$ . It then suffices to take  $U := V \cap g^{-1}(S \setminus f(X \setminus W))$ .

Suppose now that  $S$  is locally (quasi)compact Hausdorff. It only remains to prove

- (e)  $\Rightarrow$  (c) We have to show that  $f$  is closed. Let  $C \subset X$  be closed; we have to show that for all  $s \in S \setminus f(C)$ , there is an open neighborhood  $U$  of  $s$  in  $S$  contained in  $S \setminus f(C)$ . Pick an open neighborhood  $V$  of  $s$  in  $S$  whose closure  $\bar{V} \subset S$  is (quasi)compact. Then  $f^{-1}(\bar{V}) \subset X$  is quasicompact, and hence so is  $C \cap f^{-1}(\bar{V})$  by 2.4.4(e). By 2.4.4(f), the subspace  $f(C \cap f^{-1}(\bar{V})) = f(C) \cap \bar{V} \subset S$  is also quasicompact, and hence closed by 2.4.4(g). It suffices to take  $U := V \setminus (f(C) \cap \bar{V}) = V \setminus f(C)$ . ■

For an example where (e) is strictly weaker than (a)-(d), see [7, Problem 1.21]. We end this section by stating without proof the following wonderful result relating these modern and classical notions. For this recall that when  $X$  is scheme of finite type over  $\mathbb{C}$ , then we can associate to it a complex analytic space  $X^{\text{an}}$  whose underlying topological space is the set  $X(\mathbb{C})$  of complex points of  $X$  equipped with the classical (i.e., analytic) topology (c.f. [2, Appendix B]).

**Theorem 2.4.7.** Let  $X$  be a scheme of finite type over  $\mathbb{C}$ .

- (a) The scheme  $X$  is separated iff  $X(\mathbb{C})$  with the classical topology is Hausdorff.
- (b) The scheme  $X$  is complete (i.e., proper over  $Y = \text{Spec } \mathbb{C}$ ) iff  $X(\mathbb{C})$  with the classical topology is compact (i.e., quasicompact Hausdorff).

*TOCITE.* ■

<sup>11</sup>This uses the following elementary fact in point-set topology: if  $p : A \rightarrow B$  is a continuous closed map and  $C \subset B$  any subset, then the restriction  $p|_{p^{-1}(C)} : p^{-1}(C) \rightarrow C$  is also closed. Indeed, any closed subspace of  $p^{-1}(C)$  is of the form  $p^{-1}(C) \cap D$  for some closed  $D \subset A$ ; then it is easy to check that  $p(p^{-1}(C) \cap D) = C \cap p(D)$ .

## Chapter 3

# Affine Communication Lemma and Properties of Morphisms

### 3.1 The Affine Communication Lemma and Reasonable Classes of Morphisms

The wonderful theory-building tool we will repeatedly use is

**Lemma 3.1.1** (Affine Communication Lemma). Let  $X$  be a scheme and  $\mathcal{P}$  be a property satisfied by some affine open subsets of  $X$  such that for any affine open  $U \subset X$ , the following two conditions hold.

- (a) For any distinguished open  $D \subset U$ , if  $U$  has  $\mathcal{P}$  then so does  $D$ .
- (b) Suppose  $\{D_\alpha\}$  is a finite collection of distinguished opens in  $U$  that cover  $U$  (i.e., such that  $U = \bigcup_\alpha D_\alpha$ ). If each  $D_\alpha$  has  $\mathcal{P}$ , then so does  $U$ .

In this case, if  $X$  admits some affine open cover all of whose elements have  $\mathcal{P}$ , then every affine open  $U \subset X$  has  $\mathcal{P}$ .

*Proof.* Suppose we are given an affine open cover  $\{V_\beta\}$  of  $X$  such that each  $V_\beta$  has  $\mathcal{P}$ . Using 1.2.3, cover  $U$  by distinguished opens  $D_\alpha$  that are simultaneously also distinguished in *some*  $V_{\beta(\alpha)}$ . Since each  $V_\beta$  has  $\mathcal{P}$ , property (a) tells us that each  $D_\alpha$  has  $\mathcal{P}$ . To apply (b), it remains to only note that finitely many  $D_\alpha$  cover  $U$ , and this is true because  $U$  is quasicompact (1.2.1(e)). ■

The Affine Communication Lemma is best used in conjunction with the following two.

**Lemma 3.1.2.** Let  $\mathcal{P}$  be a property of morphisms of schemes that is local on the target.<sup>1</sup> If  $\mathcal{P}$  is stable under affine base change<sup>2</sup>, then  $\mathcal{P}$  is stable under base change.

*Proof.* Let  $X \rightarrow S$  have  $\mathcal{P}$  and let  $T \rightarrow S$  be a morphism. Since  $\mathcal{P}$  is local on the target, to show that the pullback  $X_T \rightarrow T$  has  $\mathcal{P}$ , it suffices to produce an affine open cover  $\mathcal{U}$  of  $T$  such that for each  $U \in \mathcal{U}$ , the morphism  $X_U \rightarrow U$  has  $\mathcal{P}$ . For this, let  $\mathcal{U}$  be the cover of  $T$  consisting of the affine opens  $U \subset T$  whose image in  $S$  lies in some affine open of  $S$ ; we show this  $\mathcal{U}$  works. Given a  $U \in \mathcal{U}$ , pick an affine open  $V$  of  $S$  such that  $U$  maps to  $V$  in  $S$ . Since  $\mathcal{P}$  is local on the target, the basechange  $X_V \rightarrow V$  has  $\mathcal{P}$ . Then the morphism  $X_U \rightarrow U$  is the affine base change of  $X_V \rightarrow V$  along  $U \rightarrow V$  and hence also has  $\mathcal{P}$ . ■

**Lemma 3.1.3.** Let  $\mathcal{P}$  be a property of morphisms of schemes that is local on the target. If  $\mathcal{P}$  is further local on the source<sup>3</sup> and stable under affine composition,<sup>4</sup> then it is stable under arbitrary composition.

*Proof.* Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be morphisms having  $\mathcal{P}$ ; we want to show that so does  $gf : X \rightarrow Z$ . Since  $\mathcal{P}$  is local on the target, we may assume without loss of generality that  $Z$  is affine. Since  $\mathcal{P}$  is local on the source, to show that  $gf$  has  $\mathcal{P}$ , it suffices to produce an affine open cover  $\mathcal{U}$  of  $X$  such that for each  $U \in \mathcal{U}$ , the composite  $U \hookrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$  has  $\mathcal{P}$ . For this, let  $\mathcal{U}$  be the cover of  $X$  consisting of affine opens  $U \subset X$  whose image in  $Y$  under  $f$  lies in some affine open of  $Y$ ; we show that this  $\mathcal{U}$  works. Given  $U \in \mathcal{U}$ , pick an affine open  $V$  of  $Y$  such that  $U \subset f^{-1}(V)$ . Since  $\mathcal{P}$  is local on the source, the restriction  $g|_V : V \rightarrow Z$  has  $\mathcal{P}$ . Since  $\mathcal{P}$  is local on the target,  $f|_{f^{-1}(V)} : f^{-1}(V) \rightarrow V$  has  $\mathcal{P}$ . Since  $\mathcal{P}$  is local on the source, the further restriction  $f|_U : U \rightarrow V$  has  $\mathcal{P}$ . Finally, since  $\mathcal{P}$  is stable under affine composition, the composite  $g|_V \circ f|_U : U \rightarrow Z$  has  $\mathcal{P}$ , as needed. ■

The following definition is surprisingly useful.

**Definition 3.1.4.** A class of morphisms  $\mathcal{P}$  of schemes is said to be *reasonable* iff it contains isomorphisms, is local on the target, and is stable under composition and base-change.

**Remark 3.1.5.** The same definition can also, of course, be applied in other contexts—e.g., of locally ringed spaces (and hence of topological spaces, manifolds, etc.). For instance, injections, embeddings, open embeddings, closed embeddings, and surjections are all reasonable classes of continuous maps.

<sup>1</sup>In other words, if  $\pi : X \rightarrow Y$  is a morphism of schemes, then  $\pi$  has the property  $\mathcal{P}$  iff for each  $y \in Y$ , there is an open  $V \subset Y$  such that the restriction  $\pi^{-1}(V) \rightarrow V$  has  $\mathcal{P}$ .

<sup>2</sup>An *affine base change* is a base change in which the source and target bases are affine.

<sup>3</sup>In other words, if  $\pi : X \rightarrow Y$  is a morphism of schemes, then  $\pi$  has the property  $\mathcal{P}$  iff for each  $x \in X$ , there is an open  $U \subset X$  such that the restriction  $\pi|_U : U \rightarrow Y$  has  $\mathcal{P}$ .

<sup>4</sup>An *affine composition* is a composition of two morphisms, both of whose sources and targets are affine.

**Lemma 3.1.6.** Let  $\mathcal{P}$  be a property of morphisms of schemes stable under composition and base-change. Let  $S$  be a scheme, and let  $f : X \rightarrow Y$  and  $g : X' \rightarrow Y'$  be two morphisms of  $S$ -schemes with property  $\mathcal{P}$ . Then  $f \times g : X \times_S X' \rightarrow Y \times_S Y'$  also has  $\mathcal{P}$ .

*Proof.* Since  $\mathcal{P}$  is stable under base-change, the product  $f \times 1_{X'} : X \times_S X' \rightarrow Y \times_S X'$  (resp.  $1_Y \times g : Y \times_S X' \rightarrow Y \times_S Y'$ ) has  $\mathcal{P}$ , because it is the base-change of  $f$  (resp.  $g$ ) along the projection  $Y \times_S X' \rightarrow Y$  (resp.  $Y \times_S Y' \rightarrow Y'$ ). Since  $\mathcal{P}$  is stable under composition, the morphism  $f \times g = (1_Y \times g) \circ (f \times 1_{X'})$  also has  $\mathcal{P}$ . ■

In this chapter, we will meet a huge collection of reasonable morphisms. Before we do that, let's give one general way of producing more

**Lemma 3.1.7.**

**Lemma 3.1.8.**

## 3.2 (Locally) Noetherian Schemes and (Locally) Finite Type and Presentation Morphisms

In this section we discuss some fundamental finiteness properties of schemes and morphisms. For this we start with a general lemma.

**Lemma 3.2.1.** Let  $A$  be a ring,  $n \in \mathbb{Z}_{\geq 1}$ , and  $f_1, \dots, f_n \in A$  such that  $(f_1, \dots, f_n) = A$ .

- (a) Let  $M$  be an  $A$ -module. If for each  $i = 1, \dots, n$ , the  $A[f_i^{-1}]$ -module  $A[f_i^{-1}] \otimes_A M$  is finitely generated, then so is  $M$ .
- (b) Let  $\mathfrak{a} \subset A$  be an ideal. If for each  $i = 1, \dots, n$ , the ideal  $\mathfrak{a}A[f_i^{-1}] \subset A[f_i^{-1}]$  is finitely generated, then so is  $\mathfrak{a}$ .

*Proof.*

- (a) For each  $i = 1, \dots, n$ , pick finitely many elements  $m_{ij} \in M$  whose images generate  $A[f_i^{-1}] \otimes_A M$  as an  $A[f_i^{-1}]$ -module. Check that the finite set  $\{m_{ij}\}_{i,j}$  generates  $M$  as an  $A$ -module.
- (b) Follows from applying (a) to  $M = \mathfrak{a}$ , and noting that for each  $i$ , we have  $A[f_i^{-1}] \otimes_A \mathfrak{a} \cong \mathfrak{a}A[f_i^{-1}]$  because localization is exact (i.e.,  $A[f_i^{-1}]$  is a flat  $A$ -algebra). ■

**Proposition/Definition 3.2.2** (Locally Noetherian Schemes). The following conditions on a scheme  $X$  are equivalent:

- (a) For every affine open  $U \subset X$ , the ring  $\mathcal{O}(U)$  is Noetherian.
- (b) There is an affine open cover  $X = \bigcup_i U_i$  such that for each  $i$ , the ring  $\mathcal{O}(U_i)$  is Noetherian.

A scheme  $X$  satisfying these equivalent conditions is said to be *locally Noetherian*. A scheme  $X$  is said to be *Noetherian* iff it is locally Noetherian and quasicompact. Further:

- (c) If  $X$  is a (locally) Noetherian scheme, then the underlying topological space of  $X$  is also (locally) Noetherian. Further, any open subscheme  $U \subset X$  is also (locally) Noetherian.

*Proof.* By 3.1.1 it suffices to show the following.

- (a) If  $A$  is a Noetherian ring, then for each  $f \in A$ , so is  $A[f^{-1}]$ . This follows, for instance, from the Hilbert Basis Theorem, since  $A[f^{-1}] \cong A[X]/(fX - 1)$ .
- (b) Suppose  $A$  is a ring,  $n \in \mathbb{Z}_{\geq 1}$ , and  $f_1, \dots, f_n \in A$  such that  $(f_1, \dots, f_n) = (1)$ . If for each  $i$  the ring  $A[f_i^{-1}]$  is Noetherian, then so is  $A$ ; this follows immediately from 3.2.1(b).

The first statement in (c) follows from 2.2.5 and 2.2.7(a). The second statement for the locally Noetherian case follows immediately from the equivalence of (a) and (b), while the in the Noetherian case follows from the locally Noetherian case coupled with the first statement and 2.2.7(b). ■

Let  $A$  be a ring. Recall that an  $A$ -algebra  $B$  is said to be of *finite type*<sup>5</sup> (resp. *finite presentation*) if  $B$  if there is an  $n \in \mathbb{Z}_{\geq 1}$  and a surjective  $A$ -algebra morphism  $A[X_1, \dots, X_n] \twoheadrightarrow B$  (resp. there is an  $n \in \mathbb{Z}_{\geq 1}$  and a surjective  $A$ -algebra morphism  $A[X_1, \dots, X_n] \twoheadrightarrow B$  with the kernel a finitely generated ideal). Recall that algebras of finite presentation satisfy the following persistence property.

**Lemma 3.2.3.** Let  $A$  be a ring and  $B$  be an algebra of finite presentation. For any  $m \in \mathbb{Z}_{\geq 1}$  and surjective  $A$ -algebra morphism  $\rho : A[X_1, \dots, X_m] \twoheadrightarrow B$ , the kernel is a finitely generated ideal.

*Proof.* Since  $B$  is of finite presentation, there is an  $n \in \mathbb{Z}_{\geq 1}$  and a surjection  $\pi : A[Y_1, \dots, Y_n] \twoheadrightarrow B$  with the kernel a finitely generated ideal, say generated by the polynomials  $f_1, \dots, f_r \in A[Y_1, \dots, Y_n]$  for some  $r \in \mathbb{Z}_{\geq 0}$ . Let  $A[X] := A[X_1, \dots, X_m]$  and  $A[Y] := A[Y_1, \dots, Y_n]$  for notational simplicity. Pick  $A$ -algebra homomorphisms  $\tilde{\pi} : A[Y] \rightarrow A[X]$  and  $\tilde{\rho} : A[X] \rightarrow A[Y]$  lifting  $\pi$  and  $\rho$  respectively. Check that the kernel of  $\pi$  is generated by the  $r + m$  elements  $\tilde{\pi}(f_1), \dots, \tilde{\pi}(f_r), X_1 - \tilde{\pi}\tilde{\rho}(X_1), \dots, X_m - \tilde{\pi}\tilde{\rho}(X_m)$ . ■

With this, we are now ready to define morphisms locally of finite type/presentation.

---

<sup>5</sup>Also known as being “finitely generated”.

**Proposition/Definition 3.2.4** (Locally Finite Type/Presentations Schemes). Let  $A$  be a ring and  $X$  be an  $A$ -scheme. Then  $X$  is said to be *locally of finite type*<sup>6</sup> (resp. *locally of finite presentation*<sup>7</sup>) over  $A$  if the following equivalent conditions hold:

- (a) For every affine open  $U \subset X$ , the  $A$ -algebra  $\mathcal{O}(U)$  is of finite type (resp. presentation).
- (b) There is an affine open cover  $X = \bigcup_i U_i$  such that for each  $i$ , the  $A$ -algebra  $\mathcal{O}(U_i)$  is of finite type (resp. presentation).

Further:

- (c) If  $X$  is lft (resp. lfp) over  $A$ , then so is any open subscheme  $U \subset X$ . Conversely, if  $X$  admits an open cover by lft (resp. lfp)  $A$ -schemes, then  $X$  is lft over  $A$ .

*Proof.* By 3.1.1, it suffices to show the following. Suppose  $B$  is an  $A$ -algebra.

- (i) If  $B$  is of finite type (resp. presentation) over  $A$ , then for each  $f \in B$ , so is  $B[f^{-1}] \cong B[T]/(Tf-1)$ ; this is clear.
- (ii) If there is an  $n \in \mathbb{Z}_{\geq 1}$  and  $f_1, \dots, f_n \in B$  such that  $(f_1, \dots, f_n) = (1)$  and for each  $i$ , the localization  $B[f_i^{-1}]$  is of finite type (resp. presentation) over  $A$ , then so is  $B$ .  
 “ft”. There is an  $N \gg 1$  such that for each  $i$ , there is a finite set of  $g_{ij} \in B$  such that  $B[f_i^{-1}]$  is generated as an  $A$ -algebra by the  $g_{ij} \cdot f_i^{-N}$ . If we pick  $h_1, \dots, h_n \in B$  such that  $\sum_{i=1}^n h_i f_i = 1$ , then  $B = A[f_i, g_{ij}, h_i]$ .  
 “fp”. By what we have already shown,  $B$  is of finite type, and so we may pick an  $m \in \mathbb{Z}_{\geq 1}$  and surjection  $\pi : A[X_1, \dots, X_m] \twoheadrightarrow B$  of  $A$ -algebras. Let  $\mathfrak{a} := \ker \pi$ . For each  $f_i$ , since localization is exact, the kernel of the localized map  $\pi_i : A[f_i^{-1}][X_1, \dots, X_m] \rightarrow B[f_i^{-1}]$  is exactly  $\mathfrak{a}A[f_i^{-1}][X_1, \dots, X_m]$ . By 3.2.3, each  $\mathfrak{a}A[f_i^{-1}][X_1, \dots, X_m]$  is finitely generated, so we conclude by 3.2.1.

The statement (c) follows immediately from the equivalence of (a) and (b). ■

The relative version of the above property is given in

**Proposition/Definition 3.2.5** (Locally Finite Type/Presentations Morphisms). A morphism  $\pi : X \rightarrow Y$  of schemes is said to be *locally of finite type* (resp. *locally of finite presentation*<sup>8</sup>) if the following equivalent conditions hold:

- (a) For every affine open  $V \subset Y$ , the  $\mathcal{O}(V)$ -scheme  $\pi^{-1}(V)$  is lft (resp. lfp) over  $\mathcal{O}(V)$ .
- (b) There is an affine open cover  $Y = \bigcup_i V_i$  such that for each  $i$ , the  $\mathcal{O}(V_i)$ -scheme  $\pi^{-1}(V_i)$  is lft (resp. lfp) over  $\mathcal{O}(V_i)$ .

Further:

- (c) Lft (resp. lfp) morphisms are affine local on both the source and target. In particular, an open immersion is lfp (and hence lft).
- (d) Lft (resp. lfp) morphisms form a reasonable class.
- (e) If  $\pi : X \rightarrow Y$  is lft and  $Y$  locally Noetherian, then so is  $X$ .

If  $A$  is a ring and  $X$  an  $A$ -scheme, then  $X$  is lft (resp. lfp) over  $A$  iff the structure morphism  $X \rightarrow \operatorname{Spec} A$  is lft (resp. lfp).

*Proof.* By 3.1.1, it suffices to show the following. Let  $A$  be a ring.

- (i) If  $X$  is an  $A$ -scheme lft (resp. lfp) over  $A$ , then for each  $f \in A$ , the scheme  $\varphi^{-1}(D_f)$  is lft (resp. lfp) over  $\mathcal{O}(D_f) = A[f^{-1}]$ . Indeed, for each affine open  $U \subset \varphi^{-1}(D_f)$ , we know that  $\mathcal{O}(U)$  is a finite type (resp. presentation)  $A$ -algebra and hence also a finite type (resp. presentation)  $A[f^{-1}]$ -algebra.
- (ii) If  $n \in \mathbb{Z}_{\geq 1}$  and  $f_1, \dots, f_n \in A$  are such that  $(f_1, \dots, f_n) = (1)$  and for each  $i$ , the preimage  $\varphi^{-1}(D_{f_i})$  is lft (resp. lfp) over  $A[f_i^{-1}]$ , then  $X$  is lft (resp. lfp) over  $A$ . Indeed, for each  $i$ , there is an affine open cover  $\varphi^{-1}(D_{f_i}) = \bigcup_j U_{ij}$  such that for each  $j$ , the  $A[f_i^{-1}]$ -algebra  $\mathcal{O}(U_{ij})$  is of finite type (resp. presentation). Then  $X = \bigcup_i \bigcup_j U_{ij}$  is an affine open cover such that for each  $(i, j)$ , the

<sup>6</sup>Abbreviated “lft”.

<sup>7</sup>Abbreviated “lfp”.

<sup>8</sup>We use the same abbreviations as for schemes.

$A$ -algebra  $\mathcal{O}(U_{ij})$  is of finite type (resp. presentation) over  $A[f_i^{-1}]$  and hence also over  $A$ , so we are done.

Now we show the remaining properties.

- (c) Affine-locality on the target follows from the equivalence of (a) and (b). To prove affine-locality on the source, use affine-locality on the target to reduce the statement to 3.2.4(c).
- (d) Stability under base-change follows from 3.1.2 and stability under composition follows from 3.1.3.
- (e) Clear.

■

### 3.3 Quasicompact and Quasiseparated Morphisms

The first order of business is to understand quasicompact morphisms.

**Proposition/Definition 3.3.1** (Quasicompact Morphisms). Let  $\pi : X \rightarrow Y$  be a morphism of schemes. The following conditions on  $\pi$  are equivalent.

- (a) For every quasicompact open  $V \subset Y$ , the preimage  $\pi^{-1}(V) \subset X$  is quasicompact.
- (b) For every affine open  $V \subset Y$ , the preimage  $\pi^{-1}(V)$  is quasicompact.
- (c) There is an affine open cover  $Y = \bigcup_i V_i$  such that for each  $i$ , the preimage  $\pi^{-1}(V_i) \subset X$  is quasicompact.

A morphism  $\pi$  of schemes satisfying these equivalent conditions is said to be *quasicompact*. Further:

- (d) Quasicompact morphisms form a reasonable class.
- (e) Every morphism out of a Noetherian scheme is quasicompact.

*Proof.*

- (a)  $\Rightarrow$  (b) Every affine open subset is quasicompact.
- (b)  $\Rightarrow$  (a) Every quasicompact open subset is a finite union of affine opens.
- (b)  $\Leftrightarrow$  (c) By 3.1.1, it remains to check the following in the case  $Y$  is affine.
  - (i) If  $X$  is quasicompact, then for any distinguished  $D \subset Y$  the preimage  $\pi^{-1}(D)$  is quasicompact. Since  $X$  is quasicompact, it is a finite union of affines; this reduces us to the case where  $X$  is also affine, but then we can invoke 1.2.1(a).
  - (ii) If  $\{D_\alpha\}$  is a finite cover of  $Y$  by distinguished opens and each preimage  $\pi^{-1}(D_\alpha) \subset X$  is quasicompact, then so is  $X$ . This is clear because  $X = \pi^{-1}(Y) = \bigcup_\alpha \pi^{-1}(D_\alpha)$ .
- (d) Stability under composition follows from (a), and the rest is clear from 3.1.2.
- (e) This follows from 2.2.7.

■

Next, we discuss a few properties of quasicompact morphisms that make them a very helpful class of morphisms. For this, we need a preparatory lemma.

**Lemma 3.3.2.** Let  $\varphi : A \rightarrow B$  be a ring homomorphism. The following conditions on  $\varphi$  are equivalent.

- (a) The map  $\varphi$  takes non-nilpotent elements of  $A$  to non-nilpotent elements of  $B$ .
- (b) We have  $\ker \varphi \subset \text{Nil}(A)$ .
- (c) Every minimal prime of  $A$  is contracted from  $B$ .
- (d) The induced morphism  $\text{Spec } \varphi : \text{Spec } B \rightarrow \text{Spec } A$  has dense image.

In particular, if  $A$  is domain, then the above equivalent conditions hold iff  $\varphi$  is injective.

*Proof.*

- (a)  $\Leftrightarrow$  (b) Clear.
- (b)  $\Rightarrow$  (c) In general, every prime minimal over  $\ker \varphi$  is contracted from  $B$ . This follows from the fact that if  $A \subset B$  is a *subring*, then every minimal prime of  $A$  is contracted from  $B$ ; this itself follows from the fact that if  $\mathfrak{p} \subset A$  is a prime, then  $B_{\mathfrak{p}} \neq 0$ , so there is a prime  $\mathfrak{q}$  of  $B$  such that  $A \cap \mathfrak{q} \subset \mathfrak{p}$ .
- (c)  $\Rightarrow$  (d) The closure of the image of  $\text{Spec } \varphi$  contains all irreducible components of  $\text{Spec } A$ .
- (d)  $\Rightarrow$  (a) For  $f \in A$ , we have  $(\text{Spec } \varphi)^{-1} D(f) = D(\varphi(f))$ ; finish by using that  $f$  is nilpotent iff  $D(f) = \emptyset$ .

■

As a consequence, we obtain

**Theorem 3.3.3.** Let  $\pi : X \rightarrow Y$  be a quasicompact morphism of schemes.

- (a) The morphism  $\pi$  is dominant (2.1.14) iff it has dense image, i.e.,  $\overline{\pi(X)} = Y$ .
- (b) The image  $\pi(X) \subset Y$  is closed iff it is stable under specialization.

*Proof.*

- (a) One direction is 2.1.15. Suppose  $\overline{\pi(X)} = Y$ , and let  $Z \subset Y$  be an irreducible component. Let  $V \subset Y$  be an affine open subset such that  $Z \cap V \neq \emptyset$ ; then  $Z \cap V$  is an irreducible component of  $V$  by 2.1.5(g). The preimage  $\pi^{-1}(V)$  is quasicompact, and hence covered by finitely many affine opens, say  $U_1, \dots, U_n$  for some  $n \in \mathbb{Z}_{\geq 1}$ . The morphism  $\coprod_{i=1}^n U_i \rightarrow \pi^{-1}(V) \rightarrow V$  also has dense image. Combining with 2.1.13 reduces us to 3.3.2.
- (b) One direction is clear, so suppose that  $\pi(X)$  is stable under specialization; we want to show it is closed. Replacing  $Y$  by the schematic image of  $X$  (3.5.10) and using (a), we may assume  $\pi$  is dominant; then we want to show that  $\pi$  is surjective. It follows from the definition 2.1.14 that  $\pi(X)$  contains the generic points of all irreducible component of  $Y$ ; then the stability of  $\pi(X)$  under specialization gives the result. ■

We end with a wonderful fact that is not as well-known as it should be.

**Theorem 3.3.4.** A universally closed morphism of schemes is quasicompact.

*Proof.* (Taken from [10].) By the implication (d)  $\Rightarrow$  (c) of 2.4.6, we reduce to showing: if  $k$  is a field, and  $\pi : X \rightarrow \text{Spec } k$  a universally closed morphism of schemes, then  $X$  is quasicompact. Let  $X = \bigcup_{i \in I} U_i$  be an open affine cover of  $X$ , and let  $A = k[X_i]_{i \in I}$  so that  $T := \text{Spec } A = \mathbb{A}_k^I$ . Let

$$Z := X \times_k T - \bigcup_{i \in I} U_i \times_k D(X_i).$$

Pick a nonzero  $f \in A$  such that  $D(f) \subset T \setminus \pi_T(Z)$ ; this is possible because  $\pi_T$  is closed and  $\pi_T(Z) \neq T$ , since say  $(1, 1, \dots) \in T \setminus \pi_T(Z)$ . Pick a finite subset  $J \subset I$  such that  $f \in k[X_j]_{j \in J}$ ; we will show that  $X = \bigcup_{j \in J} U_j$ . Indeed, if there were an  $x \in X \setminus \bigcup_{j \in J} U_j$ , and we picked a  $y \in D(f) \subset \mathbb{A}_k^n \hookrightarrow T$  (where the last closed immersion is given by setting all other variables  $X_i$  for  $i \notin J$  to zero) and a point  $z \in X \times_k T$  above  $(x, y)$ , then  $z$  would have to lie in  $Z$ . However, we would also conclude that  $f(\pi_T(z)) = f(y) \neq 0$ , which would contradict the fact that  $\pi_T(Z) \subset \mathbb{V}(f)$ . ■

The next order of business is to understand quasiseparated morphisms, but for that we need to understand quasiseparated schemes first.

**Proposition/Definition 3.3.5** (Quasiseparated Schemes). The following conditions on a scheme  $X$  are equivalent:

- (a) If  $U, V \subset X$  are quasicompact opens, then so is  $U \cap V$ .
- (b) If  $U, V \subset X$  are affine opens, then  $U \cap V$  is a finite union of affine opens in  $X$ .
- (c) There is an affine open cover  $\mathcal{U}$  of  $X$  all of whose pairwise intersections are quasicompact.

A scheme  $X$  satisfying these equivalent conditions is said to be *quasiseparated*. Further:

- (d) An open subscheme of an quasiseparated scheme is quasiseparated.
- (e) A locally Noetherian scheme is quasiseparated.

*Proof.*

- Step 1. The implications (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c) follow from the fact that an open subset of  $X$  is quasicompact iff it can be written as a finite union of affine opens of  $X$ .
- Step 2. We show that an affine scheme  $X$  satisfies (b). Indeed, let  $X$  be affine and  $U, V \subset X$  be affine opens. By 1.2.3,  $U$  (resp.  $V$ ) can be written as the finite union of affine opens  $U_i$  (resp.  $V_j$ ) which are simultaneously distinguished in  $U$  and  $X$  (resp. in  $V$  and  $X$ ). Then  $U \cap V = \bigcup_{i,j} U_i \cap V_j$ , with each intersection  $U_i \cap V_j$  being distinguished in  $X$  by 1.2.1(b).
- Step 3. Now we show (c)  $\Rightarrow$  (b). Let  $\mathcal{U}$  and  $U, V \subset X$  be as given. By 2.4.3 applied to  $\mathcal{U}$  and  $W = U$ , we conclude that for each  $U' \in \mathcal{U}$ , the intersection  $U \cap U'$  is quasicompact. Then 2.4.3 applied to  $\mathcal{U} \cup \{U\}$  and  $W = V$  shows that  $U \cap V$  is quasicompact.

The statement (d) is clear from (a). The statement (e) is clear from (b) combined with (b) and (e) of 2.2.7. ■

**Corollary 3.3.6** (Qcqs Schemes). A scheme  $X$  is quasicompact and quasiseparated (abbreviated *qcqs*) iff it admits a finite affine open cover, all of whose pairwise intersections are also covered by finitely many affine opens.

We like qcqs schemes for various reasons; one is

**Lemma 3.3.7** (Qcqs Lemma). Let  $X$  be a qcqs scheme and  $\mathcal{F} \in \mathbf{QCoh}(X)$ . Then for any  $f \in \mathcal{O}(X)$ , the natural morphism  $\mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \rightarrow \mathcal{F}(X_f)$  is an isomorphism.

*Proof.* Note that if  $U \subset X$  is any affine, then  $U \cap X_f = D(f|_U)$  is a distinguished open in  $U$ . Now cover  $X$  by finitely many affines  $U_i$  such that each pairwise intersection  $U_i \cap U_j$  is the finite union of affines  $U_{ijk}$ . Then for any sheaf  $\mathcal{F}$  on  $X$ , we have an exact sequence

$$0 \rightarrow \mathcal{F}(X) \rightarrow \bigoplus_i \mathcal{F}(U_i) \rightarrow \bigoplus_{i,j,k} \mathcal{F}(U_{ijk}). \quad (3.3.8)$$

Tensoring with the flat  $\mathcal{O}(X)$ -algebra  $\mathcal{O}(X)[f^{-1}]$  and using that tensor products commute with direct sums, along with some natural identifications, yields the analogous sequence

$$0 \rightarrow \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X) \rightarrow \bigoplus_i \mathcal{O}(U_i \cap X_f) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i) \rightarrow \bigoplus_{i,j,k} \mathcal{O}(U_{ijk} \cap X_f) \otimes_{\mathcal{O}(U_{ijk})} \mathcal{F}(U_{ijk}).$$

Map this to the sequence corresponding to 3.3.8 for the cover  $U_i \cap X_f$  of  $X_f$ , and use that the middle and last maps are then isomorphisms (because  $\mathcal{F} \in \mathbf{QCoh}(X)$ ) along with the five lemma to finish the proof.  $\blacksquare$

Now we are ready to define quasiseparated morphisms.

**Proposition/Definition 3.3.9** (Quasiseparated Morphisms). Let  $\pi : X \rightarrow Y$  be a morphism of schemes. The following conditions on  $\pi$  are equivalent.

- (a) For every quasiseparated open  $V \subset Y$ , the preimage  $\pi^{-1}(V) \subset X$  is quasiseparated.
- (b) For every affine open  $V \subset Y$ , the preimage  $\pi^{-1}(V)$  is quasiseparated.
- (c) There is an affine open cover  $Y = \bigcup_i V_i$  such that for each  $i$ , the preimage  $\pi^{-1}(V_i)$  is quasiseparated.
- (d) The diagonal morphism  $\Delta_\pi : X \rightarrow X \times_Y X$  is quasicompact.

A morphism  $\pi$  of schemes satisfying these equivalent conditions is said to be *quasiseparated*. Further:

- (e) Quasiseparated morphisms form a reasonable class.
- (f) Any morphism out of a quasiseparated scheme is quasiseparated.

*Proof.* The implications (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c) are clear.

- (c)  $\Rightarrow$  (d) We produce an affine open cover of  $X \times_Y X$  such that for each element of this cover, its preimage under  $\Delta_\pi$  is quasicompact. For each  $i$ , pick an affine open cover  $\pi^{-1}(V_i) = \bigcup_j U_{ij}$  of  $\pi^{-1}(V_i)$ . Then we claim that the affine open cover  $\{U_{ij} \times_{V_i} U_{ij'}\}_{i,j,j'}$  of  $X \times_Y X$  has this property; indeed, we have  $\Delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij'}) = U_{ij} \cap U_{ij'}$ , which is an intersection of affines in the quasiseparated  $\pi^{-1}(V_i)$  and hence quasicompact.
- (d)  $\Rightarrow$  (b) Let  $U, U' \subset \pi^{-1}(V)$  be affine opens; we want to show that  $U \cap U'$  is quasicompact. Well,  $U \times_V U' \xrightarrow{\sim} U \times_Y U' \subset X \times_Y X$  is an affine open and hence quasicompact, therefore its preimage under  $\Delta_\pi$ , namely  $U \cap U'$ , is also quasicompact.

At this point, we have shown the equivalence of (b), (c), and (d), and it is clear that a scheme  $X$  is quasiseparated in the sense of 3.3.5 iff the final morphism  $X \rightarrow \text{Spec } \mathbb{Z}$  satisfies conditions (d). Next, we note that, with definition (d) of quasiseparatedness, it is clear from the abstract formalism (3.1.8) that quasiseparated morphisms are stable under composition. Given this, we are ready to show

- (d)  $\Rightarrow$  (a) Let  $V \subset Y$  be any open subset and let  $X_V = \pi^{-1}(V)$ . Then the diagram

$$\begin{array}{ccc} X_V & \xhookrightarrow{\quad} & X \\ \downarrow \Delta & & \downarrow \Delta \\ X_V \times_V X_V & \xhookrightarrow{\quad} & X \times_Y X \end{array}$$

is a pullback, so stability of quasicompact morphisms under basechange (3.3.1(d)), we conclude that  $X_V \rightarrow V$  is also quasiseparated in the sense of (d). Since  $V$  is quasiseparated, we know that the final morphism  $V \rightarrow \text{Spec } \mathbb{Z}$  is quasiseparated in the sense of (d). Since quasiseparated morphisms in the sense of (d) are stable under composition, we conclude that the composite  $X_V \rightarrow V \rightarrow \text{Spec } \mathbb{Z}$  is quasiseparated in the sense of (d), but this is just saying that  $X_V$  is a quasiseparated scheme.

That quasiseparated morphisms are preserved by basechange follows from 3.1.8, and (f) is clear from 3.3.5(d). ■

Here's another proof of (c)  $\Rightarrow$  (b) using the Affine Communication Lemma.

*Proof 2 of (c)  $\Rightarrow$  (b).* By 3.1.1, it remains to check the following in the case  $Y$  is affine:

- (i) If  $X$  is quasiseparated, then for any distinguished  $D \subset Y$ , so is the preimage  $\pi^{-1}(D)$ ; this follows from 3.3.5(d).
- (ii) If  $\{D_\alpha\}$  is a finite cover of  $Y$  by distinguished opens and each  $\pi^{-1}(D_\alpha)$  is quasiseparated, then so is  $X$ . Indeed, suppose we are given affines  $U, V \subset X$ . Covering  $U$  by finitely many distinguished opens in  $U$ , each of which lies in  $\pi^{-1}(D_\alpha)$  for some  $\alpha$ , we reduce to the case in which  $U \subset \pi^{-1}(D_\alpha)$  for some  $\alpha$ . Then

$$U \cap V = U \cap \pi^{-1}(D_\alpha) \cap V = U \cap \pi|_V^{-1}(D_\alpha).$$

Here  $U$  is affine and hence quasicompact, and  $\pi|_V^{-1}(D_\alpha)$  is distinguished in  $V$  (by 1.2.1(a) applied to  $\pi|_V$ ) and hence quasicompact; since this intersection takes place in  $\pi^{-1}(D_\alpha)$ , which is quasiseparated, it follows that  $U \cap V$  is quasicompact. ■

A morphism is said to be *qcqs* if it is both quasicompact and quasiseparated. We like such morphisms because of the corresponding analog of the qcqs lemma:

**Lemma 3.3.10.** Let  $\pi : X \rightarrow Y$  be a qcqs morphism and  $\mathcal{F} \in \text{QCoh}(X)$ . Then  $\pi_*\mathcal{F} \in \text{QCoh}(Y)$ .

*Proof.* Since qcqs morphisms are affine-local on the target and the question of quasicohherence is likewise local on  $Y$ , we may assume that  $Y$  is affine; then  $X$  is a qcqs scheme. We have to show that if  $D \subset Y$  is distinguished, then the natural morphism  $\mathcal{O}(D) \otimes_{\mathcal{O}(Y)} \pi_*\mathcal{F}(Y) \rightarrow \pi_*\mathcal{F}(D)$  is an isomorphism. If  $g \in \mathcal{O}(Y)$  is such that  $D = D(g)$  and  $f := \mathcal{O}(\pi)(g) \in \mathcal{O}(X)$  the image of  $g$  under the ring homomorphism  $\mathcal{O}(\pi) : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ , then  $\pi^{-1}(D) = X_f$ . Since  $g$  acts by  $f$  on  $\mathcal{F}(X)$ , in light of 3.3.7, it remains to note only that the natural morphism  $\mathcal{O}(Y)[g^{-1}] \otimes_{\mathcal{O}(Y)} \mathcal{F}(X) \rightarrow \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{F}(X)$  is an isomorphism. ■

### 3.4 Affine, Integral, and Finite Morphisms

Before we define affine morphisms, we use the following consequence of the qcqs lemma taken from [2, Exercise 2.17].

**Lemma 3.4.1.** A scheme  $X$  is affine iff there is an  $n \in \mathbb{Z}_{\geq 1}$  and  $f_1, \dots, f_n \in \mathcal{O}(X)$  generating the unit ideal such that each  $X_{f_i}$  is affine.

*Proof.* If  $X$  is affine, it suffices to take  $n = 1$  and  $f_1 = 1$ . Conversely, the hypothesis implies that  $X$  is qcqs (3.3.6), so for each  $i$ , 3.3.7 applied to  $\mathcal{F} = \mathcal{O}_X$  and  $f = f_i$  tells us that  $\mathcal{O}(X)[f_i^{-1}] \simeq \mathcal{O}(X_{f_i})$ . Since  $X_{f_i}$  is affine, we get a series of isomorphisms

$$X_{f_i} \simeq \operatorname{Spec} \mathcal{O}(X_{f_i}) \simeq \operatorname{Spec} \mathcal{O}(X)[f_i^{-1}] \cong D(f_i) \subset \operatorname{Spec} \mathcal{O}(X).$$

This says exactly that the natural morphism  $X \rightarrow \operatorname{Spec} \mathcal{O}(X)$  is an isomorphism over the cover  $\{D(f_i)\}_i$  of  $\operatorname{Spec} \mathcal{O}(X)$ , so by the affine locality of isomorphisms we conclude that  $X \rightarrow \operatorname{Spec} \mathcal{O}(X)$  is also an isomorphism. ■

**Proposition/Definition 3.4.2** (Affine Morphisms). Let  $\pi : X \rightarrow Y$  be a morphism of schemes. The following conditions on  $\pi$  are equivalent.

- (a) For every affine open  $V \subset Y$ , the preimage  $\pi^{-1}(V) \subset X$  is affine.
- (b) There is an open affine cover  $Y = \bigcup_i V_i$  such that for each  $i$ , the preimage  $\pi^{-1}(V_i)$  is affine.

A morphism satisfying these equivalent conditions is said to be *affine*. Further:

- (c) Affine morphisms form a reasonable class.

*Proof.* Using 3.1.1, reduce to 1.2.1(a) and 3.4.1. For (c), use 3.1.2. ■

Recall that an algebra  $A \rightarrow B$  is said to be *finite* if  $B$  is a finitely generated  $A$ -module. The globalization of this definition is

**Proposition/Definition 3.4.3** (Integral and Finite Morphisms). Let  $\pi : X \rightarrow Y$  be a morphism of schemes. The following conditions on  $\pi$  are equivalent.

- (a) open  $V \subset Y$ , the preimage  $\pi^{-1}(V) \subset X$  is affine and  $\mathcal{O}(\pi^{-1}(V))$  is an integral (resp. a finite)  $\mathcal{O}(V)$ -algebra.
- (b) There is an open affine cover  $Y = \bigcup_i V_i$  such that for each  $i$ , the preimage  $\pi^{-1}(V_i)$  is affine and  $\mathcal{O}(\pi^{-1}(V_i))$  is an integral (resp. a finite)  $\mathcal{O}(V_i)$ -algebra.

A morphism satisfying these equivalent conditions is said to be *finite*. Further:

- (c) Integral (resp. finite) morphisms form a reasonable class.
- (d) Finite morphisms have finite discrete fibers.

*Proof.* Using that a ring homomorphism  $A \rightarrow B$  is finite iff it is integral and of finite type along with the already proven results (3.2.4), it suffices to show the integral case. Using 3.1.1, reduce to showing the following.

- (i) If  $\varphi : A \rightarrow B$  is an integral algebra, then for any  $f \in A$ , the localization  $A[f^{-1}] \rightarrow B[\varphi(f)^{-1}]$  is also. This is clear.
- (ii) If  $\varphi : A \rightarrow B$  is an  $A$ -algebra,  $n \in \mathbb{Z}_{\geq 1}$ , and  $f_1, \dots, f_n \in A$  with  $(f_1, \dots, f_n) = 1$  such that for each  $i$ , the algebra  $A[f_i^{-1}] \rightarrow B[\varphi(f_i)^{-1}]$  is integral, then  $A \rightarrow B$  is finite. To show this, replace  $A$  by  $\varphi(A)$  in  $B$  to assume that  $\varphi$  is an inclusion. Then for each  $b \in B$ , pick  $N, M \gg 1$  such that for all  $i = 1, \dots, n$ , we have  $f_i^N b^M \in \sum_{i=0}^{M-1} A b^i$ . Then use a partition-of-unity argument to check that in fact  $b^M \in \sum_{i=0}^{M-1} A b^i$  as needed.

For (c), use 3.1.2 as usual. For (d), we use (c) to reduce to the case of the target being  $\operatorname{Spec} k$  for a field  $k$ ; then  $X$  is an Artinian scheme over  $k$ . ■

### 3.5 Closed Embeddings

Let  $X$  be a locally ringed space and  $\mathcal{I} \subset \mathcal{O}_X$  an ideal sheaf. We want to define a closed subscheme  $Z = \mathbb{V}(\mathcal{I})$  of  $X$  corresponding to the ideal sheaf  $\mathcal{I}$ . For this, as a topological space we take  $Z := \text{Supp}(\mathcal{O}_X/\mathcal{I}) \subset X$ , which is a closed subset since  $\mathcal{O}_X/\mathcal{I}$  is a sheaf of rings on  $X$ . Let  $\iota : Z \hookrightarrow X$  denote the inclusion map. Define a sheaf of rings on  $Z$  by taking  $\mathcal{O}_Z := \iota^{-1}(\mathcal{O}_X/\mathcal{I})$ , so that at a point  $z \in Z$ , the stalk  $\mathcal{O}_{Z,z} = \mathcal{O}_{X,z}/\mathcal{I}_z$  is a nonzero quotient of a local ring  $\mathcal{O}_{X,z}$  and hence local. This construction therefore gives us a locally ringed space  $Z$ . Define a morphism  $\iota : Z \rightarrow X$  of locally ringed spaces by requiring the corresponding morphism of sheaves to arise from the natural surjection  $\iota^\sharp : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I} \cong \iota_*\mathcal{O}_Z$ .<sup>9</sup> In particular, there is an exact sequence of abelian sheaves on  $X$  of the form

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \iota_*\mathcal{O}_Z \rightarrow 0 \quad (3.5.1)$$

called the *ideal sheaf exact sequence*.<sup>10</sup> With this construction, it is clear that the morphism  $\iota : Z \rightarrow X$  is a monomorphism in the category of locally ringed spaces.

As with ideals in a ring, this construction satisfies the expected universal property.

**Proposition 3.5.2.** Let  $X$  be a locally ringed space and  $\mathcal{I}$  an ideal sheaf on  $X$ . Let  $Z$  and  $\iota$  be as above. If  $Y$  is any other locally ringed space and  $f : Y \rightarrow X$  a morphism such that  $\mathcal{I} \subset \ker(f^\sharp : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y)$ , then  $f$  factors uniquely through  $Z$ , i.e., there is a unique morphism of locally ringed spaces  $\tilde{f} : Y \rightarrow Z$  such that  $f = \iota \circ \tilde{f}$ .

We could take this universal property as a *definition* of  $\mathbb{V}(\mathcal{I})$ ; then the above paragraph would amount to giving a *construction*. Note that conversely if a morphism  $f : Y \rightarrow X$  admits a factorization of the form  $f = \iota \circ \tilde{f}$  for some  $\tilde{f} : Y \rightarrow Z$ , then automatically we have that  $\mathcal{I} \subset \ker(f^\sharp : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y)$ .

*Proof.* The uniqueness of such an  $f$  follows from  $\iota$  being a monomorphism. To show existence, first note that  $f^\sharp : \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$  admits a factorization as a morphism of sheaves of rings of the form

$$\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I} \rightarrow f_*\mathcal{O}_Y. \quad (3.5.3)$$

This implies that set-theoretically  $f(Y) \subset Z$ ; indeed, if  $y \in Y$ , then we have a factorization  $\mathcal{O}_{X,f(y)} \rightarrow (\mathcal{O}_X/\mathcal{I})_{f(y)} \rightarrow \mathcal{O}_{Y,y}$  of  $f_y^\sharp$ ; since the composite is a nonzero homomorphism of rings between nonzero rings, the middle ring cannot be zero. Therefore, by definition of the subspace topology, there is a unique continuous map  $\tilde{f} : Y \rightarrow Z$  such that  $f = \iota \circ \tilde{f}$  as continuous maps. To upgrade this to a morphism of locally ringed spaces, we need to give a morphism  $\tilde{f}^\sharp : \mathcal{O}_Z \rightarrow \tilde{f}_*\mathcal{O}_Y$  of sheaves of rings on  $Z$  such that the resulting composition  $f = \iota \circ \tilde{f}$  holds as morphisms of locally ringed spaces. Since  $\iota_* : \text{Ab}(Z) \rightarrow \text{Ab}(X)$  is fully faithful (and preserves ring sheaves), it suffices to produce a morphism  $\iota_*\tilde{f}^\sharp : \iota_*\mathcal{O}_Z \rightarrow \iota_*\tilde{f}_*\mathcal{O}_Y$  of sheaves on  $X$ ; take this to be the composition

$$\iota_*\mathcal{O}_Z = \iota_*\iota^{-1}(\mathcal{O}_X/\mathcal{I}) \xrightarrow{\eta_{\mathcal{O}_X/\mathcal{I}}^{-1}} \mathcal{O}_X/\mathcal{I} \rightarrow f_*\mathcal{O}_Y \cong \iota_*\tilde{f}_*\mathcal{O}_Y,$$

where the second-to-last morphism comes from the factorization (3.5.3). Then the identity  $f = \iota \circ \tilde{f}$  as morphisms of locally ringed spaces holds by construction using (3.5.3). ■

That this is the right construction in our context follows from

**Proposition 3.5.4.** Let  $X$  be a scheme and  $\mathcal{I}$  a *quasicoherent* ideal sheaf on  $X$ . Then the locally ringed space  $Z$  constructed above is a scheme and  $\iota : Z \rightarrow X$  is a morphism of schemes. Further, the quotient  $\iota_*\mathcal{O}_Z$  is also quasicoherent on  $X$ , i.e., the ideal sheaf exact sequence (3.5.1) is an exact sequence in  $\text{QCoh}(X)$ .

<sup>9</sup>Here we are using the following fact: if  $\pi : Z \rightarrow X$  is a continuous map, then we have an adjunction  $\pi^{-1} \dashv \pi_*$  between  $\text{Ab}(Z)$  and  $\text{Ab}(X)$ . Further, if  $\pi = \iota$  is a topological closed embedding (i.e., up to identifications, the inclusion of a closed subset), then the counit  $\epsilon : \iota^{-1}\iota_* \rightarrow 1$  is an isomorphism, which tells us that  $\iota_* : \text{Ab}(Z) \rightarrow \text{Ab}(X)$  is fully faithful with essential image consisting of the abelian sheaves on  $X$  supported on  $Z$ . In particular, if  $\mathcal{F}$  is an abelian sheaf on  $X$  supported on  $Z$ , then the unit  $\eta_{\mathcal{F}} : \mathcal{F} \rightarrow \iota_*\iota^{-1}\mathcal{F}$  is an isomorphism. This follows also from the fact that if  $j : U \hookrightarrow X$  denotes the open complement of  $Z$ , then we have a natural exact sequence  $0 \rightarrow j_!j^{-1}\mathcal{F} \rightarrow \mathcal{F} \rightarrow \iota_*\iota^{-1}\mathcal{F} \rightarrow 0$  on  $\mathcal{F}$ .

<sup>10</sup>Since the functor  $\iota_*$  is fully faithful, some texts drop the  $\iota_*$  from 3.5.1, writing it instead as  $0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$ , but this means exactly the same thing.

*Proof.* The question is local on  $X$ , so we may assume without loss of generality that  $X$  is affine. Let  $A := \mathcal{O}(X)$ , and let  $I := \mathcal{I}(X) \subset A$ . The quasicoherence of  $\mathcal{I}$  tells us that there is an isomorphism  $\mathcal{O}_X/\mathcal{I} \cong \widetilde{A/I}$  of sheaves of  $X$ , from which it follows that as a topological space  $Z = \text{Supp}(\widetilde{A/I}) = \text{Supp}(A/I) = \mathbb{V}(I) \subset \text{Spec } A$ . Therefore, the underlying topological space of  $Z$  is homeomorphic to  $\text{Spec}(A/I)$ , and so admits the sheaf  $\mathcal{O}_{\text{Spec}(A/I)}$  also. Now consider the adjunctions (and inclusion)

$$\begin{aligned} \text{Hom}_{A\text{-Mod}}(A/I, \Gamma(Z, \mathcal{O}_{\text{Spec}(A/I)})) &\cong \text{Hom}_{\mathcal{O}_X\text{-Mod}}(\widetilde{A/I}, \iota_* \mathcal{O}_{\text{Spec}(A/I)}) \\ &\subset \text{Hom}_{\text{Ab}(X)}(\widetilde{A/I}, \iota_* \mathcal{O}_{\text{Spec}(A/I)}) \\ &\cong \text{Hom}_{\text{Ab}(Z)}(\iota^{-1}(\widetilde{A/I}), \mathcal{O}_{\text{Spec}(A/I)}). \end{aligned}$$

The obvious isomorphism in the first term above therefore gives rise to an abelian sheaf morphism  $\iota^{-1}\widetilde{A/I} \rightarrow \mathcal{O}_{\text{Spec}(A/I)}$ , which is easily seen to be an isomorphism on stalks and hence an isomorphism. In particular,  $Z$  is a(n) (affine) scheme. Further, considering global sections shows that the morphism  $i : Z \rightarrow X$  constructed above corresponds exactly to the natural morphism  $\text{Spec } \pi$  arising from  $\pi : A \twoheadrightarrow A/I$ . ■

**Remark 3.5.5.**

- (a) This statement is *not* true in general for a scheme  $X$  if we do not require  $\mathcal{I}$  to be a quasicoherent. Here's a standard counterexample: let  $X = \mathbb{A}_{\mathbb{C}}^1$  and  $U = \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} \subset X$ , with  $j : U \hookrightarrow X$  the inclusion. Then  $j_!\mathcal{O}_U \subset \mathcal{O}_X$  is a sheaf of ideals. Applying the above construction to this sheaf yields  $Z = \{0\}$  equipped with the structure sheaf  $\mathcal{O}_{X,0} \cong \mathbb{C}[t]_{(t)}$ . This  $Z$  is certainly a locally ringed space, but not a scheme.
- (b) Different quasicoherent ideal sheaves  $\mathcal{I}$  can give rise to the same underlying topological space  $Z$  (e.g., the sheaves  $(t)$  and  $(t^2)$  on  $X = \mathbb{A}_{\mathbb{C}}^1 = \text{Spec } \mathbb{C}[t]$ ), although the *scheme*  $Z$  along with the morphism  $\iota : Z \rightarrow X$  determines  $\mathcal{I}$  uniquely, namely as the kernel of the map of sheaves  $\iota^\# : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z$ .

**Proposition/Definition 3.5.6** (Closed Immersion). The following conditions on a morphism  $\iota : Z \rightarrow X$  of schemes are equivalent:

- (a) For each open affine  $U \subset X$ , the preimage  $\iota^{-1}(U)$  is affine, and  $\mathcal{O}(U) \twoheadrightarrow \mathcal{O}(\iota^{-1}(U))$ .
- (b) There is an affine open cover  $X = \bigcup U_i$  of  $X$  such that for each  $i$ , the preimage  $\iota^{-1}(U_i)$  is affine and  $\mathcal{O}(U_i) \twoheadrightarrow \mathcal{O}(\iota^{-1}(U_i))$ .
- (c) The continuous map  $|\iota|$  is a topological closed embedding (i.e., a homeomorphism onto a closed subset of  $X$ ), and further the sheaf morphism  $\iota^\# : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z$  is surjective.
- (d) There is a quasicoherent ideal sheaf  $\mathcal{I} \subset \mathcal{O}_X$  such that the map  $\iota$  factors as an isomorphism  $Z \xrightarrow{\sim} \mathbb{V}(\mathcal{I})$  followed by the inclusion morphism  $\mathbb{V}(\mathcal{I}) \hookrightarrow X$  constructed above.

A morphism  $\iota$  satisfying these equivalent conditions is said to be a *closed embedding* or a *closed immersion*. In this case, the ideal sheaf  $\mathcal{I} \subset \mathcal{O}_X$  and the isomorphism  $Z \rightarrow \mathbb{V}(\mathcal{I})$  in (d) are determined uniquely by  $\iota$ . Further:

- (e) Closed immersions form a reasonable class.

*Proof.*

- (a)  $\Rightarrow$  (b) Clear.
- (b)  $\Rightarrow$  (c) Topological closed embeddings are a reasonable class of continuous maps (3.1.5), and the surjectivity of  $\iota^\#$  is a local question.
- (c)  $\Rightarrow$  (d) The conditions of (c) imply that  $\iota$  is qcqs; indeed, a topological closed embedding is a quasicompact, and (c) implies that  $\iota$  is a monomorphism in the category of locally ringed spaces, so that the property 3.3.9(d) for  $\iota$  is obvious. By 3.3.10, the sheaf  $\iota_* \mathcal{O}_Z$  on  $X$  is quasicoherent, and hence so is the kernel  $\mathcal{I} := \ker(\iota^\#)$  by 1.3.2(e). We claim that this  $\mathcal{I}$  works. By the universal property (3.5.2), there is a unique factorization of  $\iota$  as  $Z \xrightarrow{\tilde{\iota}} \mathbb{V}(\mathcal{I}) \hookrightarrow X$  as a morphism of schemes, and we have to show that  $\tilde{\iota} : Z \rightarrow \mathbb{V}(\mathcal{I})$  is an isomorphism. Topologically, since  $\iota$  and  $\mathbb{V}(\mathcal{I}) \hookrightarrow X$  are closed embeddings, so is  $\tilde{\iota}$ ; it is then a homeomorphism because it is also surjective: the sheaf  $\iota_* \mathcal{O}_Z$  is supported exactly on  $\iota(Z)$ . To check that  $\tilde{\iota}^\# : \mathcal{O}_{\mathbb{V}(\mathcal{I})} \rightarrow \tilde{\iota}_* \mathcal{O}_Z$  is an isomorphism, we may check it after pushing forward via the inclusion to  $X$ , at which point we have to check that the natural

morphism  $\mathcal{O}_X/\mathcal{I} \rightarrow \iota_*\mathcal{O}_Z$  induced by  $\iota^\sharp$  is an isomorphism, but indeed this is true by definition of  $\mathcal{I}$  since we assume that  $\iota^\sharp$  is surjective as a morphism of sheaves.

(d)  $\Rightarrow$  (a) Follows from (the proof of) 3.5.4.

Affine-locality on the target follows from the equivalence of (a) and (b), stability under composition is clear from (a), and stability under base change follows from 3.1.2.  $\blacksquare$

We could also prove (b)  $\Rightarrow$  (a) above directly using 3.1.1; this proof is left to the reader as an exercise.

**Definition 3.5.7** (Closed Subschemes). Let  $X$  be a scheme and  $\iota : Z \rightarrow X$  and  $\iota' : Z' \rightarrow X$  denote two closed embeddings.

(a) We say that  $Z'$  *majorizes*<sup>11</sup>  $Z$ , written  $Z \preceq Z'$ , iff there is a morphism  $\tilde{\iota} : Z \rightarrow Z'$  such that  $\iota = \iota' \circ \tilde{\iota}$ .

By 3.5.2, this happens iff the corresponding ideal sheaves  $\mathcal{I}$  and  $\mathcal{I}'$  satisfy  $\mathcal{I}' \subset \mathcal{I}$ , and in this case  $\tilde{\iota}$  is uniquely determined and also necessarily a closed embedding.

(b) We say that  $Z'$  and  $Z$  are *equivalent* if both  $Z \preceq Z'$  and  $Z' \preceq Z$ .

This notion of equivalence is an equivalence relation on closed embeddings (check!).

(c) A *closed subscheme* of  $X$  is an equivalence class of closed embeddings mapping to  $X$ .

Therefore, closed subschemes of  $X$  correspond to quasicoherent ideal sheaves on  $X$ , with the correspondence given by taking a quasicoherent ideal sheaf  $\mathcal{I}$  to the (equivalence class of) the morphism  $\mathbb{V}(\mathcal{I}) \hookrightarrow X$  constructed above.

**Remark 3.5.8.** On a locally Noetherian scheme (e.g. a scheme locally of finite type over a field  $k$ ), every quasicoherent ideal sheaf is automatically coherent; therefore, in this case, closed subschemes correspond to coherent ideal sheaves.

**Definition 3.5.9** (Schematic Image).

**Theorem 3.5.10.** Let  $\pi : X \rightarrow Y$  be a morphism of schemes. If  $\pi$  is qc or  $X$  is reduced, then the schematic image  $\text{Im}(\pi)$  of  $\pi$  can be computed affine-locally on  $Y$ : for any affine open  $V \subset Y$ , the intersection  $V \cap \text{Im}(\pi)$  is the closed subscheme of  $V$  corresponding to the ideal  $\ker \pi_V^\sharp : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(\pi^{-1}(V))$ . In particular, the underlying subspace of  $Y$  corresponding to  $\text{Im}(\pi)$  is exactly  $\overline{\pi(X)}$ .

<sup>11</sup>Being more pedantic, one could insist on saying that  $\iota'$  majorizes  $\iota$  instead, since this notion does depend on the morphisms  $\iota$  and  $\iota'$  and not just the abstract isomorphism classes of  $Z$  and  $Z'$ , but we leave this level of pedantry to the pedants.

### 3.6 Separated and Proper Morphisms

**Definition 3.6.1.** A morphism  $\pi : X \rightarrow Y$  is said to be *separated* iff the diagonal  $\Delta_\pi : X \rightarrow X \times_Y X$  is a closed embedding.

**Remark 3.6.2.**

- (a) A morphism  $\pi : X \rightarrow Y$  is separated iff  $\Delta_\pi(X) \subset X \times_Y X$  is a closed subset; this follows from the fact that  $\Delta_\pi$  is a locally closed immersion.
- (b) Since closed immersions form a reasonable class (3.5.6), it follows from the general theory (3.1.8) that separated morphisms form a reasonable class.
- (c) Affine morphisms and monomorphisms of schemes are separated, and separated morphisms are quasiseparated.
- (d) A scheme  $X$  is said to be *separated* iff the final morphism  $X \rightarrow \text{Spec } \mathbb{Z}$  is separated. If  $X$  is a separated scheme, then every morphism from  $X$  to an affine scheme is separated; conversely, if  $X$  admits a separated morphism to an affine scheme, then  $X$  is itself separated.

In the following, we let  $A$  be a valuation ring with maximal ideal  $\mathfrak{m}$ , and  $K := \text{Frac } A$ . We let  $\Delta := \text{Spec } A$ , and  $\Delta^* := \text{Spec } K$ , with the inclusion  $A \rightarrow K$  corresponding to the open embedding  $\Delta^* \hookrightarrow \Delta$ . For any morphism  $\pi : X \rightarrow Y$ , we can ask about the extension property of morphisms of the form

$$\begin{array}{ccc} \Delta^* & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow \pi \\ \Delta & \longrightarrow & Y \end{array} \quad (3.6.3)$$

In other words, give morphisms  $\Delta^* \rightarrow X$  and  $\Delta \rightarrow Y$  such that the solid diagram (3.6.3) commutes, we can ask about the existence and uniqueness of dashed lifts  $\Delta \rightarrow X$  making the diagram commute. We say the morphism  $\pi$  *satisfies the uniqueness (resp. existence, resp. existence and uniqueness) part(s) of the valuative criterion (with respect to all valuation rings)* iff for all valuation rings  $A$  and solid diagrams 3.6.3, there is at most one (resp. at least one, resp. exactly one) lift  $\Delta \rightarrow X$  making the diagram 3.6.3 commute.

**Theorem 3.6.4** (Valuative Criteria).

- (a) A morphism is separated iff it is qs and satisfies the uniqueness part of the valuative criterion.
- (b) A morphism is universally closed iff it is qc and satisfies the existence part of the valuative criterion.
- (c) A morphism is separated and universally closed iff it is qcqs and satisfies the existence and uniqueness parts of the valuative criterion.

In particular, a morphism is proper iff it is qs and finite-type, and satisfies the existence and uniqueness parts of the valuative criterion.

*Proof.* Clearly, (c) follows from combining (a) and (b). Note that in the above notation, we have a natural bijection of  $X(A)$  with triples  $(\eta, s, \varphi)$ , where  $\eta, s \in X$  are points such that  $\eta \rightsquigarrow s$  and  $\varphi : \kappa(\eta) \rightarrow K$  is an local homomorphism such that under this inclusion, the local ring  $\mathcal{O}_{\bar{\eta}, s}$  of  $\kappa(\eta)$  is dominated by the local ring  $A$  of  $K$ .<sup>12</sup>

- (a) A separated morphism is qs (3.6.2(c)), and satisfies the uniqueness part of the valuative criterion: if  $\pi$  is separated, the equalizer of two lifts  $\Delta \rightarrow X$  is a closed subscheme of the integral scheme  $\Delta$ , and so if it contains  $\Delta^*$ , it must be all of  $\Delta$ . For the converse, under the given hypotheses, it suffices to show using 3.6.2(a) and 3.3.3(b) that  $\Delta_\pi(X) \subset X \times_Y X$  is stable under specialization, so suppose that  $\eta \in \Delta_\pi(X)$  specializes to some  $s \in X \times_Y X$ . By Zorn's Lemma, there is a valuation ring  $A$  of  $\kappa(\eta)$  dominating  $\mathcal{O}_{\bar{\eta}, s}$ . As noted above, this yields a morphism  $f : \text{Spec } A \rightarrow X \times_Y X$  such that  $f([\mathfrak{m}]) = s$  and which restricts to the natural morphism  $\{\eta\} \rightarrow X \times_Y X$  on  $\Delta^* = \text{Spec } \kappa(\eta)$ . Since  $\eta \in \Delta_\pi(X)$ , composing  $f$  with the canonical projections gives two morphisms  $f_1, f_2 : \Delta \rightarrow X$  and a solid diagram (3.6.3) in which these two morphisms fit as lifts. By hypothesis, this forces  $f_1 = f_2$ , i.e., that  $f$  factors through  $\Delta_\pi$ , and hence that  $s \in \Delta_\pi(X)$  as well.
- (b) A universally closed morphism is quasicompact (3.3.4). To show that it satisfies the existence part of the valuative criterion, suppose we are given a solid diagram (3.6.3), and consider the

<sup>12</sup>Here  $\bar{\eta} \subset X$  is given the reduced closed subscheme structure.

corresponding fiber product  $X \times_Y \Delta$  to reduce to the following problem: if  $\pi : X \rightarrow \Delta$  is a closed morphism, then every section  $\sigma : \Delta^* \rightarrow X$  of  $\pi$  extends to a section over  $\Delta$ . A given section  $\sigma$  corresponds to a point  $\eta \in X$  as well as a field isomorphism  $\varphi : \kappa(\eta) \xrightarrow{\sim} K$ . The restriction  $\pi|_{\bar{\eta}}$  has closed image, so there is an  $s \in \bar{\eta}$  such that  $s \mapsto [\mathfrak{m}]$ . This gives us a local homomorphism  $A \rightarrow \mathcal{O}_{\bar{\eta},s}$  which extends to the inverse isomorphism  $\varphi^{-1} : K \rightarrow \kappa(\eta)$ . Since  $A$  is a valuation ring, it is maximal with respect to dominance in its fraction field  $K$ , so that the induced map  $A \rightarrow \mathcal{O}_{\bar{\eta},s}$  is an isomorphism. Therefore, the triple  $(\eta, s, \varphi)$  corresponds to a morphism  $\Delta \rightarrow X$  which is the extension of the section  $\sigma$ .

Conversely, suppose that  $\pi : X \rightarrow Y$  is qc and satisfies the existence part of the valuative criterion. Then every base-change of  $\pi$  also satisfies these properties; therefore, it remains to show only that  $\pi$  is closed. If  $Z \subset X$  is closed, then giving it the reduced closed subscheme structure, the composite  $Z \hookrightarrow X \rightarrow Y$  is also qc and satisfies the existence part; therefore, using 3.3.3(b), we have reduced to showing that  $\pi : X \rightarrow Y$  is qc and satisfies the existence part, then  $\pi(X) \subset Y$  is stable under specialization. For this, suppose that  $\eta \in \pi(X)$  specializes to some  $s \in Y$ . Pick a point  $\theta \in X$  mapping to  $\eta$ ; then  $\mathcal{O}_{\bar{\eta},s} \subset \kappa(\eta) \xrightarrow{\varphi} \kappa(\theta) =: K$ . As above, let  $A$  be a valuation ring of  $K$  dominating  $\mathcal{O}_{\bar{\eta},s}$ . At this point, we have enough data to produce a solid diagram 3.6.3 where the bottom arrow  $\Delta \rightarrow Y$  corresponds to  $(\eta, s, \varphi)$ . The existence of a lift  $\Delta \rightarrow X$  then tells us that  $s \in \pi(X)$  as needed. ■

**Example 3.6.5.** For any scheme  $S$  and integer  $n \in \mathbb{Z}_{\geq 0}$ , the morphism  $\mathbb{P}_S^n \rightarrow S$  is proper. Since proper morphisms form a reasonable class, it suffices to do the case of  $S = \text{Spec } \mathbb{Z}$ . Then 3.6.4 tells us we only have to check the existence and uniqueness part of the valuative criterion for  $\mathbb{P}_{\mathbb{Z}}^n \rightarrow \text{Spec } \mathbb{Z}$  with respect to valuation rings  $A$ , but existence is the standard argument on clearing denominators, whereas uniqueness follows from noting that any two points of  $\mathbb{P}_{\mathbb{Z}}^n$  lie in a common distinguished affine open of the form  $D_+(X_0)$  (up to automorphisms of  $\mathbb{P}_{\mathbb{Z}}^n$  when  $n \geq 1$ —with the case  $n = 0$  being trivial).

When we allow mild finiteness hypotheses, we can in fact check only on *discrete* valuation rings.

**Theorem 3.6.6** (Noetherian Valuative Criteria). Let  $\pi : X \rightarrow Y$  be a finite-type morphism of locally Noetherian schemes. Then  $\pi$  is separated (resp. universally closed, resp. proper) iff it satisfies the uniqueness (resp. existence, resp. existence and uniqueness) part(s) of the valuative criterion with respect to all *discrete* valuation rings.

*Proof.* ([2, Ex. II.4.11(a)]) By (the proof of) 3.6.4, it only remains to quote the following result in commutative algebra: if  $(\mathcal{O}, \mathfrak{m}, k)$  is a Noetherian local domain that is not a field, with fraction field  $\kappa$ , and  $K$  a finitely generated field extension of  $\kappa$ , then there is a *discrete* valuation ring  $A$  of  $K$  dominating  $\mathcal{O}$ . This is [6, Theorem 7.1.10]. ■

It is clear from the proof that we have proven slightly more generally that if  $\pi : X \rightarrow Y$  is an *lft* morphism of locally Noetherian schemes, then  $\pi$  is separated iff it satisfies the uniqueness part of the valuative criterion with respect to all DVRs. Even in non-Noetherian settings, this is often helpful, because we can sometimes reduce to the Noetherian case by a limiting or base-change argument.

**Example 3.6.7.** Suppose  $X$  is an irreducible Noetherian regular curve proper over a field  $k$  (resp. over  $\mathbb{Z}$ ). There is a bijection between discrete valuations on  $k(X)$  trivial on  $k$  (resp. with no conditions) and the closed points of  $X$ . For instance, for any field  $k$ , we recover a classification of the discrete valuations on a field  $k(t)$  which are trivial on  $k$  (by applying this to  $X = \mathbb{P}_k^1$ ), and similarly a classification of discrete valuations on any number field  $K$  (by applying this to  $X = \text{Spec } \mathcal{O}_K$ ).

The valuative criterion gives us an important result about extending morphisms from curves to proper schemes.

**Theorem 3.6.8** (Curve to Proper Extension). Let  $C$  be a equidimensional scheme with  $\dim C = 1$  and  $P \in C$  a regular closed point. Suppose we have a solid diagram of schemes of the form

$$\begin{array}{ccc}
 C \setminus \{P\} & \longrightarrow & Y \\
 \downarrow & \searrow \exists! & \downarrow \\
 C & \longrightarrow & S.
 \end{array}$$

If  $Y \rightarrow S$  is proper, then there is a unique dashed arrow making the diagram commute.

*Proof.* Without loss of generality, we may replace  $C$  by a small neighborhood of  $P$  to assume that  $C = \operatorname{Spec} A$  is affine, with  $A$  a Noetherian domain of dimension one, and that there is a  $t \in A$  such that  $P$  corresponds to the maximal ideal  $(t)$ , where  $t$  maps to a uniformizer of the DVR  $A_{(t)}$ . At this point, uniqueness is clear from the separatedness of  $Y \rightarrow S$ . For existence, let  $K := \operatorname{Frac} A$  and use 3.6.4 to produce a morphism  $\operatorname{Spec} A_{(t)} \rightarrow Y$  such that the composite  $\operatorname{Spec} K \rightarrow \operatorname{Spec} A_{(t)} \rightarrow Y$  agrees with the map  $\operatorname{Spec} K \rightarrow C \setminus \{P\} \rightarrow Y$ . At this point, we have to solve a gluing problem in which the properness of  $Y \rightarrow S$  is no longer needed. Replacing  $Y$  by a small affine neighborhood of the image of the closed point of  $\operatorname{Spec} A_{(t)}$  and shrinking  $C$  further as necessary, it remains to show: if  $B$  is a ring, and  $B \rightarrow A[1/t]$  and  $B \rightarrow A_{(t)}$  ring homomorphisms which “agree” on the generic point, i.e., such that the two induced maps  $B \rightarrow K$  are the same, then both factor through  $A$ . But this is clear, since  $A = A_{(t)} \cap A[1/t] \subset K$ , which follows immediately from the primality of  $(t)$  in  $A$  (check!). ■

### 3.7 Exercises

**Exercise 3.1.** Let  $A$  be a DVR with fraction field  $K$ , and let  $X := \operatorname{Spec} A$ .

- (a) Give an equivalence of categories between  $\mathcal{O}_X\text{-Mod}$  and triples  $(M, V, \varphi)$ , where  $M \in A\text{-Mod}$ ,  $V \in K\text{-Mod}$ , and  $\varphi : M \rightarrow \operatorname{Res}_A^K V$  is an  $A$ -linear map, with morphisms  $(f_A, f_K) : (M_1, V_1, \varphi_1) \rightarrow (M_2, V_2, \varphi_2)$  given by pairs  $(f_A, f_K)$  where  $f_A : M_1 \rightarrow M_2$  is an  $A$ -module homomorphism and  $f_K : V_1 \rightarrow V_2$  a  $K$ -linear map such that  $f_K \varphi_1 = \varphi_2 f_A$ . Under this equivalence, quasicoherent  $\mathcal{O}_X$ -modules correspond to triples  $(M, V, \varphi)$  where  $\varphi$  induces an isomorphism  $K \otimes_A M \rightarrow V$ , and coherent (i.e., finite-type, i.e., finite-presentation)  $\mathcal{O}_X$ -modules correspond to triples  $(M, V, \varphi)$  where in addition  $M$  is finitely generated.
- (b) Under the identification of (a), suppose  $\mathcal{F}_i \in \mathcal{O}_X\text{-Mod}$  corresponds to  $(M_i, V_i, \varphi_i)$  for  $i = 1, 2$ . Show that  $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{O}_X\text{-Mod}$  corresponds to the triple  $(\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2), \operatorname{Hom}_{K\text{-Mod}}(V_1, V_2), \operatorname{pr})$ , where  $\operatorname{pr}$  is the projection map which takes in a pair  $(f_A, f_K)$  and remembers only the map  $f_K : V_1 \rightarrow V_2$ .

**Exercise 3.2.** Let  $p$  be a prime,  $k := \mathbb{F}_p(t)$ , and  $C := \operatorname{Spec} k[X, Y]/(Y^2 - X^p + t)$ , and  $P$  the point corresponding to the prime  $(Y, X^p - t)$ .

- (a) Check that  $C$  is an integral curve and that  $P \in C$  is a regular closed point, even though  $C$  is not smooth over  $k$  at  $P$ .
- (b) The  $k$ -morphism  $C \setminus \{P\} \rightarrow \mathbb{P}_k^1$  given by  $(X, Y) \mapsto [X^p - t, Y]$  extends to a  $k$ -morphism  $C \rightarrow \mathbb{P}_k^1$  by 3.6.8. What is the image of  $P$  under this extension?

## Chapter 4

# Dimension

## 4.1 Basics

Next, we discuss *dimension*, a purely topological notion that is very important in algebraic geometry.

**Definition 4.1.1** (Dimension). Let  $X$  be a topological space.

- (a) A (finite) chain of irreducible closed subsets of  $X$  consists of an integer  $n \in \mathbb{Z}_{\geq 0}$  and irreducible closed subsets

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X.$$

Here  $n$  is called the *length* of the chain.

- (b) The *dimension* of  $X$ , denoted  $\dim X$ , is the supremum of all  $n \in \mathbb{Z}_{\geq 0}$  such that there is a chain of irreducible closed subsets of  $X$  of length  $n$ . We say that  $X$  is *equidimensional* or *pure of dimension*  $n$  if every irreducible component of  $X$  has dimension  $n \in \mathbb{Z}_{\geq 0}$ . We say that  $X$  is equidimensional if it is equidimensional of dimension  $n$  for some  $n \in \mathbb{Z}_{\geq 0}$ ; in particular, an equidimensional topological space has finite dimension.
- (c) For a point  $x \in X$ , we define the *local dimension of  $X$  at  $x$* , denoted  $\dim_x X$ , to be the infimum of  $\dim U$ , where  $U$  ranges over all open neighborhoods of  $x$  in  $X$ .<sup>1</sup>
- (d) For an irreducible subset  $Y \subset X$ , we define the *codimension of  $Y$  in  $X$* , denoted  $\operatorname{codim}_X Y$ , to be the supremum of all  $n \in \mathbb{Z}_{\geq 0}$  such that there is a chain  $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X$  of irreducible closed subsets of  $X$  of length  $n$  containing  $Y$ , i.e., such that  $Y \subset X_0$ .
- (e) For any arbitrary nonempty closed subset  $Y \subset X$ , we define the *codimension of  $Y$  in  $X$*  to be  $\operatorname{codim}_X Y := \inf_{Z \in \operatorname{Irred}(Y)} \operatorname{codim}_X Z$ . We say that  $Y$  is *pure of codimension  $r$  in  $X$*  for some  $r \in \mathbb{Z}_{\geq 0}$  if all irreducible components of  $Y$  have codimension  $r$  in  $X$ .

The dimension of a scheme is the defined to be the dimension of its underlying topological space; the codimension of a closed subscheme is the codimension of its underlying closed subspace.

**Remark 4.1.2.**

- (a) For any space  $X$ , we have  $\dim X \in \mathbb{Z}_{\geq 0} \cup \{\pm\infty\}$ , with  $\dim X = -\infty$  iff  $X = \emptyset$ .
- (b) The local dimension is a local property in the sense that if  $X$  is a space,  $x \in X$ , and  $U \subset X$  an open neighborhood of  $x$  in  $X$ , then  $\dim_x X = \dim_x U$ . Further, if  $X$  has only finitely many irreducible components and  $x \in X$  is contained in all of them, then  $\dim_x X = \dim X$ . In particular, if  $X$  is irreducible, then for all  $x \in X$  we have  $\dim_x X = \dim X$ .
- (c) When  $X$  is a sober space, the above definition(s) can be made using point specializations instead.
- (d) If  $X = \operatorname{Spec} R$  is an affine scheme, then it is clear by definition that the dimension of  $X$  is the Krull dimension  $\dim R$  of  $R$ .
- (e) If  $Y \subset X$  is an irreducible subspace of a topological space, then by definition we have that  $\operatorname{codim}_X Y = \operatorname{codim}_X \overline{Y}$ , and that  $\operatorname{codim}_X Y + \dim Y \leq \dim X$ . Unfortunately, equality does not always hold. A simple counterexample is given as follows: let  $R$  be a DVR with uniformizer  $\pi$ , and let  $X = \mathbb{A}_R^1 = \operatorname{Spec} R[x]$ , so that  $\dim X = 2$ . If  $Y := \mathbb{V}(\pi x - 1)$ , then  $Y$  is a closed subscheme with  $\operatorname{codim}_X Y = 1$  (by Krull's Hauptidealsatz) and  $\dim Y = 0$ . However, we will show below (4.1.7(a)) that equality holds in reasonable circumstances (e.g., for varieties).

**Lemma 4.1.3.**

- (a) Let  $X$  be a space and  $Y \subset X$  a subspace. Then  $\dim Y \leq \dim X$ . If  $X$  is irreducible, finite dimensional, and  $Y \subsetneq X$  is a proper closed subset, then the inequality is strict.
- (b) For any space  $X$ , we have

$$\dim X = \sup_{Y \in \operatorname{Irred}(X)} \dim Y = \sup_{x \in X} \dim_x X.$$

- (c) If  $\{U_\alpha\}$  is an open cover of  $X$ , then  $\dim X = \sup_\alpha \dim U_\alpha$ .
- (d) If  $X$  is a closure-complete space (e.g., the underlying space of a quasicompact scheme), then  $\dim X = \sup_{x \in X^{\text{cl}}} \dim_x X$ , where  $X^{\text{cl}}$  is the subset of closed points.
- (e) If  $\pi : X \rightarrow Y$  is dominant integral morphism of integral affine schemes, then  $\dim X = \dim Y$ .
- (f) If there is an  $n \in \mathbb{Z}_{\geq 0}$  such that  $X$  admits an open cover by subspaces, each equidimensional of dimension  $n$ , then also  $X$  is equidimensional of dimension  $n$ .

<sup>1</sup>This definition disagrees with that given in [2, Prop. III.9.5], which is poorly behaved, and agrees with the definition in EGA or [11, Definition 3.3.6]. For the relationship between these definitions (at least for schemes locally of finite type over a field), see 4.1.7(b).

*Proof.*

- (a) Follows from 2.1.5(g) and the observation that if  $Z \subset Y$  is any closed subset with closure  $\overline{Z}$  in  $X$ , then  $Z = \overline{Z} \cap Y$ . The second statement follows from appending  $X$  to a chain of irreducible closed subsets of  $Y$ .
- (b) By (a), these suprema are at most  $\dim X$ . The first result is clear. For the second, if  $X_0 \subsetneq \cdots \subsetneq X_n \subset X$  is a chain of irreducible closed subsets, then there is an  $x \in X_0$ ; the result follows from applying 2.1.5(g) to any open neighborhood  $U$  of  $x$  in  $X$ .
- (c) Follows from (b).
- (d) In any chain as above, the set  $X_0$  contains a closed point.
- (e) This is an immediate consequence of Cohen-Seidenberg Theory ([6, Cor. 4.2.6(a)]) combined with 3.3.2.
- (f) This follows (b) and 2.1.5(g). ■

Let's first see an algebraic interpretation of the (co)dimension for (possibly non-affine) schemes.

**Lemma 4.1.4.** Let  $X$  be a scheme.

- (a) If  $Y \subset X$  is an irreducible subset, then  $\text{codim}_X Y = \dim \mathcal{O}_{X,Y}$ , where  $\mathcal{O}_{X,Y}$  is the local ring of  $X$  at the generic point of  $\overline{Y}$ .
- (b) In particular, we have  $\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x}$ .

*Proof.*

- (a) By replacing  $Y$  by its generic point, we are immediately reduced to showing that if  $X$  is a scheme and  $x \in X$ , then  $\dim \mathcal{O}_{X,x} = \text{codim}_X \{x\}$ . By 2.1.5(d), we are then reduced to the case of affine  $X$ , where the result is clear by the inverse correspondence between prime ideals and irreducible subsets.
- (b) Clear from (a). ■

Now let us focus on the case of schemes locally of finite type over a field  $k$ ; in this section, we follow [5, §5.6]. By 4.1.3(b), we may restrict to considering irreducible schemes.<sup>2</sup> The standard result from commutative algebra we need is

**Theorem 4.1.5** (Noether Normalization). Let  $k$  be a field and  $X$  be an affine scheme of finite type over  $k$ . Then  $\dim X < \infty$ , and there is a finite surjective<sup>3</sup> morphism  $X \rightarrow \mathbb{A}_k^{\dim X}$ . In addition, if  $X$  is integral, then  $\dim X = \text{trdeg}_k k(X)$ .

*Proof.* The usual (algebraic) way of stating the Noether Normalization Lemma (say [6, Thm. 6.2.7(a)]) produces an integer  $r \in \mathbb{Z}_{\geq 0}$  and a finite surjective morphism  $X \rightarrow \mathbb{A}_k^r$  (using [6, Rmk. 3.2.2] if needed). Using this, from Cohen-Seidenberg Theory, ([6, Cor. 3.2.6]), we conclude that  $\dim X = r$ . If  $X$  is also integral, the above map expresses  $k(X)$  as a finite algebraic extension of  $k(\mathbb{A}_k^r)$ , so that  $\text{trdeg}_k k(X) = \text{trdeg}_k k(\mathbb{A}_k^r) = r$  as well. ■

**Theorem 4.1.6.** Let  $X$  be an irreducible scheme lft over a field  $k$ , and let  $\eta \in X$  be the generic point of  $X$ . Let  $k(X) := \kappa(\eta)$ .

- (a) We have  $\dim X = \text{trdeg}_k k(X)$ . In particular,  $\dim X < \infty$ .
- (b) If  $U \subset X$  is any nonempty open subscheme, then  $\dim U = \dim X$ .
- (c) More generally, let  $f : Y \rightarrow X$  be a dominant lft morphism.<sup>4</sup> Then  $\dim Y \geq \dim X$ .
- (d) If  $x \in X$  is any closed point, then  $\dim X = \dim \mathcal{O}_{X,x}$ .
- (e) If  $f : Y \rightarrow X$  is a quasi-finite morphism,<sup>5</sup> then  $\dim Y \leq \dim X$ .

<sup>2</sup>Since dimension is a topological property, we may pass to the underlying reduced subscheme to restrict ourselves to considering integral schemes. The added generality in considering irreducible and not just integral schemes is somewhat superficial.

<sup>3</sup>In this case, “finite surjective” is the same as “finite dominant” because 2.4.4 tells us that, in this context, dominance is the same as having dense image, and finite morphisms are closed [TOCITE].

<sup>4</sup>Recall that when  $f$  is quasicompact, our definition of dominant agrees with the usual one; see 2.1.15.

<sup>5</sup>Our definition of “quasi-finite” includes “finite-type”.

In particular, if  $f : Y \rightarrow X$  is a dominant quasi-finite morphism, then  $\dim Y = \dim X$ .

*Proof.*

- (a) Replacing  $X$  by  $X_{\text{red}}$ , we may assume that  $X$  is integral. Using 4.1.3(c), we are reduced to the case of affine  $X$ ; this reduces us to 4.1.5.
- (b) Clear from (a).
- (c) By 2.1.13, there is a  $\theta \in Y$  such that  $f(\theta) = \eta$ . Then  $f$  induces a  $k$ -embedding  $\kappa(\eta) \hookrightarrow \kappa(\theta)$ , and so

$$\dim X = \text{trdeg}_k \kappa(\eta) \leq \text{trdeg}_k \kappa(\theta) = \dim \overline{\{\theta\}} \leq \dim Y,$$

where we are repeatedly using (a) and 4.1.3(a).

- (d) Using (b), we may assume that  $X$  is integral and affine. This reduces us to: if  $R$  is a domain finitely generated over a field and  $\mathfrak{m} \subset R$  a maximal ideal, then  $\dim R = \text{ht } \mathfrak{m}$ ; this is a standard consequence of Noether Normalization as well ([6, Theorem 6.2.7(d)]).
- (e) By 4.1.3(b), we may assume  $Y$  to be irreducible. Replacing  $X$  and  $Y$  by their reductions, we can assume that both  $X$  and  $Y$  are integral. By (b), we may further reduce to the case when  $X$  and  $Y$  are affine. This reduces us to: if  $R$  and  $S$  are domains finitely generated over a field  $k$  and  $f : R \rightarrow S$  a  $k$ -algebra homomorphism such that for all primes  $\mathfrak{p} \subset R$ , the  $\kappa(\mathfrak{p})$ -algebra  $\kappa(\mathfrak{p}) \otimes_R S$  has only finitely many primes, then  $\dim S \leq \dim R$ . To show this, take  $\mathfrak{p} := \ker \varphi$ . Then  $\text{Spec}(\kappa(\mathfrak{p}) \otimes_R S)$  is a nonempty finite-type  $\kappa(\mathfrak{p})$ -scheme whose underlying topological space is *finite*. The subset of non-closed points is then open, and hence by 2.3.9, must then be empty, i.e., every point is closed. In particular, applying this to the prime corresponding to  $(0) \subset S$  shows us that  $\text{Frac } S$  is a *finite* extension of  $\kappa(\mathfrak{p})$ , and hence that  $\dim S = \text{trdeg}_k S = \text{trdeg}_k \kappa(\mathfrak{p}) = \text{coht } \mathfrak{p} \leq \dim R$  as needed. ■

Parts (b) and (d) of 4.1.6 combine to tell us that, in the above setting, if  $x \in X$  is any closed point, then  $\dim_x X = \dim \mathcal{O}_{X,x}$ . When  $x$  is not necessarily closed, the correct generalization is

**Theorem 4.1.7.** Let  $X$  be a scheme that is locally of finite type over a field  $k$ .

- (a) Let  $Y \subset X$  be a closed subset. Suppose that either  $X$  is equidimensional, or that  $X$  has finitely many components and  $Y$  lies in each of them. Then

$$\text{codim}_X Y + \dim Y = \dim X < \infty.$$

- (b) In general, for any  $x \in X$ , we have  $\dim \mathcal{O}_{X,x} + \text{trdeg}_k \kappa(x) = \dim_x X$ .

Note that some sort of hypothesis as in (a) is clearly necessary, as easily constructed counterexamples otherwise show.

*Proof.*

- (a) The hypotheses allow us to reduce to the case of affine integral  $X$  and  $Y$  (exercise). Let  $R := \mathcal{O}(X)$ . If  $Y$  corresponds to the prime ideal  $\mathfrak{p} \subset R$ , we have  $\text{codim}_X Y = \dim R_{\mathfrak{p}} = \text{ht } \mathfrak{p}$  by 4.1.4, and  $\dim Y = \dim R/\mathfrak{p} = \text{coht } \mathfrak{p}$ . This reduces us to the following standard result in commutative algebra: if  $R$  is a domain that is finitely generated over a field  $k$  and  $\mathfrak{p} \subset R$  a prime ideal, then  $\dim R = \text{ht } \mathfrak{p} + \text{coht } \mathfrak{p}$  ([6, Theorem 6.2.7(d)]).
- (b) We will reduce to (a). First, replacing  $X$  by an affine open containing  $x$ , we may assume that  $X$  is of finite type over  $k$ . In particular,  $X$  is Noetherian and so has only finitely many irreducible components. Removing all irreducible components of  $X$  which do not contain  $x$ , we may assume that  $x$  is contained in every irreducible component of  $X$ . In this case, we have by 4.1.2(b) that  $\dim_x X = \dim X$ . Applying (a) to  $Y := \overline{\{x\}}$ , and using 4.1.4(a) and 4.1.6(a) gives the result. ■

Next, we check that the dimension of a scheme locally of finite type over a field behaves well under changing the base field.

**Theorem 4.1.8.** Let  $k$  be a field,  $X$  a  $k$ -scheme, and  $k \subset K$  be a field extension. If either

- (a)  $X$  is affine over  $k$  and  $K/k$  algebraic, or

(b)  $X$  is lft over  $k$  (and  $K/k$  arbitrary), then

the scheme  $X$  is equidimensional iff the base-change  $X_K$  of  $X$  to  $K$  is, and then  $\dim X_K = \dim X$ .

This result is (I think) more subtle than is often given credit for, and the (only detailed) proof (I know of it) is somewhat tricky. Some sort of hypothesis as in (a) or (b) is clearly necessary; a simple counterexample otherwise is given by taking  $X = \operatorname{Spec} k(t)$  and  $K = k(s)$ , both purely transcendental; then  $X_K$  is (the spectrum of) a suitable localization of (the coordinate ring of)  $\mathbb{A}_{k(s)}^1$  and can be easily shown to be one-dimensional.<sup>6</sup> The following proof strategy has been taken from [1, Chapter 12]. For the proof, let's isolate two small results.

**Lemma 4.1.9.** Let  $X$  be an affine integral scheme over a field  $k$ , and  $K/k$  a field extension.

- (a) Every irreducible component of  $X_K$  dominates  $X$ .
- (b) If  $K/k$  is purely transcendental, then  $X_K$  is also integral.

*Proof.* Let  $X = \operatorname{Spec} R$  for a  $k$ -domain  $R$ .

- (a) For any minimal prime  $\mathfrak{p}$  of  $R_K = K \otimes_k R$ , the natural map  $R \rightarrow R_K \rightarrow R_K/\mathfrak{p}$  is injective. This follows from two facts: (i) that  $R_K$  is a free  $R$ -module and  $R$  is a domain, and (ii) if  $S$  is any ring and  $\mathfrak{p} \subset S$  a minimal prime, then any element of  $\mathfrak{p}$  is a zero divisor.<sup>7</sup>
- (b) If  $K = k(e_i)_{i \in I}$ , then  $X_K = \operatorname{Spec} R_K$  where

$$R_K = K \otimes_k R \cong k(e_i) \otimes_{k[e_i]} (k[e_i] \otimes_k R) \cong (k[e_i] \setminus \{0\})^{-1} R[e_i]$$

is a localization of the domain  $R[e_i]$  and hence also a domain. ■

*Proof of 4.1.8.* First we prove the forward statement; suppose that  $X$  is equidimensional. In both (a) and (b), we can reduce to the case of integral  $X$ . Indeed, we reduce to the case of irreducible  $X$  by noting that an irreducible component of  $X_K$  must be an irreducible component of  $Y_K$  for some irreducible component  $Y$  of  $X$ . Also, the reduction map  $X^{\text{red}} \rightarrow X$  is a surjective closed immersion and hence a universal homeomorphism; in particular, the natural map  $(X^{\text{red}})_K \rightarrow X_K$  is a homeomorphism. Hence suppose that  $X$  is integral.

- (a) Since  $K/k$  is algebraic, every component of  $X_K$  maps to  $X$  via an integral morphism. Since  $X$  is affine and integral, the result follows from 4.1.9(a) combined with 4.1.3(e).
- (b) By 4.2 and 4.1.3(f), it suffices to deal with the case of affine  $X$ . Since any  $K/k$  can be written as a composite of a purely transcendental extension followed by an algebraic extension, by part (a), it suffices to treat the case when  $K/k$  is purely transcendental. Using 4.1.9(b) and 4.1.6, it remains only to justify that  $\operatorname{trdeg}_k k(R) = \operatorname{trdeg}_K K(R_K)$ , but indeed  $K(R_K) = k(R)(e_i)$ , so the result is clear by, say, [6, Exercise 5.10].

For the converse, suppose now that  $X_K$  is equidimensional. The morphism  $X_K \rightarrow X$  is affine and hence quasicompact, and it is surjective because it is the base-change of the surjective morphism  $\operatorname{Spec} K \rightarrow \operatorname{Spec} k$ . Therefore, it follows from 3.3.3(a) that  $X_K \rightarrow X$  is dominant (2.1.14); in particular, if  $Y \subset X$  is any irreducible component, then there is an irreducible component  $Z \subset X_K$  dominating  $Y$ .

- (a) Giving  $Z$  and  $Y$  the reduced subscheme structures, they are both integral. Since  $X$  is affine, so are  $Y$  and  $Z$ , and the result follows from 4.1.3(e) applied to the induced morphism  $Z \rightarrow Y$ .
- (b) By 4.1.3(f) and 4.2, it suffices to deal with affine  $X$ . As in the forward implication in (b) above, we may reduce to the case of  $K/k$  purely transcendental. If  $Y$  and  $Z$  are as above, then we may once again give  $Y$  and  $Z$  the reduced subscheme structure to make them integral. Then  $Z \subset Y_K$ , but  $Y_K$  is integral by 4.1.9(b), so  $Z = Y_K$  by maximality. By the forward implication, we get  $\dim Y = \dim Y_K = \dim Z = \dim X_K$  as needed. ■

<sup>6</sup>I suspect similar counterexamples exist when we drop *any* of the hypotheses of the theorem.

<sup>7</sup>Indeed, since  $\mathfrak{p}$  is minimal, the ring  $S_{\mathfrak{p}}$  has a unique prime ideal  $\mathfrak{p}S_{\mathfrak{p}}$ . Therefore,  $\operatorname{Nil}(S_{\mathfrak{p}}) = \mathfrak{p}S_{\mathfrak{p}}$ , so that any element of  $\mathfrak{p}$  is nilpotent in  $S_{\mathfrak{p}}$  and hence a zero-divisor in  $S$ .

**Corollary 4.1.10.** Let  $k$  be a field and  $X$  be a  $k$ -scheme. Let  $K/k$  be a field extension. Suppose either that  $K/k$  is algebraic or that  $X$  is lft over  $k$ . Then for any  $p \in X_K$  over  $q \in X$ , we have  $\dim_p X_K = \dim_q X$ .

*Proof.* Follows from the above theorem, and is left as an exercise. ■

Finally, we deal with the dimension of the product of two  $k$ -schemes.

**Theorem 4.1.11.** Let  $k$  be a field, and  $X, Y$  be nonempty equidimensional schemes, lft over  $k$ . Then  $X \times_k Y$  is also nonempty equidimensional and lft over  $k$ , and further  $\dim X \times_k Y = \dim X + \dim Y$ .

*Proof.* That the product  $X \times_k Y$  is nonempty and lft over  $k$  is clear, so we only have to show that it is equidimensional of dimension  $\dim X + \dim Y$ . We proceed in several steps.

- Step 1. Suppose that  $k$  is algebraically closed and  $X, Y$  affine integral. Then  $X \times_k Y$  is also affine and integral ([1, 10.4.H]), and the result follows from 4.1.5: pick finite surjective morphisms  $X \rightarrow \mathbb{A}_k^{\dim X}$  and  $Y \rightarrow \mathbb{A}_k^{\dim Y}$ , and observe that their product  $X \times_k Y \rightarrow \mathbb{A}_k^{\dim X + \dim Y}$  is also finite surjective, whence the result follows from 3.3.3(a) and 4.1.3(e).
- Step 2. Suppose that  $k$  is algebraically closed and  $X, Y$  affine. Replacing  $X$  and  $Y$  by their reductions by using that  $X^{\text{red}} \times_k Y^{\text{red}} \rightarrow X \times_k Y$  is a homeomorphism, we may assume that  $X$  and  $Y$  are reduced. The result follows from noting that an irreducible component of  $X \times_k Y$  must be (contained in, and hence) of the form  $X' \times_k Y'$  for some irreducible components  $X' \subset X$  and  $Y' \subset Y$  (given the reduced subscheme structure).
- Step 3. Suppose that  $X$  and  $Y$  are affine (and  $k$  arbitrary). By 4.1.8(a) applied to an algebraic closure  $K = \bar{k}$  of  $k$ , we get that  $X_K$  (resp.  $Y_K$ ) is equidimensional of the same dimension as  $X$  (resp.  $Y$ ). By Step 2,  $X_K \times_K Y_K$  is equidimensional of dimension  $\dim X_K + \dim Y_K = \dim X + \dim Y$ . We conclude by invoking 4.1.8(a) once again by noting that  $X_K \times_K Y_K = (X \times_k Y)_K$ .
- Step 4. Suppose that  $X$  and  $Y$  are irreducible and  $k$  arbitrary. Let  $Z \subset X \times_k Y$  be an irreducible component. Pick affine opens  $U \subset X$  and  $V \subset Y$  such that  $Z \cap (U \times_k V) \neq \emptyset$ . From 4.1.6(b), we conclude that  $\dim Z = \dim Z \cap (U \times_k V)$ . Now  $Z \cap (U \times_k V)$  is an irreducible component of  $U \times_k V$  by 2.1.5(g), and so by Step 3 we have  $\dim Z \cap (U \times_k V) = \dim U + \dim V = \dim X + \dim Y$ , where in the last step we are again using 4.1.6(b).
- Step 5. The main theorem in full generality then follows easily from Step 4 and the fact (as in Step 2) that every irreducible component  $Z \subset X \times_k Y$  is an irreducible component of  $X' \times_k Y'$  for some irreducible components  $X' \subset X$  and  $Y' \subset Y$ . ■

## 4.2 Krull's Theorems

### 4.3 Smoothness

**Definition 4.3.1** (Smoothness over a Field). Let  $k$  be a field and  $X$  an lft  $k$ -scheme.

- (a) If  $X$  is irreducible, we say that  $X$  is *smooth* iff Jacobian corank function of  $X$  is constant with value  $\dim X$ .
- (b) For arbitrary  $X$ , we say that  $X$  is smooth iff the irreducible components of  $X$  are disjoint, and each component is smooth.

**Lemma 4.3.2.** Let  $X$  be lft and equidimensional. Then  $X$  is smooth iff the Jacobian corank function of  $X$  is constant with value  $\dim X$ .

*Proof.* When  $k$  is algebraically closed (or more generally perfect), this follows from the fact that smooth implies regular at closed points, and regular local rings are domains. To do the general case, base-change to an algebraic closure of  $k$ , using 4.1.8, stability of the Jacobian corank function under field extensions, and 4.7. ■

**Definition 4.3.3** (Smooth Morphisms I). Let  $n \in \mathbb{Z}_{\geq 0}$  and let  $A$  be a ring.

- An  $A$ -scheme  $X$  is *smooth of relative dimension  $n$  over  $A$  at a point  $x \in X$*  iff there is an open neighborhood  $U$  of  $x$  in  $X$ , an integer  $r \in \mathbb{Z}_{\geq 0}$ , polynomials  $f_1, \dots, f_{n+r} \in A[x_1, \dots, x_{n+r}]$ , and an isomorphism of  $A$ -schemes of the form

$$U \xrightarrow{\sim} \operatorname{Spec} A[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)[1/\det \operatorname{Jac}_{\leq r}] \quad (4.3.4)$$

where

$$\operatorname{Jac}_{\leq r} = \left[ \frac{\partial f_i}{\partial x_j} \right]_{i,j=1,\dots,r}.$$

An  $A$ -scheme  $X$  is *smooth of relative dimension  $n$*  or a *smooth  $A$ -scheme of relative dimension  $n$*  iff it is smooth of relative dimension  $n$  over  $A$  at all  $x \in X$ .

- An  $A$ -scheme  $X$  is *étale* over  $A$  (at  $x \in X$ ) iff it is smooth of relative dimension 0 (at  $x \in X$ ).
- An  $A$ -scheme  $X$  is *smooth* iff for all  $x \in X$ , there is an integer  $n \in \mathbb{Z}_{\geq 0}$  (possibly depending on  $x$ ) such that  $X$  is smooth of relative dimension  $n$  over  $A$  at  $x$ .

**Remark 4.3.5.** Let  $n \in \mathbb{Z}_{\geq 0}$ , and let  $A$  be a ring.

- (a) Let  $X$  be an  $A$ -scheme smooth of relative dimension  $n$  over  $A$  (at  $x \in X$ ). For any ring homomorphism  $A \rightarrow B$ , the base-change  $X_B$  is a  $B$ -scheme smooth of relative dimension  $n$  over  $B$  (at any  $x_B$  over  $x$ ).
- (b) If the structure morphism  $a_X : X \rightarrow \operatorname{Spec} A$  is an open embedding, then  $X$  is étale over  $A$ . This follows immediately from 1.2.1(d). The same proof shows that if  $X$  is a smooth  $A$ -scheme of relative dimension  $n$  over  $A$  (at  $x \in X$ ), then for any open subset  $U \subset X$  (containing  $x$ ), the  $A$ -scheme  $U$  is smooth of relative dimension  $n$  over  $A$  (at  $x \in U$ ). Conversely, if  $X$  is an  $A$ -scheme that admits an open cover by smooth  $A$ -schemes of relative dimension  $n$  over  $A$ , then  $X$  itself is a smooth  $A$ -scheme of relative dimension  $n$  over  $A$ .
- (c) If  $A = k$  is a field, then  $X$  is a smooth  $k$ -scheme of relative dimension  $n$  iff  $X$  is a lft  $k$ -scheme which is equidimensional of dimension  $n$  and smooth in the sense of 4.3.1.<sup>8</sup>

**Definition 4.3.6** (Smooth Morphisms II). Let  $n \in \mathbb{Z}_{\geq 0}$ .

- A morphism  $\pi : X \rightarrow Y$  of schemes is said to be *smooth of relative dimension  $n$  at a point  $x \in X$*  iff there is an open affine neighborhood  $V$  of  $\pi(x) \in Y$  such that the restriction of  $\pi$  to the preimage  $X_V = \pi^{-1}(V)$  makes the  $\mathcal{O}(V)$ -scheme  $X_V$  smooth of relative dimension  $n$  over  $\mathcal{O}(V)$  at the point  $x \in X_V$ . A morphism  $\pi : X \rightarrow Y$  is said to be *smooth of relative dimension  $n$*  iff it is smooth of relative dimension  $n$  at each  $x \in X$ .
- A morphism  $\pi : X \rightarrow Y$  is said to be *étale* (at  $x \in X$ ) iff it is smooth of relative dimension zero (at  $x \in X$ ).
- A morphism  $\pi : X \rightarrow Y$  is said to be *smooth* iff for all  $x \in X$ , there is an integer  $n \in \mathbb{Z}_{\geq 0}$  (possibly depending on  $x$ ) such that  $\pi$  is smooth of relative dimension  $n$  at  $x$ .

<sup>8</sup>To note that a  $k$ -scheme that is smooth of relative dimension  $n$  is equidimensional, it might be helpful to use Krull's theorems [TODO] to show that every irreducible component of  $X$  has dimension at least  $n$ , and then use [1, 13.1.M] along with 4.1.3(b) to prove  $\dim X \leq n$ .

**Remark 4.3.7.** If  $n \in \mathbb{Z}_{\geq 0}$  and  $\pi : X \rightarrow Y$  is a smooth morphism of relative dimension  $n$  at  $x \in X$ , then there is an open neighborhood  $U$  of  $x$  in  $X$  such that the restriction  $\pi_U : U \rightarrow Y$  can be factored as an étale map  $U \rightarrow \mathbb{A}_Y^n$  followed by the projection  $\mathbb{A}_Y^n \rightarrow Y$ .

**Lemma 4.3.8.**

- (a) (Affine Communication) Let  $n \in \mathbb{Z}_{\geq 0}$ , let  $A$  be a ring, let  $a_X : X \rightarrow \operatorname{Spec} A$  be an  $A$ -scheme, and let  $x \in X$ . Then  $X$  is smooth of relative dimension  $n$  over  $A$  (at  $x$ ) iff the structure morphism  $a_X$  is smooth of relative dimension  $n$  (at  $x$ ).
- (b) For any morphism  $\pi : X \rightarrow Y$  and integer  $n \in \mathbb{Z}_{\geq 0}$ , the locus  $X_n \subset X$  of points of  $X$  at which  $\pi$  is smooth of relative dimension  $n$  is open.
- (c) Smooth (and hence étale) morphisms are lfp (and hence lft). For any  $n \in \mathbb{Z}_{\geq 0}$ , smooth morphisms of relative dimension  $n$  are affine local on both the source and target, and stable under base-change: if  $\pi : X \rightarrow Y$  is smooth of relative dimension  $n$  (at  $x \in X$ ), then for any morphism  $Y' \rightarrow Y$ , the basechange  $\pi' : X' = X \times_Y Y' \rightarrow Y'$  is smooth of relative dimension  $n$  (at any  $x'$  over  $x$ ).
- (d) Given integers  $m, n \in \mathbb{Z}_{\geq 0}$ , if  $\pi : X \rightarrow Y$  is smooth of relative dimension  $n$  at  $x \in X$ , and  $\rho : Y \rightarrow Z$  is smooth of relative dimension  $m$  at  $y := \pi(x)$ , then the composite  $\rho \circ \pi : X \rightarrow Z$  is smooth of relative dimension  $m + n$  at  $x$ . In particular, étale morphisms are stable under composition, and hence form a reasonable class of morphisms.

*Proof.*

- (a) Follows from clearing denominators, and left as an exercise (c.f. 4.3.5(b)).
- (b) Clear from the definition.
- (c) Affine-locality on the target follows from the definition; affine-locality on the source follows from 4.3.5(b). Finally, stability under base-change follows at once from (a), 4.3.5(a), and 3.1.2.
- (d) By a minor variant of the proof of 3.1.3, it suffices to do the case when  $X, Y, Z$  are all affine and  $\pi$  and  $\rho$  are globally standard smooth (i.e., of the form (4.3.4)); then the result is clear. ■

Combining 4.3.8(c) with 4.3.5(c) tells us that smooth morphisms have smooth fibers in the sense of 4.3.1.

**Example 4.3.9.**

- (a) Open embeddings of schemes are étale.
- (b) For any  $m \in \mathbb{Z}_{\geq 2}$ , the morphism  $\mathbb{A}_{\mathbb{Z}}^1 \rightarrow \mathbb{A}_{\mathbb{Z}}^1$  corresponding to the ring homomorphism  $\mathbb{Z}[x] \rightarrow \mathbb{Z}[y]$  given by  $x \mapsto y^m$  is étale over the subset  $\operatorname{Spec} \mathbb{Z}[x, (mx)^{-1}] \cong D(mx) \subset \mathbb{A}_{\mathbb{Z}}^1$ .
- (c) For a number field  $K$  with ring of integers  $\mathcal{O}_K$ , the structure morphism  $\operatorname{Spec} \mathcal{O}_K \rightarrow \operatorname{Spec} \mathbb{Z}$  is étale over the subset  $\operatorname{Spec} \mathbb{Z}[1/\operatorname{disc}(K)] \cong D(\operatorname{disc} K) \subset \operatorname{Spec} \mathbb{Z}$ .
- (d) Let  $k$  be a field,  $X, Y$  be lft  $k$ -schemes, and  $\pi : X \rightarrow Y$  a  $k$ -morphism. Let  $x$  be a closed point with residue field  $\kappa(x)$  a finite separable extension of  $k$  (e.g., any closed point when  $k$  is perfect), and let  $y := \pi(x)$ . Suppose for simplicity that the natural morphism  $\kappa(y) \rightarrow \kappa(x)$  is an isomorphism. Then  $\pi$  is étale at  $x$  iff the differential  $d\pi_x : T_x X \rightarrow T_y Y$  is an isomorphism of  $\kappa(x)$ -vector spaces.
- (e) Let  $k$  be a field. A morphism  $\pi : X \rightarrow \operatorname{Spec} k$  is étale iff  $X$  is discrete, and each point of  $X$  is the spectrum of a finite separable field extension of  $k$ .
- (f) For any integer  $n \in \mathbb{Z}_{\geq 0}$  and scheme  $S$ , the structure morphisms of the  $S$ -schemes  $\mathbb{A}_S^n$  and  $\mathbb{P}_S^n$  are smooth of relative dimension  $n$ .

## 4.4 Exercises

**Exercise 4.1.** Show that a locally Noetherian topological space of dimension zero is discrete. Is the result true without the “locally Noetherian” hypothesis?

**Exercise 4.2.**

- (a) Give an example of a topological space  $X$  and a nonempty open subset  $U \subset X$  such that  $X$  is equidimensional but  $\dim U < \dim X$ .
- (b) Show that (a) cannot happen if  $X$  is (the underlying topological space of) a scheme locally of finite type over some field. In fact, show that, in this case, every nonempty open subset  $U \subset X$  is equidimensional of dimension  $\dim X$ .
- (c) (\*) In (b), does it suffice for  $X$  to be instead locally Noetherian and closure complete?

**Exercise 4.3.** (Adapted from [2, Ex. 3.22].) Let  $\pi : X \rightarrow Y$  be a dominant morphism of integral schemes of finite type over a field  $k$ .

- (a) Let  $W \subset Y$  be an irreducible closed subset and  $Z \subset \pi^{-1}(W)$  an irreducible component dominating  $W$ . Show that  $\operatorname{codim}_X(Z) \leq \operatorname{codim}_Y(W)$ . Show by example that the result is false in general if we do not require  $Z$  to dominate  $W$ .
- (b) Let  $r := \dim X - \dim Y = \operatorname{trdeg}_{K(Y)} K(X) \in \mathbb{N}$  denote the *relative dimension* of  $X$  over  $Y$ . Show that for every  $y \in Y$ , every irreducible component of the fiber  $X_y = \pi^{-1}(y)$  has dimension at least  $r$ .
- (c) Show that there is a dense open subset  $V \subset Y$  such that for all  $y \in V$ , the fiber  $X_y$  is equidimensional of dimension  $r$ .
- (d) (Upper Semicontinuity of Fiber Dimension) Show that the function  $X \rightarrow \mathbb{N}$  defined by  $x \mapsto \dim_x X_{\pi(x)}$  is upper semicontinuous, i.e., for each  $n \in \mathbb{N}$ , show that

$$D_n := \{x \in X : \dim_x X_{\pi(x)} \geq n\} \subset X$$

is closed. Show that  $D_r = X$ .

- (e) (Chevalley) For each  $n \in \mathbb{N}$ , let  $C_n := \{y \in Y : \dim X_y = n\}$ . Show that each  $C_n \subset Y$  is a constructible subset, and that  $C_r$  contains an open dense subset of  $Y$ .
- (f) (Upper Semicontinuity in the Base) Suppose further that  $\pi$  is closed. Show that the function  $Y \rightarrow \mathbb{N}$  defined by  $y \mapsto \dim X_y$  is upper semicontinuous. Show by example that the result is false in general if we do not require  $\pi$  to be closed.
- (g) Show that the result of (d) holds when we only assume  $X$  and  $Y$  to be finite type  $k$ -schemes (with no conditions on the  $k$ -morphism  $\pi$ ); similarly, the result of (f) holds when, in addition,  $\pi$  is closed.

**Exercise 4.4.** (Adapted from [1, 12.4.D] and [12, Theorem 11.14]) Let  $k$  be a field,  $X, Y$  be finite type  $k$ -schemes with  $Y$  irreducible, and let  $\pi : X \rightarrow Y$  be a closed  $k$ -morphism. Show that if there is an  $n \in \mathbb{N}$  such that for every  $y \in Y$ , the fiber  $X_y$  is irreducible of dimension  $n$ , then  $X$  is irreducible with  $\dim X = \dim Y + n$ .

**Exercise 4.5.** (Adapted from [12, Exercise 11.15].) Let  $k$  be a field,  $X, Y, Z$  be integral finite-type  $k$ -schemes, and let  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  be dominant  $k$ -morphisms.

- (a) Show that  $X \times_Z Y$  is nonempty, and that

$$\dim X \times_Z Y \geq \dim X + \dim Y - \dim Z.$$

- (b) Give examples to show that the inequality can be strict, and that  $X \times_Z Y$  can, in general, have irreducible components of smaller dimension than the right side, although this cannot happen if  $Z = \mathbb{A}_k^n$ .
- (c) Is the result true when we don't require integrality or dominance in the hypotheses? Produce at least one counterexample to the result without at least one of these hypotheses, and one (nontrivial) situation where these hypotheses do not hold but the result still does. Open ended: can you come up with optimum hypotheses, or equivalently the strongest version of such a result?

The following harder exercise generalizes 4.5.

**Exercise 4.6.** Let  $X, Y, Z$  be pure-dimensional lft  $k$ -schemes, and  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  be

$k$ -morphisms. Show that if  $Z$  is smooth, then every irreducible component of  $X \times_Z Y$  has dimension at least  $\dim X + \dim Y - \dim Z$ .

Here is some more practice with irreducible components under field extensions.

**Exercise 4.7.** Let  $k$  be a field,  $X$  be an lft  $k$ -scheme, and  $K/k$  a field extension.

- (a) Let  $d \in \mathbb{Z}_{\geq 0}$ . Show that if  $\dim X \leq d$ , then also  $\dim X_K \leq d$ .
- (b) Suppose that  $X$  is irreducible and  $K/k$  is algebraic. Show that every irreducible component of  $X_K$  maps surjectively onto  $X$ .
- (c) Suppose that  $X$  is pure dimensional. Show that if the irreducible components of  $X_K$  are disjoint, then so are those of  $X$ . Is the converse true?

## Chapter 5

# Divisors and Line Bundles

## 5.1 Weil Divisors

Let  $X$  be a scheme. A *prime Weil divisor* on  $X$  is a codimension 1 irreducible closed subset (or equivalently integral closed subscheme) of  $X$ . A *Weil divisor* on  $X$  is a formal finite  $\mathbb{Z}$ -linear combination of prime Weil divisors on  $X$ , and the group of Weil divisors on  $X$  is denoted  $\text{Div}(X)$ .

Suppose in the rest of this section that  $X$  is a Noetherian, integral, and  $R_1$  (e.g., normal) scheme. Let  $K = K(X)$  be the function field of  $X$ . Define the class group  $\text{Cl}(X)$  by the sequence

$$K^\times \xrightarrow{\text{div}} \text{Div}(X) \rightarrow \text{Cl}(X) \rightarrow 0.$$

In classical terminology, we say that two weil divisors  $D, D' \in \text{Div}(X)$  are *linearly equivalent* iff there is an  $f \in K^\times$  such that  $D - D' = \text{div}(f)$ ; the group  $\text{Cl}(X)$  is the group of Weil divisors modulo linear equivalence. Finally, when  $X$  is normal, algebraic Hartogs's Lemma tells us that the natural morphism  $\mathcal{O}(X)^\times \hookrightarrow \ker \text{div}$  is an isomorphism, producing the exact sequence

$$0 \rightarrow \mathcal{O}(X)^\times \rightarrow K^\times \xrightarrow{\text{div}} \text{Div}(X) \rightarrow \text{Cl}(X) \rightarrow 0.$$

**Example 5.1.1.** Let  $A$  be a Dedekind domain and  $X = \text{Spec } A$ . Then  $\text{Div}(X)$  is the free abelian group on the nonzero prime ideals of  $A$ , and  $\text{Cl}(X)$  is the ideal class group of  $A$ .

**Example 5.1.2.** Let  $A$  be a Noetherian domain and  $X = \text{Spec } A$ . Then  $A$  is a UFD iff  $X$  is normal and  $\text{Cl}(X) = 0$ . This follows from the fact that a Noetherian domain is a UFD iff every height 1 prime is principal. For instance, for any  $n \geq 0$  and UFD  $A$  such as  $A = \mathbb{Z}$  or  $A = k$  a field,  $\text{Cl}(\mathbb{A}_A^n) = 0$ .

**Example 5.1.3.** Let  $k$  be a field,  $n \in \mathbb{Z}_{\geq 1}$ , and  $X = \mathbb{P}_k^n$ . Then

$$0 \rightarrow k^\times \rightarrow K^\times \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\deg} \mathbb{Z} \rightarrow 0,$$

so we get an isomorphism  $\deg : \text{Cl}(X) \xrightarrow{\sim} \mathbb{Z}$ .

**Example 5.1.4.** Let  $Z \subset X$  be a proper closed subset and  $U := X \setminus Z$ . Write the irreducible decomposition of  $Z$  as  $Z = Z_1 \cup \cdots \cup Z_r \cup Z_{r+1} \cup \cdots \cup Z_n$  for some  $r, n \in \mathbb{Z}_{\geq 0}$  with  $0 \leq r \leq n$ , ordered so that  $\text{codim}_X(Z_i) = 1$  for  $1 \leq i \leq r$  and  $\text{codim}_X(Z_i) \geq 2$  for  $r+1 \leq i \leq n$ . Then we have a commutative diagram with exact rows and columns as below.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 K^\times & \xrightarrow{\text{div}_U} & \text{Div}(U) & \longrightarrow & \text{Cl}(U) & \longrightarrow & 0 \\
 \uparrow & & \uparrow & & \uparrow & & \\
 = \uparrow & & \uparrow & & \uparrow & & \\
 K^\times & \xrightarrow{\text{div}} & \text{Div}(X) & \longrightarrow & \text{Cl}(X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \bigoplus_{i=1}^r \mathbb{Z} \cdot [Z_i] & \xrightarrow{=} & \bigoplus_{i=1}^r \mathbb{Z} \cdot [Z_i] & & \\
 & & \uparrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

In particular, we have:

- (a) If  $\text{codim}_X(Z) \geq 2$ , then  $\text{Div}(X) \xrightarrow{\sim} \text{Div}(U)$  and  $\text{Cl}(X) \xrightarrow{\sim} \text{Cl}(U)$ .
- (b) If  $Z$  is irreducible of codimension 1, then the kernel of  $\text{Cl}(X)$  is cyclic and generated by  $[Z]$ . For instance, if  $Z \subset X = \mathbb{P}_k^n$  is an irreducible degree  $d$  hypersurface, then  $\deg : \text{Cl}(U) \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$ .

Further, there is natural isomorphism between the kernel of  $\mathbb{Z}^{\oplus r} \rightarrow \text{Cl}(X)$  and  $\ker \text{div}_U / \ker \text{div}$ ; in particular, if  $X$  is normal, then this gives us an exact sequence of abelian groups of the form

$$0 \rightarrow \mathcal{O}(U)^\times / \mathcal{O}(X)^\times \rightarrow \bigoplus_{i=1}^r \mathbb{Z} \cdot [Z_i] \rightarrow \text{Cl}(X) \rightarrow \text{Cl}(U) \rightarrow 0. \quad (5.1.5)$$

In principle, this gives us a way to compute the relations between any finite collection of prime divisors on  $X$  in the class group  $\text{Cl}(X)$ , at least when  $X$  is normal.

**Example 5.1.6.** Let  $k$  be a field and  $A := k[x, y, z] := k[X, Y, Z]/(XZ - Y^2)$  and  $X = \operatorname{Spec} A$ . This is Noetherian, integral, and normal (check!), of dimension 2. Let  $Z := \mathbb{V}(x, z) \subset X$ . Then  $\operatorname{div}(x) = 2[Z]$  and  $Z$  is not principal, showing that  $\operatorname{Cl}(X) \cong \mathbb{Z}/2$  generated by  $[Z]$ . In particular,  $A$  is not a UFD.

To produce more examples, we first note

**Lemma 5.1.7.** Let  $X$  be a Noetherian, integral, and  $R_1$  scheme. Then for any  $n \in \mathbb{Z}_{\geq 0}$ , the schemes  $\mathbb{A}_X^n = X \times \mathbb{A}^n$  and  $\mathbb{P}_X^n = X \times \mathbb{P}^n$  are also Noetherian, integral, and  $R_1$ .

*Proof.* It suffices to show that  $X \times \mathbb{A}^1$  is  $R_1$ . Let  $\pi : X \times \mathbb{A}^1 \rightarrow X$  be the projection map. If  $z \in X \times \mathbb{A}^1$  is a point of codimension 1, then its image  $x := \pi(z) \in X$  has codimension either 0 or 1 according to whether or not  $\overline{\{z\}}$  dominates  $X$ . If  $\operatorname{codim}_X(x) = 0$ , then  $x$  is the generic point of  $X$ , so  $\mathcal{O}_{X \times \mathbb{A}^1, z}$  is a DVR since  $\mathbb{A}_{K(X)}^1$  is  $R_1$ . If  $\operatorname{codim}_X(x) = 1$ , then  $z \in \pi^{-1}(\overline{\{x\}})$  is the generic point; then  $\mathcal{O}_{X, x}$  is a DVR since  $X$  is  $R_1$ , and hence so is  $\mathcal{O}_{X \times \mathbb{A}^1, z} \cong \mathcal{O}_{X, x}[t]_{\mathfrak{m}_{X, x}[t]}$ . ■

In what follows, we call a codimension one point  $z \in X$  or the corresponding prime divisor  $Z = \overline{\{z\}} \subset X$  of type 0 or type 1 according to whether or not  $Z$  dominates  $X$  via  $\pi$  respectively. Note that when  $z$  is of type 1, then a uniformizer at the projected point  $x := \pi(z)$  maps to a uniformizer at  $z$  under the natural inclusion  $\mathcal{O}_{X, x} \hookrightarrow \mathcal{O}_{X \times \mathbb{A}^1, z}$ .

**Example 5.1.8.** Let  $X$  be a Noetherian, integral, and  $R_1$  scheme. Then for any  $n \in \mathbb{Z}_{\geq 0}$ , we have  $\operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Cl}(X \times \mathbb{A}^n)$ , induced by the “pullback” map. It suffices to do the case  $n = 1$ ; fix an identification of  $K(t)$  with the function field of  $X \times \mathbb{A}^1$ . For each prime divisor  $Y \subset X$ , the pullback  $Y \times \mathbb{A}^1 \cong \pi^{-1}(Y) \subset X \times \mathbb{A}^1$  is a prime divisor of type 1. This gives us a commutative diagram with exact rows as below.

$$\begin{array}{ccccccc} K(t)^\times & \xrightarrow{\operatorname{div}} & \operatorname{Div}(X \times \mathbb{A}^1) & \longrightarrow & \operatorname{Cl}(X \times \mathbb{A}^1) & \longrightarrow & 0 \\ \uparrow & & \uparrow \pi^* & & \uparrow & & \\ K^\times & \xrightarrow{\operatorname{div}} & \operatorname{Div}(X) & \longrightarrow & \operatorname{Cl}(X) & \longrightarrow & 0 \end{array}$$

We claim that the rightmost vertical map is an isomorphism. To show injectivity, it suffices to note that if  $f \in K(t)^\times \setminus K^\times$  is any element, then there is a type 0 prime divisor  $Z \subset X \times \mathbb{A}^1$  (e.g., the vanishing locus of an irreducible factor of the numerator or denominator of  $f$  in  $K[t]$ ) such that  $v_Z(f) \neq 0$ .

To show surjectivity, it suffices to show that if  $Z \subset X \times \mathbb{A}^1$  is any prime divisor of type 0, then  $[Z]$  is in the image of  $\operatorname{Cl}(X)$ . Now  $Z$  corresponds to a nonzero prime of  $K[t]$ , so  $Z = \mathbb{V}(g)$  for some irreducible polynomial  $g \in K[t]$ . Then  $\operatorname{div}(g)$  cannot contain any other type 0 prime divisors besides  $Z$ , i.e.,  $\operatorname{div}(g) = [Z] + \sum_i n_i [Y_i \times \mathbb{A}^1]$  for some prime divisors  $Y_i \subset X$  and integers  $n_i \in \mathbb{Z}$ . This says precisely that  $[Z]$  is in the image of  $\operatorname{Cl}(X)$ , as desired.

**Example 5.1.9.** Let  $X$  be a Noetherian, integral, and *normal* scheme. For any  $m \in \mathbb{Z}_{\geq 1}$  and integers  $n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}$ , we show that

$$\operatorname{Cl}\left(X \times \prod_{i=1}^m \mathbb{P}^{n_i}\right) \cong \operatorname{Cl}(X) \oplus \bigoplus_{i=1}^m \mathbb{Z}.$$

Clearly, it suffices to do the case  $m = 1$ ; let  $n := n_1$ . The case  $n = 0$  is trivial, so suppose  $n \geq 1$ . Applying (5.1.5) to the prime divisor  $X \times \mathbb{P}^{n-1} \subset X \times \mathbb{P}^n$  gives us an exact sequence of the form

$$0 \rightarrow \mathcal{O}(X \times \mathbb{P}^{n-1})^\times / \mathcal{O}(X)^\times \rightarrow \mathbb{Z}[X \times \mathbb{P}^{n-1}] \rightarrow \operatorname{Cl}(X \times \mathbb{P}^n) \rightarrow \operatorname{Cl}(X \times \mathbb{A}^n) \rightarrow 0.$$

Clearly  $\mathcal{O}(X)^\times \xrightarrow{\sim} \mathcal{O}(X \times \mathbb{P}^{n-1})^\times$ , so the class  $[X \times \mathbb{P}^{n-1}] \in \operatorname{Cl}(X \times \mathbb{P}^n)$  is non-torsion. Finally, the “flat pullback”  $\operatorname{Cl}(X) \rightarrow \operatorname{Cl}(X \times \mathbb{P}^n)$ , defined analogously to 5.1.8, composed with the restriction  $\operatorname{Cl}(X \times \mathbb{P}^n) \rightarrow \operatorname{Cl}(X \times \mathbb{A}^n)$  is an isomorphism, again by 5.1.8. This gives us a splitting of the above exact sequence  $\operatorname{Cl}(X \times \mathbb{P}^n) \cong \operatorname{Cl}(X) \oplus \mathbb{Z}$  as desired.

Unraveling the proof gives us explicit generators: the inclusion of the  $\operatorname{Cl}(X)$  factor is again given by flat pullback, whereas a generator of the  $i^{\text{th}}$  copy of  $\mathbb{Z}$  is given by the class of the prime divisor obtained by replacing  $\mathbb{P}^{n_i}$  with a hyperplane  $\mathbb{P}^{n_i-1} \subset \mathbb{P}^{n_i}$ .

**Example 5.1.10.** Let  $k$  be a field and  $Q = \mathbb{V}(wz - xy) \subset \mathbb{P}_k^3$  be the quadric surface. Compute  $\mathrm{Cl}(Q)$  and use this to show that there is no surface  $Y \subset \mathbb{P}_k^3$  such that  $Y$  intersects  $Q$  set-theoretically in the twisted cubic  $C$ .

**Example 5.1.11.** Let  $A$  be a Noetherian UFD. Then  $\mathrm{Pic}(\mathrm{Spec} A) = 0$ . For instance,  $\mathrm{Pic}(\mathrm{Spec} \mathbb{Z}) = 0$  and  $\mathrm{Pic}(\mathbb{A}_k^n) = 0$  for any field  $k$  and integer  $n \in \mathbb{Z}_{\geq 0}$ .

**Example 5.1.12.** Let  $k$  be a field. Then  $\mathrm{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$  generated by the line bundle  $\mathcal{O}_{\mathbb{P}_k^n}(1)$ .

## 5.2 Meromorphic Functions and Cartier Divisors

**Convention 5.2.1.** In this section,  $X$  denotes a locally Noetherian scheme.

Define the ring of meromorphic (or rational) functions on  $X$  to be the colimit of  $\mathcal{O}(U)$  where  $U$  is an open subset containing all associated points of  $X$ , and denote it by  $K(X)$ . This is a subring of  $\prod_{p \in \text{Ass}(X)} \mathcal{O}_{X,p}$ . The assignment  $U \mapsto K(U)$  for  $U \subset X$  open clearly defines a presheaf of  $\mathcal{O}_X$ -algebras. In fact, this presheaf is a sheaf, and we denote this sheaf by  $\mathcal{K}_X$ , and call it the *sheaf of meromorphic functions* on  $X$ . To show that  $\mathcal{K}_X$  is a sheaf, it is helpful to construct it another way, which is what we do now (5.2.3); then we show that these two constructions agree (5.2.4). For this, we start with small lemma.

**Lemma 5.2.2.** Let  $A$  be a ring,  $X = \text{Spec } A$ , and  $U \subset X$  an open subset. Pick an ideal  $\mathfrak{a} \subset A$  such that  $U = X \setminus V(\mathfrak{a})$ . Consider the following conditions.

- (a) The ideal  $\mathfrak{a}$  contains a nonzerodivisor of  $A$ .
- (b) The open subset  $U$  contains a principal open subset  $D(f)$  for some nonzerodivisor  $f \in A$ .
- (c) The open subset  $U \subset X$  is schematically dense, i.e., the schematic closure of  $U$  in  $X$  is  $X$  itself.
- (d) The annihilator  $\text{Ann}_A(\mathfrak{a})$  is zero.
- (e) The open subset  $U$  contains all associated points of  $X$ , i.e., associated primes of  $A$ .

Then we have the implications (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d)  $\Rightarrow$  (e). If  $A$  is Noetherian, then also (e)  $\Rightarrow$  (a), so that all five conditions are equivalent, and there is a natural isomorphism  $\text{Quot}(A) \xrightarrow{\sim} K(\text{Spec } A)$ .

*Proof.* ([5, Lemma 9.23]) Clearly, (a)  $\Leftrightarrow$  (b).

- (b)  $\Rightarrow$  (c) Let  $\mathfrak{b} \subset A$  be an ideal such that  $\text{Spec } A/\mathfrak{b}$  majorizes  $U$ . In particular, for all  $b \in \mathfrak{b}$  we have  $b|_U = 0$  and so  $b|_{D(f)} = 0$ , i.e.,  $b \in \ker(A \rightarrow A[f^{-1}])$ . Since  $f$  is a nonzerodivisor, this kernel is zero, and so  $b = 0$ . Therefore,  $\mathfrak{b} = 0$ .
- (c)  $\Rightarrow$  (d) Let  $b \in \text{Ann}_A(\mathfrak{a})$ . Given an  $x \in U$ , pick an  $a \in \mathfrak{a}$  such that  $a(x) \neq 0$ . Now  $(ab)(x) = 0$  implies that  $b(x) = 0$ . Since this holds for all  $x \in U$  and  $U$  is schematically dense, we have  $b = 0$ .
- (d)  $\Rightarrow$  (e) Suppose there is an associated prime  $\mathfrak{p} \subset A$  such that  $\mathfrak{p} \supset \mathfrak{a}$ . Pick a  $b \in A \setminus \{0\}$  such that  $\mathfrak{p} = \text{Ann}_A(b)$ . Then  $b \in \text{Ann}_A(\mathfrak{a}) \setminus \{0\}$ .

Finally, when  $A$  is a Noetherian ring, then we have

- (e)  $\Rightarrow$  (a) If  $\mathfrak{a}$  does not contain a nonzerodivisor, then  $\mathfrak{a}$  lies in the union of the finitely many associated primes of  $A$ , and hence lies in one of them by prime avoidance.

To show the last statement, note that if  $f \in A$  is a nonzerodivisor, then  $D(f)$  contains all associated primes of  $A$ ; in particular, the natural morphism  $A \rightarrow K(\text{Spec } A)$  factors through  $\text{Quot}(A) \rightarrow K(\text{Spec } A)$ . This is injective, since both sides further embed into  $\prod_{\mathfrak{p} \in \text{Ass}(A)} A_{\mathfrak{p}}$ . Surjectivity follows from the implication (e)  $\Rightarrow$  (b) above.  $\blacksquare$

The above discussion allows us to globalize the definition.

**Proposition/Definition 5.2.3.** The  $\mathcal{O}_{X_{\text{Zar}}^{\text{dist}}}$ -premodule  $U \mapsto \text{Quot } \mathcal{O}_X(U)$  is a sheaf, and hence an  $\mathcal{O}_{X_{\text{Zar}}^{\text{dist}}}$ -module. In particular, sheafification gives a unique sheaf  $\mathcal{K}_X$  of  $\mathcal{O}_X$ -algebras, called the *sheaf of meromorphic functions* on  $X$ , such that for any affine  $U \subset X$  the natural morphism  $\mathcal{O}_X(U) \rightarrow \mathcal{K}_X(U)$  is given by  $\mathcal{O}_X(U) \hookrightarrow \text{Quot } \mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{K}_X(U)$ .

*Proof.* ([13, Lect. 9, p. 61]) We need to show that if  $A$  is a ring,  $n \in \mathbb{Z}_{\geq 1}$ , and  $f_1, \dots, f_n \in A$  elements such that  $(f_1, \dots, f_n) = (1)$ , then the sequence

$$\text{Quot } A \rightarrow \prod_i \text{Quot } A[f_i^{-1}] \rightarrow \prod_{i,j} \text{Quot } A[(f_i f_j)^{-1}]$$

is exact.  $\blacksquare$

Part of the point of the above discussion is

**Lemma 5.2.4.** The two constructions of  $\mathcal{K}_X$  above are naturally isomorphic as sheaves of  $\mathcal{O}_X$ -algebras. In particular, the first construction defines a sheaf of  $\mathcal{O}_X$ -algebras.

*Proof.* ■

Recall that if  $\mathcal{R}$  is a sheaf of rings on a topological space  $X$ , then the assignment  $U \mapsto \mathcal{R}(U)^\times$  defines a sheaf of abelian groups on  $X$ , denoted by  $\mathcal{R}^\times$ , and this construction is functorial in  $\mathcal{R}$ . In particular, if  $X$  is a locally Noetherian scheme, then we have a monomorphism of abelian sheaves  $\mathcal{O}_X^\times \hookrightarrow \mathcal{K}_X^\times$  on  $X$ .

**Definition 5.2.5.**

- (a) Define the *sheaf of Cartier divisors* on  $X$  to be the quotient abelian sheaf  $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ . We denote the global sections of this sheaf by  $\text{CaDiv}(X) := \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ , and call its elements *Cartier divisors* on  $X$ .
- (b) We define the *Cartier class group* to be the cokernel of the natural morphism  $K(X)^\times \rightarrow \text{CaDiv}(X)$  denoted by  $\text{div}$ , i.e., define it via the exact sequence

$$0 \rightarrow \mathcal{O}(X)^\times \rightarrow K(X)^\times \rightarrow \text{CaDiv}(X) \rightarrow \text{CaCl}(X) \rightarrow 0.$$

Super explicitly. The connecting homomorphism therefore defines an injection  $\text{CaCl}(X) \hookrightarrow H^1(X, \mathcal{O}_X^\times) \cong \text{Pic}(X)$  with image consisting of line bundles which can be embedded in the  $\mathcal{O}_X$ -module  $\mathcal{K}_X$ . In fact, there is a natural injection  $\text{CaDiv}(X) \rightarrow \widetilde{\text{Pic}(X)}$ , where  $\widetilde{\text{Pic}(X)}$  is the extension of  $\text{Pic}(X)$  by  $K(X)^\times / \mathcal{O}(X)^\times$  consisting of isomorphism classes of pairs  $(\mathcal{L}, \sigma)$ , where  $\mathcal{L}$  is a line bundle and  $\sigma$  a rational section giving rise to a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \text{CaDiv}(X) & \longrightarrow & \text{CaCl}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(X)^\times & \longrightarrow & K(X)^\times & \longrightarrow & \widetilde{\text{Pic}(X)} & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0. \end{array}$$

**Lemma 5.2.6.** When  $X$  is integral, the above morphisms  $\text{CaDiv}(X) \hookrightarrow \widetilde{\text{Pic}(X)}$  and  $\text{CaCl}(X) \hookrightarrow \text{Pic}(X)$  are isomorphisms.

Finally, suppose that  $X$  is Noetherian, integral, and  $\mathbf{R}_1$ . Then there is a commutative diagram with exact rows of the form below.

$$\begin{array}{ccccccccc} & & K^\times / \mathcal{O}(X)^\times & \xrightarrow{\text{div}} & \text{Div}(X) & \longrightarrow & \text{Cl}(X) & \longrightarrow & 0 \\ & & \uparrow = & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K^\times / \mathcal{O}(X)^\times & \longrightarrow & \text{CaDiv}(X) & \longrightarrow & \text{CaCl}(X) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & K^\times / \mathcal{O}(X)^\times & \longrightarrow & \widetilde{\text{Pic}(X)} & \longrightarrow & \text{Pic}(X) & \longrightarrow & 0. \end{array}$$

The resulting morphism  $\widetilde{\text{Pic}(X)} \rightarrow \text{Div}(X)$  is given by taking divisors, i.e., sending the pair  $(\mathcal{L}, \sigma)$  to the Weil divisor  $\text{div}(\sigma) = \sum_Z v_Z(\sigma)[Z]$ , and similarly for  $\text{Pic}(X) \rightarrow \text{Cl}(X)$ . In general, this map  $\text{Pic}(X) \rightarrow \text{Cl}(X)$  is neither injective nor surjective, as the following examples show.

**Example 5.2.7.**

- (a) Non-normal, where not injective.
- (b) Non-factorial, where not surjective.

**Lemma 5.2.8.**

- (a) When  $X$  is also normal, the map  $\text{div} : K^\times / \mathcal{O}(X)^\times \rightarrow \text{Div}(X)$  is an injection, and so are the maps  $\text{CaDiv}(X) \rightarrow \text{Div}(X)$  and  $\text{CaCl}(X) \rightarrow \text{Cl}(X)$ .
- (b) If  $X$  is also factorial, then the maps  $\text{CaDiv}(X) \rightarrow \text{Div}(X)$  and  $\text{CaCl}(X) \rightarrow \text{Cl}(X)$  are isomorphisms.

In the normal case, the image  $\text{CaDiv}(X) \rightarrow \text{Div}(X)$  consists of the so-called *locally principal* divisors, i.e., those Weil divisors  $D$  such that for each point  $x \in X$ , there is an open neighborhood

$U \subset X$  of  $x$  and an  $f \in K(X)^\times$  such that  $D|_U = \operatorname{div}_U(f)$ . In particular, when  $X$  is normal, the map  $\operatorname{CaCl}(X) \rightarrow \operatorname{Cl}(X)$  is an isomorphism onto the subgroup of locally principal Weil divisors modulo linear equivalence. The above lemma says that when  $X$  is factorial, every Weil divisor is locally principal.

**Corollary 5.2.9.** If  $X$  is a Noetherian, integral, and factorial scheme, then  $\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Cl}(X)$ .

## Chapter 6

# Intersection Theory

Following [14]. Throughout (except in the Appendices), we will fix an algebraically closed field  $k$ . By a *scheme*, we mean a separated finite-type scheme over  $k$ , and by a *variety*, we mean an integral scheme. We will denote the function field of a variety  $X$  by  $k(X)$ . In particular, all schemes are assumed to be Noetherian.

## Chapter 7

## Appendices

## 7.1 Generic Freeness

**Theorem 7.1.1** (Geometric Nakayama). Let  $X$  be a locally Noetherian scheme,  $\mathcal{F} \in \mathbf{Coh}(X)$  and  $x \in X$ . Let  $n \in \mathbb{Z}_{\geq 0}$ .

- (a) Given an epimorphism  $\psi(x) : \kappa(x)^{\oplus n} \twoheadrightarrow \mathcal{F}(x)$  of  $\kappa(x)$ -vector spaces, there is an open subset  $V \subset X$  containing  $x$  and an  $\mathcal{O}_V$ -module epimorphism  $\psi_V : \mathcal{O}_V^{\oplus n} \twoheadrightarrow \mathcal{F}|_V$  lifting  $\psi(x)$ . Further,  $V$  can be chosen to be affine and such that  $\psi(V) : \mathcal{O}_X(V)^{\oplus n} \twoheadrightarrow \mathcal{F}(V)$  is an  $\mathcal{O}_X(V)$ -module epimorphism.
- (b) Given an isomorphism  $\psi_x : \mathcal{O}_{X,x}^{\oplus n} \xrightarrow{\sim} \mathcal{F}_x$  of  $\mathcal{O}_{X,x}$ -modules, then there is an open subset  $W \subset X$  containing  $x$  and an isomorphism  $\psi_W : \mathcal{O}_W^{\oplus n} \xrightarrow{\sim} \mathcal{F}|_W$  of  $\mathcal{O}_W$ -modules lifting  $\psi_x$ .

In particular,  $\mathcal{F}$  is flat iff it is locally free of finite rank, i.e. a vector bundle.

*Proof.*

- (a) Pick an open subset  $U \subset X$  containing  $x$  and elements  $a_1, \dots, a_n \in \mathcal{F}(U)$  such that for  $i = 1, \dots, n$ , the map  $\psi(x)$  sends the  $i^{\text{th}}$  basis vector to  $a_i(x)$ . Without loss of generality, we may assume  $U = \operatorname{Spec} A$  for a Noetherian ring  $A$  and  $\mathcal{F} = \widetilde{M}$  for a finite  $A$ -module  $M$ ; then  $\mathcal{F}(U) = M$ . Let  $x$  correspond to the prime ideal  $\mathfrak{p} \subset A$ . The hypothesis when combined with Nakayama's Lemma tells us that (the classes of)  $a_1, \dots, a_n$  generate  $M_{\mathfrak{p}}$  as an  $A_{\mathfrak{p}}$ -module. Let  $M' \subset M$  be the submodule of  $M$  generated by  $a_1, \dots, a_n$ ; since  $M$  is finite over  $A$ , there is an  $s \in A \setminus \mathfrak{p}$  such that  $sM \subset M'$ . It suffices to take  $V := D(s)$ .
- (b) By (the proof of) (a), there is an affine open  $V \subset U$  and a lifting  $\psi_V : \mathcal{O}_V^{\oplus n} \twoheadrightarrow \mathcal{F}|_V$  of  $\psi_x$  to an  $\mathcal{O}_V$ -module epimorphism. Then  $\ker \psi_V \in \mathbf{Coh}(V)$ ; taking  $W := V \setminus \operatorname{supp} \ker \psi_V$  works.

The last statement follows from (b) and Lemma 7.2.1. ■

**Remark 7.1.2.** For an arbitrary scheme, the correct statement is that vector bundles, i.e. locally free sheaves of finite rank, are exactly the flat modules of finite presentation.

**Corollary 7.1.3** (Generic Freeness). Let  $X$  be a locally Noetherian integral scheme and  $\mathcal{F} \in \mathbf{Coh}(X)$ .

- (a) Let  $\eta \in X$  denote the generic point. If  $\mathcal{F}_{\eta} = 0$ , then there is a dense open  $U \subset X$  with  $\mathcal{F}|_U = 0$ .
- (b) There is a dense open  $U \subset X$  such that  $\mathcal{F}|_U$  is locally free of finite rank, i.e. a vector bundle.

## 7.2 Flatness

**Lemma 7.2.1.** . Let  $A$  be a ring, and  $M$  be a finitely presented flat module. (For instance, this holds if  $A$  is Noetherian and  $M$  finite.) Then  $M$  is locally free, i.e. projective.

*Proof.* Localizing, we may assume that  $(A, \mathfrak{m}, k)$  is a local ring; then we have to show that a finitely presented flat  $A$ -module  $M$  is free. Since  $M$  is finitely generated,  $k \otimes_A M$  is a finite-dimensional  $k$ -vector space, of dimension say  $r \in \mathbb{Z}_{\geq 0}$ . Pick  $m_1, \dots, m_r \in M$  such that the images of these in  $k \otimes_A M$  constitute a  $k$ -basis; we show that the corresponding map  $A^{\oplus r} \rightarrow M$  is an isomorphism. Indeed, it is surjective by another application of Nakayama's Lemma. For injectivity, let  $I$  denote the kernel, so that we have a short exact sequence  $0 \rightarrow I \rightarrow A^{\oplus r} \rightarrow M \rightarrow 0$ . Note that  $I$  is a finitely generated  $A$ -module.<sup>1</sup> Since  $M$  is flat, we have  $\mathrm{Tor}_1^A(k, M) = 0$ , and hence tensoring with  $k$  yields the exact sequence  $0 \rightarrow k \otimes_A I \rightarrow k^{\oplus r} \rightarrow k \otimes_A M \rightarrow 0$ . Since the corresponding map  $k^{\oplus r} \rightarrow k \otimes_A M$  is an injection, we conclude that  $k \otimes_A I = 0$ , and so by a final application of Nakayama's Lemma we get  $I = 0$ . ■

**Lemma 7.2.2.** Let  $A$  be a Noetherian domain with fraction field  $K$ , let  $B$  an irreducible Noetherian ring (i.e. a Noetherian ring such that  $\gamma := \mathrm{Nil}(B)$  is a prime ideal), and  $\varphi : A \hookrightarrow B$  be a finite flat injective morphism. The natural map  $K \otimes_A B \cong (A \setminus \{0\})^{-1}B \rightarrow B_\gamma$  is an isomorphism of  $K$ -vector spaces.

*Proof.* The idea is that in this setting, the nilpotent elements of  $K \otimes_A B$  are precisely the zero-divisors. Injectivity then comes from the fact that the localization map obtained by inverting non-zero-divisors is injective. Surjectivity comes from checking that [TODO]. ■

---

<sup>1</sup>This is due to the “finitely presented implies always finitely presented” property.

## 7.3 Counterexamples in Algebraic Geometry

**Example 7.3.1.** A locally ringed space that is not a scheme. Take  $X$  to be a point with local ring that has more than one prime ideal.

**Example 7.3.2.** Open subscheme of an affine scheme that is not affine.

**Example 7.3.3.** Scheme with no closed point.

**Example 7.3.4.** An affine open subscheme of an affine scheme that is not a distinguished affine open. (Elliptic curve - non-torsion)

**Example 7.3.5.** A non-Noetherian scheme whose underlying space is Noetherian.

## 7.4 Possible Hints to Selected Exercises

### 4.3.

- (a) Use Krull's theorems [TODO].
- (b) Combine (a) for  $W = \{y\}$  with 2.1.16, 4.1.6(a), and 4.1.7(a).
- (c) Use Noether normalization [TODO] (and a little more) to show that for some nonempty open subset  $V \subset Y$ , the map  $X_V = \pi^{-1}(V) \rightarrow V$  factors as  $X_V \rightarrow \mathbb{A}_V^r \rightarrow V$ , where the first map is finite surjective.
- (d) Combine (c) with induction on  $\dim Y$ .

**4.4.** Show that if  $Y$  is an irreducible topological space,  $r, n \in \mathbb{Z}_{\geq 1}$ , and  $f_1, f_2, \dots, f_r : Y \rightarrow \mathbb{N}$  upper semicontinuous functions such that  $\max_i f_i \equiv n$ , then there is an  $i$  such that  $f_i \equiv n$ .

**4.5.** For (a), use 2.1.13 to show the first statement. For the second statement, reduce to the case of affine  $X, Y, Z$  and surjective  $f, g$ ; you might need Chevalley's Theorem along with 2.3.4(b). First use Krull's Theorems [TODO] to prove the result when  $Z = \mathbb{A}_k^n$  for some  $n \in \mathbb{N}$ , and in fact solve the second part of (b). For the general case, use Noether normalization combined with the diagonal-base-change diagram ([1, 1.2.S]), noting that if  $Z \rightarrow \mathbb{A}_k^n$  is finite surjective, then the diagonal morphism  $Z \rightarrow Z \times_{\mathbb{A}_k^n} Z$  is the inclusion of an irreducible component for dimensional reasons (this might need 4.1.7). Since  $f, g$  are surjective, so is  $X \times_{\mathbb{A}_k^n} Y \rightarrow Z \times_{\mathbb{A}_k^n} Z$ , and so this map is dominant by 3.3.3; in particular, there is an irreducible component of  $X \times_{\mathbb{A}_k^n} Y$  dominating  $\Delta(Z)$ , and this must also be an irreducible component of  $X \times_Z Y$ . For (b), for both examples take  $X = Y$  and  $f = g$ ; for the first one, consider the blow-up of  $\mathbb{A}_k^3$  at the origin, and for the second one the normalization of the nodal cubic.

**4.6.** Show that a product of smooth  $k$ -schemes is smooth, and hence that the diagonal  $\Delta : Z \rightarrow Z \times_k Z$  is a regular locally closed embedding of codimension  $\dim Z$ . Next, using Krull's Theorems, prove that if  $S, T, U$  are locally Noetherian schemes and  $U \rightarrow T$  a regular locally closed embedding of codimension say  $d \in \mathbb{N}$  and  $p : S \rightarrow T$  any morphism, then every irreducible component of  $p^{-1}(U)$  has codimension at most  $d$  in  $S$ .

### 4.7.

- (a) Use 4.1.3 to reduce to the affine case; then use 4.1.5.
- (b) Using 4.1.8, reduce to the case of affine integral  $X$ . Now use 4.1.9(a) along with the fact that integral morphisms are closed [TOCITE].
- (c) Let  $\pi : X_K \rightarrow X$  denote the projection. If  $x \in X$  lies in an irreducible component  $Z \subset X$ , then  $\pi^{-1}(x) \subset Z_K$ . If this happens for two distinct  $Z$ , say  $Z_1$  and  $Z_2$ , then pick any point  $\tilde{x} \in \pi^{-1}(x)$  and, for  $i = 1, 2$ , an irreducible component  $Z'_i \subset Z_i$  containing  $\tilde{x}$ . Use 4.1.8 to show that  $Z'_1, Z'_2 \subset X_K$  are irreducible components which intersect.

# Bibliography

- [1] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*. Princeton University Press, 2025.
- [2] R. Hartshorne, *Algebraic Geometry*. No. 52 in Graduate Texts in Mathematics, Springer, 1977.
- [3] The Stacks Project Authors, “*Stacks Project*.” <https://stacks.math.columbia.edu>, 2018.
- [4] Q. Liu, *Algebraic Geometry and Arithmetic Curves*. No. 6 in Oxford Graduate Texts in Mathematics, Oxford University Press, 2002.
- [5] U. Görtz and T. Wedhorn, *Algebraic Geometry I: Schemes*. Springer Studium Mathematik - Master, Springer Spektrum, second ed., 2010.
- [6] D. Goel, “Commutative Algebra.” [https://gdmgoel.github.io/Writings/Commutative\\_Algebra.pdf](https://gdmgoel.github.io/Writings/Commutative_Algebra.pdf).
- [7] T. Wedhorn, *Manifolds, Sheaves, and Cohomology*. Springer Studium Mathematik - Master, Springer Spektrum, 2016.
- [8] M. S. Davis, “Quasiseparated if finitely covered by affines in appropriate way.” Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/4919351> (version: 2024-06-14).
- [9] M. Escardó, “Intersections of compactly many open sets are open.” <https://martinescardo.github.io/papers/compactness-submitted.pdf>.
- [10] B. Poonen, “Is a universally closed morphism of schemes quasi-compact?.” MathOverflow. URL: <https://mathoverflow.net/q/23528> (version: 2010-05-05).
- [11] B. Poonen, *Rational Points on Varieties*, vol. 186 of *Graduate Studies in Mathematics*. American Mathematical Society, 2017.
- [12] J. Harris, *Algebraic Geometry: A First Course*, vol. 133 of *Graduate Texts in Mathematics*. Springer, 1992.
- [13] D. Mumford, *Lectures on Curves on an Algebraic Surface*. Princeton University Press, 1996.
- [14] W. Fulton, *Intersection Theory*. Springer, second ed., 1998.