Cohomology of Curves

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Abstract

This is a short collection of lecture notes based on a mini talk-series aimed at counselors I gave at Ross/Ohio 2024. The goal of this set of notes is to show finite-dimensionality of the cohomology of invertible sheaves on nonsingular projective curves and the Riemann-Roch Theorem, following [1, Propositions V.3.16 and VI.2.7], and the Serre duality theorem for such curves using the method of adeles or "répartitions" due to Weil, following Serre's [2, Chapter II, Theorem 2].

Contents

1 Notation and Fundamentals

Throughout this article, we let X be an irreducible smooth projective curve over an algebraically closed field k. We will not assume that k has characteristic zero; however, if doing so significantly simplifies a given proof, then we will, and direct the reader to a general proof elsewhere.

(a) We denote the function field of X by K, so that if $\eta \in X$ is the generic point, then

$$K := k(X) = \mathcal{O}_{X,\eta}.$$

and K is a field extension of k of transendence degree 1.

(b) Every point in X other than η is a closed point; in what follows, we will use the notation $x \in U$ for an open subset $U \subseteq X$ to mean that x is a closed point of X contained in U, so that, for instance, the divisor group $\mathrm{Div}(X)$ of X is the free abelian group on the set of $x \in X$. Given a divisor $D = \sum_{x\in X} n_x x$, we use the notation $D(x) := n_x$ and define the degree of D to be $\deg D := \sum_{x\in X} D(x)$.

(c) For each $x \in X$, the stalk $\mathcal{O}_{X,x} \subset K$ is a regular local ring of dimension one, i.e. a discrete valuation ring (DVR). We will denote the discrete valuation on K corresponding to the point $x \in X$ by v_x, so that

$$\mathcal{O}_{X,x} = \{f \in K : v_x(f) \geq 0\} \text{ has maximal ideal } \mathfrak{m}_{X,x} := \{f \in K : v_x(f) \geq 1\},$$

where $v_x(0) = \infty$ by convention. Finally, we will denote the completion of $\mathcal{O}_{X,x}$ (resp. K) with respect to the valuation v_x by $\hat{\mathcal{O}}_{X,x}$ (resp. $\hat{K}_x$), so that $\hat{K}_x = \mathrm{Frac}\, \hat{\mathcal{O}}_{X,x}$. A choice $t \in \mathcal{O}_{X,x}$ of uniformizer for v_x yields the isomorphisms

$$\hat{\mathcal{O}}_{X,x} \cong k[\![t]\!] \text{ and } \hat{K}_x \cong k(\!(t)\!).$$

(d) For each $D \in \mathrm{Div}(X)$, we denote the associated invertible sheaf by $\mathcal{O}_X(D)$, which at $x \in X$ has stalk

$$\mathcal{O}_X(D)_x := \{f \in K : v_x(f) + D(x) \geq 0\}$$

and for any nonempty open subset $U \subseteq X$ satisfies $\mathcal{O}_X(D)(U) := \bigcap_{x\in U} \mathcal{O}_X(D)_x \subseteq K$. For any $D \in \mathrm{Div}(X)$, we have a monomorphism $\mathcal{O}(X) \hookrightarrow \underline{K}$ of $\mathcal{O}_X(D)$ into the constant sheaf $\underline{K}$, and the space of global sections of the quotient $\underline{K}/\mathcal{O}_X(D)$ can be identified via evaluation on stalks as

$$\mathrm{H}^0(X, \underline{K}/\mathcal{O}_X(D)) \cong \bigoplus_{x\in X} K/\mathcal{O}_X(D)_x. \tag{1}$$

For $i = 0, 1$, we also use the notation

$$\mathrm{H}^i(D) := \mathrm{H}^i(X, \mathcal{O}_X(D)) \text{ and } h^i(D) := \dim_k \mathrm{H}^i(D).$$

Then the short exact sequence $0 \to \mathcal{O}_X(D) \to \underline{K} \to \underline{K}/\mathcal{O}_X(D) \to 0$ gives rise to the long exact sequence

$$0 \to \mathrm{H}^0(D) \to K \to \mathrm{H}^0(\underline{K}/\mathcal{O}_X(D)) \to \mathrm{H}^1(D) \to 0, \tag{2}$$

where $\mathrm{H}^1(\underline{K}) = 0$ because a constant sheaf on an irreducible space is flasque.

(e) We say that D is effective, written $D \geq 0$, to mean that $n_x \geq 0$ for all $x \in X$, and for $D_1, D_2 \in \mathrm{Div}(X)$, write $D_1 \leq D_2$ to mean that $D_2 - D_1 \geq 0$. Clearly, $D_1 \leq D_2$ implies that $\deg D_1 \leq \deg D_2$ and $\mathrm{H}^0(D_1) \subseteq \mathrm{H}^0(D_2)$, whence $h^0(D_1) \leq h^0(D_2)$. We will also have need for the maximum of two arbitrary divisors D_1 and D_2, which is the unique divisor $D = \max\{D_1, D_2\}$ satisfying

$$D(x) = \max\{D_1(x), D_2(x)\}$$

for all $x \in X$. Similarly, we can define the minimum $\min\{D_1, D_2\}$ of D_1 and D_2; then we have $\mathrm{H}^0(D_1) \cap \mathrm{H}^0(D_2) = \mathrm{H}^0(\min\{D_1, D_2\})$.

(f) For each nonzero $f \in K$, we denote the divisor of f by

$$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f) \in \mathrm{Div}(X),$$

where

$$\mathrm{div}_0(f) := \sum_{x\in X : v_x(f)>0} v_x(f) \cdot x \text{ and } \mathrm{div}_\infty(f) := \sum_{x\in X : v_x(f)<0} -v_x(f) \cdot x$$

are the divisors of zeroes and poles of f, respectively. Then $\mathrm{div}(f) = 0$ iff f is constant; each nonconstant f gives rise to a finite morphism $f : X \to \mathbb{P}^1_k$ of degree

$$\deg f = \deg \mathrm{div}_\infty(f) = \deg \mathrm{div}_0(f) = [K : k(f)],$$

and then $\mathrm{div}_0(f) = f^*(0)$ and $\mathrm{div}_\infty(f) = f^*(\infty)$. In particular, $\deg \mathrm{div}(f) = 0$. Finally, for $D \in \mathrm{Div}(X)$, if $h^0(D) \geq 1$ then $\deg D \geq 0$ with equality iff D is principal, i.e. $D = \mathrm{div}(f)$ for some $0 \neq f \in K$.

2 Finiteness of Cohomology and Riemann-Roch

Our first order of business is to show the two Theorem 2.1 below.

Theorem 2.1 (Finiteness of Cohomology). For $D \in \mathrm{Div}(X)$, we have $h^0(D), h^1(D) < \infty$.

Remark 2.2. There are many sophisticated proofs of this result and its generalizations; the aim of this article is to give a fairly accessible direct proof following [1, Propositions V.3.16 and VI.2.7], but we list now some references to the general literature on finiteness of sheaf cohomology. On one hand, one can show algebraically that if X is a projective scheme over a Noetherian ring A and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module, then for all $i \geq 0$, the cohomology groups $\mathrm{H}^i(X, \mathscr{F})$ are finitely generated A-modules via a reduction to an explicit calculation of the cohomology of $\mathcal{O}_{\mathbb{P}^n_A}(d)$ for all n, d (see [3, Theorem 5.2] or [4, Corollary 23.2]), or more generally that if $f : X \to S$ is a proper morphism of schemes with S locally Noetherian, then for each coherent $\mathcal{O}_X$-module $\mathscr{F}$ and integer $i \geq 0$, the higher pushforward $\mathrm{R}^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module (see [4, Theorem 23.17]). On the other hand, one can show analytically for $k = \mathbb{C}$ that the cohomology of vector bundles on a compact complex manifold is finite-dimensional, via finite-dimensionality of solution spaces to elliptic PDEs (see [5, Chapter IV, Example 5.7] for the general case, or [6, Corollary 14.10] and the reduction as below for the case of line bundles on compact Riemann surfaces), then invoke Serre's GAGA principle to conclude the result for smooth proper varieties over $\mathbb{C}$ (see the discussion in [4, §23.9]), and then conclude from this the case of arbitrary algebraically closed fields of characteristic zero via the Lefschetz principle.

In light of Theorem 2.1, we define the **genus** of X to be

$$g := h^1(0) = \dim_k \mathrm{H}^1(X, \mathcal{O}_X).$$

One thing we can them immediately derive from Theorem 2.1 is

Theorem 2.3 (Riemann-Roch, Version I). For any $D \in \mathrm{Div}(X)$, we have

$$h^0(D) - h^1(D) = \deg D - g + 1.$$

Proof. For each pair of divisors $D_1 \leq D_2$ on X, there is a sheaf monomorphism $\mathcal{O}_X(D_1) \hookrightarrow \mathcal{O}_X(D_2)$ inducing maps $\mathrm{H}^i(D_1) \to \mathrm{H}^i(D_2)$ for $i = 0, 1$, such that the map $\mathrm{H}^0(D_1) \to \mathrm{H}^0(D_2)$ is injective. The comparison map of the (slightly modified) short exact sequences (2), looks like

$$\begin{tikzcd}
0 \arrow[r] & K/\mathrm{H}^0(D_1) \arrow[r] \arrow[d] & \mathrm{H}^0(\underline{K}/\mathcal{O}_X(D_1)) \arrow[r] \arrow[d] & \mathrm{H}^1(D_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K/\mathrm{H}^0(D_2) \arrow[r] & \mathrm{H}^0(\underline{K}/\mathcal{O}_X(D_2)) \arrow[r] & \mathrm{H}^1(D_2) \arrow[r] & 0,
\end{tikzcd}$$

applying the snake lemma to which, along with the observation (1), yields both the short exact sequence

$$0 \to \mathrm{H}^0(D_2)/\mathrm{H}^0(D_1) \to \bigoplus_{x \in X} \mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x \to \mathrm{H}^1(D_1/D_2) \to 0, \tag{3}$$

where $\mathrm{H}^1(D_1/D_2) := \ker\left(\mathrm{H}^1(D_1) \to \mathrm{H}^1(D_2)\right)$, as well as the fact that $\mathrm{H}^1(D_1) \to \mathrm{H}^1(D_2)$ is surjective. Since for each $x \in X$, the ring $\mathcal{O}_{X,x}$ is a DVR, it follows that the vector space $\mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x$ has dimension $D_2(x) - D_1(x)$ over k: indeed if $t \in K$ is a uniformizer at x, then $\mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x$ has basis given by the classes of $t^{-D_1(x)-1}, t^{-D_1(x)-2}, \ldots, t^{-D_2(x)}$. Since $D_1(x) = D_2(x) = 0$ for all but finitely many $x \in X$, the middle term in (3) is finite-dimensional over k, whence so are the left and right terms, and

$$\deg D_2 - \deg D_1 = \dim_k \left(\mathrm{H}^0(D_2)/\mathrm{H}^0(D_1)\right) + \dim_k \mathrm{H}^1(D_1/D_2). \tag{4}$$

If we assume, as guaranteed by Theorem 2.1 that $h^i(D_j) < \infty$ for $i = 0, 1$ and $j = 1, 2$, then

$$\dim_k \left(\mathrm{H}^0(D_2)/\mathrm{H}^0(D_1)\right) = h^0(D_2) - h^0(D_1) \text{ and } \dim_k \mathrm{H}^1(D_1/D_2) = h^1(D_1) - h^1(D_2),$$

where we are using in the last step that $\mathrm{H}^1(D_1) \twoheadrightarrow \mathrm{H}^1(D_2)$. It then follows from (4) after some rearrangement that for any divisors $D_1, D_2 \in \mathrm{Div}(X)$ with $D_1 \leq D_2$, we have

$$h^0(D_1) - h^1(D_1) - \deg D_1 = h^0(D_2) - h^1(D_2) - \deg D_2.$$

Since the partial order $\leq$ on $\mathrm{Div}(X)$ is directed, it follows that the quantity $h^0(D) - h^1(D) - \deg D$ is a constant independent of $D \in \mathrm{Div}(X)$, and this constant can be evaluated by taking $D = 0$ to be

$$h^0(0) - h^1(0) - \deg 0 = 1 - g,$$

where $h^0(0) = 1$ by projectivity of X (see [3, Theorem I.3.4]), $h^1(0) = g$ by definition, and $\deg 0 = 0$. ■

We are ready to prove Theorem 2.1, following [1, Propositions V.3.16 and VI.2.7].

Proof. The proof of Theorem 2.3 suggests a proof strategy for Theorem 2.1. Indeed, the sequence (3) along with the directedness of the partial order $\geq$ on $\mathrm{Div}(X)$ tells us that to show that $h^0(D) < \infty$ for all $D \in \mathrm{Div}(X)$, it suffices to exhibit one $D_0 \in \mathrm{Div}(X)$ such that $h^0(D_0) < \infty$ (and similarly also for h^1). Since $h^0(0) = 1$ by projectivity of X, we are done for h^0. For h^1, we divide the proof into four steps. In steps (a) through (c) below, we will fix a function $f \in K \smallsetminus k$, and let $D := \mathrm{div}_\infty(f)$ be its divisor of poles and $d := \deg f = \deg D$.

(a) For any divisor $E \in \mathrm{Div}(X)$, there is an integer $m \geq 1$ and a $0 \neq g \in k[f] \subset K$ such that

$$E - \mathrm{div}(g) \leq mD.$$

To show this, let $x_1, \ldots, x_n$ be the (closed) points of X such that both $E(x_i) \geq 1$ and $v_{x_i}(f) \geq 0$ for each $i = 1, \ldots, n$. Then taking $g := \prod_{i=1}^n (f - f(x_i))^{E(x_i)}$ suffices, where, for each i, the number $f(x_i) = [f] \in \mathcal{O}_{X,x_i}/\mathfrak{m}_{X,x_i} \cong k$ is the result of evaluating f at x_i, the element $g \in K$ is nonzero because $f \notin k$, and $\mathrm{div}_\infty(g) = N \cdot D$ where $N = \sum_{i=1}^n E(x_i)$.

(b) There is an integer $m_0 \geq 1$ such that for all $m \geq m_0$, we have

$$h^0(mD) \geq (m - m_0 + 1)d.$$

In particular, there is an integer $M \geq 0$ such that for all $m \geq 1$ we have

$$md - h^0(mD) \leq M.$$

The second statement follows from the first by taking $M = (m_0 - 1)d$. To show the first, note that since $d = [K : k(f)]$, we may find $h_1, \ldots, h_d \in K$ that are linearly independent over $k(f)$. For each $j = 1, \ldots, d$, applying (a) to $E = -\mathrm{div}(h_j)$ yields an integer $m_j \geq 1$ and a $0 \neq g_j \in k[f] \subset K$ such that $g_j h_j \in \mathrm{H}^0(m_j D)$. Let $m_0 := \max_{i=1}^d \{m_i\}$. If $m \geq m_0$, then for each $i = 0, \ldots, m - m_0$ and $j = 1, \ldots, d$, we have $f^i g_j h_j \in \mathrm{H}^0(mD)$, where we are using that $f \in \mathrm{H}^0(D)$. Finally, these $(m - m_0 + 1)d$ functions $(f^i g_j h_j)_{i,j}$ are linearly independent over k, because the h_j are linearly independent over $k(f)$, each g_j is nonzero, and f is transcendental over k (this uses that k is algebraically closed).

(c) The same bound M in (b) works not just for all nonnegative multiples of D, but in fact all $E \in \mathrm{Div}(X)$, i.e. for all $E \in \mathrm{Div}(X)$, we have

$$\deg E - h^0(E) \leq M.$$

Indeed, given any E, let m and g be as in (1). If we let $F := E - \mathrm{div}(g)$, then $\deg E = \deg F$ because $\deg \mathrm{div}(g) = 0$. Also, multiplication by g gives the sheaf isomorphism $\mathcal{O}_X(E) \cong \mathcal{O}_X(F)$, so that $h^0(E) = h^0(F)$, and hence $\deg E - h^0(E) = \deg F - h^0(F)$, but this time $F \leq mD$. It follows from (4) and the finite-dimensionality of h^0, already proven, that for any divisors $D_1, D_2 \in \mathrm{Div}(X)$ with $D_1 \leq D_2$, we have

$$h^1(D_1/D_2) := \dim_k H^1(D_1/D_2) = (\deg D_2 - h^0(D_2)) - (\deg D_1 - h^0(D_1)) \geq 0. \tag{5}$$

Taking $D_1 := F$ and $D_2 := mD$ in (5) then yields

$$\deg F - h^0(F) = md - h^0(mD) - h^1(F/mD) \leq M.$$

(d) Step (c) tells us that there is some $D_0 \in \mathrm{Div}(X)$ such that $\deg D_0 - h^0(D_0)$ is maximal. To finish the proof, it suffices to show that $h^1(D_0) = 0$, for which it suffices to show that for any divisor $D \in \mathrm{Div}(X)$, we have

$$\mathrm{H}^1(D) = \bigcup_{D' \geq D} \mathrm{H}^1(D/D'). \tag{6}$$

Indeed, it would follow that if $h^1(D_0) \neq 0$, then there is a nonzero $\alpha \in \mathrm{H}^1(D_0)$; then by (6), there would be a $D' \geq D_0$ such that $\alpha \in \mathrm{H}^1(D_0/D')$, whence $h^1(D_0/D') \geq 1$, and then (5) would contradict our choice of D_0. To show (6), in light of (1) and (2), it suffices to show that given finitely many points $x_1, \ldots, x_n \in X$ and $f_1, \ldots, f_n \in K$, there is a divisor $D' \geq D$ such that for all $i = 1, \ldots, n$, we have $f_i \in \mathcal{O}_X(D')_{x_i}$, which is clear.

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Incidentally, this proof illustrates that showing the analogous result about finite dimensionality (and then Riemann-Roch) for a compact Riemann surface X is directly equivalent to producing a nonconstant global meromorphic function on X.

3 Serre Duality

The power of the Riemann-Roch Theorem comes from being able to identify the "correction term" h^1 explicitly as the h^0 of a different divisor. This is enabled by Serre Duality. For our treatment of Serre Duality, we will need a more explicit description of the group $\mathrm{H}^1(D)$ for a $D \in \mathrm{Div}(X)$–such as is afforded by the adelic interpretation–as well as machinery to deal with differentials on curves. Let us develop both tools now. In this section, we closely follow the argument used in the proof of [2, Chapter II, Theorem 2].

3.1 Adele Ring

We first define and discuss the basic properties of, the adele ring $\mathbb{A}_X$ associated to a curve X.

Definition 3.1. The adele ring of X is the restricted direct product ring

$$\mathbb{A}_X := \prod_{x \in X} (K, \mathcal{O}_{X,x}).$$

In other words, an element of $\mathbb{A}_X$ is a family $r = (r_x)_{x \in X}$ of elements of K indexed by the closed points $x \in X$ such that for all but finitely many $x \in X$ we have $r_x \in \mathcal{O}_{X,x}$.

Remark 3.2. Conventions differ on what this ring should be called–older texts like Serre's [2] call this the algebra of répartitions of X,[1] and most sources define the adele ring to be the restricted direct products of the completions, i.e. $\prod_{x \in X}(\hat{K}_x, \hat{\mathcal{O}}_{X,x})$; we will stick to the convention in Definition 3.1.

The diagonal embedding $K \hookrightarrow \mathbb{A}_X$ makes $\mathbb{A}_X$ a K-algebra. For each divisor $D \in \mathrm{Div}(X)$, we define the k-subspace

$$\mathbb{A}_X(D) := \{r \in \mathbb{A}_X : \text{for all } x \in X, \text{ we have } v_x(r_x) + D(x) \geq 0\} \subset \mathbb{A}_X.$$

Some basic properties of these spaces are listed in the following lemma, the proof of which is clear.

Lemma 3.3.
(a) For divisors $D_1, D_2 \in \mathrm{Div}(X)$, we have
  (i) $D_1 \leq D_2$ iff $\mathbb{A}_X(D_1) \subseteq \mathbb{A}_X(D_2)$,
  (ii) $\mathbb{A}_X(D_1) \cap \mathbb{A}_X(D_2) = \mathbb{A}_X(\min\{D_1, D_2\})$, and
  (iii) $\mathbb{A}_X(D_1) \cdot \mathbb{A}_X(D_2) = \mathbb{A}_X(D + E)$.
(b) We have $\bigcap_{D \in \mathrm{Div}(X)} \mathbb{A}_X(D) = 0$, while $\bigcup_{D \in \mathrm{Div}(X)} \mathbb{A}_X(D) = \mathbb{A}_X$.
(c) For any $D \in \mathrm{Div}(X)$, we have $\mathbb{A}_X(D) \cap K = \mathrm{H}^0(D)$ and

$$\mathrm{H}^0(\underline{K}/\mathcal{O}_X(D)) \cong \bigoplus_{x \in X} K/\mathcal{O}_X(D)_x \cong \mathbb{A}_X/\mathbb{A}_X(D),$$

where the last isomorphisms are compatible with the diagonal embedding of K into these. In particular, from (2), we obtain the isomorphism

$$\frac{\mathbb{A}_X}{\mathbb{A}_X(D) + K} \cong \mathrm{H}^1(D).$$

It follows from Lemma 3.3 that the subsets of $\mathbb{A}_X$ of the form $\mathbb{A}_X(D)$ for $D \in \mathrm{Div}(X)$ form the base for a Hausdorff topology on $\mathbb{A}_X$, called the Tate topology. This topology turns $\mathbb{A}_X$ into a topological ring: if $\pi : \mathbb{A}_X \times \mathbb{A}_X \to \mathbb{A}_X$ is the multiplication map, then for any divisor $D \in \mathrm{Div}(X)$, we have

$$\pi^{-1}\mathbb{A}_X(D) = \bigcup_{D_1 + D_2 = D} \mathbb{A}_X(D_1) \times \mathbb{A}_X(D_2).$$

We then give the quotient ring $\mathbb{A}_X/K$ the quotient topology coming from the natural surjection $\pi : \mathbb{A}_X \to \mathbb{A}_X/K$. If we give the base field k the discrete topology, $\mathbb{A}_X/K$ is a topological vector space over k, and we denote the topological dual of this space by

$$J := (\mathbb{A}_X/K)^\vee.$$

In other words, J is the set of all k-linear maps $\alpha : \mathbb{A}_X \to k$ such that $\ker \alpha \supseteq \mathbb{A}_X(D) + K$ for some divisor D. Then we can give J the structure of a vector space over K by defining for $f \in K$ and $\alpha \in J$, the element $f\alpha \in J$ by

$$\langle f\alpha, r \rangle := \langle \alpha, fr \rangle.$$

[1] This explains our choice for the letter r for an element of $\mathbb{A}_X$.

Indeed, this is in J because if $f \in \mathrm{H}^0(D)$ and $\ker \alpha \supseteq \mathbb{A}_X(D') + K$, then

$$\ker(f\alpha) \supseteq \mathbb{A}_X(D' - D) + K.$$

To phrase the above construction slightly differently, we denote for each $D \in \mathrm{Div}(X)$, the (ordinary) linear dual of $\mathrm{H}^1(D)$ by

$$J(D) := \mathrm{H}^1(D)^\vee \cong \{\alpha : \ker \alpha \supseteq \mathbb{A}_X(D) + K\} \subset \mathrm{Hom}_k(\mathbb{A}_X, k).$$

Then $\dim_k J(D) = h^1(D)$ and

$$J = \bigcup_{D \in \mathrm{Div}(X)} J(D),$$

where the union is taken in $\mathrm{Hom}_k(\mathbb{A}_X, k)$. The K-vector space structure on J comes from the natural one on $\mathrm{Hom}_k(\mathbb{A}_X, k)$ and the observation that for any divisors $D, D' \in \mathrm{Div}(X)$, we have

$$\text{if } f \in \mathrm{H}^0(D) \text{ and } \alpha \in J(D'), \text{ then } f\alpha \in J(D' - D). \tag{7}$$

In fact, we can say a lot about J as a K-vector space.

Theorem 3.4. We have $\dim_K J \leq 1$.

Proof. Suppose not, so that we can find $\alpha, \beta \in J$ that are linearly independent over K. Let D be a divisor such that $\alpha, \beta \in J(D)$; then for any $x \in X$ and integer $n \geq 0$, the map

$$\mathrm{H}^0(nx) \oplus \mathrm{H}^0(nx) \to J(D - nx), \quad (f, g) \mapsto f\alpha + g\beta$$

is injective, so that Theorem 2.3 gives us

$$n + (g - \deg D - 1) + h^0(D - nx) = h^1(D - nx) \geq 2h^0(nx) \geq 2(n - g + 1).$$

However, for $n > \deg D$, we have $h^0(D - nx) = 0$, so this inequality is false for $n \gg 0$. ■

At this point, to show $\dim_K J = 1$, all we need to do is exhibit a nonzero element of J; in stead of doing this, we will exhibit J later as the dual to the K-vector space $\Omega_{K/k}$ of Kähler differentials of K over k, which we now discuss.

3.2 Differentials and the Main Proof

For each $x \in X$ (and also for the generic point $x = \eta$), we let

$$\Omega_x := \Omega_{\mathcal{O}_{X,x}/k}$$

be the $\mathcal{O}_{X,x}$-module of Kähler differentials of the local ring $\mathcal{O}_{X,x}$ over the base field k, so that the Ω_x glue together to give us a locally free sheaf Ω on X, called the canonical sheaf of X, which has generic stalk $\Omega_\eta =: \Omega_{K/k}$ the K-vector space of meromorphic differentials on k. Each Ω_x is a free $\mathcal{O}_{X,x}$-module of rank 1; if $x \in X$ is a closed point, then a basis of Ω_x over $\mathcal{O}_{X,x}$ is given by taking $\mathrm{d}t$ for any uniformizer t at x, which then also serves as basis for $\Omega_{K/k}$ over K.[2]

To each $x \in X$ and $\omega \in \Omega_{K/k}$, we may associate the valuation of ω at x, denoted $v_x(\omega)$, as follows. Pick a uniformizer t at x; by the above discussion, $\mathrm{d}t$ is a K-basis for $\Omega_{K/k}$, and so we may write $\omega = f\mathrm{d}t$ for some unique $f \in K$, denoted $f := \omega/\mathrm{d}t$. We define

$$v_x(\omega) := v_x(f).$$

The first result we need is

Theorem 3.5.

(a) The valuation of a nonzero meromorphic differential at a point is well-defined, i.e. if $x \in X$ and $\omega \in \Omega_{K/k}$, if $t, u \in K$ are any two uniformizers at x, and we write $\omega = f\,\mathrm{d}t = g\,\mathrm{d}u$ for some $f, g \in K$, then we have

$$v_x(f) = v_x(g).$$

(b) Given any nonzero $\omega \in \Omega_{K/k}$, we have $v_x(\omega) = 0$ for all but finitely many $x \in X$.

[2] Indeed, since t is a uniformizer, it follows that $\{t\}$ is a separating transcendence basis of K/k. This is automatic in characteristic 0, and in characteristic $p > 0$ it uses that $t \notin K^p$ and [7, Exercise 2.15]. See also [3, II.8].

We will prove Theorem 3.5 in §3.3. It follows from it that for any $x \in X$, we can also describe Ω_x as

$$\Omega_x = \{\omega \in \Omega_{K/k} : v_x(\omega) \geq 0\}$$

and for a nonempty open subset $U \subset X$, the space of sections $\Omega(U)$ as

$$\Omega(U) = \bigcap_{x \in U} \Omega_x \subset \Omega_{K/k}.$$

Theorem 3.5 tells us that given a nonzero $\omega \in \Omega_{K/k}$, we may define the **divisor** associated to ω to be

$$\operatorname{div}(\omega) := \sum_{x \in X} v_x(\omega) x \in \operatorname{Div}(X).$$

Then we have for any nonzero $f \in K$ that

$$\operatorname{div}(f\omega) = \operatorname{div}(f) + \operatorname{div}(\omega).$$

A divisor of the form ω for a nonzero $\omega \in \Omega_{K/k}$ is called a **canonical divisor**; since $\Omega_{K/k}$ is a K-vector space of rank 1, it follows that for any two nonzero $\omega_1, \omega_2 \in \Omega_{K/k}$, the difference $\operatorname{div}(\omega_1) - \operatorname{div}(\omega_2)$ is principal, so that any two canonical divisors are linearly equivalent. The class of any canonical divisor in the Picard group $\operatorname{Pic}(X) = \operatorname{Div}(X)/K^*$ is called the **canonical class of** X, and is often written as ω_X or K_X.[3] Of course, under the isomorphism $\operatorname{Pic}(X) \cong \mathrm{H}^1(X, \mathcal{O}_X^\times)$, the canonical class ω_X is the class of the invertible sheaf Ω. We will also use ω_X to denote any canonical divisor on X; any assertion involving ω_X will then be be independent of the choice of canonical divisor, as usual.

For any divisor $D \in \operatorname{Div}(X)$, the sheaf $\Omega(D) := \Omega \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ then has stalk

$$\Omega(D)_x := \{\omega \in \Omega_{K/k} : v_x(\omega) + D(x) \geq 0\}$$

for any $x \in X$, and again for a nonempty open subset $U \subset X$, we have

$$\Omega(D)(U) = \bigcap_{x \in U} \Omega(D)_x \subset \Omega_{K/k}.$$

For each divisor $D \in \operatorname{Div}(X)$, we will let

$$\Omega^0(D) := \mathrm{H}^0(X, \Omega(D)).$$

Clearly, for any nonzero $\omega \in \Omega_{K/k}$ and divisor D, we have a sheaf isomorphism $\Omega(D) \xrightarrow{\sim} \mathcal{O}_X(\operatorname{div}(\omega) + D)$ given by sending $f\omega \mapsto f$, whence we get an isomorphism

$$\Omega^0(D) \cong \mathrm{H}^0(\omega_X + D).$$

The Serre Duality Theorem (Theorem 3.8) then asserts that for each divisor D, we have an isomorphism $\Omega^0(-D) \to J(D)$, whence an equality of dimensions $h^1(D) = h^0(\omega_X - D)$.

To show Serre Duality, we need one more notion. Suppose again that we have a closed point $x \in X$ and a $\omega \in \Omega_{K/k}$, and we have written $\omega = f\mathrm{d}t$ for some uniformizer t at x and $f \in K$. Since $\hat{K}_x \cong k(\!(t)\!)$, we can expand $f \in K \subset \hat{K}_x$ uniquely as

$$f = \sum_{n \gg -\infty} a_n t^n$$

for $a_n \in k$, where the notation $n \gg -\infty$ means that $a_n = 0$ for all but finitely many negative n. Then we define the residue of ω to at x to be

$$\operatorname{Res}_x \omega := a_{-1}.$$

Again, we will need

Theorem 3.6 (Invariance of Residue). The residue of a meromorphic differential at a point is well-defined: if $x \in X$ and $\omega \in \Omega_{K/k}$, if $t, u \in K$ are uniformizers at x, and

$$\omega = \sum_{n \gg -\infty} a_n t^n \mathrm{d}t = \sum_{m \gg -\infty} b_m u^m \mathrm{d}u,$$

then $a_{-1} = b_{-1}$.

[3] We will avoid the latter terminology to prevent confusion with the function field K.

We will prove this result, which is slightly harder, in §3.4. Note that in this notation, $v_x(\omega)$ is the smallest integer n such that $a_n \neq 0$, so Theorem 3.5 can also be seen as a result about this power series expansion of f. Note finally also that if $v_x(\omega) \geq 0$, then $\operatorname{Res}_x \omega = 0$, whence, by Theorem 3.5(b), the quantity

$$\sum_{x \in X} \operatorname{Res}_x \omega$$

is well-defined. The final ingredient in the soup is then

Theorem 3.7 (Residue Theorem)**.** For any $\omega \in \Omega_{K/k}$, we have

$$\sum_{x \in X} \operatorname{Res}_x \omega = 0.$$

This can be proved using transcendental methods in characteristic zero (use the Lefschetz principle to reduce to $k = \mathbb{C}$, where we can check that $\operatorname{Res}_x \omega = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \omega$, where γ is a small counterclockwise loop around x, and then finish with Stokes' Theorem); but a purely algebraic proof can also be given, and we do this in the final section §3.5. Given these ingredients, we are now ready to prove

Theorem 3.8 (Serre Duality)**.** For any divisor $D \in \operatorname{Div}(X)$, there is an isomorphism

$$\theta_D : \Omega^0(-D) \to J(D).$$

In particular,

$$h^1(D) = h^0(\omega_X - D).$$

Proof. Consider the k-bilinear pairing

$$\Omega_{K/k} \times \mathbb{A}_X \to k, \quad (\omega, r) \mapsto \langle \omega, r \rangle := \sum_{x \in X} \operatorname{Res}_x(r_x \omega).$$

Now we make three observations:
(i) For any $f \in K, \omega \in \Omega_{K/k}$ and $r \in \mathbb{A}_X$, we have $\langle f\omega, r \rangle = \langle \omega, fr \rangle$.
(ii) From Theorem 3.7, it follows that for any $f \in K \subset \mathbb{A}_X$ and $\omega \in \Omega_{K/k}$ we have that $\langle \omega, f \rangle = 0$.
(iii) For any $D \in \operatorname{Div}(X)$, if $\omega \in \Omega^0(-D)$ and $r \in \mathbb{A}_X(D)$, then $\langle \omega, r \rangle = 0$.
It follows from these observations that the pairing above induces a K-linear map

$$\theta : \Omega_{K/k} \to J, \quad \omega \mapsto \langle \omega, \cdot \rangle,$$

which for any divisor $D \in \operatorname{Div}(X)$ takes $\Omega^0(-D) \subset \Omega_{K/k}$ to $J(D)$; denote the restriction of θ to $\Omega^0(-D)$ by θ_D. We claim that this θ_D is the required isomorphism, which we prove in two steps:
(a) For any $D \in \operatorname{Div}(X)$, we have
$$\theta^{-1} J(D) = \Omega^0(-D).$$
Indeed, the inclusion $\Omega^0(-D) \subseteq \theta^{-1} J(D)$ is clear from the above discussion; to show the opposite inclusion, suppose that $\omega \notin \Omega^0(-D)$. It follows that there is an $x \in X$ such that $v_x(\omega) < D(x)$. Pick a uniformizer t at x, and consider the element $r \in \mathbb{A}_X(D)$ given by $r_y = 0$ for $y \neq x$ and $r_x = t^{-v_x(\omega)-1}$. Then $v_x(r_x\omega) = -1$, whence $\theta(\omega)(x) = \operatorname{Res}_x(r_x\omega) \neq 0$, showing $\theta(\omega) \notin J(D)$.
(b) Now we finish the proof. To show the injectivity of θ_D, it suffices to show that θ is injective, and indeed if $\omega \in \Omega_{K/k}$ is such that $\theta(\omega) = 0$, then by (a) we have that
$$\omega \in \bigcap_{D \in \operatorname{Div}(X)} \theta \in J(D) = \bigcap_{D \in \operatorname{Div}(X)} \Omega^0(-D) = \{0\}.$$
To show θ_D is surjective, it again suffices to show that θ is surjective: indeed, then if we pick an $\alpha \in J(D)$, and we find an $\omega \in \Omega_{K/k}$ such that $\theta(\omega) = \alpha$, then by (a) we get that $\omega \in \Omega^0(-D)$. But, finally, the surjectivity of θ follows from the fact that it is an injective K-linear map from a nonzero K-vector space $\Omega_{K/k}$ to the K-vector space J which, by Theorem 3.4, has dimension at most 1.

■

In fact, the above proof shows that the map $\theta : \Omega_{K/k} \to J = (\mathbb{A}_X/K)^\times$ is an isomorphism, giving us a different perspective on meromorphic differentials and adèles. Let's now fill in all the missing details.

3.3 Invariance of Valuation

The purpose of the next three subsections is to fill in the gaps in the proofs from the previous section. For this, we need to examine the DVR $\mathcal{O}_{X,x}$ for a closed $x \in X$, its fraction field K, and their x-adic completions $\hat{\mathcal{O}}_{X,x} \subset \hat{K}_x$ more carefully.

For the proofs of Theorems 3.5 and 3.6, we will fix an $x \in X$. If $t \in K$ is any uniformizer at x, then t is transcendental over k and for each $n \geq 0$, the natural map

$$k[t]/(t^n) \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^n$$

is easily seen to be injective, and hence surjective for dimension reasons. In particular, we get an isomorphism of completions

$$k[\![t]\!] \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x},$$

the inverse of which is given by "expansion in powers of t". Let us denote this inverse map by $\psi_t : \hat{\mathcal{O}}_{X,x} \to k[\![t]\!]$. Then ψ_t extends to an isomorphism of the corresponding fraction fields

$$\psi_t : \hat{K}_x \to k(\!(t)\!).$$

The first observation to be made here is then

Lemma 3.9. Let x, t and ψ_t be as above.
(a) For any $f \in K \subset \hat{K}_x$, we have $f \in \mathcal{O}_{X,x}$ iff $\psi_t(f) \in k[\![t]\!]$.
(b) For any $f \in \mathcal{O}_{X,x}$, we have $\mathrm{d}f/\mathrm{d}t \in \mathcal{O}_{X,x}$.

Recall that $\mathrm{d}f/\mathrm{d}t$ is by definition the unique element of K such that $\mathrm{d}f = (\mathrm{d}f/\mathrm{d}t) \cdot \mathrm{d}t \in \Omega_{K/k}$, which exists because $\Omega_{K/k}$ has K-basis $\mathrm{d}t$.

Proof.
(a) Clear because the map ψ_t preserves the valuation on both sides by construction, i.e. the valuation v_x on $\hat{K}_x$ coincides under ψ_t with the t-adic valuation in $k(\!(t)\!)$.
(b) On the one hand, the map $K \to k(\!(t)\!)$ given by $f \mapsto \psi_t(\mathrm{d}f/\mathrm{d}t)$ is a k-derivation taking $t \mapsto 1$. On the other hand, if we consider the k-derivation ∂_t of $k(\!(t)\!)$ defined by

$$\partial_t : \sum_{n \gg -\infty} a_n t^n \mapsto \sum_{n \gg -\infty} n a_n t^{n-1},$$

then map $f \mapsto \partial_t \psi_t(f)$ is *also* a k-derivation $K \to k(\!(t)\!)$ that takes $t \mapsto 1$. Since $\Omega_{K/k} = K \cdot \mathrm{d}t$, it follows from the universal property defining $\Omega_{K/k}$ that we must have

$$\psi_t\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = \partial_t \psi_t(f) \tag{8}$$

for any $f \in K$. In particular, if $f \in \mathcal{O}_{X,x}$, then by (a) we have $\psi_t(f) \in k[\![t]\!]$, from which it follows that $\partial_t \psi_t(f) \in k[\![t]\!]$, from which it follows by part (a) and Equation (8) that $\mathrm{d}f/\mathrm{d}t \in \mathcal{O}_{X,x}$ as needed.

■

Let us now proceed to the proof of Theorem 3.5.

Theorem 3.5.
(a) The valuation of a nonzero meromorphic differential at a point is well-defined, i.e. if $x \in X$ and $\omega \in \Omega_{K/k}$, if $t, u \in K$ are any two uniformizers at x, and we write $\omega = f\,\mathrm{d}t = g\,\mathrm{d}u$ for some $f, g \in K$, then we have
$$v_x(f) = v_x(g).$$
(b) Given any nonzero $\omega \in \Omega_{K/k}$, we have $v_x(\omega) = 0$ for all but finitely many $x \in X$.

Proof.
(a) Write $u = ct$ for some $c \in \mathcal{O}_{X,x}^\times$ to get

$$f\mathrm{d}t = g\,\mathrm{d}u = g\left(t\frac{\mathrm{d}c}{\mathrm{d}t} + c\right)\mathrm{d}t,$$

and so

$$f = g\left(t\frac{\mathrm{d}c}{\mathrm{d}t} + c\right).$$

Since $\mathrm{d}c/\mathrm{d}t \in \mathcal{O}_{X,x}$ by Lemma 3.9(b) and $c \in \mathcal{O}_{X,x}^\times$, it follows that $t(\mathrm{d}c/\mathrm{d}t) + c \in \mathcal{O}_{X,x}^\times$ as well, from which it follows that $v_x(f) = v_x(g)$ as needed.

(b) Note that if $t \in K$ is a uniformizer at $x \in K$, then there is an open subset V in X such that for all $y \in V$ the function $t - t(y)$ is a uniformizer at t. Indeed, if $t : X \to \mathbb{P}^1$ denotes the corresponding map to $\mathbb{P}^1$, then V is the locus where t is unramified, i.e. the complement of the support of the relative cotangent sheaf $\Omega_{X/\mathbb{P}^1}$, which is nonempty since it contains x. Since $\mathrm{d}t = \mathrm{d}(t - t(y))$, the result follows immediately from the corresponding result for nonzero $f \in K$.

■

3.4 Invariance of Residue

Let's now move to Theorem 3.6. For this, fix as before a point $x \in X$. For convenience, let $\mathcal{O}, \mathfrak{m}, \hat{\mathcal{O}}, \hat{\mathfrak{m}}, \hat{K}$ denote $\mathcal{O}_{X,x}, \mathfrak{m}_{X,x}, \hat{\mathcal{O}}_{X,x}, \hat{\mathfrak{m}}_{X,x}, \hat{K}_x$ respectively. For the invariance theorem, we need to analyze the differentials of $\hat{K}$, but it turns out that $\Omega_{\hat{K}/k}$ is too large for this purpose. The right thing to look at in this case is the quotient $\hat{\Omega}$.

Definition 3.10. In the above setting, let $\hat{\Omega} := \hat{\Omega}_x := \Omega_{\hat{K}/k} / \bigcap_{N \geq 0} \hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$.

With this, we can now differentiate term-by-term; this we really did already in Lemma 3.9, but we did not have a coordinate-invariant notion of a differential in the completed setting there.

Lemma 3.11. Let t be a uniformizer, and let $\psi_t : \hat{K} \xrightarrow{\sim} k(\!(t)\!)$ be the isomorphism above. Then for any $f \in \hat{K}$, we have in $\hat{\Omega}$ the identity

$$\mathrm{d}f = f'_t \mathrm{d}t,$$

where $f'_t := \psi_t^{-1} \partial_t \psi_t$. In particular, $\hat{\Omega}$ is a 1-dimensional $\hat{K}$-vector space spanned by $\mathrm{d}t$.

In the following, we will make the identification ψ_t implicitly to simplify the notation. Then, given any $f \in K$, if $f = \sum_{n \gg -\infty} a_n t^n$, then $f'_t = \sum_{n \gg -\infty} n a_n t^{n-1}$. This equality $\mathrm{d}f = f'_t \mathrm{d}t$ clearly holds (already in $\Omega_{\hat{K}/k}$) if $a_n = 0$ for all but finitely many n; the point of this lemma is to show that it remains true in general in $\hat{\Omega}$.

Proof. We have to show that $\mathrm{d}f - f'_t \mathrm{d}t \in \hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$ for each $N \geq 0$. For this, fix an N and write $f = g + t^N h$, where $g = \sum_{-\infty \ll n < N} a_n t^n$ and $h \in \hat{\mathcal{O}}$. A straightforward computation shows that

$$\mathrm{d}f - f'_t \mathrm{d}t = (\mathrm{d}g - g'_t \mathrm{d}t) + t^N (\mathrm{d}h - h'_t \mathrm{d}t).$$

Since, as observed before the proof, g is Laurent polynomial in t and hence $\mathrm{d}g - g'_t \mathrm{d}t = 0$, this shows that $\mathrm{d}f - f'_t \mathrm{d}t \in \hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$.

This shows that $\hat{\Omega}$ is spanned by $\mathrm{d}t$; it only remains to show that $\hat{\Omega}$ is not zero, and for this it suffices to show that there is a k-derivation $D : \hat{K} \to \hat{K}$ that is not identically zero such that its extension to $\Omega_{\hat{K}/k} \to \hat{K}$ vanishes identically on $\bigcap \hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$. But now the map D given by $Df := f'_t$ is such a map. Indeed, this is a k-derivation taking $t \mapsto 1$, such that the extension to $\Omega_{\hat{K}/k}$ takes $\mathrm{d}\hat{\mathcal{O}}$ to $\hat{\mathcal{O}}$. It follows that the extension takes $\hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$ to $\hat{\mathfrak{m}}^N$ for each $N \geq 0$, and hence $\bigcap \hat{\mathfrak{m}}^N \mathrm{d}\hat{\mathcal{O}}$ to $\bigcap \hat{\mathfrak{m}}^N = 0$, where in the last step we have used the Krull Intersection Theorem on the Noetherian local ring $\hat{\mathcal{O}}$.

■

From now on, by a differential on $\hat{K}$ we will mean, by default, an element of $\hat{\Omega}$, and use the K-embedding $\Omega_{K/k} \hookrightarrow \hat{\Omega}$. Given this, for any uniformizer t we can define a k-linear map

$$\mathrm{Res}_t : \hat{\Omega} \to k$$

given by writing any element $\omega \in \hat{\Omega}$ as $\omega = f \mathrm{d}t$ for some unique $f \in \hat{K}$ (using Lemma 3.11), expanding $\psi_t(f) = \sum_{n \gg -\infty} a_n t^n$ and then defining $\mathrm{Res}_t(\omega) := a_{-1}$. Here are some basic properties of this operation:

Lemma 3.12. In the above setting, fix a uniformizer t.

(a) The map $\mathrm{Res}_t : \hat{\Omega} \to k$ is k-linear, and $\mathrm{Res}_t(\omega) = 0$ if $\omega \in \Omega_{K/k}$ with $v(\omega) \geq 0$.

(b) For any $g \in \hat{K}$, we have $\mathrm{Res}_t(\mathrm{d}g) = 0$.

(c) For any $g \in K^\times$, we have

$$\mathrm{Res}_t\left(g^{-1} \mathrm{d}g\right) = v(g).$$

Proof. The statement (a) is clear, as is (b) thanks to Lemma 3.11, since the coefficient of t^{-1} in g'_t is zero. For (c), write $g = t^n u$, where $n = v(g)$ and $u \in \mathcal{O}^\times$; then

$$g^{-1}\mathrm{d}g = \left(nt^{-1} + u^{-1}\frac{\mathrm{d}u}{\mathrm{d}t}\right)\mathrm{d}t.$$

The result follows, since $u^{-1} \in \mathcal{O}^\times$ and $\mathrm{d}u/\mathrm{d}t \in \mathcal{O}$ (Lemma 3.9(b)) tells us along with part (a) that

$$\mathrm{Res}_t\left(u^{-1}\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t\right) = 0.$$

■

Remark 3.13. In fact, the function $v : \Omega_{K/k} \to \mathbb{Z}$ extends to a function $v : \hat{\Omega} \to \mathbb{Z}$, and then we have $\mathrm{Res}_t(\omega) = 0$ if $\omega \in \hat{\Omega}$ with $v(\omega) \geq 0$, slightly generalizing the statement in (a). Similarly, the statement in (c) holds for all $g \in \hat{K}^\times$. However, we do not need this in what follows, so this can be safely left as an exercise to the diligent reader.

We are now ready to prove the main result of this section.

Theorem 3.6 (Invariance of Residue)**.** The residue of a meromorphic differential at a point is well-defined: if $x \in X$ and $\omega \in \Omega_{K/k}$, if $t, u \in K$ are uniformizers at x, and

$$\omega = \sum_{n \gg -\infty} a_n t^n \mathrm{d}t = \sum_{m \gg -\infty} b_m u^m \mathrm{d}u,$$

then $a_{-1} = b_{-1}$.

Proof. Let $\omega_0 := \sum_{m \geq 0} b_m u^m \mathrm{d}u$, so that

$$\omega = \sum_{-\infty \ll m < 0} b_m u^m \mathrm{d}u + \omega_0$$

with $v(\omega_0) \geq 0$. Then $b_{-1} = \mathrm{Res}_u(\omega)$ and by Lemma 3.12(a) we have

$$a_{-1} = \mathrm{Res}_t(\omega) = \sum_{-\infty \ll m < 0} b_m \,\mathrm{Res}_t(u^m \mathrm{d}u).$$

Since $\mathrm{Res}_t(u^{-1}\mathrm{d}u) = 1$ by Lemma 3.12(c), it suffices to show that $\mathrm{Res}_t(u^{-n}\mathrm{d}u) = 0$ for $n \geq 2$. When $\mathrm{ch}\, k = 0$, we can use that

$$u^{-n}\mathrm{d}u = \mathrm{d}\left(\frac{u^{1-n}}{1-n}\right)$$

along with Lemma 3.12(b) to finish the proof. In positive characteristic, we may multiply u by a scalar to write u as

$$u = t(1 + \alpha_1 t + \alpha_2 t^2 + \cdots)$$

for some $\alpha_i \in k$. Then for each $n \geq 1$ we have

$$u^{-n} = t^{-n}(1 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \cdots),$$

where $\beta_j \in \mathbb{Z}[\alpha_1, \ldots, \alpha_j]$ for each $j \geq 1$ (so, for instance, $\beta_1 = -n\alpha_1$ and $\beta_2 = \binom{n+1}{2}\alpha_1^2 - n\alpha_2$). Next,

$$\mathrm{d}u = (1 + 2\alpha_1 t + 3\alpha_2 t^2 + \cdots)\mathrm{d}t$$

and hence

$$u^{-n}\mathrm{d}u = t^{-n}\mathrm{d}t \cdot (1 + \gamma_1 t + \gamma_2 t^2 + \cdots)$$

for some $\gamma_j \in \mathbb{Z}[\alpha_1, \ldots, \alpha_j]$ for each $j \geq 1$. Then

$$\mathrm{Res}_t(u^{-n}\mathrm{d}u) = \gamma_{n-1}.$$

Now, by what we have already shown, $\gamma_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) = 0$ whenever the α_i lie in a field of characteristic zero; this shows that the polynomial γ_{n-1} is identically zero, independent of the characteristic. ■

Briefly, in the last step, we have used what is called the "principle of prolongation of algebraic identities" to go from zero characteristic to positive characteristic.

3.5 The Residue Theorem

The final piece of the puzzle now remaining is the Residue Theorem; again we closely follow [2].

Theorem 3.7 (Residue Theorem)**.** For any $\omega \in \Omega_{K/k}$, we have

$$\sum_{x \in X} \operatorname{Res}_x \omega = 0.$$

The proof strategy will be to first show this for the curve $X = \mathbb{P}^1_k$ using explicit calculations, and then reduce to this case by studying morphisms $X \to \mathbb{P}^1_k$ for arbitrary X.

Lemma 3.14. If $X = \mathbb{P}^1_k$, then for any $\omega \in \Omega_{K/k}$ we have

$$\sum_{x \in X} \operatorname{Res}_x \omega = 0.$$

Proof. Let t be a local coordinate on $\mathbb{P}^1_k$, so that $K = k(t)$, and $\Omega_{K/t} = K\mathrm{d}t$. Write $\omega = f(t)\,\mathrm{d}t$ for $f(t) \in k(t)$. By expanding $f(t)$ in partial fractions (and using that $k = \overline{k}$), it suffices to show the result when $f(t) = t^n$ or $f(t) = (t-a)^{-n}$ for $n \geq 1$ and $a \in k$. In the first case, f has a unique pole at $t = \infty$; using the local coordinate $u = 1/t$ near this point, we see that $\omega = u^{-n-2}\mathrm{d}u$, and hence $\operatorname{Res}_\infty \omega = 0$. Similarly, if $f(t) = (t-a)^{-n}$ and $n = 1$, then there are poles at $t = a$ and $t = \infty$ with residues 1 and -1 respectively, and if $n \geq 2$, then there is a unique pole at $t = a$ with residue 0; in all cases, we are done. ∎

Now let X, Y be any curves, and $\varphi : X \to Y$ a nonconstant separable morphism of curves. If $K = k(X)$ and $L = k(Y)$ are the corresponding function fields, then the morphism φ expresses K/L as a finite separable extension, from which we get a corresponding isomorphism

$$K \otimes_L \Omega_{L/k} \xrightarrow{\sim} \Omega_{K/k}.$$

The trace map $\operatorname{Tr}^K_L : K \to L$ is L-linear, and therefore also gives rise to a corresponding trace map $\operatorname{Tr}^K_L \otimes \mathrm{id}_{\Omega_{L/k}} : \Omega_{K/k} \cong K \otimes_L \Omega_{L/k} \to L \otimes_L \Omega_{L/k} \cong \Omega_{L/k}$, which we will denote also by

$$\operatorname{Tr}^K_L : \Omega_{K/k} \to \Omega_{L/k}.$$

For instance, if $Y = \mathbb{P}^1_k$ with coordinate t, so that $\varphi = t$ considered as a function on X is a separating transcendence basis for K/k, and any $\omega \in \Omega_{K/k}$ can be written as $f\,\mathrm{d}t$ for some $f \in K$, then

$$\operatorname{Tr}^K_L(\omega) = \operatorname{Tr}^K_L(f\,\mathrm{d}t) = \operatorname{Tr}^K_L(f)\,\mathrm{d}t.$$

The last remaining step is.

Lemma 3.15. In the above set-up, for any $\omega \in \Omega_{K/k}$ and $y \in Y$ we have

$$\operatorname{Res}_y \operatorname{Tr}^K_L \omega = \sum_{x \in \varphi^{-1}(y)} \operatorname{Res}_x \omega.$$

We shall prove this lemma momentarily; first, we prove Theorem 3.7 assuming this result.

Proof of Theorem 3.7, assuming Lemma 3.15. Any curve X admits a nonconstant separable morphism to $Y = \mathbb{P}^1_k$ (e.g. take a uniformizer at a point $x \in X$ and consider it as a rational function on X); let $\varphi : X \to Y$ denote one such morphism, and consider the finite separable extension of function fields as above with $K = k(X)$ and $L = k(Y) \cong k(t)$. Now φ takes the generic point of X to that of Y, and hence takes closed points of X to closed points of Y, and the corresponding map on closed points is surjective. Therefore, dividing the sum into fibers, we conclude that for any $\omega \in \Omega_{K/k}$ we have

$$\sum_{x \in X} \operatorname{Res}_x \omega = \sum_{y \in Y} \sum_{x \in \varphi^{-1}(y)} \operatorname{Res}_x \omega.$$

By Lemma 3.15, this last sum can be written as

$$\sum_{y \in Y} \operatorname{Res}_y \operatorname{Tr}^K_L \omega,$$

and this is zero because of Lemma 3.14 applied to the form $\operatorname{Tr}^K_L \omega$. ∎

It remains only to prove Lemma 3.15.

Proof of Lemma 3.15. Fix a $y \in Y$, and let $\hat{L}_y$ denote the y-adic completion of L, i.e. the completion of L with respect to the valuation v_y. By the standard theory of extensions of Dedekind domains in finite separable extensions, we know that the valuations of K dividing v_y correspond bijectively to the points $x \in \varphi^{-1}(y)$ in the sense that for each $x \in \varphi^{-1}(y)$ there is an integer $e_x \geq 1$ such that $v_x|_L = e_x \cdot v_y$, the completion $\hat{K}_x$ is an extension of $\hat{L}_y$ of degree e_x, and there is an isomorphism

$$\hat{K}_y := K \otimes_L \hat{L}_y \cong \prod_{x \in \varphi \in (y)} \hat{K}_x,$$

whence $\sum_{x \in \varphi^{-1}(y)} e_x = [K : L]$.[4] From this, it follows that for any $f \in K$ we have

$$\mathrm{Tr}_L^K(f) = \mathrm{Tr}_{\hat{L}_y}^{\hat{K}_y}(f) = \sum_{x \in \varphi^{-1}(y)} \mathrm{Tr}_{\hat{L}_y}^{\hat{K}_x}(f).$$

Now if $u \in L$ is a separating transcendence basis over k, then $u \circ \varphi \in K$ is a separating transcendence basis as well, and the map

$$\mathrm{Tr}_L^K : \Omega_{K/k} \to \Omega_{L/k} \text{ takes } f\, \mathrm{d}(u \circ \varphi) \mapsto \mathrm{Tr}_L^K(f)\, \mathrm{d}u.$$

Therefore, to show the result, it suffices to show that for any $y \in Y$, any point $x \in \varphi^{-1}(y)$ over y, and any $f \in K$, we have

$$\mathrm{Res}_y \mathrm{Tr}_{\hat{L}_y}^{\hat{K}_x}(f)\, \mathrm{d}u = \mathrm{Res}_x f\, \mathrm{d}(u \circ \varphi),$$

where $\mathrm{Tr}_{\hat{L}_y}^{\hat{K}_x}(f) \in \hat{L}_y$, and we are considering $\mathrm{Tr}_{\hat{L}_y}^{\hat{K}_x}(f)\, \mathrm{d}t$ to be an element of $\hat{L}_y \otimes_L \Omega_{L/k} \cong \hat{\Omega}_y$.

Now picking a uniformizer u for Y at y and t for X at x reduces the problem to the following local computation[5]: if $\hat{L} = k(\!(u)\!)$ and $\hat{K}/\hat{L}$ is a finite separable extension with $\hat{K} = k(\!(t)\!)$, then for any $f \in \hat{K}$ we have to show that

$$\mathrm{Res}_t(f\, \mathrm{d}u) = \mathrm{Res}_u \left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(f)\, \mathrm{d}u\right).$$

Note that if $e := [\hat{K} : \hat{L}]$, then $e = v_{\hat{L}}(u)$ and the extension $\hat{K}/\hat{L}$ is totally ramified; by scaling u, we may assume that u is of the form

$$u = t^e + \sum_{j > e} \alpha_j t^j$$

for some $\alpha_j \in k$. In particular, $\mathrm{d}u = u_t' \mathrm{d}t$ with $v_t(u_t') \geq 0$. This allows us one further reduction: writing $f = g + h$ for a Laurent *polynomial* g in t and h with $v_t(h) \geq 0$, then $v_u\left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(h)\right) \geq 0$ as well, and so we see that

$$\mathrm{Res}_t(f\, \mathrm{d}u) = \mathrm{Res}_t(g\, \mathrm{d}u) + \mathrm{Res}_t(h u_t'\, \mathrm{d}t) = \mathrm{Res}_t(g\, \mathrm{d}u) + 0,$$

and

$$\mathrm{Res}_u \left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(f)\, \mathrm{d}u\right) = \mathrm{Res}_u \left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(g)\, \mathrm{d}u\right) + \mathrm{Res}_u \left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(h)\, \mathrm{d}u\right) = \mathrm{Res}_u \left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(g)\, \mathrm{d}u\right) + 0,$$

so it suffices to show the result for the Laurent polynomial g. By linearity, it suffices to show the result for the monomial $f = t^n$ for $n \in \mathbb{Z}$.

First suppose that $\mathrm{ch}\, k \nmid e$ (e.g. if $\mathrm{ch}\, k = 0$), so that the extension is tamely ramified; then there is a uniformizer t' of $\hat{K}$ such that $u = (t')^e$ (see [8, Chapter II, Proposition 12, p. 52-53]), and using Theorem 3.6 we may replace t by t' in the whole of the above discussion to assume that $u = t^e$ (i.e. $\alpha_j = 0$ for each $j > e$) and that $f = t^n$ for some $n \in \mathbb{Z}$, so that

$$\mathrm{d}u = e t^{e-1}\, \mathrm{d}t.$$

Then, on the one hand we have

$$\mathrm{Res}_t(f\, \mathrm{d}u) = \mathrm{Res}_t(e t^{n+e-1} \mathrm{d}t) = \begin{cases} e & \text{if } n = -e, \text{ and} \\ 0, & \text{else.} \end{cases}$$

On the other hand, $1, t, \ldots, t^{e-1}$ is a basis for $\hat{K}/\hat{L}$, and so it is easy to see that

$$\mathrm{Tr}_{\hat{L}}^{\hat{K}}(t^n) = \begin{cases} e u^{n/e}, & \text{if } e \mid n, \text{ and} \\ 0, & \text{else,} \end{cases}$$

[4] Here we are using that the base field k is algebraically closed, so there are no nontrivial extensions of the residue fields; in general, in Lemma 3.15, we need to weight each $\mathrm{Res}_x\, \omega$ by the degree $[\kappa(x) : \kappa(y)]$ of the corresponding residue fields.

[5] Here we are suppressing the letter φ for convenience.

and we are done. This finishes the proof in characteristic zero. In positive characteristic, we argue as in the proof of Theorem 3.6 as follows.

As before, $1, t, \ldots, t^{e-1}$ is a basis for $\hat{K}/\hat{L}$. For a fixed $n \in \mathbb{Z}$ and each p with $0 \leq p \leq e-1$, we can write

$$t^n t^p = \sum_{q=0}^{e-1} \beta_{n,p,q} t^q$$

for some $\beta_{n,p,q} \in k(\!(u)\!)$; if we expand $\beta_{n,p,q} = \sum_{-\infty \ll r} \beta_{n,p,q,r} u^r$ with $\beta_{n,p,q,r} \in \mathbb{Z}[\alpha_j]_{j>e}$, then

$$\mathrm{Tr}_{\hat{L}}^{\hat{K}}(t^n) = \sum_{p=0}^{e-1} \beta_{n,p,p}$$

and

$$\mathrm{Res}_u\left(\mathrm{Tr}_{\hat{L}}^{\hat{K}}(t^n)\,\mathrm{d}u\right) = \sum_{p=0}^{e-1} \beta_{n,p,p,-1}.$$

But now

$$\mathrm{d}u = \left(et^{e-1} + \sum_{j>e} j\alpha_j t^{j-1}\right)\mathrm{d}t$$

so that

$$\mathrm{Res}_t(t^n \mathrm{d}u) = -n\alpha_{-n}$$

with the convention that $\alpha_e = 1$ and $\alpha_j = 0$ if $j < e$. The required identity is then

$$-n\alpha_{-n} = \sum_{p=0}^{e-1} \beta_{n,p,p,-1}.$$

Therefore, the result follows as before from the "principle of prolongation of algebraic identities" applied to the polynomial

$$n\alpha_{-n} + \sum_{p=0}^{e-1} \beta_{n,p,p,-1} \in \mathbb{Z}[\alpha_j]_{j>e}.$$

Namely, this polynomial vanishes identically for all values of α_j in characteristic zero, and hence must be the zero polynomial. This finishes the proof of the lemma, and of all the results in this article.

■

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