Errata and Addenda to Harris's *Algebraic Geometry: A First Course*

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The following errata and addenda are labelled via triples (N, S, L), where N stands for page number, S for the minimal subsection (e.g. exercise, example, etc.), and L for the line number (or description of location). Negative line numbers indicated lines from the bottom of the page. Sometimes S or L are omitted if they do not add to the clarity of expressing the location. If the reader encounters any further errors in the book, or errors in this errata list, they are highly encouraged to (electronically) write to me at either this address or this one.

Lecture 1

- (9, 1.10, -3) "...be a two-dimensional subspace of those vanishing at r...". This statement is equivalent to saying $\ell \not\subset C$, which is a consequence, say, of Exercise 1.12.
- (12, 1.17, 4) "..., which may be any points not on the coordinate hyperplanes **such that the collection $\{\mathrm{e}_0, \ldots, \mathrm{e}_d, \mu, \nu\}$ is in linear general position, where e_i is the (class of the) i^{th} basis vector**". Indeed, this is exactly the condition that the $[\mu_i, \nu_i] \in \mathbb{P}^1$ are distinct.
- (13, 1.22, 1.22+1) "(and, on at least one occasion, useful)" This refers to the application of Steiner constructions to Castelnuovo's Lemma. See the section labelled Steiner Constructions in [1, §4.3, pp. 528 ff].
- (14, after 1.23, 8-9) "...such parametrizations, subject to the condition that for each λ the planes $H_1(\lambda), \ldots, H_d(\lambda)$ are independent." One way to do this is described at the bottom of [1, p. 529-530].

Lecture 2

- (23, 2.3, -) Since morphisms of objects that are not varieties haven't really been defined, perhaps the statement of this exercise can be clarified by writing something to the effect of: In other words, show that for $n \geq 2$ and any $N \geq 0$, there is no injective regular morphism $\mathbb{A}^n \smallsetminus \{0\} \to \mathbb{A}^N$ with image a subvariety of $\mathbb{A}^N$.
- (28, 2.19, -) It is perhaps worth emphasizing there this is only a **one-parameter family**; curves $C_{\alpha,\beta}$ and $C_{\alpha',\beta'}$ are projectively equivalent iff $\alpha\beta = \alpha'\beta'$.

Lecture 3

- (37, 3.8, 1-3) "If you're feeling energetic, ..." This amounts to noting that there are three orbits of the relevant PGL_2 action on $\mathbb{P}^3$. Note also that the similar PGL_2 action on $\mathbb{P}^d$ for $d \geq 4$ has infinitely many orbits–this is the theory of invariants of binary forms of degree d.
- (38, 3.14 and 3.15, -) "If $X \subset \mathbb{P}^n$ is any connected **projective** variety ...".
- (40, after 3.16, 10) "...we may realize the map f as the restriction to a **locally** closed subset $U \subset \mathbb{A}^n$ of a linear projection $\mathbb{A}^n \to \mathbb{A}^m$...".

Lecture 4

- (42, after 4.3, 15) "This is always constructible, though we cannot prove that here." Let $\mathcal{V} \subset B \times \mathbb{P}^n$ be the family, and let $\pi : \mathcal{V} \to B$ denote the projection map. The subset $Z := \mathcal{V} \cap (B \times (\mathbb{P}^n \smallsetminus X)) \subset \mathcal{V}$ is constructible, so so is $\pi(Z) \subset B$ (because of Chevalley's Theorem, Theorem 3.16), and hence so is the required locus $B \smallsetminus \pi(Z)$.
- (46, 4.11, -) This is a straightforward computation in local coordinates; here's another (more advanced) perspective on the matter: consider the universal hyperplane section $\Gamma \subset \mathbb{P}^{n*} \times \mathbb{P}^n$ with projections π_1, π_2, and for each $d \geq 0$ consider the vector bundle $E_d \to \mathbb{P}^{n*}$ given as $E_d := \pi_{1,*}\pi_2^*\mathcal{O}_{\mathbb{P}^n}(d)$ which has fiber $E_d|_\Lambda = \mathrm{H}^0\mathcal{O}_\Lambda(d)$. For each $a, b \geq 1$ such that $a + b = d$, consider the multiplication map $\mu_{a,b} : \mathbb{P}E_a \times_{\mathbb{P}^{n*}} \mathbb{P}E_b \to \mathbb{P}E_d$ and let $\sigma_X : \mathbb{P}^{n*} \to \mathbb{P}E_d$ be the section given by $\Lambda \mapsto (\Lambda, \Lambda \cap X)$. Then the required subvariety in $\mathbb{P}^{n*}$ is given by
$$\pi\left(\sigma_X(\mathbb{P}^{n*}) \cap \bigcup_{a+b=d} \mu_{a,b}(\mathbb{P}E_a \times_{\mathbb{P}^{n*}} \mathbb{P}E_b)\right),$$
which is closed thanks to completeness.
- (46, 4.12(c) and Footnote 1, -) Here's the general statement.

Theorem 1. Let $n, d \geq 1$ be integers and $\mathfrak{X} \subset \mathbb{P}^N \times \mathbb{P}^n$ be the universal hypersurface of degree d on $\mathbb{P}^n$, where $N = \binom{n+d}{d} - 1$. Consider sections of the first projection map $\pi_1 : \mathfrak{X} \to \mathbb{P}^N$.
(a) This projection admits a global (regular) section iff $d = 1$ and n is odd.
(b) This projection admits a *rational* section iff $d = 1$.

Proof. For simplicity, we work over $k = \mathbb{C}$ and use singular cohomology with coefficients in $\mathbb{Z}$, and prove the "only if" direction, leaving the construction of the section when $d = 1$ to the reader. Say $\mathrm{H}^*(\mathbb{P}^N \times \mathbb{P}^n) \cong \mathbb{Z}[x, y]/(x^{N+1}, y^{n+1})$. Then using the Lefschetz Hyperplane Theorem/the Leray-Hirsch Theorem/the Whitney Product Formula, we can see that $\mathrm{H}^*(\mathfrak{X})$ is the further quotient
$$\mathrm{H}^*(\mathfrak{X}) \cong \mathbb{Z}[x, y]/(x^{N+1}, y^{n+1}, x^N - dx^{N-1}y + d^2x^{N-2}y^2 + \cdots + (-d)^n x^{N-n}y^n).$$

(a) Suppose there is a global regular section $\sigma : \mathbb{P}^N \to \mathfrak{X}$, and let $\varphi = \pi_2 \circ \sigma : \mathbb{P}^N \to \mathbb{P}^n$. Then $\varphi^*(y) = cx$ for some $c \in \mathbb{Z}$, and since φ factors through $\mathfrak{X}$, we must have
$$1 - dc + d^2c^2 + \cdots + (-dc)^n = 0.$$
This forces $d = c = 1$ and n to be odd.

(b) Suppose $\sigma : \mathbb{P}^N \dashrightarrow \mathfrak{X}$ is a rational section, and let $W \subset \mathfrak{X}$ be the image of σ, so that W is a possibly singular subvariety of $\mathfrak{X}$ of codimension $n-1$. By our computation of $\mathrm{H}^*(\mathfrak{X})$ above, the (Poincaré dual to the fundamental) class of W must be

$$\mathrm{PD}_{\mathfrak{X}}[W] = \sum_{i=0}^{n-1} a_i x^{n-1-i} y^i$$

for some $a_i \in \mathbb{Z}$. It then follows from the push-pull formula[1] that the class of W in $\mathbb{P}^N \times \mathbb{P}^n$ must be

$$\mathrm{PD}_{\mathbb{P}^N \times \mathbb{P}^n}[W] = \left(\sum_{i=0}^{n-1} a_i x^{n-1-i} y^i \right)(x + dy).$$

Now since σ is a section, it follows that this class interects the general fiber of $\pi_1 : \mathbb{P}^N \times \mathbb{P}^n \to \mathbb{P}^N$ in a point, and so we compute

$$1 = [x^N y^n]\, x^N \cdot \mathrm{PD}_{\mathbb{P}^N \times \mathbb{P}^n}[W] = a_{n-1} d,$$

which forces $d = 1$.

■

- (46, 4.14, -)

Theorem 2. For each integer $n \geq 1$, let B_n the moduli space of all rational normal curves in $\mathbb{P}^n$, and let $\mathfrak{X}_n \subset B_n \times \mathbb{P}^n$ denote the universal rational normal curve. The first projection $\pi_1 : \mathfrak{X}_n \to B_n$ admits a rational section iff n is odd.

Proof. Think of $\mathfrak{X}_n$ as a single rational normal curve over $L := K(B_n)$ (so over $\overline{L}$ it is $\mathbb{P}^1$). The anticanonical bundle gives us a L-rational divisor on $\mathfrak{X}_n$ of degree 2, and the hyperplane section gives us a L-rational divisor of degree d. Therefore, when n is odd, we get a L-rational divisor on $\mathfrak{X}_n$ of degree 1, which by Riemann-Roch is linearly equivalent to an effective divisor of degree 1. This is the rational section. The case of even n can be handled by reducing this to Exercise 4.13, since any rational normal curve of even degree is a Veronese embedding of a plane conic. ■

Lecture 5

- (51, 5.4, -) "This is cruel." - Harris.
- (56, 5.13, 5) "In case $n \leq 2d+1$, exactly what open subset of $(\mathbb{P}^2)^n$ is implicitly referred to?" The claim is that $2d+1$ points fail to impose independent conditions iff some $d+2$ of them are collinear. (For $n \geq 2d+2$, having some $2d+1$ on a conic would be problematic too, and in general the conditions become more and more difficult to make explicit.)

Lecture 6

- (64, 6.1, 15-16) "We begin with a basic observation: given a multivector $\omega \in \Lambda^k V$ and a **nonzero** vector $v \in V$, the vector v with divide ω–that is …".

[1]Specifically, if $X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} M$ are inclusions of subvarieties (or submanifolds) and $\theta \in \mathrm{H}^*(M)$ such that $\mathrm{PD}_Y\, i_*[X] = j^*(\theta)$, then $\mathrm{PD}_M (ji)_*[X] = \mathrm{PD}_M\, j_*[Y] \cup \theta$.

- (68, 6.9(ii), -) "Use part (i) to show that any maximal linear subspace $\Phi \subset G \subset \mathbb{P}^N$ is either the set of k-planes containing a fixed linear subspace **of dimension** $k-1$ of V or the set of k-planes contained in a fixed linear subspace **of dimension** $k+1$ of V."
- (70, 6.16(ii), -) If a_{ij} denote Plücker coordinates on $\mathbb{G}(1,3) \subset \mathbb{P}^5$, then $\mathscr{C}_1(C)$ is the vanishing locus of
$$\det \begin{bmatrix} a_{01} & a_{02} & a_{12} \\ a_{02} & a_{03}+a_{12} & a_{13} \\ a_{12} & a_{13} & a_{23} \end{bmatrix}.$$

  One way to prove this result is via a straightforward computation in local coordinates. Three other proofs may be given as follows.

  (a) Over $\mathbb{G}(1,3)$, we have a restriction morphism of bundles $\underline{\mathrm{H}^0(\boldsymbol{\mathcal{I}}_C(2))} \to \operatorname{Sym}^2 \boldsymbol{\mathcal{S}}^\vee$. The locus of ranks $3,2,1$ of this map corresponds exactly to $\ell \cap C = \emptyset$, $\ell \in \mathscr{C}_1(C) \smallsetminus \mathscr{S}(C)$ and $\ell \in \mathscr{S}(C)$ respectively; therefore, the locus $\mathscr{C}_1(C)$ is a determinantal subvariety. This determinant always works locally.

  (b) It is easy to show that $\mathscr{C}_1(C) \subset \mathbb{V}(F)$ for this determinant F. For the reverse inclusion, use that $\mathscr{C}_1(C)$ is irreducible of dimension 3. By the Lefschetz Hyperplane Theorem, any 3-fold in $\mathbb{G}(1,3)$ is homologous to a complete intersection, and hence linearly equivalent to a complete intersection, since the Chow and cohomology groups coincide. Since $\mathbb{G}(1,3)$ is cubically normal in $\mathbb{P}^5$ (from $\boldsymbol{\mathcal{I}}_{\mathbb{G}}(3) \cong \boldsymbol{\mathcal{O}}_{\mathbb{P}^5}(1)$), it follows that every divisor D on $\mathbb{G}(1,3)$ is linearly equivalent to $\boldsymbol{\mathcal{O}}_{\mathbb{G}}(3)$ is in fact an intersection of a $\mathbb{G}(1,3)$ with a hypersurface in $\mathbb{P}^5$. Therefore, $\mathscr{C}_1(C)$ is the intersection of $\mathbb{G}$ with an irreducible cubic hypersurface, whence $\deg \mathscr{C}_1(C) = 6$. Since $\mathbb{V}(F)$ is also irreducible of degree 6, it must be irreducible and we must have equality.

  (c) Note that $\mathscr{S}(C) \subset \mathbb{G}(1,3)$ is the Veronese surface X. Consider explicitly the blow-up $\mathrm{Bl}_X \mathbb{G}(1,3)$ as the graph of a suitable map $\varphi : \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^4$. It can then be shown that φ maps $\mathscr{C}_1(C)$ to a rational normal quartic in $\mathbb{P}^4$ with fibers exactly the $\mathbb{P}^2$'s corresponding to the projections of $\mathbb{P}^3$ from points $p \in C$. This, with a little more analysis involving explicit equations, gives the result.[2]

[2] Here are the explicit equations. Let $Q := a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}$ be the Plücker relation, and F be the above determinant. Then the ideal of X is generated by the homogenous quadratic polynomials
$$\begin{aligned}
\varphi_0 &:= a_{02}^2 - a_{01}(a_{03}+a_{12}),\\
\varphi_1 &:= a_{02}a_{12} - a_{01}a_{13},\\
\varphi_2 &:= a_{12}^2 - a_{01}a_{23},\\
\varphi_3 &:= a_{12}a_{13} - a_{02}a_{23},\\
\varphi_4 &:= a_{13}^2 - a_{23}(a_{03}+a_{12}).
\end{aligned}$$
These polynomials satisfy that $(\varphi_i, Q) \supset F \cdot I$, where I is the irrelevant ideal; explicity:
$$\begin{aligned}
\varphi_0\varphi_2 - \varphi_1^2 &= a_{01}F,\\
\varphi_1\varphi_3 - \varphi_2(\varphi_2+Q) &= a_{12}F,\\
\varphi_2\varphi_4 - \varphi_3^2 &= a_{23}F,\\
\varphi_0\varphi_3 - \varphi_1(\varphi_2+Q) &= a_{02}F,\\
\varphi_1\varphi_4 - \varphi_3(\varphi_2+Q) &= a_{13}F,\\
\varphi_0\varphi_4 - \varphi_1\varphi_3 - Q(\varphi_2+Q) &= a_{03}F.
\end{aligned}$$

Lecture 8

- (91, 8.12, -) "Find an example of a variety $X \subset \mathbb{P}^n$ and integers l and $k \geq l+1$ such that $\mathcal{V}_{l,k}(X)$ **is not the closure of the locus of l-planes containing distinct k-tuples** $p_1, \dots, p_k \in X$**.**" This exercise illustrates that the loci described by "containing k points of X" and "containing k points and being their linear span" in the Grassmannian $\mathbb{G}(\ell, n)$ are in general distinct.
- (94, 8.28, -) "(An interesting question to ask in general: ...)" See [2].
- (95, 8.33, -) Hint: If V, W have dimensions $2, k+1$, then a nonzero linear functional $\lambda : V \otimes W \to K$ vanishes on a subspace of the form $L \otimes W$ for a line $L \subset V$ iff $\lambda \in V^* \otimes W^*$ is a rank 1 tensor. The main claim being made is that the Segre variety $\Sigma_{k,1}$ is self-dual; this follows from the above by noting that a hyperplane $\Lambda \subset \mathbb{P}(V \otimes W)$ contains a tangent plane to the Segre variety $\Sigma_{k,1}$ iff it contains a k-plane lying on $\Sigma_{k,1}$.[3]
- (97, 8.36, 2): "(they are isomorphic to the scroll $X_{1,1,0,\dots,0}$ or $X_{2,0,\dots,0}$); or simply as ..."

Lecture 9

- (110, after 9.16, -6) "as the zero locus of the **maximal minors of the** 2×3 matrix ..." or equivalently "as the **rank** ≤ 1 locus of the 2×3 matrix ...".

Lecture 10

- (118, 10.11, -) The main claim here is

  Lemma 3. Let $C \subset \mathbb{P}^3$ be the twisted cubic. The singular quadric surfaces containing C are precisely the cones over C with vertices on C.

  Proof. If Q is a singular quadric surface containing C and the vertex v of Q is not on C, then projection from v shows that C would have to have even degree in $\mathbb{P}^3$. Conversely, the projection of C from a $p \in C$ is a smooth conic. ∎

  In the space $\mathbb{P}\mathrm{H}^0\mathcal{O}_{\mathbb{P}^3}(2) \cong \mathbb{P}^9$ of quadrics on $\mathbb{P}^3$, there is a quartic hypersurface $\Phi_3 \subset \mathbb{P}^9$ of quadric cones. If $C \subset \mathbb{P}^3$ is a twisted cubic and $\Lambda \subset \mathbb{P}^9$ the 2-plane of quadrics containing C, then the above lemma shows that $\Lambda \cap \Phi_3$ is an everywhere nonreduced double conic.
- (123, Half-Proof of 10.19, 2) "but it is beyond our means at present." Note that $\operatorname{Pic} \mathbb{G} = A^1\mathbb{G} = \mathbb{Z}\sigma_1$, so every automorphism must take $\sigma_1 \mapsto \pm\sigma_1$. But only σ_1

If $\varphi : \mathbb{G}(1,3) \dashrightarrow \mathbb{P}^4$ is given by $[\varphi_0, \dots, \varphi_4]$, then a calculation shows that given an $\ell \in \mathbb{G}(1,3) \setminus X$ and $p \in \mathbb{P}^3$, we have $v_3(p) \in \ell$ iff $\varphi(\ell) = v_4(p)$.

[3] This follows from the following facts:
(a) Every k-plane on $\Sigma_{k,1}$ is of the form $\mathbb{P}(L \otimes W)$ for some line $L \subset W$.
(b) The projective tangent space to $\Sigma_{k,1} = \mathbb{P}V \times \mathbb{P}W$ at the point $L \otimes M$ as a subset of $\mathbb{P}(V \otimes W)$ is exactly $\mathbb{P}(V \otimes M + L \otimes W)$.
(c) If $k \geq 2$, then for any hyperplane $\Lambda \subset \mathbb{P}(V \otimes W)$, there is a $(k-1)$-dimensional subspace $U \subset W$ such that $\Lambda \supset \mathbb{P}(V \otimes U)$.

is ample, so in fact we must have $\sigma_1 \mapsto \sigma_1$. This, along with the fact that the Plücker embedding is the complete linear system $|\Sigma_1| = |\Lambda^k \boldsymbol{S}^\vee|$, tells us that the Grassmannian under the Plücker embedding is projectively normal.

Lecture 11

- (140, 11.15, -) It is not clear where to find counterexamples, so here are some hints: for an example of strict inequality, consider the self fiber product of the blowup of $\mathbb{P}^n$ at a point, and for an example of a component of smaller dimension consider the self fiber product of the normalization of a plane nodal cubic.
- (143, 11.19, -) The Schubert variety $X_{\text{sing}} = \Phi = \Sigma_1(\ell_0)$ is a rank 4 quadric in $\mathbb{P}^4$, and it has two rulings by two planes which can be seen explicitly as

$$\Sigma_1(\ell_0) = \bigcup_{p \in \ell_0} \Sigma_p = \bigcup_{H \in \ell_0^*} \Sigma_H.$$

The two resolutions Ψ and Ω (denoted $X^{\pm}$) of X_{sing} are obtained by blowing up two planes of two different rulings, and so an isomorphism between Ψ and Ω cannot commute with the projection map. The hint shows explicitly how to realize $X_{2,2,1}$ as a resolution of $X_{1,1,0} \cong \Phi$. In fact, the birational map $X^+ \dashrightarrow X^-$ is the famed Atiyah flop: there is no minimal resolution of X_{sing}, but rather two resolutions related by this flop, and this can be used to show that the fine moduli space $\boldsymbol{\mathcal{M}}_\Lambda$ of marked K3 surfaces is not Hausdorff. See [3, Remark 12.2].
- (143, 11.20, -) "Show by example that no analogous formula exists if we replace "two" by "three", even if we require $l \geq 2$." Take $k = 1, l = 2, n = 5$, and consider (a) the rational normal quintic curve $C \subset \mathbb{P}^5$, and (b) a complete intersection of the scroll $X_{2,2}$ with a cubic hypersurface.
- (143, 11.21, -) Possible Hint: Proposition 18.10.
- (145, 11.28(ii), -) Hint: To say that X has deficiency δ amounts to saying that if r is a general point on a general secant line to X, then there is a δ-dimensional family $\Sigma^d = \{(p_\lambda, q_\lambda)\} \subset X \times X$ such that $r \in \overline{p_\lambda q_\lambda}$. One would expect $\Sigma \cap Y \times Y$ to drop dimension by 2, but in fact if for some λ we have $p_\lambda, q_\lambda \in H$, so $r \in H$, then $\Sigma \cap Y \times Y = \Sigma \cap X \times Y$.
- (146, 11.31, -) The secant plane map s_l is a rational map, and hence should be notated

$$s_l : X^{l+1} \dashrightarrow \mathbb{G}(l, n).$$

Lecture 12

- (153, 12.5, 24) "It is not hard to see by the construction ..." Consider the projection from L.
- (154, 12.7, -) This exercise is too hard. See [4].
- (154, after 12.8, -3) "... can be worked out; ..." For a survey, see [5].

Lecture 13

- (164, after 13.1, 13.1+2) "…, say the zero locus of the **squarefree** polynomial $F(Z)$ of degree d."
- (167, 13.7, 5-6) "…and the map $S(X)_m \to \boldsymbol{\mathcal{L}}(m \cdot p_1 + \cdots + m \cdot p_d)$ becomes and isomorphism." This is Serre's theorem that $\mathrm{H}^1(\boldsymbol{\mathcal{I}}_X(m)) = 0$ for $m \gg 0$; see, for instance, [6, Theorem 5.2(b)].
- (167, after 13.9, 13.9+4) "…we would need cohomology to prove this." The Hilbert function is $h_X(m) = \mathrm{H}^0(X, \boldsymbol{\mathcal{O}}_X(m))$, and the Hilbert polynomial is $p_X(m) = \chi(X, \boldsymbol{\mathcal{O}}_X(m))$. In particular, $p_X(0) = \chi(X, \boldsymbol{\mathcal{O}}_X)$ is the Euler characteristic of the structure sheaf and hence independent of embedding (or equivalently $\boldsymbol{\mathcal{O}}_X(1)$).
- (169, *Syzygies*, 20-21) Here we need the uniqueness of minimal free resolutions to define the Betti numbers; see [7, §20.1].

Lecture 14

- (175, *The Zariski Tangent Space to a Variety*, 4) "Then it's not hard to see that the rank of M is at most $n-k$ at every point of X, and …." For irreducible X, this can be done either via $\operatorname{trdeg}_K K(X) = \dim_{K(X)} \operatorname{Der}_K(K(X))$ (see [8, Proposition 1.12]; recall that we are in characteristic zero), or using that every variety is birational to a hypersurface (see [9, Theorem 6.10]). In general, the tangent space to a variety at a point contains the linear spans of those of its irreducible components.
- (176, 14.4, -) Here's a proof over $\mathbb{C}$. The locus where $\mathrm{d}f_p$ is not surjective is closed in the domain, and hence its image is constructible (Theorem 3.16). Finally, a constructible set of measure zero is contained in a proper subvariety, since it cannot be dense. See also [8, 3.7].
- (179, 14.9, -) It is perhaps worth emphasizing that we need some sort of finiteness or projectivity in the hypothesis. A counterexample otherwise would be: take X to be the disjoint union of W and Z, where W is the union of the coordinate axes in $\mathbb{A}^2$ and Z is a line with a deleted point, and take Y to be the union of the three coordinate axes in $\mathbb{A}^3$.
- (179, Proof of 14.9, -2) Let $R \to S$ be a ring map and $\mathfrak{p} \subset R$ a prime such that there is a unique prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and going up holds for $R \to S$. Then $S_\mathfrak{p} = S_\mathfrak{q}$. See [10, Lemma 10.41.11].

Lecture 15

- (186, *A Note About Characteristic*, -11) "…to a **nondegenerate** plane curve $C \subset \mathbb{P}^2$…" or equivalently "…to a plane curve $C \subset \mathbb{P}^2$ **other than a line**…".
- (186, *A Note About Characteristic*, -10) "easy to prove in characteristic 0" Consider the projection of C from this point.
- (188, 15.2, -1) "the map $\mathcal{G}_X$ cannot be constant along a curve." Here's another way to say this. To say that a linear system on a curve induces the constant map to projective space is saying there is only one divisor in that system, but as long as the partials do not have the same zero locus (this uses $\operatorname{char} K = 0$), there are at least two divisors in the linear system of partials.

- (191, after 15.8, 8) "The projection π_2 is surjective, with fibers isomorphic to $\mathbb{P}^{k-1}$, so that the dimension of Σ is $2k-1$." The intention was to apply Theorem 11.14, but we cannot quite do that here since Σ is not projective. Rather we need the slightly stronger version where only the map π is assumed to be closed (which is automatically satisfied if the domain is complete); see, for instance, [11, Proposition 9.11].
- (193, after 15.12, 11) There is no Exercise 14.16; we are using that the projection π_p does not contract tangent vectors iff p lies on no projective tangent plane to X.
- (193, after 15.13, 15.13+6) The parametrization should read

$$t \mapsto [1, t^n, t^{n+1}, \dots, t^{2n-\mathbf{1}}].$$

- (196, 15.22, -14) "What is true is that such hyperplanes never form an irreducible component of the locus of tangent hyperplanes." This is perhaps easiest to see by thinking in terms of divisors in a one-parameter family.
- (198, 15.23, -) See [12].
- (198, 15.24, -) The variety Φ_X is the conormal variety to X. It is perhaps worth mentioning that the reflexivity (i.e. $\Phi_X = \Phi_{X^*}$) is equivalent to the separability/generic smoothness of the projection map $\pi : \Phi_X \to X^*$ (from the conormal *scheme*). This is the Monge-Segre-Wallace criterion; see [13, §(4)].

Lecture 16

- (200, 16.1, -) Here's another perspective on the tangent space to the Grassmannian that is sometimes helpful. Fix a (finite dimensional) vector space V, an integer $k \geq 1$, and let $\mathbb{G} := \mathrm{Gr}(k, V) = \mathbb{G}(k-1, \mathbb{P}V)$. If $\rho : \mathbb{G} \to \mathbb{P}\Lambda^k V$ is the Plücker map $L \mapsto \Lambda^k L$, then the differential

$$\mathrm{Hom}(L, V/L) = T_L\mathbb{G} \xrightarrow{\mathrm{d}\rho_L} T_{\Lambda^k L}\mathbb{P}\Lambda^k V = \mathrm{Hom}(\Lambda^k L, \Lambda^k V/\Lambda^k L)$$

is given by

$$\mathrm{d}\rho_L(\varphi)\,(\ell_1 \wedge \cdots \wedge \ell_k) = \sum_{i=1}^{k} \ell_1 \wedge \cdots \wedge \varphi(\ell_i) \wedge \cdots \wedge \ell_k.$$

Given a short exact sequence of vector spaces $0 \to L \to V \to Q \to 0$ and an integer $r \geq 0$, there is a filtration

$$\Lambda^r L = F_0\Lambda^r V \subset F_1\Lambda^r V \subset \cdots \subset F_r\Lambda^r V$$

such that $F_i\Lambda^r V$ is the image of $\Lambda^{r-i}L \otimes \Lambda^i V$ in $\Lambda^r V$ and the successive subquotients are canonically

$$F_i\Lambda^r V/F_{i-1}\Lambda^r V \cong \Lambda^{r-i}L \otimes \Lambda^i Q.$$

In our setting, the differential $\mathrm{d}\rho_L$ maps $T_L\mathbb{G}$ isomorphically to the subspace

$$\mathrm{Hom}(\Lambda^k L, F_1\Lambda^k V/\Lambda^k L) \subset \mathrm{Hom}(\Lambda^k L, \Lambda^k V/\Lambda^k L)$$

whence the projective tangent space under the Plücker embedding is exactly

$$\mathbb{T}_L\mathbb{G} = \mathbb{P}F_1^L\Lambda^k V \subset \mathbb{P}\Lambda^k V.$$

In the special case of $k = 2$, this can be made even more explicit; see [14, §10.1].

- (204, 16.7, -) For the reverse implication, the version of Zariski's Main Theorem provided certainly shows that if $\mathscr{C}_k(X)$ is smooth and $\Lambda \cap X$ is finite, then $\Lambda \cap X = \{p\}$. However, we probably do not have enough tools here to show $p \in X_{\mathrm{sm}}$ and $\Lambda \cap \mathbb{T}_p X = \{p\}$; for that we would need a slightly stronger version of Zariski's Main Theorem. Basically, the way the disconnected hypothesis enters into the proof of the result is via a certain $h^0(\mathcal{O}) > 1$, and this is achieved also in the nonreduced case, such as what happens when X is singular at p or if there is a deformation along a tangent vector (i.e. $\Lambda \cap \mathbb{T}_p X \supsetneq \{p\}$). [TODO: Make more precise, and give references.]
- (204, after 16.8,16.8+1) "This is immediate over $\mathbb{C}$". The idea is that a variety is locally irreducible at each smooth point; this is the geometric version of the alegbraic fact that a Noetherian regular local ring is a domain. Now if $q \in Y$ is smooth, then by constructibility the image of a (classical) open neighborhood of each connected component of $f^{-1}(q)$ contains an open neighborhood of q, so if f is birational then there can be only one such component.
- (204, 16.9(a), 16.9(a)-7) "Show that the subvarieties $\mathscr{C}_1(X_i)$ intersect transversely." At this point in the book, it has not been explained what transversal intersection of subvarieties means. Here's the definition: subvarieties $X_1, \ldots, X_n$ of a smooth variety X are transversal at a point $x \in \bigcap X_i$ iff

$$\operatorname{codim}_{T_x X} \bigcap_{i=1}^{n} T_x X_i = \sum_{i=1}^{n} \operatorname{codim}_X X_i.$$

  This implies that each X_i is smooth at x and the vector subspaces $T_x X_i \subset T_x X$ intersect transversally; this implies that the intersection $\bigcap_i X_i$ is smooth at x as well. Subvarieties $X_1, \ldots, X_n$ are said to intersect *transversally* if they intersect transversally at each $x \in \bigcap X_i$; *generically transversally* if they intersect transversally at a general point of each component of $\bigcap X_i$.
- (205, 16.12, 18) The displayed equation should say $T_{(p,\Lambda)}(\Psi)$ instead of $T_{(p,\Lambda)}(\Phi)$.
- (205, 16.13, -) "Let $\Gamma \subset \mathbb{G}(k, n)$ be any subvariety ...". It would also be helpful to specify that $j(p)$ really means $j(\tilde{p})$ for any $\tilde{p} \in p$.
- (206, 16.16, -) Counterexamples in positive characteristic are furnished by singular strange curves such that (t, t^p, t^{2p}) in $\mathbb{A}^2$ in characteristic p.
- (206, 16.17, -) Hint: Apply Theorem 16.13 to the (smooth locus) of the image of $s_l : X^{l+1} \dashrightarrow \mathbb{G}(l, n)$. See also [15, Proposition V.1.4]. For more on these exercises and connection to the interpolation problem, see Terracini's Lemma and the discussion in [16, Proposition 10.10].
- (209, 16.20, 2) "The symmetry in the statement of Theorem **15.24** now emerges."
- (210, 16.22, -) See again [4].

Lecture 17

- (212, 17.1, 3) There is no Example 18.4. This should refer to Example 17.14 instead.
- (212, 17.3, −5) "Let $X \subset \mathbb{P}^n$ be a smooth curve **other than a line** and ...". This needs to be included to even make sense of the term "tangential surface".
- (214, after Exercise 17.7, 23) "(which we will not prove here)" This result is purely numerical, and a consequence of the Plücker formulae; see the Proposition in the section titled *Global Plücker Formula* on [1, p. 270].

- (216, 17.14-15, -) It is perhaps worth noting here that for space curves (or more generally curves in $\mathbb{P}^n$ for $n \geq 3$), the notions of flex points/lines and inflectionary points do not coincide, unlike for plane curves. In particular, if $X \subset \mathbb{P}^n$ is smooth curve and $n \geq 3$, then X is expected to have no flex lines; however, as noted on p. 214, if it is not the rational normal curve, then it will necessarily have inflectionary points.
- (216, 17.16, -) In the following proof, it is only shown that $Y_{\text{sing}} \subset X_{\text{sing}} \cap Y$, and this is sufficient for most applications. The theorem should also include the statement, as is shown in the proof and is needed in applications, that the restriction $f|_{X_{\text{sm}}} : X_{\text{sm}} \to \mathbb{P}^n$ is transverse to the general hyperplane H.
- (217, after 17.16, 8) "…every tangent **hyperplane** to the …"
- (217, Proof of 17.16, -6) "Now look at the restricted map $\tilde{\pi}_2 : \Gamma_{\text{sm}} \to \mathbb{P}^{n*}$. Bertini's theorem …" As presented in the book, Bertini's Theorem follows from applying Proposition 14.4 to $\pi_2 : \Gamma \to \mathbb{P}^{n*}$ directly.
- (217, Proof of 17.16, -5) There is no Exercise 14.6.
- (218, after 17.17, 17.17+4) "…from the fact that $(F_1, \ldots, F_k)$ has no **embedded** primes…". In a Noetherian ring, a proper ideal is radical iff the primary components of a reduced primary decomposition of it are all prime; in this case, there are no embedded primes and the reduced primary decomposition is unique.

Lecture 18

- (224, -, -13) "…the projection map $\pi_p : X \to \mathbb{P}^{n-1}$ from a general point $p \in \mathbb{P}^n$ is birational onto its image; in fact, if we choose $q \in X$ any point, it's enough for p to lie outside the cone $\overline{q, X}$." Recall that the projection has finite fibers: any line through $p \notin X$ meets X in only finitely many points. If a general fiber of the projection $\pi_p : X \to \pi_p(X)$ has at least two points, then p lies on a general secant to X, and hence on every secant to X. (Recall that X is irreducible, and p lying on a secant is a closed condition on $X \smallsetminus X \smallsetminus \Delta$.) In particular, p must lie on the cone $\overline{q, X}$ for any $q \in X$. Therefore, in characteristic 0, we are done by Proposition 7.16.
- (225, 18.2, -) "Assuming $\text{char}(K) = 0$, use **Proposition 7.16 and Exercise 11.44** to show that …"
- (227, -, 6) "In fact, complex subvarieties of $\mathbb{P}^n$ have minimal area among cycles in their homology class." This is Wirtinger's Inequality combined with Chow's Theorem; see [8, Theorem 5.35].
- (228, 18.7, -) Either X has to be assumed to be irreducible, or the statement of the exercise changed to: Let $X \subset \mathbb{P}^n$ be a subvariety of degree 1. Show that X has a unique component of $\dim X$, which is a linear subspace of $\mathbb{P}^n$.
- (229, 18.8, -15) "…they must intersect generically transversely,…" If there were two other components or if the intersection were generically nonreduced along some component, then the general hyperplane section would consist of two 2 distinct conics intersecting with multiplicity at least 5, which is not possible.
- (230, Proof of 18.10, *)
  - (Proof+2) "…for any H intersecting X generically transversely." A general H intersects X generically transversally thanks to the strengthened version of Bertini's Theorem mentioned above, combined with Exercise 11.16 and Exer-

cise 14.3(b).

  - (Proof+5) "Since X is not contained in any finite union of hyperplanes , it will intersect a general member H_λ of this pencil in at least one point p not lying on Y;" As observed above (see label (42, after 4.3, 15)), the set $\{\lambda : X \cap H_\lambda \subset Y\} \subset \mathbb{P}^1$ is constructible, and a constructible subset of $\mathbb{P}^1$ is either open or closed.
  - (Proof+15) "for a general hyperplane containing a given **general** $(n-3)$-plane Λ; so ...".
  - (Proof+19) "This is a proper closed subset ..." This is Proposition 14.4.
  - (Page−8) "(in fact, there will be no 0-dimensional components of B, but we don't need to know this)." This is the Zariski-Nagata purity of the branch locus. If X is smooth, this can be shown by locally embedding X as a hypersurface $\mathbb{P}^2 \times \mathbb{P}^1$ and using Proposition 7.16: the branch locus B is exactly the vanishing equation of the discriminant of the local defining equation for X.

- (233, after 18.14, 4-5) "...only case in which the general hyperplane section of a Veronese variety is again a Veronese variety." Perhaps this should be marked as an exercise!
- (237, Proof of Bézout's Theorem, 17) "...tangent space to J given in Exercise 16.14 (and the hypothesis that X and Y transversely) this intersection is transverse." Since X and Y intersect transversely, each point of intersection is smooth on both and we may apply Exercise 16.14 to X_{sm} and Y_{sm}. The main claim (left to the reader to work out the details) is that L intersections $J(X, Y)$ transversely along a line iff the corresponding point of intersection of X and Y is transverse.
- (238, 18.23, -) This is a result due to Morin. See, for instance, the excellent treatment in [17, §4].

Lecture 19

- (241, 19.6, -) This exercise is too vague, but the general answer is "the degree is lower than expected, due to splitting off of a component in the Grassmannian $\mathbb{G}(1, n)$." For instance, if $X_t, Y_t \subset \mathbb{P}^3$ is a one-parameter family of smooth curves with isomorphisms $\varphi_t : X_t \to Y_t$ such that $X_t \cap Y_t = \emptyset$ for $t \neq 0$, $X_0 \cap Y_0 = \{p\}$ with $\mathbb{T}_p X_0 \neq \mathbb{T}_p Y_0$, and $\varphi_0(p) = p$, then the flat limit of the $K_t = K(\varphi_t)$ as $t \to 0$ will include more than just K_0. As a concrete example, if $\mathbb{P}^3$ has coordinates x, y, z, w, consider the lines $X_t := \{[u, v, 0, 0] : [u, v] \in \mathbb{P}^1\}$ and $Y_t = \{[0, v, tv, u] : [u, v] \in \mathbb{P}^1\}$. Then the join K_t is the quadric given by the equation $ywt - z(x + w) = 0$, which for $t \neq 0$ is smooth and for $t = 0$ is the rank two quadric which is the union of $K_0 = \mathbb{V}(z)$ along with the plane $\mathbb{V}(x + w)$. The corresponding curve $Z_t \subset \mathbb{G}(1, 3)$ for $t \neq 0$ is a smooth plane conic, the flat limit of which as $t \to 0$ is a rank 2 plane conic, only one line of which is the curve $Z_0 \subset \mathbb{G}(1, 3)$.
- (247, 19.17, −19) "...curve $X \subset \mathbb{P}^2$ is smooth." This can be clarified by replacing $\mathbb{P}^2$ by $\mathbb{P}^2_{\mathbb{C}}$, emphasizing that the base field here is $\mathbb{C}$ (or any algebraically closed field containing $\mathbb{R}$).
- (248, Proof of 19.18, -) This is more or less the original proof due to Harnack. For the missing details, as well as the other direction of the proof, see [18, 11.6.1-3]; see also [19, Chapter 1]. The given proof needs some corrections as well; firstly, we may assume that $X(\mathbb{C})$ is irreducible of degree $d > 2$, since if $g(d) = \binom{d-1}{2}$, then

$g(d_1)+1+g(d_2)+1 \leq g(d_1+d_2)+1$ for $d_1, d_2 > 1$. This ensures that the curve Y so produced cannot have any components in common with X, which allows us to apply Bézout's Theorem to bound the number of points of intersection of X and Y. Finally, it is worth mentioning the more standard, topological proof of the result (which would involve introducing the genus of a curve–a topic I think well worth including in this book). Here's how it goes: If $X(\mathbb{R})$ has m components, then $(X(\mathbb{C}) - X(\mathbb{R}))/\sigma$ is a surface with m boundary components, and hence has topological Euler characteristic at most $2-m$. But now, since, $\chi(S^1)=0$, it follows that

$$\chi(X(\mathbb{C}) \smallsetminus X(\mathbb{R})) = \chi(X(\mathbb{C})) = 2-2g,$$

and so, since the action of σ on $X(\mathbb{C}) \smallsetminus X(\mathbb{R})$ is fixed-point free, it follows that

$$1-g = \chi\left(\frac{X(\mathbb{C}) \smallsetminus X(\mathbb{R})}{\sigma}\right) \leq 2-m.$$

For another perspective on this using the intersection form on the middle cohomology of $X(\mathbb{C})$, see [20, Appendix 6]. Finally, Harnack's Theorem can also be interpreted as a purely topological statement, as follows. Let S be a hyperbolic Riemann surface with an orientation reversing involution σ; then Harnack's theorem says that $\mathrm{Fix}(\sigma)$ is a set of disjoint simple complete geodesics of size at most $g+1$. See [21].

- (250, -, 1) "On the other hand, it is less clear (but true) ...." This statement is false, as can be seen by deforming an intersection of two ellipses appropriately. As an explicit example, consider the quartic of the form

$$(x^2+cy^2-1)(cx^2+y^2-1) - \varepsilon y = 0$$

for large $c \gg 1$ and small positive $0 < \varepsilon \ll 1$.

Lecture 20

- (256, 20.3, Line 2) "...the image in X of an arc $\gamma : \Delta \to \tilde{X}$ with $\gamma(0) = \mathbf{p}$ will have ..."
- (257, 20.6, -8) "What can happen if X and Y are disjoint but do not lie in disjoint linear spaces (so that, for example $\mathbb{T}C_p(X)$ may intersect Y)? What happens if X actually meets Y?" Similar to (241, 19.6, -), this exercise is too vague, but the general answer is the same: the degree can be lower than expected due to splitting off of components. As a concrete example, suppose $\mathbb{P}^4$ has coordinates x, y, z, u, v, take the conic $X := \mathbb{V}(x^2 - yz, u, v)$ and the two-parameter family of lines $Y_{s,t}$ where $Y_{s,t}$ is given parametrically by $[\mu, \nu]$ as

$$[\mu, \nu, t\nu, t(\mu+\nu), t((1+s)\mu+\nu)].$$

Then the line $Y_{s,t}$ is contained in the plane of X iff $t=0$; when $t \neq 0$, it meets the plane iff $s=0$, in which case the point of intersection lies on the conic X iff $t=1$. The join $J(X, Y_{s,t})$ for $st \neq 0$ is a rank 3 quadric hypersurface; this has equation

$$st\left[st\left(x^2 - y(z-u)\right) + su(z-u) + ty(u-v) + (2x+z-2u)(u-v)\right] + (1-t)(u-v)^2.$$

It is then easy to see the degenerations of this hypersurface according to the above specializations to $(0,0)$. The point of this exercise is exploration rather than concrete answers.

- (257, 20.7, -3) "Is the same true if C is an arbitrary space curve, if we assume that the point $p \in C$ **is smooth** and lies on no tangent line to C other than $\mathbb{T}_p C$?" Yes. Note also that a general plane section of TC for the twisted cubic C is a cardiod, which is the Steiner quartic with three cusps (two of which are at the circular points at infinity for a real cardiod).
- (259, 20.10, -) Harris says the intention of this exercise is conveyed by restricting to the case when X and Y are **smooth**. In this case, the vertices must map to each other, and the isomorphism of the tangent spaces to the cones at the vertices gives us the linear forms that induce the required automorphism of $\mathbb{P}^n$. The general case is trickier, for instance, if X and Y are already cones.
- (259, 20.12, -1) "As in Exercise 20.6, what can happen ..." The multiplicity can be lower than expected; see again the label (257, 20.6, -8).
- (260, 20.13, -) "Is the same true if C is an arbitrary space curve, if we assume that the **smooth** point $p \in C$ lies on no tangent line to C other than $\mathbb{T}_p C$?" No! Consider flex points of rational quartics in $\mathbb{P}^3$. As a concrete example, if C is given parametrically as (t, t^3, t^4) in $\mathbb{A}^3$, then the tangent cone of C is the surface with equation
$$16x^3y^3 - 27x^4z^2 + 6x^2y^2z - 27y^4 + 48xyz^2 - 16z^3 = 0.$$
The $16z^3$ term shows that (in $\operatorname{char} K \neq 2$!), the tangent cone at the flex points $t = 0, \infty$ has multiplicity 3.
- (260-261, *Examples of Singularities*, -) Much more on the unproven claims of this section can be found in [22].
- (264, *Resolution of Singularities for Curves*, -) This method was discovered by Albanese from 1924; see [23].
- (264, *Resolution of Singularities for Curves*, -11) The value of N should be **one less**; the line should read
$$N = \dim(\overline{C_0}) = dn - \binom{d-1}{2}.$$
This way of writing N also provides a hint for the next exercise.
- (265, 20.18, -) "Show that **when d $\geq$ 5** we can choose $n = d - 2$ in the preceding argument." The condition $N > D_0/2 + 1$ is equivalent to $n > d - 3 + (4/d)$, so for $d = 2, 3, 4$, respectively, the smallest n we can choose is $n = 2, 2, 3$.

Lecture 21

This lecture, particularly page 267, needs a lot of reworking and referencing. A possible rewording of this chapter could serve as an introduction to (or sneak peek into) scheme theory, and why schemes solve many of the problems faced by classical algebraic geometers.

- (269, *Chow Varieties*, 16) "map $\pi : \Gamma \to \mathbb{P}^{n*} \times \cdots \times \mathbb{P}^{n*}$ will be birational **onto its image**."

- (272, 21.5, -) For $d = 2$, there are two components of dimension 8; for $d = 3$, there are four components of dimension 12; for $d = 4$, there are six components, five of dimension 16 and one of dimension 17. See also [24].

Lecture 22

- (284, 22.1, 1) "with equality **holding in the first inequality** if and only if ...". This exercise should also be marked with a (*).
- (285, *Quadric Surfaces*, Figure 2) The figure switches notation from the text, and should have M_λ and L_μ. Perhaps the diagram can be improved by marking λ and μ in it as well.
- (292, after 22.8, -9) Either remove the "of Q", or insert the words "the ambient projective space of" before Q.
- (293, 22.13, -2) The statement should say

$$\dim(\mathbf{F}_{k,m}) = (k+1)\left(m - \frac{3k}{2}\right).$$

- (294, 22.16, -) The inequality $\max\{0, 2k+1-n\} \leq l \leq k+1$ gives $\min\{n-k+1, k+1\}$ values of l; for each l, the locus has codimension $\binom{k-l+2}{2}$, is singular along the smaller rank locus, and is irreducible except in the Fano case of $l = 0$ with $n = 2k+1$. Note that here $\operatorname{rank}(\Lambda \cap Q) \leq 0$ is supposed to be interpreted as $\Lambda \subset Q$.
- (294, 22.20, -) Harris says that this is too difficult to be an exercise and should have been a remark; alternatively, more substantial hints should be provided.
- (298, 22.29, -) See [16, Chapter 8].
- (299, 22.31, 2) The value of N should be

$$N = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}.$$

- (299, 22.31, 12): "... be the Grassmannian of $(\mathbf{n}-\mathbf{k})$**-planes** in $\mathbb{P}^n$."
- (300, 22.32, -): There is a $+1$ missing; the result should read

$$\deg(\Phi_k) = \prod_{\alpha=0}^{n-k} \frac{\binom{n+\alpha+1}{n+\mathbf{1}-k-\alpha}}{\binom{2\alpha+1}{\alpha}}.$$

- (300, after 22.33, -4) "... to that given in Examples **14.16** and 20.5 ...".
- (303, proof of 22.34, 15-16) "... either $\mathbf{Q} \in \Phi_{n-1} = (\Phi_n)_{\mathrm{sing}}$ or ...." To say more, note that either $Q \in \Phi_{n-1} = (\Phi_n)_{\mathrm{sing}}$ or $Q \in \Phi_n \setminus \Phi_{n-1}$, in which case Q' contains the unique singular point of Q. In the latter case, L is tangent to Φ_n at Q, thanks to Theorem 22.33 again.
- (303, 22.35, -) Although it is true in this case[4], it is not true in general that if $X \subset \mathbb{P}^N$ is a (reduced) hypersurface of degree $d \geq 1$, then there is a line $\ell \subset \mathbb{P}^N$

[4]Consider the matrix with 0's along the diagonal (except for the last entry) and λ's and μ's alternating in a parallel fashion along the sub- and super-diagonals. For instance, this means the matrices $\begin{bmatrix} 0 & \lambda \\ \lambda & \mu \end{bmatrix}$ and $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \mu \\ 0 & \mu & \lambda \end{bmatrix}$ for $n = 1, 2$ respectively.

such that ℓ is disjoint from X_{sing}, the intersection $\ell \cap X$ is zero-dimensional and $\#(\ell \cap X)^{\mathrm{red}} = 1$. The simplest counterexample is obtained by noting that a general plane quartic curve has no hyperflexes.[5] Note that the result is clearly true when $d = 1$, true when $d = 2$ if the quadric X has rank at least 3 (i.e. $\operatorname{codim}_X X_{\mathrm{sing}} \geq 2$), and true when $d = 3$ if the cubic X is smooth in codimension 2 (i.e. $\operatorname{codim}_X X_{\mathrm{sing}} \geq 3$) by Bertini, since every smooth plane cubic has nine flexes! See also [26].

- (304, Proof of 22.38, Proof+7) "Observe first that Y is smooth, since projection of Y into the second factor maps Y onto the plane curve $\overline{X}$ viewed as a subvariety of ...". This is really subsumed by the following discussion where it is shown that $Y \cong X$ and $Y \cong E$, but the idea is that the projection of Y onto the second factor is birational (as is shown in the next paragraph), and, as established in Exercise 22.36, $\overline{X}$ is a smooth curve; this forces Y to be smooth and the second projection to be an isomorphism to $\overline{X}$.
- (305, 22.40, -) This is a messy computation with the tools we have so far; the more modern perspective involves applying Riemann-Hurwitz to the projection of X from the vertex of Q, and recalling the "geometric addition law" on the elliptic normal curve with an inflectionary point chosen as the origin: the hyperplane class sums to zero in the group law.
- (304, Proof of 22.38, -9) "...meet X at two points, **counted with multiplicity** (that is, the two points ...)". It is possible for such an l to be tangent to X, and indeed, as the proof shows, given a point $P \in X$ there is a unique $Q \in L$ containing the tangent line $l = \mathbb{T}_P X$.
- (305, 22.42, -) The statement should mention that the λ_i for $i = 1, \ldots, n+1$ are **pairwise distinct**. It follows also from the proof and the discussion that the set $\{\lambda_i\}$ is determined by the pencil (up to automorphisms of $\mathbb{P}^1$).

Hints for Selected Exercises

- (310, 8.12, -) "Try for example $k = 4, l = 2$ and X the projection of a rational normal curve $C \subset \mathbb{P}^n$ from a **line $\ell \subset \mathbb{P}^n$ lying in a 4-secant 3-plane to C but not on a 3-secant 2-plane**."[6]
- (310, 10.7, -) The idea is that rational space curves of degree 4 or more have inflectionary points preserved by $\operatorname{Aut}(X, \mathbb{P}^n)$. However, this is perhaps not a fair exercise at this stage because the concept of inflectionary points has not been introduced yet.

[5] The locus of plane quartics with a hyperflex is a Cartier divisor on the moduli space $\mathcal{M}_3$; see [25].

[6] Here's the solution: consider a rational normal curve $C \subset \mathbb{P}^n$ for $n \geq 6$, and pick 4 points $a_1, \ldots, a_4 \in C$. Let $\Lambda = \overline{a_1 a_2 a_3 a_4} \cong \mathbb{P}^3$ be their linear span, and pick a line $\ell \subset \Lambda$ not contained in any of the four 3-secant 2-planes. Finally, consider $X := \pi_\ell(C) \subset \mathbb{P}^{n-2}$, the projection of C from ℓ. If

$$\mathcal{V}' := \overline{\{\Lambda : \Lambda \ni p_1, p_2, p_3, p_4 \text{ for distinct } p_i \in X\}} \subset \mathbb{G}(2, n-2)$$

is the closure of the locus of 2-planes containing 4 distinct points of X, then the claim is that

$$\mathcal{V}_{2,4}(X) \subsetneq \mathcal{V}'.$$

Indeed, on the one hand, the latter contains all 2-planes containing the projected line $\mu := \pi_\ell(\Lambda)$, and hence has dimension at least $n - 4$. On the other hand, the rational map $X \dashrightarrow \mathcal{V}_{2,4}(X)$ given by $x \mapsto \overline{\mu, x}$, the linear span of μ and x, is dominant, so the left hand side has dimension 1.

- (311, 11.19, -) "To show the isomorphism of Ψ with the scroll $X_{2,2,1}$, **fix a line** $M \subset \mathbb{P}^3$ **disjoint from** l_0 **and points** $q, r \in M$ and consider the image under the Segre embedding of $\mathbb{G}(1,3) \times l_0$ of the three curves

$$\{(l_0, s)\}_{s \in l_0}, \{(\overline{qs}, s)\}_{s \in l_0}, \text{ and } \{(\overline{rs}, s)\}_{s \in l_0}."$$

  The idea is that both representations express this variety as a $\mathbb{P}^2$-bundle over $l_0 \cong \mathbb{P}^1$, namely the bundle $\mathbb{P}\mathcal{Q}|_{\ell_0}$, where $\mathcal{Q} \to \mathbb{P}^3$ is the tautological quotient bundle.
- (311, 12.26, -) "but I don't know an elementary proof". This is a result due to Poporov; see [27].
- (311, 14.9, -) The label should read **14.11**.
- (312, 17.21, -) See also [16, Proposition 5.6].
- (312, 20.15, -) Another way would be to consider general sections of the (projective closure of the) hypersurface given by $z = (x^2 - y)^2$ in $\mathbb{A}^3$. See [28].

References

[1] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. No. 10 in Wiley Classics Library, John Wiley & Sons, Inc., 1994.

[2] A. Conca and D. Faenzi, "A Remark on Hyperplane Sections of Rational Normal Scrolls." `https://arxiv.org/abs/1709.08332`, 2017.

[3] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Ven, *Compact Complex Surfaces*, vol. 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer-Verlag Berlin Heidelberg GmbH, 2 ed., 2004.

[4] A. Waldron, "Fano Varieties of Low-Degree Smooth Hypersurfaces and Unirationality." `https://abel.math.harvard.edu/theses/senior/waldron/alex_waldron.pdf`, 2008.

[5] C. Ciliberto and M. Zaidenberg, "Lines, Conics, and All That." `https://arxiv.org/abs/1910.11423v3`, 2020.

[6] R. Hartshorne, *Algebraic Geometry*. No. 52 in Graduate Texts in Mathematics, Springer, 1977.

[7] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, vol. 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[8] D. Mumford, *Algebraic Geometry I: Complex Projective Varieties*. Classics in Mathematics, Springer, 1995.

[9] M. Reid, *Undergraduate Algebraic Geometry*, vol. 12 of *London Mathematical Society Student Texts*. Cambridge University Press, 1988.

[10] The Stacks project authors, "The Stacks Project." `https://stacks.math.columbia.edu`, 2024.

[11] J. S. Milne, "Algebraic geometry (v6.03)," 2023. Available at `www.jmilne.org/math/`.

[12] H. Kaji, "On the Duals of Segre Varieties," *Geometriae Dedicata*, vol. 99, no. 1, pp. 221–229, 2003.

[13] S. L. Kleiman, "Tangency and Duality," in *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, CMS Conf. Proc.*, vol. 6, pp. 163–225, Amer. Math. Soc., Providence, RI, 1986.

[14] I. V. Dolgachev, *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012.

[15] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, vol. 127 of *Translations of Mathematical Monographs*. American Mathematical Society, 1993.

[16] D. Eisenbud and J. Harris, *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[17] M. Mustaţă, "Lecture 4: Unirationality of General Hypersurfaces of Small Degree." Available at https://public.websites.umich.edu/~mmustata/lecture4_rationality.pdf.

[18] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, vol. 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge.* Springer-Verlag Berlin Heidelberg GmbH, 1998.

[19] O. Viro, "Introduction to Topology of Real Algebraic Varieties." Available at https://www.pdmi.ras.ru/~olegviro/es/es.html.

[20] G. Fischer, *Plane Algebraic Curves*, vol. 15 of *Student Mathematical Library.* American Mathematical Society, 2001.

[21] A. Costa and H. Parlier, "On Harnack's Theorem and Extensions: A Geometric Proof and Applications," *Conformal Geometry and Dynamics*, vol. 12, pp. 174–186, Oct. 2008.

[22] J. W. Milnor, *Singular Points of Complex Hypersurfaces.* No. 61 in Annals of Mathematics Studies, Princeton University Press, 1968.

[23] G. Albanese, "Transformazione birazionale di una curva algebrica qualunque in un'altra priva di punti multipli," *Rend. della R. Acc. Nazionale dei Lincei*, vol. 33, no. 5, pp. 13–14, 1924.

[24] D. Eisenbud and J. Harris, "The Dimension of the Chow Variety of Curves," *Compositio Mathematica*, vol. 83, no. 3, pp. 291–310, 1992. http://www.numdam.org/item/CM_1992__83_3_291_0/.

[25] A. M. Vermeulen, "Weierstrass points of weight two on curves of genus three." Doctor Thesis, Universiteit van Amsterdam, 1983.

[26] A. Ikeda, "The Varieties of Tangent Lines to Hypersurfaces in Projective Spaces." https://arxiv.org/abs/1012.2186, 2010.

[27] I. Coskun, M. Hadian, and D. Zakharov, "Dense PGL-Orbits in Products of Grassmannians." https://arxiv.org/abs/1405.0079v2, 2015.

[28] L. T. Moore and J. H. Neelley, "Rational Tacnodal and Oscnodal Quartic Curves Considered as Plane Sections of Quartic Surfaces," *American Journal of Mathematics*, vol. 50, no. 3, pp. 467–472, 1928. http://www.jstor.org/stable/2370814.