1

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \rightsquigarrow A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

cofactor along second column

$$2-\lambda \cdot \underbrace{\det\left(\begin{bmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}\right)}_{(-\lambda)(3-\lambda)+2}$$

$$(2-\lambda)(\lambda^2 - 3\lambda + 2) = \underbrace{-\lambda^3 + 5\lambda^2 - 8\lambda + 4}$$

set $\lambda = 0$

$$(2-\lambda)(\lambda-2)(\lambda-1)$$

EVs $\lambda = 2$, $\lambda = 1$

$\underline{\lambda = 2 \text{ case}}$

$$M_2 = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$[M_2 \mid 0]^{\text{ref}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow x + z = 0 \qquad y, z, \text{ free}$$

2 eigenvectors for 2

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{2D eigenspace.}$$

$\lambda = 1$ case $\quad [M_1 | 0]^{\text{ref}}$

$$M_1 = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x \quad + 2z = 0 \\ y - z = 0 \end{matrix}$$

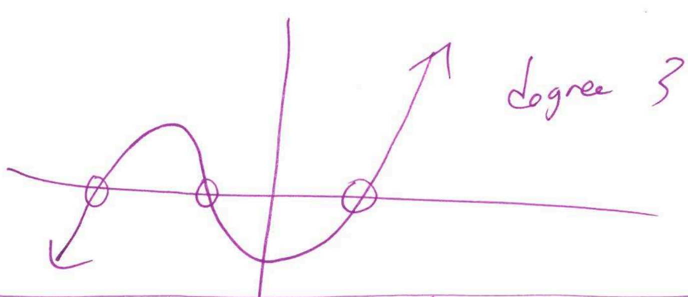
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} \rightsquigarrow \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \quad 1D.$$

~~Eigenvalues for diagonal matrices~~

~~① EVs for~~ $\cancel{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$

Fact A is $n \times n$, then $\chi_A(\lambda)$ is a polynomial of degree n.

so $\chi_A(\lambda) = 0$ has at most n- solutions.



Theorem
A matrix has $\leq n$ eigenvalues!

Theorem $\rightsquigarrow$ Constant term of χ_A is $\det(A)$.

cor: A inv $\iff$ 0 not an e.v.

EVs of diagonal matrices

E.g./

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -8 \end{bmatrix} \leftarrow \text{Find the EVs.}$$

3, 2, -8 & the Evects

$M\hat{\imath} = 3\hat{\imath}$ $M\hat{\jmath} = 2\hat{\jmath}$ $M\hat{k} = -8\hat{k}$

Espace 3 $\rightsquigarrow$ span$\{\hat{\imath}\}$
Espace 2 $\rightsquigarrow$ span$\{\hat{\jmath}\}$
espace -8 $\rightsquigarrow$ span$\{\hat{k}\}$.

Diag Matrices are nice in a lot of ways.

$$M = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$\det M = xyz$

$$M^{-1} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$M^n = \begin{bmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \end{bmatrix}$$

E.V.s x, y, z

E vects $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Geometrically: Scale in $\hat{e}_i$ direction by a_{ii}

Diagonalization

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Eigenvectors

$$\left.\begin{matrix} \underbrace{v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\lambda = 2}, \quad \underbrace{v_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\lambda = 1} \end{matrix}\right\}$$

Form a basis!

Consider $\tilde{A} \longleftarrow$ same function but in new basis.

Question: What is $\tilde{A}$?

Matrix Recon. Theorem

$$\tilde{A} = \left(\tilde{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ \ \tilde{A}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ \ \tilde{A}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= \left(\tilde{A}\vec{v}_1 \ \ \tilde{A}\vec{v}_2 \ \ \tilde{A}\vec{v}_3 \right)$$

$$= \left(2\vec{v}_1 \ \ 2\vec{v}_2 \ \ \vec{v}_3 \right)$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftarrow \text{diagonal matrix!!}$$

So... if $B = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

$$B^{-1}AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is diagonal!!

Cool application

A^{10} seems really hard...

but...

$$\tilde{A}^{10} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{A}^{10} = (B^{-1}AB)^{10} = \overbrace{(B^{-1}AB)(B^{-1}AB)\cdots(B^{-1}AB)}^{10 \text{ times}}$$

cancel

$$= B^{-1}A^{10}B$$

Solving $$A^{10} = B\tilde{A}^{10}B^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\cancel{= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1536 & 0 & 1024 \\ -512 & 512 & -512 \\ -0.5 & 0 & 0 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1024 & 0 & 2048 \\ 1024 & 1024 & 1024 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{bmatrix}$$

Question:

Is there a basis for A when A is diagonal?

Partial Answer

If there is a basis for $\mathbb{R}^n$ of eigenvectors... yes!

Pf/

$$\lambda_1 \rightsquigarrow v_1 \qquad B = [v_1 \cdots v_2]$$

$$\lambda_2 \rightsquigarrow v_2 \qquad B^{-1}AB = \tilde{A} = [\tilde{A}v_1 \cdots \tilde{A}v_n]$$

$$\vdots \qquad\qquad = [\lambda_1 v_1 \cdots \lambda_n v_n]$$

$$\lambda_n \rightsquigarrow v_n \qquad = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Def^n A matrix is diagonalizable if $\exists$ B s.t. $\tilde{A} = B^{-1}AB$ is diagonal. $\tilde{A}$ is called diagonalization of A.

~~Example~~

Example: Suppose A diagonalizable, & all eigenvalues are 1. Then $A = I$

Pf/

$$\tilde{A} = B^{-1}AB = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = I$$

so $B^{-1}AB = I$

so $A = BIB^{-1} = I$

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix}$$

$$\rightsquigarrow \quad \lambda(A) = (1-\lambda)^2 \qquad \text{so} \quad \lambda = 1 \quad \text{only E.V.}$$

Eigenvector:

$\lambda = 1$

$$M = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Null M $\quad \left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{red}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \quad y = 0 \quad x \text{ free}$

Eigen space $\quad \begin{matrix} x = t \\ y = 0 \end{matrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = \text{span}\{\hat{\imath}\}$

$\uparrow$ 1D

SO! 1 eigenspace of dim 1

$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ $\quad \lambda = 3 \quad V_1 = \text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} \longleftarrow$ 1D

$\lambda = -1 \quad V_2 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \longleftarrow$ 1D

$\underline{+}$

2

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Basis $\mathbb{R}^2$ $\Longrightarrow$ diag

$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ $\quad \lambda = 2 \quad V_1 = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \longleftarrow$ 2D

$V_2 = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\} \longleftarrow$ + 1D

3

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ Basis $\mathbb{R}^3$ $\Longrightarrow$ diag

$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda = 1 \quad V_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \leftarrow$ 1D

$1 \neq 2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ not a basis.

Theorem

A is diagonalizable exactly when the sums of the dimensions of the eigenspaces is n.

Cor: If A has n distinct e.v's. It's diag. | $\dim V_1 \geq 1$, $\dim V_2 \geq 1$ $\cdots$ $\dim V_n \geq n$ $\Rightarrow$ $\sum \dim V_i \geq n$ so $\sum \dim V_i = n$

Application

Finding a matrix from eigendata.

Know an eigenbasis for A

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad \lambda = 2$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \lambda = 5$$

$$\rightsquigarrow \tilde{A} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

So $A = B\tilde{A}B^{-1}$

$$= \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ -25 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 6 \\ -45 & 20 \end{bmatrix}$$

$$\begin{bmatrix} -13 & 6 \\ -45 & 20 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} -13 & 6 \\ -45 & 20 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$