Homework 11

Due Monday, May 4th

In this assignment we fill the proofs of a few crucial lemmas from lecture and takehome 3. Studying semidirect products reduces to the study of automorphism groups, so our first goal is to get a good way to decompose them. Here is the goal:

Lemma 1. *Let H and K be finite groups whose orders are coprime. Then*

$$\operatorname{Aut}(H \times K) \cong \operatorname{Aut} H \times \operatorname{Aut} K.$$

The following definition will be useful.

Definition 1. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

1. Let's prove Lemma 1.

   (a) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup* (recall the definition from HW9 Problem 1). Fix any automorphism $\varphi \in \operatorname{Aut} G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).

   Proof. A priori, we only have that $\varphi|_H : H \to G$. Nevertheless, since $H \operatorname{char} G$, we know $\varphi(H) = H$. In particular, for all $h \in H$ we know $\varphi(h) \in H$, so that we may view φ as a map from H to itself. It is an injective homomorphism since φ is, and surjectivity follows because $\varphi(H) = H$. $\square$

   (b) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\operatorname{Aut} G \to \operatorname{Aut} H$.

   Proof. The fact that it is well defined is part (a). It remains to show that if $\varphi, \psi, \in \operatorname{Aut} G$, then

   $$(\varphi \circ \psi)|_H = \varphi|_H \circ \psi|_H.$$

   One immediately checks this by evaluating both sides on an arbitrary element of H. $\square$

   (c) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.

   Proof. We show $H \operatorname{char} H \times K$ and remark that situation for K is identical. Fix an automorphism $\varphi : H \times K \to H \times K$, and a nontrivial element $(h, 1) \in H \leq H \times K$, and note that $|(h, 1)| = m$ divides $|H|$. Consider $(h', k) = \varphi(h, k)$. Since φ is an isomorphism, we know (HW3 Problem 1(e)) that $|(h', k)| = m$ as well. In particular, we know that $k^m = 1$ in K, so that $|k|$ divides m. Thus $|k|$ divides both $|H|$ and $|K|$. Since they are coprime, their only common divisor is 1, so $k = 1$ and $(h', k) \in H$. This shows that $\varphi(H) \leq H$. Since H is finite, and φ is injective, the order of H and $\varphi(H)$ must agree, so that in fact $\varphi(H) = H$ as desired. $\square$

(d) With H, K as in (c), construct an isomorphism $\operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$.

Proof. We define $\Phi : \operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$ via the rule:

$$\Phi(\varphi) = (\varphi|_H, \varphi|_K).$$

This is well defined by (c) and (a), and is a homomorphism by (b). We construct an inverse, $\Psi : \operatorname{Aut} H \times \operatorname{Aut} K \to \operatorname{Aut}(H \times K)$ via the rule. If $\varphi \in \operatorname{Aut} H$ and $\psi \in \operatorname{Aut} K$ then $\Psi(\varphi, \psi) = \varphi \times \psi$, where φ acts on the H coordinate and ψ acts on the K coordinate. That is

$$\begin{aligned} \varphi \times \psi : H \times K &\longrightarrow H \times K \\ (h, k) &\mapsto (\varphi(h), \psi(k)). \end{aligned}$$

One easily checks that $(\varphi \times \psi)|_H = \varphi$ and $(\varphi \times \psi)|_K = \psi$ so that $\Phi(\Psi(\varphi, \psi)) = (\varphi, \psi)$. On the other hand, for $\varphi \in \operatorname{Aut}(H, \times K)$, one notices that

$$\Psi(\Phi(\varphi)) = \varphi|_H \times \varphi|_K = \varphi,$$

and so $\Psi = \Phi^{-1}$ and we are done. $\square$

Recall that any homomorphism $\varphi : K \to \operatorname{Aut} H$ allows us to build a semidirect product $H \rtimes_\varphi K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used the following lemma.

Lemma 2. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \operatorname{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:*

$$\begin{tikzcd} K \arrow[dd, "\gamma"'] \arrow[dr, "\varphi"] & \\ & \operatorname{Aut} H \\ K \arrow[ur, "\psi"'] & \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 3. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitely, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_\alpha : \beta \mapsto \alpha\beta\alpha^{-1}$, and suppose that $\psi = \sigma_\alpha \circ \varphi$:*

$$\begin{tikzcd} & \operatorname{Aut} H \arrow[dd, "\sigma_\alpha"] \\ K \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

2. The lemmas say that if we alter φ by an automorphism of K, or an inner automorphism of $\operatorname{Aut} H$, (or both), we don't change the semidirect products. Let's prove this.

   (a) Consider the setup of Lemma 2. Show that the map:

$$\begin{array}{rcl} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (h,\gamma(k)) \end{array}$$

   is an isomorphism, thereby proving the lemma. Oddly enough it's not really updating.

   Proof. Call the map we are studying Φ. Since γ is bijective, so is Φ. So it suffices to show that Φ is a homomorphism. Fix $(h_i,k_i) \in H \rtimes_\varphi K$ for $i = 1,2$.

$$\begin{array}{rcl} \Phi(h_1,k_1)\Phi(h_2,k_2) & = & \Big(h_1,\gamma(k_1)\Big)\Big(h_2,\gamma(k_2)\Big) \\ & = & \Big(h_1(\psi(\gamma(k_1))(h_2)),\gamma(k_1)\gamma(k_2)\Big) \\ & = & \Big(h_1(\varphi(k_1)(h_2)),\gamma(k_1k_2)\Big) \\ & = & \Phi\Big(h_1(\varphi(k_1)(h_2)),k_1k_2\Big) \\ & = & \Phi\Big((h_1,k_1)(h_2,k_2)\Big), \end{array}$$

   and the result follows. □

   (b) Consider the setup of Lemma 3. Show that the map:

$$\begin{array}{rcl} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (\alpha(h),k) \end{array}$$

   is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_α is an automorphism of $\operatorname{Aut} H$, given by conjugation by α. In unweildy notation, this says $\sigma_\alpha \in \operatorname{Aut}(\operatorname{Aut} H)$.)

   Proof. Call the map we are studying Ψ. Since α is bijective, so is Ψ, so it suffices to show that Ψ is an automorphism. Fix $(h_i,k_i) \in H \rtimes_\varphi K$ for $i = 1,2$.

$$\begin{array}{rcl} \Psi(h_1,k_1)\Psi(h_2,k_2) & = & \Big(\alpha(h_1),k_1\Big)\Big(\alpha(h_2),k_2\Big) \\ & = & \Big(\alpha(h_1)(\psi(k_1)(\alpha(k_2)),k_1k_2\Big) \\ & = & \Big(\alpha(h_1)(\sigma_\alpha(\varphi((k_1)))(\alpha(k_2)),k_1k_2\Big) \\ & = & \Big(\alpha(h_1)(\alpha\varphi(k_1)\alpha^{-1})(\alpha(h_2)),k_1k_2\Big) \\ & = & \Big(\alpha(h_1)\alpha(\varphi(k_1)(h_1)),k_1k_2\Big) \\ & = & \Big(\alpha(h_1(\varphi(k_1)(h_1)),k_1k_2\Big) \\ & = & \Psi\Big(h_1(\varphi(k_1)(h_1)),k_1k_1\Big) \\ & = & \Psi\Big((h_1,k_1)(h_2,k_2)\Big). \end{array}$$

   □

(c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\ K \arrow[r, "\psi"] & \operatorname{Aut} H. \end{tikzcd}$$

That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$. (Hint: This should follow formally from Lemmas 2 and 3, so you shouldn't have to do any lengthy computations).

Proof. We give the function $\sigma \circ \varphi = \psi \circ \gamma$ the name $\xi : K \to \operatorname{Aut} H$. That is, ξ fits into the following diagram:

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] \arrow[dr, dotted, "\xi"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\ K \arrow[r, "\psi"'] & \operatorname{Aut} H. \end{tikzcd}$$

By part (b), we know that

$$H \rtimes_\varphi K \cong H \rtimes_\xi K,$$

and by part (a) we know that

$$H \rtimes_\xi K \cong H \rtimes_\psi K.$$

Combining these two gives the result. $\square$

To prove the uniqueness part of the fundamental theorem of finite abelain groups in Takehome 3, we made use of the following lemma.

Lemma 4. *Let M, M', N, N' groups, and suppose $M \times N \cong M' \times N'$. If M and M' are finite and $M \cong M'$ then $N \cong N'$.*

Remark. *This is a slightly more general restatement of the lemma we used in the takehome. In particular, before we identified M and M' as equal rather than isomorphic, and we assumed that N, N' were finite as well. We will see that this greater generality makes it a bit easier to prove.*

3. Let's explore and prove Lemma 4, and thereby fill the remaining hole in the fundamental theorem of finite abelian groups. It is actually more subtle then you might think.

(a) You will need to make use of the following fact, so we prove it first. If G_1, G_2 are groups and $H_i \trianglelefteq G_i$ for $i = 1, 2$. Then under the usual identifications, $H_1 \times H_2 \trianglelefteq G_1 \times G_2$ and:

$$(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2).$$

Proof. We proved something more general in the solutions to takehome 3 (see TH3 Solutions Lemma 4). We will reprove this special case here. Define a map

$$\Psi : G_1 \times G_2 \to (G_1/H_1) \times (G_2/H_2).$$

It is plainly surjective and the kernel is elements (g_1, g_2) with each $g_i \in H_i$. But this is precisely $H_1 \times H_2$ and the result follows by the first isomorphism theorem. $\square$

(b) Give an example to show that Lemma 4 is not true without the finiteness assumption. (Hint: Let G a nontrivial group and $M = G \times G \times G \times \cdots$ an infinite product of copies of G).

Proof. Let $M = G \times G \times \cdots$ as suggested, and notice that $M \times G$ is also an infinite product of copies of G. Therefore:

$$M \times G \cong M \cong M \times \{1\},$$

where $\{1\}$ is the trivial group. Since $G \ncong \{1\}$ (because by assumption G is nontrivial), we cannot cancel the M's on the left and right side of the equation. □

(c) Identify $M \times N$ and $M' \times N'$ as the same group G. Show that if either $M' \cap N = 1$, or if $M \cap N' = 1$ then Lemma 4 holds. (Hint: 2nd isomorphism theorem).

Proof. First identify $M, N, M', N' \leq G$ as subgroups. In particular, they are all factors of G, and hence normal. We will focus on the case where $M' \cap N = 1$ and remark that the other case is completely symmetric. We consider the following diamond:

$$\begin{tikzcd}
 & G \arrow[d, dash] & & & & & \\
 & M'N \arrow[dl, dash] \arrow[dr, dash] & & & & G \arrow[dl, dash] \arrow[dr, dash] & \\
M' \arrow[dr, dash] & & N \arrow[dl, dash] & = & M' \arrow[dr, dash] & & N \arrow[dl, dash] \\
 & M' \cap N \arrow[d, dash] & & & & 1 & \\
 & 1 & & & & &
\end{tikzcd}$$

Let's say a few words about why these two diamonds are equal. $M' \cap N = 1$ by assumption, so the bottom being 1 is clear. On the other hand, since $M' \cong M$, we know that $|M'| = |M|$ so that:

$$|M'N| = \frac{|M'||N|}{|M' \cap N|} = |M||N| = |M \times N| = |G|,$$

so that $M'N = G$. But now we apply the second isomorphism theorem (remarking that everything in sight is normal):

$$N \cong M'N/M' = G/M' = (M' \times N')/M' \cong N',$$

and we are done. □

(d) Prove Lemma 4 by induction on $|M|$. (Hint: The base case is easy (why?). For the general case, notice that if $H = M \cap N'$ or $K = M' \cap N$ are trivial, we are done by part (b). Otherwise, try manipulating $G/(H \times K)$ to apply induction).

Proof. Under the identifications $H \leq M \leq M \times N = G$ and $K \leq N \leq M \times N = G$, we can also view $H \times K \leq M \times N \leq G$ as well. Then applying part (a), we have:

$$G/(H \times K) = (M \times N)/(H \times K) \cong (M/H) \times (N/K).$$

On the other hand, again applying part (a), we also have:

$$G/(K \times H) = (M' \times N')/(K \times H) \cong (M'/K) \times (N'/H).$$

In summary:

$$(M/H) \times (N/K) \cong (M'/K) \cong (N'/H). \tag{1}$$

We'd like to use induction, because either H or K are trivial (in which case we are done by part (c)), or both M/H and M'/K are smaller then M, so we can apply induction. But they are not the same, so we cannot cancel yet. We have to do something clever to put this in the correct form. We use the assumption here that $M \cong M'$, and "multiply both sides" by M' on the right and M on the left.

$$M \times \Big((M/H) \times N/K)\Big) \cong M' \times \Big((M'/K) \times (N'/H)\Big). \tag{2}$$

We first make the following observations, using part (a) yet again.

$$\begin{aligned} M \times (N/K) &= (M/1) \times (N/K) \\ &\cong (M \times N)/K \\ &= G/K \\ &= (M' \times N')/K \\ &\cong (M'/K) \times N' \end{aligned}$$

And similarly:

$$\begin{aligned} M' \times (N'/H) &\cong (M' \times N')/H \\ &= (M \times N)/H \\ &\cong (M/H) \times N. \end{aligned}$$

Therefore we can adjust the left hand side of Equation 2 to:

$$M \times \Big((M/H) \times (N/K)\Big) = \Big(M \times (N/K)\Big) \times M/H \cong \Big((M'/K) \times N)\Big) \times M/H.$$

And the right hand side to:

$$M' \times \Big((M'/K) \times (N'/H)\Big) = \Big(M' \times (N'/H)\Big) \times M'/K \cong \Big((M/H) \times N')\Big) \times M'/K.$$

Reordering the terms, Equation 2 becomes:

$$M'/K \times M/H \times N \cong M'/K \times M/H \times N'.$$

Since $|M'/K| < |M|$, by induction we can cancel:

$$M/H \times N \cong M/H \times N'.$$

Since $|M/H| < |M|$, by induction we can cancel again:

$$N \cong N',$$

completing the argument. $\square$

4. Let's finish with a classification problem. Classify all groups of order 20 up to isomorphism. How many are there total? (You may use that if p is prime, then $\mathrm{Aut}(Z_p) \cong Z_{p-1}$).

Proof. Let $|G| = 20$, and note that $20 = 2^2 * 5$. Let $P \leq G$ by a Sylow 2-subgroup and $Q \leq G$ a Sylow 5-subgroup. We now reference our *table of stuff*, which tells us that if $|G| = p^2 q$ and $q > p$ then either the Sylow q-subgroup is normal, or else $G \cong A_4$. Since $|G| \neq 12$, we know it cannot be A_4, so we conclude that $Q \trianglelefteq G$ is normal. This immediately implies $PQ \leq G$ is a subgroup. Furthermore, since P and Q have coprime orders (4 and 5 respectively), we know by Lagrange's theorem that $P \cap Q = 1$. Finally:

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = 20,$$

so that:

$$G = PQ \cong Q \rtimes P.$$

So it remains to classify all possible maps:

$$P \to \mathrm{Aut}\, Q.$$

Since $|Q| = 5$, we know it is isomorphic to Z_5 so that $\mathrm{Aut}(Q) \cong Z_4$. On the other hand, $|P| = 4$. There are 2 groups of order 4, the cyclic group and the Klein 4 group. We must treat each case seperately.

Case 1: $P \cong Z_4$

We are now classifying maps $\varphi : Z_4 \to Z_4$. Let $Z_4 = \langle x \rangle$. Then φ is determined by the image of x, and $|\varphi(x)|$ must divide $|x| = 4$. But every element of Z_4 has order dividing 4. Therefore there are 4 maps, $\varphi_i : x \mapsto x^i$, for $i = 0, 1, 2, 3$. We let $G_i = Z_5 \rtimes_{\varphi_i} Z_4$.

Case 1a: $i = 0$

In this case the map is trivial and we have:

$$G_0 = Z_5 \rtimes_{\varphi_0} Z_4 \cong Z_5 \times Z_4 \cong Z_{20}.$$

Case 1b: $i = 1, 3$

We claim that $G_1 \cong G_3$. Indeed, if we let $\gamma : Z_4 \to Z_4$ be the automorphism $x \mapsto x^3$, the following diagram commutes:

$$\begin{tikzcd}
Z_4 \arrow[dr, "\varphi_1"] \arrow[dd, "\gamma"'] & \\
& \mathrm{Aut}(Z_5) \cong Z_4 \\
Z_4 \arrow[ur, "\varphi_3"'] &
\end{tikzcd}$$

Indeed, since $\varphi_1 = id$ and $\varphi_3 = \gamma$, this is essentially immediate. Therefore by Lemma 2, the desired isomorphism follows.

Although this isn't strictly necessary to get an accurate count we also include how to find generators and relations for this group we let x be a generator for the Z_4 and y be the generator for the Z_5. We must understand what xyx^{-1} is. This should be $\varphi_1(x)(y)$. Since $\varphi_1(x)$ is a order 4 automorphism, for instance $y \mapsto y^2$ (the other one is $y \mapsto y^3$, but as we just argued, by Lemma 2 the choice is symmetric). Therefore:

$$G_1 = \langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^2 \rangle .$$

Finally we compute $Z(G_1)$. Notice that every element in G_1 looks like $x^i y^j$, and that:

$$x(x^i y^j)x^{-1} = x^i y^{2j},$$

which is equal to what we started with if and ony if $j = 0$. Thus if $j \neq 0$, $x^i y^j \notin Z(G_1)$. On the other hand,

$$x^i y x^{-i} = y^{2i},$$

which is only y if $i = 0$. Thus if $i \neq 0$ we have $x^1 \notin Z(G_1)$, so we conclude that $Z(G_1) = 1$.

Case 1c: $i = 2$

We claim that G_2 is not isomorphic to G_i for $i = 0, 1, 3$. We will do this by computing its center. We first notice that $\varphi_2(x) = x^2$ has order 2, and is the unique order 2 element of $Z_4 = \mathrm{Aut}(Z_5)$. Since $\iota : Z_5 \to Z_5$ which is the inversion map $y \mapsto y^{-1}$ has order 2, the action induced by φ_2 must be this inversion action. Therefore G_2 has the following presentation:

$$G_2 = \langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^{-1} \rangle$$

We then notice that $x^2 \in Z(G_2)$. Indeed,

$$x^2 y x^{-2} = \iota(\iota(y)) = (y^{-1})^{-1} = y,$$

and certainly $x^2 x^i x^{-2} = x^i$. Since x^2 commutes with both generators, it is in the center. Since $Z(G_2)$ is nontrivial, G_2 cannot be isomorphic to G_1 (or G_3), and since G_2 is nonabelian it is not G_0. Although we did not need it here, we remark that arguing as in case 1b we can show that nothing else is in the center, so that $Z(G_2) = \langle x^2 \rangle = \{1, x^2\}$.

Case 2: $P \cong Z_2 \times Z_2$

We are now classifying maps $\psi : Z_2 \times Z_2 \to Z_4$. As above we are letting $Z_4 = \langle x \rangle$, and let

$$Z_2 \times Z_2 = \langle a \rangle \times \langle b \rangle = \langle a, b \rangle,$$

where a, b have order 2. Then ψ is determined by where it sends a and b. Since $|\psi(g)|$ must divide $|g|$ (HW3 Problem 1(c)), we see that both $\psi(a), \psi(b) \in \{1, x^2\}$. As before, there are four options, which we will denote by $\psi_{j,k}$ for $j, k \in \{0, 1\}$.

$$\psi_{0,0} : \begin{array}{c} a \mapsto 1 \\ b \mapsto 1 \end{array} \qquad \psi_{1,0} : \begin{array}{c} a \mapsto x^2 \\ b \mapsto 1 \end{array}$$

$$\psi_{0,1} : \begin{array}{c} a \mapsto 1 \\ b \mapsto x^2 \end{array} \qquad \psi_{1,1} : \begin{array}{c} a \mapsto x^2 \\ b \mapsto x^2 \end{array}$$

We let $G_{j,k} = Z_5 \rtimes_{\psi_{j,k}} (Z_2 \times Z_2)$.

Case 2a: $j = k = 0$

In this case $\psi_{0,0}$ is trivial so we have:

$$G_{0,0} \cong Z_5 \times (Z_2 \times Z_2) \cong Z_{10} \times Z_2.$$

Case 2b: j, k **not both** 0

We claim that in this case all the $G_{j,k}$ are isomorphic. We remark that

$$Z_2 \times Z_2 \cong \langle a \rangle \times \langle b \rangle \cong \langle a \rangle \times \langle ab \rangle \cong \langle ab \rangle \times \langle b \rangle,$$

and each nontrivial $\psi_{j,k}$ takes 2 generators to x^2 and the third to 1. But $\mathrm{Aut}(Z_2 \times Z_2) = GL_2(\mathbb{F}_2)$ includes all the *change of basis* matrices which takes a pair of generators to any other pair of generators. In particular, if we fix any nontrivial $\psi_{j,k}$ and view $Z_2 = \langle g \rangle \times \langle h \rangle$ as generated by g and h for the two generators sent to x^2 (i.e., where $\psi_{j,k}(g) = \psi_{j,k}(h) = x^2$ and $\psi_{j,k}(gh) = 1$) then there exists some $\eta \in \mathrm{Aut}(Z_2 \times Z_2)$ where $\eta(a) = g$ and $\eta(b) = h$. That is, we have the following:

$$\begin{tikzcd}
Z_2 \times Z_2 \arrow[d, "\eta"] \arrow[dr, "\psi_{1,1}"] & \\
& \mathrm{Aut}(Z_5) \cong Z_4 \\
Z_2 \times Z_2 \arrow[ur, "\psi_{j,k}"'] &
\end{tikzcd}$$

By Lemma 2 this shows $G_{1,1} \cong G_{i,j}$, so that all three nontrivial $G_{i,j}$'s must be isomorphic. Let's extract generators and relations for $G_{1,1}$. We notice that a and b are both sent via $\psi_{1,1}$ to the inversion automorphism $\iota : y \mapsto y^{-1}$ of Z_5 (arguing as above in Case 1c that ι is the only automorphism of order 2). Therefore conjugating the generator y of Z_5 corresponds to inversion. Thus we have:

$$G_{1,1} \cong \left\langle y, a, b \mid y^5 = a^2 = b^2 = 1,\ ab = ba,\ aya^{-1} = byb^{-1} = y^{-1} \right\rangle.$$

We did it!!! There are 5 groups (up to isomorphism): let's summarize in the following table.

Group	Product notation	Presentation
G_0	Z_{20}	$\langle x \mid x^{20} = 1 \rangle$
$G_{0,0}$	$Z_{10} \times Z_2$	$\langle x, y \mid x^{10} = y^2 = 1,\ xy = yx \rangle$
G_1	$Z_5 \rtimes_{\varphi_1} Z_4$	$\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^2 \rangle$
G_2	$Z_5 \rtimes_{\varphi_2} Z_4$	$\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^{-1} \rangle$
$G_{1,1}$	$Z_5 \rtimes_{\psi_{1,1}} (Z_2 \times Z_2)$	$\langle y, a, b \mid y^5 = a^2 = b^2 = 1,\ ab = ba,\ aya^{-1} = byb^{-1} = y^{-1} \rangle$

$\square$