Homework Assignment 5

Due Friday, February 25

In this assignment we answer the following question:

Question 1. *Let G be a group, and $H \leq G$ a subgroup. When is G/H a group?*

More specifically we are asking when the set of a cosets a group under the mutliplication rule given by $(g_1H)(g_2H) = g_1g_2H$. We want a way to answer the question *intrinsically* to G and H. For this we recall the following definition from the 2/22 lecture.

Definition 2. *Let G be a group and $H \leq G$ a subgroup. For $g \in G$ the* ***conjugate of** H **by** g *is the set:*

$$gHg^{-1} = \{ghg^{-1} \ : \ h \in H\}.$$

We say that H is a ***normal subgroup*** *if for every $g \in G$ we have $gHg^{-1} = H$. If $H \leq G$ is a normal subgroup, we write $H \trianglelefteq G$.*

An intrinsic answer to Question 1 is given by the following theorem.

Theorem 3. *Let G be a group and $H \leq G$ a subgroup. The following are equivalent.*

(i) $H \trianglelefteq G$

(ii) G/H *is a group under the rule* $(g_1H)(g_2H) = g_1g_2H$.

(iii) H *is the kernel of a group homomorphism with domain* G.

1. There is only one goal in this assignment: to prove Theorem 3. To achieve this goal, we will prove $(i) \implies (ii) \implies (iii) \implies (i)$.

   (a) Suppose $H \trianglelefteq G$. Show that G/H is a group under the rule $(g_1H)(g_2H) = g_1g_2H$. This shows that $(i) \implies (ii)$.

   (b) Suppose that G/H is a group under the rule $(g_1H)(g_2H) = g_1g_2H$. Produce a group homomorphism from G to some group whose kernel is H. This shows that $(ii) \implies (iii)$. (Note: we essentially gave this argument in the 2/22 lecture, but do include a full proof here as well).

   (c) Let $\varphi : G \to G'$ be a group homomorphism with kernel H. Show that $H \trianglelefteq G$. This proves $(iii) \implies (i)$, thereby completing the proof of Theorem 3.