Homework Assignment 8

Due Friday, March 19

Recall the following important Lemma from the March 11th lecture.

Lemma 1. *Let G be a finite group, and $H \trianglelefteq G$ a normal subgroup. Let $P \leq H$ be a Sylow p subgroup of H. If $P \trianglelefteq H$ then $P \trianglelefteq G$.*

We noted in class that this feels like a normal Sylow subgroup is somehow *strongly* normal, in such a way that we get transitivity of normal subgroups. The following definition makes this precise.

Definition 1 (Characteristic Subgroups)**.** *A subgroup $H \leq G$ is called characteristic in G if for every automorphism $\varphi \in \operatorname{Aut} G$, we have $\varphi(H) = H$. This is denoted by $H \operatorname{char} G$.*

1. Let's prove some basic facts about characteristic subgroups and use them to prove Lemma 1.
   (a) Show that characteristic subgroups are normal. That is, if $H \operatorname{char} G$ then $H \trianglelefteq G$.
   (b) Let $H \leq G$ be the unique subgroup of G of a given order. Then $H \operatorname{char} G$.
   (c) Let $K \operatorname{char} H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$. (This is the transitivity statement alluded to, and justifies the feeling that a characteristic subgroup is somehow *strongly normal*).
   (d) Let G be a finite group and P a Sylow p-subgroup of G. Show that $P \trianglelefteq G$ if and only if $P \operatorname{char} G$.
   (e) Put all this together to deduce Lemma 1.

Sylow's theorem and some of the work you did last week makes it easy to prove Cauchy's theorem:

Theorem 1 (Cauchy's Theorem)**.** *Let G be a finite group and p a prime number dividing the order of G. Show that G has an element of order p.*

2. (a) Prove the following strong version of Cauchy's theorem: Suppose G is a finite group of order n, and that p a prime number such that $p^d | n$ for some $d \geq 0$. Prove that G has a subgroup H of order p^d.
   (b) Deduce Cauchy's theorem as a special case of part (a).

3. Let G be a group of order $p^2 q$ for primes $p \neq q$. We will show that G always has a nontrivial *normal* Sylow subgroup.
   (a) Suppose $p > q$. Show that G has a normal subgroup of order p^2.
   (b) Suppose $q > p$. Show that either G has a normal subgroup of order q, or else $G \cong A_4$.
   (c) Explain why a group of order $p^2 q$ for primes $p \neq q$ can never be simple.

4. In class we've alluded many times to the fact that if G is an abelian group of order pq for primes $p \neq q$, then $G \cong Z_{pq}$. Let's prove it.
   (a) Let $x, y \in G$ be two elements of finite order and suppose that $xy = yx$. Conclude that $|xy|$ divides the least common multiple of $|x|$ and $|y|$.
   (b) Let G be an abelian group of order pq for primes $p < q$. Use Cauchy's theorem and part (a) to conclude that G is cyclic. (This completes the argument from class about groups of order pq).

5. Next lets poke and prod $GL_2(\mathbb{F}_p)$.

   (a) Recall the order of $GL_2(\mathbb{F}_p)$ from HW5 problem 3(d). What is the maximal p divisor of $|GL_2(\mathbb{F}_p)|$?

   (b) The subset of *upper triangular matrices* of $GL_2(\mathbb{F}_p)$ is:

   $$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

   The subset of *strictly upper triangular matrices* is:

   $$\overline{T} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

   Show that T and $\overline{T}$ are subgroups of $GL_2(\mathbb{F}_p)$. We will see that they are not normal.

   (c) Show that $\overline{T}$ is a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$ and of T.

   (d) Show that $GL_2(\mathbb{F}_p)$ has $p+1$ Sylow p-subgroups.

   (e) Prove that T is not normal in $GL_2(\mathbb{F}_p)$. (Hint: use Lemma 1).

6. Prove that a group of order 200 cannot be simple.

7. Let $G_1, G_2, \cdots, G_n$ be groups. Show that:

   $$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdot \times Z(G_n).$$

   Conclude that a product of groups is abelian if and only if the factors are.

Let's finish with an important cancellation lemma for direct products.

Lemma 2. *Let M, M', N, N' groups, and suppose $M \times N \cong M' \times N'$. If M and M' are finite and $M \cong M'$ then $N \cong N'$.*

8. Let's explore and prove Lemma 2. It is actually more subtle then you might think.

   (a) You will need to make use of the following fact, so we prove it first. If G_1, G_2 are groups and $H_i \trianglelefteq G_i$ for $i = 1, 2$. Then under the usual identifications, $H_1 \times H_2 \trianglelefteq G_1 \times G_2$ and:

   $$(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2).$$

   (b) Give an example to show that Lemma 2 is not true without the finiteness assumption. (Hint: Let G a nontrivial group and $M = G \times G \times G \times \cdots$ an infinite product of copies of G).

   (c) Identify $M \times N$ and $M' \times N'$ as the same group G. Show that if either $M' \cap N = 1$, or if $M \cap N' = 1$ then Lemma 2 holds. (Hint: 2nd isomorphism theorem).

   (d) Prove Lemma 2 by induction on $|M|$. (Hint: The base case is easy (why?). For the general case, notice that if $H = M \cap N'$ or $K = M' \cap N$ are trivial, we are done by part (b). Otherwise, try manipulating $G/(H \times K)$ to apply induction).