Homework Assignment 10

Due Friday, April 15

Recall the following important Lemma from the April 5th lecture.

Lemma 1. *Let G be a finite group, and $H \trianglelefteq G$ a normal subgroup. Let $P \leq H$ be a Sylow p subgroup of H. If $P \trianglelefteq H$ then $P \trianglelefteq G$.*

We noted in class that this feels like a normal Sylow subgroup is somehow *strongly* normal, in such a way that we get transitivity of normal subgroups. The following definition makes this precise.

Definition 1 (Characteristic Subgroups)**.** *A subgroup $H \leq G$ is called characteristic in G if for every automorphism $\varphi \in \operatorname{Aut} G$, we have $\varphi(H) = H$. This is denoted by $H \operatorname{char} G$.*

1. Characteristic subgroups will turn out to be the right type of subgroup to guarantee a transitive property for normality. In this exercise we will establish basic facts about characteristic subgroups, and use it prove Lemma 1. First we will make sure this definition is even necessary.
   (a) Give an example to show that the relation *is a normal subgroup of* is not transitive. That is, give a chain of subgroups $H \leq K \leq G$ such that $H \trianglelefteq K$ and $K \trianglelefteq G$ but $H \not\trianglelefteq K$.
   (b) Show that characteristic subgroups are normal. That is, if $H \operatorname{char} G$ then $H \trianglelefteq G$.
   (c) Let $H \leq G$ be the unique subgroup of G of a given order. Then $H \operatorname{char} G$.
   (d) Let $K \operatorname{char} H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$. (This is the transitivity statement alluded to, and justifies the feeling that a characteristic subgroup is somehow *strongly normal*).
   (e) Let G be a finite group and P a Sylow p-subgroup of G. Show that $P \trianglelefteq G$ if and only if $P \operatorname{char} G$.
   (f) Put all this together to deduce Lemma 1.
2. Next let's poke and prod $GL_2(\mathbb{F}_p)$.
   (a) Recall the order of $GL_2(\mathbb{F}_p)$ from HW6 problem 7(d). What is the maximal p divisor of $|GL_2(\mathbb{F}_p)|$?
   (b) The subset of *upper triangular matrices* of $GL_2(\mathbb{F}_p)$ is:
   $$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$
   The subset of *strictly upper triangular matrices* is:
   $$\overline{T} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$
   Show that T and $\overline{T}$ are subgroups of $GL_2(\mathbb{F}_p)$. We will see that they are not normal.
   (c) Show that $\overline{T}$ is a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$ and of T.
   (d) Show that $N_{GL_2(\mathbb{F}_p)}(\overline{T}) = T$.
   (e) Show that $GL_2(\mathbb{F}_p)$ has $p+1$ Sylow p-subgroups.
   (f) Prove that T is not normal in $GL_2(\mathbb{F}_p)$. (Hint: you could do this directly, or you could use Lemma 1).

3. Let's establish a few fundamentals about direct products.

   (a) Suppose $M \cong M'$ and $N \cong N'$. Show that $M \times N \cong M' \times N'$.

   (b) Let $G_1, G_2, \cdots, G_n$ be groups. Show that:

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdot \times Z(G_n).$$

   Conclude that a product of groups is abelian if and only if the factors are.

The notion of characteristic subgroups will allow us to compute automorphism groups of certain direct products.

Lemma 2. *Let H and K be finite groups whose orders are coprime. Then*

$$\operatorname{Aut}(H \times K) \cong \operatorname{Aut} H \times \operatorname{Aut} K.$$

The following definition will be useful.

Definition 2. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

4. Let's study and prove Lemma 2.

   (a) Give an example to show that the condition on the orders of H and K are necessary. That is, give an example of an H and K whose order is not coprime, and where $\operatorname{Aut}(H \times K) \ncong \operatorname{Aut} H \times \operatorname{Aut} K$.

   (b) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup*. Fix any automorphism $\varphi \in \operatorname{Aut} G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).

   (c) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\operatorname{Aut} G \to \operatorname{Aut} H$.

   (d) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.

   (e) With H, K as in (c), construct an isomorphism $\operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$.

To understand groups, it is often useful to break them down into direct products. The following theorem allows us to do this.

Theorem 3 (Recognition Theorem for Direct Products)**.** *Suppose G is a group and $H, K \trianglelefteq G$ are normal subgroups such that $H \cap K = 1$. Then $HK \cong H \times K$. In particular, if we further assume $HK = G$ then $G \cong H \times K$.*

(Recall from the 2nd Isomorphism Theorem that because $H, K \trianglelefteq G$ then $HK \leq G$ is a subgroup).

5. Let's prove Theorem 3

   (a) Let G be a group and $H, K \leq G$ subgroups. Fix $g \in HK$. Show that there are precisely $|H \cap K|$ distinct ways to write $g = hk$ for $h \in H$ and $k \in K$. Deduce that if $H \cap K = 1$ then g can be written uniquely as a product hk for $h \in H$ and $k \in K$.

(b) Suppose that $H, K \trianglelefteq G$ are normal subgroups, and that $H \cap K = 1$. Show that for any $h \in H$ and $k \in K$, $hk = kh$. (*Hint:* show that the commutator $[h, k] = k^{-1}h^{-1}kh$ is in both H and K).

(c) Deduce that the function $\varphi : H \times K \to HK$ given by $\varphi(h, k) = hk$ is an isomorphism, thereby proving Theorem 3.

One way to state the fundamental theorem of finite abelian groups is that a finite abelian group is a product of cyclic groups. A consequence of Theorem 3 is that an abelian group is the product of it's Sylow subgroups, which reduces the proof of the fundamental theorem to the case of p-groups.

6. Let G be an abelian group

(a) Explain why G has a *unique* Sylow p-subgroup for each prime p. This justifies our use of the word *the* in the following.

(b) Suppose G has order $p^\alpha q^\beta$ for distinct primes p and q. Let P be the Sylow p-subgroup, and Q the Sylow q-subgroup. Show that $G \cong P \times Q$.

(c) In general the prime factorization of $|G|$ is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Show by induction on t that G is the product of its Sylow subgroups. Explicitly, this means that if P_i is the Sylow p_i-subgroup for $i = 1, \cdots, t$, then

$$G \cong P_1 \times P_2 \times \cdots \times P_t.$$