

## Homework Assignment 7

Due Friday, March 12

1. Let  $n \geq 3$ . Show that  $Z(S_n) = \{(1)\}$ .
2. Let  $G$  be a group. Prove that  $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$ . The quotient  $\text{Aut}(G)/\text{Inn}(G)$  is called the *outer automorphism group* of  $G$ , and is denoted by  $\text{Out}(G)$ .
3. The converse to Lagrange's theorem holds for groups of prime power order. To prove this we will need to strengthen the fourth isomorphism theorem (HW5#1).
  - (a) Let  $G$  be a group and  $N \trianglelefteq G$ . Let  $N \leq H \leq K \leq G$ , and let  $\overline{H}, \overline{K}$  be the corresponding subgroups of  $G/N$  as in HW5#1. Show that  $|K : H| = |\overline{K} : \overline{H}|$ . (*Hint*: There is an obvious map  $K/H \rightarrow \overline{K}/\overline{H}$ . Prove it is bijective. Be careful though, we don't know that  $K/H$  is a group, just a set of cosets.)
  - (b) Suppose  $|G| = p^d$  for a prime  $p$  and  $d \geq 1$ . Show that for every  $a = 1, 2, \dots, d$ ,  $G$  has a subgroup of order  $p^a$ . (*Hint*: Use what we know about the center of a group of  $p$ -power order and proceed by induction using part (a)).
4. Find all groups with exactly 2 conjugacy classes. (*Hint*: Use the class equation.)

For the next question we remind the reader of the following definitions from linear algebra.

**Definition 1.** Let  $F$  be a field, with additive identity 0 and multiplicative identity 1. An  $F$ -vector space  $V$  is an abelian group  $(V, +)$  together with a scalar multiplication function  $F \times V \rightarrow V$  denoted  $(\lambda, v) \mapsto \lambda v$  such that for all  $u, v \in V$  and  $\lambda, \tau \in F$ :

- (1)  $0v = 0$ .
- (2)  $1v = v$ .
- (3)  $\lambda(\tau v) = (\lambda\tau)v$ .
- (4)  $\lambda(u + v) = \lambda u + \lambda v$ .

Let  $V, W$  be two  $F$ -vector spaces. A function  $\varphi : V \rightarrow W$  is called  $F$ -linear if for all  $u, v \in V$  and  $\lambda \in F$ :

- (1)  $\varphi(u + v) = \varphi(u) + \varphi(v)$ .
- (2)  $\varphi(\lambda v) = \lambda\varphi(v)$ .

5. Fix a prime  $p$  and let

$$V = \underbrace{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \cdots \times \mathbb{Z}/p\mathbb{Z}}_{n \text{ times}}.$$

For  $v = (x_1, \dots, x_n) \in V$ , and  $\lambda \in \mathbb{F}_p$ , define  $\lambda v = (\lambda x_1, \dots, \lambda x_n)$  where multiplication in the coordinates is defined modulo  $p$ .

- (a) Show that  $V$  with scalar multiplication as defined above is an  $\mathbb{F}_p$ -vector space.
- (b) Show that a function  $\varphi : V \rightarrow V$  is a group homomorphism if and only if it is  $\mathbb{F}_p$ -linear.
- (c) Show that  $\text{Aut}(V) \cong GL_n(\mathbb{F}_p)$ . (*Hint*: You may cite Proposition 1 from HW5.)

- (d) Let  $p$  be a prime number and  $G$  a group of order  $p^2$ . What are the possible values for  $|\text{Aut}(G)|$ ? (Use the classification of groups of order  $p^2$  and HW#5 3(d).)
6. We can apply part (5) as follows. Let  $G$  be a group of order  $63 = 3^2 * 7$  and suppose that there is a normal subgroup  $P \trianglelefteq G$  of order 9. We will show that  $G$  is abelian.
- (a) Construct an injective map  $G/C_G(P) \rightarrow \text{Aut } P$ . (*Hint:* Since  $P$  consider the action of  $G$  on  $P$  by conjugation).
- (b) Use 5(d) and Lagrange's theorem to show that  $C_G(P) = G$ . Conclude that  $G$  is abelian. (*Hint:* HW6#2b may be helpful).
7. Let's finish by computing the automorphism group of a  $D_8$ .
- (a) For  $n \in \mathbb{Z}$ , define a homomorphism  $\iota : D_{2n} \rightarrow D_{4n}$  on the generators of  $D_{2n}$  by sending  $\iota(r) = r^2$  and  $\iota(s) = s$ . Show that  $\iota$  is injective and its image is a normal subgroup of  $D_{4n}$ . We abuse notation by saying  $D_{2n} \trianglelefteq D_{4n}$ .
- (b) Show that  $|\text{Aut}(D_8)| \leq 8$ . (*Hint:* If  $\varphi : D_8 \rightarrow D_8$  is an isomorphism, how many options are there for  $\varphi(r)$ . What about for  $\varphi(s)$ ?)
- (c) By part (a),  $D_{16}$  acts on  $D_8$  by conjugation. Use the associated permutation representation to prove  $\text{Aut}(D_8) \cong D_8$ . (*Hint:* This last part requires a couple of steps. Rather than have parts (d),(e),(f),..., let's see if you can follow your nose! If you get stuck you can always ask for hints on the discord.)