Homework Assignment 10

Due Friday, April 15

Recall the following important Lemma from the April 5th lecture.

Lemma 1. *Let G be a finite group, and $H \trianglelefteq G$ a normal subgroup. Let $P \leq H$ be a Sylow p subgroup of H. If $P \trianglelefteq H$ then $P \trianglelefteq G$.*

We noted in class that this feels like a normal Sylow subgroup is somehow *strongly* normal, in such a way that we get transitivity of normal subgroups. The following definition makes this precise.

Definition 1 (Characteristic Subgroups)**.** *A subgroup $H \leq G$ is called characteristic in G if for every automorphism $\varphi \in \operatorname{Aut} G$, we have $\varphi(H) = H$. This is denoted by $H \operatorname{char} G$.*

1. Characteristic subgroups will turn out to be the right type of subgroup to guarantee a transitive property for normality. In this exercise we will establish basic facts about characteristic subgroups, and use it prove Lemma 1. First we will make sure this definition is even necessary.

   (a) Give an example to show that the relation *is a normal subgroup of* is not transitive. That is, give a chain of subgroups $H \leq K \leq G$ such that $H \trianglelefteq K$ and $K \trianglelefteq G$ but $H \ntrianglelefteq G$.

   Proof. Let's give 2 examples. Both use a similar trick of finding a normal abelian subgroup between a non-normal subgroup and the whole group.

   Example 1: We argue that $\langle s\rangle \trianglelefteq \langle r^2, s\rangle \trianglelefteq D_8$ but $\langle s\rangle \ntrianglelefteq D_8$. Indeed, letting $K = \langle r^2, s\rangle$, we see that $[D_8 : K] = 2$ proving that $K \trianglelefteq D_8$. Furthermore, $|K| = 4$ so that it is abelian (by TH1 Problem 4), so any subgroup of it is normal, including $\langle s\rangle \trianglelefteq K$. But we've seen that $\langle s\rangle$ is not normal in D_8 (for example $rsr^{-1} = sr^2$).

   Example 2: Let $G = A_4$, and $K = \{(12)(34), (13)(24), (14)(23), (1)\}$. Then we have seen that $K \trianglelefteq G$ (it is the unique Sylow 2-subgroup), and it is also abelian (isomorphic to V_4), so every subgroup of K is normal. Take $H = \{(12)(34), (1)\} \trianglelefteq K$. We want to observe that H is not normal in G. We can see this by computing:

   $$(123)(12)(34)(123)^{-1} = (23)(14) \notin H.$$

   □

   (b) Show that characteristic subgroups are normal. That is, if $H \operatorname{char} G$ then $H \trianglelefteq G$.

   Proof. Fix $g \in G$. Then $x \mapsto gxg^{-1}$ is an automorphism of G. In particular, it fixes H. Thus $gHg^{-1} = H$ and H is normal. □

   (c) Let $H \leq G$ be the unique subgroup of G of a given order. Then $H \operatorname{char} G$.

   Proof. Let $\varphi \in \operatorname{Aut}(G)$. Then $\varphi(H) \leq G$ is a subgroup of G isomorphic to H. In particular $|\varphi(H)| = |H|$. Since H is the unique subgroup of G with that order, we have that $\varphi(H) = H$. But φ was arbitrary, so $H \operatorname{char} G$. □

(d) Let $K \operatorname{char} H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$. (This is the transitivity statement alluded to, and justifies the feeling that a characteristic subgroup is somehow *strongly normal*).

Proof. Fix $g \in G$. The normality of H implies that $gHg^{-1} = H$, so that conjugation by g induces an automorphism of H. Since K is fixed by automorphisms of H, this means $gKg^{-1} = K$. But $g \in G$ was arbitrary, so K is normal in G. □

(e) Let G be a finite group and P a Sylow p-subgroup of G. Show that $P \trianglelefteq G$ if and only if $P \operatorname{char} G$.

Proof. If $P \operatorname{char} G$ then P is normal by part (b). Conversely, if P is a normal p-Sylow subgroup of G, it is the unique p-Sylow subgroup of G, so that it is the unique subgroup of G with order $|P|$. By part (c) then $P \operatorname{char} G$. □

(f) Put all this together to deduce Lemma 1.

Proof. Let $P \leq H \trianglelefteq G$ as in the statement of the Lemma. If $P \trianglelefteq H$ then by part (e) we have $P \operatorname{char} H$. Therefore by part (d) $P \trianglelefteq G$, completing the proof. □

2. Next let's poke and prod $GL_2(\mathbb{F}_p)$.

(a) Recall the order of $GL_2(\mathbb{F}_p)$ from HW6 problem 7(d). What is the maximal p divisor of $|GL_2(\mathbb{F}_p)|$?

Proof. $p^4 - p^3 - p^2 + p = p(p^3 - p^2 - p + 1)$ and since the second term is one more than a multiple of p, p cannot divide it. So the maximal p divisor of $|GL_2(\mathbb{F}_p)|$ is p itself. □

(b) The subset of *upper triangular matrices* of $GL_2(\mathbb{F}_p)$ is:

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

The subset of *strictly upper triangular matrices* is:

$$\overline{T} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

Show that T and $\overline{T}$ are subgroups of $GL_2(\mathbb{F}_p)$. We will see that they are not normal.

Proof. It is clear that they are both nonempty (for example, they both contain the identity matrix). To show T is a subgroup notice:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay + bz \\ 0 & dz \end{pmatrix} \in T,$$

and

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \frac{1}{ad} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \in T.$$

Similarly, to show that $\overline{T}$ is a subgroup notice:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + y \\ 0 & 1 \end{pmatrix} \in \overline{T},$$

and

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \in \overline{T}.$$

□

(c) Show that $\overline{T}$ is a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$ and of T.

Proof. It's straightforward to see that $|\overline{T}| = p$, which shows the first statement applying part (a). By Lagrange's theorem, p divides the order of T, which divides the order of $GL_2(\mathbb{F}_p)$, so that p is a maximal p divisor of T, proving the second statement. □

(d) Show that $N_{GL_2(\mathbb{F}_p)}(\overline{T}) = T$.

Proof. Since $\overline{T}$ has order p, is cyclic, and any nontrivial element is a generator, so $\overline{T}$ is generated by

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The normalizer of $\overline{T}$ is therefore precisely the elements which conjugate g an element of $\overline{T}$. Let's conjugate g, by some arbitrary matrix τ.

$$\begin{aligned}
\tau g \tau^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{ad - bc} \begin{pmatrix} ad - bc - ac & a^2 \\ c^2 & ad - bc + ac \end{pmatrix}
\end{aligned}$$

If this is in $\overline{T}$ then $c = 0$, and conversely, if $c = 0$ then this is:

$$\frac{1}{ad} \begin{pmatrix} ad & a^2 \\ 0 & ad \end{pmatrix} = \begin{pmatrix} 1 & a^2 \\ 0 & 1 \end{pmatrix} \in \overline{T}.$$

In particular, τ nomralizes σ if and only if $c = 0$ if and only if τ is upper triangular, so the normalizer of $\overline{T}$ is precisely T. □

(e) Show that $GL_2(\mathbb{F}_p)$ has $p + 1$ Sylow p-subgroups.

Proof. Sylow's theorem says that $n_p = |GL_2(\mathbb{F}_p) : N(\overline{T})|$ which by part (d) is $|GL_2(\mathbb{F}_p) : T|$. Then we can compute $|T| = (p-1)^2 * p$. Indeed, given an upper triangular matrix

$$\tau = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

we see it is invertible if and only if $ad = 0$. This means $a, d \neq 0$, giving $p - 1$ choices for each, and b can be any element of $\mathbb{F}_p$. Thus:

$$n_p = |GL_2(\mathbb{F}_p) : T| = \frac{p(p-1)^2(p+1)}{p(p-1)^2} = p + 1.$$

□

(f) Prove that T is not normal in $GL_2(\mathbb{F}_p)$. (Hint: you could do this directly, or you could use Lemma 1).

Proof. Any subgroup is normal in its normalizer, so that by part (d) we know $\overline{T} \trianglelefteq T$. But part (c) said $\overline{T}$ is also a p-Sylow subgroup of T, so in fact $\overline{T} \operatorname{char} T$. Therefore if T were normal, $\overline{T}$ would have to be as well, which we saw in part (e) it is not. □

3. Let's establish a few fundamentals about direct products.

(a) Suppose $M \cong M'$ and $N \cong N'$. Show that $M \times N \cong M' \times N'$.

Proof. Let $\varphi : M \to M'$ and $\psi : N \to N'$ be isomorphisms. Then define $(\varphi \times \psi) : M \times N \to M' \times N'$ via the rule $(\varphi \times \psi)(m, n) = (\varphi(m), \psi(n))$. One checks coordinstewise that it is a homomorphism (at the risk of being overly pedantic we include the computation...)

$$\begin{aligned}\big((\varphi \times \psi)(m,n)\big) \cdot \big((\varphi \times \psi)(m',n')\big) &= \big(\varphi(m), \psi(n)\big) \cdot \big(\varphi(m'), \psi(n')\big) \\ &= \big(\varphi(m)\varphi(m'), \psi(n)\psi(n')\big) \\ &= \big(\varphi(mm'), \psi(nn')\big) \\ &= (\varphi \times \psi)(mm', nn') \\ &= (\varphi \times \psi)\big((m,n) \cdot (m',n')\big).\end{aligned}$$

One can also check that $(\varphi \times \psi)^{-1} = (\varphi^{-1} \times \psi^{-1})$. Indeed:

$$\left((\varphi^{-1} \times \psi^{-1}) \circ (\varphi \times \psi)\right)(m,n) = \left(\varphi^{-1}(\varphi(m)), \psi^{-1}(\psi(n))\right) = (m,n),$$

and similarly for the other direction of composition. □

(b) Let $G_1, G_2, \cdots, G_n$ be groups. Show that:

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdot \times Z(G_n).$$

Conclude that a product of groups is abelian if and only if the factors are.

Proof. If $x = (x_1, \cdots, x_n) \in Z(G_1) \times \cdots Z(G_n)$, so that each $x_i \in Z(G_i)$, then:

$$\begin{aligned} xy &= (x_1, \cdots, x_n)(y_1, \cdots, y_n) \\ &= (x_1 y_1, \cdots, x_n y_n) \\ &= (y_1 x_1, \cdots, y_n x_n) \\ &= (y_1, \cdots, y_n)(x_1, \cdots, x_n) \\ &= yx.\end{aligned}$$

This shows the right side is a subset of the left one. On the other hand, if $x = (x_1, \cdots, x_n) \in Z(G_1 \times \cdots \times G_n)$. Notice that the projection maps $\pi : G_1 \times \cdots \times G_n \to G_i$ is surjective for each i. In particular, each element of $y_i \in G_i$ is $\pi(y)$ for some y in the product group. Notice that:

$$\pi(x) y_i = \pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x) = y_i \pi(x).$$

Thus $\pi(x) = x_i \in Z(G_i)$. Since each coordinate of x is in the center of its respective group, we have $x \in Z(G_1) \times \cdots \times Z(G_n)$, proving the left side includes in the right one, completing the proof.
Now notice that if every G_i is abelian, then:

$$Z(G_1 \times \cdots \times G_n) = Z(G_1) \times \cdots \times Z(G_n) = G_1 \times \cdots \times G_n,$$

so that the product group is abelian. Conversely, if the product group is abelian, fix some $g_i \in G_i$, then $(1, \cdots, g_i, \cdots, 1)$ is in the center of the product group (everything is!), so g_i is in the center of G_i. Since g_i was arbitrary, G_i is abelian. $\square$

The notion of characteristic subgroups will allow us to compute automorphism groups of certain direct products.

Lemma 2. *Let H and K be finite groups whose orders are coprime. Then*

$$\operatorname{Aut}(H \times K) \cong \operatorname{Aut} H \times \operatorname{Aut} K.$$

The following definition will be useful.

Definition 2. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

4. Let's study and prove Lemma 2.

   (a) Give an example to show that the condition on the orders of H and K are necessary. That is, give an example of an H and K whose order is not coprime, and where $\operatorname{Aut}(H \times K) \ncong \operatorname{Aut} H \times \operatorname{Aut} K$.

   Proof. Let $H = K = Z_3$. Then $\operatorname{Aut}(Z_3) \cong (\mathbb{Z}/3\mathbb{Z})^* \cong Z_2$, so that $\operatorname{Aut}(Z_3) \times \operatorname{Aut}(Z_3) \cong Z_2 \times Z_2 = V_4$, and in particular only has 4 elements. On the other hand, by HW8 Problem 7 we know $\operatorname{Aut}(Z_3 \times Z_3) \cong GL_2(\mathbb{F}_3)$ which by HW6 Problem 7(d) has $3^4 - 3^3 - 3^2 + 3 = 48$ elements. $\square$

   (b) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup*. Fix any automorphism $\varphi \in \operatorname{Aut} G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).

   Proof. A priori, we only have that $\varphi|_H : H \to G$. Nevertheless, since $H \operatorname{char} G$, we know $\varphi(H) = H$. In particular, for all $h \in H$ we know $\varphi(h) \in H$, so that we may view φ as a map from H to itself. It is an injective homomorphism since φ is, and surjectivity follows because $\varphi(H) = H$. $\square$

   (c) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\operatorname{Aut} G \to \operatorname{Aut} H$.

Proof. The fact that it is well defined is part (b). It remains to show that if $\varphi, \psi, \in \operatorname{Aut} G$, then

$$(\varphi \circ \psi)|_H = \varphi|_H \circ \psi|_H.$$

One immediately checks this by evaluating both sides on an arbitrary element $h \in H$, and obtaining $\varphi(\psi(h))$ in each case. □

(d) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.

Proof. We show $H \operatorname{char} H \times K$ and remark that situation for K is identical. Fix an automorphism $\varphi : H \times K \to H \times K$, and a nontrivial element $(h, 1) \in H \leq H \times K$, and note that $|(h, 1)| = m$ divides $|H|$. Consider $(h', k) = \varphi(h, 1)$. Since φ is an isomorphism, we know that $|(h', k)| = m$ as well. In particular, we know that $k^m = 1$ in K, so that $|k|$ divides m. Thus $|k|$ divides both $|H|$ and $|K|$. Since they are coprime, their only common divisor is 1, so $k = 1$ and $(h', k) = (h', 1) \in H$. This shows that $\varphi(H) \leq H$. Since H is finite, and φ is injective, the order of H and $\varphi(H)$ must agree, so that in fact $\varphi(H) = H$ as desired. □

(e) With H, K as in (c), construct an isomorphism $\operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$.

Proof. We define $\Phi : \operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$ via the rule:

$$\Phi(\varphi) = (\varphi|_H, \varphi|_K).$$

This is well defined by (d) and (b), and is a homomorphism by (c) (strictly speaking, (c) shows it is a homomorphism in each coordinate, but as we saw in 3(a), this implies that it is a homomorphism). We construct an inverse, $\Psi : \operatorname{Aut} H \times \operatorname{Aut} K \to \operatorname{Aut}(H \times K)$ via the rule. If $\varphi \in \operatorname{Aut} H$ and $\psi \in \operatorname{Aut} K$ then $\Psi(\varphi, \psi) = \varphi \times \psi$, where φ acts on the H coordinate and ψ acts on the K coordinate. That is

$$\begin{aligned} \varphi \times \psi : H \times K &\longrightarrow H \times K \\ (h, k) &\mapsto (\varphi(h), \psi(k)). \end{aligned}$$

One easily checks that $(\varphi \times \psi)|_H = \varphi$ and $(\varphi \times \psi)|_K = \psi$ so that $\Phi(\Psi(\varphi, \psi)) = (\varphi, \psi)$. On the other hand, for $\varphi \in \operatorname{Aut}(H, \times K)$, one notices that

$$\Psi(\Phi(\varphi)) = \varphi|_H \times \varphi|_K = \varphi,$$

and so $\Psi = \Phi^{-1}$ and we are done. □

To understand groups, it is often useful to break them down into direct products. The following theorem allows us to do this.

Theorem 3 (Recognition Theorem for Direct Products)**.** *Suppose G is a group and $H, K \trianglelefteq G$ are normal subgroups such that $H \cap K = 1$. Then $HK \cong H \times K$. In particular, if we further assume $HK = G$ then $G \cong H \times K$.*

(Recall from the 2nd Isomorphism Theorem that because $H, K \trianglelefteq G$ then $HK \leq G$ is a subgroup).

5. Let's prove Theorem 3

(a) Let G be a group and $H, K \leq G$ subgroups. Fix $g \in HK$. Show that there are precisely $|H \cap K|$ distinct ways to write $g = hk$ for $h \in H$ and $k \in K$. Deduce that if $H \cap K = 1$ then g can be written uniquely as a product hk for $h \in H$ and $k \in K$.

Proof. Fix $g = h_0 k_0 \in HK$ with $h_0 \in H$ and $k_0 \in K$ We define a bijection:

$$H \cap K \longleftrightarrow \big\{(h, k) \in H \times K \text{ such that } hk = h_0 k_0\big\}$$

The right side of this equation is precisely the set of ways to write $h_0 k_0$ for some $h \in H$ and $k \in K$, so this bijection will immediately prove the result. We first define a map Φ from the left to the right. Let $x \in H \cap K$, so that $x^{-1} \in H \cap K$. Then we define $\Phi(x) = (h_0 x, x^{-1} k_0) \in H \times K$, and observe that $(h_0 x)(x^{-1} k_0) = h_0 k_0$ so that the map does indeed land in the correct place.

To define an inverse we try to *solve for* x in the construction of the previous paragraph. This suggests we can Ψ from the right side to the left by taking $\Psi(h, k) = h_0^{-1} h$. It is clear that $h_0 h^{-1} \in H$, but also notice that since $hk = h_0 k_0$ then $h_0^{-1} h = k_0 k^{-1} \in K$. Therefore the image of Ψ lands in $H \cap K$.

Finally we'd like to show that these are inverse constructions. Fix $x \in H \cap K$. Then

$$\Psi(\Phi(x)) = \Psi(h_0 x, x^{-1} k_0) = h_0^{-1}(h_0 x) = x.$$

Conversely, fix $(h, k) \in H \times K$ such that $hk = h_0 k_0$, and let $x = h_0^{-1} h = k_0 k^{-1} = (k k_0^{-1})^{-1}$, we can compute that:

$$\Phi(\Psi(h, k)) = \Phi(x) = (h_0 x, x^{-1} k_0) = \big(h_0(h_0^{-1} h), (k k_0^{-1}) k_0\big) = (h, k).$$

The second statement follows immediately from this bijection, as if $H \cap K = 1$ there is only 1 way to write any element of HK as an element in h times one in k. $\square$

(b) Suppose that $H, K \trianglelefteq G$ are normal subgroups, and that $H \cap K = 1$. Show that for any $h \in H$ and $k \in K$, $hk = kh$. (*Hint:* show that the commutator $[h, k] = k^{-1} h^{-1} k h$ is in both H and K).

Proof. Notice that if $[h, k] = k^{-1} h^{-1} k h = 1$, then multiplying on the left by k and then by h produces the equation $kh = hk$, so it suffices to prove that $[h, k] = 1$. Since $K \trianglelefteq G$ we know that $h^{-1} k h \in K$, so that $[h, k] = k^{-1}(h^{-1} k h) \in K$. Similarly, since $H \trianglelefteq G$ we know that $k^{-1} h^{-1} k \in H$, so that $[h, k] = (k^{-1} h^{-1} k) h \in H$. Therefore $[h, k] \in H \cap K = \{1\}$, so that $[h, k] = 1$, as desired. $\square$

(c) Deduce that the function $\varphi : H \times K \to HK$ given by $\varphi(h, k) = hk$ is an isomorphism, thereby proving Theorem 3.

Proof. The fact that φ is injective is exactly follows from part (a). Indeed if

$$\varphi(h, k) = hk = x = h'k' = \varphi(h', k'),$$

then, because there is $|H \cap K| = 1$ way to write x as an element of H times one of K, we can conclude that $h = h'$ and $k = k'$. It is also surjective by definition, any $hk \in HK$

is $\varphi(h,k)$. It remains to show it is a homomorphism. For this we use part (b), elements of h and k commute. Therefore:

$$\varphi(h,k)\varphi(h',k') = hkh'k' = hh'kk' = \varphi(hh',kk') = \varphi\big((h,k)\cdot(h',k')\big).$$

$\square$

One way to state the fundamental theorem of finite abelian groups is that a finite abelian group is a product of cyclic groups. A consequence of Theorem 3 is that an abelian group is the product of it's Sylow subgroups, which reduces the proof of the fundamental theorem to the case of p-groups.

6. Let G be an abelian group

(a) Explain why G has a *unique* Sylow p-subgroup for each prime p. This justifies our use of the word *the* in the following.

Proof. Let $P \leq G$ be a Sylow p-subgroup. Since G is abelian, $P \trianglelefteq G$. All Sylow p-subgroups are conjugate, and P is the only conjugate of P, so it is unique. $\square$

(b) Suppose G has order $p^\alpha q^\beta$ for distinct primes p and q. Let P be the Sylow p-subgroup, and Q the Sylow q-subgroup. Show that $G \cong P \times Q$.

Proof. Notice that $P \cap Q = 1$ by Lagrange's theorem, and that $P, Q \trianglelefteq G$ since G is abelian. Therefore by Theorem 3, we have that $PQ \cong P \times Q$. Also:

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{p^\alpha q^\beta}{1} = |G|,$$

so that $PQ = G$, and the result follows. $\square$

(c) In general the prime factorization of $|G|$ is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$. Show by induction on t that G is the product of its Sylow subgroups. Explicitly, this means that if P_i is the Sylow p_i-subgroup for $i = 1, \cdots, t$, then

$$G \cong P_1 \times P_2 \times \cdots \times P_t.$$

Proof. Let $H_i = P_1P_2\cdots P_i$. We show that $H_i \cong P_1 \times \cdots \times P_i$ by induction. The base case is part (b) (in fact, the base case where $i = 1$ is trivial). For the induction step, notice that:

$$H_i = P_1P_2\cdots P_{i-1}P_i = H_{i-1}P_i.$$

By induction,

$$|H_{i-1}| = |P_1 \times P_2 \cdots \times P_{i-1}| = |P_1||P_2|\cdots|P_{i-1}| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{i-1}^{\alpha_{i-1}}$$

and the order of $P_i = p_i^{\alpha_i}$. Since all the p_i are distinct, these are coprime, so that by Lagrange's theorem, $H_{i-1} \cap P_i = 1$. They are both normal in G since G is abelian so that:

$$H_i = H_{i-1}P_i \cong H_{i-1} \times P_i \cong P_1 \times \cdots \times P_{i-1} \times P_i,$$

where the last step follows by induction. Therefore we see that:

$$|H_t| = |P_1 \times \cdots \times P_t| = p_1^{\alpha_1} \cdots p_t^{\alpha_t} = |G|,$$

so that $H_t = G$ and the result follows. $\square$