Homework 9

Due Monday, April 20th

Recall the following important Lemma from the April 8th lecture.

Lemma 1. *Let G be a finite group, and $H \trianglelefteq G$ a normal subgroup. Let $P \leq H$ be a Sylow p subgroup of H. If $P \trianglelefteq H$ then $P \trianglelefteq G$.*

We noted in class that this feels like a normal Sylow subgroup is somehow *strongly* normal, in such a way that we get transitivity of normal subgroups. The following definition makes this precise.

Definition 1 (Characteristic Subgroups)**.** *A subgroup $H \leq G$ is called characteristic in G if for every automorphism $\varphi \in \operatorname{Aut} G$, we have $\varphi(H) = H$. This is denoted by $H \operatorname{char} G$.*

1. Let's prove some basic facts about characteristic subgroups and use them to prove Lemma 1.
   (a) Show that characteristic subgroups are normal. That is, if $H \operatorname{char} G$ then $H \trianglelefteq G$.
   (b) Let $H \leq G$ be the unique subgroup of G of a given order. Then $H \operatorname{char} G$.
   (c) Let $K \operatorname{char} H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$. (This is the transitivity statement alluded to, and justifies the feeling that a characteristic subgroup is somehow *strongly normal*).
   (d) Let G be a finite group and P a Sylow p-subgroup of G. Show that $P \trianglelefteq G$ if and only if $P \operatorname{char} G$.
   (e) Put all this together to deduce Lemma 1.
2. Recall from HW7 exercise 5 the definition of the subgroup $SL_n(F) \leq GL_n(F)$, which consists of matrices whose determinant is 1. Let's use the tools we've developted to study $SL_2(\mathbb{F}_3)$.
   (a) Compute the order of $SL_2(\mathbb{F}_3)$ (*cf.* HW7 problem 5e).
   (b) Show that the matrices:
   $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
   generate a subgroup $H \leq SL_2(\mathbb{F}_3)$ which is isomorphic to Q_8.
   (c) Conclude (*cf.* takehome 2 problem 3) that $SL_2(\mathbb{F}_3)$ and S_4 are 2 nonisomorphic groups of the same order. (We point out that this is in contrast to $GL_2(\mathbb{F}_2)$ being isomorphic to S_3.)
   (d) Compute the number of Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$.
   (e) Show that the subgroup defined in part (b) is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$. (Hint, use a counting argument together with part (d)).
   (f) Show that $Z(SL_2(\mathbb{F}_3)) = \{\pm I\}$ where I is the identity matrix. (You will need to use what you learned in parts (d) and (e) together with the computation of $Z(Q_8)$ from the takehome).
   (g) Prove that $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3)) \cong A_4$. (Hint: Use what we know about groups of order 12).

3. Next lets poke and prod $GL_2(\mathbb{F}_p)$.

    (a) Recall the order of $GL_2(\mathbb{F}_p)$ from HW7 problem 4(d). What is the maximal p divisor of $|GL_2(\mathbb{F}_p)|$?

    (b) The subset of *upper triangular matrices* of $GL_2(\mathbb{F}_p)$ is:

    $$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

    The subset of *strictly upper triangular matrices* is:

    $$\overline{T} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

    Show that T and $\overline{T}$ are subgroups of $GL_w(\mathbb{F}_p)$. We will see that they are not normal.

    (c) Show that $\overline{T}$ is a Sylow p-subgroup of $GL_2(\mathbb{F}_p)$ and of T.

    (d) Show that $GL_2(\mathbb{F}_p)$ has $p+1$ Sylow p-subgroups (Hint: you only need to exhibit one more than you already have). Conclude that $\overline{T}$ is not normal in $GL_2(\mathbb{F}_p)$.

    (e) Show that $\overline{T} \trianglelefteq T$.

    (f) Conclude that T is not normal in $GL_2(\mathbb{F}_p)$. (Hint: use Lemma 1).

Skip question 4. We haven't talked about finite fields of order 4, so it was silly of me to assign.

4. Let's study $SL_2(\mathbb{F}_4)$.

    (a) Compute the order of $SL_2(\mathbb{F}_4)$.

    (b) Give 2 subgroups of order 5 in $SL_2(\mathbb{F}_4)$

    (c) Conclude that $SL_2(\mathbb{F}_4)$ is simple and isomorphic to A_5.

5. Next let's study the dihedral group.

    (a) Let P be a Sylow 2-subgroup of D_{2n}. Show that $N_{D_{2n}}(P) = P$.

    (b) Suppose that $2n = 2^a k$ for some odd k. Show that the number of Sylow 2-subgroups is k.

    (c) List all the Sylow 2-subgroups of D_{2n} if n is odd.

    (d) Give an example of a Sylow 2-subgroup of D_{12}.