Homework Assignment 10

Due Friday, April 16

1. Let R be a ring. Recall that for $a \in R$ we denote the *additive* identity of a by $-a$. Establish the following identities.
   (a) $(-a)b = a(-b) = -ab$
   (b) $(-a)(-b) = ab$
   (c) If $1 \in R$ then $(-1)a = -a$.
   (d) Suppose R is an integral domain. Show that if $a^2 = 1$ then $a = \pm 1$.

2. Let R be a ring with $1 \neq 0$.
   (a) Let $R^\times \subseteq R$ be the set of units of R. Show that $R^\times$ is a group under the multiplication operation of R.
   (b) Suppose that $a \in R$ is a zero divisor. Show that $a \notin R^\times$.
   (c) Suppose R is a subring of some ring S. Show that if $a \in R^\times$ then $a \in S^\times$. Give an example to show the converse is false.

3. Let R be a commutative ring. An element $r \in R$ is called *nilpotent* if there exists a positive n such that $r^n = 0$. A commutative ring is called *reduced* if it has no nonzero nilpotent elements.
   (a) Show that a nilpotent element of a ring is either 0 or a zero divisor.
   (b) Give an example of a ring with a nonzero nilpotent element.
   (c) Show that the sum of nilpotent elements is nilpotent.
   (d) Suppose r is nilpotent. Show that rx is nilpotent for all $x \in R$. (*Note*, in future terminology, (c) and (d) prove that the set of nilpotent elements is an *ideal* of R, which we will call the *nilradical*).
   (e) Suppose R is a commutative ring with $1 \neq 0$, and suppose $r \in R$ is nilpotent. Show that $1 + r \in R^\times$.

4. (a) Let $\{S_i \subseteq R\}$ be a nonempty collection of subrings of R. Show that $\bigcap_i S_i$ is a subring of R.
   (b) Suppose S is a subring of R, and R is a subring of T. Show that S is a subring of T.

5. For a ring R, define the *center* of R to be:

$$Z(R) = \{r \in R \mid ra = ar \text{ for all } a \in R\}.$$

   (a) Show that $Z(R)$ is a subring of R.
   (b) Suppose R has $1 \neq 0$. Show that $R^\times \cap Z(R) \subseteq Z(R^\times)$. (The converse is *not true* in general, but I don't consider this to be obvious. Perhaps we will see an example later).
   (c) Show that the center of a division ring is a field.
   (d) Let $\mathbb{H}$ be Hamilton's quaternions (defined in Lecture 21 or [DF] Example 5 on Page 224). Compute $Z(\mathbb{H})$. (Notice that $\mathbb{H}$ contains a copy of $\mathbb{C}$, is this the center?)

6. Let R be ring, and X any set. Define
$$\mathrm{Maps}(X,R) = \{f : X \to R \mid f \text{ is a function}\}.$$
Define binary operations $+$ and $\times$ as follows.
$$(f+g)(x) = f(x)+g(x) \qquad (f\times g)(x) = f(x)g(x).$$
   (a) Show that $\mathrm{Maps}(X,R)$ is a ring.
   (b) Suppose R is commutative, show that $\mathrm{Maps}(X,R)$ is too.
   (c) Suppose R is unital, show that $\mathrm{Maps}(X,R)$ is too.
   (d) Suppose R is reduced (defined in Problem 3), show that $\mathrm{Maps}(X,R)$ is too.
   (e) Give an example to show that even if R is a field, $\mathrm{Maps}(X,R)$ need not be.
   (f) Give an example to show that even if R is an integral domain, $\mathrm{Maps}(X,R)$ need not be.

7. We now develop an example of rings that appear along the intersection of the algebraic and analytic theory (for example in *functional analysis*). You may use without proof the following facts from elementary calculus: **(1)** If f,g are continuous so are their sum and product. **(2)** If f,g are differentiable then they are continuous and:
$$(f+g)' = f'+g' \qquad (fg)' = f'g+fg'$$
   (a) Let $\mathscr{P}$ be a property of maps from $X \to R$, and let
$$\mathrm{Maps}_{\mathscr{P}}(X,R) = \{f : X \to R \mid f \text{ has property } \mathscr{P}\}.$$
   Suppose that if f and g have property $\mathscr{P}$, then so do $f-g$ and $f\times g$. Show that $\mathrm{Maps}_{\mathscr{P}}(X,R)$ is a subring of $\mathrm{Maps}(X,R)$.
   (b) Let $X = R = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ have property $\mathscr{C}^0$ if f is continuous, and define $C^0(\mathbb{R}) = \mathrm{Maps}_{\mathscr{C}^0}(\mathbb{R},\mathbb{R})$ to be the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Use part (a) to show that $C^0(\mathbb{R})$ is a subring of $\mathrm{Maps}(\mathbb{R},\mathbb{R})$.
   (c) For each $n > 0$ let $f : \mathbb{R} \to \mathbb{R}$ have property $\mathscr{C}^n$ if f has a derivative everywhere, and df/dx has propery $\mathscr{C}^{n-1}$. (So for example, f is $\mathscr{C}^1$ if it is differentiable and its derivative is continuous). Show by induction on n that $C^n(\mathbb{R}) = \mathrm{Maps}_{\mathscr{C}^n}(\mathbb{R},\mathbb{R})$ is a subring of $C^{n-1}(\mathbb{R})$.
   (d) A funcion $f : \mathbb{R} \to \mathbb{R}$ is has property $\mathscr{C}^\infty$ if for each positive n the n'th derivative of f exists and is continuous. (Such a function is also often called *smooth*). Show that $C^\infty(\mathbb{R}) = \mathrm{Maps}_{\mathscr{C}^\infty}(\mathbb{R},\mathbb{R})$ is a subring of $C^n(\mathbb{R})$ for each n. (Hint: rather than prove this directly, you could use (4)).

8. Let A be an abelian group (written additively). Define the *endomorphism ring* of A as follows:
$$\mathrm{End}(A) = \{f : A \to A \mid f \text{ is a homomorpism}\}.$$
Give $\mathrm{End}(A)$ 2 binary operations $+$ and $\times$ as follows:
$$(f+g)(a) = f(a)+g(a) \qquad (f\times g)(a) = f(g(a)).$$
   (a) Prove that $\mathrm{End}(A)$ is a ring.
   (b) Prove that $(\mathrm{End}(A))^\times \cong \mathrm{Aut}(A)$.
   (c) Let E be an elementary abelian p-group of order p^n. Show that $\mathrm{End}(E) \cong M_n(\mathbb{F}_p)$ (You may use that $n\times n$ matrices over a field F correspond to linear maps $F^n \to F^n$. Compare to HW7 Problem 5).