Homework Assignment 7

Due Friday, March 12

1. Let $n \geq 3$. Show that $Z(S_n) = \{(1)\}$.

2. Let G be a group. Prove that that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G, and is denoted by $\mathrm{Out}(G)$

3. The converse to Lagrange's theorem holds for groups of prime power order. To prove this we will need to strengthen the fourth isomorphism theorem (HW5#1).

   (a) Let G be a group and $N \trianglelefteq G$. Let $N \leq H \leq K \leq G$, and let $\overline{H}, \overline{K}$ be the corresponding subgroups of G/N as in HW5#1. Show that $|K : H| = |\overline{K} : \overline{H}|$. (*Hint*: There is an obvious map $K/H \to \overline{K}/\overline{H}$. Prove it is bijective. Be careful though, we don't know that K/H is a group, just a set of cosets.)

   (b) Suppose $|G| = p^d$ for a prime p and $d \geq 1$. Show that for every $a = 1, 2, \cdots, d$, G has a subgroup of order p^a. (*Hint*: Use what we know about the center of a group of p-power order and proceed by induction using part (a)).

4. Find all groups with exactly 2 conjugacy classes. (*Hint*: Use the class equation.)

For the next question we remind the reader of the following definitions from linear algebra.

Definition 1. *Let F be a field, with additive identity 0 and multiplicative identity 1. An F-vector space V is an abelian group $(V, +)$ together with a scalar multiplication function $F \times V \to V$ denoted $(\lambda, v) \mapsto \lambda v$ such that for all $u, v \in V$ and $\lambda, \tau \in F$:*

(1) $0v = 0$.

(2) $1v = 1$.

(3) $\lambda(\tau v) = (\lambda\tau)v$.

(4) $\lambda(u + v) = \lambda u + \lambda v$.

Let V, W be two F-vector spaces. A function $\varphi : V \to W$ is called F-linear if for all $u, v \in V$ and $\lambda \in F$:

(1) $\varphi(u + v) = \varphi(u) + \varphi(v)$.

(2) $\varphi(\lambda v) = \lambda\varphi(v)$.

5. Fix a prime p and let
$$V = \underbrace{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \cdots \times \mathbb{Z}/p\mathbb{Z}}_{n \text{ times}}.$$
   For $v = (x_1, \cdots, x_n) \in V$, and $\lambda \in \mathbb{F}_p$, define $\lambda v = (\lambda x_1, \cdots, \lambda x_n)$ where multiplication in the coordinates is defined modulo p.

   (a) Show that V with scalar multiplication as defined above is an $\mathbb{F}_p$-vector space.

   (b) Show that a function $\varphi : V \to V$ is a group homomorphism if and only if it is $\mathbb{F}_p$-linear.

   (c) Show that $\mathrm{Aut}(V) \cong GL_n(\mathbb{F}_p)$. (*Hint:* You may cite Proposition 2 from HW5.)

(d) Let p be a prime number and G a group of order p^2. What are the possible values for for $|\operatorname{Aut}(G)|$? (Use the classification of groups of order p^2 and HW#5 3(d).)

6. We can apply part (5) as follows. Let G be a group of order $63 = 3^2 * 7$ and suppose that there is a normal subgroup $P \trianglelefteq G$ of order 9. We will show that G is abelian.

   (a) Construct an injective map $G/C_G(P) \to \operatorname{Aut} P$. (*Hint:* Since P consider the action of G on P by conjugation).

   (b) Use 5(d) and Lagrange's theorem to show that $C_G(P) = G$. Conclude that G is abelian. (*Hint:* HW6#2b may be helpful).

7. Let's finish by computing the automorphism group of a D_8.

   (a) For $n \in \mathbb{Z}$, define a homomorphism $\iota : D_{2n} \to D_{4n}$ on the generators of D_{2n} by sending $\iota(r) = r^2$ and $\iota(s) = s$. Show that ι is injective and its image is a normal subgroup of D_{4n}. We abuse notation by saying $D_{2n} \trianglelefteq D_{4n}$.

   (b) Show that $|\operatorname{Aut}(D_8)| \leq 8$. (*Hint*: If $\varphi : D_8 \to D_8$ is an isomorphism, how many options are there for $\varphi(r)$. What about for $\varphi(s)$?)

   (c) By part (a), D_{16} acts on D_8 by conjugation. Use the associated permutation representation to prove $\operatorname{Aut}(D_8) \cong D_8$. (*Hint:* This last part requires a couple of steps. Rather than have parts (d),(e),(f),..., let's see if you can follow your nose! If you get stuck you can always ask for hints on the discord.)