Homework Assignment 10 Due Friday, April 16

1. Let R be a ring. Recall that for $a \in R$ we denote the additive identity of a by -a. Establish the following identities.

- (a) (-a)b = a(-b) = -ab
- (b) (-a)(-b) = ab
- (c) If $1 \in R$ then (-1)a = -a.
- (d) Suppose R is an integral domain. Show that if $a^2 = 1$ then $a = \pm 1$.
- 2. Let R be a ring with $1 \neq 0$.
 - (a) Let $R^{\times} \subseteq R$ be the set of units of R. Show that R^{\times} is a group under the multiplication operation of R.
 - (b) Suppose that $a \in R$ is a zero divisor. Show that $a \notin R^{\times}$.
 - (c) Suppose R is a subring of some ring S. Show that if $a \in R^{\times}$ then $a \in S^{\times}$. Give an example to show the converse is false.
- 3. Let R be a commutative ring. An element $r \in R$ is called *nilpotent* if there exists a positive n such that $r^n = 0$. A commutative ring is called *reduced* if it has no nilpotent elements.
 - (a) Show that a nilpotent element of a ring is either 0 or a zero divisor.
 - (b) Give an example of a ring with a nonzero nilpotent element.
 - (c) Show that the sum of nilpotent elements is nilpotent.
 - (d) Suppose r is nilpotent. Show that rx is nilpotent for all $x \in R$. (Note, in future terminology, (c) and (d) prove that the set of nilpotent elements is an *ideal* of R, which we will call the *nilradical*).
 - (e) Suppose R is a commutative ring with $1 \neq 0$, and suppose $r \in R$ is nilpotent. Show that $1 + r \in R^{\times}$.
- 4. Let $\{S_i \subseteq R\}$ be a nonempty collection of subrings of R. Show that $\bigcap_i S_i$ is a subring of R.
- 5. For a ring R, define the center of R to be:

$$Z(R) = \{ r \in R \mid ra = ar \text{ for all } a \in R \}.$$

- (a) Show that Z(R) is a subring of R.
- (b) Suppose R has $1 \neq 0$. Show that $R^{\times} \cap Z(R) \subseteq Z(R^{\times})$. (The converse is *not true* in general, but I don't consider this to be obvious. Perhaps we will see an example later).
- (c) Show that the center of a division ring is a field.
- (d) Let \mathbb{H} be Hamilton's quaternions (defined in Lecture 21 or [DF] Example 5 on Page 224). Compute $Z(\mathbb{H})$. (Notice that \mathbb{H} contains a copy of \mathbb{C} , is this the center?)
- 6. Let R be ring, and X any set. Define

$$\operatorname{Maps}(X, R) = \{ f : X \to R \mid f \text{ is a function} \}.$$

Define binary operations + and \times as follows.

$$(f+g)(x) = f(x) + g(x) \qquad (f \times g)(x) = f(x)g(x).$$

- (a) Show that Maps(X, R) is a ring.
- (b) Suppose R is commutative, show that Maps(X, R) is too.
- (c) Suppose R is unital, show that Maps(X, R) is too.
- (d) Suppose R is reduced (defined in Problem 3), show that Maps(X, R) is too.
- (e) Give an example to show that even if R is a field, Maps(X, R) need not be.
- (f) Give an example to show that even if R is an integral domain, Maps(X, R) need not be.
- 7. We now develop an example of rings that appear along the intersection of the algebraic and analytic theory (for example in *functional analysis*). You may use without proof the following facts from elementary calculus: (1) If f, g are continuous so are their sum and product. (2) If f, g are differentiable then they are continuous and:

$$(f+g)' = f' + g'$$
 $(fg)' = f'g + fg'$

(a) Let \mathscr{P} be a property of maps from $X \to R$, and let

$$\operatorname{Maps}_{\mathscr{P}}(X,R) = \{f: X \to R \mid f \text{ has property } \mathscr{P}\}.$$

Suppose that if f and g have property \mathscr{P} , then so do f-g ne $f\times g$. Show that Maps $_{\mathscr{P}}(X,R)$ is a subring of Maps (X,R).

- (b) Let $X = R = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ have property \mathscr{C}^0 if f is continuous, and define $C^0(\mathbb{R}) = \operatorname{Maps}_{\mathscr{C}^0}(\mathbb{R}, \mathbb{R})$ to be the set of continuous functions from \mathbb{R} to \mathbb{R} . Use part (a) to show that $C^0(\mathbb{R})$ is a subring of $\operatorname{Maps}(\mathbb{R}, \mathbb{R})$.
- (c) For each n > 0 let $f : \mathbb{R} \to \mathbb{R}$ have property \mathscr{C}^n if f has a derivative everywhere, and df/dx has propery \mathscr{C}^{n-1} . (So for example, f is \mathscr{C}^1 if it is differentiable and its derivative is continuous). Show by induction on n that $C^n(\mathbb{R}) = \operatorname{Maps}_{\mathscr{C}^n}(\mathbb{R}, \mathbb{R})$ is a subring of $C^{n-1}(\mathbb{R})$.
- (d) A funcion $f: \mathbb{R} \to \mathbb{R}$ is has property \mathscr{C}^{∞} if for each positive n the n'th derivative of f exists and is continuous. (Such a function is also often called smooth). Show that $C^{\infty}(\mathbb{R}) = \operatorname{Maps}_{\mathscr{C}^{\infty}}(\mathbb{R}, \mathbb{R})$ is a subring of $C^{n}(\mathbb{R})$ for each \mathbb{R} . (Hint: rather than prove this directly, you could use (4)).
- 8. Let A be an abelian group (written additively). Define the endomorphism ring of A as follows:

$$\operatorname{End}(A) = \{ f : A \to A \mid f \text{ is a homomorpism} \}.$$

Give End(A) 2 binary operations + and \times as follows:

$$(f+g)(a) = f(a) + g(a)$$
 $(f \times g)(a) = f(g(a)).$

- (a) Prove that End(A) is a ring.
- (b) Prove that $(\operatorname{End}(A))^{\times} \cong \operatorname{Aut}(A)$.
- (c) Let E be an elementary abelian p-group of order p^n . Show that $\operatorname{End}(E) \cong M_n(\mathbb{F}_p)$ (You may use that $n \times n$ matrices over a field F correspond to linear maps $F^n \to F^n$. Compare to HW7 Problem 5).