Def A composition series is a sequences
$1=N_0\le N_1\le\cdots\le N_k=G$
s.t
* $N_i \trianglelefteq N_{i+1}$
* N_{i+1}/N_i simple

Ex
$1\le\langle r^2\rangle\le\langle r\rangle\le D_8$
$D_8/\langle r\rangle\simeq\mathbb{Z}_2$
$\langle r\rangle/\langle r^2\rangle\simeq\mathbb{Z}_2$
$\langle r^2\rangle/1\simeq\mathbb{Z}_2$

$1\le\langle s\rangle\le\langle s,r^2\rangle\le D_8$
$D_8/\langle s,r^2\rangle\simeq\mathbb{Z}_2$
$\langle s,r^2\rangle/\langle s\rangle\simeq\mathbb{Z}_2$
$\langle s\rangle/1\simeq\mathbb{Z}_2$

Theorem $|G|=n>1$
1) G has a comp. series
2) Comp factors are unique. ie.
$1=N_0\le N_1\le\cdots\le N_k=G$
$1=M_0\le M_1\le\cdots\le M_s=G$
Then $k=s$ &
$(N_1/N_0, N_2/N_1,\ldots,N_k/N_{k-1})$
$(M_1/M_0, M_2/M_1,\ldots,M_s/M_{s-1})$
are same groups but reordered

Lemma 1 G simple
$1=N_0\le N_1=G$
is a composition series
Pf/ $N_0\trianglelefteq N_1$ | $N_1/N_0\simeq G/1\simeq G$ = simple

Lemma 2 $H\trianglelefteq G$.
G/H simple
$\iff H\le N\trianglelefteq G$ then $N=H$ or $N=G$ | H max'l normal subgroup
Pf/ $\left\{\begin{matrix}\text{Normal subgroups } N\\ H\le N\trianglelefteq G\end{matrix}\right\}\longleftrightarrow\left\{\begin{matrix}\text{Normal subs}\\ \bar N\trianglelefteq G/H\end{matrix}\right\}$
H G ∅ 1, G/H

Prop 1 G finite group & $|G|>1$, then G has a comp series.
Pf/ Induct on $|G|$
Base: $|G|=2$.
$\Rightarrow G\simeq\mathbb{Z}_2$ simple
So lemma 1 $\Rightarrow$ done
General Case
Assume prop1 holds $\forall$ gps order $<|G|$.

Find $H\trianglelefteq G$. $H\ne 1, G$
* If none exist, G simple so Lemma1 $\Rightarrow$ done

Claim: We may assume G/H simple.
Pf/ $|G/H|\le|G|\Rightarrow\exists$ C.S.
$1=\widetilde M_0\le\widetilde M_1\le\cdots\le\widetilde M_s=G/H$
Let $H'=\pi^{-1}(\widetilde M_{s-1})$
$\pi:G\to G/H$

Notice
$H\trianglelefteq H'\trianglelefteq G$
& $\dfrac{G}{H'}\overset{3rd}{\simeq}\dfrac{G/H}{H'/H}\simeq\dfrac{\widetilde M_s}{\widetilde M_{s-1}}$ simple
Replace H w/ H' ∅

Have $H\trianglelefteq G$ w/ G/H simple
$|H|<|G|\Rightarrow$ induction
$1=N_0\le N_1\le\cdots\le N_k=H$
so
$1=N_0\le N_1\le\cdots\le N_k\le N_{k+1}=G$
is a comp series for G.
∅

Prop 2 JH w/ $s=2$
$1=N_0\le N_1\le\cdots\le N_k=G$
$1=M_0\le M_1\le M_2=G$
$\Rightarrow k=2$. &
$(N_1/N_0, N_2/N_1)\leftrightarrow(M_1/M_0, M_2/M_1)$

Lemma 3 $H,K\trianglelefteq G$.
Then $HK\trianglelefteq G$.
Pf/ $HK\le G$ by 2nd iso thm. $h\in H$, $k\in K$, $g\in G$
$ghkg^{-1}=(ghg^{-1})(gkg^{-1})$
$=h'\cdot k'\in H\cdot K$
$H\trianglelefteq G$ $K\trianglelefteq G$ ∅

Pf of prop 2

$$\begin{tikzcd}[column sep=small] & G \arrow[d, dash] & \\ & M_1\cdot N_{k-1} \arrow[dl, dash] \arrow[dr, dash] & \\ M_1 \arrow[dr, dash] & & N_{k-1} \arrow[dl, dash] \\ & M_1\cap N_{k-1} \arrow[d, dash] & \\ & \{1\} & \end{tikzcd}$$

Get
i) $M_1\trianglelefteq M_1N_{k-1}$
ii) $N_{k-1}\trianglelefteq M_1N_{k-1}$
iii) $M_1\cap N_{k-1}\trianglelefteq M_1$
iv) $M_1\cap N_{k-1}\trianglelefteq N_{k-1}$
v) $M_1N_{k-1}/M_1\simeq N_{k-1}/M_1\cap N_{k-1}$
iv) $M_1N_{k-1}/N_{k-1}\simeq M_1/M_1\cap N_{k-1}$

Notice Lemma 2 & 3
G/M_1 simple : G/N_{k-1} simple
1) $\bullet\, M_1N_{k-1}=M_1 \leftrightarrow \bullet\, M_1N_{k-1}=N_{k-1}$
2) $\bullet\, M_1N_{k-1}=G \leftrightarrow \bullet\, M_1N_{k-1}=G$

1) $M_1=M_1\cdot N_{k-1}=N_{k-1}$
$N_{k-2}\trianglelefteq N_{k-1}$ ← simple
$1=N_0$
$\Longrightarrow k=2$
$1=N_0\le N_1\le N_2=G$
$1=M_0\le M_1\le M_2=G$ ($N_1 = M_1$)
Same series done!!

2)

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ M_1 \arrow[dr, dash] & & N_{k-1} \arrow[dl, dash] \\ & M_1\cap N_{k-1} & \end{tikzcd}$$

~~M_1~~ or 1
$M_1\cap N_{k-1} = 1$
$\Rightarrow k=2$
$1=N_0\le N_1\le N_2=G$
$1=M_0\le M_1\le M_2=G$
Done ∅

$G/N_{k-1}\simeq M_1/M_1\cap N_{k-1}$
$G/N_{k-1}\simeq M_1$
$N_{k-1}/M_1\cap N_{k-1}=N_{k-1}$
$\| \quad$ simple
G/M_1
simple
$(N_1/N_0,\ N_2/N_1)$
N_1 G/N_1
M_1 G/M_1
$(M_1/M_0,\ M_2/M_1)$

Prop 3
By induction, JH holds.
Pf/ Induction on $\min\{k,s\}$
Base case
min = 2 (b)
Inductive Step
$H=N_{k-1}\cap M_{s-1}$
* $N_{k-1}=M_{s-1}$
$1=N_0\le\cdots\le N_{k-1}=H$
$1=M_0\le\cdots\le M_{s-1}=H$
Ind $k-1=s-1$ $(k=s)$
$(M_{i+1}/M_i)\longleftrightarrow(N_{i+1}/N_i)$
Adding G/H to each list proves result.

Else $N_{k-1}\ne M_{s-1}$.
Claim $N_{k-1}\cdot M_{s-1}=G$.
Pf/ $N_{k-1}\trianglelefteq N_{k-1}M_{s-1}\trianglelefteq G$ ← Lemma3
Lemma 2 since G/N_{k-1} simple done ∅

$$\begin{tikzcd}[column sep=small] & G=N_{k-1}M_{s-1} \arrow[dl, dash] \arrow[dr, dash] & \\ N_{k-1} \arrow[dr, dash] & & M_{s-1} \arrow[dl, dash] \\ & H=N_{k-1}\cap M_{s-1} & \end{tikzcd}$$

Lemma 4
* $N_{k-1}/H\simeq G/M_{s-1}$
* $M_{s-1}/H\simeq G/N_{k-1}$

simple

Assume $k\le s$
Induction to H
$1\le H_0\le\cdots\le H_t=H$
is a CS for H
so
$1\le H_0\le\cdots\le H_t\le N_{k-1}$
Is a CS for N_{k-1}
So is
$1\le N_0\le\cdots\le N_{k-1}$ is too
Ind $t+1=k-1$ *
$(N_{i+1}/N_i)\longleftrightarrow(H_{i+1}/H_i)\cup(N_{k-1}/H)$ (+)

$1=H_0\le H_1\le\cdots\le H_t\le M_{s-1}$
$1=M_0\le M_1\le\cdots\le M_{s-1}$
2 C.S. $\Rightarrow$ by ind.
$s-1=t+1$ *
$(M_{i+1}/M_i)\longleftrightarrow(H_{i+1}/H_i)\cup(M_{s-1}/H)$ (H)

① $s=k$
Need
$(M_{i+1}/M_i)\overset{(H)}{=}(H_{i+1}/H_i)\cup(M_{s-1}/H,\ G/M_{s-1})$
$=(H_{i+1}/H_i)\cup(G/N_{k-1},\ N_{k-1}/H)$
$\overset{+}{=}(N_{i+1}/N_i)$ DONE
∅