Homework 10

Due Friday, April 24th

Recall the definition of the semidirect product.

Definition 1. *Let H, K be groups, and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism. Denote the induced action of K on H by:*

$$k \cdot h = \varphi(k)(h).$$

The semidirect product of H and K with respect to φ is the set $H \rtimes K = \{(h, k) : h \in H, k \in K\}$, where multiplication is defined by the rule:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1 k_2).$$

1. Let's make sure that $H \rtimes K$ is a group.
   (a) Show that $(1, 1) \in H \rtimes K$ is the identity. (Remember you have to check both sides).
   (b) Show that $(h, k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$. (As above, you have to check both sides).
   (c) Prove that multiplication is associative.

2. Let $G_1, G_2, \cdots, G_n$ be groups. Show that:

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdot \times Z(G_n).$$

   Conclude that a product of groups is abelian if and only if the factors are.

3. Let's classify some abelian groups! List all *abelian* groups of the following orders, in elementary divisor and invariant factor forms.
   (a) 100
   (b) 243
   (c) 9801

4. Which of the following groups of order 80 are isomorphic?
   (a) $Z_5 \times Z_4 \times Z_4$
   (b) $Z_{10} \times Z_8$
   (c) $Z_4 \times Z_{20}$
   (d) $Z_8 \times Z_5 \times Z_2$

5. Let A be an abelian group of (invariant factor) type $(n_1, n_2, \cdots, n_s)$. Show that there exists some element in A of order m if and only if $m | n_1$. Conclude that the exponenet of A is n_1.