Homework Assignment 7

Due Friday, March 13

There are two parts to this homework. The first part outlines a proof of the Jordan-Hölder theorem, while the second introduces a new class of examples of finite groups.

1 Jordan-Hölder

Recall the following defintion from class.

Definition 1. *Let G be a group. A sequence of subgroups:*

$$1 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_{k-1} \leq N_k = G,$$

is called a composition series if for each i $N_i \trianglelefteq N_{i+1}$ and the quotient N_{i+1}/N_i is normal.

The important point is that composition series exist, and are in some sense unique.

Theorem 1 (Jordan-Hölder)**.** *Let G be a finite group with $G \neq 1$. Then,*

(1) G has a composition series.

(2) The composition factors of the composition series are unique. Specifically, this means that if

$$1 = N_0 \leq N_1 \leq \cdots \leq N_k = G,$$

$$1 = M_0 \leq M_1 \leq \cdots \leq M_s = G,$$

are two composition series', then $s = k$ and there is a permutation π of $\{1, \cdots, k\}$ such that

$$M_{\pi(i)}/M_{\pi(i)-1} \cong N_i/N_{i-1},$$

for each i.

1. This first exercise proves the Jordan-Hölder theorem.

   (a) Prove part (1) of the Jordan-Hölder theorem by induction on $|G|$.

   Proof. We should first make a simple observation, but we state is as a lemma so that we can easily refer to it.

   Lemma 1. *If G is a simple group, the then $1 = N_0 \leq N_1 = G$ is a composition series for G.*

   Proof. This is immediate, as $1 \trianglelefteq G$ and the quotient is $G/1 = G$ which is simple by assumption. ☐

   Now lets take care of the base case. This is where $|G| = 2$. Then $G \cong Z_2$ which is simple, and therefore has a composition series given by Lemma 1.

   Now to prove the general case, we are allowed to make the following assumption (the inductive hypothesis): if K is any group and $|K| < |G|$ then K has a composition series. We must now show G has a composition series. If G is simple, then it already has a composition series given by Lemma 1. Otherwise, we can find some nontrivial normal subgroup $H \trianglelefteq G$. Let's first notice that we may assume that G/H is simple. If not, noticing that $|G/H| < |G|$, we may apply the inductive hypothesis and observe that there is a composition series:

$$1 = M_0 \leq M_1 \leq \cdots \leq M_r = G/H$$

Let $\pi : G \to G/H$ be the projection map and let $H' = \pi^{-1}(M_{r-1})$. Then by the fourth isomorphism theorem (the meat of which was homework 6 problem 4), $H \leq H' \trianglelefteq G$, and by the second isomorphism theorem:

$$G/H' = (G/H)/(H'/H) = M_r/M_{r-1},$$

which is simple because the M_i form a composition series. Therefore, replacing H with H', we assume that G/H is simple.
Since $|H| < |G|$, we may apply the inductive hypothesis and assert that H has a composition series. Call it

$$1 = N_0 \leq N_1 \leq \cdots \leq N_m = H.$$

Since G/H is simple, then

$$1 = N_0 \leq N_1 \leq \cdots \leq N_m = H \leq G,$$

is a composition series for G and we are done. □

(b) Prove part (2) if the Jordan-Hölder theorem in the case that $s = 2$. (Hint: Show if H,K are normal subgroups, then so is HK, then use the second isomorphism theorem with M_1 and N_{k-1}).

Proof. Let's begin by proving the lemma suggested in the hint.

Lemma 2. *Let $H, K \trianglelefteq G$ be two normal subgroups. Then HK is a normal subgroup.*

Proof. HK is a subgroup by the second isomorphism theorem. Fix $hk \in HK$ (so $h \in H$ and $k \in K$). Then for any $g \in G$,

$$g(hk)g^{-1} = (ghg^{-1})(gkg^{-1}) = h'k' \in HK,$$

where $h' \in H$ and $k' \in K$ since H, K are normal. Thus HK is normal as desired. □

Let's draw the diamond suggested in the hint.

$$\begin{tikzcd}
 & G \arrow[d, dash] & \\
 & M_1N_{k-1} \arrow[dl, dash] \arrow[dr, dash] & \\
M_1 \arrow[dr, dash] & & N_{k-1} \arrow[dl, dash] \\
 & M_1 \cap N_{k-1} \arrow[d, dash] & \\
 & 1 &
\end{tikzcd}$$

The second isomorphism theorem applies in both directions sinc everything in sight is normal, so this gives us a lot of information. Let's list it so it can be easily referenceed:

i. $M_1 \trianglelefteq M_1 N_{k-1}$
ii. $N_{k-1} \trianglelefteq M_1 N_{k-1}$
iii. $M_1 \cap N_{k-1} \trianglelefteq M_1$
iv. $M_1 \cap N_{k-1} \trianglelefteq N_{k-1}$
v. $M_1 N_{k-1}/M_1 \cong N_{k-1}/M_1 \cap N_{k-1}$
vi. $M_1 N_{k-1}/N_{k-1} \cong M_1/M_1 \cap N_{k-1}$

The first thing I want to zero in on is part iii. Since

$$1 = M_0 \leq M_1 \leq M_2 = G,$$

is a composition series for G, we have that M_1 is simple. Therefore every normal subgroup of M_1 is either the trivial subgroup or all of M_1, that is one of the following two things hold:

- $M_1 \cap N_{k-1} = M_1$
- $M_1 \cap N_{k-1} = 1$

We will deal with these two cases separately, starting with the first case. If this holds then by $M_1/M_1 \cap N_{k-1} = 1$, so that applying part v from the list above so does $M_1 N_{k-1}/N_{k-1}$, so that $N_{k-1} = M_1 N_{k-1}$. In particular, we have $M_1 \leq N_{k-1} \leq G$. By the fourth isomorphism theorem, normal subgroups between M_1 and G correspond to normal subgroups of G/M_1, but as G/M_1 is simple, there are no nontrivial ones. Therefore $M_1 = N_{k-1}$. But as M_1 is simple, the normal subgroup $N_{k-2} \trianglelefteq N_{k-1}$ must be the trivial subgroup 1 which was N_0. In particular, $k = 2$ and the composition series

$$1 = N_0 \leq N_1 \leq N_2 = G,$$

is the same as the composition series

$$1 = M_0 \leq M_1 \leq M_2 = G,$$

so that in the first case above the Jordan Hölder theorem holds. Now we tackle the second case, where $M_1 \cap N_{k-1} = 1$ is trivial. Notice that $M_1 \leq M_1 N_{k-1} \trianglelefteq G$. By the fourth isomorphism theorem then $M_1 N_{k-1}$ corresponds to a normal subgroup $K \trianglelefteq G/M_1$, but G/M_1 is simple so that $K = 1 or G/M_1$. Thus $M_1 N_{k-1}$ is ether M_1 or G. But part v above shows that

$$M_1 N_{k-1}/M_1 \cong N_{r-1}/M_1 \cap N_{k-1} \cong N_{k-1} \neq 1,$$

so that $M_1 N_{k-1} \neq M_1$. Thus $M_1 N_{k-1} = G$. Notice that this implies that $N_{k-1} \cong G/M_1$ so that N_{k-1} is simple. Thus $N_{k-2} \trianglelefteq N_{k-1}$ must be the trivial subgroup $1 = N_0$. Thus $k = 2$ Finally, applying the second isomorphism theorem again (using part vi above) we see that:

$$G/N_1 = M_1 N_{k-1}/N_{k-1} \cong M_1/M_1 \cap N_{k-1} \cong M_1,$$

Therefore, although the two composition series

$$1 = M_0 \leq M_1 \leq M_2 = G,$$

$$1 = N_0 \leq N_1 \leq N_2 = G,$$

may not be exactly the same, we have shown that:

$$M_2/M_1 \cong G/M_1 \cong N_1 \cong N_1/N_0,$$

and,

$$M_1/M_0 \cong M_1 \cong G/N_1 \cong N_2/N_1,$$

so that the two series have the same composition factors, and so the Jordan Hölder theorem holds. □

(c) Prove part (2) of the Jordan-Hölder theorem by induction on the minimum of k and s. (Apply the inductive hypothesis to $H = N_{k-1} \cap M_{s-1}$).

Proof. We will do induction on $\min\{k, s\}$. The base case is where the minimum is 2, and was handled (with some difficulty) in part (b). For the general case, we we assume by induction that if a group has a composition series of length smaller than the minimum of k and s, then the Jordan Hölder theorem holds for that group.

Consider $H = N_{k-1} \cap M_{s-1}$. If $N_{k-1} = M_{k-1}$ the the M_i and the N_i form two composition series for H, which therefore have the same length and composition factors. In particular, $k-1 = s-1$ so that $k = s$ and the sets $\{N_{i+1}/N_i\}$ and $\{M_{i+1}/M_i\}$ agree for $i < k$, but for $i = k$ they are both

$$M_k/M_{k-1} = G/H = N_k/N_{k-1}.$$

Thus the theorem holds. Otherwise $N_{k-1} \neq M_{s-1}$. By the fourth isomorphism theorem (since G/N_{k-1}) is simple there are no normal subgroups strictly, between N_{k-1} and G, so that $N_{k-1}M_{s-1} = G$. Thus we can consider the diamond:

$$\begin{tikzcd}
 & N_{k-1}M_{s-1} = G \arrow[dl, dash] \arrow[dr, dash] & \\
N_{k-1} \arrow[dr, dash] & & M_{s-1} \arrow[dl, dash] \\
 & N_{k-1} \cap M_{s-1} = H &
\end{tikzcd}$$

Therefore we know the following

$$N_{k-1}/H = G/M_{s-1} \tag{1}$$

$$M_{s-1}/H = G/N_{k-1}$$

are simple. We know that H has a composition series:

$$1 = H_0 \leq H_1 \leq \cdots \leq H_t = H.$$

And since N_{k-1}/H is simple then:

$$1 = H_0 \leq H_1 \leq \cdots \leq H_t = H \leq N_{k-1},$$

is a composition series for N_{k-1}. But so is:

$$1 = N_0 \leq N_1 \leq \cdots \leq N_{k-1}.$$

By the induction hypothesis, $t+1 = k-1$, and the composition factors agree, that is the following sets are equal.

$$\{N_{i+1}/N_i\}_{i<k} = \{H_{i+1}/H_i\} \cap \{N_{k-1}/H\}. \tag{2}$$

But an identical argument shows that:

$$1 = H_0 \leq H_1 \leq \cdots \leq H_t = H \leq M_{s-1}$$

and

$$1 = M_0 \leq M_1 \leq \cdots \leq M_{s-1}.$$

are two composition series for M_{s-1}, so that $t+1 = s-1$ and the we have equality

$$\{M_{i+1}/M_i\}_{i<s} = \{H_{i+1}/H_i\} \cap \{M_{s-1}/H\}. \tag{3}$$

Therefore $s = t+2 = k$, and applying the equalities in Equations 1 above we have:

$$\begin{aligned}\{M_{i+1}/M_i\}_{i\leq s} &= \{H_{i+1}/H_i\} \cap \{M_{s-1}/H, G/M_{s-1}\} \\ &= \{H_{i+1}/H_i\} \cap \{G/N_{k-1}, N_{k-1}/H\} \\ &= \{N_{i+1}/N_i\}_{i\leq k},\end{aligned}$$

completing the proof. □

2 Matrix Groups

The rest of the homework introduces a new family of finite groups. So far we've only studies a few examples of finite groups: D_{2n}, S_n and direct products of cyclic groups. As we start defining more exotic properties of groups we will need to expand our library of finite groups to exhibit some of these interesting properties. In this homework we will introduce a new example: finite matrix groups. We will need a definition.

Definition 2. *A field is a set F together with two commutative binary operations, $+$ and $\cdot$ (addition and multiplication), such that $(F, +)$ and $(F\backslash\{0\}, \cdot)$ are abelian groups, and such that the distributive law holds. That is, for all $a, b, c \in F$ we have:*

$$a \cdot (b+c) = a \cdot b + a \cdot c.$$

For any field we let $F^\times = F \setminus \{0\}$ be its mutliplicative group. A field F is called a finite field if $|F| < \infty$.

It turns out that vector space theory over F is pretty much identical to vector space theory over R. We can define the first matrix group we hope to study.

Definition 3. *Let F be a field. If M, N are matrices with entries in F, we can compute their product MN and the determinant $\det(M) \in F$ using the same formulas as if $F = \mathbb{R}$. Then the general linear group of degree n over F is,*

$$GL_n(F) = \{A \mid A \text{ is an } n \times n \text{ matrix with entries in } F \text{ and } \det(A) \neq 0\}.$$

2. It turns out that we have seen examples of finite fields already.

   (a) Let p be a prime number. Show that $\mathbb{Z}/p\mathbb{Z}$ with the operations $+$ and $\times$ is a field. This is the *finite field of order* p and will be denoted by $\mathbb{F}_p$.

   Proof. We already have seen that $\mathbb{Z}/p\mathbb{Z}$ is an abelian group under addition and that $(\mathbb{Z}/p\mathbb{Z})^\times = \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ is an an abelian group under multiplication. Furthermore, multiplication and additiion are inherited from the same operations on $\mathbb{Z}$ which satisfy the distributive law. $\square$

   (b) Show that if n is not prime, $\mathbb{Z}/n\mathbb{Z}$ is not a field.

   Proof. In homework 2 problem 1(a) we showed that $a \in \mathbb{Z}/n\mathbb{Z}$ has a multiplicative inverse if and only if $\gcd(a, n) = 1$. Therefore, letting $a|n$ and $a \neq 1$, we have $a \in \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$ without a multiplicative inverse, so that $\mathbb{Z}/n\mathbb{Z}$ cannot be a field. $\square$

3. Let's study the simplest example of general linear groups: $GL_2(F)$.

   (a) Let $A, B \in GL_2(F)$. Show that $\det(AB) = \det(A)\det(B)$.

   Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

   Then,

$$AB = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix},$$

   And we can compute directly that

$$\begin{aligned} \det(A)\det(B) &= (ad - bc)(xw - zy) \\ &= (adxw - adzy - bcxw + bczy). \end{aligned}$$

   and

$$\begin{aligned} \det(AB) &= (ax + bz)(cy + dw) - (cx + dz)(ay + bw) \\ &= axcy + axdw + bzcy + bzdw - cxay - cxbw - dzay - dzbw \\ &= axdw + bzcy - cxbw - dzay \end{aligned}$$

   observing that they are the same. $\square$

   (b) Show that $\det(A) = 0$ if and only if one row is a multiple of the other.

   Proof. Let A be as in the previous problem, so that $\det(A) = ad - bc$. Then $ad - bc = 0$ if and only if $ad = bc$ if and only if $a/b = c/d$ if the pair (c, d) is a multiple of (a, b) $\square$

   (c) Show that $A^{-1} = \frac{1}{\det(A)}\tilde{A}$ where $\tilde{A}$ is defined by the rule:

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof. This we check directly, noting that the product:

$$\frac{1}{ad-bc}A\tilde{A} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+da \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

(d) Conclude that $GL_2(F)$ is a group, and that $\det : GL_2(F) \to F^\times$ is a homomorphism.

Proof. We see that $GL_2(F)$ has inverses, and that the identity matrix acts as the identity element. Since matrix multiplication is associative, it is a group. Then det is a homomorphism by part (a). □

(e) Prove that $GL_2(F)$ is isomorphic to the group G of linear ismorphisms from $F^2 \to F^2$. (Hint, use matrix multiplication to get a map $GL_2(F) \to G$.)

Proof. Define $\varphi : GL_2(F) \to G$ by $\varphi(M)$ is the linear map $\vec{v} \mapsto M\vec{v}$ where $\vec{v}$ is viewed as a column vector and we do standard matrix multiplication. Then $\varphi(M)$ is a linear map. Furthermore, multiplication in $GL_2(F)$ corresponds to composition in G, so that this is a homomorphism. To construct an inverse, let $f \in G$ be a linear map. Then define $\varphi^{-1}(f)$ to be the matrix whose first column is $f(1,0)$ and whose second column is $f(0,1)$. □

As you may have noticed, this proof went through the same way it does for $F = \mathbb{R}$ in linear algebra. The proofs can get more computationally intense for $GL_n(F)$ as n increases, so for now lets take on faith that $GL_n(F)$ is a group and $\det : GL_n(F) \to F^\times$ is a homomorphism, and that $GL_n(F)$ parametrizes linear automorphisms of F^n.

4. Now let's study $GL_2(\mathbb{F}_p)$.

(a) Prove that $|GL_2(\mathbb{F}_2)| = 6$.

Proof. We will do parts (a) and (b) together, listing all the elements to see that there are 6 of them. A general matrix looks like

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Notice first that a and b can't both be 0. In each case, we fix (a,b) and leverage the fact that (c,d) must not be a multiple of (a,b). There are 3 cases, each with two possibilities. First, $a=1$ and $b=0$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The case wehre $a=0$ and $b=1$ is similar.

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Finally, we have the case $a=b=1$.

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus there are six elements. □

(b) Write all the elements of $GL_2(\mathbb{F}_2)$ and compute the order of each element.

Proof. $A = I$ is the identity element and therefore has order 1. One can compute directly that $B^2 = C^2 = E^2 = I$ are the identity as well, thus they have order 2. But notice that $D^2 = F$ and that F^2 is D. Nevertheless, we notice that $F = D^{-1}$ so that $DF = FD = I$ so that $D^3 = F^3 = I$ so that they have order 3. □

(c) Show that $GL_2(\mathbb{F}_2)$ is not abelian. (We will later see that it is isomorphic to S_3).

Proof. We can check directly that $BC = D$ and that $CB = F$.
This isn't part of the assignment, but one can directly check now that $\varphi : GL_2(\mathbb{F}_2) \to S_3$ given by the rule:

$$\begin{aligned} A &\mapsto (1) \\ B &\mapsto (12) \\ C &\mapsto (23) \\ D &\mapsto (123) \\ E &\mapsto (13) \\ F &\mapsto (132) \end{aligned}$$

is an isomorphism. □

(d) Generalizing part (a), show that if p is prime then

$$|GL_2(\mathbb{F}_p)| = p^4 - p^3 - p^2 + p.$$

Use exercise 3(b).

Proof. As usual, we fix some:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We want to count how many choices of (a, b, c, d) we have. Let's notice that the total number should be the product of the choices for (a, b) with the choices for (c, d) having fixed (a, b). Let's begin by counting the number of choices fo (a, b). They must be selected from $\mathbb{F}_p$, so that we have p choices each for a and p, so there are p^2 total choices, but since they cannot both be 0, $p^2 - 1$ allowable ones. Now all we have to say is that (c, d) is not a multiple of (a, b). That is, there are p^2 choices for (c, d), but p of them are $(\lambda a, \lambda b)$ for all the different $\lambda \in \mathbb{F}_p$. In particular, there are $p^2 - p$ allowable choices. Thus the total is:

$$(p^2 - 1)(p^2 - p) = p^4 - p^3 - p^2 + p.$$

□

5. The general linear group has lots of interesting subgroups and quotients.

(a) Show that the constant diagonal matrices are a normal subgroup of $GL_n(F)$ isomorphic to $F^\times$

Proof. Define a homomorphism $\varphi : F^\times \to GL_n(F)$ given by the rule:

$$\varphi(\lambda) = \lambda \cdot I = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

This map is certainly injective, and its image is precisely the constant diagonal matrices, so that the constant diagonal matrices are a subgroup isomorphic to $F^\times$. To see they are normal, notice:

$$M\varphi(\lambda)M^{-1} = M\lambda \cdot IM^{-1} = \lambda \cdot (MIM^{-1}) = \lambda \cdot I = \varphi(\lambda).$$

(In fact, we just showed that the constant diagonal matrices are contained in the *center* of $GL_n(F)$). $\square$

We will often abuse notation and denote this by $F^\times \trianglelefteq GL_n(F)$. The quotient group $GL_n(F)/F^\times$ is called the *projective general linear group* and denoted $PGL_n(F)$.

(b) The *special linear group* $SL_n(F)$ is defined

$$SL_n(F) = \{A \in GL_n(F) \mid \det(A) = 1.\}$$

Show that $SL_n(F)$ is a normal subgroup of $GL_n(F)$.

Proof. It is the kernel of the homomorphism det so it is automatically a normal subgroup. $\square$

(c) Prove the following isomorphism.

$$GL_n(F)/SL_n(F) \cong F^\times.$$

Proof. If *det* is a surjective homomorphism, this follows immediately from the first isomorphism theorem. But fix $\lambda \in F^\times$. Then letting A be the matrix

$$A = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we notice that $\det(A) = \lambda$. $\square$

(d) List all the elements of $SL_2(\mathbb{F}_2)$.

Proof. This is a bit of a trick question. Indeed, in $\mathbb{F}_2$ the only element not equal to 0, is 1. Thus if $\det(A) \neq 0$ then $\det(A) = 1$. Thus $SL_n(\mathbb{F}_2) = GL_n(F_2)$ and in partilar, the answer is the same as 4(b). $\square$

(e) Use problem 3(d) to compute $|SL_2(\mathbb{F}_p)|$.

Proof. By part (c), we have that:

$$\frac{|GL_n(F)|}{|SL_n(F)|} = |F^\times|,$$

so that:

$$|SL_n(F)| = \frac{|GL_n(F)|}{|F^\times|} = \frac{p^4 - p^3 - p^2 + p}{p - 1} = p^3 - p - 1.$$

□

(f) Let I be the identity matrix. Show that $\{\pm I\} \leq SL_n(F)$ if and only if n is even.

Proof. Notice that $I \in GL_n(F)$, we have $\det(I) = (-1)^n$, giving the result. □

(g) Use the second isomorphism theorem to construct an isomorphism:

$$PGL_2(F) \cong SL_2(F)/\{\pm I\}.$$

Proof. This is false. Let's look at why. The goal was to consider the following diamond.

$$\begin{tikzcd}
 & GL_2(F) \arrow[d, dash] & \\
 & SL_2(F) \cdot F^\times \arrow[dl, dash] \arrow[dr, dash] & \\
SL_2(F) \arrow[dr, dash] & & F^\times \arrow[dl, dash] \\
 & SL_2(F) \cap F^\times &
\end{tikzcd}$$

The second isomorphism theorem immediately tells us:

$$SL_2(F) \cdot F^\times / F^\times \cong SL_2(F)/(SL_2(F) \cap F^\times). \tag{4}$$

Furthermore, it is not hard to see that $SL_2(F) \cap F^\times = \{\pm I\}$. Indeed, if I have a constant diagonal matrix λI its determinant is λ^2, which is 1 if and only if $\lambda = \pm 1$. So that means the left hand side of Equation 4 is the left hand side of the isomorphism we are trying to prove. The thing that is not true in general is that:

$$SL_2(F) \cdot F^\times = GL_2(F).$$

If this is true, notice that we are done. I will now point out that whether or not this is true actually depends on the aritmetic of F. For example, let $F = \mathbb{R}$. Then we have the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then notice that $\det(A) = -1$, and that for all $\lambda \in \mathbb{R}^\times$, we have

$$\det(\lambda A) = \lambda^2 \det A = -\lambda^2 \leq 0.$$

In particular, there is no $F^\times$ scaling of A to get positive determinant, much less determinant 1. In $\mathbb{C}$ the situation is different, we can scale A by i and get

$$iA = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

whose determinant is $i * -i = 1$. In fact, Given any $M \in GL_2(\mathbb{C})$, let $d = \det(M)$. Then $\lambda = 1/\sqrt{d}$ does exist as a complex number, so that

$$\det(\lambda M) = \lambda^2 \det M = \frac{1}{d} d = 1.$$

Thus in particular, $\lambda M \in SL_2(\mathbb{C})$. Thus we have seen that $SL_2(\mathbb{C}) \cdot \mathbb{C}^\times = GL_2(\mathbb{C})$, so that the theorem holds when $F = \mathbb{C}$, but the same cannot be said for when $F = \mathbb{R}$. $\square$