Takehome 3

Due Monday, April 27th

This assignment will walk you through a proof of the structure theorem for finite abelian groups. We will prove the following:

Theorem 1 (Fundamental Theorem for Finite Abelian Groups). *Let G be a finite abelian group. Then:*

$$G \cong Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s},$$

for a unique sequence of integers $(n_1, n_2, \cdots, n_s)$ with each $n_i \geq 2$ and $n_{i+1} | n_i$.

Recall that we call the decomposition from Theorem 1 the *invariant factor decomposition*. We will deal with the existence and uniqueness of such a decomposition separately. Our first goal is the following proposition, which does most of the heavy lifting.

Proposition 1. *Every finite abelian group is the direct product of cyclic groups.*

1. Step one is to reduce the problem to p-groups. Let G be a *finite abelian* group.
   (a) Explain why G has a *unique* Sylow p-subgroup for each prime p. This justifies our use of the word *the* in the following.
   (b) Suppose G has order $p^\alpha q^\beta$ for distinct primes p and q. Let P be the Sylow p-subgroup, and Q the Sylow q-subgroup. Show that $G \cong P \times Q$.
   (c) In general the prime factorization of $|G|$ is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Show by induction on t that G is the product of its Sylow subgroups. Explicitly, this means that if P_i is the Sylow p_i-subgroup for $i = 1, \cdots, t$, then
   $$G \cong P_1 \times P_2 \times \cdots \times P_t.$$
   (d) Explain why if we prove Proposition 1 for each of the P_i, then we have proved Proposition 1 for G.

By Exercise 1, we have reduced the proof of Proposition 1 to following:

Proposition 2. *Let A be an abelian p-group i.e., one of prime power order p^α. Then A is a product of cyclic groups.*

We will do this by induction on α but first we must develop an auxiliary tool.

2. Let A be a nontrivial abelian p-group. Define the p-power map $\varphi : A \to A$ by the rule $\varphi(x) = x^p$.
   (a) Show that φ is a homomorphism.
   (b) Let $A_p = \ker \varphi = \{a : a^p = 1\} \trianglelefteq A$ be the p-torsion of A (first studied in HW4 Problem 2). Show that A_p is an elementary abelian p-group (recall the definition from HW8 Problem 5).
   (c) Let $A^p = \operatorname{im} \varphi = \{a^p : a \in A\} \leq A$. Show that $A/A^p \cong A_p$. (Hint, show they are elementary abelian p-groups of the same order, then apply HW8 Problem 5).
   (d) Conclude $|A^p| < |A|$. This will be a crucial ingredient for our induction step.

3. We will now prove Proposition 2 by induction on $|A|$.
   (a) First the base case: show that Proposition 2 is true if $|A| = p$.
   (b) The induction step is more involved, begin by showing that A^p is the product of cyclic groups. That is $A^p = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_t \rangle$. (Use 2(d)).
   (c) Show that $A^p \cap A_p$ is an elementary abelian group of order p^t. (Hint: it is clear that it is elementary abelian (why?), so it remains to show it contains p^t elements.)
   (d) We now split into two cases. For the first case, assume that $A_p \leq A^p$
      i. For each generator x_i of A^p, show that there is some $y_i \in A$ with $y_i^p = x_i$.
      ii. Let $A_0 = \langle y_1, \cdots, y_t \rangle$. Show that $A_0 \cong \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_t \rangle$. (It might be useful to use induction on t).
      iii. Show that $A^p \trianglelefteq A_0$ and that A_0/A^p is an elementary abelian group of order p^t.
      iv. Use part (c) and (d)(iii) to show that $|A_0| = |A|$. Conclude that Proposition 2 holds for A.
   (e) For the second case $A_p \not\leq A^p$, so we know there is some $x \in A_p$ with $x \notin A^p$.
      i. Let $\overline{A} = A/A^p$, and let $\pi : A \to \overline{A}$ be the natural projection. Let $\overline{x} = \pi(x)$. Show that $|x| = |\overline{x}| = p$.
      ii. Show that $\overline{A} \cong \langle \overline{x} \rangle \times \overline{E}$ for some subgroup $\overline{E} \leq \overline{A}$. (Hint: first notice $\overline{A}$ is elementary abelian (why?). Now this should look a lot like the induction step of proof of HW8 Problem 5, in particular, it may be useful to consider the fibers of the projection $\overline{A} \to \overline{A}/\langle \overline{x} \rangle$).
      iii. Let $E = \pi^{-1}(\overline{E}) \leq A$. Show that $A \cong E \times \langle x \rangle$. Conclude that Proposition 2 holds true for A.

We have now proved Proposition 2, which by 1(d) immediately implies Proposition 1. In class we described a process which put a product of cyclic groups into an *elementary divisor form*. We also described a process that took a finite abelian group in elemetary divisor form, and produced its *invariant factor decomposition*. We will not reproduce that here, and instead assert that this implies the existence part of Theorem 1. Therefore only the uniqueness statement remains. As a useful tool, we provide you with the following lemma which you may use without proof.

Lemma 1 (Cancellation Property for Products of Finite Groups)**.** *Let M, N, K be finite groups and suppose $K \times M \cong K \times N$. Then $M \cong N$.*

Remark. *This lemma is more subtle then one might think, and it is not true without assuming the groups are finite. There is a lot to explore here that is beyond the scope of this assignment. For now feel free to use the lemma as a black box, and we will study this problem more deeply in a future assignment.*

Finally, we remind ourselves of the following definition.

Definition 1. *Let G be a group. The exponent of G is the minimum n such that $x^n = 1$ for all $x \in G$.*

4. We finish by proving the uniqueness part of Theorem 1. Let G be a group, and suppose it has 2 invariant factor decompositions. That is:
$$G \cong Z_{n_1} \times \cdots \times Z_{n_s} \cong Z_{m_1} \times \cdots \times Z_{m_t}.$$
   Where each $n_i, m_i \geq 2$, and $n_{i+1}|n_i$ and $m_{i+1}|m_i$. Use HW10 Problem 5 and Lemma 1 in descending induction to show that $s = t$ and $n_i = m_i$ for every i.