Homework Assignment 4

Due Friday, February 19

1. Let G be a group and H a *nonempty* subset of G. Let's introduce a few tricks to speed up testing if something is a subgroup.
   (a) *(Subgroup Criterion)* Suppose that for all $x, y \in H$, $xy^{-1} \in H$. Show that H is a subgroup of G.
   (b) *(Finite Subgroup Criterion)* Show that if H is finite and closed under multiplication, then H is a subgroup of G.
   (c) Suppose now that H is a subgroup of G, and that K is another subgroup of G. Show that if $K \subseteq H$, then $K \leq H$.

2. Let G be a group. Let $H, K \leq G$ be two subgroups.
   (a) Show that the intersection $H \cap K$ is a subgroup of G.
   (b) Give an example to show that the union $H \cup K$ need not be a subgroup of G.
   (c) Show that $H \cup K$ is a subgroup of G if and only if $H \subset K$ or $K \subset H$.
   (d) Adjust your proof from part (a) to show that the intersection of an arbitrary collection of subgroups is a subgroup. That is, let $\mathcal{A}$ be a collection of subgroups of G. Show that

$$\bigcap_{H \in \mathcal{A}} H$$

   is a subgroup of G. This completes the proof that the subgroup generated by a subset is in fact a subgroup.

   Hint. *For part (d), the proof should be very similar to part (a), with only cosmetic modifications. You won't need to use induction. In fact, since $\mathcal{A}$ is could in principle be uncountable, induction won't work without modifications (think about why this is).*

3. Let G be a group, and let A be a subset of G. Let's establish some facts about centralizers and normalizers.
   (a) Let A be a subset of G. Prove that $N_G(A) \leq G$.
   (b) Deduce the following chain of inclusions.

$$Z(G) \leq C_G(A) \leq N_G(A) \leq G.$$

   (c) Show that $C_G(A) = C_G(\langle A \rangle)$.
   (d) Give an example to show the analog of part (c) for normalizers is not true. That is, give $A \subseteq A$ where $N_G(A) \neq N_G(\langle A \rangle)$.
   (e) Show that if H is a subgroup of G, then $H \leq N_G(H)$.
   (f) Show that $H \leq C_G(H)$ if and only if H is abelian.

4. Compute the center of the dihedral group. Explicitly, let n be an integer ≥ 3. Compute $Z(D_{2n})$. (Note: you will need to split into the two cases, where n is even or n is odd).

5. In this exercise we study products of finite cyclic groups. Recall that we denote by Z_n the cyclic group of order n (written multiplicatively).

(a) Prove that $Z_2 \times Z_2$ is not a cyclic group.

(b) Prove that $Z_2 \times Z_3 \cong Z_6$. Conclude that $Z_2 \times Z_3$ is a cyclic group.

Those two examples really cover all the bases. Use the intuition you gained from them to prove the following classification result.

(c) Show that $Z_n \times Z_m$ is cyclic if and only if $\gcd(n, m) = 1$. (Hint: recall that up to isomorphism there is only one cyclic group of order N for every positive integer N).

6. For $n \geq 2$ let $G = S_n$ be the symmetric group equipped with it's natural action on $\Omega_n = \{1, 2, \cdots, n\}$ by permutations. For $i \in \Omega_n$, let $G_i = \{\sigma \in G | \sigma(i) = i\}$ be the stabilizer of i. Describe an isomorphism between G_i and S_{n-1}.

7. In this problem we will introduce the following very important class of subgroups. A subgroup $H \leq G$ is called *normal* if $N_G(H) = G$. Recall that this means, for $x \in G$, the set $xHx^{-1} = H$. If H is a normal subgroup, we write $H \trianglelefteq G$.

   (a) Let H be a subgroup, and $x \in G$. Give a bijection between H and xHx^{-1}.

   (b) Part (a) makes it easy to check if something is normal. In particular, suppose that for every $h \in H$, the element $xhx^{-1} \in H$ for every $x \in G$. Show that H is normal.

   (c) Let $\varphi : G \to G'$ be a homomorphism with kernel K. Show that K is a normal subgroup of G.

   (d) Give an example of a subgroup that is not normal. Conclude that not every subgroup can be the kernel of some homomorphism.

8. Let's study the converse of the previous question. We will give an intrinsic definitnion of quotient groups along the way. A lot of this problem is covered in class (with some details for you to fill in), but I think it is very important to work through these constructions carefully for yourself. This should feel very similar to the construction of $\mathbb{Z}/n\mathbb{Z}$.

   Recall the following definition from class: Let $K \leq G$ be a subgroup. For $x, y \in G$ we say that x and y are congruent mod K, $x \equiv y \mod K$ if $y^{-1}x \in K$ (or equivalently if $x = yk$ for some $k \in K$).

   (a) Show that congruence modulo K is an equivalence relation on G. Observe that the the equivalence classes of congruence mod K are the sets

   $$xK = \{xk : k \in K\}.$$

   We call these the *cosets* of K.

   (b) Suppose $K \trianglelefteq G$. If $x \equiv x_1 \mod K$ and $y \equiv y_1 \mod K$, show $xy \equiv x_1y_1 \mod K$. (You will need normality here. Be careful not to assume your group is abelian).

   (c) Define G/K to be the set of cosets of K.

   $$G/K = \{xK : x \in X\}.$$

   If K is normal, show that the operation $(xK)(yK) = xyK$ is a well defined binary operation making G/K into a group. What is the identity element? (Note: You already did the work to show it's well defined.)

(d) Suppose K is a normal subgroup. Let $\pi : G \to G/K$ be the map $x \mapsto xK$. Show that π is a group homomorphisms with kernel K. This is often called *the natural projection*.

(e) Suppose that G/K is a group under the operation described in part (c). Show that K must be normal (*Hint:* Rather than trying to explicitly compute things with elements, use the then natural projection together with 7(c)).

(f) Putting everything together, conclude the following are equivalent for a subgroup $K \leq G$.

(i) K is normal in G.

(ii) K is the kernel of a homomorphism.

(iii) G/K is a group.

Hint. *You've already done all the work for this. Each implication should be easily accessible appealing to something proven in question 7 or 8.*