Homework Assignment 9

Due Friday, April 9

This assignment will fill in many details from lecture, and do a few hands on classifications. To begin we will confirm that the semidirect product is indeed a group. First recall the definition.

Definition 1. *Let H, K be groups, and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism. Denote the induced action of K on H by:*

$$k \cdot h = \varphi(k)(h).$$

The semidirect product of H and K with respect to φ is the set $H \rtimes K = \{(h, k) : h \in H, k \in K\}$, where multiplication is defined by the rule:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1 k_2).$$

1. Let's make sure that $H \rtimes K$ is a group.

   (a) Show that $(1, 1) \in H \rtimes K$ is the identity. (Remember you have to check both sides).

   Proof. Fix $h \in H$ and $k \in H$. Then we check,

   $$(1, 1)(h, k) = (1(1 \cdot h), 1k) = (h, k).$$

   On the other hand:

   $$(h, k)(1, 1) = (h(k \cdot 1), k1) = (h, k),$$

   where we remark that $k \cdot 1 = 1$ because K acts *by automorphisms* (i.e. $x \mapsto k \cdot x$ is not merely a bijection, but also a homomorphism which in particular sends 1 to 1 [by HW3 Problem 1a]). □

   (b) Show that $(h, k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$. (As above, you have to check both sides).

   Proof. We check both directly:

   $$\begin{aligned} (h, k)(k^{-1} \cdot h^{-1}, k^{-1}) &= (h(k \cdot (k^{-1} \cdot h^{-1})), kk^{-1}) \\ &= (h(kk^{-1} \cdot h^{-1})), 1) \\ &= (h(1 \cdot h^{-1}), 1) \\ &= (hh^{-1}, 1) \\ &= (1, 1). \end{aligned}$$

   For the other hand, we remark that because K acts by automorphisms, we have that for each $\ell \in K$, we have $(\ell \cdot x)(\ell \cdot y) = \ell \cdot (xy)$ (because $x \mapsto \ell \cdot x$ is a homomorphism). In particular

   $$\begin{aligned} (k^{-1} \cdot h^{-1}, k^{-1})(h, k) &= ((k^{-1} \cdot h^{-1})(k^{-1} \cdot h), k^{-1}k) \\ &= (k^{-1} \cdot (h^{-1}h), 1) \\ &= (k^{-1} \cdot 1, 1) \\ &= (1, 1). \end{aligned}$$

   □

(c) Prove that multiplication is associative.

Proof. We consider $h_1, h_2, h_3 \in H$ and $k_1, k_2, k_3 \in K$. We passing from line 2 to 3 that K acts by automorphism. Then:

$$\begin{aligned}
((h_1, k_1)(h_2, k_2))(h_3, k_3) &= (h_1(k_1 \cdot h_2), k_1k_2)(h_3, k_3) \\
&= (h_1(k_1 \cdot h_2)(k_1k_2 \cdot h_3), k_1k_2k_3) \\
&= (h_1(k_1 \cdot (h_2(k_2 \cdot h_3)), k_1k_2k_3) \\
&= (h_1, k_1)(h_2(k_2 \cdot h_3), k_2k_3) \\
&= (h_1, k_1)((h_2, k_2)(h_3, k_3)).
\end{aligned}$$

□

Studying semidirect products reduces to the study of automorphism groups, so it is useful to be able to to decompose them.

Lemma 2. *Let H and K be finite groups whose orders are coprime. Then*

$$\operatorname{Aut}(H \times K) \cong \operatorname{Aut} H \times \operatorname{Aut} K.$$

The following definition will be useful.

Definition 3. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

2. Let's prove Lemma 2.

(a) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup* (recall the definition from HW8 Definition 1). Fix any automorphism $\varphi \in \operatorname{Aut} G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).

Proof. A priori, we only have that $\varphi|_H : H \to G$. Nevertheless, since $H \operatorname{char} G$, we know $\varphi(H) = H$. In particular, for all $h \in H$ we know $\varphi(h) \in H$, so that we may view φ as a map from H to itself. It is an injective homomorphism since φ is, and surjectivity follows because $\varphi(H) = H$. □

(b) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\operatorname{Aut} G \to \operatorname{Aut} H$.

Proof. The fact that it is well defined is part (a). It remains to show that if $\varphi, \psi, \in \operatorname{Aut} G$, then

$$(\varphi \circ \psi)|_H = \varphi|_H \circ \psi|_H.$$

One immediately checks this by evaluating both sides on an arbitrary element of H. □

(c) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.

Proof. We show $H \operatorname{char} H \times K$ and remark that situation for K is identical. Fix an automorphism $\varphi : H \times K \to H \times K$, and a nontrivial element $(h, 1) \in H \leq H \times K$, and note that $|(h, 1)| = m$ divides $|H|$. Consider $(h', k) = \varphi(h, 1)$. Since φ is an isomorphism, we know (HW3 Problem 1(e)) that $|(h', k)| = m$ as well. In particular, we know that $k^m = 1$ in K, so that $|k|$ divides m. Thus $|k|$ divides both $|H|$ and $|K|$. Since they are coprime, their only common divisor is 1, so $k = 1$ and $(h', k) = (h', 1) \in H$. This shows that $\varphi(H) \leq H$. Since H is finite, and φ is injective, the order of H and $\varphi(H)$ must agree, so that in fact $\varphi(H) = H$ as desired. □

(d) With H, K as in (c), construct an isomorphism $\operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$.

Proof. We define $\Phi : \operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$ via the rule:

$$\Phi(\varphi) = (\varphi|_H, \varphi|_K).$$

This is well defined by (c) and (a), and is a homomorphism by (b). We construct an inverse, $\Psi : \operatorname{Aut} H \times \operatorname{Aut} K \to \operatorname{Aut}(H \times K)$ via the rule. If $\varphi \in \operatorname{Aut} H$ and $\psi \in \operatorname{Aut} K$ then $\Psi(\varphi, \psi) = \varphi \times \psi$, where φ acts on the H coordinate and ψ acts on the K coordinate. That is

$$\begin{aligned} \varphi \times \psi : H \times K &\longrightarrow H \times K \\ (h, k) &\mapsto (\varphi(h), \psi(k)). \end{aligned}$$

One easily checks that $(\varphi \times \psi)|_H = \varphi$ and $(\varphi \times \psi)|_K = \psi$ so that $\Phi(\Psi(\varphi, \psi)) = (\varphi, \psi)$. On the other hand, for $\varphi \in \operatorname{Aut}(H, \times K)$, one notices that

$$\Psi(\Phi(\varphi)) = \varphi|_H \times \varphi|_K = \varphi,$$

and so $\Psi = \Phi^{-1}$ and we are done. □

Recall that any homomorphism $\varphi : K \to \operatorname{Aut} H$ allows us to build a semidirect product $H \rtimes_\varphi K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used the following lemma.

Lemma 4. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \operatorname{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:*

$$\begin{tikzcd} K \arrow[dr, "\varphi"] \arrow[dd, "\gamma"'] & \\ & \operatorname{Aut} H \\ K \arrow[ur, "\psi"'] & \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 5. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitely, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_\alpha : \beta \mapsto \alpha\beta\alpha^{-1}$, and suppose that $\psi = \sigma_\alpha \circ \varphi$:*

$$\begin{tikzcd} & \operatorname{Aut} H \arrow[dd, "\sigma_\alpha"] \\ K \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

3. Lemmas 4 and 5 say that if we alter φ by an automorphism of K, or an inner automorphism of $\operatorname{Aut} H$, (or both), we don't change the semidirect products. Let's prove this.

   (a) Consider the setup of Lemma 4. Show that the map:

$$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (h, \gamma(k)) \end{array}$$

   is an isomorphism, thereby proving the lemma.

   Proof. Call the map we are studying Φ. Since γ is bijective, so is Φ. So it suffices to show that Φ is a homomorphism. Fix $(h_i, k_i) \in H \rtimes_\varphi K$ for $i = 1, 2$.

$$\begin{aligned} \Phi(h_1,k_1)\Phi(h_2,k_2) &= \Big(h_1, \gamma(k_1)\Big)\Big(h_2, \gamma(k_2)\Big) \\ &= \Big(h_1(\psi(\gamma(k_1))(h_2)), \gamma(k_1)\gamma(k_2)\Big) \\ &= \Big(h_1(\varphi(k_1)(h_2)), \gamma(k_1k_2)\Big) \\ &= \Phi\Big(h_1(\varphi(k_1)(h_2)), k_1k_2\Big) \\ &= \Phi\Big((h_1,k_1)(h_2,k_2)\Big), \end{aligned}$$

   and the result follows. □

   (b) Consider the setup of Lemma 5. Show that the map:

$$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (\alpha(h), k) \end{array}$$

   is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_α is an automorphism of $\operatorname{Aut} H$, given by conjugation by α. In unweildy notation, this says $\sigma_\alpha \in \operatorname{Aut}(\operatorname{Aut} H)$.)

Proof. Call the map we are studying Ψ. Since α is bijective, so is Ψ, so it suffices to show that Ψ is an automorphism. Fix $(h_i, k_i) \in H \rtimes_\varphi K$ for $i = 1, 2$.

$$\begin{aligned}
\Psi(h_1, k_1)\Psi(h_2, k_2) &= \Big(\alpha(h_1), k_1\Big)\Big(\alpha(h_2), k_2\Big) \\
&= \Big(\alpha(h_1)(\psi(k_1)(\alpha(k_2)), k_1k_2\Big) \\
&= \Big(\alpha(h_1)(\sigma_\alpha(\varphi((k_1)))(\alpha(k_2)), k_1k_2\Big) \\
&= \Big(\alpha(h_1)(\alpha\varphi(k_1)\alpha^{-1})(\alpha(h_2)), k_1k_2\Big) \\
&= \Big(\alpha(h_1)\alpha(\varphi(k_1)(h_1)), k_1k_2\Big) \\
&= \Big(\alpha(h_1(\varphi(k_1)(h_1)), k_1k_2\Big) \\
&= \Psi\Big(h_1(\varphi(k_1)(h_1)), k_1k_1\Big) \\
&= \Psi\Big((h_1, k_1)(h_2, k_2)\Big).
\end{aligned}$$

$\square$

(c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\
K \arrow[r, "\psi"] & \operatorname{Aut} H.
\end{tikzcd}$$

That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$. (Hint: This should follow formally from Lemmas 4 and 5, so you shouldn't have to do any lengthy computations).

Proof. We give the function $\sigma \circ \varphi = \psi \circ \gamma$ the name $\xi : K \to \operatorname{Aut} H$. That is, ξ fits into the following diagram:

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] \arrow[dr, dotted, "\xi"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\
K \arrow[r, "\psi"'] & \operatorname{Aut} H.
\end{tikzcd}$$

By part (b), we know that

$$H \rtimes_\varphi K \cong H \rtimes_\xi K,$$

and by part (a) we know that

$$H \rtimes_\xi K \cong H \rtimes_\psi K.$$

Combining these two gives the result. $\square$

4. A lot of studying semidirect products comes down to enumerating and classifying homomorphisms. Let's record a useful fact.

(a) Show that giving a homomorphism $Z_n \to G$ is the same as selecting an element $g \in G$ with $|g|$ dividing n. That is, give a bijection between the following sets:

$$\left\{ \begin{array}{c} \text{Homomorphisms} \\ Z_n \to G \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Elements } g \in G \\ \text{where } |g| \text{ divide } n \end{array} \right\}$$

Proof. Fix once and for all a generator x of Z_n. Then given a map $\varphi : Z_n \to G$, we know that $g = \varphi(x) \in G$ has order dividing $|x| = n$ (HW3 Probelem 1(c)). Conversely, given $g \in G$ of order dividing n, the map $\psi : x^i \mapsto g^i$ is a homomorphism from $Z_n \to G$. One readily checks that these are inverse constructions. □

(b) If p is prime show that giving a *nontrivial* map $Z_p \to G$ is the same as choosing an element of order p in G. (Note: the trivial map is the one that sends every element to the identity of G).

Proof. In part (a) we saw that a map $Z_p \to G$ is the same as an element of G whose order divides p. The trivial map corresponds to 1_G, so all other maps correspond to elements of order p. □

(c) Show that giving a homomorphism $Z_{n_1} \times \cdots \times Z_{n_r} \to G$ is the same as chosing elements $g_1, \cdots, g_r \in G$ such that all the g_i commute with eachother and each $|g_i|$ divides n_i.

Proof. This is essentially identical to part (a). Fix generators x_i of Z_{n_1}. Given a homomorphism φ we let $g_i = \varphi(x_i)$ and remark that its order must divide $|x_i| = n_i$ (again HW3 1(c)). Furthermore, we notice that:

$$g_i g_j = \varphi(x_i)\varphi(x_j) = \varphi(x_i x_j) = \varphi(x_j x_i) = \varphi(x_j)\varphi(x_i) = g_j g_i,$$

so that they commute. Conversely, given such g_i, we define ψ on the generators of $Z_{n_1} \times \cdots \times Z_{n_r}$ via the rule

$$\psi(x_1^{j_1}, \cdots, x_r^{j_r}) = g_1^{j_1} \cdots g_r^{j_r},$$

noting that ψ is a homomorphism precisely because the g_i commute and have order dividing x_i. □

(d) Suppose G is abelian and p is prime. Describe the set of homomorphisms $Z_p \times Z_p \to G$ as a subset of $G \times G$.

Proof. By part (c) this should correspond to pairs $(a, b) \in G \times G$ such that $a^p = b^p = 1$. In particular, we remark that this is the p-torsion of $G \times G$, i.e., in the notation of Takehome 2 it is $G_p \times G_p$. □

We finish with a couple of classification problems. You will find HW8#3 useful, as well as the following facts (you proved the third one in HW7, the other two you can freely use).

Facts (Automorphisms of abelian groups of order p and p^2)**.** *Let p a prime number. Then:*

- $\operatorname{Aut} Z_p \cong Z_{p-1}$

- Aut $Z_{p^2} \cong Z_{p(p-1)}$.
- Aut $(Z_p \times Z_p) \cong GL_2(\mathbb{F}_p)$.

We'll walk through the first one together and then leave the second one to you!

5. In this problem we classify all groups of order 75 up to isomorphism. (There should be 3 total).

   (a) List all the abelian groups of order 75 using the fundamental theorem of finite abelian groups.

   Proof. Notice $75 = 3 * 5^2$. Therefore, these decompose into elementary divisor form as:

   $$Z_3 \times Z_{25} \qquad \text{and} \qquad Z_3 \times Z_5 \times Z_5.$$

   In invariant factor form these correspond to Z_{75} and $Z_{15} \times Z_5$ respectively. □

   (b) Prove that a group of order 75 is isomorphic to $P \rtimes Q$ where P is a Sylow 5-subgroup and Q is a Sylow 3-subgroup.

   Proof. Let $|G| = 75 = 3 \cdot 5^2$. By Sylow III we know $n_5 \in \{1, 6, 11, \cdots\}$ and that n_5 divides 3. Therefore $n_5 = 1$ and so we can fix the *unique* Sylow 5-subgroup P, which by necessity is normal. Let Q be any Sylow 3-subgroup. Then $P \cap Q = \{1\}$ by Lagrange's theorem. We then compute:

   $$|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{25 \cdot 3}{1} = 75 = |G|.$$

   Therefore $PQ = G$. The recognition theorem for semidirect products now gives the result. □

   (c) Prove that if a group of order 75 has a *cyclic* Sylow 5-subgroup, then it is abelian.

   Proof. By part (b), we know $G \cong P \rtimes_\varphi Q$ for some map $\varphi : Q \to \text{Aut}(P)$. By assumption, $P \cong Z_{25}$, and by TH1 Problem 4 we know $Q \cong Z_3$. Therefore we may identify φ with map

   $$Z_3 \to \text{Aut}(Z_{25}) \cong Z_{20}.$$

   By 4(a) this corresponds to selecting an element in Z_{20} whose order divides 3. By Lagrange's theorem this can only be the identity element, so in fact φ is the trivial map. Therefore $G \cong Z_{25} \times Z_3 \cong Z_{75}$ which is abelian. □

   (d) Show that there is a unique nonabelian group of order 75. (*Hint:* Show that 3 is a maximal 3-divisor of $|GL_2(\mathbb{F}_5)|$. Then use Sylow's theorems and 3(c).)

   Proof. By 5(c), to be nonabelian we must have $P \cong Z_5 \times Z_5$. Therefore we must study nontivial maps

   $$\psi : Z_3 \longrightarrow \text{Aut}(Z_5 \times Z_5) \cong GL_2(\mathbb{F}_5).$$

   In HW 5 problem 3(d) we computed:

   $$|GL_2(\mathbb{F}_5)| = 5^4 - 5^3 - 5^2 + 5 = 480 = 3 * 160.$$

Since $3|480$ then by Cauchy's theorem there exists an element $M \in GL_2(\mathbb{F}_3)$ of order 3. If y is a generator of Z_3, then we let $\varphi(y) = M$ and get a nonabelian group

$$G_\varphi = (Z_5 \times Z_5) \rtimes_\varphi Z_3$$

of order 75. In fact, any such group comes from choosing some N of order 3 in $|GL_2(\mathbb{F}_5)|$ and letting $\psi : y \mapsto N$ and building G_ψ as above. We finish the proof by showing $G_\psi \cong G_\varphi$.
Since $3 \not| 160$, we know $\langle M \rangle$ and $\langle N \rangle$ are both Sylow 3-subgroups of $GL_2(\mathbb{F}_5)$. Therefore they are conjugate. That is, there is some $\alpha \in GL_2(\mathbb{F}_5)$ such that

$$\alpha \langle M \rangle \alpha^{-1} = \langle N \rangle.$$

Denote by $\sigma_\alpha \in \mathrm{Inn}(GL_2(\mathbb{F}_5))$ the associated inner automorphism. In particular, we see that $\sigma(M)$ is either N or N^2. Define $\gamma : Z_3 \to Z_3$ by the following rule. If $\sigma(M) = N$ then γ is the identity, and if $\sigma(M) = N^2$ then $\gamma : y \mapsto y^2$. In either case, $\gamma \in \mathrm{Aut}(Z_3)$. In particular, we have the following commutative diagram:

$$\begin{tikzcd} Z_3 \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & GL_2(\mathbb{F}_5) \arrow[d, "\sigma"] \\ Z_3 \arrow[r, "\psi"'] & GL_2(\mathbb{F}_5). \end{tikzcd}$$

Applying Problem 4(c) immediately implies:

$$G_\varphi = (Z_5 \times Z_5) \rtimes_\varphi Z_3 \cong (Z_5 \times Z_5) \rtimes_\psi Z_3 = G_\psi,$$

and so we are done. □

6. Classify all groups of order 20 up to isomorphism. (There should be 5 total).

Proof. Let $|G| = 20$, and note that $20 = 2^2 * 5$. Let $P \leq G$ be a Sylow 2-subgroup and $Q \leq G$ a Sylow 5-subgroup. Homework 8 problem 3 tells us that if $|G| = p^2 q$ and $q > p$ then either the Sylow q-subgroup is normal, or else $G \cong A_4$. Since $|G| \neq 12$, we know it cannot be A_4, so we conclude that $Q \trianglelefteq G$ is normal. This immediately implies $PQ \leq G$ is a subgroup. Furthermore, since P and Q have coprime orders (4 and 5 respectively), we know by Lagrange's theorem that $P \cap Q = 1$. Finally:

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = 20,$$

so that:

$$G = PQ \cong Q \rtimes P.$$

So it remains to classify all possible maps:

$$P \to \mathrm{Aut}\, Q.$$

Since $|Q| = 5$, we know it is isomorphic to Z_5 so that $\mathrm{Aut}(Q) \cong Z_4$. On the other hand, $|P| = 4$. There are 2 groups of order 4, the cyclic group and the Klein 4 group. We must treat each case seperately.

Case 1: $P \cong Z_4$

We are now classifying maps $\varphi : Z_4 \to Z_4$. Let $Z_4 = \langle x \rangle$. Then φ is determined by the image of x, and $|\varphi(x)|$ must divide $|x| = 4$. But every element of Z_4 has order dividing 4. Therefore there are 4 maps, $\varphi_i : x \mapsto x^i$, for $i = 0, 1, 2, 3$. We let $G_i = Z_5 \rtimes_{\varphi_i} Z_4$.

Case 1a: $i = 0$

In this case the map is trivial and we have:

$$G_0 = Z_5 \rtimes_{\varphi_0} Z_4 \cong Z_5 \times Z_4 \cong Z_{20}.$$

Case 1b: $i = 1, 3$

We claim that $G_1 \cong G_3$. Indeed, if we let $\gamma : Z_4 \to Z_4$ be the automorphism $x \mapsto x^3$, the following diagram commutes:

$$\begin{tikzcd}
Z_4 \arrow[dd, "\gamma"] \arrow[dr, "\varphi_1"] & \\
& \mathrm{Aut}(Z_5) \cong Z_4 \\
Z_4 \arrow[ur, "\varphi_3"'] &
\end{tikzcd}$$

Indeed, since $\varphi_1 = id$ and $\varphi_3 = \gamma$, this is essentially immediate. Therefore by Lemma 4, the desired isomorphism follows.

Although this isn't strictly necessary to get an accurate count we also include how to find generators and relations for this group we let x be a generator for the Z_4 and y be the generator for the Z_5. We must understand what xyx^{-1} is. This should be $\varphi_1(x)(y)$. Since $\varphi_1(x)$ is a order 4 automorphism, for instance $y \mapsto y^2$ (the other one is $y \mapsto y^3$, but as we just argued, by Lemma 4 the choice is symmetric). Therefore:

$$G_1 = \left\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^2 \right\rangle.$$

Finally we compute $Z(G_1)$. Notice that every element in G_1 looks like $x^i y^j$, and that:

$$x(x^i y^j)x^{-1} = x^i y^{2j},$$

which is equal to what we started with if and ony if $j = 0$. Thus if $j \neq 0$, $x^i y^j \notin Z(G_1)$. On the other hand,

$$x^i y x^{-i} = y^{2i},$$

which is only y if $i = 0$. Thus if $i \neq 0$ we have $x^1 \notin Z(G_1)$, so we conclude that $Z(G_1) = 1$.

Case 1c: $i = 2$

We claim that G_2 is not isomorphic to G_i for $i = 0, 1, 3$. We will do this by computing its center. We first notice that $\varphi_2(x) = x^2$ has order 2, and is the unique order 2 element of $Z_4 = \mathrm{Aut}(Z_5)$. Since $\iota : Z_5 \to Z_5$ which is the inversion map $y \mapsto y^{-1}$ has order 2, the action induced by φ_2 must be this inversion action. Therefore G_2 has the following presentation:

$$G_2 = \langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^{-1} \rangle$$

We then notice that $x^2 \in Z(G_2)$. Indeed,

$$x^2 y x^{-2} = \iota(\iota(y)) = (y^{-1})^{-1} = y,$$

and certainly $x^2 x^i x^{-2} = x^i$. Since x^2 commutes with both generators, it is in the center. Since $Z(G_2)$ is nontrivial, G_2 cannot be isomorphic to G_1 (or G_3), and since G_2 is nonabelian it is not G_0. Although we did not need it here, we remark that arguing as in case 1b we can show that nothing else is in the center, so that $Z(G_2) = \langle x^2 \rangle = \{1, x^2\}$.

Case 2: $P \cong Z_2 \times Z_2$

We are now classifying maps $\psi : Z_2 \times Z_2 \to Z_4$. As above we are letting $Z_4 = \langle x \rangle$, and let

$$Z_2 \times Z_2 = \langle a \rangle \times \langle b \rangle = \langle a, b \rangle,$$

where a, b have order 2. Then ψ is determined by where it sends a and b. Since $|\psi(g)|$ must divide $|g|$ (HW3 Problem 1(c)), we see that both $\psi(a), \psi(b) \in \{1, x^2\}$. As before, there are four options, which we will denote by $\psi_{j,k}$ for $j, k \in \{0, 1\}$.

$$\psi_{0,0} : \begin{array}{c} a \mapsto 1 \\ b \mapsto 1 \end{array} \qquad \psi_{1,0} : \begin{array}{c} a \mapsto x^2 \\ b \mapsto 1 \end{array}$$

$$\psi_{0,1} : \begin{array}{c} a \mapsto 1 \\ b \mapsto x^2 \end{array} \qquad \psi_{1,1} : \begin{array}{c} a \mapsto x^2 \\ b \mapsto x^2 \end{array}$$

We let $G_{j,k} = Z_5 \rtimes_{\psi_{j,k}} (Z_2 \times Z_2)$.

Case 2a: $j = k = 0$

In this case $\psi_{0,0}$ is trivial so we have:

$$G_{0,0} \cong Z_5 \times (Z_2 \times Z_2) \cong Z_{10} \times Z_2.$$

Case 2b: j, k not both 0

We claim that in this case all the $G_{j,k}$ are isomorphic. We remark that

$$Z_2 \times Z_2 \cong \langle a \rangle \times \langle b \rangle \cong \langle a \rangle \times \langle ab \rangle \cong \langle ab \rangle \times \langle b \rangle,$$

and each nontrivial $\psi_{j,k}$ takes 2 generators to x^2 and the third to 1. But $\mathrm{Aut}(Z_2 \times Z_2) = GL_2(\mathbb{F}_2)$ includes all the *change of basis* matrices which takes a pair of generators to any other pair of generators. In particular, if we fix any nontrivial $\psi_{j,k}$ and view $Z_2\times = \langle g \rangle \times \langle h \rangle$ as generated by g and h for the two generators sent to x^2 (i.e., where $\psi_{j,k}(g) = \psi_{j,k}(h) = x^2$ and $\psi_{j,k}(gh) = 1$) then there exists some $\eta \in \mathrm{Aut}(Z_2 \times Z_2)$ where $\eta(a) = g$ and $\eta(b) = h$. That is, we have the following:

$$\begin{tikzcd}
Z_2 \times Z_2 \arrow[dd, "\eta"] \arrow[dr, "\psi_{1,1}"] & \\
& \mathrm{Aut}(Z_5) \cong Z_4 \\
Z_2 \times Z_2 \arrow[ur, "\psi_{j,k}"'] &
\end{tikzcd}$$

By Lemma 4 this shows $G_{1,1} \cong G_{i,j}$, so that all three nontrivial $G_{i,j}$'s must be isomorphic. Let's extract generators and relations for $G_{1,1}$. We notice that a and b are both sent via $\psi_{1,1}$ to the inversion automorphism $\iota : y \mapsto y^{-1}$ of Z_5 (arguing as above in Case 1c that ι is the only automorphism of order 2). Therefore conjugating the generator y of Z_5 corresponds to inversion. Thus we have:

$$G_{1,1} \cong \left\langle y, a, b \mid y^5 = a^2 = b^2 = 1,\ ab = ba,\ aya^{-1} = byb^{-1} = y^{-1} \right\rangle .$$

We did it!!! There are 5 groups (up to isomorphism): let's summarize in the following table.

Group	Product notation	Presentation
G_0	Z_{20}	$\langle x \mid x^{20} = 1 \rangle$
$G_{0,0}$	$Z_{10} \times Z_2$	$\langle x, y \mid x^{10} = y^2 = 1,\ xy = yx \rangle$
G_1	$Z_5 \rtimes_{\varphi_1} Z_4$	$\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^2 \rangle$
G_2	$Z_5 \rtimes_{\varphi_2} Z_4$	$\langle x, y \mid x^4 = y^5 = 1,\ xyx^{-1} = y^{-1} \rangle$
$G_{1,1}$	$Z_5 \rtimes_{\psi_{1,1}} (Z_2 \times Z_2)$	$\langle y, a, b \mid y^5 = a^2 = b^2 = 1,\ ab = ba,\ aya^{-1} = byb^{-1} = y^{-1} \rangle$

□