Linear Algebra

Math 217: Saint Lawrence University

Gabriel Dorfsman-Hopkins

GABRIEL DORFSMAN-HOPKINS
SAINT LAWRENCE UNIVERSITY
DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE

These are growing course notes developed for a first course in Linear Algebra at Saint Lawrence University in Canton, NY. They are written and maintained by Professor Gabriel Dorfsman-Hopkins. The notes have been typset in LaTeXwith a template that was adapted from the Legrand Orange Book Template by Mathias Legrand (`legrand.mathias@gmail.com`) with modifications by Vel (`vel@latextemplates.com`).

First release, Spring 2023

Contents

1. Introduction

1.1 January 24, 2023

1.1.1 Functions

Almost any course in mathematics is centered around studying types of *functions*. For example, in *Calculus* we study the behavior of functions of a single variable, that is, functions whose input is a single real number and whose output is a single real number, looking especially closely at functions which are *continous* or *differentiable*.

■ **Example 1.1 — Functions of a single variable.** Consider the function

$$f(x) = 3x.$$

Its input is a real number, x, and the output is computed by multiplying the input by 3. To see what this function does to a real number, say, 11, we can compute:

$$f(11) = 3 \times 11 = 33.$$

Explicitly, f takes an input of eleven an *transforms it* into an output of 33. ■

■ **Example 1.2** Consider the function:

$$g(x) = x^2 - 2x + 1.$$

What does this function do to the number 2? ■

The study of calculus looks closely at these functions of a single variable, establishing concepts like *derivatives* and *integrals*, and connecting them to many real world questions and situations. A shorthand that we will adopt to describe a function f of a single variable is the following

$$f : \mathbb{R} \to \mathbb{R}$$

This can be read aloud as *f is a function from $\mathbb{R}$ to $\mathbb{R}$*. It signifies that f takes a real number (on the left of the arrow), and runs it through the arrow to produce another real number (on the right of the arrow). *Note: The set before the arrow is called the **domain** of the function. It is also sometimes called the **source**. The set after the arrow is called the **co-domain**. It is sometimes also called the **target**.*

In *Mulltivariable Calculus* we develop similar ideas, **but the types of functions we study are different**. In particular, we allow for functions which take more than one real number as an input. Allowing for mutli-variable inputs allows calculus to be applied to our multi-dimensional world, and vastly expands the applications of derivatives, integrals, and related ideas.

■ Example 1.3 — Functions of 2 variables. In multivariable calculus you may encounter a function like:

$$f(x,y) = x - y.$$

It takes as input a *pair* of real numbers (x,y), and outputs their difference. For example, to see what the function does to the pair of number $(5,2)$ we can compute:

$$f(5,2) = 5 - 2 = 3.$$

In partiuclar, f will *transform* the pair of numbers $(5,2)$ into the single number 3. ■

■ Example 1.4 — Functions of 3 variables. Consider the function of 3 variables:

$$f(x,y,z) = xyz + 1.$$

What does this function do to the triple $(1,2,3)$? ■

The *arrow notation* of a function introduced above carries over here as well. For example, if f is a function of two variables, (whose input is 2 real numbers) we may write:

$$f : \mathbb{R}^2 \to \mathbb{R},$$

which we read as *f is a function from $\mathbb{R}^2$ to $\mathbb{R}$*. Here $\mathbb{R}^2$ denotes the collection of *pairs of real numbers*. Similarly, if g is a function of 3 variables (like in Example 1.4), we may write

$$g : \mathbb{R}^3 \to \mathbb{R}.$$

Notice that for each function we've describe so far, the output is *1-dimensional*. That is, we may have a function into which takes multiple real numbers as an input, but in each case the output is *a single real number*.[1] But just as allowing a multi-dimensional input massively expanded the scope of calculus, allowing functions to have a multidimensional output can be very useful as well.

■ Example 1.5 — Analyzing Ocean Currents. A group of oceanographers are measuring the movement of the water in the Atlantic, by studying where a collection of sensors start and end over the course of two weeks. They compile their data into a function C whose input is the GPS coordinates of a location in the Atlantic, and whose output of where the water at that location ends up 2 weeks later. For example,

$$C(40.47, -68.73) = (41.71, -64.07),$$

[1] You may recall that $\mathbb{R}$ can be thought of as a line, $\mathbb{R}^2$ as a plane, and $\mathbb{R}^3$ as 3-dimensional space. We will eventually adopt this notion of dimensionality, and explore it more carefully.

means that a drop of water whose GPS Coordinates are 40.47N 68.73W will move over the course of two weeks to the location 41.71N 64.07W. Observe that this is a function that takes as input two real numbers, and outputs *2 real numbers* as well! That is, both the input and the output are *2-dimensional*. In our arrow notation, we would write:

$$C : \mathbb{R}^2 \to \mathbb{R}^2.$$

TODO: put an image here!

■

■ **Example 1.6 — Casting Shadows.** Shadows are cast when a body in space blocks the sun from hitting the ground. If we'd like to study the shape of shadows mathematically, it is worth modelling shadows with a function, say S. Here:

$$S(\text{A point in space}) = \text{The spot on the ground where it casts a shadow.}$$

Modelling 3-dimensional space with $\mathbb{R}^3$ and the 2-dimensional ground with $\mathbb{R}^2$, this gives a function:

$$S : \mathbb{R}^3 \to \mathbb{R}^2.$$

In fact, this will be a *projection function*, a certain kind of *linear transformation* that we will study in TBA.

TODO: put an image here!

■

As we can see, functions with multivariable outputs are not hard to come up with, and model many different situations we would hope to study with mathematics. Let us begin by looking at a very special case:

1.1.2 Functions $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose you wanted to describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. How would you go about it? Both the input and output of f consist of pairs of numbers, so to be explicit with our notation, let's give the first $\mathbb{R}^2$ the coordinates (x, y), and the second $\mathbb{R}^2$ the coordinates (u, v). In particular, our function will look something like

$$f(x, y) = (u, v).$$

The function should be a rule so that, given a pair (x, y) of real numbers, we return with another pair of numbers, (u, v). In particular, we have to say what u is, and what v is. But each of these coordinates depend on both x and y, so in essense this is just *two functions* whose output is a real number:

$$u = u(x, y)$$

$$v = v(x, y).$$

■ **Slogan 1.1** To describe a function whose output is two real numbers, you can give 2 functions which output a single real number each.

Let's see how this works with an example.

■ **Example 1.7** Lets define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ via the rule $f(x,y) = (u,v)$ where:

$$u = u(x,y) = xy + 1,$$

$$v = v(x,y) = x + 2y^2$$

The input of this function is a pair of numbers (x,y), and the output is *another* pair of number (u,v). So, for example, if we feed the function the pair $(-1,3)$, we can compute:

$$u = u(-1,3) = -1 \times 3 + 1 = -3 + 1 = -2$$

$$v = v(-1,3) = -1 + 2 \times 3^2 = -1 + 18 = 17.$$

Therefore, this function transforms the pair $(-1,3)$ to the pair $(-2,17)$:

$$f(-1,3) = (-2,17).$$

■

■ **Example 1.8** Define a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (x,y) to (u,v) via the rule

$$u = u(x,y) = 2x - 2y,$$

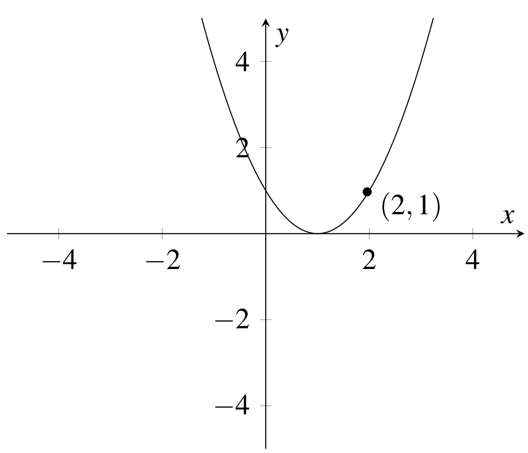
$$v = v(x,y) = \frac{1}{2}x + y.$$

Where does g take the point $(1,1)$? ■

It is often useful to think about a function as something that *moves* the point (x,y) to the point (u,v), and to emphasize this intuition, we will often refer a function $\mathbb{R}^2 \to \mathbb{R}^2$ as a *trnasformation of the plane*.

1.1.3 Visualizing Transformations of the Plane

How do we visualize these types of functions? Since these will be central objects of study, let's start by spending some time developing techniques for how to think about and imagine a function from $\mathbb{R}^2$ to itself. Recall that in calculus you often visualize functions $g : \mathbb{R} \to \mathbb{R}$ using their graphs in the xy-plane. Here the x axis plays the role of the domain, and the y-axis the role of the co-domain, and the graph is generally a curve consisting of the points $(x, g(x))$. For example, the graph of the function $g(x) = x^2 - 2x + 1$ from Example 1.2 is below.



The fact that $f(2)=1$ is captured by the fact that $(2,1)$ lies on the curve. A similar approach is used in multivariable functions, where now the domain is the entire xy-plane, and the co-domain is the z-axis. Then a function $f:\mathbb{R}^2\to\mathbb{R}$ can be graphed in 3-dimensional space, coloring in the points $(x,y,f(x,y))$, generally giving rise to a surface in 3-dimensional space.

■ **Question 1.1** Can we take a similar approach to a function $f:\mathbb{R}^2\to\mathbb{R}^2$? Why or why not?

Given the dimensional constraints, we have to come up with another way to represent a function $f:\mathbb{R}^2\to\mathbb{R}^2$. One way to do this is to get to the heart of what a function really does: *it transforms a point in* $\mathbb{R}^2$ *to another point in* $\mathbb{R}^2$. In particular, we can think about such a function as *something that transforms the plane*, moving the points of the plane around.

■ **Slogan 1.2** Think about a function $\mathbb{R}^2\to\mathbb{R}^2$ as something that moves around the points on a single plane. The input (x,y) is where the point starts, and the output $(u,v)=f(x,y)$ is where the point ends.

In fact, this is exactly what the function from Example 1.5 does, it keeps track of of where a drop of water in the Atlantic moves over the course of two weeks! Let's try to visualize the functions from Examples 1.7 and 1.8. First with Example 1.7, which was function $f(x,y)=(u,v)$ where:

$$u=u(x,y)=xy+1,$$

$$v=v(x,y)=x+2y^2$$

To get a sense of what kind of movement, let's keep track of what happens to a few points:

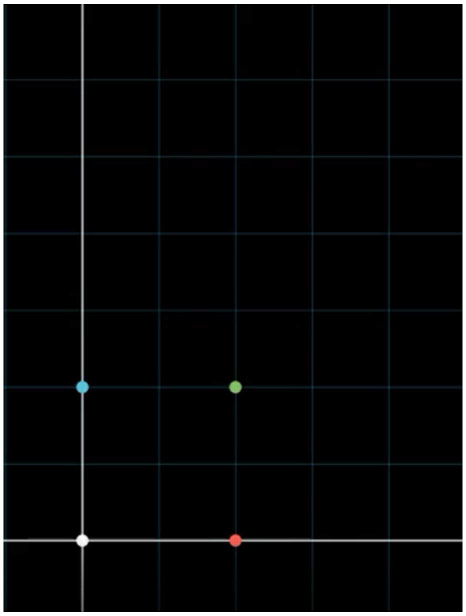
$$(0,0),(1,0),(0,1),(1,1).$$

Using the formulas we can compute where f takes these points, just like in Example 1.7.

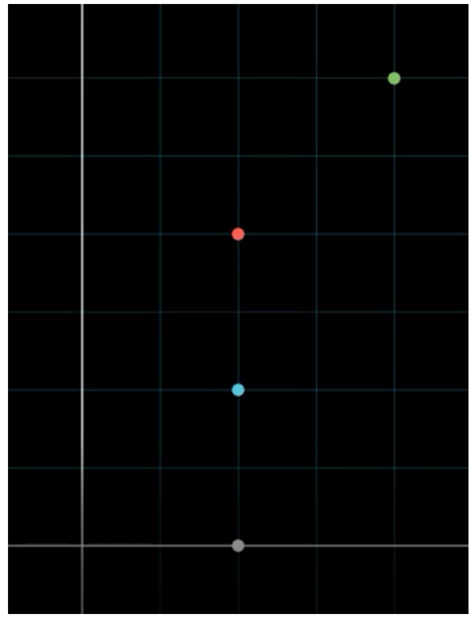
$$f(0,0)=(1,0),\ \ f(1,0)=(1,2),$$

$$f(0,1)=(1,1),\ \ f(1,1)=(2,3).$$

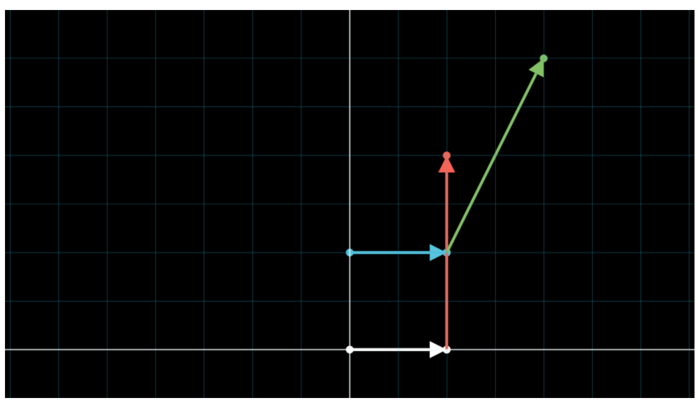
Instead of a single graph of the function, we can represent what f does with two pictures of the plane, a *before* shot and an *after* shot. On the left, we see the 4 points before applying f, and on the right, we see them after.



$\Longrightarrow$



The *movement* of the situation can be captured nicely by an animation, which you can see by clicking `here`.[2] You can also emphasize that it is movement on a single page by using arrows that point from the start to the finish of the various points:



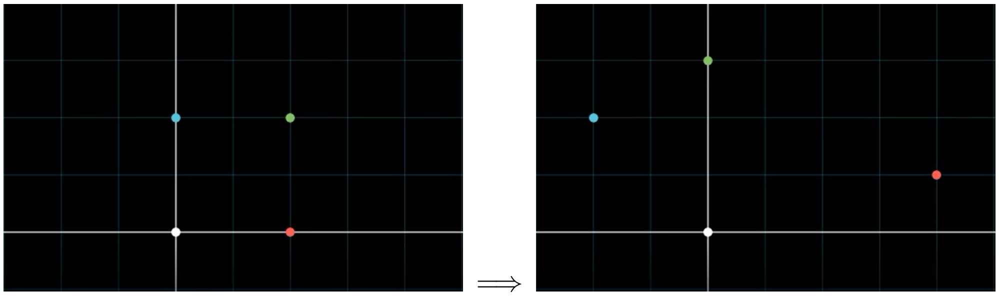
Exercise 1.1 Do the same thing for the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 1.8. In particular, compute where g takes the four points:

$$(0,0), {\color{red}(1,0)}, {\color{blue}(0,1)}, {\color{green}(1,1)},$$

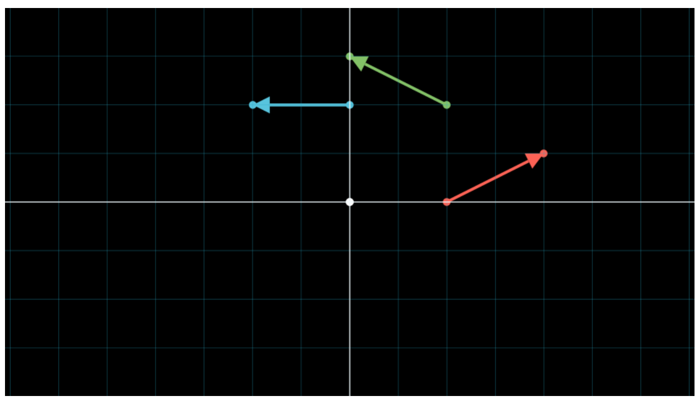
and graph their positions before and after applying g. Try also plotting them on the same plane with arrows tracing their movement. ■

The exercise is meant to be done in class. I've included the images I made below.

[2]TODO: Make this link live.



And here is a version with arrows.



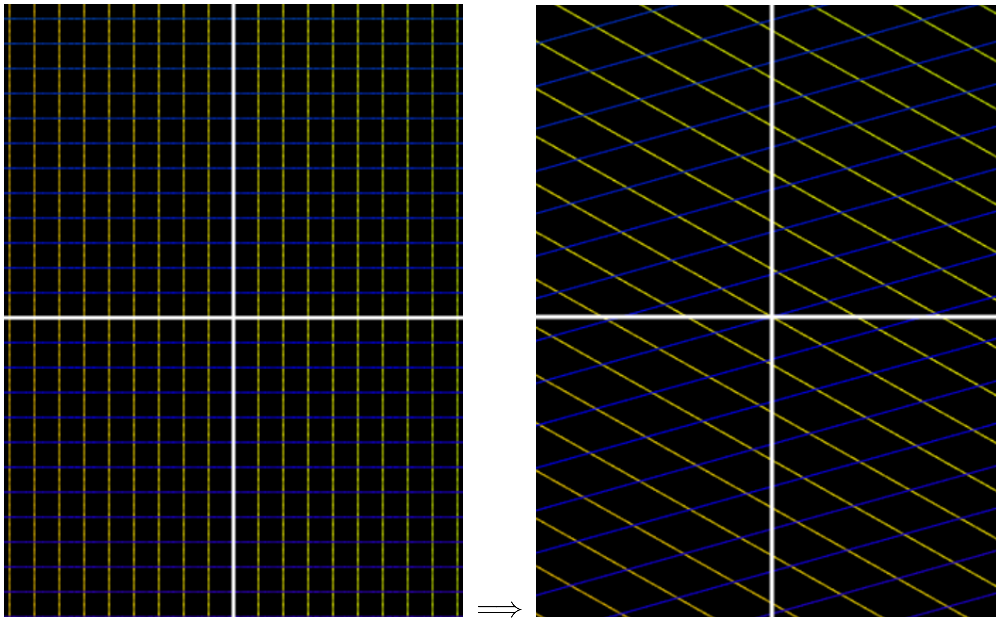
Again, we should imagine this function something *moving around the points on the plane*, a perspective that is emphasized when animating the function. You find an animation of the moving points `here`[3].

While seeing where a few points go begins to give a sense of how a function moves the plane, but only drawing where a few points go gives an incomplete picture. Of course, as we start to fill in more and more points, the image can start to get cluttered and it will be difficult to infer much from the picture. [4]. That being said, if you carefully pic which points to keep track of, you can get a nice sense of the *geometric* properties of a function. One way to do this, is by keeping track of what the function does to the *gridlines* of the plane.

Let's keep track of what the function g from Example 1.8 does to the gridlines of the plane.

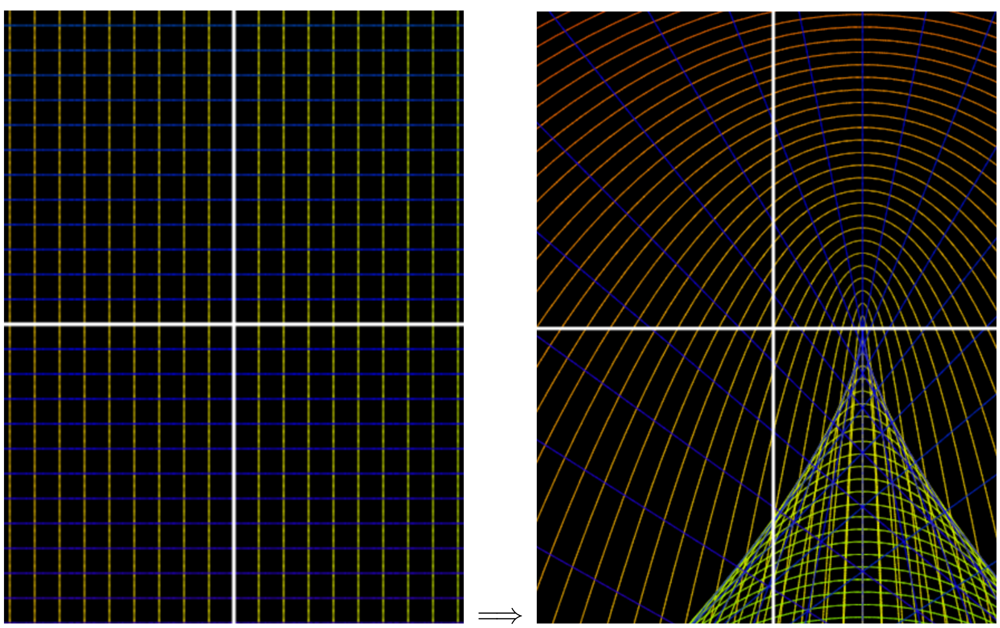
[3] TODO: Make this link live

[4] *try this! For some functions you can actually get a nice picture!*



One can really get a sense for how g *moves* the plane by looking at animation `here`[5]. In particular, we see that it sort of *stretches* and *rotates* the plane, distorting it slightly but not too much. In this course we will develop a vocabulary to mathematically describe terms like *stretching the plane*, and ways to extract that information from the equations given in Exampel 1.8, but for now we're trying to get a qualitative sense of what's going on.

Let's also look at what the function f from Example 1.7 does to the gridlines of the plane.



The animation is actually quite nice to look at: `here`[6]. Before drawing too many conclusions, it is

[5] TODO: Make this link live
[6] TODO: Make this link live

fair to say that this function appears far more complicated than the previous one. This turns out to be true, and in fact, in some sense it is complicated in a way that puts it beyond the purview of *linear algebra*[7] For the context of linear algebra, we will have to restrict ourselves to functions more like that of Example 1.8, functions that we will call *linear transformations*. Before describing exactly what these are, it might be worth while to ponder the following question. Qualitative answers are always welcome!

■ **Question 1.2** What are some differences between what happens to the gridlines in the two examples on the previous page?

1.1.4 Linear Transformations of the Plane

One answer to Question 1.2 could be: *In example 1.8 the gridlines remain as lines after applying g, but in Example 1.7 the gridlines become curvy*. This is a good observation. Recall that lines played a special role in calculus. Not only where they the simplest functions, we used them to model more complicated functions locally, by taking *tangent lines*. We do something similar in multivariable calculus, modelling more complicated functions with linear ones by taking the *tangent plane*. Not only were these functions simple *geomterically* (being lines and planes), but they were also simple *algebracially*. For example, a line usually has the following equation:

$$f(x) = {\color{red}mx} + {\color{blue}b}.$$

Above we highlighted the *linear term* in red, and the *constant term* in blue. Similarly, a plane had a simple equation as well:

$$h(x,y) = {\color{red}mx} + {\color{red}ny} + {\color{blue}b},$$

where again the linear terms are highlighted in red, and the constant term in blue. Looking at the function $g(x,y) = (u,v)$ from Example 1.8, we see that the equations for both u and v have only linear terms (and no constant terms).

$$u = u(x,y) = 2x - 2y,$$

$$v = v(x,y) = \frac{1}{2}x + y.$$

This will turn out to be a good definition for a linear function.

Definition 1.1.1 — Linear Transformations of the Plane. A *linear transformation of the plane*, also called a *linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $L(x,y) = (u,v)$, where u and v are given by linear equations with no constant term:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

where a,b,c, and d are real numbers.

■ **Warning 1.1** A linear transformation is not quite the same as a linear function from Calculus, because a linear function from calculus can have a constant term, and a linear transformation

[7] This is the kind of function studied in *algebraic geometry*.

cannot. This is an unfortunate inconsistency in terminology, but perhaps you can think about a linear transformation as being more *purely linear* since the only terms it has are linear terms, and no constant terms.

■ **Warning 1.2** In light of Question 1.2, you may want a geometric definition of a linear transformation of the plane to be something like: *it takes gridlines to lines*. This isn't quite the case (we will see some examples of this). To be completely precise, we also need the gridlines to remain parallel and evenly spaced, and we need $L(0,0) = (0,0)$. We will discuss this geometric reformulation more later, but for now I just wanted to mention that a this first guess is not quite enough.

You might be getting this far and thinking *wait...I thought linear algebra was about matrices? Where do those fit in?* This is a good question, so let's give a preliminary answer. Take a linear tranformation $L(x,y) = (u,v)$ where:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

This function is completely determined by the coefficients of x, and the coefficeints of y. That is, to know L, it is enough to know a, b, c, and d. So, we can completely capture all the data for L in the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For now we should just think of a matrix as a rectangular array of numbers, so that a linear transformation of the plane corresponds to a 2×2 matrix.

Definition 1.1.2 The matrix associated to the linear transformation in Definition 1.1.1 is the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■ **Example 1.9** Consider the function $g(x,y) = (u,v)$ from example 1.8. Observe that the coefficient of y in the first equation is -2, because adding $-2y$ is the same as subtracting $2y$. Also, the coefficient of y in the second equation is a 1 becuase $y = 1 \times y$.

$$u = u(x,y) = 2x + -2y,$$

$$v = v(x,y) = \frac{1}{2}x + 1y.$$

The matrix associated to this fuction is therefore:

$$\begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1. \end{bmatrix}$$

■

So far we've only seen how a correspondance between linear transformations of the plane and 2×2 matrices. We will work out in the coming weeks how this fits in to notions of matrix multiplication, determinants, and other matrix operations. For now, we the main take away should be the following.

■ **Slogan 1.3** A matrix is a function.