Linear Algebra

Math 217: Saint Lawrence University

Gabriel Dorfsman-Hopkins

GABRIEL DORFSMAN-HOPKINS
SAINT LAWRENCE UNIVERSITY
DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE

These are growing course notes developed for a first course in Linear Algebra at Saint Lawrence University in Canton, NY. They are written and maintained by Professor Gabriel Dorfsman-Hopkins. The notes have been typset in LaTeXwith a template that was adapted from the Legrand Orange Book Template by Mathias Legrand (`legrand.mathias@gmail.com`) with modifications by Vel (`vel@latextemplates.com`).

First release, Spring 2023

Contents

1. Introduction

1.1 January 24, 2023

1.1.1 Functions

Almost any course in mathematics is centered around studying types of *functions*. For example, in *Calculus* we study the behavior of functions of a single variable, that is, functions whose input is a single real number and whose output is a single real number, looking especially closely at functions which are *continous* or *differentiable*.

■ **Example 1.1 — Functions of a single variable.** Consider the function

$$f(x) = 3x.$$

Its input is a real number, x, and the output is computed by multiplying the input by 3. To see what this function does to a real number, say, 11, we can compute:

$$f(11) = 3 \times 11 = 33.$$

Explicitly, f takes an input of eleven an *transforms it* into an output of 33. ■

■ **Example 1.2** Consider the function:

$$g(x) = x^2 - 2x + 1.$$

What does this function do to the number 2? ■

The study of calculus looks closely at these functions of a single variable, establishing concepts like *derivatives* and *integrals*, and connecting them to many real world questions and situations. A shorthand that we will adopt to describe a function f of a single variable is the following

$$f : \mathbb{R} \to \mathbb{R}$$

This can be read aloud as *f is a function from $\mathbb{R}$ to $\mathbb{R}$*. It signifies that f takes a real number (on the left of the arrow), and runs it through the arrow to produce another real number (on the right of the arrow). *Note: The set before the arrow is called the* ***domain*** *of the function. It is also sometimes called the* ***source****. The set after the arrow is called the* ***co-domain****. It is sometimes also called the* ***target****.*

In *Mulltivariable Calculus* we develop similar ideas, **but the types of functions we study are different**. In particular, we allow for functions which take more than one real number as an input. Allowing for mutli-variable inputs allows calculus to be applied to our multi-dimensional world, and vastly expands the applications of derivatives, integrals, and related ideas.

■ Example 1.3 — Functions of 2 variables. In multivariable calculus you may encounter a function like:

$$f(x,y) = x - y.$$

It takes as input a *pair* of real numbers (x,y), and outputs their difference. For example, to see what the function does to the pair of number $(5,2)$ we can compute:

$$f(5,2) = 5 - 2 = 3.$$

In partiuclar, f will *transform* the pair of numbers $(5,2)$ into the single number 3. ■

■ Example 1.4 — Functions of 3 variables. Consider the function of 3 variables:

$$f(x,y,z) = xyz + 1.$$

What does this function do to the triple $(1,2,3)$? ■

The *arrow notation* of a function introduced above carries over here as well. For example, if f is a function of two variables, (whose input is 2 real numbers) we may write:

$$f : \mathbb{R}^2 \to \mathbb{R},$$

which we read as *f is a function from $\mathbb{R}^2$ to $\mathbb{R}$*. Here $\mathbb{R}^2$ denotes the collection of *pairs of real numbers*. Similarly, if g is a function of 3 variables (like in Example 1.4), we may write

$$g : \mathbb{R}^3 \to \mathbb{R}.$$

Notice that for each function we've describe so far, the output is *1-dimensional*. That is, we may have a function into which takes multiple real numbers as an input, but in each case the output is *a single real number*.[1] But just as allowing a multi-dimensional input massively expanded the scope of calculus, allowing functions to have a multidimensional output can be very useful as well.

■ Example 1.5 — Analyzing Ocean Currents. A group of oceanographers are measuring the movement of the water in the Atlantic, by studying where a collection of sensors start and end over the course of two weeks. They compile their data into a function C whose input is the GPS coordinates of a location in the Atlantic, and whose output of where the water at that location ends up 2 weeks later. For example,

$$C(40.47, -68.73) = (41.71, -64.07),$$

[1] You may recall that $\mathbb{R}$ can be thought of as a line, $\mathbb{R}^2$ as a plane, and $\mathbb{R}^3$ as 3-dimensional space. We will eventually adopt this notion of dimensionality, and explore it more carefully.

means that a drop of water whose GPS Coordinates are 40.47N 68.73W will move over the course of two weeks to the location 41.71N 64.07W. Observe that this is a function that takes as input two real numbers, and outputs *2 real numbers* as well! That is, both the input and the output are *2-dimensional*. In our arrow notation, we would write:

$$C : \mathbb{R}^2 \to \mathbb{R}^2.$$

TODO: put an image here!

■

■ **Example 1.6 — Casting Shadows.** Shadows are cast when a body in space blocks the sun from hitting the ground. If we'd like to study the shape of shadows mathematically, it is worth modelling shadows with a function, say S. Here:

$$S(\text{A point in space}) = \text{The spot on the ground where it casts a shadow.}$$

Modelling 3-dimensional space with $\mathbb{R}^3$ and the 2-dimensional ground with $\mathbb{R}^2$, this gives a function:

$$S : \mathbb{R}^3 \to \mathbb{R}^2.$$

In fact, this will be a *projection function*, a certain kind of *linear transformation* that we will study in TBA.

TODO: put an image here!

■

As we can see, functions with multivariable outputs are not hard to come up with, and model many different situations we would hope to study with mathematics. Let us begin by looking at a very special case:

1.1.2 Functions $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose you wanted to describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. How would you go about it? Both the input and output of f consist of pairs of numbers, so to be explicit with our notation, let's give the first $\mathbb{R}^2$ the coordinates (x, y), and the second $\mathbb{R}^2$ the coordinates (u, v). In particular, our function will look something like

$$f(x, y) = (u, v).$$

The function should be a rule so that, given a pair (x, y) of real numbers, we return with another pair of numbers, (u, v). In particular, we have to say what u is, and what v is. But each of these coordinates depend on both x and y, so in essense this is just *two functions* whose output is a real number:

$$u = u(x, y)$$

$$v = v(x, y).$$

■ **Slogan 1.1** To describe a function whose output is two real numbers, you can give 2 functions which output a single real number each.

Let's see how this works with an example.

■ **Example 1.7** Lets define a function $f:\mathbb{R}^2\to\mathbb{R}^2$ via the rule $f(x,y)=(u,v)$ where:

$$u=u(x,y)=xy+1,$$

$$v=v(x,y)=x+2y^2$$

The input of this function is a pair of numbers (x,y), and the output is *another* pair of number (u,v). So, for example, if we feed the function the pair $(-1,3)$, we can compute:

$$u=u(-1,3)=-1\times 3+1=-3+1=-2$$

$$v=v(-1,3)=-1+2\times 3^2=-1+18=17.$$

Therefore, this function transforms the pair $(-1,3)$ to the pair $(-2,17)$:

$$f(-1,3)=(-2,17).$$

■

■ **Example 1.8** Define a function $g:\mathbb{R}^2\to\mathbb{R}^2$ which takes (x,y) to (u,v) via the rule

$$u=u(x,y)=2x-2y,$$

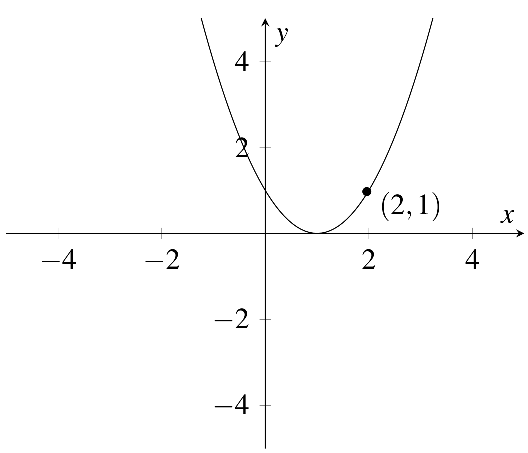
$$v=v(x,y)=\frac{1}{2}x+y.$$

Where does g take the point $(1,1)$? ■

It is often useful to think about a function as something that *moves* the point (x,y) to the point (u,v), and to emphasize this intuition, we will often refer a function $\mathbb{R}^2\to\mathbb{R}^2$ as a *trnasformation of the plane*.

1.1.3 Visualizing Transformations of the Plane

How do we visualize these types of functions? Since these will be central objects of study, let's start by spending some time developing techniques for how to think about and imagine a function from $\mathbb{R}^2$ to itself. Recall that in calculus you often visualize functions $g:\mathbb{R}\to\mathbb{R}$ using their graphs in the xy-plane. Here the x axis plays the role of the domain, and the y-axis the role of the co-domain, and the graph is generally a curve consisting of the points $(x,g(x))$. For example, the graph of the function $g(x)=x^2-2x+1$ from Example 1.2 is below.



The fact that $f(2)=1$ is captured by the fact that $(2,1)$ lies on the curve. A similar approach is used in multivariable functions, where now the domain is the entire xy-plane, and the co-domain is the z-axis. Then a function $f:\mathbb{R}^2\to\mathbb{R}$ can be graphed in 3-dimensional space, coloring in the points $(x,y,f(x,y))$, generally giving rise to a surface in 3-dimensional space.

■ **Question 1.1** Can we take a similar approach to a function $f:\mathbb{R}^2\to\mathbb{R}^2$? Why or why not?

Given the dimensional constraints, we have to come up with another way to represent a function $f:\mathbb{R}^2\to\mathbb{R}^2$. One way to do this is to get to the heart of what a function really does: *it transforms a point in* $\mathbb{R}^2$ *to another point in* $\mathbb{R}^2$. In particular, we can think about such a function as *something that transforms the plane*, moving the points of the plane around.

■ **Slogan 1.2** Think about a function $\mathbb{R}^2\to\mathbb{R}^2$ as something that moves around the points on a single plane. The input (x,y) is where the point starts, and the output $(u,v)=f(x,y)$ is where the point ends.

In fact, this is exactly what the function from Example 1.5 does, it keeps track of of where a drop of water in the Atlantic moves over the course of two weeks!

■ **Example 1.9** Let's visualizae the function from Example 1.7, which was function $f(x,y)=(u,v)$ where:

$$u=u(x,y)=xy+1,$$
$$v=v(x,y)=x+2y^2$$

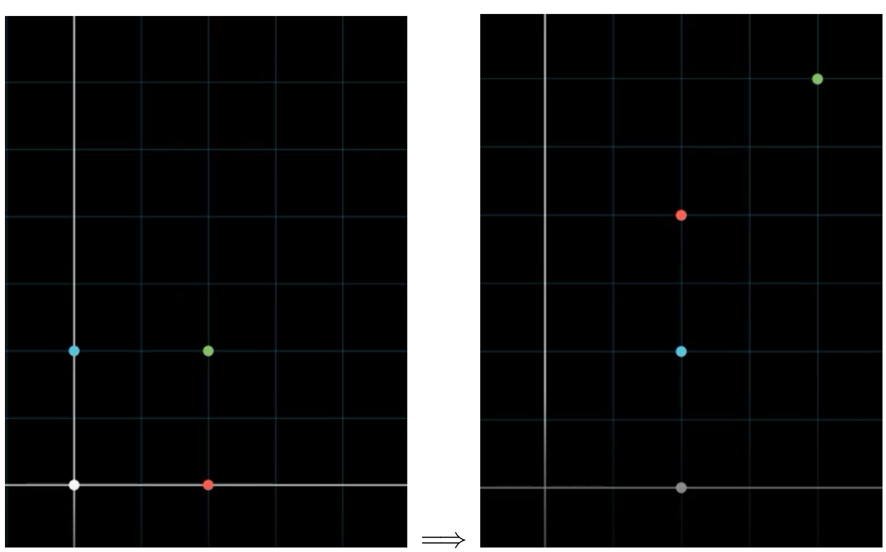
To get a sense of what kind of movement, let's keep track of what happens to a few points:

$$(0,0),(1,0),(0,1),(1,1).$$

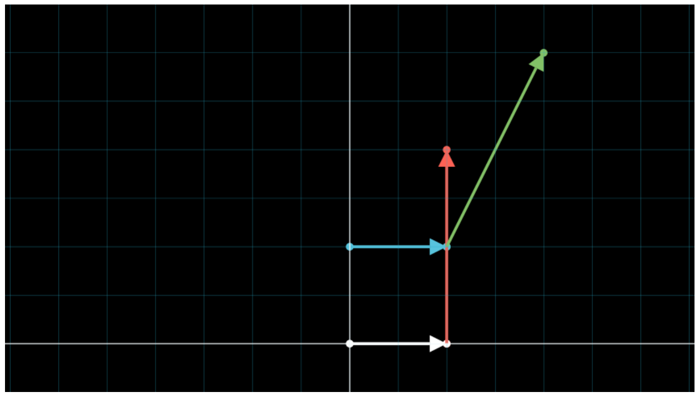
Using the formulas we can compute where f takes these points, just like in Example 1.7.

$$f(0,0)=(1,0),\ \ f(1,0)=(1,2),$$
$$f(0,1)=(1,1),\ \ f(1,1)=(2,3).$$

Instead of a single graph of the function, we can represent what f does with two pictures of the plane, a *before* shot and an *after* shot. On the left, we see the 4 points before applying f, and on the right, we see them after.



The *movement* of the situation can be captured nicely by an animation linked below.[2] You can also emphasize that it is movement on a single page by using arrows that point from the start to the finish of the various points:



■

Exercise 1.1 — January 24th Checkin. Consider the transformation $L(x,y) = (u,v)$ of the plane $\mathbb{R}^2$, given by the following two equations:

$$u = u(x,y) = y$$

$$v = v(x,y) = -x.$$

On a single coordinate plane, draw what the function does to a number of points. Do this by plotting a point (x,y), its image $L(x,y)$, and connecting them with an arrow. Use a few sentences to describe what the transformation L is doing to the plane. This can be a *qualitative* description. *What does it look like is happening?* ■

1.2 January 26, 2023

Let's begin by recalling some of the techniques discussed last time to visualize a function $\mathbb{R}^2 \to \mathbb{R}^2$ as a *transformation* of the plane.

■ **Example 1.10** Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 1.8. In particular, it is given by the rule $g(x,y) = (u,v)$ where:

$$u = 2x - 2y \text{ and,}$$

$$v = \frac{1}{2}x + y.$$

We compute compute where g takes the four points: $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. For example, $g(0,0)$, we may compute the u coordinate to be $2 \times 0 - 2 \times 0 = 0$ and the v coordinate to be $\frac{1}{2} \times 0 + 0 = 0$, so that $g(0,0) = 0$. Similar computations show that:

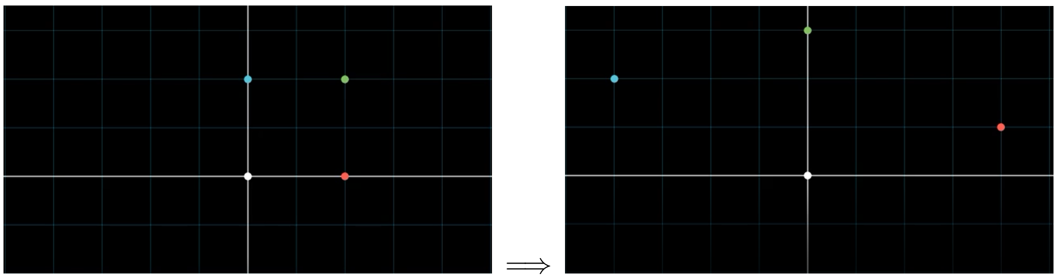
$$g(1,0) = (2,0.5)$$

[2] www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan24_Quad.mp4

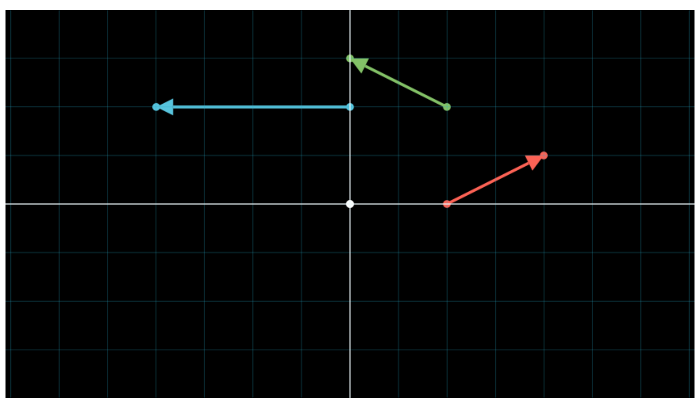
$$g(0,1) = (-2,1)$$

$$g(1,1) = (0,1.5).$$

Plotting the points before and after applying g gives:



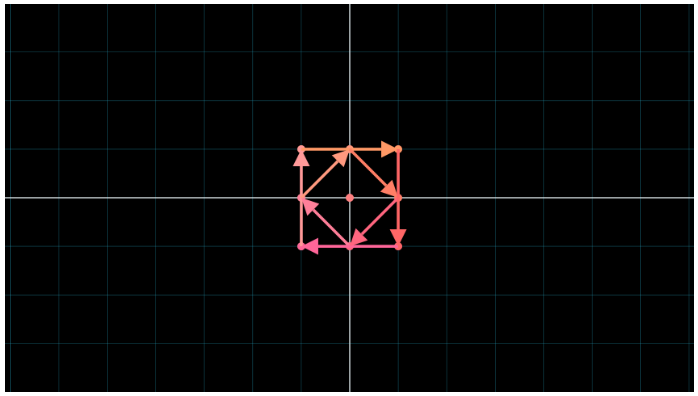
Plotting the before and after on the same plane, connecting (x,y) with $g(x,y)$ using arrows gives the following picture.



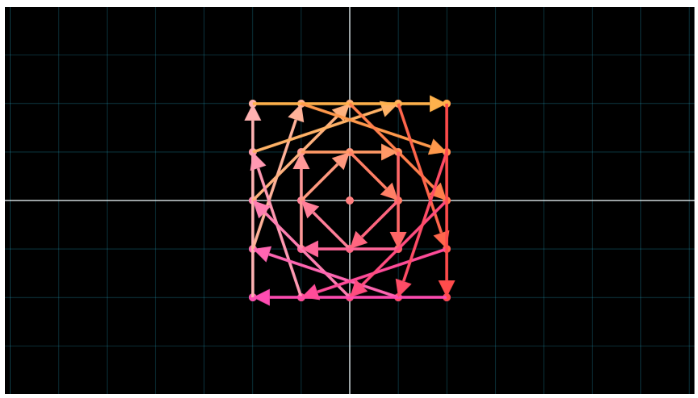
We should imagine this function something *moving around the points on the plane*, a perspective that is emphasized when animating the function. You find an animation of the moving points below.[3]. Try to give a qualitative description of what this function is doing to the plane. Plotting moer points may give a better picture. ∎

∎ **Example 1.11** At the end of our last class, we did a similar exercise using the function $L(x,y) = (y,-x)$ (cf. Exercise 1.1). Let's draw a few pictures and see if we can arrive at a description of what is happening to the plane. First, we plot all the points whose x and y coordinate's are between -1 and 1, connecting the points before and after applying L with an arrow.

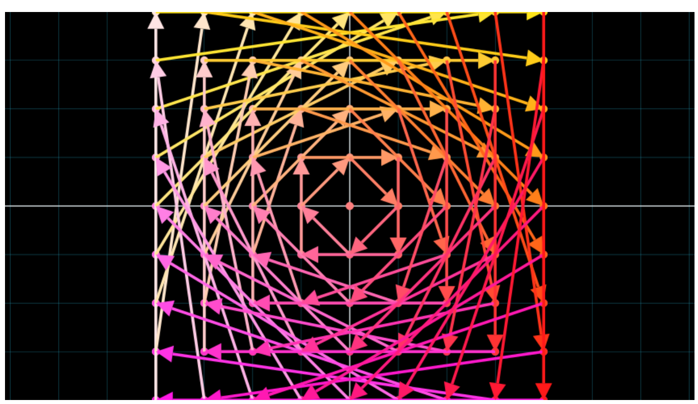
[3] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26_Linear.mp4`



Can you begin to describe what L is doing to the plane? Let's throw in a few more points, now letting the coordinates range between -2 and 2.



As you can see, it appears that L is *rotating the plane clockwise!*. We include one more image now with coordinates ranging between -5 and 5.



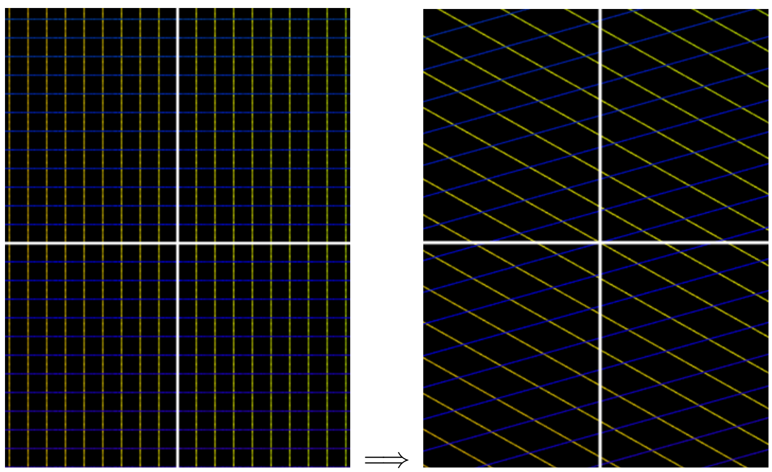
Althought the image is starting to get cluttered, this definitely appears to be a rotation, and indeed, replacing the arrows with an animation makes this clear (see the animation below[4]). ■

To summarize, plotting where points go under a function can give a sense qualitative sense of how a function moves the plane. That said, Examples 1.9 and 1.10 suggest that only drawing where

[4] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26_Rotate.mp4`

a few points go gives an incomplete picture. On the other hand, as we saw at the end of Example 1.11, if we to fill in more and more points, the image can start to get cluttered and it may become difficult to infer much from the picture. [5]. That being said, if you carefully pic which points to keep track of, you can get a nice sense of the *geometric* properties of a function. One way to do this, is by keeping track of what the function does to the *gridlines* of the plane.

■ **Example 1.12** To get a better picture of the function g from Examples 1.8 and 1.10, let's analyze what it does does to the gridlines of the plane.

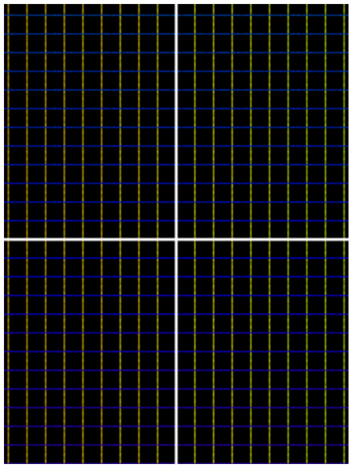
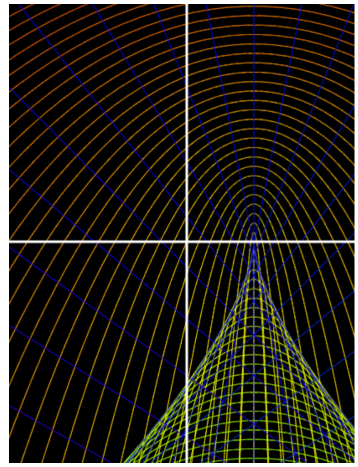


One can really get a sense for how g *moves* the plane by playing around with the tool linked below[6]. In particular, we see that it sort of *stretches* and *rotates* the plane, distorting it slightly but not too much. In this course we will develop a vocabulary to mathematically describe terms like *stretching the plane*, and ways to extract that information from the equations given in Exampel 1.8, but for now we're trying to get a qualitative sense of what's going on. ■

■ **Example 1.13** Let's also look at what the function f from Example 1.7 does to the gridlines of the plane.

[5] *Try this! For some functions you can actually get a nice picture! In fact, the situation in Example 1.11 is a particularly nice one. In general it will be much more complicated*

[6] Click the *linear* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26Gridlines/index.html`

 $\Longrightarrow$ 

The animation is actually quite nice to look at[7]. ■

It is fair to say that the function in Example 1.13 appears far more complicated than the one in Example 1.12. In fact, in some sense it is complicated in a way that puts it beyond the purview of *linear algebra*[8] For the context of linear algebra, we will have to restrict ourselves to functions more like that of Example 1.12, functions that we will call *linear transformations*. Before describing exactly what these are, it might be worth while to ponder the following question. Qualitative answers are always welcome!

■ **Question 1.2** What are some differences between what happens to the gridlines in the two examples on the previous page?

1.2.1 Linear Transformations of the Plane

One answer to Question 1.2 could be: *In example 1.8 the gridlines remain as lines after applying g, but in Example 1.7 the gridlines become curvy*. This is a good observation. Recall that lines played a special role in calculus. Not only where they the simplest functions, we used them to model more complicated functions locally, by taking *tangent lines*. We do something similar in multivariable calculus, modelling more complicated functions with linear ones by taking the *tangent plane*. Not only were these functions simple *geomterically* (being lines and planes), but they were also simple *algebracially*. For example, a line usually has the following equation:

$$f(x) = {\color{red}mx} + {\color{blue}b}.$$

Above we highlighted the *linear term* in red, and the *constant term* in blue. Similarly, a plane had a simple equation as well:

$$h(x,y) = {\color{red}mx} + {\color{red}ny} + {\color{blue}b},$$

where again the linear terms are highlighted in red, and the constant term in blue. Looking at the function $g(x,y) = (u,v)$ from Example 1.8, we see that the equations for both u and v have only linear terms (and no constant terms).

$$u = u(x,y) = 2x - 2y,$$

[7] Click the *Quadratic* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26Gridlines/index.html`

[8] This is the kind of function studied in *algebraic geometry*.

$$v = v(x,y) = \frac{1}{2}x + y.$$

This will turn out to be a good definition for a linear function.

Definition 1.2.1 — Linear Transformations of the Plane. A *linear transformation of the plane*, also called a *linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $L(x,y) = (u,v)$, where u and v are given by linear equations with no constant term:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

where a, b, c, and d are real numbers.

■ **Warning 1.1** A linear transformation is not quite the same as a linear function from Calculus, because a linear function from calculus can have a constant term, and a linear transformation cannot. This is an unfortunate inconsistency in terminology, but perhaps you can think about a linear transformation as being more *purely linear* since the only terms it has are linear terms, and no constant terms.

■ **Warning 1.2** In light of Question 1.2, you may want a geometric definition of a linear transformation of the plane to be something like: *it takes gridlines to lines*. This isn't quite the case (we will see some examples of this). To be completely precise, we also need the gridlines to remain parallel and evenly spaced, and we need $L(0,0) = (0,0)$. We will discuss this geometric reformulation more later, but for now I just wanted to mention that a this first guess is not quite enough.

You might be getting this far and thinking *wait...I thought linear algebra was about matrices? Where do those fit in?* This is a good question, so let's give a preliminary answer. Take a linear tranformation $L(x,y) = (u,v)$ where:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

This function is completely determined by the coefficients of x, and the coefficeints of y. That is, to know L, it is enough to know a, b, c, and d. So, we can completely capture all the data for L in the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For now we should just think of a matrix as a rectangular array of numbers, so that a linear transformation of the plane corresponds to a 2×2 matrix.

Definition 1.2.2 The matrix associated to the linear transformation in Definition 1.2.1 is the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■ **Example 1.14** Consider the function $g(x,y) = (u,v)$ from example 1.8. Observe that the coefficient of y in the first equation is -2, because adding $-2y$ is the same as subtracting $2y$. Also, the coefficient of y in the second equation is a 1 becuase $y = 1 \times y$.

$$u = u(x,y) = 2x + -2y,$$

$$v = v(x,y) = \frac{1}{2}x + 1y.$$

The matrix associated to this fuction is therefore:

$$\begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1. \end{bmatrix}$$

■

This correspondance goes in both directions. That is, given a matrix, you can extract a linear function.

Definition 1.2.3 Consider a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The *linear function associated to this matrix* is the function $L(x,y) = (u,v)$ where:

$$u = ax + by \text{ and,}$$

$$v = cx + dy.$$

Let's run through an example of applying a function, given only a matrix.

■ **Example 1.15** We compute $T(1,-2)$ where $T(x,y)$ is the function associated to the matrix

$$\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Applying the definition we see that $T(x,y) = (u,v)$ where:

$$u = 3x + 1y = 3x + y,$$

$$v = -1x + 0y = -x.$$

Plugging in $(x,y) = (1,-2)$ gives:

$$u = 3 \times 1 + (-2) = 1, \qquad \text{and} \qquad v = -1$$

Therefore $T(1,-2) = (1,-1)$. ■

R Later we will streamline this process using *matrix multiplication*.

Exercise 1.2 Consider the matrix:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.

2. Evaluate L at $(0,0),(0,1),(1,0)$ and $(1,1)$.
3. Plot the four points of part (b), before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

So far we've only seen how a correspondance between linear transformations of the plane and 2×2 matrices. We will work out in the coming weeks how this fits in to notions of matrix multiplication, determinants, and other matrix operations. For now, we the main take away should be the following.

■ **Slogan 1.3** A matrix is a function.

1.3 Homework 1

Exercise 1.3 Consider the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate L at all the points with integer coordinates are between -1 and 1. (There should be nine such points).
3. Plot the 9 points from part (b) before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

Exercise 1.4 Consider the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $T(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate T at 5 points of your choice.
3. Plot the 5 points from part (b) before and after applying T, and connect them with arrows.
4. Give a qualitative description of what you think T is doing to the plane.

■

Exercise 1.5 For this problem, adopt the notation of Exercises 1.3 and 1.4. Also consider the matrix N associated to the function $g(x,y)$ from Example **??**:

$$N = \begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1 \end{bmatrix}$$

1. Can you identify any relationships between the outputs $L(1,0), L(0,1)$ and the matrix M?
2. Can you identify any relationships between the outputs $T(1,0), T(0,1)$ and the matrix I?
3. Can you identify any relationships between the outputs $g(1,0), g(0,1)$ and the matrix N?

4. Now let's treat the general case: pet $\ell(x,y)$ be a linear transformation associated to a general matrix:

$$P = \begin{bmatrix} a & b \\ c & d. \end{bmatrix}$$

Describe the relationship between $\ell(1,0)$, $\ell(0,1)$ and the matrix P. Give reasoning for your answer.

■

Exercise 1.6 Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Do you agree or disagree with the following statement?

Once I know $\ell(1,0)$ and $\ell(0,1)$, I can determine $\ell(x,y)$ for any pair (x,y).

Explain your reasoning. ■

Exercise 1.7 Consider a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, and suppose that $F(0,0) = (1,1)$. Is it possible for F to be a linear transformation? Why or why not? ■

Exercise 1.8 Adopt the notation of Problem 1.5. Define a rule for adding two points as follows:

$$(x_0,y_0) + (x_1,y_1) = (x_0 + x_1, y_0 + y_1).$$

Let $P = (1,2)$ and $Q = (3,-2)$.

1. Can you identify any relationship between $L(P), L(Q)$, and $L(P+Q)$?
2. Can you identify any relationship between $g(P), g(Q)$, and $g(P+Q)$?
3. To see that it's not a fluke, do parts (a) and (b) again, but with new points P and Q of your choice.
4. Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a general linear transformation, and let P and Q be two random points in $\mathbb{R}^2$. Make a conjecture for the relationship between $\ell(P)$, $\ell(Q)$, and $\ell(P+Q)$. (There is no need to prove this yet, but you can extrapolate from the evidence collected in (a) through (c)).

■

Exercise 1.9 To give a function $\mathbb{R}^2 \to \mathbb{R}^2$ we needed to give 2 functions of 2 variables which output a single number each (for more detail, see Section 1.1.2 in the course notes). Let's see if we can work out what to do in higher dimensions. In particular, adapt Section 1.1.2 in the course notes to describe a function $F : \mathbb{R}^3 \to \mathbb{R}^3$. You can make up any funcion you like, just make sure that you describe it fully. Evaluate this function at the points $(0,0,0), (1,0,0)$, and $(1,2,3)$. ■

2. Vectors

Acknowledgment: I'd like to attribute this approach to vectors in part to Grant Sanderson, author of the delightful youtube channel `3Blue1Brown`. In particular, I borrow heavily his description of the three perspectives of vectors presented below as the *physicist's perspective*, the *computer scientist's perspective*, and the *mathematician's perspective*.

2.1 January 31, 2023

We suggested in the Introduction that the field of Linear Algebra is centered around the study of *linear transformations*. Furthermore, Exercise 1.8 suggests that a linear transformation L satisfies the following equation:[1]

$$L(P+Q) = L(P) + L(Q).$$

It is worth taking some time to unpack what $+$ is doing here. If P and Q are points, what is their sum? In Exercise 1.8, we defined the sum of 2 points to be a third point, whose coordinates correspond to adding the coordinates of P and Q. In $\mathbb{R}^2$ this is written as follows:

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1).$$

This is a perfectly valid formula, but it may also seem a bit strange. In particular, we should unpack a concrete interpretation of this algebraic operation to answer the following question:

■ **Question 2.1** What exactly is the meaning of adding coordinates of points in $\mathbb{R}^2$?

By trying to answer this question, we naturally encounter the notion of a *vector*. In fact, a first definition of a vector is pretty much exactly the idea of *points you can add*.

[1] At least when $L : \mathbb{R}^2 \to \mathbb{R}^2$

Definition 2.1.1 — 2-dimensional vectors: the computer scientist's perspective. A two dimensional vector is an array of two numbers aligned vertically:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

These can be added coordinatewise:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix}.$$

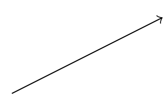
Notation 2.1. *Aligning vectors* vertically *allows them to fit into the matrix theory we will develop in the coming weeks. We will sometimes use the term* column vector *when writing it this way.*

We call this *the computer scientist's perspective*, because it remembers a vector as a light-weight data type stored in a way that easily allows for vector operations (like addition and applying linear maps) to be computed efficiently by a computer in almost any programming language. This perspective also has the advantage of generalizing very easily to higher dimensions, *can you see how?* That being said, it doesn't really get us any closer to answering Question 2.1. In order to do this, we give another perspective on vectors you may have seen in a physics course.

Definition 2.1.2 — 2-dimensional vectors: the physicist's perspective. A two dimensional vector is a quantity specifying a *magnitude* together with a *direction* in the two dimensional plane. This can be represented by an arrow, pointing in the given direction, whose length is the given magnitude.

The physicist's vectors can be added too, essentially by *concatenating the two arrows*. We define this more carefully in Definition **??** below. In particular, we have encountered two different perspectives on the notion of a vector. Below to the left is an example of a computer scientist's vector, and to the right is an example of a physicist's vector.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

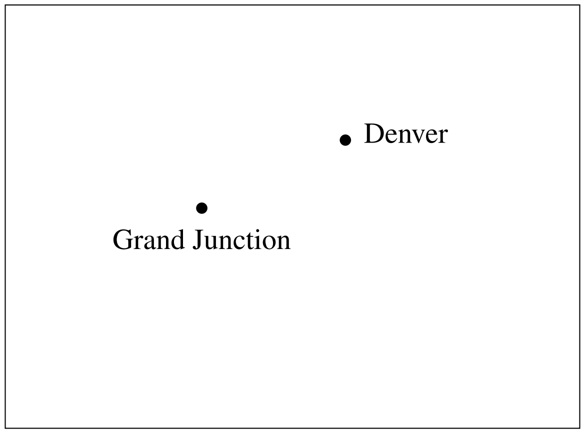


In fact, one could say that these are two perspectives on the same vector, *can you explain why?*. An important part of linear algebra involves learning how to pass seemlessly between these two perspectives, as one lends itself better to computations, while the other lends itself better to interpretations. In this section, we will explore these two perspectives, and start developing a dictionary between them, keeping track of what information can get lost in translation. Along the way we will extract algebraic properties of vectors, and have a first encounter the notions of *linear combinations* and *spans*, which are among the most important in this course.

R Another thing we can do from both perspectives is *scale* vectors by numbers. In fact, there is a third perspective on vectors, which we can call *the mathematician's perspective*, which essentially defines vectors as: *theoretical objects which can be added together and scaled*. We will postpone discussion of this third, more abstract, perspective until the end of the semester. The attentive reader may want to pay attention to how most properties of vectors can be expressed in terms of of these two operations (addition and scaling).

2.1.1 The Physicist's Perspective

A vector is a natural quantity to describe relationships between objects in the physical world. For example, suppose that a pilot is hoping to fly from Denver, Colorado to Grand Junction, Colorado, and asks you for directions.[2]

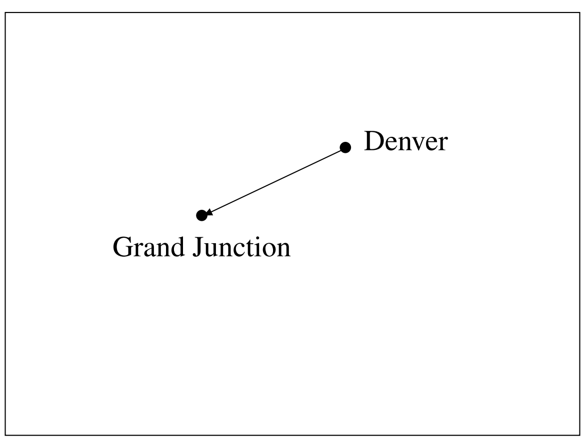


The distance between Denver and Grand Junction is 220 miles. That be but it is *not enough* to tell the pilot to fly 220 miles. If they don't want to end up in Wyoming, they must also know which direction to fly. So for example, you may tell the pilot to fly 220 miles *west-southwest*. These two peices of information constitute a quantity which we call a *vector*:

Magnitude: 220 miles,

Direction: West-Southwest.

All of the defining this quantity information can be exhibited on a map, by drawing an arrow from the start to the finish. One can recover the direction from the direction of the arrow, and the magnitude from the length of the arrow.



This gives us another way to think about Definition 2.1.2.

[2]Map not to scale.

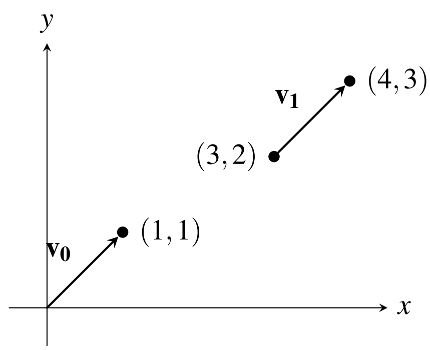
Definition 2.1.3 [Vectors as Displacement] Given two points P and Q, the *displacement vector from P to Q* is the arrow whose tip is at the point Q whenever its tail is placed on P.

Notation 2.2. *Given a vector* $\mathbf{v}$*, we denote its magnitude by* $||\mathbf{v}||$.

With this definition we think of a *displacement vector* as something you can apply to a point. To apply a vector to a point P, we put its tail at P, and the output is wherever its tip points. Importantly, the same vector can be placed in different locations. Let us take this perspective to our vector which takes us from Denver to Grand Junction. This vector is completely determined by the fact that it goes 220 miles west-southwest. It isn't necessary for its to lie on Grand Junction. For example, we could apply the *same vector* starting from Canton, and we would end up somewhere near Toronto. We would still be using the same magnitude (220 miles) and direction (west-southwest), and therefore following the same vector. This is an important point: *a vector is determined by magnitude and direction*. 2 vectors of the same length and pointing in the same direction are the same vector, even if they are drawn at different places.

■ **Slogan 2.1** The vector is the arrow, not where the arrow is.

■ **Example 2.1** Consider following vectors. $\mathbf{v_0}$ connects $(0,0)$ and $(1,1)$, while $\mathbf{v_1}$ connects $(3,2)$ and $(4,3)$. Does $\mathbf{v_0} = \mathbf{v_1}$?



■

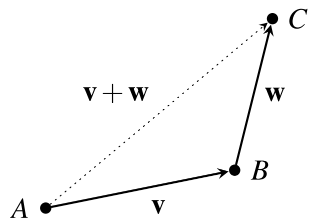
It is common in linear algebra to distinguish between *vectors*, which have a direction and a magnitude, numbers without any direction. The latter is just a number, but we will also often refer to it as a *scalar*.

Adding and Scaling Arrow Vectors

Recall that Question 2.1 asked what adding vectors meant. By thinking of vectors as measuring displacement, we can get a geometrically and physically meaningful understanding how they add, subtract, and scale. We will explore this with the following thought experiment in mind:

You are programming autonomous vehicles. To command a vehicle to move, you give it a vector. The vehicle will then move along the vector: in the given direction for the given magnitude.

Addition: Suppose you give your vehicle a vector $\mathbf{v}$ to follow, and it moves from point A to point B. Once it has arrived at point B, you give it another vector $\mathbf{w}$, and it moves from point B to point C. At this point, the net displacement that the vehicle has travelled is from point A to point C. Define $\mathbf{v}+\mathbf{w}$ to be this displacement vector from A to C.



We can summarize in the following definitoin.

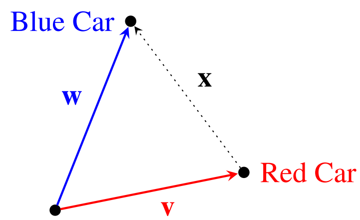
Definition 2.1.4 The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$, is the combined displacement resulting from first applying $\mathbf{v}$, and then applying $\mathbf{w}$ to the result.

■ **Question 2.2** Given 2 vectors $\mathbf{v}$ and $\mathbf{w}$, is it always true that

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}?$$

With addition now defined, we can move on to:

Subtraction: Suppose you control two vehicles, a red vehicle and a blue one, both starting at the same point. You send the red one along vector $\mathbf{v}$, and send the blue one along vector $\mathbf{w}$. After they arrive, the blue vehicle breaks down, so you must send the red vehicle to rescue it. What vector $\mathbf{x}$ must you command the red car to follow?



In particular, if the red car first does $\mathbf{v}$, and then does $\mathbf{x}$, it should overall be following $\mathbf{w}$, and therefore should end up alongside the blue car. We translate this by Definition 2.1.4 to the statement,

$$\mathbf{w}=\mathbf{v}+\mathbf{x}.$$

If subtraction of vectors were to make any sense, then we could subtract $\mathbf{v}$ from both sides and discover that $\mathbf{x}$ really should be the difference of $\mathbf{w}$ and $\mathbf{v}$:

$$\mathbf{w}-\mathbf{v}=\mathbf{x}.$$

Therefore, that is the definition we will make.

Definition 2.1.5 The difference $\mathbf{w}-\mathbf{v}$ of two vectors $\mathbf{w}$ and $\mathbf{v}$, is the vector which, when added to $\mathbf{v}$, gives $\mathbf{w}$.

We can now add and subtract vectors in a geometrically meaningful way. Bringing us closer to getting meaningful answer to Question 2.1. Before moving on, though, we'd like to introduce a special vector.

The Zero Vector: If your vehicle doesn't move at all, what vector does it follow? Since magnitude is the net distance covered, the magnitude is 0. As for direction, this isn't really well defined, since if you move 0 units in any direction, you've stayed put. We will call the vector from a point to itself the *zero vector*.

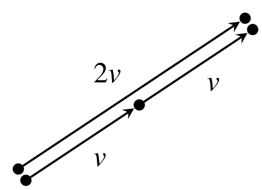
Definition 2.1.6 The vector whose magnitude is zero is called the *zero vector*, and is denoted $\mathbf{0}$.

R The zero vector is the only vector whose direction is unspecified.

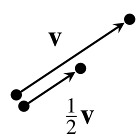
Exercise 2.1 Let $\mathbf{v}$ be any vector. What is $\mathbf{v}+\mathbf{0}$? What about $\mathbf{v}-\mathbf{v}$? ■

Let's introduce one more important operation that vectors allow.

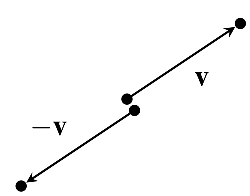
Scalar Multiplication: Suppose you'd like to send your car in the same direction as a vector $\mathbf{v}$, but twice as far as the vector $\mathbf{v}$ allows. You could acheive this by applying $\mathbf{v}$, and then doing so again. With Definition 2.1.4 in mind, you send your car along $\mathbf{v}+\mathbf{v}$, which we can write as $2\mathbf{v}$.



Similarly, if you wanted your car to go in the same direction as $\mathbf{v}$, but half as far, you could follow a vector $\mathbf{w}$ which satisfied $\mathbf{v}=\mathbf{w}+\mathbf{w}$. Since $2\mathbf{w}=\mathbf{v}$ we could reasonably say that $\mathbf{w}=\frac{1}{2}\mathbf{v}$.



Alternatively, suppose you wanted to go the same distance as $\mathbf{v}$, but in the opposite direction. You could follow a vector $\mathbf{x}$. Notice that if the car first does $\mathbf{v}$ and then does $\mathbf{x}$, it will travel along $\mathbf{v}$, and move the same distance in the opposite direction until it gets back to where it started. In particular, we have that $\mathbf{v}+\mathbf{x}=\mathbf{0}$, so it is reasonable to write $\mathbf{x}=-\mathbf{v}$.



Following this logic we can deduce that to scale a vector by a positive number, you scale its magnitude. The negative of a vector reverses direction. What about scaling by a negative number? The following formula should shed some light.

$$-2\mathbf{v}=-(2\mathbf{v}).$$

It appears that to scale by negative 2, you can first scale by 2, and then reverse direction.

Exercise 2.2 With $\mathbf{v}$ as in the figures above, sketch $-2\mathbf{v}$ ■

We can put all this together into the following definition.

Definition 2.1.7 Let $\mathbf{v}$ be a vector and c any scalar. Then the vector $c\mathbf{v}$ is defined by the following data.

- If c is positive, the direction of $c\mathbf{v}$ is the same as $\mathbf{v}$. Otherwise, the direction of $c\mathbf{v}$ is opposite to that of $\mathbf{v}$.
- The magnitude of $c\mathbf{v}$ is:

$$||c\mathbf{v}|| = |c| \cdot ||\mathbf{v}||,$$

where $|c|$ denotes the absolute value of the scalar c.

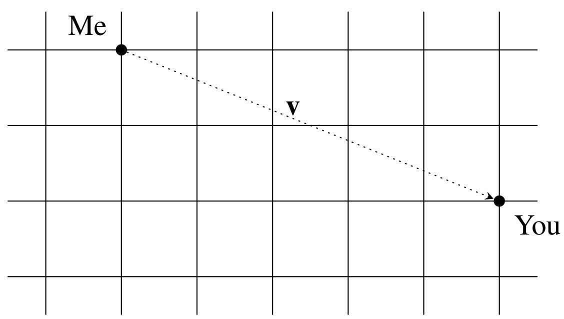
Exercise 2.3 Let $\mathbf{v}$ be any vector. What is $0\mathbf{v}$? ■

A nice output of this geometric approach is that we can give geometric names to certain algebraic operations. For example, we should call 2 vectors parallel, if when we draw their arrows are parallel as lines segments. Our definition of scalar multiplication tells us that this is equivalent to one being a scalar multiple of the other, giving us a algebraic notion of parallel-ness (that can and will extend to higher dimensions).

Definition 2.1.8 Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if there is some constant c such that $\mathbf{v} = c\mathbf{w}$.

2.1.2 Decomposing a Vector into Components

We'd like to connect the ideas described in the *physicist's perspective* on vectors, to the ordered pair numbers which we called the *computer scientist's perspective*. To do this, we'll take the example of giving directions in a city.



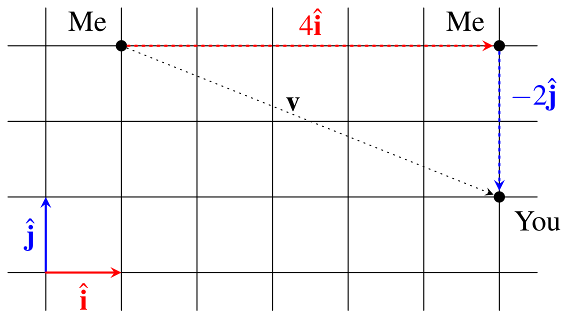
Above is a map depiction of a city, with the thick vertical and horizontal lines roads. You'd like to tell me how to find you. In particular, you need to tell what vector $\mathbf{v}$ I need to follow, in order to get to your location. One way to describe this to me is to give a magnitude and direction. But it is unlikely that you'll tell me something like *go about 530 meters in a direction that is mostly east but somewhat south*. In fact, even if you were more precise with the angles and distance, it is unlikely that I would be able to follow the directions (without walking through buildings).

Instead, you'd probably say something like *walk 5 blocks east, and then 2 blocks south*. Indeed, the regular gridlines of the city give us two natural vectors which we can all agree on:

$$\hat{\mathbf{i}} = \text{ one block east,}$$

$$\hat{\mathbf{j}} = \text{ one block north.}$$

With this in hand, everyone can agree on a set of navigational rules. Let's put them on our plot.



So we can see that in order to get to you, I first have to follow $4\hat{\mathbf{i}}$ and then $-2\hat{\mathbf{j}}$. In particular:

$$\mathbf{v} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}}.$$

Once we all agree on a definition of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we can represent the vector $\mathbf{v}$, just in terms of numbers 4 and 2. This is what the computer scientist would call:

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

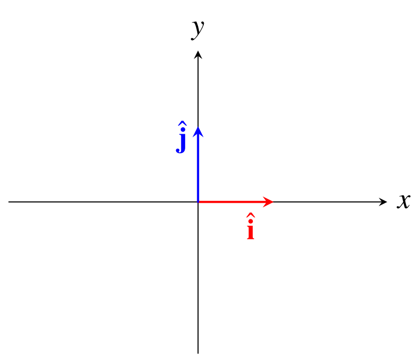
R It is important here that we all agree on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. One can imagine a city which is not grided parallel to north an south, and instead has *updown* and *downtown* directions. Then you may represent a vector using the coefficient of $\hat{\mathbf{j}}$ to describe how many units it goes uptown or downtown, but someone else might represent the vector using the coefficient of $\hat{\mathbf{j}}$ to represent how many units it goes north or south. The coordinates the computer scientist would write down would be different in each case. This is an important subtlety, but one that we will table until we are discussing *bases* and *change of bases*. For now, just remember that the coordinates that you might right down for a vector depend on a *choice* of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. This is one advantage of the physicist's perspective over the computer scientist's perspective.

The general setup (in 2-dimensions) is essentially the same

Definition 2.1.9 Given a cartesian coordinate system (that is, an *xy*-plane with coordinates), we can define the *standard basis vectors* $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to be the vectors:

$$\hat{\mathbf{i}} = \text{ one unit in the positive } x \text{ direction,}$$

$$\hat{\mathbf{j}} = \text{ one unit in the positive } y \text{ direction.}$$



Given any vector $\mathbf{v}$ in the coordinate plane, we can *resolve* it into its components:

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}}.$$

Here v_1 is a *scalar*, representing how far $\mathbf{v}$ goes in the x-direction, and v_2 is a *scalar* representing how far $\mathbf{v}$ goes in the y-direction. They are unique.

Notation 2.3. *Given a vector in component form:*

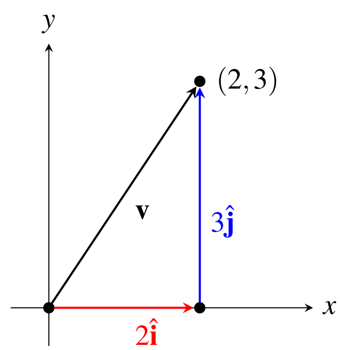
$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}},$$

we call the coefficients v_1 and v_2 the components *of* $\mathbf{v}$. *We can take these components and write* $\mathbf{v}$ *as a* column vector*:*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

It is important to note that the coordinates of the column vector depend on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. In particular, we should think of this column vector as meaning, first do $v_1\hat{\mathbf{i}}$, then do $v_2\hat{\mathbf{j}}$.

■ **Example 2.2** Consider the displacement vector $\mathbf{v}$ from $(0,0)$ to $(2,3)$.



Therefore we see that:

$$\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

■

Notice that the displacement vector from $(0,0)$ to (x,y) can always be written

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 2.4 Consider the vectors $\mathbf{v_0}$ and $\mathbf{v_1}$ from Example 2.1. Write them both in component form and as column vectors. Use your result to decide whether they are equivalent. ■

Exercise 2.5 Let $P = (x_0, y_0)$, $Q = (x_1, y_1)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ in component form and as a column vector, in terms of the coordinates of P and Q. ■