

## Homework 7

Due Thursday, October 29

## Implementation Part

For several of the implementation parts of the problem you will need to choose large secret primes, so make sure you have access to `findPrime` and all the functions it depends on from the takehome project.

1. Let's begin by writing functions that efficiently compute Legendre and Jacobi symbols.
  - (a) Write a function `legendreSymbol(a,p)` which takes as input an integer  $a$  and an *odd* prime  $p$ , and returns the Legendre symbol  $\left(\frac{a}{p}\right)$  in  $\mathcal{O}(\log(p))$  time.
  - (b) Write a function `jacobiSymbol(a,b)` which takes as input integers  $a$  and  $b$  where  $b$  is odd and positive and returns the Jacobi symbol  $\left(\frac{a}{b}\right)$  *without factoring*  $b$ . We remind you of the following properties of Jacobi symbols which should help with your computation.
    - This only depends on the residue of  $a$  modulo  $b$ .
    - If  $a \equiv -1, 0, 1, 2 \pmod b$  this is easy to compute directly (using quadratic reciprocity for  $-1$  and  $2$ ).
    - If  $b$  is prime then this is a Legendre symbol! (`probablyPrime` will help determine this quickly).
    - You can use quadratic reciprocity to relate  $\left(\frac{a}{b}\right)$  and  $\left(\frac{b}{a}\right)$ . Since we can reduce  $b$  modulo  $a$  this gives us a strictly smaller problem! (**Warning:** if  $a$  is even the  $\left(\frac{b}{a}\right)$  doesn't make sense! You will have to factor out the 2's from  $a$  and use the multiplicativity of the Jacobi function to deal with this case!)
  - (c) Compute the following Jacobi symbols. For the first 3 you can check your work by hand.

$$\left(\frac{8}{15}\right), \left(\frac{11}{15}\right), \left(\frac{12}{15}\right), \left(\frac{171337608}{536134436237}\right).$$

2. Implement the Goldwasser-Micali probabilistic encryption scheme to securely send 1 bit of data.
  - (a) Create a function `GenerateGMKey(b)` which creates a Goldwasser-Micali key from  $b$ -bit primes. In particular, it should output a *public key*  $[N, a]$  where  $N = pq$  is a product of (secret)  $b$  bit primes and  $a \in \mathbb{Z}/N\mathbb{Z}$  is a quadratic *nonresidue* modulo  $p$  and  $q$ , as well as a *private key* which should just consist of one of the secret primes. (You can use part 1 to compute Legendre symbols.)
  - (b) Write functions `GMEncrypt(publicKey,m)` and `GMDecrypt(privateKey,c)`. The first takes a Goldwasser-Micali public key, and a bit  $m = \{0, 1\}$ , and returns a ciphertext  $c \in \mathbb{Z}/N\mathbb{Z}$  which is a square modulo  $N$  if and only if  $m = 0$ . The second recovers  $m$  from  $c$  and the private key and Legendre symbols. (*Note:* When you compute the random integer in the encryption function make sure it is larger than the square root of  $N$ . For extremely large  $N$ , you should make use of the sage integer square root function `isqrt()` since floats may round up to infinity and have trouble reconverting to ints).
  - (c) Generate and print a Goldwasser-Micali key from 16 bit primes. Use it to encrypt and then decrypt both 0 and 1. Confirm that you recover the bit correctly.

- (d) Implement a Goldwasser-Micali key from primes  $p = 151$  and  $q = 233$ . Recover the bit from the cipher  $c = 33482$ .
3. Implement the RSA Digital Signature algorithm. It will look very similar to the first project, and you are welcome to reuse code from that assignment (especially for key generation).
  - (a) Write a function `generateRSAKey(b)` which generates an RSA private signing key and public verification key from primes  $b$  bits long. The verification key will be a pair  $[N, e]$  where  $N = pq$  is a product of secret primes  $b$  bits long and  $e$  is an integer prime to  $(p-1)(q-1)$ , and the signing key will be a pair  $[N, d]$  for the same  $N$  and  $d$  the inverse of  $e$  modulo  $(p-1)(q-1)$  (sound familiar?).
  - (b) Write functions

```
RSASign(signingKey,document) and  
RSASign(signingKey,document,signedDocument).
```

The former will sign a document with the signing key (by taking an appropriate root), and the second will verify that a document is correctly signed (by exponentiating).

- Generate an RSA digital signature key from 16 bit primes. Use it to sign the document  $D = 314159$ . Run your verification algorithm for twice, once with the signed document and once with the unsigned document, and confirm you get the expected results.
- My RSA key has not changed since the project. In this exercise we will use it as a public verification key.

$$N \equiv$$

10522131111414083920142713395050769617442433657695255146537311896434311144361  
52105697372008556663566615508174430418471897265040190399740387723795071615822  
78783519296999687194872224184723574952162166440945091505292215584920700394034  
42226365661647537411508608651137113856704007797414387581567049968246853446643  
9.

$$e =$$

21016836287029986747723759800774101835281275854873180166245543058525887395193  
96307766860254990165396054303259228348297307889093719835313656626522445205557  
65407005232827058670837160067093166648952976725071881739083691398929472902959  
38964443139784969754551470562127704086246141892044152375206351855610254115515.

You receive two documents:

$$D = 44591585690519734445193105605299933531568892342090748601970008137$$
$$D' =$$

33737713092926002711997965938686781423985198340985423980678740668540837920165  
7143087595936042616688490456529718758762313

Run `intToText` to read this documents (note: they are not encrypted). Each claims to be signed by me, and comes with a digital signature:

$$D^{sig} =$$

84266656881633759645931434414589646373474140299696318568846558871354683992664  
91115775348447213721512083388365998130055074496366371336684453834515335491064  
69940898673531353559586336758802075651425026668697230132550765551653945548321  
68424776641559903698464211162982834359695243021419912674971036001530626269241

$D^{sig} =$

90592809313509991477767898543561252818730285233946708276918316631997361944737  
 64728663367105923665686312882726901254765910591149065525748838442609274074694  
 58299675381621160449150556922441295777712973489019131240798265816551541474157  
 14324848399155672059930064485203220452985824881282662030076919344957027591627  
 Which message is truthful?

4. Implement Elgamal digital signatures.

- (a) Write a function `generateElgamalKey(p,g)` which takes as input a prime  $p$  and a primitive root  $g \in \mathbb{F}_p^*$ , chooses a secret exponent  $a$  (at random) and returns a private signing key  $[a, p, g]$  and a public verification key  $[A, p, g]$ .
- (b) Write functions

`elgamalSign(signingKey,document)` and  
`elgamalVerify(verificationKey,document,signedDocument)`.

The former will sign a document with the given signing key, returning the signed document as a pair  $[S_1, S_2]$  where  $0 \leq S_1 < p$  and  $0 \leq S_2 < p-1$ . The second will verify that the document was correctly signed. Both should follow the elgamal digital signature protocol defined in the October 22 lecture and described in Table 4.2 of [HPS].

- (c) If you sign 2 different documents with the same random element  $a$ , your Elgamal signing key becomes insecure. In problem 9 you will describe an algorithm to steal that signing key. Implement that algorithm here. In particular, implement an algorithm

`stealElgamalSignature(verificationKey,D,Dsig,D',D'sig)`.

which takes as input verification key associated to an Elgamal signature (which is public information), as well as 2 distinct documents signed with that key. It will first check if those 2 documents were signed with the same random value  $k$ , and if they were it will return the signers secret exponent  $a$ . (This attack was used to steal Sony's digital signature in 2013!).

- (d) Let  $p = 3700273081$ , and  $g = 7$ . Create an Elgamal key, and use it to sign the document  $D = 314159$ . Then verify that the signature was valid.
- (e) Suppose Samantha has a public verification key  $[A, p, g] = [185149, 348149, 113459]$ . Suppose Samantha signed the following 2 documents:

$$\begin{aligned} D &= 153405 & D^{sig} &= (S_1, S_2) = (208913, 209176) \\ D' &= 127561 & D'^{sig} &= (S'_1, S'_2) = (208913, 217800). \end{aligned}$$

Use `stealElgamalSignature` to steal Samantha's signing exponent.

5. Implement DSA.

- (a) Write a function `generateDSA(p,q,g)` which takes as input primes  $p, q$  with  $p \equiv 1 \pmod q$ , and an element  $g \in \mathbb{F}_p^*$  of order  $q$ . It then chooses a secret exponent  $a$  (at random) and returns a private signing key  $[a, p, q, g]$  and a public verification key  $[A, p, q, g]$ .
- (b) Write functions

DSASign(signingKey,document) and  
 DSAVerify(verificationKey,document,signedDocument).

The former will sign a document with the given signing key, returning the signed document as a pair  $[S_1, S_2]$  where  $0 \leq S_i < q$  for  $i = 1, 2$ . The second will verify that the document was correctly signed. Both should follow the DSA protocol defined in the October 22 lecture and described in Table 4.3 of [HPS].

- (c) Let  $p = 48731$  and  $q = 443$ . Assume that 7 is a primitive root for  $\mathbb{F}_p^*$ . Use this to find an element of order 443 in  $\mathbb{F}_p^*$ . Call this element  $g$ .
- (d) Generate a DSA key for  $p, q, g$  in part (c). Use it to sign the document  $D = 314$  and verify that this signature is valid.

## Written Part

- 6. Let's do a few checks from the implementation part.
  - (a) Compute the Jacobi symbols  $(\frac{8}{15})$ ,  $(\frac{11}{15})$ ,  $(\frac{12}{15})$  by hand and confirm your solutions from 1(c) are correct.
  - (b) In the Goldwasser-Micali algorithm it was suggested that the random number be chosen as greater than  $\sqrt{N}$ . Why?
- 7. Let  $p$  be an odd prime and  $g \in \mathbb{F}_p^*$  a primitive root. Fix any  $h \in \mathbb{F}_p^*$ . In this problem we study how to get information about  $\log_g(h)$ .
  - (a) Describe how to easily tell  $\log_g(h)$  is even or odd.
  - (b) We can write  $\log_g a$  in binary:

$$\log_g a = \varepsilon_0 + \varepsilon_1 \cdot 2 + \varepsilon_2 \cdot 2^2 + \varepsilon_3 \cdot 2^3 + \cdots \quad \varepsilon_i \in \{0, 1\}.$$

Explain why (a) means that we know  $\varepsilon_0$ . This property is summarized as saying that the *first bit* of the discrete log problem over  $\mathbb{F}_p$  is insecure.

- (c) If  $p - 1$  is divisible by higher powers of 2, we can recover more bits! Factor  $p - 1 = 2^k m$ . Describe an algorithm to compute the first  $k$  bits of  $\log_g h$ , that is, to recover  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{k-1}$ . You may assume that there is a fast algorithm to compute square roots modulo  $p$  (if  $p \equiv 3 \pmod{4}$  we described such an algorithm in class, but there is a general fast algorithm which we may encounter in the coming weeks).
- 8. Let  $p$  be a prime number and  $g \in \mathbb{F}_p^*$  a primitive root. Let  $i$  and  $j$  be integers such that  $\gcd(j, p - 1) = 1$ . Let  $A$  be arbitrary. Set:

$$\begin{aligned} S_1 &\equiv g^i A^j \pmod{p} \\ S_2 &\equiv -S_1 j^{-1} \pmod{p-1} \\ D &\equiv -S_1 i j^{-1} \pmod{p-1} \end{aligned}$$

- (a) Show that the pair  $(S_1, S_2)$  is a valid Elgamal signature for the document  $D$ . In particular, this means Eve can produce valid Elgamal signatures.
- (b) Explain why this doesn't mean that Eve can forge Sam's signature on a given document. What extra information would allow Eve to do this?

9. In this exercise we describe a potential security flaw in the Elgamal digital signature algorithm. Suppose that Samantha made the mistake of signing two documents  $D$  and  $D'$  using the same random value  $k$ .
- (a) Explain how Eve can immediately recognize that Samantha has made this blunder.
  - (b) Let the signature for  $D$  be  $D^{sig} = (S_1, S_2)$  and the signature for  $D'$  be  $D'^{sig} = (S'_1, S'_2)$ . Explain how Eve can recover Samantha's secret signing key  $k$ .