

Homework Assignment 9

Due Friday, April 8

1. Let G be a finite group, p a prime, and $P \in \text{Syl}_p(G)$ a Sylow p -subgroup. Use **(Sylow 2)** to show that $P \trianglelefteq G$ if and only if $n_p = \#\text{Syl}_p(G) = 1$.
2. Use Sylow's theorems to prove that a group of order 200 can never be simple.
3. Let G be a group of order p^2q for primes p and q . We will show that G always has a nontrivial *normal* Sylow subgroup.
 - (a) Suppose $p > q$. Show that G has a normal subgroup of order p^2 .
 - (b) Suppose $q > p$. Show that either G has a normal subgroup of order q , or else $G \cong A_4$. (You may use the result from the April 5 lecture that if $|G| = 12$ and $n_3 \neq 1$ then $G \cong A_4$).
 - (c) Explain why a group of order p^2q can never be simple. (You may need to treat the cases where $G = A_4$ or $p = q$ separately).
4. Let G be a group of order $99 = 3^2 * 11$. Let's show that G is abelian.
 - (a) Let $P \leq G$ be a Sylow 3-subgroup. Show that $P \trianglelefteq G$.
 - (b) Construct an injective homomorphism $G/C_G(P) \hookrightarrow \text{Aut}(P)$. (Can you use group actions and part (a)?)
 - (c) Deduce from part (b) and Lagrange's theorem that $G = C_G(P)$. Leverage this fact to prove that G is abelian.