Math 120 B Midterm 2 (80 minutes) May 12, 2016

Name Key

Directions:

- You have 80 minutes to complete this exam.
- No graphing calculators are allowed.
- You are allowed one hand-written sheet (two sided is ok) of notes on regular 8.5-11 paper.
- You must show ALL your work.
- Leave answers in EXACT FORM or record up to 2 DECIMAL PLACES.
- If you have any questions, raise your hand.

Question	Points	Score
1	20	
2	15	
3	15	
4	10	
5	15	
Total:	75	

1. (a) (5 points) Solve for x.

$$4^{2^{3x+1}} - 7 = 249.$$

$4^{2^{3x+1}} = 256$

Apply $\log_4(\cdot)$

$2^{3x+1} = 4$

Apply $\log_2(\cdot)$

$3x + 1 = 2$

$3x = 1$

$x = \frac{1}{3}$

(b) (5 points) Solve for t.

$$7^{\log_5(\ln(t))} = \frac{1}{49}.$$

Apply $\log_7(\cdot)$

$\log_5(\ln(t)) = -2$

Apply 5^x

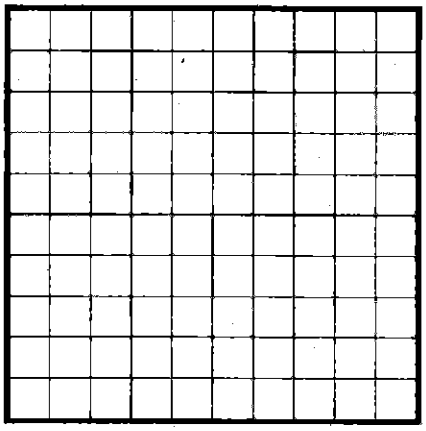
$\ln t = \frac{1}{25}$

Apply e^x

$t = e^{1/25}$

(c) (5 points) Give a sketch of the graph of the following linear rational function labeling all asymptotes, as well as your x and y axes.

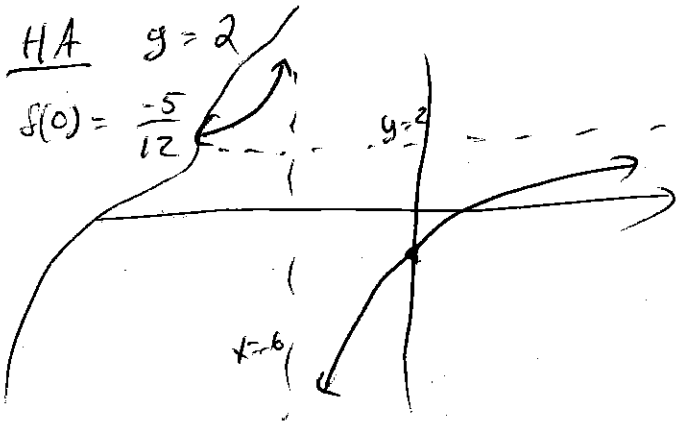
$$f(x) = \frac{4x-5}{2x+12}.$$



VA $x = -6$

HA $y = 2$

$f(0) = \frac{-5}{12}$



(d) (5 points) I invest \$1500 at an 8% interest rate. What is the value of my investment after five years if it is compounded monthly? What if it were compounded continuously?

Monthly

$V(t) = 1500\left(1 + \frac{.08}{12}\right)^{12t}$

$V(5) = 2234.77$

Ctsly

$V(t) = 1500e^{.08t}$

$V(5) = 2237.74$

2. The high price of diamonds is derived from relative rarity of the stone. Indeed, if more and more diamonds pour into the market, the price of a 1-carat diamond will drop and drop, but of course never below 0. Therefore, it is natural to model the price of diamonds as a function of the number of diamonds on the market, as a linear rational function $f(x)$.

(a) (6 points) When there were only about 5,000 diamonds on the market, the price (adjusted for inflation) was about \$100,000 each. Now there are closer to 200,000 diamonds on the market and the price is about \$4,000. Find an equation for $f(x)$.

2 pts

① $(5000, 100000)$

② $(200000, 4000)$

Horiz. Asym:

$y = 0$

$\Rightarrow a = 0$

① $\frac{b}{5000 + d} = 100000 \rightsquigarrow b = 500000000 + 100000d$

② $\frac{b}{200000 + d} = 4000 \rightsquigarrow b = 400000000 + 4000d$

So $5 \times 10^8 + 100000d = 4 \times 10^8 + 4000d$

$10^8 = -96000d$

$d = -\frac{10^8}{96000} \approx -1041.67$

So $b = 3,958 \times 10^8$

$f(x) = \frac{-1041.67}{x + 3.958 \times 10^8}$

(b) (5 points) Write a function $g(x)$ which will output the number of diamonds on the market if their current price is x. What is the domain of this function? (Assume f from part (a) has domain: $x > 0$).

$f^{-1}(x) = \frac{-3.958 \times 10^8 x - 1041.67}{x}$

(c) (4 points) Leonardo can make artificial diamonds, and wants to flood the market with diamonds so that they become almost worthless. How many diamonds would Leonardo have to make in order for diamonds to be worth less than \$1000? What about less than \$100? (Don't forget that there are already 100,000 diamonds on the market).

Compute $f^{-1}(1000)$

$f^{-1}(100)$

3. The city of Gotham has a serious pest problem.

(a) (5 points) The number of rats in the city doubles every 2 years. Last year there were ten million rats. Write a function $r(t)$ which gives the number of rats t years from now. (Note, last year is $t = -1$).

See

(b) (5 points) After a failed experiment by the Gotham military scientists, 100,000 super cockroaches were released into the city last year ($t = -1$). This year the population has exploded to and there are 500,000 cockroaches in the city. Write a function $c(t)$ for the number of cockroaches t years from now.

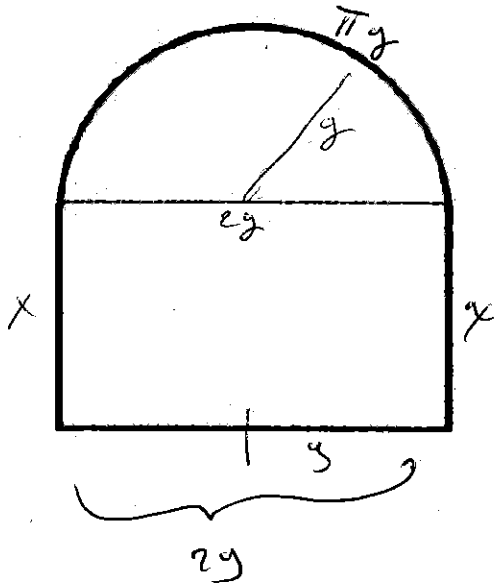
Quiz

4

(c) (5 points) When will there be 10 cockroaches for every rat?

Solutions

4. (10 points) A farmer is creating an enclosure for her goats. It is shaped like a rectangle, with a semi-circle attatched at one edge, where sick and weak and younger goats can rest and recover. She has 800 ft of fencing. What is the maximum area she can enclose? In this case, what are the dimensions of the rectangular part of the enclosure? (Recall, area of a circle is πr^2 and the circumference of a circle is $2\pi r$).



Perimeter

$2y + 2x + \pi y + 2x = 800$

$x = \frac{800 - 4y - \pi y}{2}$

$y(4+\pi) + 800 - 2x$

$y = \frac{800 - 2x}{4+\pi}$

Area $2y \cdot x + \frac{\pi y^2}{2}$

$= 2y\left(\frac{800 - y(4+\pi)}{2}\right) + \frac{\pi y^2}{2}$

$= 800y - 4y^2 - \pi y^2 + \frac{\pi y^2}{2}$

$= \left(-4 - \frac{1}{2}\pi\right)y^2 + 800y$

Perimeter

$x = \frac{800 - y(4+\pi)}{2}$

max @

$y = \frac{-b}{2a} = \frac{-800}{(-4 - \frac{\pi}{2})2} = \frac{-800}{-8-\pi} \approx 71.8$

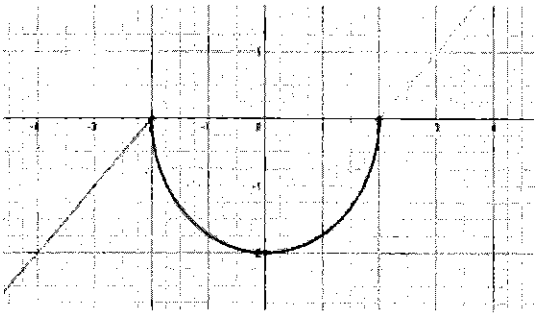
$A(71.8) \approx 28721.21$

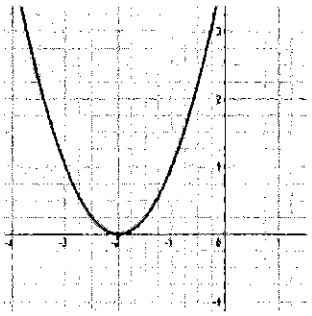
5. Consider the following two functions.

$$f(x) = \begin{cases} x+2 & : x < -2 \\ -\sqrt{4-x^2} & : -2 \le x \le 2 \\ x-2 & : x > 2 \end{cases}$$

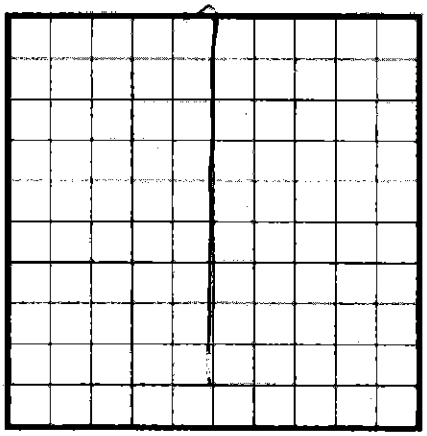
$$g(x) = x^2 + 4x + 4.$$

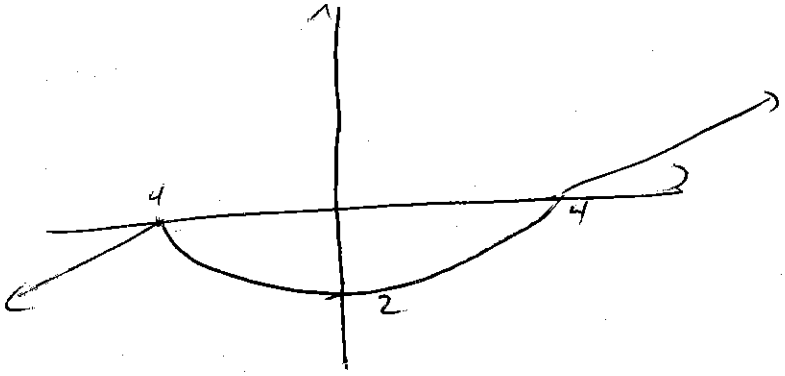
Here are their graphs.





(a) (5 points) Expand f horizontally by a factor of 2. Draw it and write the multi-part rule.





$$f\left(\frac{x}{2}\right) = \begin{cases} \frac{x}{2}+2 & x \le -4 \\ -\sqrt{4-\left(\frac{x}{2}\right)^2} & -4 \le x \le 4 \\ \frac{x}{2}-2 & x \ge 4 \end{cases}$$

(b) (5 points) Restrict the domain of g so that it is one-to-one. Then compute an inverse. (There are multiple acceptable solutions).

$g(x) = (x+2)^2$

Restrict so $x \ge -2$

$y = (x+2)^2$

$x = -2 + \sqrt{y}$

(c) (5 points) Write the multi-part rules for $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = f((x+2)^2) = \begin{cases} -\sqrt{4-(x+2)^4} & -\sqrt{2}-2 \le x \le \sqrt{2}-2 \\ (x+2)^2 - 2 & x \ge \sqrt{2}-2, \text{ or } x \le -\sqrt{2}-2 \end{cases}$$

$$g(f(x)) = \begin{cases} (x+4)^2 & x \le -2 \\ (-\sqrt{4-x^2}+2)^2 & -2 \le x \le 2 \\ (x)^2 & x \ge 2 \end{cases}$$

(d) **BONUS: 5 points**
Write the multi-part rule for $f(f(x))$.