Homework Assignment 9

Due Friday, April 8

1. Let G be a finite group, p a prime, and $P \in Syl_p(G)$ a Sylow p-subgroup. Use **(Sylow 2)** to show that $P \trianglelefteq G$ if and only if $n_p = \#Syl_p(G) = 1$.

   Proof. Let P be a Sylow p-subgroup. By **(Sylow 2)** (or TH2 Problem 2(c)), we know that the set of Sylow p-subgroups are precisely the conjugates of P, that is:

$$Syl_P(G) = \{gPg^{-1} : g \in G\}.$$

   In particular, $n_p = 1$ if and only if $\{gPg^{-1} : g \in G\} = \{P\}$, which holds if and only if $gPg^{-1} = P$ for all $g \in G$, that is, if and only if $P \trianglelefteq G$. □

2. Use Sylow's theorems to prove that a group of order 200 can never be simple.

   Proof. We show that a group of order 200 has a unique Sylow 5-subgroup, which must therefore be normal by Problem 1. Notice that 200 factors as $25 * 8 = 5^2 * 8$. By **(Sylow 3)**, the number of Sylow 5 subgroups is one more than a multiple of 5:

$$n_5 = 1 + 5 * k \in \{1, 6, 11, 16, \cdots\}.$$

   But we also know by **(Sylow 3)** that n_5 must divide 8. The only number in that list dividing 8 is 1, so $n_5 = 1$. Thus there is a unique Sylow 5-subgroup P which is normal by Problem 1. Since $|P| = 25$, we have produced a nontrivial normal subgroup, so that G cannot be simple. □

3. Let G be a group of order p^2q for primes p and q. We will show that G always has a nontrivial *normal* Sylow subgroup.

   (a) Suppose $p > q$. Show that G has a normal subgroup of order p^2.

   Proof. Let n_p be the number of p-Sylow subgroups. By (Sylow 3) $n_p = 1 + kp$ for some $k \geq 0$. Also by (Sylow 3), we know that n_p divides q, so that in particular $1 + kp \leq q$. Since $p > q$ this means $k = 0$ so that $n_p = 1$. Then the unique Sylow p-subgroup $P \leq G$ must be normal by Problem 1. □

   (b) Suppose $q > p$. Show that either G has a normal subgroup of order q, or else $G \cong A_4$. (You may use the result from the April 5 lecture that if $|G| = 12$ and $n_3 \neq 1$ then $G \cong A_4$).

   Proof. Let n_q be the number of q-Sylow subgroups. Then $n_q = 1 + kq$ for some $k \geq 0$. We also know that n_q divides p^2, so that $n_q \in \{1, p, p^2\}$. If $n_q = 1$ we are done (as the unique Sylow q-subgroup would be normal by Problem 1). Else $k \geq 1$ so that $n_q = 1 + kq > q > p$, so that $n_q = p^2$. In particular:

$$kq = p^2 - 1 = (p + 1)(p - 1).$$

   By Euclid's lemma, q must divide either $p + 1$ or $p - 1$, but since $q > p$ it cannot divide $p - 1$. This means that q divides $p + 1$, and so $p < q \leq p + 1$. We may therefore conclude that $q = p+1$. The only primes that are one apart are 2 and 3. So this means $q = 3$ and $p = 2$, and our group has order 12. We now cite the April 5 lecture where we proved that a group of order 12 either has a normal subgroup of order q = 3, or else is isomorphic to A_4. □

(c) Explain why a group of order p^2q can never be simple. (You may need to treat the cases where $G = A_4$ or $p = q$ separately).

Proof. If $p > q$ we part (a) gives a normal Sylow p-subgroup so G isn't simple.

If $q > p$ we either have a normal Sylow q-subgroup (so G isn't simple), or else $G \cong A_4$.

One can observe that A_4 has a normal subgroup of order 4 in a number of ways. We outline two proofs. The first uses the lattice of A_4, which contains $H = \langle (12)(34), (13)(24) \rangle$ as the unique subgroup of order 4. Since this is the unique Sylow 2-subgroup of A_4 it is normal by Problem 1. A second proof is to notice that H is the subgroup of 1 and all of the permutations of cycle type (2,2). Let $\sigma \in H$, and consider $\tau * \sigma = \tau\sigma\tau^{-1}$. If $\sigma = 1$ then $\tau * \sigma = \sigma \in H$. Else $\tau * \sigma$ still has cycle type (2,2) so it remains in H, directly showing the normality of H.

Finally if $q = p$ we have that $|G| = p^3$ is a p-group. One of the first consequences of the class equation we proved was that a group of prime power order has nontrivial center. So $|Z(G)| \in \{p, p^2, p^3\}$. Also, by HW6 Problem 3, $Z(G) \trianglelefteq G$, so if it has order p or p^2, it is itself a nontrivial normal subgroup. Otherwise, $Z(G) = G$, in which case G is abelian. By Cauchy's theorem it has a subgroup $H \leq G$ of order p, which is automatically normal since G is abelian. In either case, we have observed G cannot be simple. □

4. Let G be a group of order $99 = 3^2 * 11$. Let's show that G is abelian.

(a) Let $P \leq G$ be a Sylow 3-subgroup. Show that $P \trianglelefteq G$.

Proof. Applying **(Sylow 3)**: $n_3 \equiv 1 \mod 3$ so that $n_3 \in \{1, 4, 7, 10, 13, ...\}$, and $n_3 | 11$. Therefore $n_3 = 1$ and so P is normal by Problem 1. □

(b) Construct an injective homomorphism $G/C_G(P) \hookrightarrow Aut(P)$. (Can you use group actions and part (a)?)

Proof. Since P is normal, G acts on it by conjugation, and any action by conjugation is an action by automorphisms, so the permutation representation is a homomorphism $G \to \mathrm{Aut}(P)$. The kernel of this permutation representation is the kernel of the action, which by definition is:

$$\{g \in G : gpg^{-1} = p \text{ for all } p \in P\} = C_G(P).$$

Therefore the first isomorphism theorem gives us an injection $G/C_G(P) \hookrightarrow \mathrm{Aut}(P)$, as desired. □

(c) Deduce from part (b) and Lagrange's theorem that $G = C_G(P)$. Leverage this fact to prove that G is abelian.

Proof. Since $|P| = 9 = 3^2$, we know that $P \cong Z_9$ or $P \cong Z_3 \times Z_3$. In each case, P is abelian, so that (by HW6 Problem 1(f)), we know $P \leq C_G(P) \leq G$. In particular, $|C_G(P)|$ is divisible by 9 and divides 99, so it must have order 9 or 99. By Lagrange's theorem we see that $|G/C_G(P)|$ is either 1 or 11. We will show it cannot be 11.

Applying part (b), we can deduce that $|G/C_G(P)|$ must divide $|\operatorname{Aut}(P)|$. If $P \cong Z_9$ then $\operatorname{Aut}(P) \cong (\mathbb{Z}/9\mathbb{Z})^*$, which has 6 elements. If $P \cong Z_3 \times Z_3$, then (by HW8 Problem 7), $\operatorname{Aut}(P) \cong GL_2(\mathbb{F}_3)$, which (by HW6 Problem 7(d)) has 48 elements. Since 11 divides neither 6 nor 48, we must conclude that $|G/C_G(P)| = 1$, so that $G = C_G(P)$.

Since all of G centralizes P, we deduce that $P \leq Z(G) \leq G$. Arguing as above, we know that $Z(G)$ has either 9 or 99 elements. If it has 99 elements, G is abelian and we win. Otherwise it has 9 elements than $G/Z(G)$ has eleven elements, and is therefore isomorphic to Z_11 (by TH1 Problem 4(a)). But this is cyclic, so that we may conclude by HW6 Problem 3(b) that G is abelian, and again, we win! $\square$