# Homework Assignment 11

Due Friday, April 23

1. Let $R$ and $S$ be rings and $\varphi : R \to S$ a ring homomorphism.
   (a) Show that $\operatorname{im}\varphi$ is a subring of $S$.
   (b) Show that $\ker\varphi$ is a (two-sided) ideal of $R$.
   (c) Suppose $J \subseteq S$ is an ideal. Show that $\varphi^{-1}(J)$ is an ideal of $R$.
   (d) Suppose $R$ and $S$ are unital rings with *nonzero* identities $1_R$ and $1_S$ respectively. Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is either zero, or a zero divisor in $S$.
   (e) Deduce that if $S$ is an integral domain and $\varphi$ is nonzero then $\varphi(1_R) = 1_S$. (*Remark:* many authors require rings to be unital, and also require ring homomorphisms to take the identity to the identity.)

2. In this exercise we prove the third and fourth isomorphism theorems for rings.
   (a) We start with the fourth isomorphism theorem. Let $R$ be a ring and $I \subseteq R$ an ideal. In particular (since $R$ is abelian), $I$ is a normal subgroup. Therefore, applying the fourth isomorphism theorem for groups (HW5 Problem 1), there is a bijection:

$$\left\{ \begin{array}{c} \text{Subgroups } A \leq R \\ \text{such that } I \leq A \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Subgroups} \\ \overline{A} \leq R/I \end{array} \right\}$$

   Prove the following ring theoretic enhancements hold:
      i. $A$ is a subring of $R$ if and only if $\overline{A}$ is a subring of $R/I$.
      ii. If $A$ is a subring of $R$, then $I$ is an ideal of $A$ and that $A/I \cong \overline{A}$.
      iii. $A$ is a left ideal of $R$ if and only if $\overline{A}$ is a left ideal of $R/I$.
      iv. $A$ is a right ideal of $R$ if and only if $\overline{A}$ is a right ideal of $R/I$.
      v. $A$ is an ideal of $R$ if and only if $\overline{A}$ is an ideal of $R/I$.
   (b) We now prove the third isomorphism theorem for rings. Let $J \subseteq I \subseteq R$, with $J, I$ ideals of a ring $R$. By part (a) we know that $I/J$ is an ideal of $R/J$. Prove that:

$$\frac{R/J}{I/J} \cong \frac{R}{I}.$$

   (c) We finish with a ring theoretic analog of *passing to the quotient*. Suppose $\varphi : R \to S$ is a ring map, and suppose that $I \subseteq \ker\varphi$. Prove that there is a unique map $\overline{\varphi} : R/I \to S$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\pi"] & S \\ R/I \arrow[ur, dotted, "\overline{\varphi}"'] & \end{tikzcd}$$

   That is, $\overline{\varphi}$ is the unique map so that $\overline{\varphi} \circ \pi = \varphi$. (*Hint*: We already know from group theory that there is a unique such map on the level of group homomorphisms. What remains is to confirm that map is a ring homomorphism.)

3. Let $R$ be a ring.

(a) Suppose $\{I_j\}$ is a collection of left ideals of $R$. Show that the intersection $\cap I_j$ is a left ideal of $R$.

(b) Show that part (a) also holds for right ideals and two-sided ideals.

(c) Let $R$ be a ring with $1 \neq 0$. Show that:

$$RA = \bigcap_{A \subset I \text{ left ideal}} I.$$

(d) State the analog for part (c) for right ideals. (The proof will be identical, so I won't make you repeat yourself.)

4. Let $I$ and $J$ be ideals of a ring $R$.

   (a) Prove that $I + J$ is the smallest ideal of $R$ containing both $I$ and $J$.

   (b) Show that $IJ$ is an ideal contained in $I \cap J$

   (c) Give an example where $IJ \neq I \cap J$

   (d) Suppose $R$ is commutative and unital, and that $I + J = R$. Show $IJ = I \cap J$.

5. Let $R$ be a commutative ring with $1 \neq 0$.

   (a) Fix $a \in R$. Show that $(a) = R$ if and only if $a \in R^\times$.

   (b) Fix $a, b \in R$, and suppose that $a$ is not a zero divisor. Show that $(a) = (b)$ if and only if $a = ub$ for some unit $u \in R^\times$.

   (c) Let $I$ be any ideal. Show that $I = R$ if and only if $I$ contains a unit $u \in R^\times$.

   (d) Prove that $R$ is a field if and only if the only ideals in $R$ are $(0)$ and $R$ itself.

   (e) Now suppose $S$ is a (not necessarily commutative) ring with $1 \neq 0$. Show that $S$ is a division ring if and only if the only all left, right, and 2-sided ideals are one of $S$ or $(0)$. (*Hint*: Start by proving a version of part (c) for noncommutative rings.)

6. Let $R$ be any ring. We define the *$n$ by $n$ matrix ring* of $R$: $M_n(R)$, to be the set of $n$ by $n$ matrices whose entries are elements of $R$. We often denote an element of $M$ as a $n^2$-tuple of entries indexed by $i$ and $j$ between 1 and $n$:

$$M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij}).$$

   We make $M_n(R)$ into a ring under usual matrix multiplication and addition. That is, given $M = (a_{ij})$ and $N = (b_{ij})$ then $M + N = (a_{ij} + b_{ij})$, and the $ij$th entry of $MN$ is $\sum_{k=1}^{n} a_{ik} b_{kj}$.

   (a) Prove that $M_n(R)$ is a ring.

   (b) Suppose $R$ is a ring with $1 \neq 0$, and that $n \geq 2$. Show that $M_n(R)$ always has a left ideal that is not a right ideal, and vice versa.

   (c) Let $I$ be a left (respectively right) ideal of $R$. Show that $M_n(I)$ is a left (respectively right) ideal of $M_n(R)$.

(d) Suppose $R$ is unital. Show that the 2-sided ideals of $M_n(R)$ are precisely $M_n(J)$ for two sided ideals $J \subseteq R$. (*Hint*: Think about mutliplication by the matrices $E_{ij}$ which have a 1 in the $ij$ entry and are are 0 everywhere else).

(e) The determinant $\det : M_n(R) \to R$ is a function. Is it always a ring homomorphism? If yes, prove it. If no, give a counterexample?

7. Recall that a group was called *simple* if it had no normal subgroups, or equivalently, if it has no nontrivial quotients. There is a similar notion for rings. A ring $R$ is called *simple* if the only quotients of $R$ are $R$ itself and the the zero ring.

   (a) Give an equivalent formulation of simplicity in terms of ideals.

   (b) Show that a commutative ring is simple if and only if it is a field.

   (c) Give an example to show that a noncommutative ring may be simple even but not a division ring.

8. Let $R$ be a ring. The *nilradical* of $R$ is $\mathfrak{N}(R) = \{r \in R : r \text{ is nilpotent}\}$. By HW10 Problem 3 we know that $\mathfrak{N}(R)$ is an ideal of $R$.

   (a) Show that $R/\mathfrak{N}(R)$ is reduced. This is often called the *reduction of R,* and is denoted $R_{red}$.

   (b) Let $\varphi : R \to S$ be any ring homomorphism. Show that $\varphi(\mathfrak{N}(R)) \subseteq \mathfrak{N}(S)$. Deduce that if $S$ is reduced then $\mathfrak{N}(R)$ is contained in the kernel of $\varphi$.

   (c) Let $S$ be a reduced ring. Show that there is a bijection:

$$\{\text{Ring homomorphisms } \varphi : R \to S\} \Longleftrightarrow \{\text{ Ring homomorphisms } \tilde{\varphi} : R_{red} \to S\}.$$

   *Hint:* Use passing to the quotient! *Remark:* This should feel reminicient of the *abelianization* from HW6 Problem 4. In fact, both are examples of something more general, called a *universal property.* Keep your eyes open for things like this, they appear all over mathematics!