Take Home Assignment 2

Due Monday, March 23

With everthing going on right now the Monday deadline is flexible. That being said, if you are going to need extra time *please let me know*, I will be granting extensions no questions asked but I need to know when to expect your assignment so that nothing falls between the cracks. Good luck and stay safe.

In this set of problems we will study the quaternion group Q_8. It is a nonabelian group with very interesting properties.

Definition 1. *The quaternion group of order 8, denoted Q_8 is the group of the following 8 elements:*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

subject to the relations:

$$i^2 = j^2 = k^2 = -1,$$

$$\begin{aligned} ij &= k, & ji &= -k, \\ jk &= i, & kj &= -i, \\ ki &= j, & ik &= -j. \end{aligned}$$

1. Let's start with a few simple facts. Much of this is worked out in the book.

   (a) Write the entire multiplication table for Q_8.

   Proof. The group is nonabelian, so we make sure to stick to the convention that in row a and column b we are writing ab (rather than ba),

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

   □

   (b) Find a presentation for Q_8 with 2 generators and 3 relations.

   Proof. In the presentation I gave above is hidden the notation that $(-1)^2 = 1$ and that $-1 * i = -i$. If we keep these assumptions, then i and j generate everything. Indeed:

$$\begin{array}{ccc} -1 = i^2 & -i = i^3 & -j = j^3 \\ 1 = i^4 & k = ij & -k = ji. \end{array}$$

   Therefore we have

$$\langle i, j \mid i^2 = j^2 = -1, ij = -ji \rangle.$$

This answer is acceptable on this assignment, but not precisely correct. We probably want to assume in our presentation that we don't know what -1 is (i.e., that its square is 1). In this case the generators are still i and j, but the relations are $i^4 = j^4 = 1$, $i^2 = j^2$, and $ji = i^3 j$. $\square$

(c) Prove that Q_8 is not isomorphic to D_8.

Proof. The easiest way to see this is to notice that if they were isomorphic, they would need to have the same number of elements of order n for each n. Then we can consider the order of every element in each group.

Q_8	order	D_8	order
1	1	1	1
-1	2	r	4
i	4	r^2	2
$-i$	4	r^3	4
j	4	s	2
$-j$	4	sr	2
k	4	sr^2	2
$-k$	4	sr^3	2

In particular, Q_8 only has one element of order 2 whereas D_8 has 5. $\square$

(d) Find all the subgroups of Q_8, and draw its lattice. (Hint: there are 6 total subgroups).

Proof. The nontrivial subgroups (i.e., Q_8 and $\{1\}$) must have orders 2 or 4 by lagranges theorem. The order 2 subgroups must be cyclic, generated by an element of order 2. The only element of order 2 is -1, so the only subgroup of order 2 is $\{\pm 1\}$. As for subgroups of order four, they are either cyclic or isomorphic to the Klein 4 group V_4. But V_4 must be generated by 2 elements of order 2, and Q_8 only has one. Thus each subgroup of order 4 is cyclic. There are 6 elements of order 4, but $-i = i^3$, and similarly for j and k, so there 3 subgroups of order 4 generated by i and j and k. As $i^2 = j^2 = k^2 = -1$, the subgroup $\{\pm 1\}$ is contained in all of them. thus the lattice is as follows.

$$
\begin{tikzcd}[arrows=dash]
 & Q_8 \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle i \rangle \arrow[dr] & \langle j \rangle \arrow[d] & \langle k \rangle \arrow[dl] \\
 & \{\pm 1\} \arrow[d] & \\
 & \{1\} &
\end{tikzcd}
$$

$\square$

(e) Prove that every subgroup of Q_8 is normal.

Proof. Q_8 and $\{1\}$ are automatically normal. Next notice that $-1 * a = a * -1$ for each $a \in Q_8$ (use the multiplication table). Thus $\{\pm 1\}$ is contained in the center of Q_8 and is therefore automatically normal.
The cases for $\langle i \rangle, \langle j \rangle$ and $\langle k \rangle$ are completely symmetric, so we just treat the case of $H = \langle i \rangle$. Notice that

$$H \leq N_{Q_8}(H) \leq Q_8.$$

Also $|H| = 4$ and $|N_{Q_8}(H)|$ divides 8 by Lagrange's theorem, so that $N_{Q_8}(H)$ is either H or all of Q_8. Thus if we exhibit one element of the normalizer which is not in H, the normalizer is all of Q_8, which precisely means that $H \trianglelefteq Q_8$. Notice that:

$$jij^{-1} = ji(-j) = (-k)(-j) = kj = -i \in \langle i \rangle.$$

Thus $j \in N_{D_8}(H)$ and we are done. □

(f) Prove that every subgroup and quotient group of Q_8 is abelian (Hint: use the classification of groups of order 4 and 2, as well as Lagrange's theorem).

Proof. Precisely speaking, since Q_8 is not abelian, it is only true that every *proper* subgroup and quotient of Q_8 is abelian. Let H be a proper subgroup or quotient of Q_8. Then by Lagranges theorem, $|H| = 1, 2$ or 4. In the first case H is the trivial group which is abelian, in the second it is isomorphic to Z_2 which is abelian, and in the third it is isomorphic to either Z_4 or V_4 which are abelian. □

(g) Compute $Z(Q_8)$ and $Q_8/Z(Q_8)$ (Hint for the second part: you can do this by hand, but it might be slicker to apply Homework 6 problem 5(b)).

Proof. It is readily checked using the multiplication table in part (a) that $Z(Q_8) = \{\pm 1\}$. Then

$$|Q_8/Z(Q_8)| = |Q_8|/|\{\pm 1\}| = 8/2 = 4.$$

Then in particular, it is either cyclic or isomorphic to V_4. If it is cyclic, Homework 6 part 5(b) says that Q_8 is abelian, which is false. So the quotient is V_4. (Note, one could also use the lattice from part (d) together with the fourth isomorphism theorem to see that the lattice of the quotient has to be the lattice above $\{\pm 1\}$, which is the lattice of V_4). □

(h) Write a composition series for Q_8.

Proof. Any path up the complete lattice is a composition series. So for example:

$$1 \leq \{\pm 1\} \leq \langle i \rangle \leq Q_8.$$

This is a composition series because each subgroup has index 2 and is therefore normal, and the subquotients (composition factors) all have order 2, which are therefore isomorphic to Z_2 and thus simple. □

2. Now let's follow the proof of Cayley's theorem to exhibit Q_8 as a subgroup of S_8.

(a) Label $\{1, -1, i, -i, j, -j, k, -k\}$ as the numbers $\{1, 2, \cdots, 8\}$. Then the action of Q_8 on itself by left multiplication gives an injective map $Q_8 \to S_8$. Write the permutation representations for -1 and i as elements $\sigma_{-1}, \sigma_i \in S_8$, and verify that $\sigma_i^2 = \sigma_{-1}$. (Using the multiplication table from question 1 will make this easier).

Proof. Let's first compute σ_{-1}.

$$\begin{array}{ccc} -1 * 1 = -1 & \leftrightarrow & \sigma_{-1}(1) = 2 \\ -1 * -1 = 1 & \leftrightarrow & \sigma_{-1}(2) = 1 \\ -1 * i = -i & \leftrightarrow & \sigma_{-1}(3) = 4 \\ -1 * -i = i & \leftrightarrow & \sigma_{-1}(4) = 3 \\ -1 * j = -j & \leftrightarrow & \sigma_{-1}(5) = 6 \\ -1 * -j = j & \leftrightarrow & \sigma_{-1}(6) = 5 \\ -1 * k = -k & \leftrightarrow & \sigma_{-1}(7) = 8 \\ -1 * -k = k & \leftrightarrow & \sigma_{-1}(8) = 7 \end{array}$$

Thus σ_{-1} swaps 1 and 2, 3 and 4, 5 and 6, 7 and 8. That is:

$$\sigma_{-1} = (12)(34)(56)(78) \in S_8.$$

Let's do a similar computation for σ_i.

$$\begin{array}{ccc} i * 1 = i & \leftrightarrow & \sigma_i(1) = 3 \\ i * -1 = -i & \leftrightarrow & \sigma_i(2) = 4 \\ i * i = -1 & \leftrightarrow & \sigma_i(3) = 2 \\ i * -i = 1 & \leftrightarrow & \sigma_i(4) = 1 \\ i * j = k & \leftrightarrow & \sigma_i(5) = 7 \\ i * -j = -k & \leftrightarrow & \sigma_i(6) = 8 \\ i * k = -j & \leftrightarrow & \sigma_i(7) = 6 \\ i * -k = j & \leftrightarrow & \sigma_i(8) = 5 \end{array}$$

Thus σ_i takes 1 to 3 to 2 to 4 to 1, while taking 5 to 7 to 6 to 8 and back to 5. Thus we have:

$$\sigma_i = (1324)(5768) \in S_8.$$

Next we compute the square of σ_i by hand, using in the first equality that disjoint cycles commute.

$$\begin{aligned} (\sigma_i)^2 &= (1324)^2(5768)^2 \\ &= (1324)(1324)(5768)(5768) \\ &= (12)(34)(56)(78). \end{aligned}$$

$\square$

(b) Use the generators from question 1(b) to give two elements of S_8 which generate a subgroup $H \leq S_8$ isomorphic to Q_8.

Proof. Since i and j generate Q_8, the permutations σ_i and σ_j generate the isomorphic subgroup of S_8. Thus we must also compute σ_j like we did for i and -1 in part (a).

$$\begin{array}{rcl} j * 1 = j & \leftrightarrow & \sigma_j(1) = 5 \\ j * -1 = -j & \leftrightarrow & \sigma_j(2) = 6 \\ j * i = -k & \leftrightarrow & \sigma_j(3) = 8 \\ j * -i = k & \leftrightarrow & \sigma_j(4) = 7 \\ j * j = -1 & \leftrightarrow & \sigma_j(5) = 2 \\ j * -j = 1 & \leftrightarrow & \sigma_j(6) = 1 \\ j * k = i & \leftrightarrow & \sigma_j(7) = 3 \\ j * -k = -i & \leftrightarrow & \sigma_j(8) = 4 \end{array}$$

Therefore we get:

$$\sigma_j = (1526)(3847).$$

Thus we have:

$$Q_8 \cong \langle \sigma_i, \sigma_j \rangle = \langle (1324)(5768), (1526)(3847) \rangle \leq S_8.$$

□

(c) Is σ_i even or odd?

Proof. Let's compute the sign.

$$\epsilon((1324)(5768)) = \epsilon((1324))\epsilon((5768)) = (1)(1) = 1.$$

Thus σ_i is even. □

(d) $A_8 \cap H$ is isomorphic to a subgroup of Q_8. Which one?

Proof. One can compute that σ_j is even as well, so that the entire subgroup they generate is contained in A_8. Thus $A_8 \cap H = H \cong Q_8$. □

3. Cayley's theorem says that if $|G| = n$ then G embeds at S_n. But might not be the smallest symmetric group that G embeds in. For example, D_8 embeds in S_4 (thinking about symmetries of the square as permutations of the vertices). Nevertheless, for Q_8 the symmetric group given by Cayley's theorem is the smallest.

(a) Let Q_8 act an a set A with $|A| \leq 7$. Let $a \in A$. Show that the stabilizer of a, $(Q_8)_a \leq Q_8$ must contain the subgroup $\{\pm 1\}$.

Proof. Let $a \in A$, and denote the stabilizer of a by the subgroup $(Q_8)_a \leq Q_8$. Then recall that the index of the stabilizer of a is Q_8 is the same as the size of the orbit of a $Q_8 \cdot a$ which is a subset of A. That is:

$$|Q_8 : (Q_8)_a| = |Q_8 \cdot a| \leq |A| \leq 7 < 8.$$

The left hand size is $8/|(Q_8)_a|$ by Lagrange's theorem, so that $(Q_8)_a$ cannot be the trivial subgroup of Q_8. But in the lattice from 1(d), we saw that every nontrivial subgroup of Q_8 contains $\{\pm 1\}$, completing the proof. □

(b) Deduce that the kernel of the action of Q_8 on A contains $\{\pm 1\}$.

Proof. $\{\pm 1\}$ is contained in the stabilizer of every element of A by part (a), and so it acts trivially on all of A. This is precisely what it means to be in the kernel. □

(c) Conclude that Q_8 cannot embed into S_n for $n \leq 7$.

Proof. An embedding $Q_8 \hookrightarrow S_n$ corresponds induces a faithful action on the set $\{1, 2, \cdots, n\}$. But we just saw that if $n \leq 7$, any action on $\{1, 2, \cdots, n\}$ has a nontrivial kernel. □

4. Finally let's say a few things about the automorphism group of Q_8.

(a) By counting possible places where the generators may go, show that $|\operatorname{Aut}(Q_8)| \leq 24$.

Proof. Let $\varphi : Q_8 \to Q_8$ be an automorphism. It is completely determined by where the generators i and j go (for example, $\varphi(k) = \varphi(i)\varphi(j)$ and so on). But $\varphi(i)$ must have order 4 since i does, so it must be one of $\pm i, \pm j$ or $\pm k$, giving 6 choices. Since j is not a power of i, $\varphi(j)$ cannot be a power of $\varphi(i)$, so that 4 choices remain for $\varphi(j)$. That is, if $\varphi(i) = k$, then $\varphi(j) \neq \pm k$ since both those are powers of k, and similarly for the rest of the cases. Thus there are a maximum of $6 * 4 = 24$ choices for the images of i and j. (Note: we didn't prove that all these 24 choices are automorphisms, just that there can be at most 24). □

(b) What is $\operatorname{Inn}(Q_8)$? (Hint: You already did this in question 1(g)!)

Proof. Using $1(g)$ we know that $\operatorname{Inn}(Q_8) \cong Q_8/Z(Q_8) \cong V_4$. □

(c) Use parts (a) and (b) to conclude that $|\operatorname{Aut}(Q_8)|$ must be one of $\{4, 8, 12, 16, 20, 24\}$. (Note: it will turn out that it is 24, but the proof of this fact is more involved).

Proof. Since $\operatorname{Aut}(Q_8)$ has a subgroup of order 4, it's order must be divisible by 4 (by Lagrange's theorem). The list given is all the numbers divisible by 4 which are less than or equal to 24. □