Homework 11

Due Monday, May 4th

In this assignment we fill the proofs of a few crucial lemmas from lecture and takehome 3. Studying semidirect products reduces to the study of automorphism groups, so our first goal is to get a good way to decompose them. Here is the goal:

Lemma 1. *Let H and K be finite groups whose orders are coprime. Then*

$$\mathrm{Aut}(H \times K) \cong \mathrm{Aut}\, H \times \mathrm{Aut}\, K.$$

The following definition will be useful.

Definition 1. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

1. Let's prove Lemma 1.
   (a) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup* (recall the definition from HW9 Problem 1). Fix any automorphism $\varphi \in \mathrm{Aut}\, G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).
   (b) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\mathrm{Aut}\, G \to \mathrm{Aut}\, H$.
   (c) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.
   (d) With H, K as in (c), construct an isomorphism $\mathrm{Aut}(H \times K) \to \mathrm{Aut}\, H \times \mathrm{Aut}\, K$.

Recall that any homomorphism $\varphi : K \to \mathrm{Aut}\, H$ allows us to build a semidirect product $H \rtimes_\varphi K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used the following lemma.

Lemma 2. *Let $\varphi, \psi : K \to \mathrm{Aut}\, H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \mathrm{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:*

$$\begin{tikzcd} K \arrow[d, "\gamma"'] \arrow[dr, "\varphi"] & \\ K \arrow[r, "\psi"'] & \mathrm{Aut}\, H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 3. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitely, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_\alpha : \beta \mapsto \alpha\beta\alpha^{-1}$, and suppose that $\psi = \sigma_\alpha \circ \varphi$:*

$$\begin{tikzcd} & \operatorname{Aut} H \arrow[dd, "\sigma_\alpha"] \\ K \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

2. The lemmas say that if we alter φ by an automorphism of K, or an inner automorphism of $\operatorname{Aut} H$, (or both), we don't change the semidirect products. Let's prove this.

   (a) Consider the setup of Lemma 2. Show that the map:

   $$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (h, \gamma(k)) \end{array}$$

   is an isomorphism, thereby proving the lemma.

   (b) Consider the setup of Lemma 3. Show that the map:

   $$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (\alpha(h), k) \end{array}$$

   is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_α is an automorphism of $\operatorname{Aut} H$, given by conjugation by α. In unweildy notation, this says $\sigma_\alpha \in \operatorname{Aut}(\operatorname{Aut} H)$.)

   (c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

   $$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\ K \arrow[r, "\psi"] & \operatorname{Aut} H. \end{tikzcd}$$

   That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$. (Hint: This should follow formally from Lemmas 2 and 3, so you shouldn't have to do any lengthy computations).

To prove the uniqueness part of the fundamental theorem of finite abelain groups in Takehome 3, we made use of the following lemma.

Lemma 4. *Let M, M', N, N' groups, and suppose $M \times N \cong M' \times N'$. If M and M' are finite and $M \cong M'$ then $N \cong N'$.*

Remark. *This is a slightly more general restatement of the lemma we used in the takehome. In particular, before we identified M and M' as equal rather than isomorphic, and we assumed that N, N' were finite as well. We will see that this greater generality makes it a bit easier to prove.*

3. Let's explore and prove Lemma 4, and thereby fill the remaining hole in the fundamental theorem of finite abelian groups. It is actually more subtle then you might think.

   (a) You will need to make use of the following fact, so we prove it first. If G_1, G_2 are groups and $H_i \trianglelefteq G_i$ for $i = 1, 2$. Then under the usual identifications, $H_1 \times H_2 \trianglelefteq G_1 \times G_2$ and:
   $$(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2).$$

   (b) Give an example to show that Lemma 4 is not true without the finiteness assumption. (Hint: Let G a nontrivial group and $M = G \times G \times G \times \cdots$ an infinite product of copies of G).

   (c) Identify $M \times N$ and $M' \times N'$ as the same group G. Show that if either $M' \cap N = 1$, or if $M \cap N' = 1$ then Lemma 4 holds. (Hint: 2nd isomorphism theorem).

   (d) Prove Lemma 4 by induction on $|M|$. (Hint: The base case is easy (why?). For the general case, notice that if $H = M \cap N'$ or $K = M' \cap N$ are trivial, we are done by part (b). Otherwise, try manipulating $G/(H \times K)$ to apply induction).

4. Let's finish with a classification problem. Classify all groups of order 20 up to isomorphism. How many are there total? (You may use that if p is prime, then $\mathrm{Aut}(Z_p) \cong Z_{p-1}$).