

Homework Assignment 8

Due Friday, March 19

Recall the following important Lemma from the March 11th lecture.

Lemma 1. *Let G be a finite group, and $H \trianglelefteq G$ a normal subgroup. Let $P \leq H$ be a Sylow p subgroup of H . If $P \trianglelefteq H$ then $P \trianglelefteq G$.*

We noted in class that this feels like a normal Sylow subgroup is somehow *strongly* normal, in such a way that we get transitivity of normal subgroups. The following definition makes this precise.

Definition 1 (Characteristic Subgroups). *A subgroup $H \leq G$ is called characteristic in G if for every automorphism $\varphi \in \text{Aut } G$, we have $\varphi(H) = H$. This is denoted by $H \text{ char } G$.*

1. Let's prove some basic facts about characteristic subgroups and use them to prove Lemma 1.
 - (a) Show that characteristic subgroups are normal. That is, if $H \text{ char } G$ then $H \trianglelefteq G$.
 - (b) Let $H \leq G$ be the unique subgroup of G of a given order. Then $H \text{ char } G$.
 - (c) Let $K \text{ char } H$ and $H \trianglelefteq G$, then $K \trianglelefteq G$. (This is the transitivity statement alluded to, and justifies the feeling that a characteristic subgroup is somehow *strongly normal*).
 - (d) Let G be a finite group and P a Sylow p -subgroup of G . Show that $P \trianglelefteq G$ if and only if $P \text{ char } G$.
 - (e) Put all this together to deduce Lemma 1.

Sylow's theorem and some of the work you did last week makes it easy to prove Cauchy's theorem:

Theorem 1 (Cauchy's Theorem). *Let G be a finite group and p a prime number dividing the order of G . Show that G has an element of order p .*

2. (a) Prove the following strong version of Cauchy's theorem: Suppose G is a finite group of order n , and that p a prime number such that $p^d | n$ for some $d \geq 0$. Prove that G has a subgroup H of order p^d .
 - (b) Deduce Cauchy's theorem as a special case of part (a).
3. Let G be a group of order p^2q for primes $p \neq q$. We will show that G always has a nontrivial *normal* Sylow subgroup.
 - (a) Suppose $p > q$. Show that G has a normal subgroup of order p^2 .
 - (b) Suppose $q > p$. Show that either G has a normal subgroup of order q , or else $G \cong A_4$.
 - (c) Explain why a group of order p^2q for primes $p \neq q$ can never be simple.
4. In class we've alluded many times to the fact that if G is an abelian group of order pq for primes $p \neq q$, then $G \cong Z_{pq}$. Let's prove it.
 - (a) Let $x, y \in G$ be two elements of finite order and suppose that $xy = yx$. Conclude that $|xy|$ divides the least common multiple of $|x|$ and $|y|$.
 - (b) Let G be an abelian group of order pq for primes $p < q$. Use Cauchy's theorem and part (a) to conclude that G is cyclic. (This completes the argument from class about groups of order pq).

5. Next lets poke and prod $GL_2(\mathbb{F}_p)$.

- (a) Recall the order of $GL_2(\mathbb{F}_p)$ from HW7 problem 4(d). What is the maximal p divisor of $|GL_2(\mathbb{F}_p)|$?
- (b) The subset of *upper triangular matrices* of $GL_2(\mathbb{F}_p)$ is:

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

The subset of *strictly upper triangular matrices* is:

$$\bar{T} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{F}_p) \right\}.$$

Show that T and \bar{T} are subgroups of $GL_2(\mathbb{F}_p)$. We will see that they are not normal.

- (c) Show that \bar{T} is a Sylow p -subgroup of $GL_2(\mathbb{F}_p)$ and of T .
 - (d) Show that $GL_2(\mathbb{F}_p)$ has $p + 1$ Sylow p -subgroups.
 - (e) Prove that T is not normal in $GL_2(\mathbb{F}_p)$. (Hint: use Lemma 1).
6. Prove that a group of order 200 cannot be simple.
7. Let G_1, G_2, \dots, G_n be groups. Show that:

$$Z(G_1 \times G_2 \times \dots \times G_n) = Z(G_1) \times Z(G_2) \times \dots \times Z(G_n).$$

Conclude that a product of groups is abelian if and only if the factors are.

Let's finish with an important cancellation lemma for direct products.

Lemma 2. *Let M, M', N, N' groups, and suppose $M \times N \cong M' \times N'$. If M and M' are finite and $M \cong M'$ then $N \cong N'$.*

8. Let's explore and prove Lemma 2. It is actually more subtle than you might think.

- (a) You will need to make use of the following fact, so we prove it first. If G_1, G_2 are groups and $H_i \trianglelefteq G_i$ for $i = 1, 2$. Then under the usual identifications, $H_1 \times H_2 \trianglelefteq G_1 \times G_2$ and:

$$(G_1 \times G_2)/(H_1 \times H_2) \cong (G_1/H_1) \times (G_2/H_2).$$

- (b) Give an example to show that Lemma 2 is not true without the finiteness assumption. (Hint: Let G a nontrivial group and $M = G \times G \times G \times \dots$ an infinite product of copies of G).
- (c) Identify $M \times N$ and $M' \times N'$ as the same group G . Show that if either $M' \cap N = 1$, or if $M \cap N' = 1$ then Lemma 2 holds. (Hint: 2nd isomorphism theorem).
- (d) Prove Lemma 2 by induction on $|M|$. (Hint: The base case is easy (why?). For the general case, notice that if $H = M \cap N'$ or $K = M' \cap N$ are trivial, we are done by part (b). Otherwise, try manipulating $G/(H \times K)$ to apply induction).