Homework Assignment 8

Due Friday, March 18

1. Cayley's theorem says that if $|G| = n$ then G embeds into S_n (that is, is isomorphic to a subgroup of S_n). One could ask if this n is *sharp*, or if perhaps G can embed in some smaller symmetric group.

   (a) Give an example to show that Cayley's theorem isn't always sharp. That is, give a group of order n which embeds into S_d for some $d < n$.

   Nevertheless, we are about to see that for Q_8 the symmetric group given by Cayley's theorem is the smallest. This shows that there can be no strengthening of Cayley's theorem in general.

   (b) Let Q_8 act on a set A with $|A| \leq 7$. Let $a \in A$. Show that the stabilizer of a, $(Q_8)_a \leq Q_8$ must contain the subgroup $\{\pm 1\}$. (*Hint:* It might be helpful to use the orbit stabilizer theorem and the lattice from HW6 Problem 5(d).)

   (c) Deduce that the kernel of the action of Q_8 on A contains $\{\pm 1\}$.

   (d) Conclude that Q_8 cannot embed into S_n for $n \leq 7$. That is, show there is no injective homomorphisms $Q_8 \hookrightarrow S_n$ for $n \leq 7$.

2. Find all groups with exactly 2 conjugacy classes. (*Hint*: Use the class equation.)

3. Compute all the conjugacy classes for the following groups, and verify that the class equation holds in each case.

   (a) S_3

   (b) Q_8

For the next problem it may be useful to recall the following fact we proved in class.

Theorem 1 (Cauchy's Theorem for Abelian Groups)**.** *Let G be an abelian group of order n. If p is a prime dividing n, then G has a subgroup of order p.*

This will turn out to be true for all groups, so so far we only have it in the abelian case.

4. The converse to Lagrange's theorem holds for groups of prime power order. To prove this we will need to strengthen the fourth isomorphism theorem (HW5#1).

   (a) Let G be a group and $N \trianglelefteq G$. Let $N \leq H \leq K \leq G$, and let $\overline{H}, \overline{K}$ be the corresponding subgroups of G/N as in HW5#1. Show that $|K : H| = |\overline{K} : \overline{H}|$. (*Hint*: There is an obvious map $K/H \to \overline{K}/\overline{H}$. Prove it is bijective. Be careful though, we don't know that K/H is a group, just a set of cosets.)

   (b) Suppose $|G| = p^d$ for a prime p and $d \geq 1$. Show that G has a normal subgroup of order p. In particular, we have extended Cauchy's theorem to nonabelian p-groups! (*Hint:* What did the class equation say about the center of a p-group?)

   (c) Suppose $|G| = p^d$ for a prime p and $d \geq 1$. Show that for every $a = 1, 2, \cdots, d$, G has a subgroup of order p^a. (Use parts (a) and (b) to proceed by induction).

5. Here we classify all abelian groups of order pq for $p \neq q$ prime.

   (a) Let G be a group of finite order and suppose that $x, y \in G$ are commuting elements, i.e., that $xy = yx$. Show that that $|xy|$ divides the least common multiple of x and y.

(b) Let G be an abelian group of order pq for primes $p \neq q$. Show that $G \cong Z_{pq}$.

(c) Classify all groups of order 6 up to isomorphism.

6. Let V be an abelian group of order p^n for some prime p and $n > 0$. Suppose that every element of V has order $\leq p$. Show by induction on n that:

$$V \cong \underbrace{Z_p \times Z_p \times \cdots Z_p}_{n \text{ times}}.$$

We will call such a V the *elementary abelian group of order p^n*. We will see in the following question that these are the same as finite dimensional $\mathbb{F}_p$ vector spaces!

7. Let V be an elementary abelian group of order p^n. And identify it with

$$V \cong \underbrace{(\mathbb{Z}/p\mathbb{Z}) \times \cdots (\mathbb{Z}/p\mathbb{Z})}_{n \text{ times}}.$$

For $\lambda \in \mathbb{F}_p$ and $v = (v_1, \cdots, v_n) \in V$, we can let:

$$\lambda v = (\lambda v_1, \cdots, \lambda v_n).$$

(a) Explain why the scalar multiplication giving above makes V into an $\mathbb{F}_p$-vector space.

(b) Show that a function $\varphi : V \to V$ is a homomorphism if and only if it is a linear map of vector spaces.

(c) Using Proposition 1 from HW6, identify the set of isomorphisms from V to itself with a group we have already seen.