Searching for Rigidity in Algebraic Starscapes

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Joint work with Shuchang Xu

ST. LAWRENCE UNIVERSITY

Land Acknowledgement

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The University of California Berkeley sits on the territory of xučyun, the ancestral and unceded land of the Chochenyo speaking Ohlone people. This land was and continues to be of great importance to the Muwekma Ohlone Tribe.

It started with a picture...

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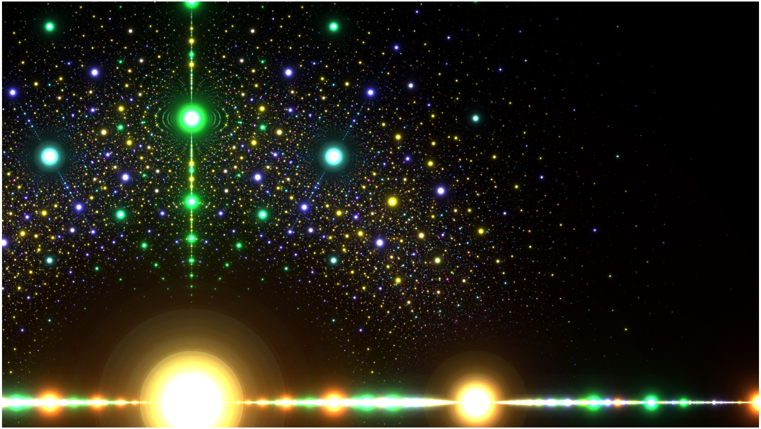
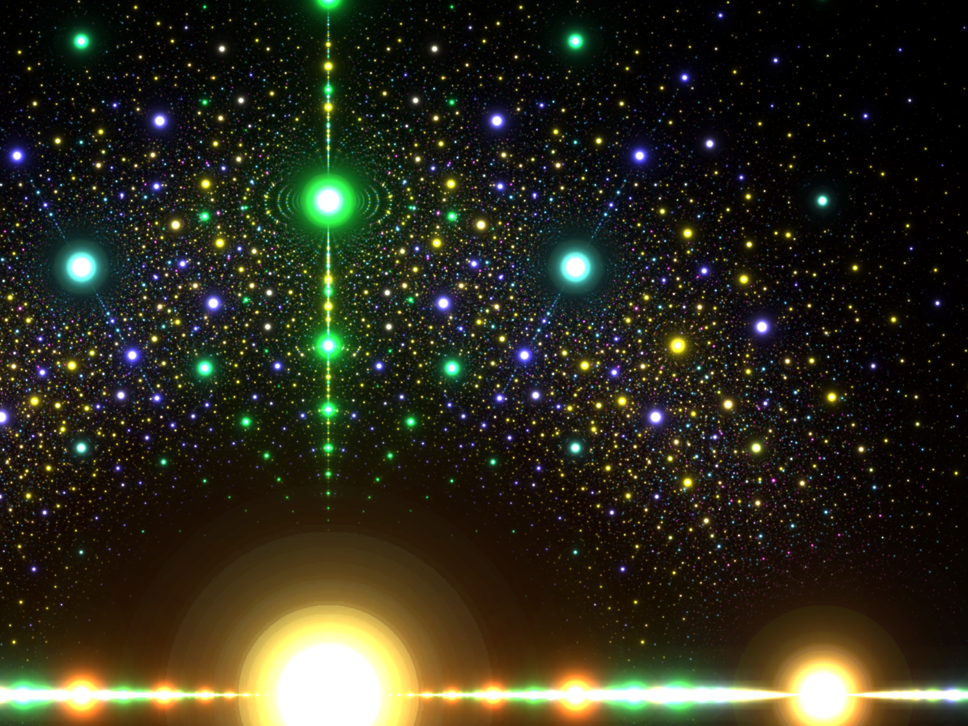


Image Credit: Stephen Brooks and David Moore



Call this an Algebraic Starscape

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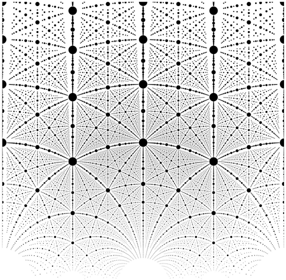
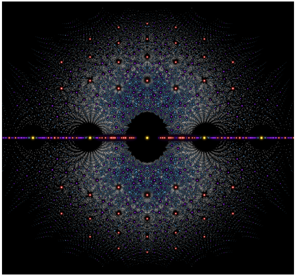
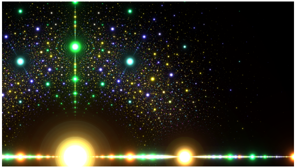
- The golden ratio is a root of $x^2 - x - 1$.



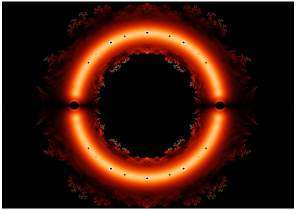
Harris, Stange, Trettel



DH, Xu



Brooks, Moore



Derbyshire

Bohemian Eigenvalue Plots are Starscapes!

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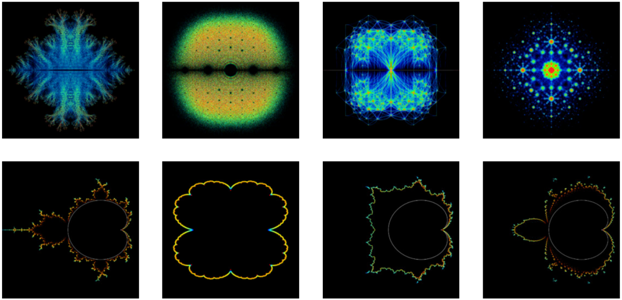

http://www.bohemianmatrices.com/gallery/

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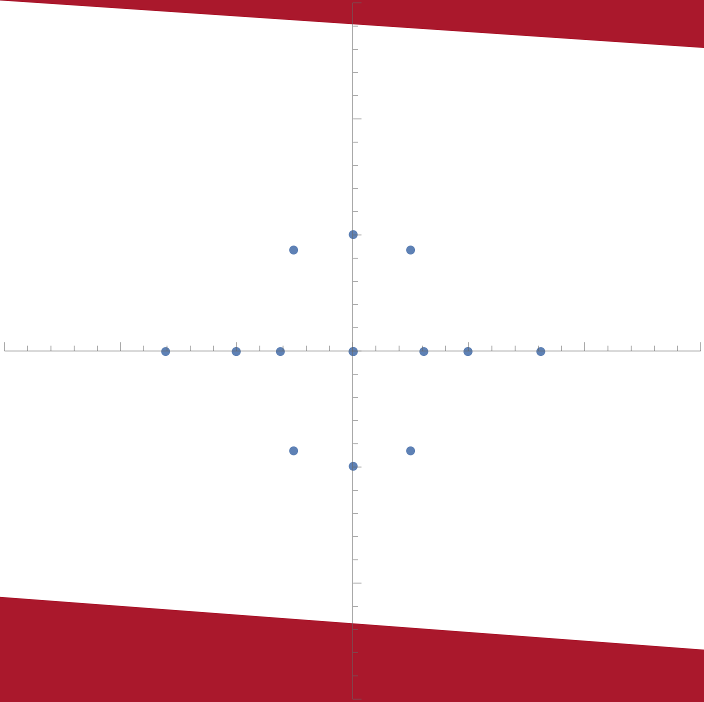
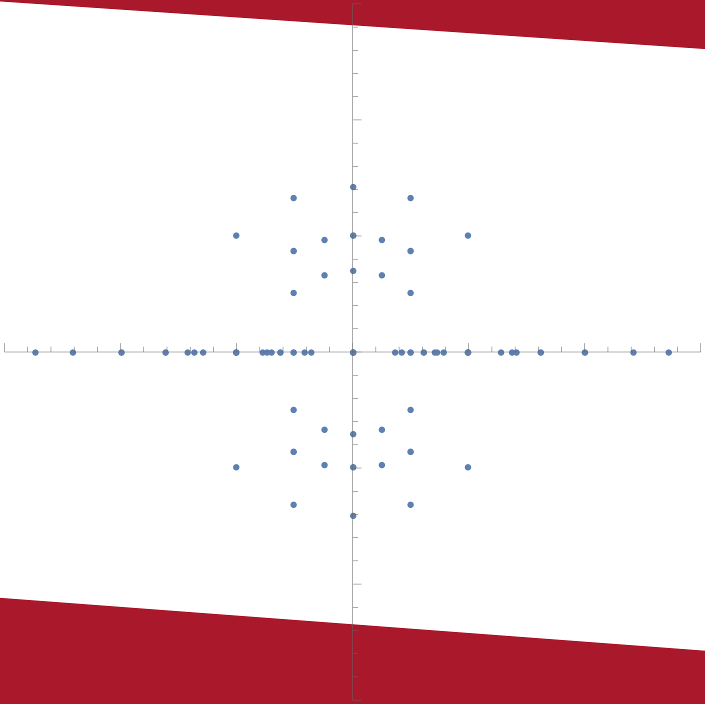
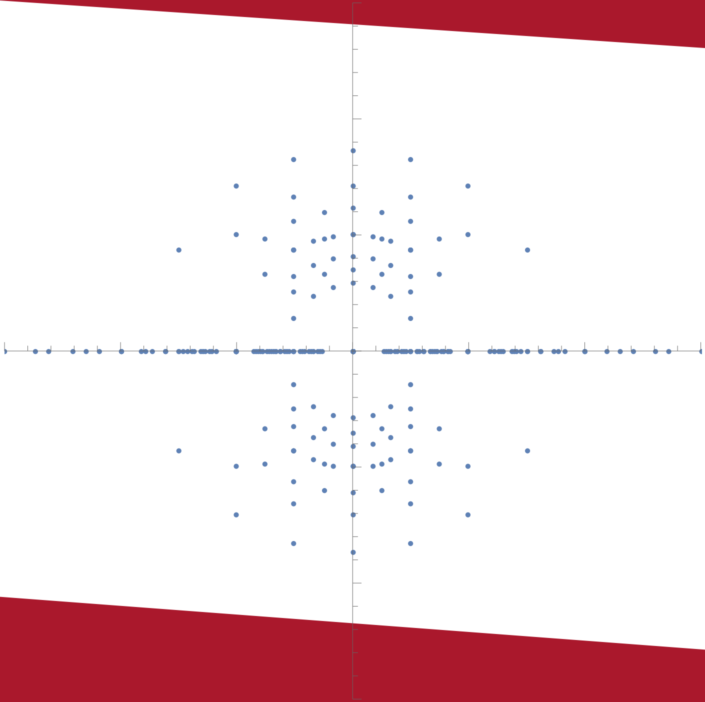
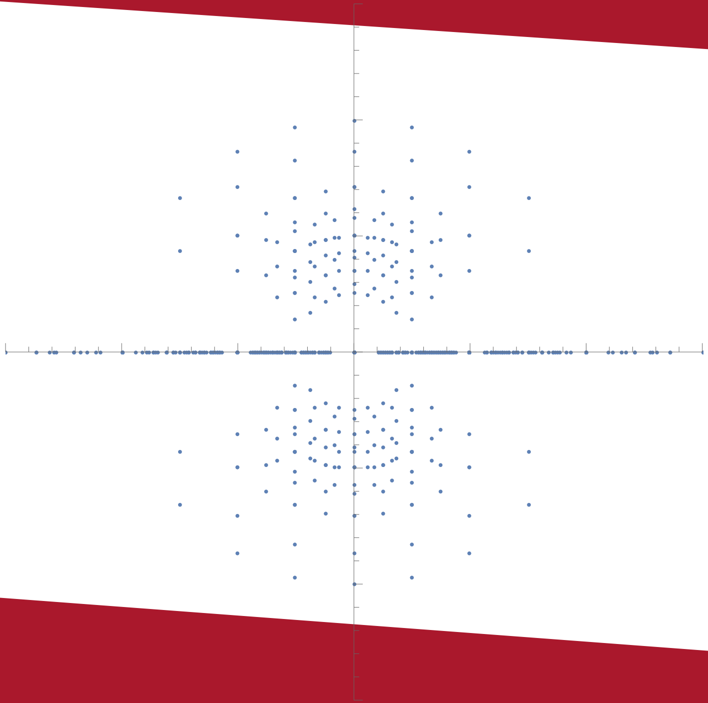
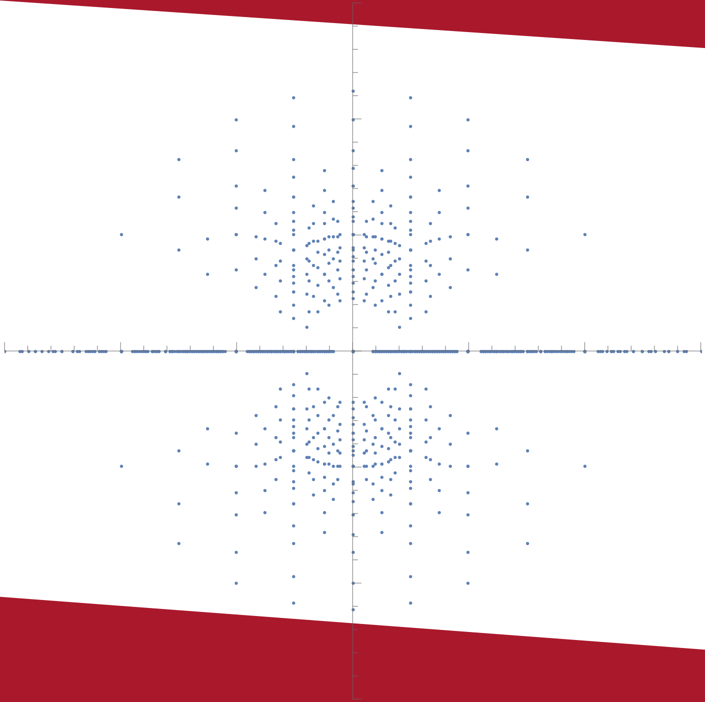
To start let’s naïvely plot quadratics in Sagemath.

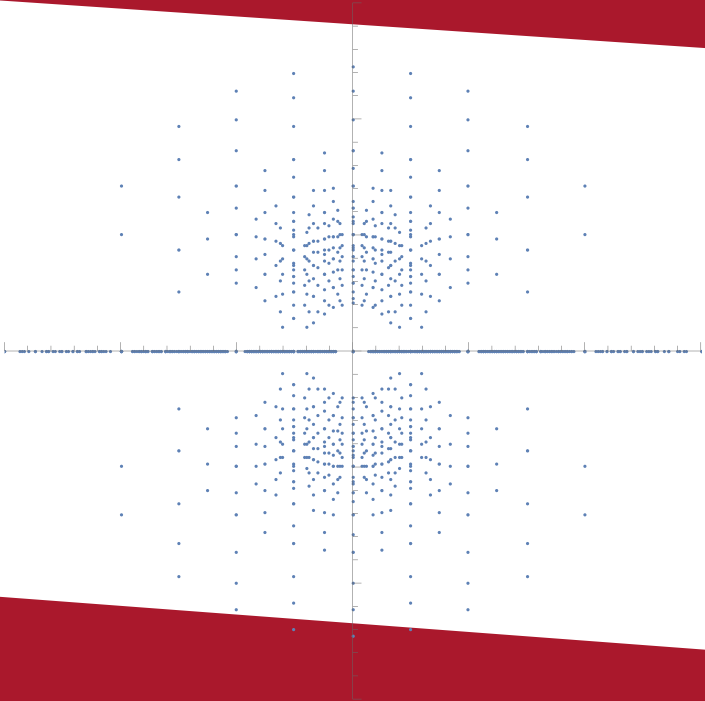


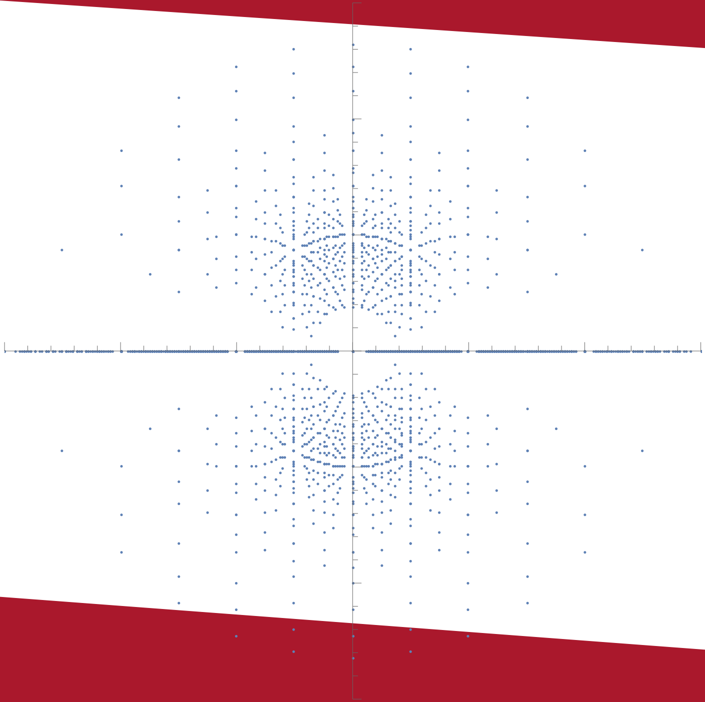


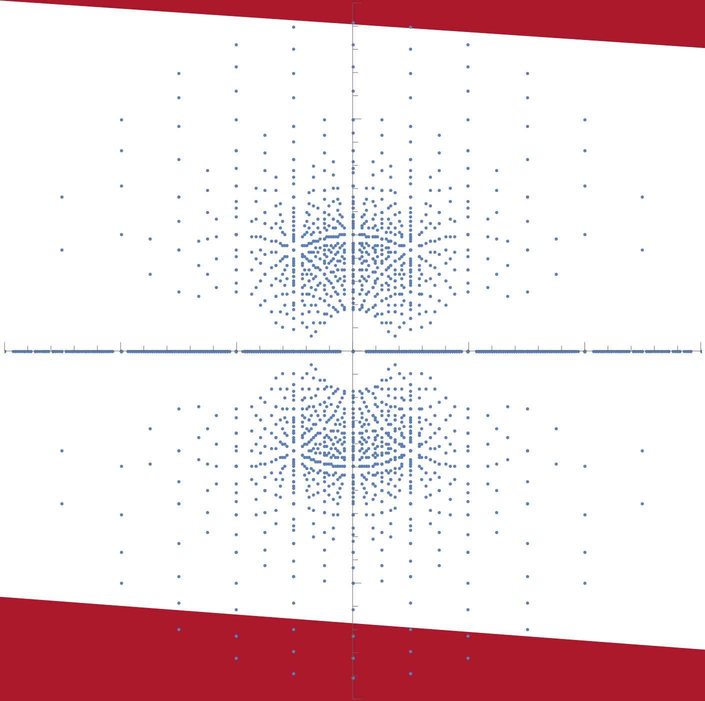


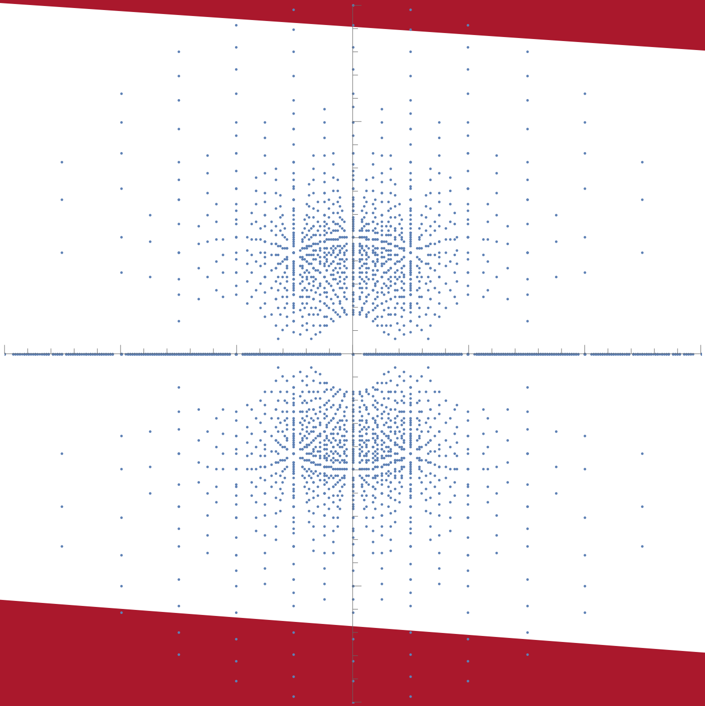


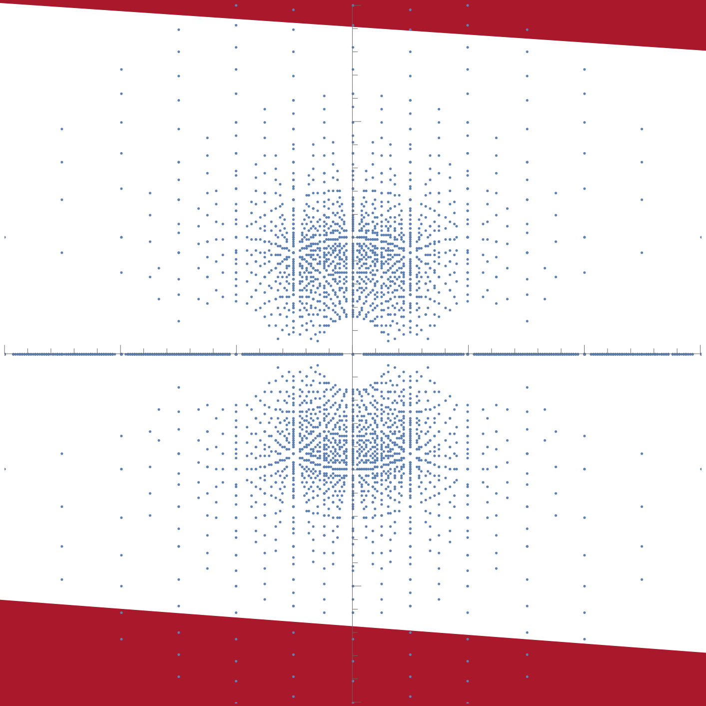


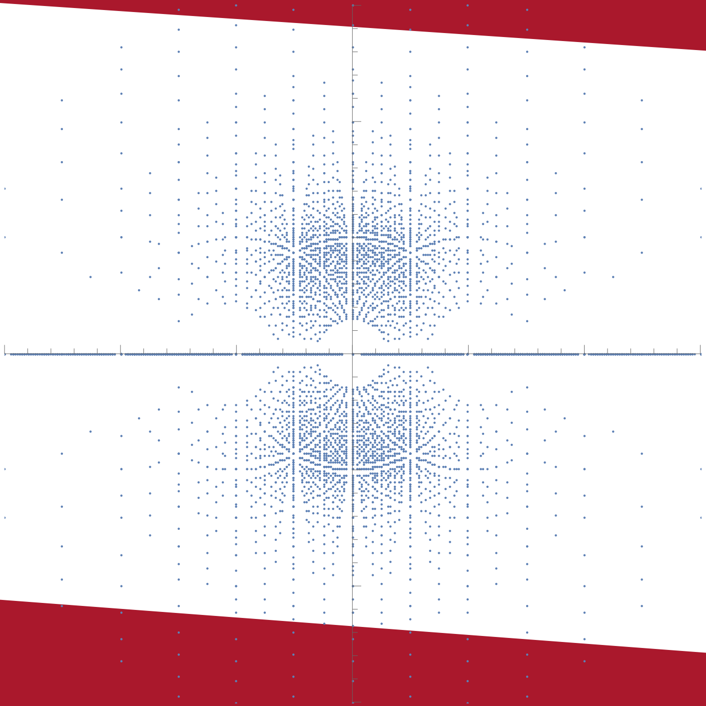


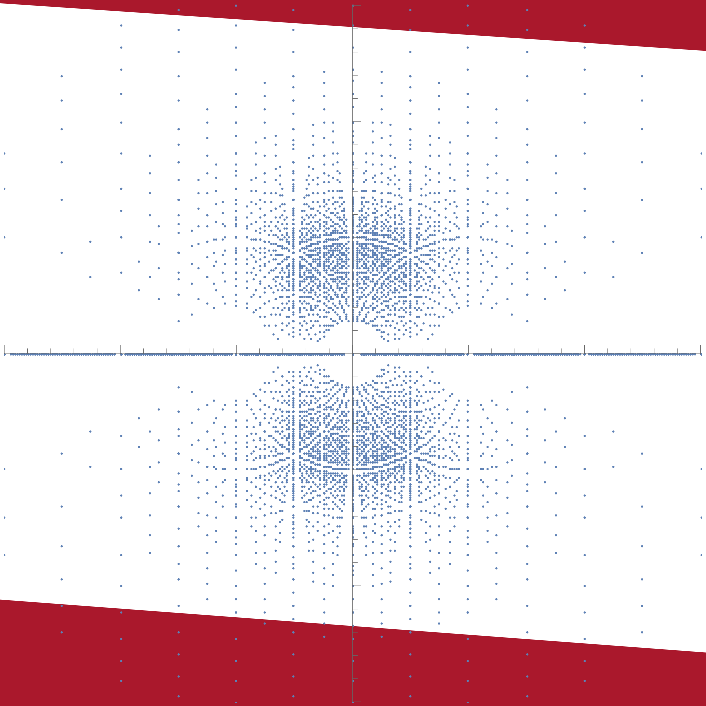


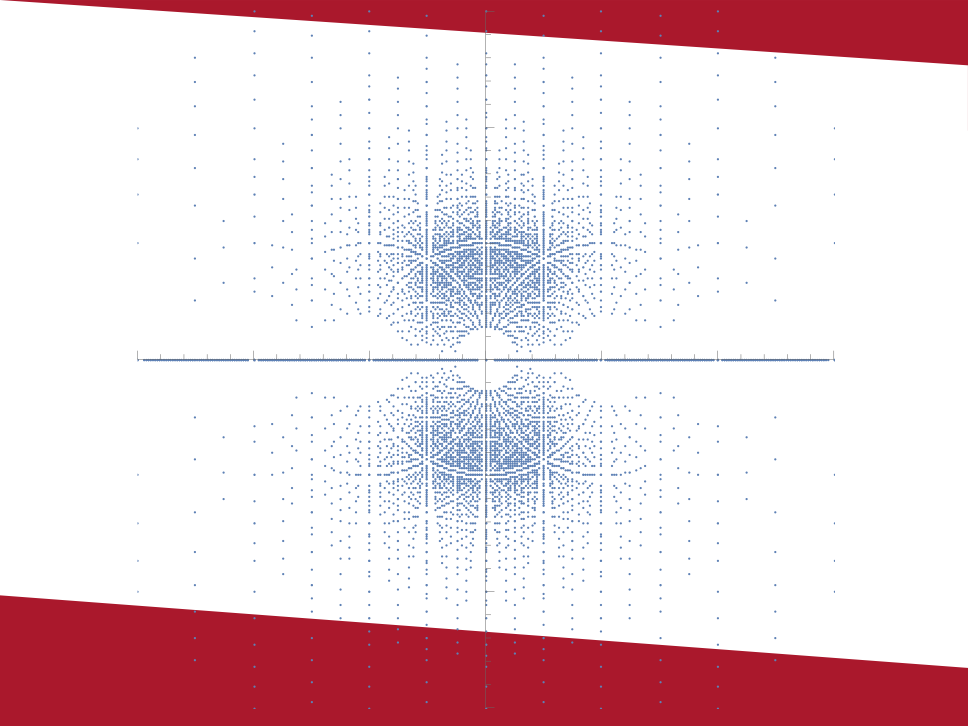


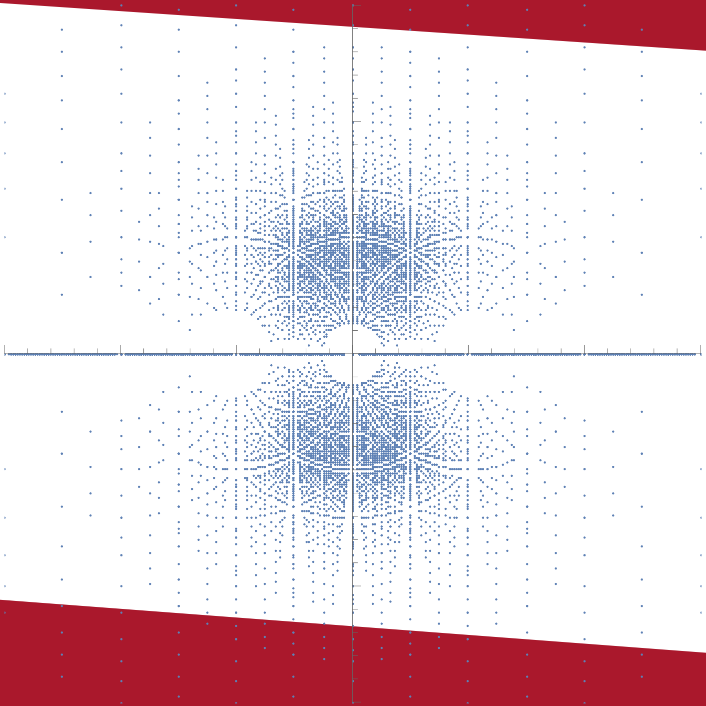


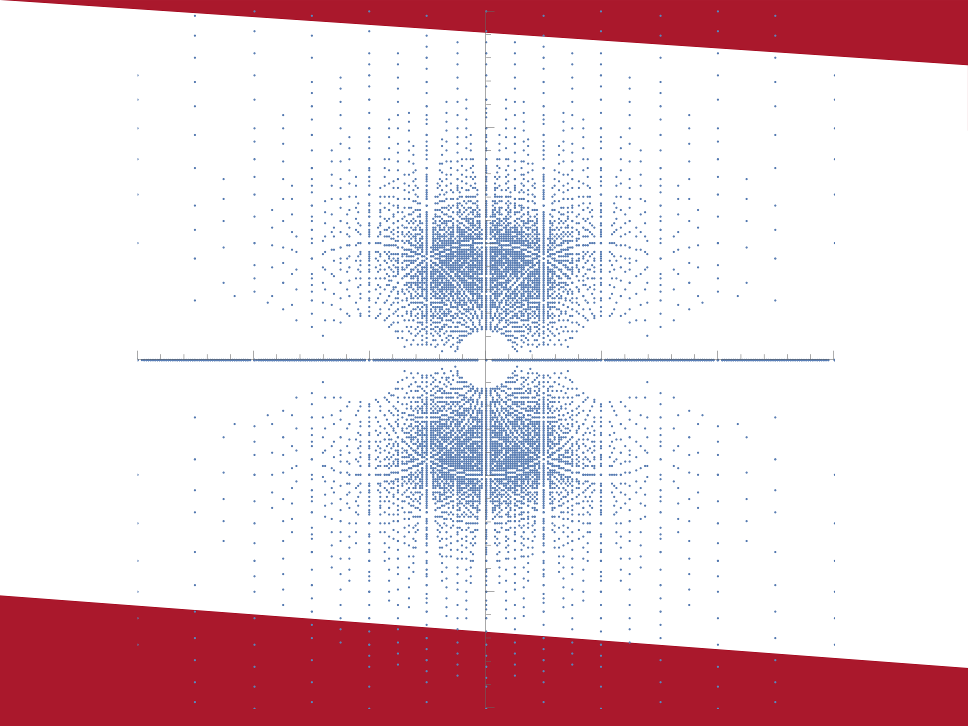


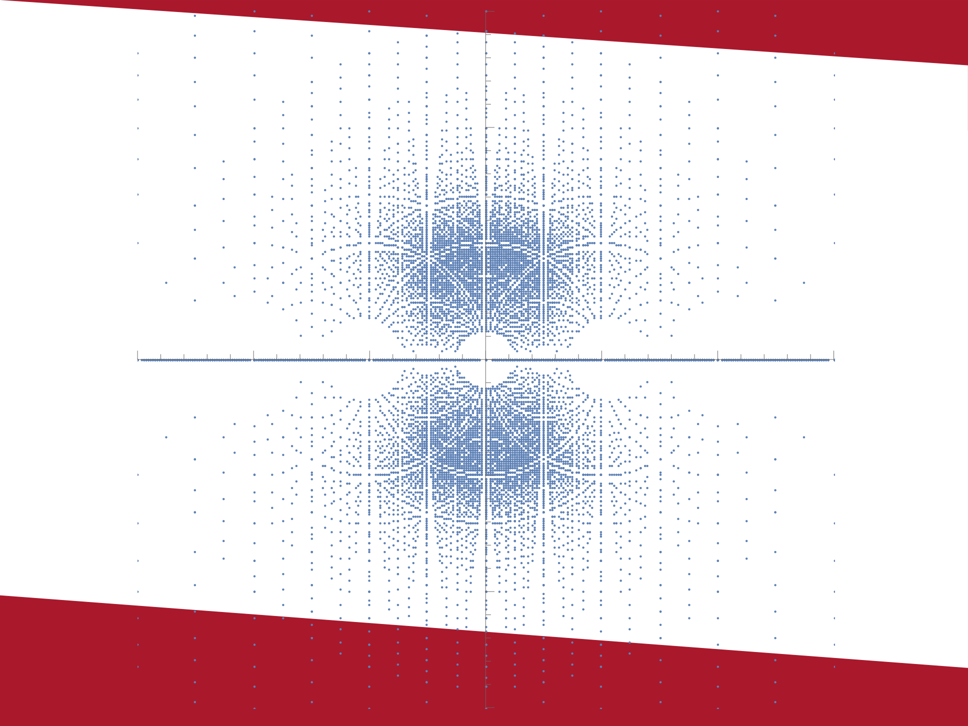


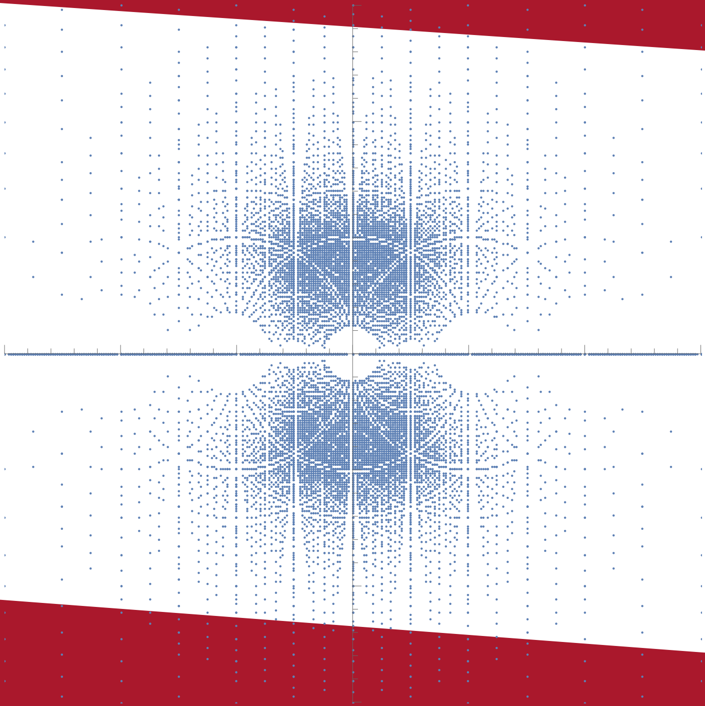


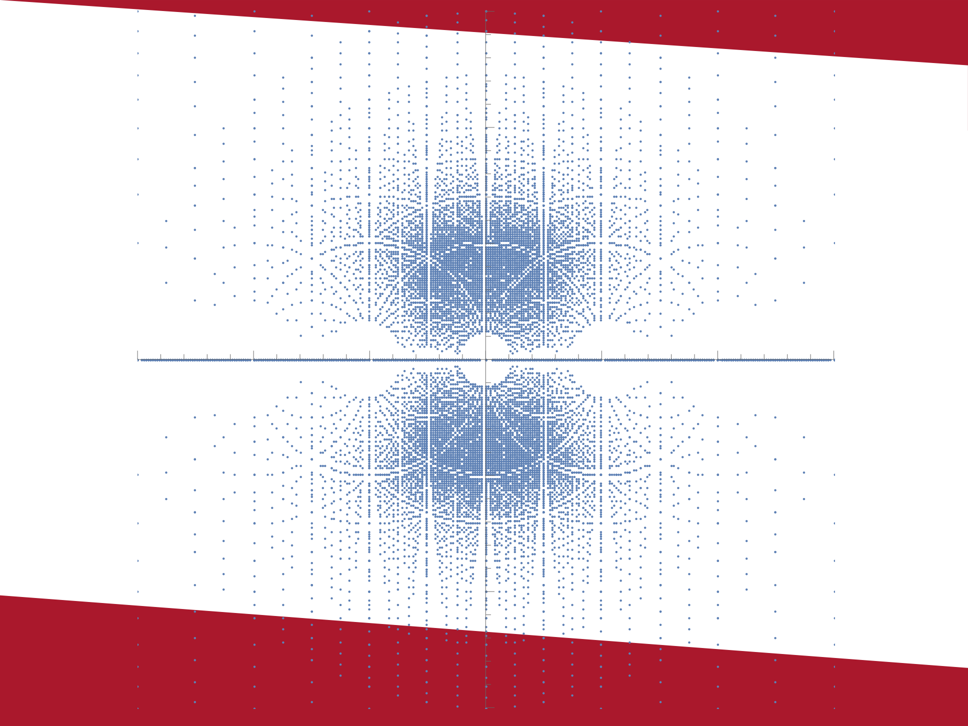


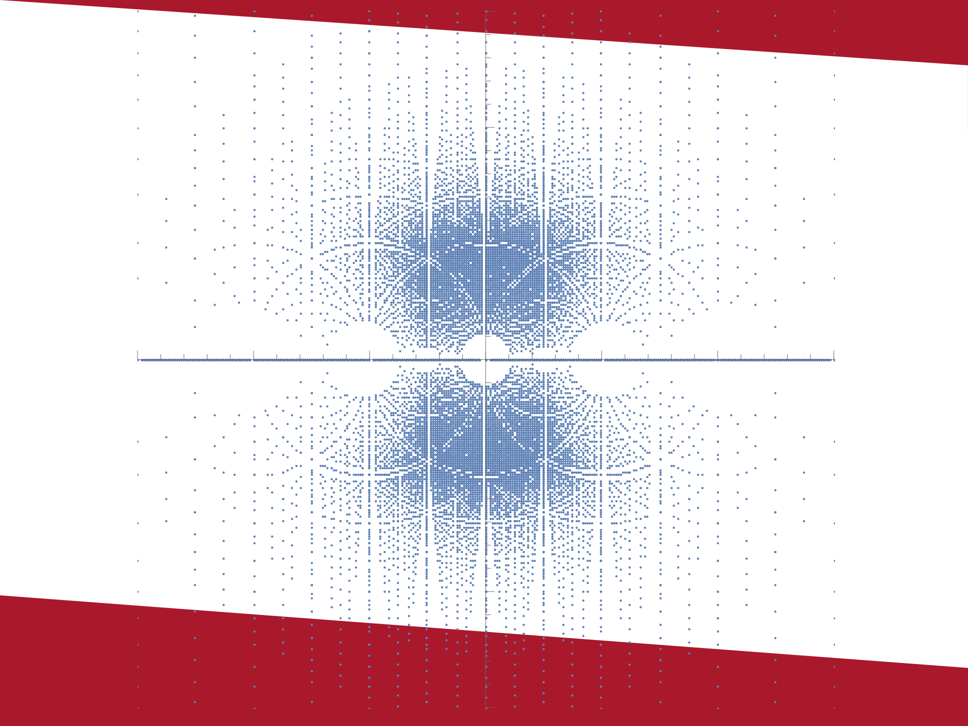


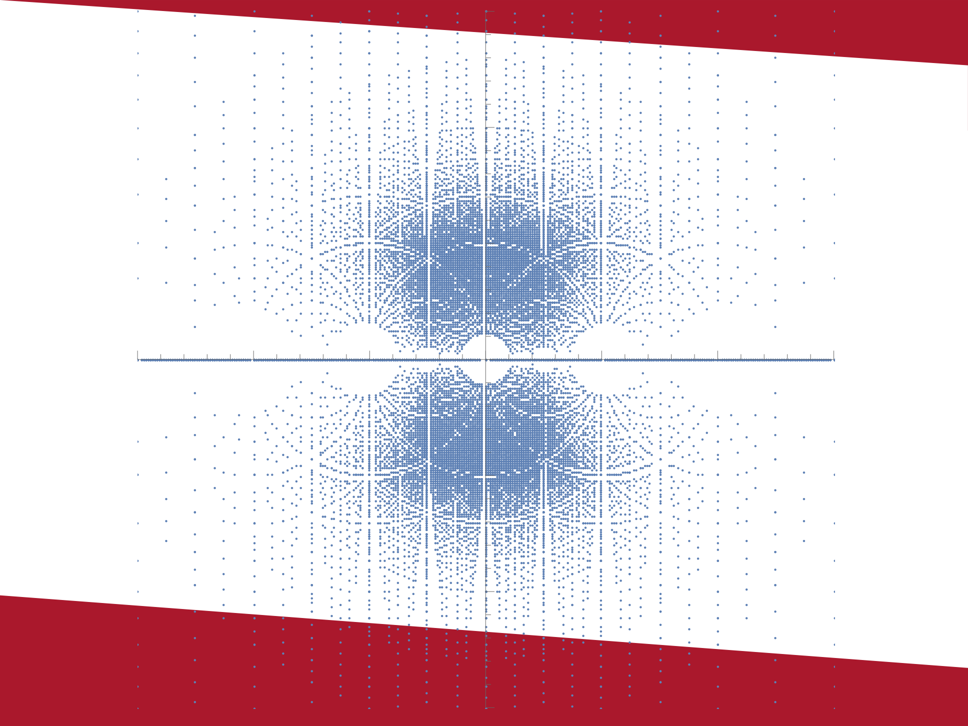












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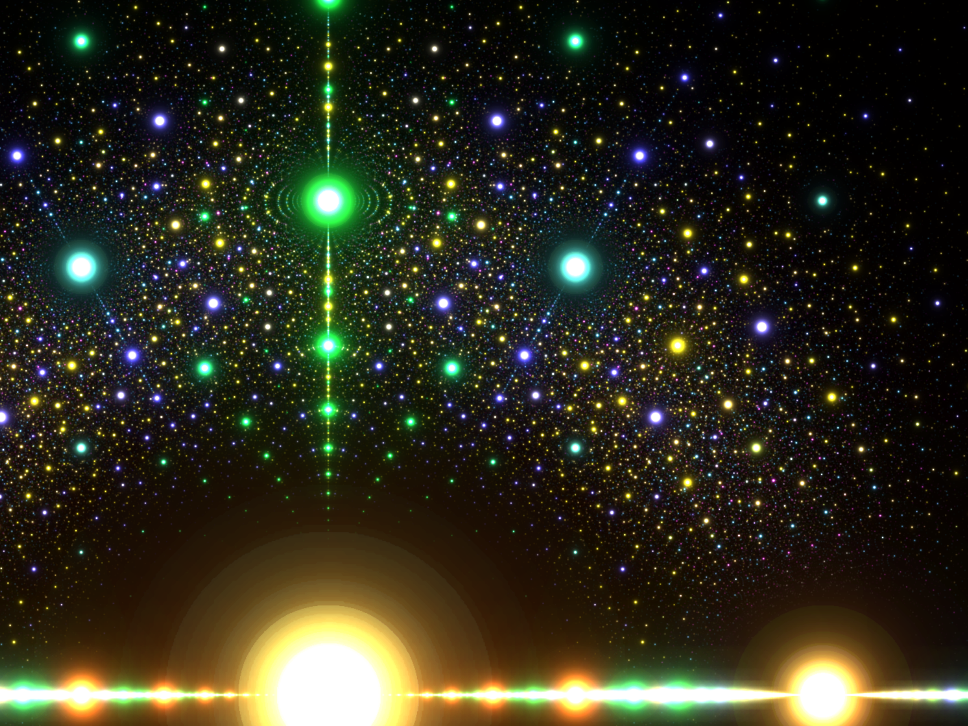
As you add more roots the space seemed to fill and lose some sharpness and clarity. Can we deal with this in a meaningful way?



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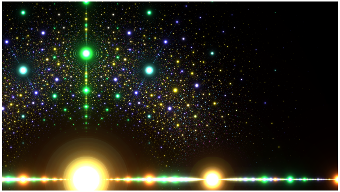

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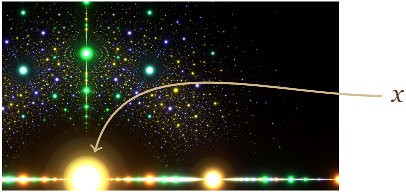

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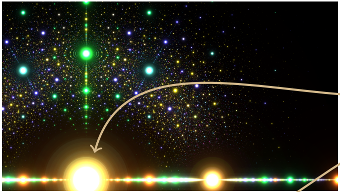

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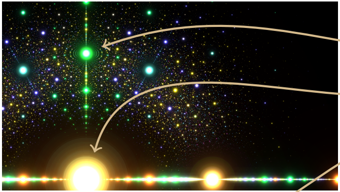

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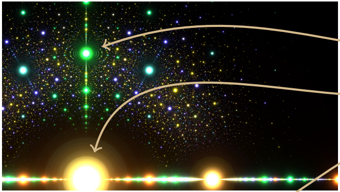

$x^2 + 1$

x

$x - 1$

Simpler Polynomial

$=$

Bigger Point

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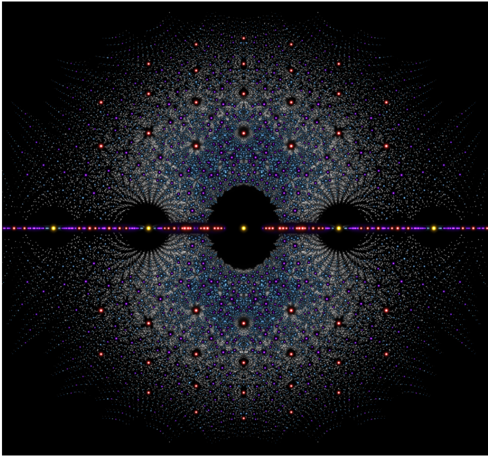
Δ is a measure of the irrationality of x.

Small discriminant

=

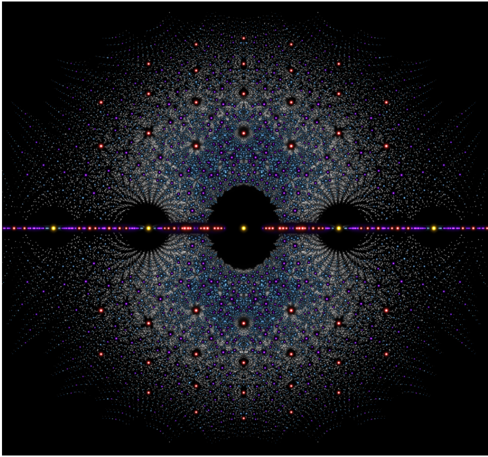
Larger point



Less irrational $\approx$ Small discriminant

$=$

Larger point



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$$\begin{array}{ccc} \text{More Integers} & \longrightarrow & \text{Brighter} \\ \text{Fewer Integers} & \longrightarrow & \text{Darker} \end{array}$$

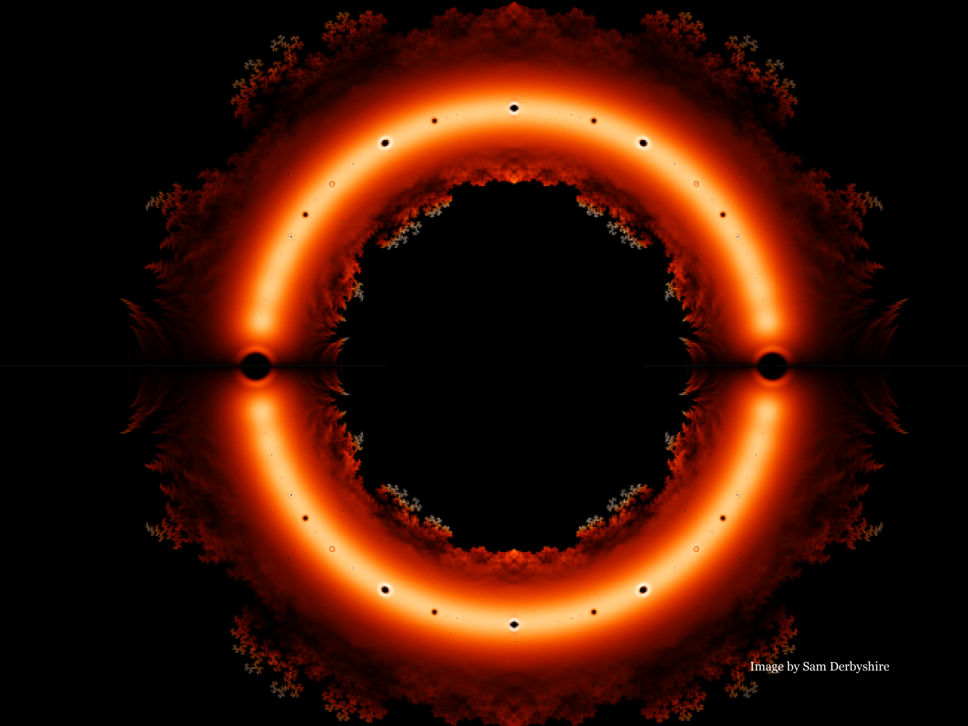
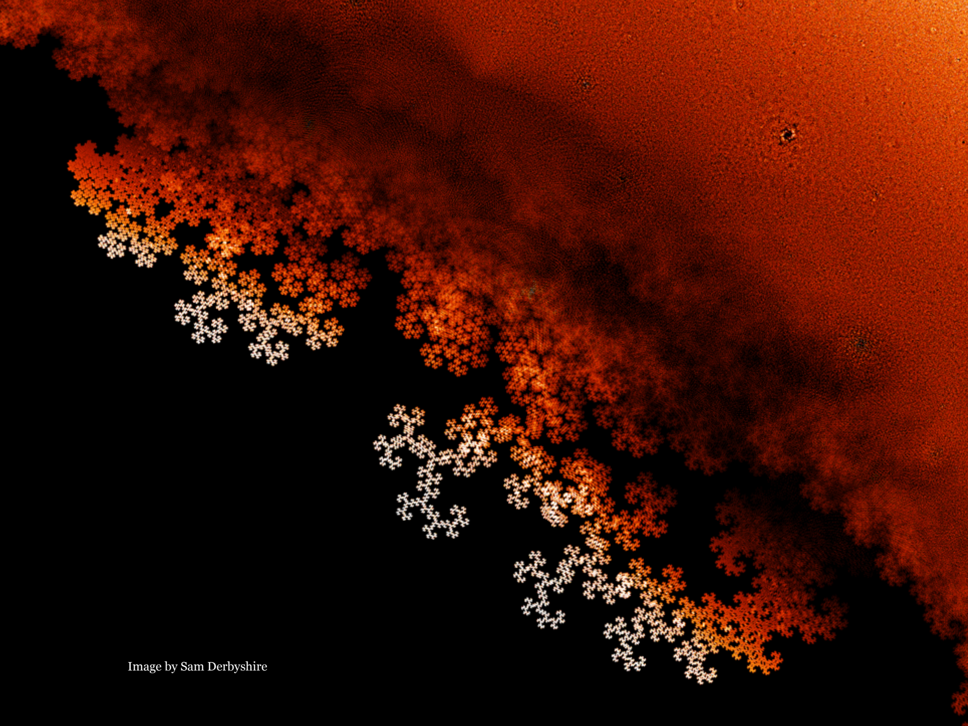


Image by Sam Derbyshire



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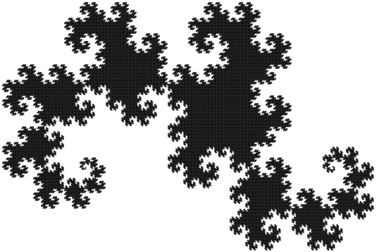
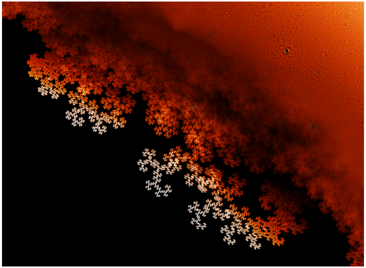
Looks a lot like a dragon fractal!





Density Plots Reveal Analytic Properties

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On the other hand…

When you plot individual points sized by algebraic invariants (like the discriminant), you seem to get algebraic curves, like circles, parabolas and lines.

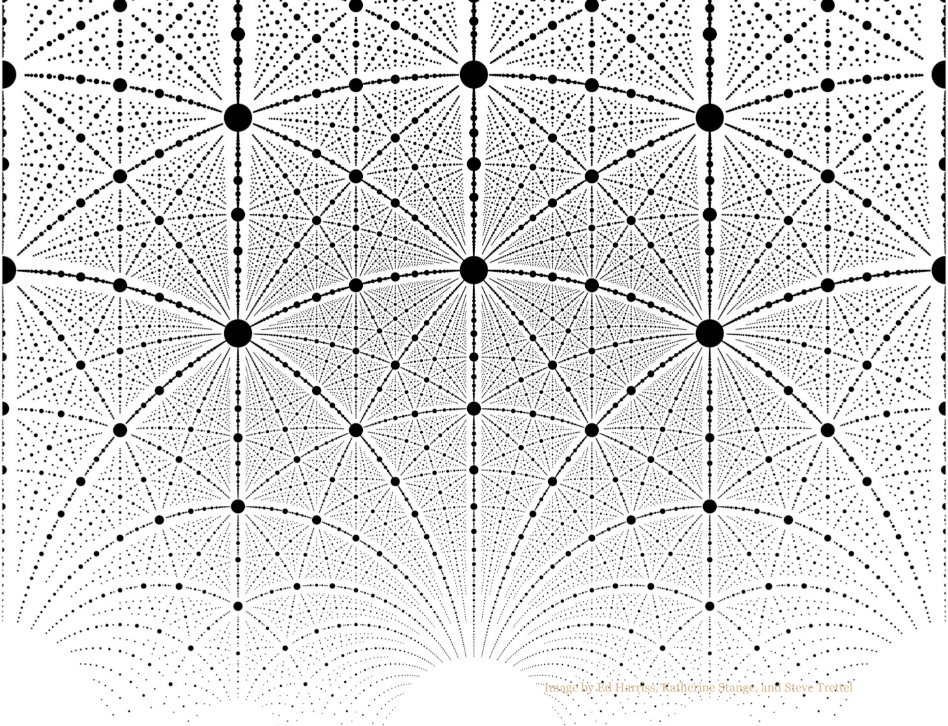

Image by Ed Harriss, Katherine Stange, and Steve Trettel

Paradigm Shifts

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Compare these two perspectives on drawing a collection of algebraic numbers.

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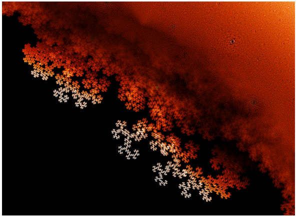

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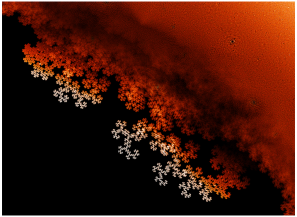

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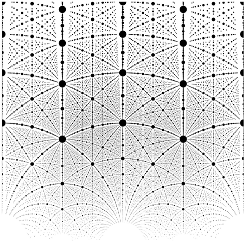

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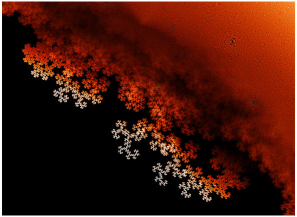

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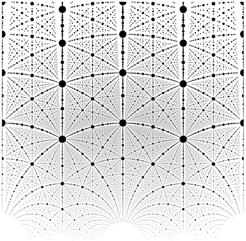

Let’s explain the image on the right.

Rational Geodesics

What exactly are these curves and why do they appear?

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Proposition [HST21]

The set of complex numbers $z \in \mathbb{C}$ such that $\{1, z + \bar{z}, z\bar{z}\}$ are linearly dependent over $\mathbb{Q}$

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Proposition [HST21]

The set of complex numbers $z \in \mathbb{C}$ such that $\{1, z + \bar{z}, z\bar{z}\}$ are linearly dependent over $\mathbb{Q}$ form a collection of vertical lines with rational x-coordinates, and circles of centered at rational points with rational radii.

Rational Geodesics

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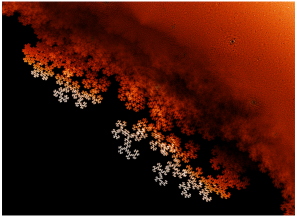
$\{1, z + \bar{z}, z\bar{z}\}$ are 3 elements of a 2 dimensional $\mathbb{Q}$-vector space $\mathbb{Q}(z)$. $\square$

Paradigm Shifts

We saw two perspectives on drawing a collection of algebraic numbers.

Density coloring revealed analytic properties.



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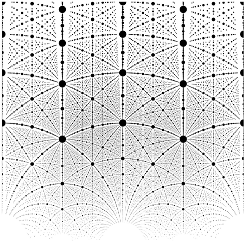

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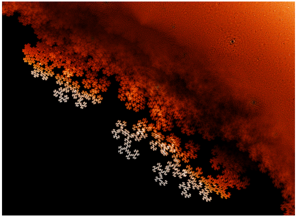

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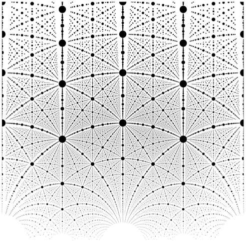

In each case, the images inspired new research and new theorems!

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The field of algebraic number theory essentially studies how to understand and classify these invariants. Let's focus on Galois theory.

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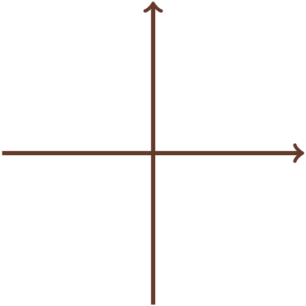

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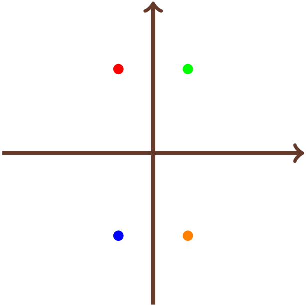

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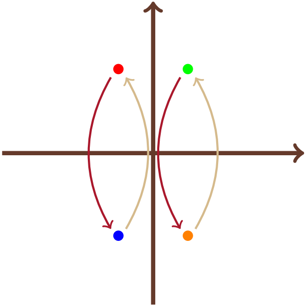

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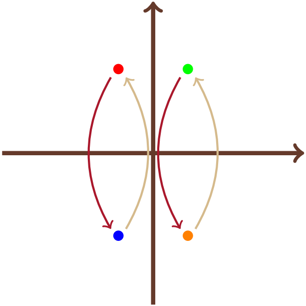

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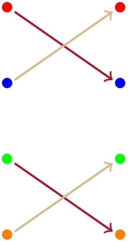

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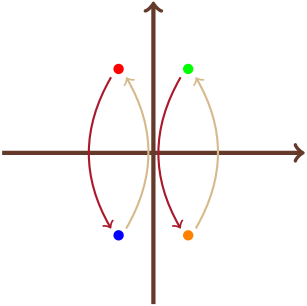

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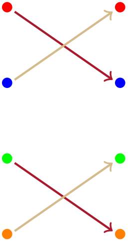

Slogan: An algebraic symmetry of the roots gives rise to a permutation of the roots.

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σ does not respect multiplication, and is therefore not algebraic. Since not every permutation of the roots of $x^5 - 1$ is algebraic, we will call the roots rigid.

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Do rigid algebraic integers satisfy interesting geometric relationships? To explore this we introduce an invariant which measures how far G is from S_n, called rigidity.

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Notice that if $G = S_n$ then $\#G = n!$ so that $\mathrm{rig}(z) = 0$.

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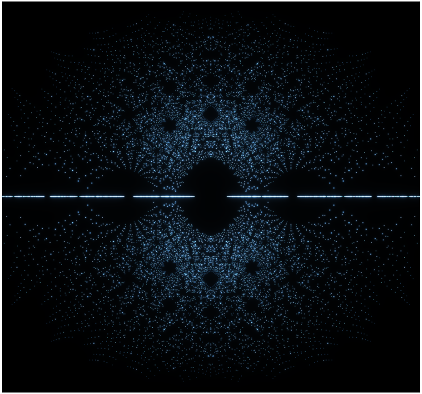
Quartics sized by discriminant.



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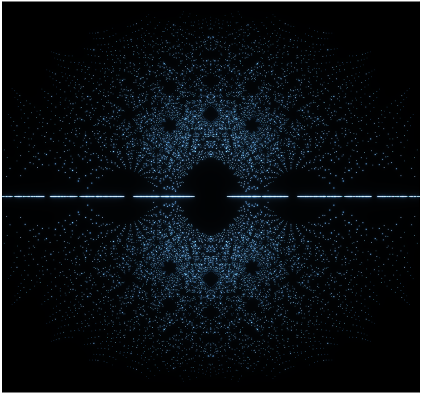

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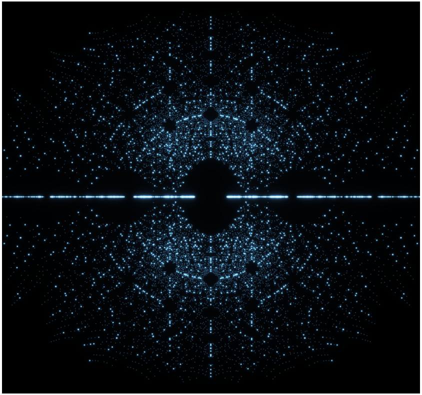

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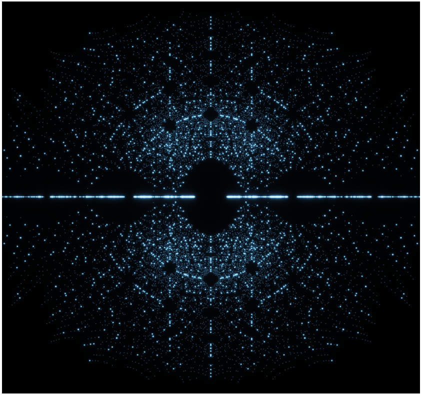

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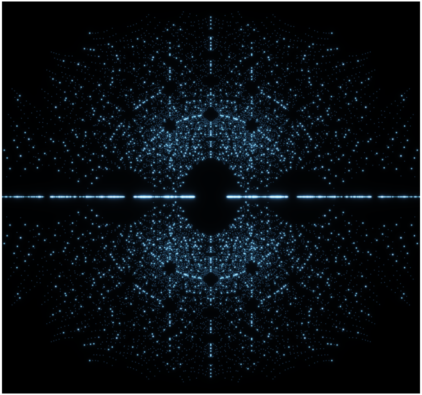

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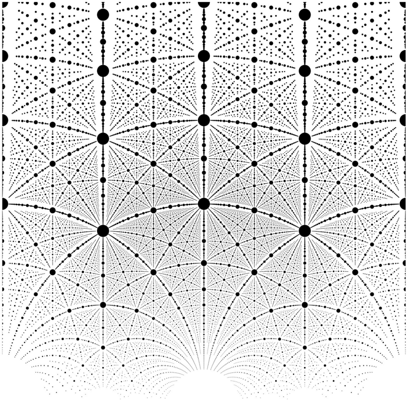

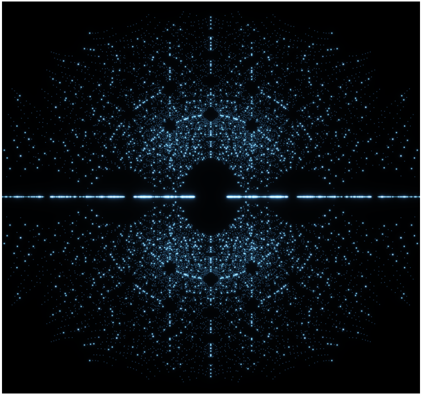

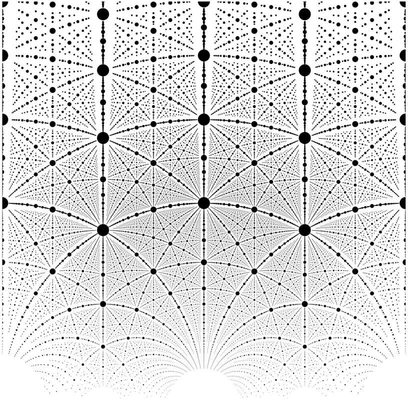

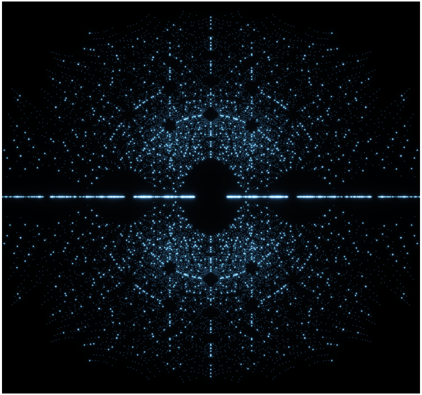

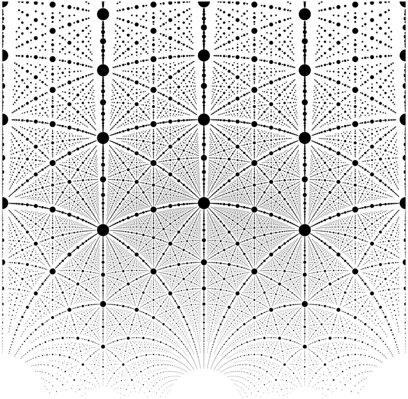

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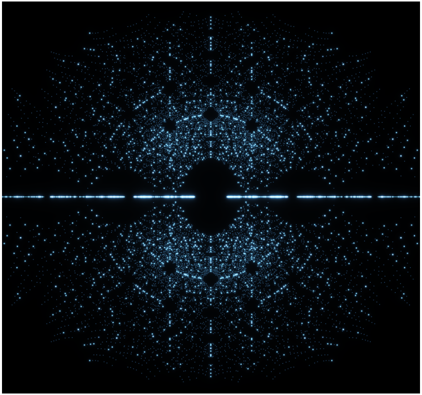

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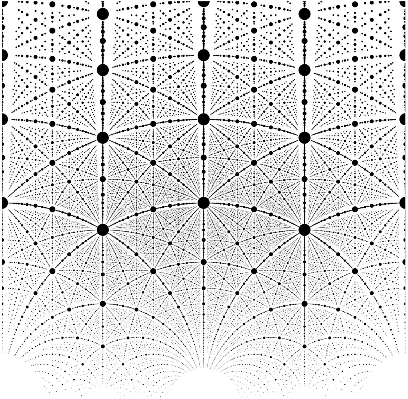

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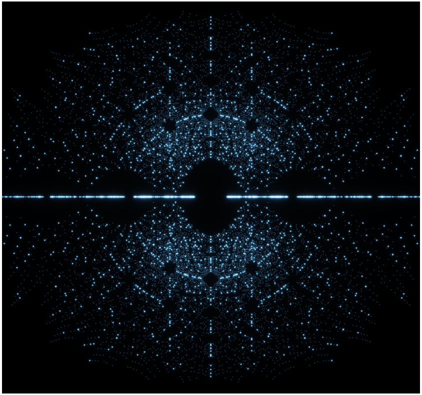

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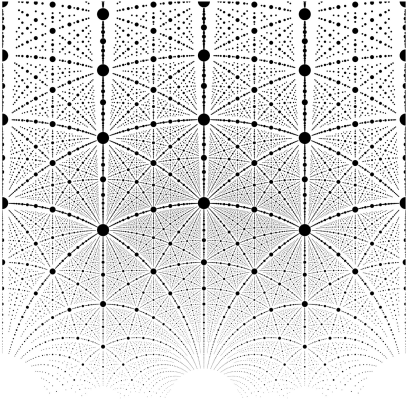

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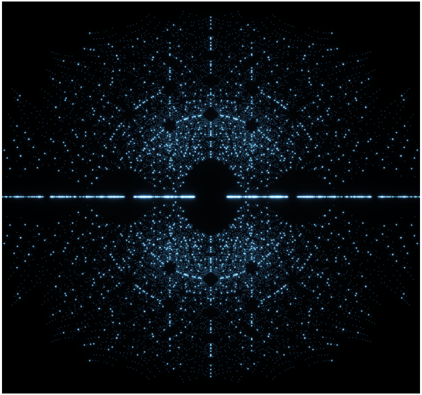

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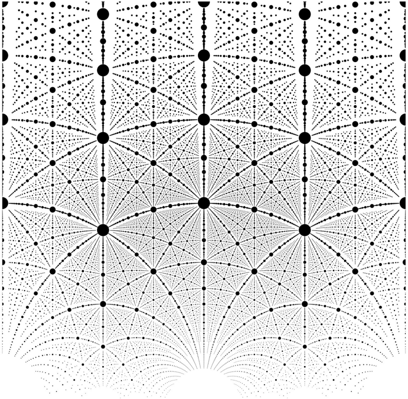

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- Ignores Galois theory.

And yet…

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Theorem (DH, Xu)

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- *If a quartic lies on a rational geodesic, it is rigid*

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This result explains the hidden patterns in the image of the rigid quartics, and was completely inspired by that image. It also exhibits a previously hidden geometric connection between quadratics and rigid quartics.

Conclusion

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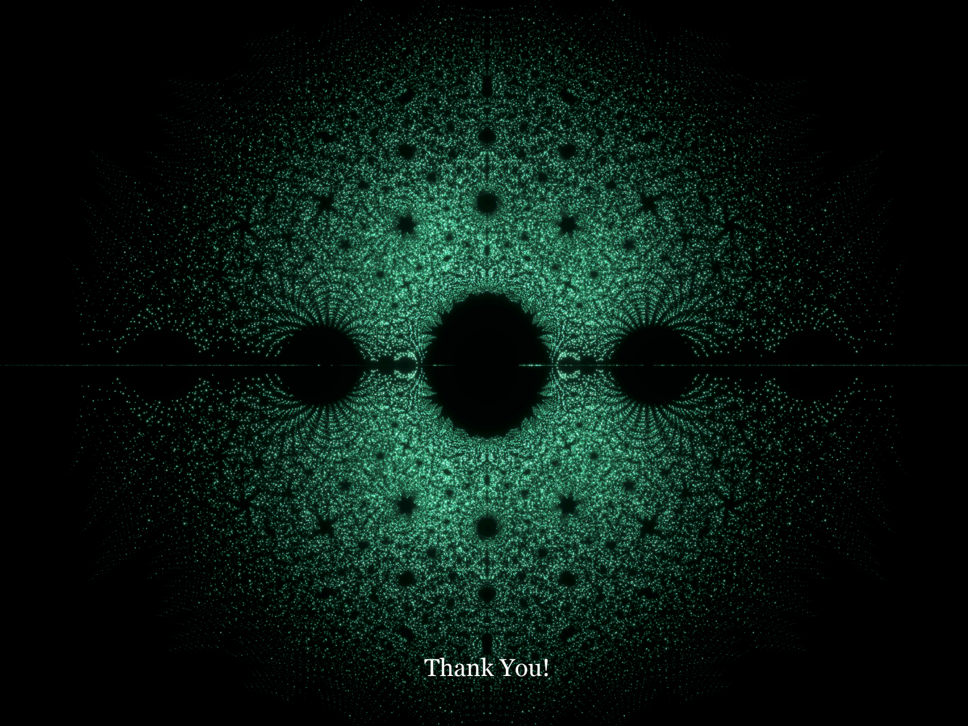
Question

What choices can we make when drawing Bohemian starscapes that could reveal interesting patterns in matrix theory?


Thank You!