Homework Assignment 7

Due Friday, March 11

1. Let G be a group and let $H, K \leq G$ be subgroups. Recall that we defined the set:

$$HK = \{hk : h \in H, k \in K\} \subseteq G.$$

The second isomorphism theorem relied on the following two facts, which you will now verify.

(a) Show that HK is a subgroup of G if and only if $HK = KH$.

(b) Use part (a) to show that that if $H \leq N_G(K)$, then HK is a subgroup of G. Explain why this means that if either H or K are normal subgroups, then $HK \leq G$.

2. Let G be a group, and $M, N \trianglelefteq G$ normal subgroups such that $MN = G$. Use the first and second isomorphism theorems to establish the following facts.

(a) Show $G/(M \cap N) \cong (G/M) \times (G/N)$

(b) Suppose further that $M \cap N = \{1\}$. Show that $G \cong M \times N$.

3. We continue by proving the fourth isomorphism theorem. Let $N \trianglelefteq G$ be a normal subgroup of a group G. Let $\pi : G \to G/N$ be the natural projection.

(a) Let $H \leq G/N$. Show that the preimage $\pi^{-1}(H) = \{g \in G : \pi(g) \in H\}$ is a subgroup of G containing N.

(b) Let $H \leq G$. Show that its image $\pi(H)$ is a subgroup of G/N.

(c) These constructions do not in general give a bijection between subgroups of G and subgroups of G/N. Give an example showing why.

(d) If we restrict our attention to certain subgroups of G we do get a bijection. Show that the constructions in parts (a) and (b) give a bijection:

$$\left\{ \begin{array}{c} \text{Subgroups } H \leq G \\ \text{such that } N \leq H \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Subgroups} \\ \overline{H} \leq G/N \end{array} \right\}$$

(e) This bijection satisfies certain properties. First let's establish some notation. Let $H, K \in G$ be two subgroups containing N, and denote the corresponding subgroups of G/N by $\overline{H}$ and $\overline{K}$. Prove the following properties.

i. $H \leq K$ if and only if $\overline{H} \leq \overline{K}$.

ii. $H \trianglelefteq K$ if and only if $\overline{H} \trianglelefteq \overline{K}$.

iii. $\overline{H \cap K} = \overline{H} \cap \overline{K}$

iv. $\overline{\langle H, K \rangle} = \langle \overline{H}, \overline{K} \rangle$.

Hint. *You can do (iii) and (iv) directly, but if you want to be really slick use that the intersection of two subgroups is the largest subgroup contained in both, (and the dual notion for the subgroup generated by two subgroups). Notice that this means that being the intersection of two subgroups (or generated by two subgroups) is a condition on the lattice of G (or G/N). Then the result should easily follow from part (i).*

4. By Cayley's theorem, the group Q_8 from HW6 Problem 5 is isomorphic to a subgroup of S_8. Let's write down such a subgroup explicitly!

(a) Label $\{1, -1, i, -i, j, -j, k, -k\}$ as the numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Then the action of Q_8 on itself by left multiplication gives an injective map $Q_8 \to S_8$. Write the permutation representations for -1 and i as elements $\sigma_{-1}, \sigma_i \in S_8$, and verify that $\sigma_i^2 = \sigma_{-1}$. (Using the multiplication table from HW6 Problem 5 may make this easier).

(b) Use the generators from HW6 Problem 5(b) to give two elements of S_8 which generate a subgroup $H \leq S_8$ isomorphic to Q_8.

5. Let G be a group. Let $[G, G] = \langle x^{-1}y^{-1}xy | x, y \in G\rangle$.

   (a) Show that $[G, G]$ is a normal subgroup of G.

   (b) Show that $G/[G, G]$ is abelian.

   $[G, G]$ is called the *commutator subgroup* of G, and $G/[G, G]$ is called the *abelianization* of G, denoted G^{ab}. The rest of this exercise explains why.

   (c) Let $\varphi : G \to H$ be a homomorhism with H abelian. Show $[G, G] \subseteq \ker \varphi$.

   (d) Conclude that for H an abelian group there is a bijection:

$$\left\{ \text{ Homomorphisms } \varphi : G \to H \right\} \Longleftrightarrow \left\{ \text{ Homomorphisms } \tilde{\varphi} : G^{\mathrm{ab}} \to H \right\}$$

   Hint. *Recall the technique of passing to the quotient described in the 5/3 lecture.*

6. Let's now compute D_{2n}^{ab}. We should begin computing $xyx^{-1}y^{-1}$. There are 3 cases.

   (a) Compute $x^{-1}y^{-1}xy$ in each of the following 3 cases. (*Hint:* HW2#9(e) gives the inverse for a reflection.)

      (i) x, y both reflections. So $x = sr^i$ and $y = sr^j$.

      (ii) x a reflection and y not a reflection. So $x = sr^i$ and $y = r^j$.

      (iii) Neither x nor y are reflections. So $x = r^i$ and $y = r^j$.

   (b) Prove that $[D_{2n}, D_{2n}] = \langle r^2\rangle$. If n is odd one could choose another generator. What is it?

   (c) Now prove that D_{2n}^{ab} is either V_4 or Z_2 depending on whether n is odd or even. Note that since this is so small we should interpret this as suggesting that D_{2n} is far from abelian.

Let F be a field. The general linear group $GL_n(F)$ from HW6 Problem 7 has lots of interesting subgroups and quotients, which we study in the following problem. You may use the following fact without proof, as it is a standard result of linear algebra.

Proposition 1. *If* $A, B \in GL_n(F)$, *then* $\det(AB) = \det(A)\det(B)$. *In particular,* $\det : GL_n(F) \to F^\times$ *is a group homomorphism.*

7. (a) Show that the constant diagonal matrices are a normal subgroup of $GL_n(F)$ isomorphic to $F^\times$.

   We will often abuse notation and denote this by $F^\times \trianglelefteq GL_n(F)$. The quotient group $GL_n(F)/F^\times$ is called the *projective general linear group* and denoted $PGL_n(F)$.

   (b) The *special linear group* $SL_n(F)$ is defined

$$SL_n(F) = \{A \in GL_n(F) \mid \det(A) = 1.\}$$

Show that $SL_n(F)$ is a normal subgroup of $GL_n(F)$ and prove that

$$GL_n(F)/SL_n(F) \cong F^\times.$$

(*Hint:* Use the First Isomorphism Theorem and Proposition 1)

(c) List all the elements of $SL_2(\mathbb{F}_2)$.

(d) Compute $|SL_2(\mathbb{F}_p)|$ (*Hint*, between 4(b) and HW6 Problem 7(d) you've already done all the work).

(e) Let I be the identity matrix. Show that $\{\pm I\} \leq SL_n(F)$ if and only if n is even.

(f) Use the second isomorphism theorem to construct an isomorphism:

$$PGL_2(\mathbb{C}) \cong SL_2(\mathbb{C})/\{\pm I\}.$$

(As a bonus, think about why this is not true for a general field. For example, it is false over $\mathbb{R}$, or over $\mathbb{F}_p$ for $p \neq 2$.)