Homework Assignment 1

Due: Friday, January 28

- 1. Let S and T be sets, and suppose that $T \subseteq S$. Describe the following sets, proving the correctness of your answers.
 - (a) $T \cap S$.
 - (b) $T \cup S$.
 - (c) $T \cap (S \setminus T)$
 - (d) $T \cup (S \setminus T)$.
- 2. Let S be a set with 3 elements (say $\{0,1,2\}$) and T be a set with 5 elements (say $\{a,b,c,d,e\}$).
 - (a) Give an example of an injection $f: S \to T$.
 - (b) Give an example of a surjection $g: T \to S$.
 - (c) Can there be a bijection between S and T? Why or why not?
- 3. A subset $T \subset S$ is called a *proper subset* if $T \neq S$. This is often denoted $T \subsetneq S$. Give an example of a set S and a bijection between S and a *proper* subset of S.
- 4. Let S and T be two nonempty sets, and $f: S \to T$ a function between them.
 - (a) Show that f is injective if and only if it has a left inverse.
 - (b) Show that f is surjective if and only if it has a right inverse
 - (c) Show that f is bijective if and only if it has an inverse.
 - (d) Show that if f has a (two-sided) inverse, that inverse is unique.

Remark. Because of part (c) and (d) of the question 4, we see that if f is bijective, then f has a unique inverse, which we call the inverse of f and denote by f^{-1} .

5. Let S and T be finite sets and suppose that |S| = |T|. Let $f: S \to T$ be a function. Prove that

$$f$$
 is injective $\Leftrightarrow f$ is surjective $\Leftrightarrow f$ is bijective.

- 6. Show that equivalence relations are partitions are equivalent. Explicitly, let S be a set, construct a natural bijection between the partitions on S and the equivalence relations on S in the following way.
 - (a) Let \sim be an equivalence relation. Show that the equivalence classes of \sim form a partition of S.
 - (b) Conversely, let $\{X_i\}$ be a partition of S. Show that the relation \sim given by the rule

$$x \sim y$$
 if $x, y \in X_i$ for the same i

is an equivalence relation for S.

(c) Show that parts (a) and (b) give a bijection between the sets:

{Equivalence relations on
$$S$$
} \longleftrightarrow {Partitions of S }.

(Hint: Part (a) gives a function from the left to the right. Part (b) gives a function from the right to the left. Show that these are inverses to eachother).

- 7. Let $a, b, c \in \mathbb{Z}$. Prove the following divisibility facts.
 - (a) If a|b and a|c then a|(b+c)
 - (b) If a|b then a|bc.
- 8. In this exercise we prove the existence and uniqueness of division with remainder. Let $a, b \in \mathbb{Z}$, and suppose that $b \neq 0$. We start with existence.
 - (a) We begin by considering the set of numbers a-bq as q varies over the integers. Prove that the set

$$S = \{a - bq : q \in \mathbb{Z}\},\$$

has at least one nonnegative element.

- (b) Let r be the minimal nonnegative element of S. Show that $0 \le r < |b|$.
- (c) Use (b) to conclude that a = bq + r for some $q, r \in \mathbb{Z}$ with $0 \le r < |b|$. This proves existence.
- (d) Show that the division with remainder from part (c) is unique. That is, suppose there are $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that

$$a = bq_1 + r_1$$
 and $a = bq_2 + r_2$.

Suppose further that $0 \le r_i < |b|$ for i = 1, 2. Then show $q_1 = q_2$ and $r_1 = r_2$.

- 9. In this exercise we prove the Euclidean algorithm works.
 - (a) Suppose $a, b \in \mathbb{N}$ are two positive integers, and let a = bq + r for $0 \le r < b$ (as in the previous exercise). Show that:

$$gcd(a, b) = gcd(b, r).$$

- (b) Let $a \neq 0$ be an integer. What is gcd(a, 0)? Justify your answer.
- (c) Prove the correctness of the Euclidean algorithm. That is, suppose $a, b \in \mathbb{N}$ are two positive integers, and suppose you iterate the division algorithm as follows:

$$\begin{array}{rcl} a & = & bq_0 + r_0 & 0 \leq r_0 < b \\ b & = & r_0q_1 + r_1 & 0 \leq r_1 < r_0 \\ r_0 & = & r_1q_2 + r_2 & 0 \leq r_2 < r_1 \\ & \vdots & \\ \vdots & & \\ r_{n-2} & = & r_{n-1}q_n + r_n & 0 \leq r_n < r_{n-1} \\ r_{n-1} & = & r_nq_{n+1}. \end{array}$$

Show that $gcd(a, b) = r_n$.

10. Let d be the greatest common divisor of 792 and 275. Using Euclid's algorithm, find d and write d = 792x + 275y for some x and y.