

## Homework Assignment 6

Due Friday, March 5

1. Let  $G$  be a group, and  $M, N \trianglelefteq G$  normal subgroups such that  $MN = G$ .
  - (a) Show  $G/(M \cap N) \cong (G/M) \times (G/N)$
  - (b) Suppose further that  $M \cap N = \{1\}$ . Show that  $G \cong M \times N$ .
2. Let  $G$  be a group and  $Z(G)$  its center.
  - (a) Suppose  $H \leq Z(G)$ . Show that  $H$  is a normal subgroup. (In particular,  $Z(G)$  is normal).
  - (b) Show that if  $G/Z(G)$  is cyclic, then  $G$  is abelian.
  - (c) Let  $p$  and  $q$  be prime numbers (not necessarily distinct), and  $G$  a group of order  $pq$ . Show that if  $G$  is not abelian, then  $Z(G) = \{1\}$ .
3. Let's classify all groups of order 6. To begin, let  $G$  be a nonabelian group of order 6. We will show  $G \cong S_3$ .
  - (a) Show that there is an element  $x \in G$  of order 2. (Once we have Cauchy's theorem for nonabelian groups this part becomes easy, but since  $G$  has 6 elements, one can do this by inspection using Lagrange's theorem).
  - (b) Let  $x \in G$  have order 2, and let  $H = \langle x \rangle$ . Show that  $H$  is not normal in  $G$ . (*Hint*: Show that if  $H$  is normal then  $H \leq Z(G)$ , then apply 2(c) to find a contradiction.)
  - (c) Consider the action of  $G$  on  $A = G/H$  by left multiplication. Show that the associated permutation representation is injective. Conclude that  $G \cong S_3$ .
  - (d) Complete the classification of all groups of order 6 by showing that if  $Z$  is an abelian group of order 6 then  $Z \cong Z_6$ . (*Hint*: We do have Cauchy's theorem for abelian groups.)  
*We've now classified groups of order  $\leq 7$ .*
4. Let  $G$  be a group. Let  $[G, G] = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$ .
  - (a) Show that  $[G, G]$  is a normal subgroup of  $G$ .
  - (b) Show that  $G/[G, G]$  is abelian.

$[G, G]$  is called the *commutator subgroup* of  $G$ , and  $G/[G, G]$  is called the *abelianization* of  $G$ , denoted  $G^{\text{ab}}$ . The rest of this exercise explains why.

  - (c) Let  $\varphi : G \rightarrow H$  be a homomorphism with  $H$  abelian. Show  $[G, G] \subseteq \ker \varphi$ .
  - (d) Conclude that for  $H$  an abelian group there is a bijection:
 
$$\{ \text{Homomorphisms } \varphi : G \rightarrow H \} \iff \{ \text{Homomorphisms } \tilde{\varphi} : G^{\text{ab}} \rightarrow H \}$$

**Hint.** Recall the technique of passing to the quotient described at the beginning of the 2/23 lecture
5. Let's now compute  $D_{2n}^{\text{ab}}$ . We should begin computing  $xyx^{-1}y^{-1}$ . There are 3 cases.
  - (a) Compute  $x^{-1}y^{-1}xy$  in each of the following 3 cases. (*Hint*: HW2#9(e) gives the inverse for a reflection.)
    - (i)  $x, y$  both reflections. So  $x = sr^i$  and  $y = sr^j$ .

- (ii)  $x$  a reflection and  $y$  not a reflection. So  $x = sr^i$  and  $y = r^j$ .
- (iii) Neither  $x$  nor  $y$  are reflections. So  $x = r^i$  and  $y = r^j$ .
- (b) Prove that  $[D_{2n}, D_{2n}] = \langle r^2 \rangle$ . If  $n$  is odd one could choose another generator. What is it?
- (c) Now prove that  $D_{2n}^{\text{ab}}$  is either  $V_4$  or  $Z_2$  depending on whether  $n$  is odd or even. Note that since this is so small we should interpret this as suggesting that  $D_{2n}$  is far from abelian.

For the remainder we will study the quaternion group  $Q_8$ . It is a nonabelian group with very interesting properties.

**Definition 1.** *The quaternion group of order 8, denoted  $Q_8$  is the group of the following 8 elements:*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

*subject to the relations:*

$$\begin{aligned} (-1)^2 &= 1 \\ i^2 &= j^2 = k^2 = -1, \\ (-1)x &= -x = x(-1) \text{ for all } x, \\ ij &= k, & ji &= -k, \\ jk &= i, & kj &= -i, \\ ki &= j, & ik &= -j. \end{aligned}$$

6. Let's start with a few simple facts. Much of this is worked out in the book.
  - (a) Write the entire multiplication table for  $Q_8$ .
  - (b) Find 2 elements which generate all of  $Q_8$ . (*Bonus:* Can you give a presentation of  $Q_8$ ?)
  - (c) Prove that  $Q_8$  is not isomorphic to  $D_8$ .
  - (d) Find all the subgroups of  $Q_8$ , and draw its lattice. (*Hint:* there are 6 total subgroups).
  - (e) Prove that every subgroup of  $Q_8$  is normal.
  - (f) Prove that every *proper* subgroup and quotient group of  $Q_8$  is abelian (*Hint:* TH1#4).
  - (g) Compute  $Z(Q_8)$  and  $Q_8/Z(Q_8)$  (*Hint for the second part:* you can do this by hand, but it might be slicker to apply 2(b)).
7. Now let's follow the proof of Cayley's theorem to exhibit  $Q_8$  as a subgroup of  $S_8$ .
  - (a) Label  $\{1, -1, i, -i, j, -j, k, -k\}$  as the numbers  $\{1, 2, \dots, 8\}$ . Then the action of  $Q_8$  on itself by left multiplication gives an injective map  $Q_8 \rightarrow S_8$ . Write the permutation representations for  $-1$  and  $i$  as elements  $\sigma_{-1}, \sigma_i \in S_8$ , and verify that  $\sigma_i^2 = \sigma_{-1}$ . (Using the multiplication table from question 1 will make this easier).
  - (b) Use the generators from question 1(b) to give two elements of  $S_8$  which generate a subgroup  $H \leq S_8$  isomorphic to  $Q_8$ .
  - (c) Is  $\sigma_i$  even or odd?
  - (d)  $A_8 \cap H$  is isomorphic to a subgroup of  $Q_8$ . Which one?

8. Cayley's theorem says that if  $|G| = n$  then  $G$  embeds at  $S_n$ . One could ask if this  $n$  is *sharp*, or if perhaps  $G$  can embed in some smaller symmetric group. For example,  $D_8$  embeds in  $S_4$  (thinking about symmetries of the square as permutations of the vertices, cf HW3#5). Nevertheless, for  $Q_8$  the symmetric group given by Cayley's theorem is the smallest.
- (a) Let  $Q_8$  act on a set  $A$  with  $|A| \leq 7$ . Let  $a \in A$ . Show that the stabilizer of  $a$ ,  $(Q_8)_a \leq Q_8$  must contain the subgroup  $\{\pm 1\}$ . (*Hint:* The orbit stabilizer theorem might help.)
  - (b) Deduce that the kernel of the action of  $Q_8$  on  $A$  contains  $\{\pm 1\}$ .
  - (c) Conclude that  $Q_8$  cannot embed into  $S_n$  for  $n \leq 7$ . That is, show there is no injective homomorphisms  $Q_8 \hookrightarrow S_n$  for  $n \leq 7$ .