Linear Algebra

Math 217: Saint Lawrence University

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These are growing course notes developed for a first course in Linear Algebra at Saint Lawrence University in Canton, NY. They are written and maintained by Professor Gabriel Dorfsman-Hopkins. The notes have been typset in LaTeXwith a template that was adapted from the Legrand Orange Book Template by Mathias Legrand (legrand.mathias@gmail.com) with modifications by Vel (vel@latextemplates.com).

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1. Introduction

1.1 Sept 3, 2024

1.1.1 Functions

Almost any course in mathematics is centered around studying types of *functions*. For example, in *Calculus* we study the behavior of functions of a single variable, that is, functions whose input is a single real number and whose output is a single real number, looking especially closely at functions which are *continous* or *differentiable*.

■ **Example 1.1 — Functions of a single variable.** Consider the function

$$f(x) = 3x.$$

Its input is a real number, x, and the output is computed by multiplying the input by 3. To see what this function does to a real number, say, 11, we can compute:

$$f(11) = 3 \times 11 = 33.$$

Explicitly, f takes an input of eleven an *transforms it* into an output of 33. ■

■ **Example 1.2** Consider the function:

$$g(x) = x^2 - 2x + 1.$$

What does this function do to the number 2? ■

The study of calculus looks closely at these functions of a single variable, establishing concepts like *derivatives* and *integrals*, and connecting them to many real world questions and situations. A shorthand that we will adopt to describe a function f of a single variable is the following

$$f : \mathbb{R} \to \mathbb{R}$$

This can be read aloud as *f is a function from $\mathbb{R}$ to $\mathbb{R}$*. It signifies that f takes a real number (on the left of the arrow), and runs it through the arrow to produce another real number (on the right of the arrow). *Note: The set before the arrow is called the* ***domain*** *of the function. It is also sometimes called the* ***source****. The set after the arrow is called the* ***co-domain****. It is sometimes also called the* ***target****.*

In *Mulltivariable Calculus* we develop similar ideas, **but the types of functions we study are different**. In particular, we allow for functions which take more than one real number as an input. Allowing for mutli-variable inputs allows calculus to be applied to our multi-dimensional world, and vastly expands the applications of derivatives, integrals, and related ideas.

■ Example 1.3 — Functions of 2 variables. In multivariable calculus you may encounter a function like:

$$f(x,y) = x - y.$$

It takes as input a *pair* of real numbers (x,y), and outputs their difference. For example, to see what the function does to the pair of number $(5,2)$ we can compute:

$$f(5,2) = 5 - 2 = 3.$$

In partiuclar, f will *transform* the pair of numbers $(5,2)$ into the single number 3. ■

■ Example 1.4 — Functions of 3 variables. Consider the function of 3 variables:

$$f(x,y,z) = xyz + 1.$$

What does this function do to the triple $(1,2,3)$? ■

The *arrow notation* of a function introduced above carries over here as well. For example, if f is a function of two variables, (whose input is 2 real numbers) we may write:

$$f : \mathbb{R}^2 \to \mathbb{R},$$

which we read as f *is a function from* $\mathbb{R}^2$ *to* $\mathbb{R}$. Here $\mathbb{R}^2$ denotes the collection of *pairs of real numbers*. Similarly, if g is a function of 3 variables (like in Example 1.4), we may write

$$g : \mathbb{R}^3 \to \mathbb{R}.$$

Notice that for each function we've describe so far, the output is *1-dimensional*. That is, we may have a function into which takes multiple real numbers as an input, but in each case the output is *a single real number*.[1] But just as allowing a multi-dimensional input massively expanded the scope of calculus, allowing functions to have a multidimensional output can be very useful as well.

■ Example 1.5 — Analyzing Ocean Currents. A group of oceanographers are measuring the movement of the water in the Atlantic, by studying where a collection of sensors start and end over the course of two weeks. They compile their data into a function C whose input is the GPS coordinates of a location in the Atlantic, and whose output of where the water at that location ends up 2 weeks later. For example,

$$C(40.47, -68.73) = (41.71, -64.07),$$

[1] You may recall that $\mathbb{R}$ can be thought of as a line, $\mathbb{R}^2$ as a plane, and $\mathbb{R}^3$ as 3-dimensional space. We will eventually adopt this notion of dimensionality, and explore it more carefully.

means that a drop of water whose GPS Coordinates are 40.47N 68.73W will move over the course of two weeks to the location 41.71N 64.07W. Observe that this is a function that takes as input two real numbers, and outputs *2 real numbers* as well! That is, both the input and the output are *2-dimensional*. In our arrow notation, we would write:

$$C : \mathbb{R}^2 \to \mathbb{R}^2.$$

■

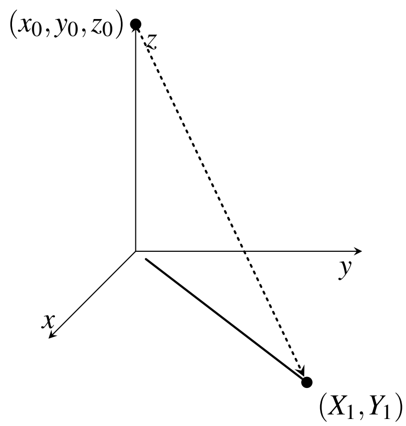
■ **Example 1.6 — Casting Shadows.** Shadows are cast when a body in space blocks the sun from hitting the ground. If we'd like to study the shape of shadows mathematically, it is worth modelling shadows with a function, say S. Here:

$$S(\text{A point in space}) = \text{The spot on the ground where it casts a shadow.}$$

Modelling 3-dimensional space with $\mathbb{R}^3$ and the 2-dimensional ground with $\mathbb{R}^2$, this gives a function:

$$S : \mathbb{R}^3 \to \mathbb{R}^2.$$

In fact, this will be a *projection function*, a certain kind of *linear transformation* that we will study in Section 3.2.4.



■

As we can see, functions with multivariable outputs are not hard to come up with, and model many different situations we would hope to study with mathematics. Let us begin by looking at a very special case:

1.1.2 Functions $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose you wanted to describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. How would you go about it? Both the input and output of f consist of pairs of numbers, so to be explicit with our notation, let's give the first $\mathbb{R}^2$ the coordinates (x, y), and the second $\mathbb{R}^2$ the coordinates (u, v). In particular, our function will look something like

$$f(x, y) = (u, v).$$

The function should be a rule so that, given a pair (x, y) of real numbers, we return with another pair of numbers, (u, v). In particular, we have to say what u is, and what v is. But each of these coordinates depend on both x and y, so in essense this is just *two functions* whose output is a real number:

$$u = u(x,y)$$

$$v = v(x,y).$$

■ **Slogan 1.1** To describe a function whose output is two real numbers, you can give 2 functions which output a single real number each.

Let's see how this works with an example.

■ **Example 1.7** Lets define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ via the rule $f(x,y) = (u,v)$ where:

$$u = u(x,y) = xy + 1,$$

$$v = v(x,y) = x + 2y^2$$

The input of this function is a pair of numbers (x,y), and the output is *another* pair of number (u,v). So, for example, if we feed the function the pair $(-1,3)$, we can compute:

$$u = u(-1,3) = -1 \times 3 + 1 = -3 + 1 = -2$$

$$v = v(-1,3) = -1 + 2 \times 3^2 = -1 + 18 = 17.$$

Therefore, this function transforms the pair $(-1,3)$ to the pair $(-2,17)$:

$$f(-1,3) = (-2,17).$$

■

■ **Example 1.8** Define a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (x,y) to (u,v) via the rule

$$u = u(x,y) = 2x - 2y,$$

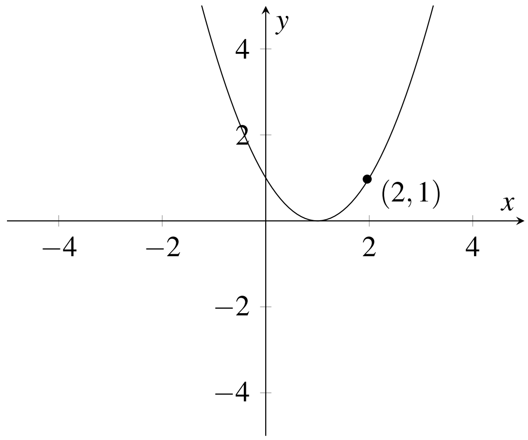
$$v = v(x,y) = \frac{1}{2}x + y.$$

Where does g take the point $(1,1)$? ■

It is often useful to think about a function as something that *moves* the point (x,y) to the point (u,v), and to emphasize this intuition, we will often refer a function $\mathbb{R}^2 \to \mathbb{R}^2$ as a *trnasformation of the plane*.

1.1.3 Visualizing Transformations of the Plane

How do we visualize these types of functions? Since these will be central objects of study, let's start by spending some time developing techniques for how to think about and imagine a function from $\mathbb{R}^2$ to itself. Recall that in calculus you often visualize functions $g : \mathbb{R} \to \mathbb{R}$ using their graphs in the xy-plane. Here the x axis plays the role of the domain, and the y-axis the role of the co-domain, and the graph is generally a curve consisting of the points $(x, g(x))$. For example, the graph of the function $g(x) = x^2 - 2x + 1$ from Example 1.2 is below.



The fact that $f(2) = 1$ is captured by the fact that $(2,1)$ lies on the curve. A similar approach is used in multivariable functions, where now the domain is the entire xy-plane, and the co-domain is the z-axis. Then a function $f : \mathbb{R}^2 \to \mathbb{R}$ can be graphed in 3-dimensional space, coloring in the points $(x, y, f(x, y))$, generally giving rise to a surface in 3-dimensional space.

■ **Question 1.1** Can we take a similar approach to a function $f : \mathbb{R}^2 \to \mathbb{R}^2$? Why or why not?

Given the dimensional constraints, we have to come up with another way to represent a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. One way to do this is to get to the heart of what a function really does: *it transforms a point in* $\mathbb{R}^2$ *to another point in* $\mathbb{R}^2$. In particular, we can think about such a function as *something that transforms the plane*, moving the points of the plane around.

■ **Slogan 1.2** Think about a function $\mathbb{R}^2 \to \mathbb{R}^2$ as something that moves around the points on a single plane. The input (x, y) is where the point starts, and the output $(u, v) = f(x, y)$ is where the point ends.

In fact, this is exactly what the function from Example 1.5 does, it keeps track of of where a drop of water in the Atlantic moves over the course of two weeks!

■ **Example 1.9** Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 1.8. In particular, it is given by the rule $g(x, y) = (u, v)$ where:

$$u = 2x - 2y \text{ and,}$$

$$v = \frac{1}{2}x + y.$$

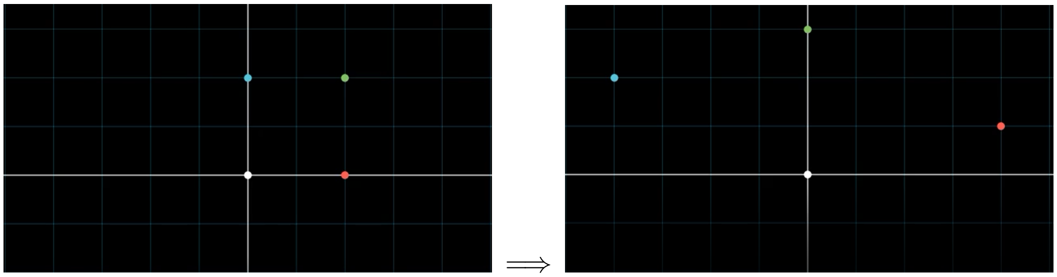
We compute compute where g takes the four points: $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. For example, $g(0,0)$, we may compute the u coordinate to be $2 \times 0 - 2 \times 0 = 0$ and the v coordinate to be $\frac{1}{2} \times 0 + 0 = 0$, so that $g(0,0) = 0$. Similar computations show that:

$$g(1,0) = (2, 0.5)$$

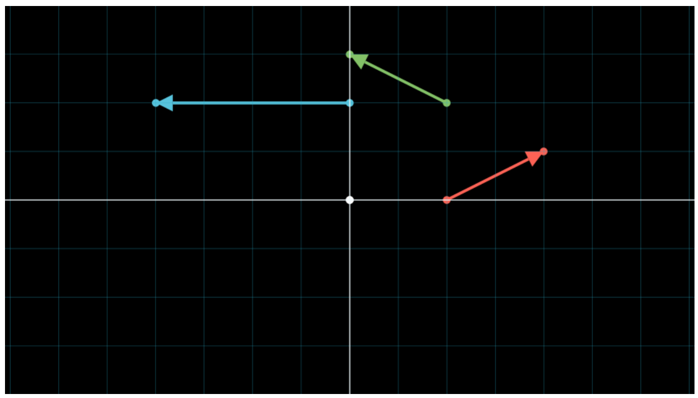
$$g(0,1) = (-2, 1)$$

$$g(1,1) = (0, 1.5).$$

Plotting the points before and after applying g gives:



Plotting the before and after on the same plane, connecting (x,y) with $g(x,y)$ using arrows gives the following picture.



We should imagine this function something *moving around the points on the plane*, a perspective that is emphasized when animating the function. You find an animation of the moving points below.[2]. Try to give a qualitative description of what this function is doing to the plane. Plotting moer points may give a better picture. ■

■ **Example 1.10** Let's visualize the function from Example 1.7, which was function $f(x,y) = (u,v)$ where:

$$u = u(x,y) = xy + 1,$$

$$v = v(x,y) = x + 2y^2$$

To get a sense of what kind of movement, let's keep track of what happens to a few points:

$$(0,0), (1,0), (0,1), (1,1).$$

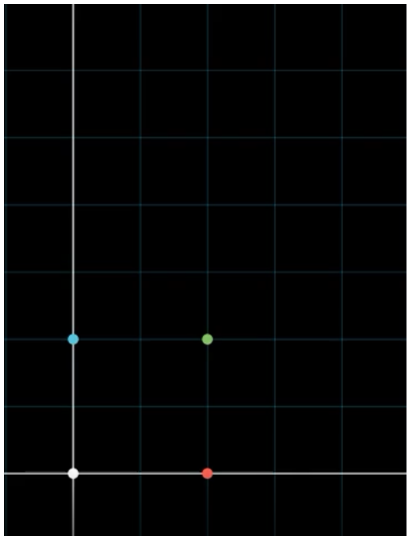
Using the formulas we can compute where f takes these points, just like in Example 1.7.

$$f(0,0) = (1,0), \quad f(1,0) = (1,2),$$

$$f(0,1) = (1,1), \quad f(1,1) = (2,3).$$

Instead of a single graph of the function, we can represent what f does with two pictures of the plane, a *before* shot and an *after* shot. On the left, we see the 4 points before applying f, and on the right, we see them after.

[2] www.gabrieldorfsmanhopkins.com/LinearAlgebraNotesFall24/animationsAndTools/LinearAnimation.mp4



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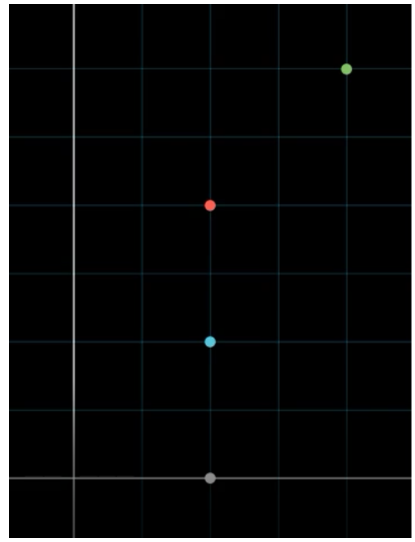
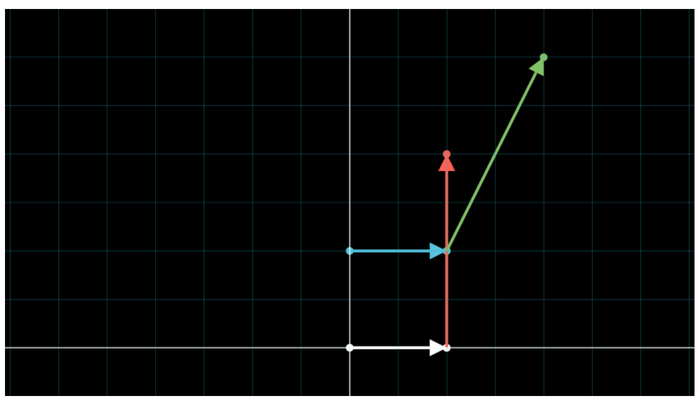


The *movement* of the situation can be captured nicely by an animation linked below.[3] You can also emphasize that it is movement on a single page by using arrows that point from the start to the finish of the various points:



■

Exercise 1.1 Consider the transformation $L(x,y) = (u,v)$ of the plane $\mathbb{R}^2$, given by the following two equations:

$$u = u(x,y) = y$$

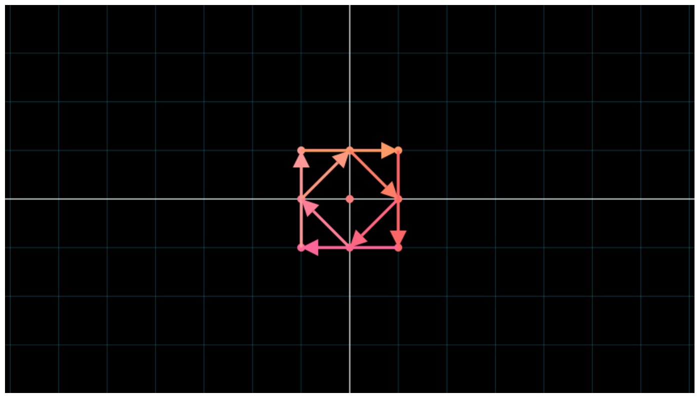
$$v = v(x,y) = -x.$$

On a single coordinate plane, draw what the function does to a number of points. Do this by plotting a point (x,y), its image $L(x,y)$, and connecting them with an arrow. Use a few sentences to describe what the transformation L is doing to the plane. This can be a *qualitative* description. *What does it look like is happening?* ■

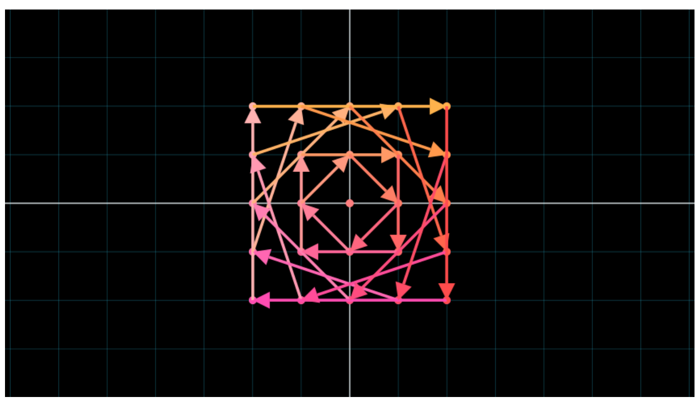
[3] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotesFall24/animationsAndTools/QuadraticAnimation.mp4`

1.2 Sept 5, 2024

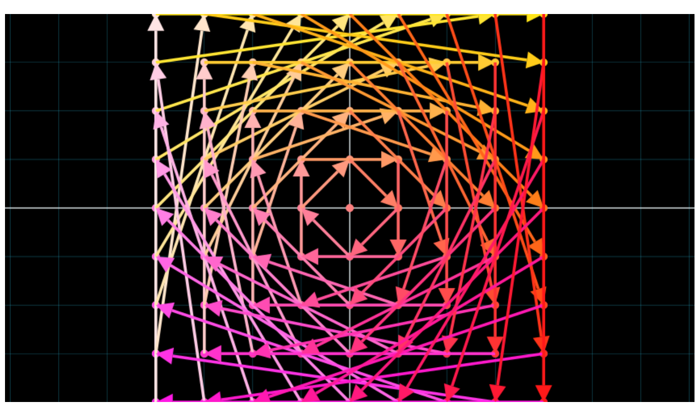
■ **Example 1.11** At the end of our last class, we visualized the function $L(x,y) = (y,-x)$ (cf. Exercise 1.1). Let's draw a few pictures and see if we can arrive at a description of what is happening to the plane. First, we plot all the points whose x and y coordinate's are between -1 and 1, connecting the points before and after applying L with an arrow.



Can you begin to describe what L is doing to the plane? Let's throw in a few more points, now letting the coordinates range between -2 and 2.



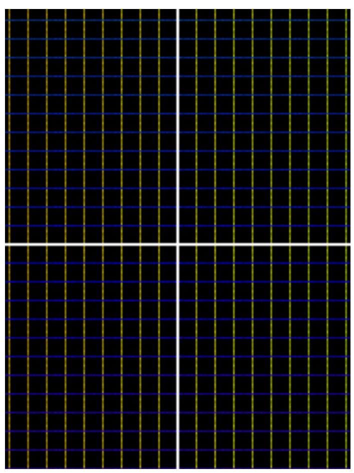
As you can see, it appears that L is *rotating the plane clockwise!*. We include one more image now with coordinates ranging between -5 and 5.



Althought the image is starting to get cluttered, this definitely appears to be a rotation, and indeed, replacing the arrows with an animation makes this clear (see the animation below[4]). ■

To summarize, plotting where points go under a function can give a sense qualitative sense of how a function moves the plane. That said, Examples 1.10 and 1.9 suggest that only drawing where a few points go gives an incomplete picture. On the other hand, as we saw at the end of Example 1.11, if we to fill in more and more points, the image can start to get cluttered and it may become difficult to infer much from the picture. [5] That being said, if you carefully pick which points to keep track of, you can get a nice sense of the *geometric* properties of a function. One way to do this, is by keeping track of what the function does to the *gridlines* of the plane.

■ **Example 1.12** To get a better picture of the function g from Examples 1.8 and 1.9, let's analyze what it does does to the gridlines of the plane.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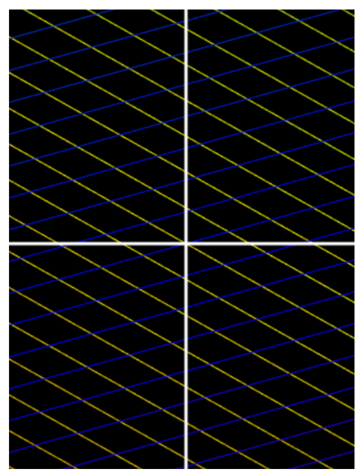


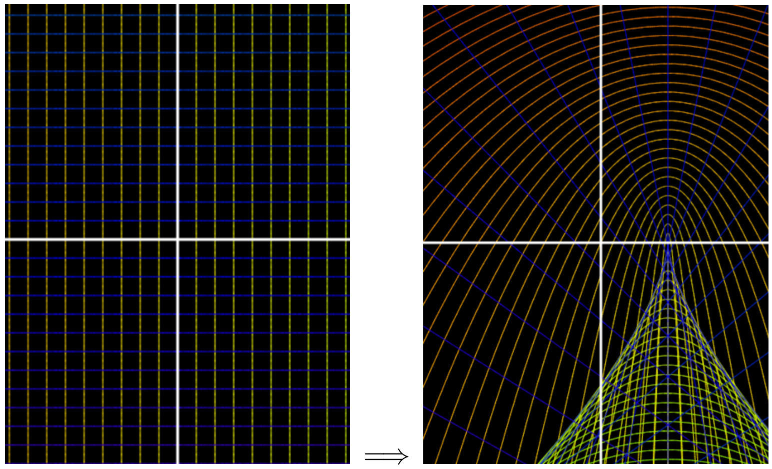
One can really get a sense for how g *moves* the plane by playing around with the tool linked below[6]. In particular, we see that it sort of *stretches* and *rotates* the plane, distorting it slightly but not too much. In this course we will develop a vocabulary to mathematically describe terms like *stretching the plane*, and ways to extract that information from the equations given in Exampel 1.8, but for now we're trying to get a qualitative sense of what's going on. ■

■ **Example 1.13** Let's also look at what the function f from Example 1.7 does to the gridlines of the plane.

[4] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotesFall24/animationsAndTools/RotationAnimation.mp4`

[5] *Try this! For some functions you can actually get a nice picture! In fact, the situation in Example 1.11 is a particularly nice one. In general it will be much more complicated*

[6] Click the *linear* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotesFall24/animationsAndTools/Gridlines/index.html`



The animation is actually quite nice to look at[7]. ■

It is fair to say that the function in Example 1.13 appears far more complicated than the one in Example 1.12. In fact, in some sense it is complicated in a way that puts it beyond the purview of *linear algebra*[8] For the context of linear algebra, we will have to restrict ourselves to functions more like that of Example 1.12, functions that we will call *linear transformations*. Before describing exactly what these are, it might be worth while to ponder the following question. Qualitative answers are always welcome!

■ **Question 1.2** What are some differences between what happens to the gridlines in the two examples on the previous page?

1.2.1 Linear Transformations of the Plane

One answer to Question 1.2 could be: *In example 1.8 the gridlines remain as lines after applying g, but in Example 1.7 the gridlines become curvy*. This is a good observation. Recall that lines played a special role in calculus. Not only where they the simplest functions, we used them to model more complicated functions locally, by taking *tangent lines*. We do something similar in multivariable calculus, modelling more complicated functions with linear ones by taking the *tangent plane*. Not only were these functions simple *geomterically* (being lines and planes), but they were also simple *algebracially*. For example, a line usually has the following equation:

$$f(x) = \color{red}{mx} + \color{blue}{b}.$$

Above we highlighted the *linear term* in red, and the *constant term* in blue. Similarly, a plane had a simple equation as well:

$$h(x,y) = \color{red}{mx} + \color{red}{ny} + \color{blue}{b},$$

where again the linear terms are highlighted in red, and the constant term in blue. Looking at the function $g(x,y) = (u,v)$ from Example 1.8, we see that the equations for both u and v have only linear terms (and no constant terms).

$$u = u(x,y) = 2x - 2y,$$

[7] Click the *Quadratic* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotesFall24/animationsAndTools/Gridlines/index.html`

[8] This is the kind of function studied in *algebraic geometry*.

$$v = v(x,y) = \frac{1}{2}x + y.$$

This will turn out to be a good definition for a linear function.

Definition 1.2.1 — Linear Transformations of the Plane. A *linear transformation of the plane*, also called a *linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $L(x,y) = (u,v)$, where u and v are given by linear equations with no constant term:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

where a, b, c, and d are real numbers.

■ **Warning 1.1** A linear transformation is not quite the same as a linear function from Calculus, because a linear function from calculus can have a constant term, and a linear transformation cannot. This is an unfortunate inconsistency in terminology, but perhaps you can think about a linear transformation as being more *purely linear* since the only terms it has are linear terms, and no constant terms.

■ **Warning 1.2** In light of Question 1.2, you may want a geometric definition of a linear transformation of the plane to be something like: *it takes gridlines to lines*. This isn't quite the case (we will see some examples of this). To be completely precise, we also need the gridlines to remain parallel and evenly spaced, and we need $L(0,0) = (0,0)$. We will discuss this geometric reformulation more later, but for now I just wanted to mention that a this first guess is not quite enough.

You might be getting this far and thinking *wait...I thought linear algebra was about matrices? Where do those fit in?* This is a good question, so let's give a preliminary answer. Take a linear tranformation $L(x,y) = (u,v)$ where:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

This function is completely determined by the coefficients of x, and the coefficeints of y. That is, to know L, it is enough to know a, b, c, and d. So, we can completely capture all the data for L in the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For now we should just think of a matrix as a rectangular array of numbers, so that a linear transformation of the plane corresponds to a 2×2 matrix.

Definition 1.2.2 The matrix associated to the linear transformation in Definition 1.2.1 is the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■ **Example 1.14** Consider the function $g(x,y) = (u,v)$ from example 1.8. Observe that the coefficient of y in the first equation is -2, because adding $-2y$ is the same as subtracting $2y$. Also, the coefficient of y in the second equation is a 1 becuase $y = 1 \times y$.

$$u = u(x,y) = 2x + -2y,$$

$$v = v(x,y) = \frac{1}{2}x + 1y.$$

The matrix associated to this fuction is therefore:

$$\begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1. \end{bmatrix}$$

■

This correspondance goes in both directions. That is, given a matrix, you can extract a linear function.

Definition 1.2.3 Consider a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The *linear function associated to this matrix* is the function $L(x,y) = (u,v)$ where:

$$u = ax + by \text{ and},$$

$$v = cx + dy.$$

Let's run through an example of applying a function, given only a matrix.

■ **Example 1.15** We compute $T(1,-2)$ where $T(x,y)$ is the function associated to the matrix

$$\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Applying the definition we see that $T(x,y) = (u,v)$ where:

$$u = 3x + 1y = 3x + y,$$

$$v = -1x + 0y = -x.$$

Plugging in $(x,y) = (1,-2)$ gives:

$$u = 3 \times 1 + (-2) = 1, \qquad \text{and} \qquad v = -1$$

Therefore $T(1,-2) = (1,-1)$. ■

R Later we will streamline this process using *matrix multiplication*.

■ **Example 1.16** Let f be the function from Example 1.7 and g the function from Example 1.8. Let's define an addition rule for points in $\mathbb{R}^2$ as follows:

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1).$$

Let $P = (4,2)$ and $Q = (2,5)$, so that $P+Q = (6,7)$. Observe that

$$g(P) = (4,4), g(Q) = (-6,6), \text{ and } g(P+Q) = (-2,10),$$

so that $g(P)+g(Q) = g(P+Q)$. On the other hand:

$$f(P) = (9,12), f(Q) = (11,52), \text{ and } f(P+Q) = (43,104),$$

so the analogous formula doesn't hold for f. It appears that linear functions play well with addition, while other functions do not! ■

Exercise 1.2 Consider the matrix:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate L at $(0,0),(0,1),(1,0)$ and $(1,1)$.
3. Plot the four points of part (b), before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

So far we've only seen how a correspondance between linear transformations of the plane and 2×2 matrices. We will work out in the coming weeks how this fits in to notions of matrix multiplication, determinants, and other matrix operations. For now, we the main take away should be the following.

■ **Slogan 1.3** A matrix is a function.

1.3 Exercises

Exercise 1.3 Consider the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate L at all the points with integer coordinates are between -1 and 1. (There should be nine such points).
3. Plot the 9 points from part (b) before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

Exercise 1.4 Consider the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $T(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate T at 5 points of your choice.
3. Plot the 5 points from part (b) before and after applying T, and connect them with arrows.
4. Give a qualitative description of what you think T is doing to the plane.

■

Exercise 1.5 For this problem, adopt the notation of Exercises 1.3 and 1.4. Also consider the matrix N associated to the function $g(x,y)$ from Example **??**:

$$N = \begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1 \end{bmatrix}$$

1. Can you identify any relationships between the outputs $L(1,0), L(0,1)$ and the matrix M?
2. Can you identify any relationships between the outputs $T(1,0), T(0,1)$ and the matrix I?
3. Can you identify any relationships between the outputs $g(1,0), g(0,1)$ and the matrix N?
4. Now let's treat the general case: let $\ell(x,y)$ be a linear transformation associated to a general matrix:

$$P = \begin{bmatrix} a & b \\ c & d. \end{bmatrix}$$

Describe the relationship between $\ell(1,0)$, $\ell(0,1)$ and the matrix P. Give reasoning for your answer.

■

Exercise 1.6 Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Do you agree or disagree with the following statement?

Once I know $\ell(1,0)$ and $\ell(0,1)$, I can determine $\ell(x,y)$ for any pair (x,y).

Explain your reasoning. ■

Exercise 1.7 Consider a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, and suppose that $F(0,0) = (1,1)$. Is it possible for F to be a linear transformation? Why or why not? ■

Exercise 1.8 Adopt the notation of Problem 1.5. Define a rule for adding two points as follows:

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1).$$

Let $P = (1,2)$ and $Q = (3,-2)$.
1. Can you identify any relationship between $L(P), L(Q)$, and $L(P+Q)$?
2. Can you identify any relationship between $g(P), g(Q)$, and $g(P+Q)$?
3. To see that it's not a fluke, do parts (a) and (b) again, but with new points P and Q of your choice.
4. Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a general linear transformation, and let P and Q be two random points

in $\mathbb{R}^2$. Make a conjecture for the relationship between $\ell(P)$, $\ell(Q)$, and $\ell(P+Q)$. (There is no need to prove this yet, but you can extrapolate from the evidence collected in (a) through (c)).

Exercise 1.9 To give a function $\mathbb{R}^2 \to \mathbb{R}^2$ we needed to give 2 functions of 2 variables which output a single number each (for more detail, see Section 1.1.2 in the course notes). Let's see if we can work out what to do in higher dimensions. In particular, adapt Section 1.1.2 in the course notes to describe a function $F : \mathbb{R}^3 \to \mathbb{R}^3$. You can make up any funcion you like, just make sure that you describe it fully. Evaluate this function at the points $(0,0,0), (1,0,0)$, and $(1,2,3)$.

2. Vectors

Acknowledgment: I'd like to attribute this approach to vectors in part to Grant Sanderson, author of the delightful youtube channel `3Blue1Brown`. In particular, I borrow heavily his description of the three perspectives of vectors presented below as the *physicist's perspective*, the *computer scientist's perspective*, and the *mathematician's perspective*.

2.1 September 10, 2024

We suggested in the Introduction that the field of Linear Algebra is centered around the study of *linear transformations*. Furthermore, Exercise 1.8 suggests that a linear transformation L satisfies the following equation:[1]

$$L(P+Q) = L(P) + L(Q).$$

It is worth taking some time to unpack what $+$ is doing here. If P and Q are points, what is their sum? In Exercise 1.8, we defined the sum of 2 points to be a third point, whose coordinates correspond to adding the coordinates of P and Q. In $\mathbb{R}^2$ this is written as follows:

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1).$$

This is a perfectly valid formula, but it may also seem a bit strange. In particular, we should unpack a concrete interpretation of this algebraic operation to answer the following question:

■ **Question 2.1** What exactly is the meaning of adding coordinates of points in $\mathbb{R}^2$?

By trying to answer this question, we naturally encounter the notion of a *vector*. In fact, a first definition of a vector is pretty much exactly the idea of *points you can add*.

[1] At least when $L : \mathbb{R}^2 \to \mathbb{R}^2$

Definition 2.1.1 — 2-dimensional vectors: the computer scientist's perspective. A two dimensional vector is an array of two numbers aligned vertically:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

These can be added coordinatewise:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix}.$$

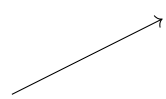
Notation 2.1. *Aligning vectors* vertically *allows them to fit into the matrix theory we will develop in the coming weeks. We will sometimes use the term* column vector *when writing it this way.*

We call this *the computer scientist's perspective*, because it remembers a vector as a light-weight data type stored in a way that easily allows for vector operations (like addition and applying linear maps) to be computed efficiently by a computer in almost any programming language. This perspective also has the advantage of generalizing very easily to higher dimensions, *can you see how?* That being said, it doesn't really get us any closer to answering Question 2.1. In order to do this, we give another perspective on vectors you may have seen in a physics course.

Definition 2.1.2 — 2-dimensional vectors: the physicist's perspective. A two dimensional vector is a quantity specifying a *magnitude* together with a *direction* in the two dimensional plane. This can be represented by an arrow, pointing in the given direction, whose length is the given magnitude.

The physicist's vectors can be added too, essentially by *concatenating the two arrows*. We define this more carefully in Definition **??** below. In particular, we have encountered two different perspectives on the notion of a vector. Below to the left is an example of a computer scientist's vector, and to the right is an example of a physicist's vector.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

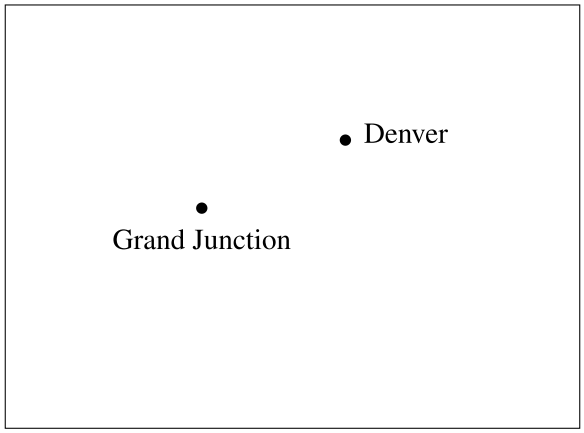


In fact, one could say that these are two perspectives on the same vector, *can you explain why?*. An important part of linear algebra involves learning how to pass seemlessly between these two perspectives, as one lends itself better to computations, while the other lends itself better to interpretations. In this section, we will explore these two perspectives, and start developing a dictionary between them, keeping track of what information can get lost in translation. Along the way we will extract algebraic properties of vectors, and have a first encounter the notions of *linear combinations* and *spans*, which are among the most important in this course.

R Another thing we can do from both perspectives is *scale* vectors by numbers. In fact, there is a third perspective on vectors, which we can call *the mathematician's perspective*, which essentially defines vectors as: *theoretical objects which can be added together and scaled*. We will postpone discussion of this third, more abstract, perspective until the end of the semester. The attentive reader may want to pay attention to how most properties of vectors can be expressed in terms of of these two operations (addition and scaling).

2.1.1 The Physicist's Perspective

A vector is a natural quantity to describe relationships between objects in the physical world. For example, suppose that a pilot is hoping to fly from Denver, Colorado to Grand Junction, Colorado, and asks you for directions.[2]

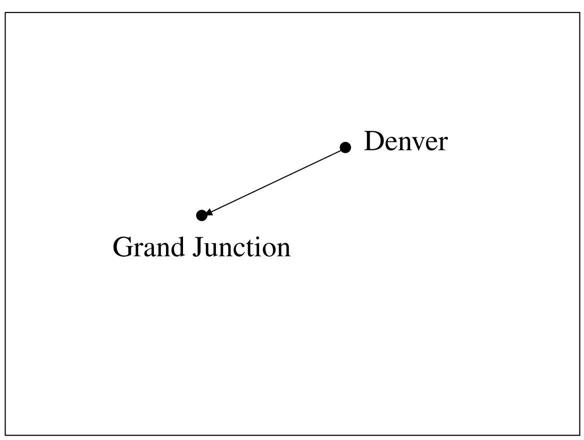


The distance between Denver and Grand Junction is 220 miles. That be but it is *not enough* to tell the pilot to fly 220 miles. If they don't want to end up in Wyoming, they must also know which direction to fly. So for example, you may tell the pilot to fly 220 miles *west-southwest*. These two peices of information constitute a quantity which we call a *vector*:

Magnitude: 220 miles,

Direction: West-Southwest.

All of the defining this quantity information can be exhibited on a map, by drawing an arrow from the start to the finish. One can recover the direction from the direction of the arrow, and the magnitude from the length of the arrow.



This gives us another way to think about Definition 2.1.2.

[2]Map not to scale.

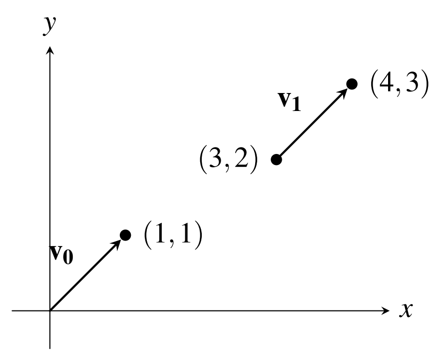
Definition 2.1.3 [Vectors as Displacement] Given two points P and Q, the *displacement vector from P to Q* is the arrow whose tip is at the point Q whenever its tail is placed on P.

Notation 2.2. *Given a vector* $\mathbf{v}$, *we denote its magnitude by* $||\mathbf{v}||$.

With this definition we think of a *displacement vector* as something you can apply to a point. To apply a vector to a point P, we put its tail at P, and the output is wherever its tip points. Importantly, the same vector can be placed in different locations. Let us take this perspective to our vector which takes us from Denver to Grand Junction. This vector is completely determined by the fact that it goes 220 miles west-southwest. It isn't necessary for its to lie on Grand Junction. For example, we could apply the *same vector* starting from Canton, and we would end up somewhere near Toronto. We would still be using the same magnitude (220 miles) and direction (west-southwest), and therefore following the same vector. This is an important point: *a vector is determined by magnitude and direction*. 2 vectors of the same length and pointing in the same direction are the same vector, even if they are drawn at different places.

■ **Slogan 2.1** The vector is the arrow, not where the arrow is.

■ **Example 2.1** Consider following vectors. $\mathbf{v_0}$ connects $(0,0)$ and $(1,1)$, while $\mathbf{v_1}$ connects $(3,2)$ and $(4,3)$. Does $\mathbf{v_0} = \mathbf{v_1}$?



■

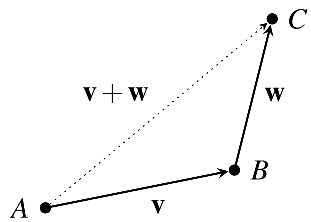
It is common in linear algebra to distinguish between *vectors*, which have a direction and a magnitude, numbers without any direction. The latter is just a number, but we will also often refer to it as a *scalar*.

Adding and Scaling Arrow Vectors

Recall that Question 2.1 asked what adding vectors meant. By thinking of vectors as measuring displacement, we can get a geometrically and physically meaningful understanding how they add, subtract, and scale. We will explore this with the following thought experiment in mind:

You are programming autonomous vehicles. To command a vehicle to move, you give it a vector. The vehicle will then move along the vector: in the given direction for the given magnitude.

Addition: Suppose you give your vehicle a vector $\mathbf{v}$ to follow, and it moves from point A to point B. Once it has arrived at point B, you give it another vector $\mathbf{w}$, and it moves from point B to point C. At this point, the net displacement that the vehicle has travelled is from point A to point C. Define $\mathbf{v}+\mathbf{w}$ to be this displacement vector from A to C.



We can summarize in the following definitoin.

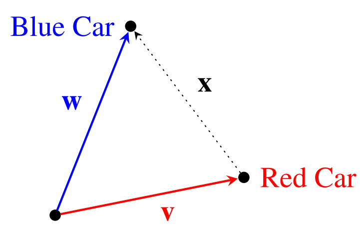
Definition 2.1.4 The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$, is the combined displacement resulting from first applying $\mathbf{v}$, and then applying $\mathbf{w}$ to the result.

■ **Question 2.2** Given 2 vectors $\mathbf{v}$ and $\mathbf{w}$, is it always true that

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}?$$

With addition now defined, we can move on to:

Subtraction: Suppose you control two vehicles, a red vehicle and a blue one, both starting at the same point. You send the red one along vector $\mathbf{v}$, and send the blue one along vector $\mathbf{w}$. After they arrive, the blue vehicle breaks down, so you must send the red vehicle to rescue it. What vector $\mathbf{x}$ must you command the red car to follow?



In particular, if the red car first does $\mathbf{v}$, and then does $\mathbf{x}$, it should overall be following $\mathbf{w}$, and therefore should end up alongside the blue car. We translate this by Definition 2.1.4 to the statement,

$$\mathbf{w}=\mathbf{v}+\mathbf{x}.$$

If subtraction of vectors were to make any sense, then we could subtract $\mathbf{v}$ from both sides and discover that $\mathbf{x}$ really should be the difference of $\mathbf{w}$ and $\mathbf{v}$:

$$\mathbf{w}-\mathbf{v}=\mathbf{x}.$$

Therefore, that is the definition we will make.

Definition 2.1.5 The difference $\mathbf{w}-\mathbf{v}$ of two vectors $\mathbf{w}$ and $\mathbf{v}$, is the vector which, when added to $\mathbf{v}$, gives $\mathbf{w}$.

We can now add and subtract vectors in a geometrically meaningful way. Bringing us closer to getting meaningful answer to Question 2.1. Before moving on, though, we'd like to introduce a special vector.

The Zero Vector: If your vehicle doesn't move at all, what vector does it follow? Since magnitude is the net distance covered, the magnitude is 0. As for direction, this isn't really well defined, since if you move 0 units in any direction, you've stayed put. We will call the vector from a point to itself the *zero vector*.

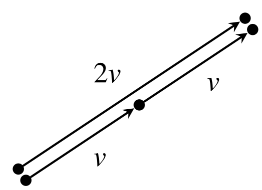
Definition 2.1.6 The vector whose magnitude is zero is called the *zero vector*, and is denoted $\mathbf{0}$.

R The zero vector is the only vector whose direction is unspecified.

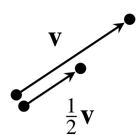
Exercise 2.1 Let $\mathbf{v}$ be any vector. What is $\mathbf{v}+\mathbf{0}$? What about $\mathbf{v}-\mathbf{v}$? ■

Let's introduce one more important operation that vectors allow.

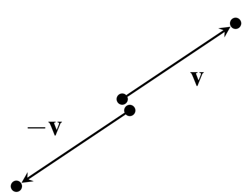
Scalar Multiplication: Suppose you'd like to send your car in the same direction as a vector $\mathbf{v}$, but twice as far as the vector $\mathbf{v}$ allows. You could acheive this by applying $\mathbf{v}$, and then doing so again. With Definition 2.1.4 in mind, you send your car along $\mathbf{v}+\mathbf{v}$, which we can write as $2\mathbf{v}$.



Similarly, if you wanted your car to go in the same direction as $\mathbf{v}$, but half as far, you could follow a vector $\mathbf{w}$ which satisfied $\mathbf{v}=\mathbf{w}+\mathbf{w}$. Since $2\mathbf{w}=\mathbf{v}$ we could reasonably say that $\mathbf{w}=\frac{1}{2}\mathbf{v}$.



Alternatively, suppose you wanted to go the same distance as $\mathbf{v}$, but in the opposite direction. You could follow a vector $\mathbf{x}$. Notice that if the car first does $\mathbf{v}$ and then does $\mathbf{x}$, it will travel along $\mathbf{v}$, and move the same distance in the opposite direction until it gets back to where it started. In particular, we have that $\mathbf{v}+\mathbf{x}=\mathbf{0}$, so it is reasonable to write $\mathbf{x}=-\mathbf{v}$.



Following this logic we can deduce that to scale a vector by a positive number, you scale its magnitude. The negative of a vector reverses direction. What about scaling by a negative number? The following formula should shed some light.

$$-2\mathbf{v}=-(2\mathbf{v}).$$

It appears that to scale by negative 2, you can first scale by 2, and then reverse direction.

Exercise 2.2 With $\mathbf{v}$ as in the figures above, sketch $-2\mathbf{v}$ ■

We can put all this together into the following definition.

Definition 2.1.7 Let $\mathbf{v}$ be a vector and c any scalar. Then the vector $c\mathbf{v}$ is defined by the following data.

- If c is positive, the direction of $c\mathbf{v}$ is the same as $\mathbf{v}$. Otherwise, the direction of $c\mathbf{v}$ is opposite to that of $\mathbf{v}$.
- The magnitude of $c\mathbf{v}$ is:

$$||c\mathbf{v}|| = |c| \cdot ||\mathbf{v}||,$$

where $|c|$ denotes the absolute value of the scalar c.

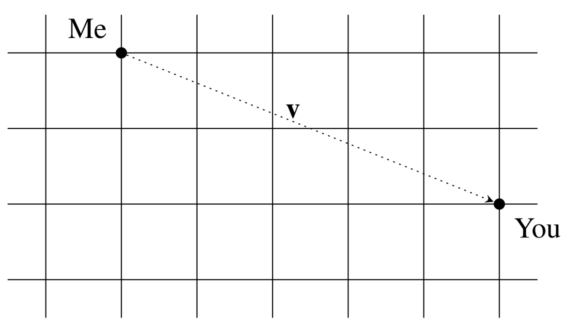
Exercise 2.3 Let $\mathbf{v}$ be any vector. What is $0\mathbf{v}$? ■

A nice output of this geometric approach is that we can give geometric names to certain algebraic operations. For example, we should call 2 vectors parallel, if when we draw their arrows are parallel as lines segments. Our definition of scalar multiplication tells us that this is equivalent to one being a scalar multiple of the other, giving us a algebraic notion of parallel-ness (that can and will extend to higher dimensions).

Definition 2.1.8 Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if there is some constant c such that $\mathbf{v} = c\mathbf{w}$.

2.1.2 Decomposing a Vector into Components

We'd like to connect the ideas described in the *physicist's perspective* on vectors, to the ordered pair numbers which we called the *computer scientist's perspective*. To do this, we'll take the example of giving directions in a city.



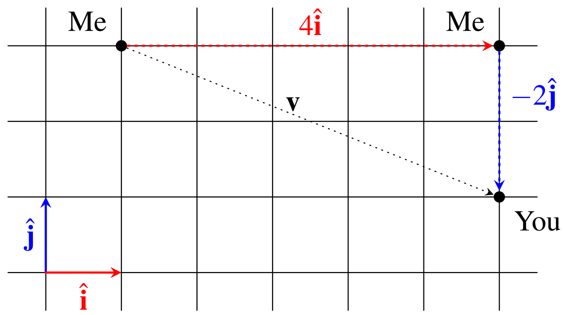
Above is a map depiction of a city, with the thick vertical and horizontal lines roads. You'd like to tell me how to find you. In particular, you need to tell what vector $\mathbf{v}$ I need to follow, in order to get to your location. One way to describe this to me is to give a magnitude and direction. But it is unlikely that you'll tell me something like *go about 530 meters in a direction that is mostly east but somewhat south*. In fact, even if you were more precise with the angles and distance, it is unlikely that I would be able to follow the directions (without walking through buildings).

Instead, you'd probably say something like *walk 5 blocks east, and then 2 blocks south*. Indeed, the regular gridlines of the city give us two natural vectors which we can all agree on:

$$\hat{\mathbf{i}} = \text{ one block east,}$$

$$\hat{\mathbf{j}} = \text{ one block north.}$$

With this in hand, everyone can agree on a set of navigational rules. Let's put them on our plot.



So we can see that in order to get to you, I first have to follow $4\hat{\mathbf{i}}$ and then $-2\hat{\mathbf{j}}$. In particular:

$$\mathbf{v} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}}.$$

Once we all agree on a definition of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we can represent the vector $\mathbf{v}$, just in terms of numbers 4 and 2. This is what the computer scientist would call:

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

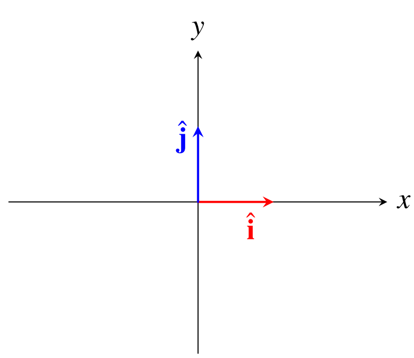
R It is important here that we all agree on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. One can imagine a city which is not grided parallel to north an south, and instead has *updown* and *downtown* directions. Then you may represent a vector using the coefficient of $\hat{\mathbf{j}}$ to describe how many units it goes uptown or downtown, but someone else might represent the vector using the coefficient of $\hat{\mathbf{j}}$ to represent how many units it goes north or south. The coordinates the computer scientist would write down would be different in each case. This is an important subtlety, but one that we will table until we are discussing *bases* and *change of bases*. For now, just remember that the coordinates that you might right down for a vector depend on a *choice* of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. This is one advantage of the physicist's perspective over the computer scientist's perspective.

The general setup (in 2-dimensions) is essentially the same

Definition 2.1.9 Given a cartesian coordinate system (that is, an xy-plane with coordinates), we can define the *standard basis vectors* $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to be the vectors:

$$\hat{\mathbf{i}} = \text{ one unit in the positive } x \text{ direction},$$

$$\hat{\mathbf{j}} = \text{ one unit in the positive } y \text{ direction}.$$



Given any vector $\mathbf{v}$ in the coordinate plane, we can *resolve* it into its components:

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}}.$$

Here v_1 is a *scalar*, representing how far $\mathbf{v}$ goes in the x-direction, and v_2 is a *scalar* representing how far $\mathbf{v}$ goes in the y-direction. They are unique.

Notation 2.3. *Given a vector in component form:*

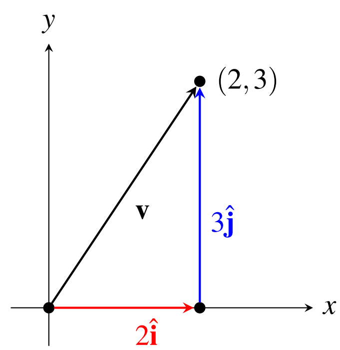
$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}},$$

we call the coefficients v_1 and v_2 the components *of* $\mathbf{v}$. *We can take these components and write* $\mathbf{v}$ *as a* column vector*:*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

It is important to note that the coordinates of the column vector depend on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. In particular, we should think of this column vector as meaning, first do $v_1\hat{\mathbf{i}}$, then do $v_2\hat{\mathbf{j}}$.

■ **Example 2.2** Consider the displacement vector $\mathbf{v}$ from $(0,0)$ to $(2,3)$.



Therefore we see that:

$$\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

■

Notice that the displacement vector from $(0,0)$ to (x,y) can always be written

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 2.4 Consider the vectors $\mathbf{v_0}$ and $\mathbf{v_1}$ from Example 2.1. Write them both in component form and as column vectors. Use your result to decide whether they are equivalent. ■

Exercise 2.5 Let $P = (x_0, y_0)$, $Q = (x_1, y_1)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ in component form and as a column vector, in terms of the coordinates of P and Q. ■

Exercise 2.6 For problems 1 through 3, fix the following vectors:

$$\mathbf{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} \qquad \mathbf{w} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}.$$

1. Suppose you start at the origin, first travel along the displacement vector $\mathbf{v}$, and then travel along the displacement vector $\mathbf{w}$. Sketch your overall path to determine your endpoint. Use this to write down the vector $\mathbf{v} + \mathbf{w}$ in terms of its components.
2. Suppose you start at the origin and travel in the direction of $\mathbf{v}$, but twice as far. Sketch the path you travel to determine your endpoint. Use this to write down the vector $2\mathbf{v}$ in terms of its components.
3. Suppose you start at the origin and travel along the displacement vector $\mathbf{v}$, and then along the displacement vector $-\mathbf{w}$. Sketch the path you travel to determine your endpoint. Use this to write down the vector $\mathbf{v} - \mathbf{w}$ in terms of its components.
4. Now fix generic vectors

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{w} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}}$$

   and let c be a scalar. Use the intuition developed on the previous page to write down formulas for $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, and $c\mathbf{v}$.

■

Exercise 2.7 Let $P = (-1, 2)$ and $Q = (3, 4)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ as a column vector. ■

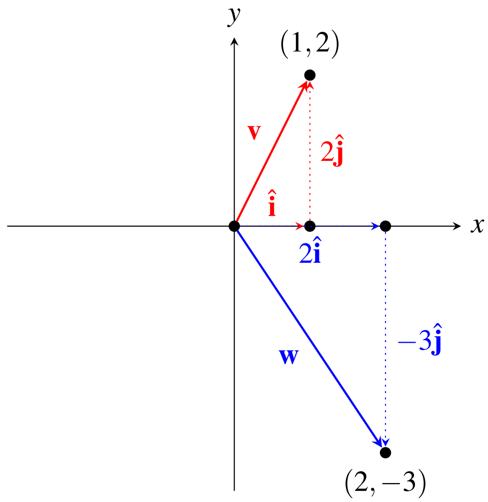
Exercise 2.8 Now let $P = (x_0, y_0)$ and $Q = (x_1, y_1)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ as a column vector (in terms of x_0, x_1, y_0, and y_1). ■

In Exercise 2.6, we explored what addition, subtraction, and scalar multiplication looked like when we put vectors in component form. Let's start by working through an example summarizing our observations.

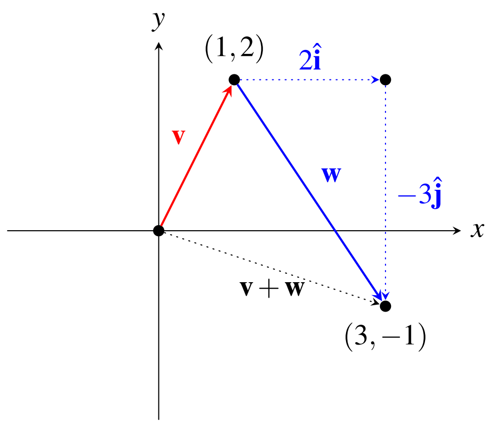
■ **Example 2.3** Consider the following two vectors given in component form:

$$\color{red}{\mathbf{v}} \color{black}{= \hat{\mathbf{i}} + 2\hat{\mathbf{j}}} \qquad \color{blue}{\mathbf{w}} \color{black}{= 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}.}$$

Let's plot them both starting at the origin.



Let's add them together! To do this, we move the tail of $\mathbf{w}$ to the tip of $\mathbf{v}$, so that we can see what happens when you iterate them.



In particular, we have computed that: $\mathbf{v}+\mathbf{w} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Let's compare this to adding together these two vectors *as computer scientists*. The column vector forms of $\mathbf{v}$ and $\mathbf{w}$ are (as in Notation 2.3):

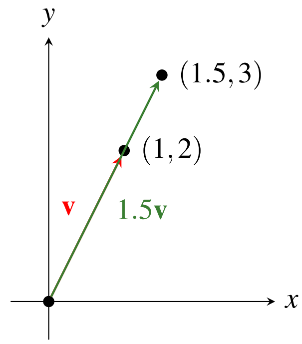
$$\mathbf{v} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{w} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Then if we add coordinatewise we get:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

which translates to a the vector $3\hat{\mathbf{i}} - \hat{\mathbf{j}}$ in component form. We get the same answer! Let's say a word as to why this makes sense. Since $\mathbf{v}$ moves 1 unit in the x direction, and $\mathbf{w}$ moves 2 units in the x direction, then doing both moves $1+2=3$ units in the x direction. This is exactly what the computer scientist's approach does, adding together the x-coordinates. An identical argument explains why the $\hat{\mathbf{j}}$ components agree with both perspectives!

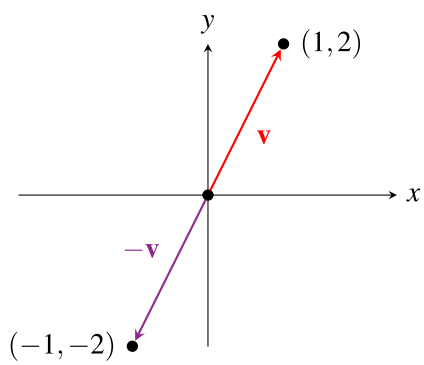
Now let's look at scalar multiplication, comparing $\mathbf{v}$ and $1.5\mathbf{v}$.



Since $1.5\mathbf{v}$ is 1.5 times longer, it (in particular), goes 1.5 times further in the x-direction, and 1.5 times further in the y-direction, so that:

$$1.5\mathbf{v} = 1.5(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) = 1.5\hat{\mathbf{i}} + 1.5 * 2\hat{\mathbf{j}} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}.$$

Let's also compare $\mathbf{v}$ and $-\mathbf{v}$.



Notice that $-\mathbf{v}$ goes the same distance along the x-axis $\mathbf{v}$ but in the opposite direction, and similarly along the y-axis. Therefore:

$$-\mathbf{v} = -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} \qquad \text{or} \qquad -\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

■

This example suggests that to add a pair of vectors, we can merely add the $\hat{\mathbf{i}}$ components and the $\hat{\mathbf{j}}$ components, and to scale a vector, we can just scale the components. The second statement in fact gives us a formula for scaling column vectors as well: scaling a column vector can be acheived by scaling each entry. Let's record this:

Theorem 2.1.1 Fix two general vectors in component form (and as column vectors).

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mathbf{w} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

1. We can compute the sum using the formula:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1)\hat{\mathbf{i}} + (v_2 + w_2)\hat{\mathbf{j}}.$$

In particular, the physicist's perspective on vector addition agrees with the computer scientist's formula:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

2. Let c be any constant. We can compute the scalar multiple $c\mathbf{v}$ using the formula:

$$c\mathbf{v} = (cv_1)\hat{\mathbf{i}} + (cv_2)\hat{\mathbf{j}}.$$

From this we can derive a formula for scaling a column vector which agrees with the physicist's perspective on vector scaling:

$$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

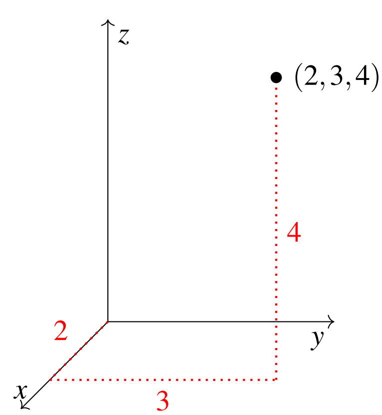
Proof. The argument for why Theorem 2.1.1 is true is essentially identical to the arguments appearing in Example 2.3 For example, $\mathbf{v}$ goes v_1 units in the x-direction, and $\mathbf{w}$ goes w_1 units in the x-direction, so doing both in succession results in an overall movement of $v_1 + w_1$ units in the x-direction. See if you can adapt the remaining arguments from Example 2.3 to the general setup. ■

This gives a complete answer to Question 2.1. To concretely interpret what *adding points* means, we should think about the points as vectors. Then the physicist's perspective on vector addition gives us a concrete way to think about the addition geometrically.

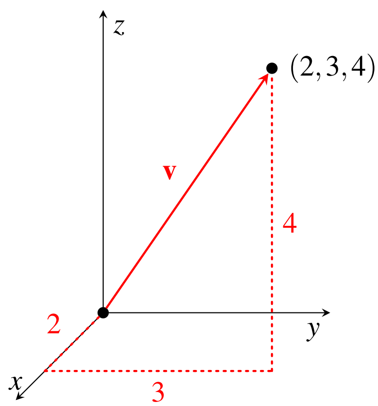
Exercise 2.9 Adopt the notation of Exercise 2.7. Suppose the tail for $-2.5\mathbf{v}$ is placed at P, what are the coordinates of its tip? ■

2.1.3 Extending this to 3 dimensions...and beyond

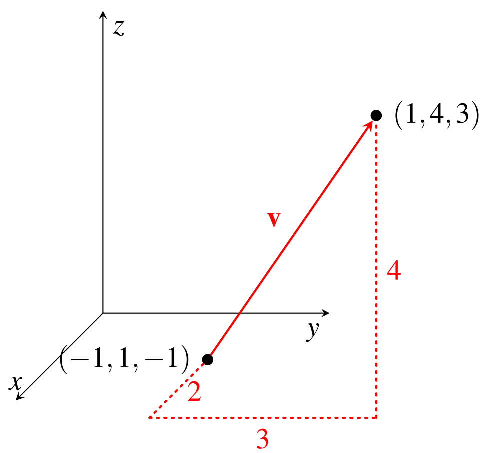
Everything described so far also works in 3 dimensions with a few cosmetic adjusments. Much like before, we can take the physicist's perspective or the computer scientist's perspective, and use coordinates to connect the two. The main difference is that we pass from our 2 dimensional xy-plane, to 3 dimensional space, whose points are now specified by 3 coordinates (x,y,z). We will usually adopt the convention of viewing the x axis as coming forward out of the page, the y axis as going to the right, and the vertical z-axis as going up. Here, for example, is how you plot the point $(2,3,4)$.



The Physicist's Perspective in 3d: As in 2d, the physicist's vector is a quantity $\mathbf{v}$ specifying a *magnitude* and *direction*, though the direction is now in 3d. We can again represent a vector by an arrow in 3-dimensional space: it should point in the *direction* of $\mathbf{v}$ and its length should be the *magnitude* of $\mathbf{v}$. Given two points P and Q in 3-space, we can obtain the vector from P to Q as an arrow connecting P and Q. For example, here we draw the vector $\mathbf{v}$ connecting $(0,0,0)$ and $(2,3,4)$.



As before *the vector is the arrow*, not the location. If we put the tail of $\mathbf{v}$ somewhere else, the vector remains the same.



In this way, we can think about a vector as something we can *apply* to a point. For example $\mathbf{v}$ takes $(0,0,0)$ to $(2,3,4)$, and also takes $(-1,1,-1)$ to $(1,4,3)$.

With this perspective, the definitions of addition, subtraction, and scalar multiplicate defined in Definitions 2.1.4, 2.1.5, and 2.1.7 go through unchanged. For example, to add two vectors $\mathbf{v}$ and $\mathbf{w}$, we take the vector obtained by *first applying* $\mathbf{v}$*, and then applying* $\mathbf{w}$.

The computer scientist's perspective in 3d: For a computer scientist, a 3-dimensional vector is an array of 3 numbers, aligned vertically:

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The can be added and scaled coordinatewise:

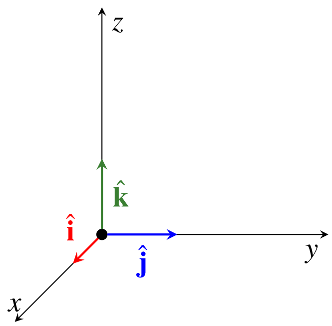
$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \\ z_0 + z_1 \end{bmatrix} \qquad \text{and} \qquad c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

Combining the two perspectives in 3d: To go between these two perspectives in 3 dimensions, we follow a similar approach to what we did in 2d. To start, we define the standard unit vectors by their directions and magnitudes:

$\hat{\mathbf{i}} =$ one unit in the positive x direction

$\hat{\mathbf{j}} =$ one unit in the positive y direction

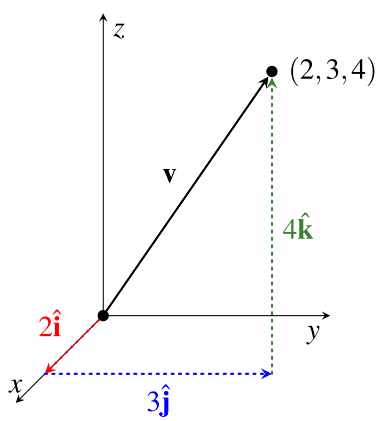
$\hat{\mathbf{k}} =$ one unit in the positive z direction



We can now express any vector in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. For example, the vector from $(0,0,0)$ to $(2,3,4)$ can be expressed as

$$\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}},$$

because it goes 2 units in the x direction (achieved by applying $\hat{\mathbf{i}}$ twice), 3 units in the y direction (acheived by applying $\hat{\mathbf{j}}$ 3 times), and 4 units in the z direction (achieved by applying $\hat{\mathbf{k}}$ four times).



To get this in the form of a column vector, just arrange the coefficients of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vertically.

$$2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Exercise 2.10 Let $\mathbf{w}$ be the vector from $(5,-3,9)$ to $(1,1,2)$. Write $\mathbf{w}$ as a column vector. ■

We can generalize 2.8 to 3d as well.

Proposition 2.1.2 Let $P=(x_0,y_0,z_0)$, $Q=(x_1,y_1,z_1)$, and let $\mathbf{w}$ be the vector from P to Q. Then $\mathbf{w}$ can be written as a column vector in the following way:

$$\mathbf{w}=\begin{bmatrix} x_1-x_0 \\ y_1-y_0 \\ z_1-z_0 \end{bmatrix}.$$

Exercise 2.11 Explain why Proposition 2.1.2 is true. ■

Higher dimensions: The computer scientist's approach to vectors generalizes to higher dimensions. To the computer scientist, a 2d vector was an array of 2 numbers, and a 3d vector was an array of 3 numbers. Following the pattern, an n-dimensional vector should be an array of n-numbers.

Definition 2.1.10 — Higher Dimensional Vectors: The Computer Scientist's Approach. Let n be a positive integer. A *n-dimensional column vector* is an array of n numbers, arranged vertically:

$$\mathbf{v}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The can be added and scaled coordinatewise:

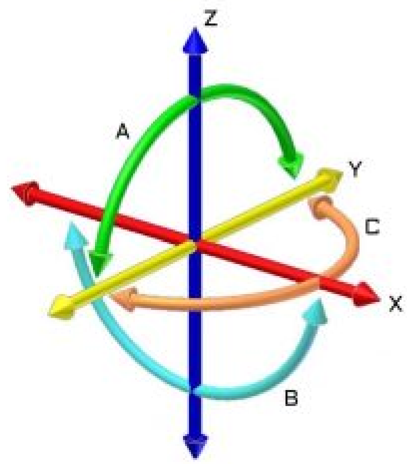
$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}+\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}=\begin{bmatrix} x_0+y_0 \\ x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} \quad \text{and} \quad c\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} cx_0 \\ cx_1 \\ \vdots \\ cx_n \end{bmatrix}.$$

At first we might be wary of straying beyond 3-dimensions, especially our interest is limited to applications within the physical world. Nevertheless, there are many reasons one might be interested in having more than 3 axes of data in a vector, even for things that get modelled in the physical world. Let's see a couple of examples.

■ **Example 2.4 — 5-axis CNC drilling.** Where a drilling machine drills a hole depends not only on the location of the tip of the drill bit, but also on its orientation in space (what direction is the drill bit pointing?). For these reasons, programmable drills control the movement of the drilling head using a five dimensional vector:

$$\mathbf{v}=\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}.$$

The first three entries, x, y, z, control the location of the machine-head in space, while θ and ϕ control rotation, in order to angle the drill to the necessary position. In particular, θ controls rotation in the xy-plane (about the z-axis), while ϕ controls rotatoin in the yz-plane (about the x-axis).



[3]

The drill can be controlled much like the autonomous vehicles of Section **??**, by being sent vectors. For example, if you would like to send the drill 1 cm in the x-direction and 2 cm down, without changing rotation, you may send it the vector:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

If the drill is pointing down and you would like to point it up, and lift it 5 cm, you could then send it:

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 180 \end{bmatrix}.$$

To do these in succession, we would first do $\mathbf{v}$ (say to drill a hole), and then do $\mathbf{w}$ (say, to lift it and drill another hole from below), which a physicist may say should add the vectors. And indeed, tracing the overall movement of the printhead would have it end up following the vector

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 180 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 180 \end{bmatrix} = \mathbf{v} + \mathbf{w}.$$

In particular, the computer scientist's definition of vector addition has a physical interpretation as well! ■

[3]From Autodesk: `https://blogs.autodesk.com/inventor/understanding-process-5-axis-machining/`

This example shows that adding even 5-dimensional vectors can concretely understood once we specify the data that the entries store. The following exercise shows you can do something similar for scalar multiplication.

Exercise 2.12 Suppose you have a 5-axis drill, at its *home position*. You program it to move along the 5-dimensional $\mathbf{v}$ to drill a hole. If you'd like it to return to home position, what vector should you ask it to follow? ■

■ **Example 2.5 — A 171,000 dimensional data set.** The study of *stylometry* studies variations in literary style using statistical analysis, and part of this work involves measuring how frequently certain words appear. There are approximately 171,000 words in the english language, so if you would like to record how many times each word appears in a given book, you could do so in a vector:

$$\mathbf{v} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{171,000} \end{bmatrix},$$

where w_1 counts the number of times the first word in the dictionary appears, and w_2 counts how many times the second word in the dictionary appears, and so on. Adding the vectors associated to a 2 different books will count how many times each word appears in both. Having a mathematical model grounded in geometry allows one to use linear algebraic techniques to study questions about variations in word frequency. One could also use scalar multiples to weight the importance or prominence of certain sources over others. ■

2.2 Exercises

Exercise 2.13 Let $P = (1,2), Q = (2,1)$ and $R = (-3,-1)$. Let $\mathbf{v}$ be the vector from P to Q, and let $\mathbf{w}$ be the vector from Q to R.

1. Do you agree or disagree with the following statement:

   $$\mathbf{v}+\mathbf{w} \text{ is the vector from } P \text{ to } R.$$

   Justify your answer by drawing a picture.
2. Write $\mathbf{v}, \mathbf{w}$ and $2\mathbf{v}-\mathbf{w}$ in component form, and as column vectors.
3. Draw $2\mathbf{v}-\mathbf{w}$ on the plane, with its tail starting at R. What are the coordinates of where the tip lands?

■

Exercise 2.14 Write the standard 3-dimensional unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ as column vectors. ■

Exercise 2.15 To a physicist, a vector was a quantity with *magnitude* and *direction*. We saw how to turn such a quantity into a *column vector*. Let's start going in the other direction.

1. Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the standard unit vectors in 2d. What are their magitudes: $||\hat{\mathbf{i}}||$ and $||\hat{\mathbf{j}}||$? (*Recall:* When representing a vector with an arrow magnitude, is the length of the arrow.)
2. Let $\mathbf{v}$ and $\mathbf{w}$ be the same vectors from Problem 1. Use the Pythagorean theorem to compute

their magnitudes: $||\mathbf{v}||$ and $||\mathbf{w}||$.

3. Consider a general column vector:

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Write formula for the magnitude $||\mathbf{u}||$ in terms of x and y. Explain your reasoning.

An artist is trying to mix the perfect color. One way to measure colors is in RGB, where:

$$\begin{array}{rc} & \text{r units of red} \\ & \text{g units of green} \\ + & \text{b units of blue} \\ \hline & \text{a rich spectrum of colors} \end{array}$$

The artist has two pigments, which they are hoping to mix together into paint to try and get the color they want.

Pigment X: Adding 1 ounce of Pigment X to paint adds:

1 unit of red, 2 units of green, 3 units of blue

Pigment Y: Adding 1 ounce of Pigment Y to paint adds:

7 units of red, 5 units of green, 2 units of blue

Exercise 2.16 The effect of adding an ounce **pigment X** can be represented by a vector $\mathbf{x}$ and the effect of adding **pigment Y** by a vector $\mathbf{y}$.

1. Give a concrete interpretation of what the vector $\mathbf{x}+\mathbf{y}$ represents.
2. Express $\mathbf{x}$ and $\mathbf{y}$ as column vectors.

Exercise 2.17 The artist wants to create a color the called **fancy gold**, consisting of:

24 units of red, 21 units of green, 15 units of blue

1. Write **fancy gold** as a column vector $\mathbf{f}$.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X and b units of Pigment Y produce fancy gold.

3. Translate the single vector equation from part 2 into a system of linear equations. Solve this system of equations to see how many ounces of **Pigment X** and how many ounces of **Pigment Y** the artist needs to mix to get **fancy gold**.

Exercise 2.18 The artist also whats to create **super green**, which consists of:

3 units of red, 90 units of green, 3 units of blue.

1. Write **super green** as a column vector **s**.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X and b units of Pigment Y produce super green.

3. Translate the single vector equation from part 2 into a system of linear equations. Try to solve this system of equations to determine if it is possible to mix **super green** from **Pigment X** and **Pigment Y**.

■

Exercise 2.19 The artist got their hands on a fancy new pigment called **Greenifier**, which, perhaps suprisingly, doesn't actually contain any green. Instead, it works by absorbing red and blue light. The net effect of adding 1 ounce of **Greenifier** is

subtract 5 units of red, subtract 5 units of blue

1. Write **Greenifier** as a column vector **g**.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X, b units of Pigment Y, and c units of Greenifier produce super green.

3. Translate the single vector equation from part 2 into a system of linear equations. Solve this system of equations to see how many ounces of **Pigment X**, how many ounces of **Pigment Y**, and how many ounces of **Greenifier** the artist needs to mix to get **fancy gold**.

■

2.3 September 12, 2024

2.3.1 A summary of this weeks groupwork and chalktalk.

Let's start by unpacking Exercises 2.17 and 2.18, and in doing so, get a quick review of how we solve systems of linear equations. Our goal in Exercise 2.17 is to mix fancy gold from the two pigments, Pigment X and Pigment Y. How much of each pigment do we add?

To introduce mathematical notation, we let the vector $\mathbf{x}$ represent the effect of adding an ounce of Pigment X and $\mathbf{y}$ represent adding an ounce of Pigment Y. In particular, if we add a ounces of Pigment X and b ounces of Pigment Y, the overall effect on color can be represented by the vector:

$$a\mathbf{x} + b\mathbf{y}.$$

If we let $\mathbf{f}$ represent the color fancy gold, we are therefore looking for integers a and b so that:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{f}. \tag{2.1}$$

Plugging in the column vectors from Exercises 2.16.2 and 2.17.1 turns this into:

$$a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}7\\5\\2\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}.$$

But we know how to scale and add column vectors. So this becomes:

$$\begin{bmatrix}a+7b\\2a+5b\\3a+2b\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}$$

Now we can just remove the matrix brackets and obtain:

$$\begin{aligned} a+7b &= 24 \qquad (2.2)\\ 2a+5b &= 21 \qquad (2.3)\\ 3a+2b &= 15, \qquad (2.4) \end{aligned}$$

which is a system of linear equations with 3 equations and 2 unknowns. We want to find values a and b so that *all 3 equations hold*! Many readers have probably solved systems of linear equations before, but it may have been a long time, so let's briefly review how one might solve this. We will establish a completely systematic way of doing this in Section **??**, but for now, let's just follow our noses. A first step can be to solve for a in Equation (2.2):

$$a = 24 - 7b. \tag{2.5}$$

We can now plug this value of a into Equation (2.3). The left hand side is:

$$2a + 5b = 2(24 - 7b) + 5b = 48 - 9b.$$

So Equation (2.3) turns into:

$$48 - 9b = 21.$$

We can therefore solve for $b = 3$. Plugging this value into Equation (2.5) gives:

$$a = 24 - 7(3) = 3.$$

So in order for the first Equations (2.2) and (2.3) to hold, we need $a = 3$ and $b = 3$. What about Equation (2.4)? Plugging in $a = 3$ and $b = 3$ gives:

$$3a + 2b = 3*3 + 2*3 = 15.$$

So we have determined that $a = 3$ and $b = 3$ solves all three equations in our system. Translating back into our vector equations we have:

$$3\mathbf{x} + 3\mathbf{y} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}7\\5\\2\end{bmatrix} = \begin{bmatrix}3*1+3*7\\3*2+3*5\\3*3+3*2\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix} = \mathbf{f}.$$

So $a = 3$ and $b = 3$ also form solutions to the vector equation (2.1)! Translating back into plain english gives:

3 ounces of Pigment X and 3 ounces of Pigment Y results in Fancy Gold

One main takeaway from this example is that, the *single* linear equation of vectors (2.1), corresponds exactly to the *system* of linear equation*s* (2.2), (2.3), and (2.3).

■ **Slogan 2.2** Solving *one* linear equation of vectors is the same as solving *a system* of linear equations.

What about Super Green? The question becomes one of solving the single vector equation:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{s},$$

which we can unpack to the system of equations:[4]

$$\begin{aligned} a + 7b &= 3 \\ 2a + 5b &= 90 \\ 3a + 2b &= 3 \end{aligned}$$

Following the same script as before, we can use the first two equations to solve for $a = \frac{199}{3}$ and $b = \frac{-28}{3}$ (the reader fill should in the missing steps!). This already presents a problem, how can we add a negative amount of Pigment Y? Let's ignore this for a moment, and pretend the artist had some tool to remove pigment. Would this solve it? Well, we see that to get 3 units of red and 90 units of green we are forced to add $\frac{199}{3}$ ounces of Pigment X and to (somehow) subtract $\frac{28}{3}$ ounces of Pigment Y. But we haven't even checked if we have the correct amount of blue. This is measured in the third equation, let's see if our values of a and b work:

$$3a + 2b = 3 \times \left(\frac{199}{3}\right) + 2\left(\frac{-28}{3}\right) = \frac{541}{3}.$$

That is way more than 3 units of blue! So our attempt at making Super Green is going to come out more blue than anything else, and we will fail. What we've encountered here is the following fact:

[4] the reader should fill in the details here!

sometimes a system of linear equations has no solutions. Indeed, if there are more equations than unknowns, this is rather common: notice we were able to find an a and b that satisfied the first two equations, but the third equation that caused us trouble. When we translate Exercise 2.19 into a system of equations, we have a third unknown, giving us added flexibility we can exploit. For now, let's just record the following observation:

■ **Slogan 2.3** Sometimes a linear equation of vectors can have no solution.

For the rest of this section, we will put these observations into a more general context, introducing some language with which we can more concisely describe these observations.

2.3.2 Linear Combinations and Spans: A First Pass

In our color mixing example, we were studying which colors we could mix from Pigment X and Pigment Y. Assigning variables a and b to the amount of each pigment added, we were able to translate this question into one which studies whether we can write a vector in the form:[5]

$$a\mathbf{x} + b\mathbf{y},$$

In a more general language, we are asking to know which vectors can be written *in terms of* $\mathbf{x}$ *and* $\mathbf{y}$ with the operations of scalar multiplication and addition. A vector that can be written in that form is called a *linear combination* of $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.3.1 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. A *linear combination* of $\mathbf{u}$ and $\mathbf{v}$ is a vector $\mathbf{w}$ which can be written:

$$\mathbf{w} = c\mathbf{u} + d\mathbf{v},$$

for constants c and d.

■ **Example 2.6** The vector $\mathbf{f}$ from Exercise 2.17 is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ because:

$$\mathbf{f} = 3\mathbf{x} + 3\mathbf{y}.$$

■

■ **Question 2.3** Is the vector $\mathbf{s}$ a linear combination of $\mathbf{x}$ and $\mathbf{y}$.

We already encountered linear combinations of vectors before this color example. Indeed, it was the tool we used to translate between the *physicist's perspective* on vectors to the *computer scientist's perspective*.

■ **Example 2.7** The column vector:

$$\mathbf{v} = \begin{bmatrix} -11 \\ 9 \end{bmatrix},$$

is a linear combination of the standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ because

$$\mathbf{v} = -11\hat{\mathbf{i}} + 9\hat{\mathbf{j}}.$$

■

[5] In fact, one could interpret the example as having us restrict to positive values of a and b, but we will not make that restriction.

■ **Question 2.4** Let $\mathbf{w}$ be a 2-dimension vector. Is $\mathbf{w}$ a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$?

Indeed, this is what the column vector *means*. The column vector:

$$\mathbf{w} = \begin{bmatrix} c \\ d \end{bmatrix}$$

exactly says that $\mathbf{w}$ can be acheived by scaling $\hat{\mathbf{i}}$ by c, and scaling $\hat{\mathbf{j}}$ by d. That is $\mathbf{w} = c\hat{\mathbf{i}} + d\hat{\mathbf{j}}$.

We can also consider linear combinations of more than 2 vectors. We have already done so to represent 3-dimensional vectors as column matrices, and also in Exercise 2.19.

Definition 2.3.2 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be a collection of n-vectors (of the same dimension). A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a vector $\mathbf{w}$ which can be written in the form:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

for constants $c_1, c_2, \cdots, c_n$.

■ **Example 2.8** Exercise 2.19 asks whether the vector $\mathbf{s}$ is a linear combination of $\mathbf{x}, \mathbf{y}$ and $\mathbf{g}$ where $\mathbf{g}$ represents the effect of adding one ounce of greenify. ■

■ **Example 2.9** Every 3d vector is a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. ■

■ **Question 2.5** Consider the column vector:

$$\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

If I know that $\mathbf{w}$ is a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$, what can we say about its middle entry b?

■ **Example 2.10** Let $\mathbf{v}$ be a single vector. A linear combination of $\mathbf{v}$ is a vector of the form $c\mathbf{v}$ for some constant c. That is, a linear combination of $\mathbf{v}$ is the same as a multiple of $\mathbf{v}$. ■

As we are starting to see, many questions in linear algebra boil down to variants of the following type of question:

■ **Question 2.6** When is one vector a linear combination of another collection of vectors?

We will see many variations of this question, so let's introduce some terminology to simplify the exposition.

Definition 2.3.3 Let $\mathbf{u}$ and $\mathbf{v}$ be a pair of vectors. The *span* of $\mathbf{u}$ and $\mathbf{v}$ is the collection of vectors that can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\text{span}\{\mathbf{u}, \mathbf{v}\} = \{\text{The collection } c\mathbf{u} + d\mathbf{v} \text{ for constants } c \text{ and } d\}.$$

■ **Example 2.11** Returning to our color mixing example: $\text{span}\{\mathbf{x}, \mathbf{y}\}$ is the collection of colors that can be mixed from Pigment X and Pigment Y. In particular, $\mathbf{f}$ is in $\text{span}\{\mathbf{x}, \mathbf{y}\}$, while $\mathbf{s}$ is not. ■

■ **Example 2.12** Denote the entire collection of 2d vectors by $\mathbb{R}^2$. Then $\mathbb{R}^2 = \text{span}\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$. ■

We can also consider the spans of more than 2 vectors.

Definition 2.3.4 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be a collection of n-vectors (of the same dimension). The span of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is the collection of vectors that are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

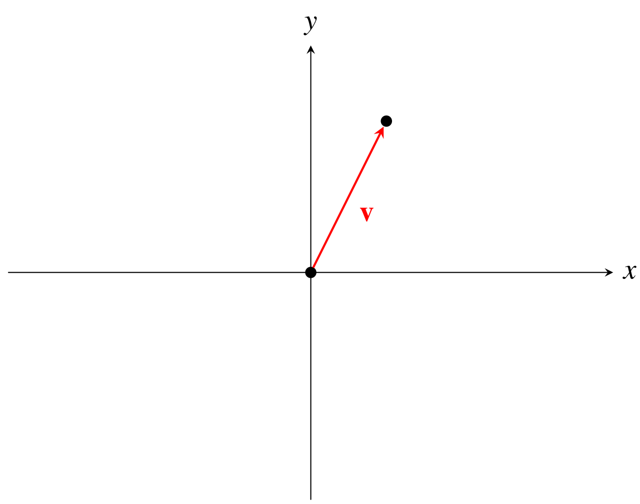
$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \{\text{ the collection } c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \text{ for constants } c_1, c_2, \cdots, c_n\}.$$

■ **Example 2.13** Exercise 2.19 asks whether the vector $\mathbf{s}$ is in $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{g}\}$. ■

Visualizing Spans

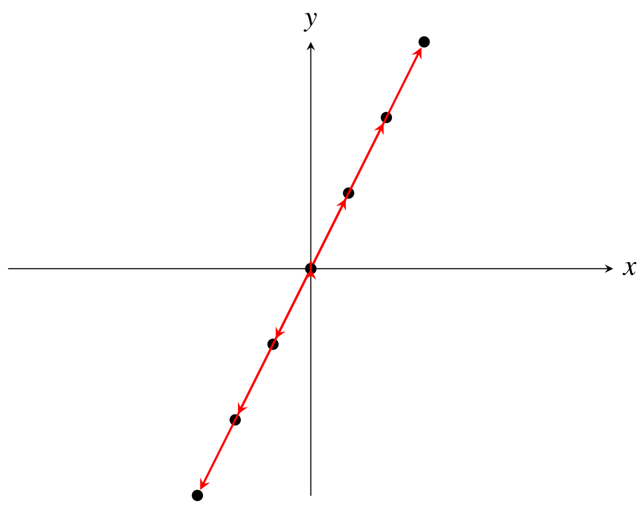
Spans can be a tricky thing to intuit on a first pass through linear algebra, but they are also an important and fundamental part of the theory. We will delay the general practice of explicitly computing spas until Section **??**, when we have developed a bit of matrix theory.[6] But if we can develop a visual intuition of what spans look like, we recognize where they arise and get some intuition about what they mean before doing any computations.

■ **Example 2.14 — The Span of a Single Vector.** Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$:

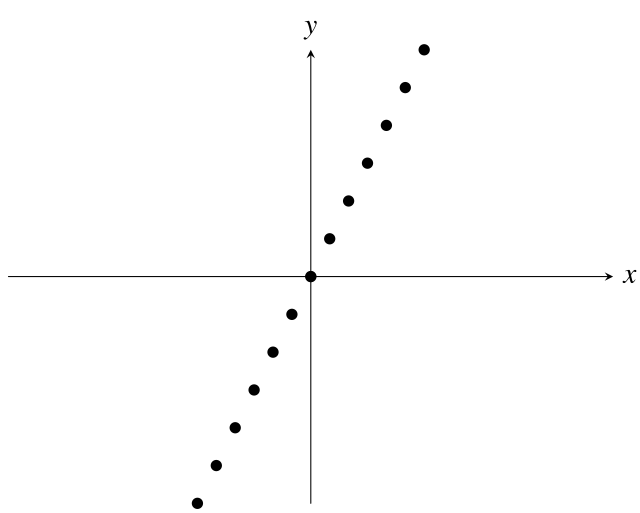


The span of $\mathbf{v}$ is the collection of linear combinations of $\mathbf{v}$. This is precisely the collectoin of multiples of $\mathbf{v}$ (cf. Example **??**). Let's plot a few of these. In particular, we'll plot: all the vectors $-1.5\mathbf{v}, -\mathbf{v}, -0.5\mathbf{v}, 0\mathbf{v}, .5\mathbf{v}, 1\mathbf{v}, 1.5\mathbf{v}$, with their tails starting at the origin.

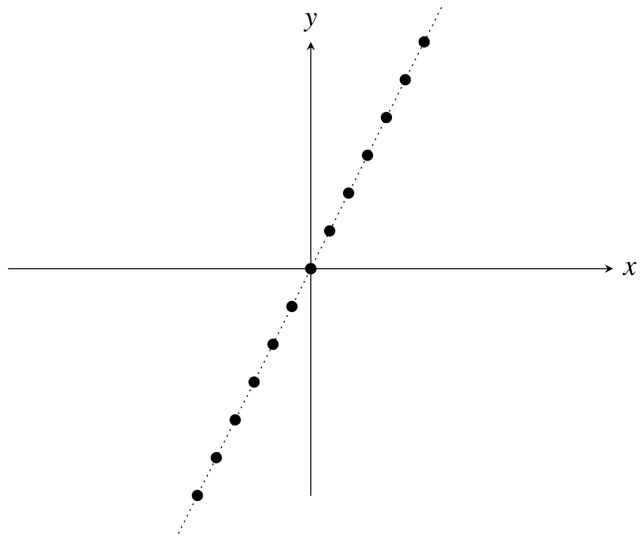
[6] though notice that Exercises 2.17 through 2.19 suggest a relationship with systems of linear equations



As we start to fill in more and more multiples, the arrows start to crowd the picture, so let's just draw the where the tips lie.



As the picture fills in, we start to see that the tips of all the multiples of **v** trace out a straight line.



■

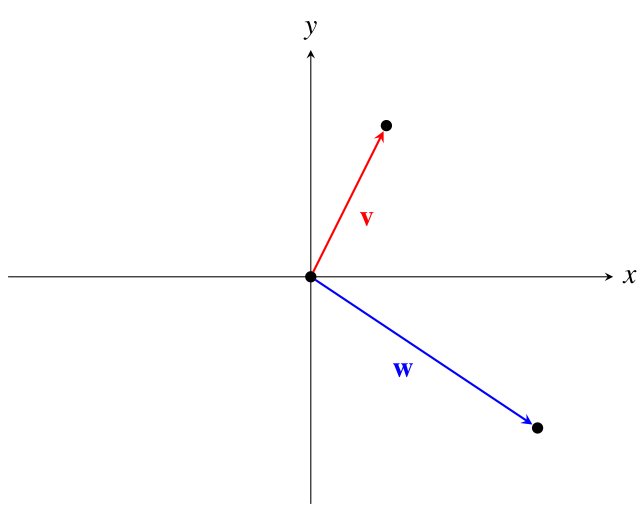
In the previous example, we deduced (without any explicit computations) some geometric facts about the span of a single vector!

■ **Slogan 2.4** The span of a single vector is a straight line through the origin.

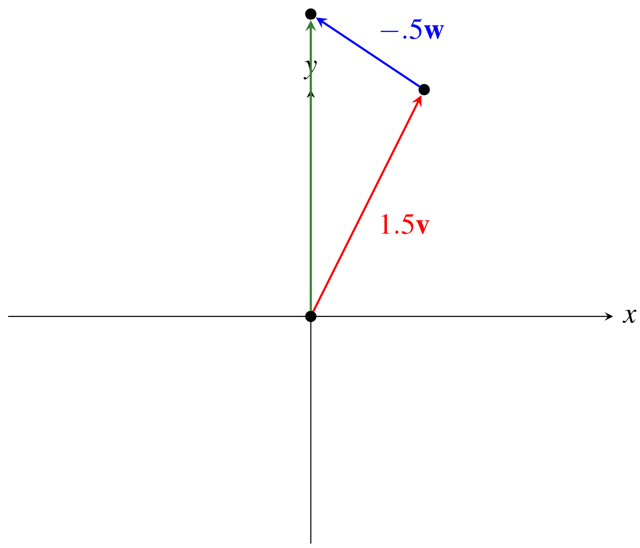
2.3.3 Spans of multiple vectors, and in higher dimensions.

Here are more geometric examples of spans. We didn't cover them in class, but they may be helpful to the reader hoping to develop their intuition further. We will look at these more carefully further down the line.

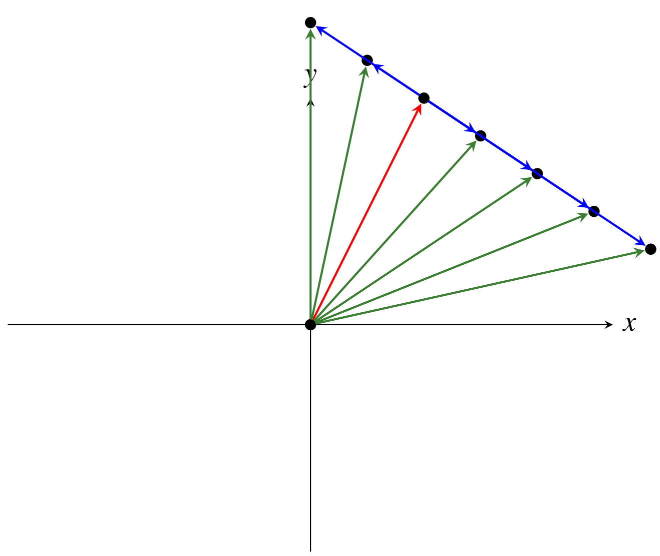
■ **Example 2.15 — The Span of 2 Vectors.** Let $\mathbf{v}$ and $\mathbf{w}$ be 2 nonzero vectors in $\mathbb{R}^2$. For now let's assume that they aren't parallel.



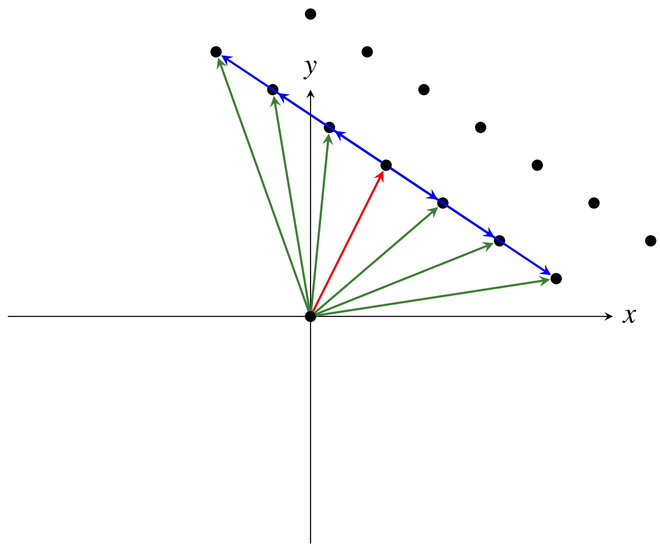
The span of $\mathbf{v}$ and $\mathbf{w}$ is the collection of vectors that can be written $c\mathbf{v}+d\mathbf{w}$. For example, $1.5\mathbf{v}-0.5\mathbf{w}$:



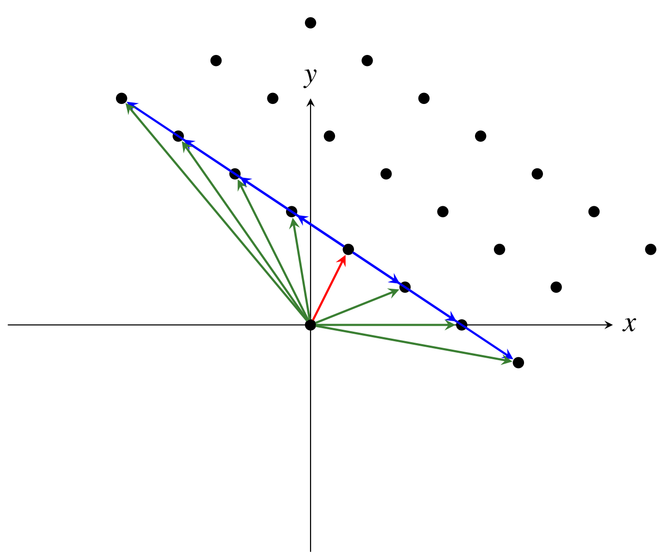
Let's plot a few more, fixing $c = 1.5$. That is, let's look at vectors of the form $1.5\mathbf{v} + d\mathbf{w}$ for various values of d.



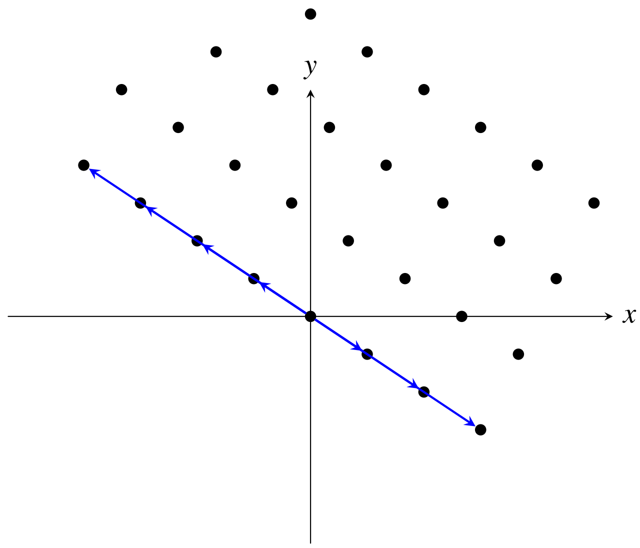
We can do the same, for $c = 1$.

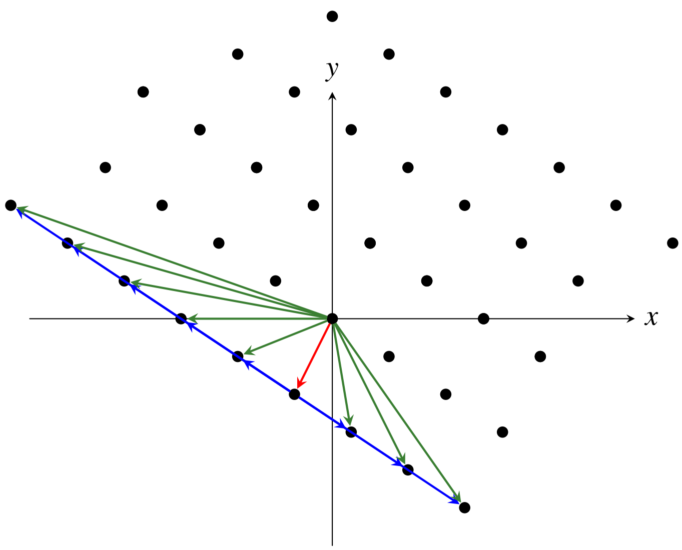


Keeping the points the tip has hit marked, let's also include $c = .5$



Before moving on, Let's throw in $c = 0$ (which are just multiples of $\mathbf{w}$) and $c = -.5$.





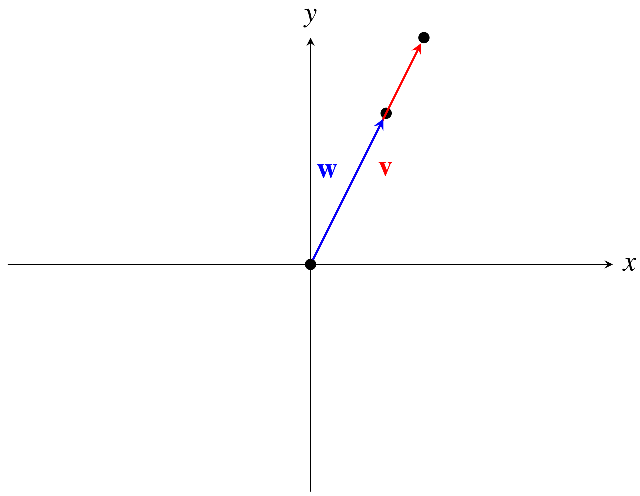
It starting to look like we can get anywhere in the plane. Indeed, imagine having two dials, one which modifies c and another which modifies d. Then c is changing the length of the red arrow, and d and changing the length of the blue arrow, tacked on to the tip of the red arrow. Then just by turning these dials, we should be able to get anywhere we want. This actually becomes even more clear when looking at an animation.[7] ■

In the previous example, we deduced (without any explicit computations) some geometric facts about the span of a pair of vectors!

■ **Slogan 2.5** The span of 2 nonzero and nonparallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$.

■ **Question 2.7** Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors which are parallel. What can you say about their span?

[7]For example `https://3b1b-posts.us-east-1.linodeobjects.com//content/lessons/2016/span/linear_combination.mp4`



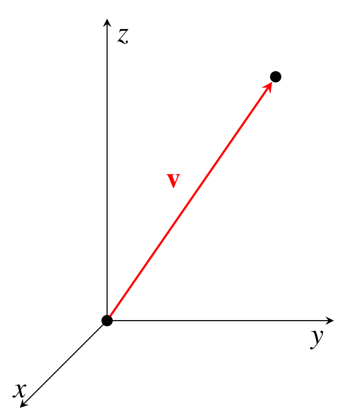
■ **Question 2.8** Can you say anything about the span of the zero vector $\mathbf{0}$?

To summarize, it looks like 3 things can happen:

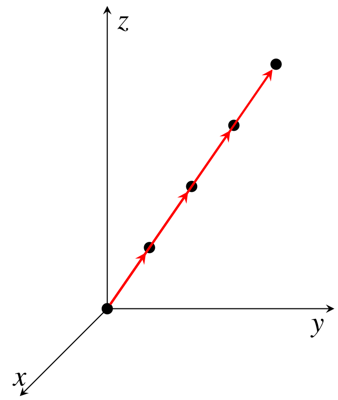
1. The span is just the zero vector. This happens if you are taking the span of the zero vector.
2. The span is a line. This happens if you are taking the span of a single nonzero vector, or of parallel vectors at least one of which is nonzero.
3. The span is all of $\mathbb{R}^2$. This happens if you take the span of at least two nonzero vectors which are not parallel.

What is remarkable is that that we were able to deduce all of this without doing any explicit computations. The story gets a bit more interesting if we move into 3-dimensions. As before, the span of the 0 vector will be just the origin, and with enough vectors it is possible to get all of $\mathbb{R}^3$ (for example, $\text{span}\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$). We can also also have again that the span of a single nonzero vector gives a straight line.

■ **Example 2.16** Let $\mathbf{v}$ be a nonzero vector.



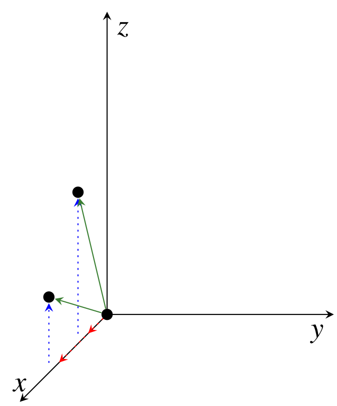
Taking a few multiples we have:



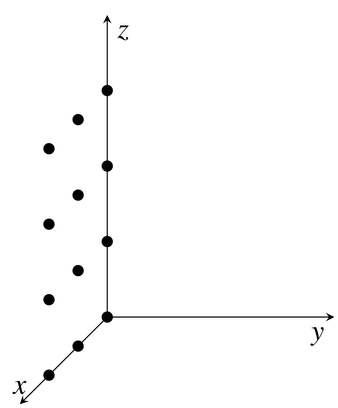
And we can observe that we are tracing out a line through the origin in $\mathbb{R}^3$. ■

So we can get 0, everything, and a straight line. But something different can happen as well, when considering the span of two parallel vectors.

■ **Example 2.17** Let's see if we can work out the span of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$. Below we plot a few different values.



It they lie in the xz-plane. Let's plot where the tips of $a\hat{\mathbf{i}} + b\hat{\mathbf{k}}$ for a few more values of a and b.



It's starting to appear that we can get anywhere in the xz-plane, but also that we can't escape from it.

This is actually not too hard to work out explicitly:

$$\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{k}} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\0\\b\end{bmatrix}.$$

So if we put the tail of $\mathbf{v}$ at the origin, then we can decide the x-coordinate of where its tip lands by choosing a, a similarly we can decide the z-coordinate of its tip by choosing b. The y-coordinate, on the other hand, must always stay at 0. ■

If you let $\mathbf{v}$ and $\mathbf{w}$ be two non-parallel vectors in $\mathbb{R}^3$, you can proceed as in Example 2.15 and think about where $c\mathbf{v} + d\mathbf{w}$. I encourage you to do this, and maybe you can convince yourself of the following fact

■ **Slogan 2.6** The span of 2 nonzero and nonparallel vectors in $\mathbb{R}^3$ trace out a plane in $\mathbb{R}^3$.

R Linear combinations are one of the most central and most important concepts in linear algebra. Even before doing any computations, we benefit greatly from thinking about how to visualize them and, and how to think about them in terms of concrete problems. This can be aided by well-made images and good animations. If you find 10 minutes to spare over break, I highly recommend Grant Sanderson's video on visualizing linear combinations: `https://www.3blue1brown.com/lessons/span`.

2.3.4 Exercises

Exercise 2.20 A group of exo-ecologists are experimenting with mixtures of gasses for a greenhouse in space. The gasses is a mix of Nitrogen, Oxygen, Carbon Dioxide, and Argon. They have 3 gas mixtures, whose compositions are given below (measured by mass).

- **Gas X:** 80% Nitrogen and 20% Oxygen.
- **Gas Y:** Pure Oxygen
- **Gas Z:** 60% Nitrogen, 30% Oxygen, 2% Carbon Dioxide, and 8% Argon

1. We represent a mixture containing a grams of Nitrogen, b grams of Oxygen, c grams of Carbon Dioxide, and d grams of Argon, by the column vector

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix}.$$

Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be the vectors representing the a mixture of *one gram* of gasses X, Y, and Z respectively. Write $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as column vectors.

2. In plain english, describe what $\operatorname{span}\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ represents.

3. The exo-ecologists would like a gas mixture $\mathbf{e}$ modelled on the concentration of gasses in earth's atmosphere, which they call *the eartlike mixture*. Fill in the blanks in the following sentence so that it means: *The earthlike mixture can be mixed from gasses X, Y, and Z.*

   The vector(s) ___________ is/are a linear combination of the vector(s) ___________

4. Do you think the exo-biologists can mix pure Carbon Dioxide from their gas mixtures? Why or why not?

Exercise 2.21 We introduced vectors because in HW1 Problem 6 it looked like linear maps played well with adding points, suggesting that it might be more meaningful to think about the inputs and outputs as vectors rather than points. Let's try to make this more precise. To do this, we will shift our perspective slightly by letting $\mathbb{R}^2$ be the collection of 2-dimensional *vectors*:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ then is a rule:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

and can therefore be described by an equation for u and one for v. I want to emphasize that this is merely a shift of perspective (and notation), but the content is the same as in HW1.

1. Let $\mathbf{w} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ via the rule:

   $$f(\mathbf{x}) = \mathbf{x} + \mathbf{w}.$$

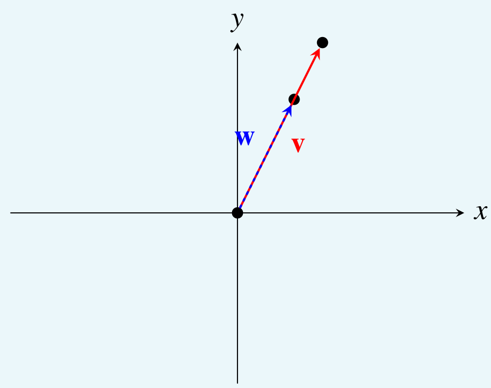
   (a) Write the two equations $u = u(x,y)$ and $v = v(x,y)$ representing f.
   (b) Is f a linear function? Why or why not?
   (c) If you determined f is a linear function, write down the 2×2 matrix associated to f. Otherwise skip this part.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the rule $g(\mathbf{x}) = 2\mathbf{x}$.
   (a) Write the two equations $u = u(x,y)$ and $v = v(x,y)$ representing g.
   (b) Is g is linear function? Why or why not?
   (c) If you determined g is a linear function, write down the 2×2 matrix associated to g. Otherwise skip this part.

Exercise 2.22 Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^3$. Do you agree or disagree with the following statements? Explain your reasoning for each.

1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are in* $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, *the so is* $\mathbf{x} + \mathbf{y}$.
2. *If* $\mathbf{x}$ *is in* $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, *then so is* $c\mathbf{x}$ *for any constant* c.
3. *If* $\mathbf{x}$ *is in* $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, *then so is* $\mathbf{x} + \hat{\mathbf{i}}$.

Exercise 2.23 Let's think about a couple more spans in 2d.

1. Let $\mathbf{0}$ be the zero vector. Give a description of $\operatorname{span}\{\mathbf{0}\}$.
2. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors which are parallel. What can you say about their span? In particular, would you say their span is all of $\mathbb{R}^2$? A line? A single point? Something else entirely? Explain your reasoning.



Exercise 2.24 Which of the following vectors in $\mathbb{R}^3$ are linear combinations of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$? Explain your reasoning.

1. The zero vector $\mathbf{0}$.
2. The column vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$
3. The column vector $\begin{bmatrix}0\\-5\\11\end{bmatrix}$
4. The vector $\hat{\mathbf{i}}$.

Exercise 2.25 We saw that the span of a single nonzero vector in $\mathbb{R}^2$ traces out a line. Let $\mathbf{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$, so that the tips of every vector in $\operatorname{span}\{\mathbf{v}\}$ trace out a line. Find the equation of that line. (Recall, the equation of a line can be written $y = mx + b$ where m is the slope and b is the y-intercept. Can you find m and b?)

Exercise 2.26 Let's see if we can get some intution about the relationship between spans and dimension, without needing to do any explicit computation.

1. We saw that 2 vectors can span $\mathbb{R}^2$. Can fewer than 2 vectors do this? Why or why not?
2. Give an example of 3 vectors that can span $\mathbb{R}^3$. Do you think fewer that 3 vectors do this? Explain your answer (this explanation can be informal, it doesn't have to be a proof).

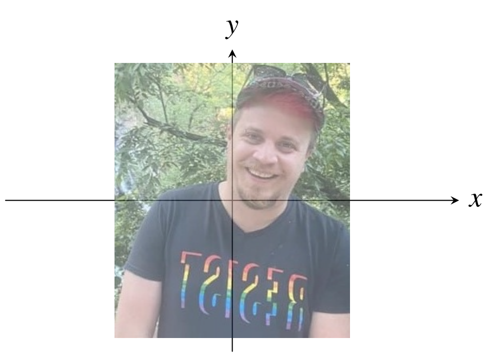
3. Let $\mathbb{R}^n$ be the collection of n-dimensional column vectors. How many vectors do you think are necessary to span all of $\mathbb{R}^n$?

3. Linear Transformations and Matrices

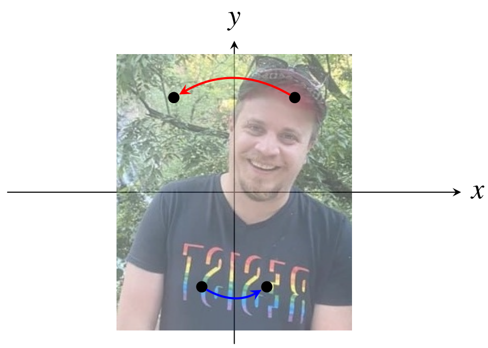
3.1 September 17, 2024

I was traveling recently, and took the following selfie.

I didn't notice anything right away, but when I was reflecting back on my vacation photos, I realized that the logo on my shirt was backwards! *Oh no! I got a mirror image!* To fix this, I had to apply a horizontal reflection to the image. *But how?* First, let's introduce some coordinate axes.



With these coordinates chosen, the goal is to reflect the plane over the y-axis. For example, the point where my hair is colored pink is currently at $(1,2)$, but we should instead color my pink hair at the reflected point, $(-1,2)$. Similarly, the part of the letter S that is colored blue at $(-.5,-2)$, should instead be drawn blue at the reflected point $(.5,2)$.



We should do this for every pixel in the image. If the pixel at (x,y) is colored a certain way, we should instead color the pixel $(-x,y)$ that way. Doing this pixel by pixel, we recover the corrected image, with the text appearing legibly!

To summarize, we applied a certain *function* or *transformation* to the plane. In Chapter 1, we studied functions from the plane to itself, and this one certainly fits. Using that language, we defined a tranformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $R(x,y) = (u,v)$ where:

$$u = -x,$$

$$v = y.$$

Notice that this is even a *linear transformation* (cf. Defintion 1.2.1). We applied this linear transformation to the image, meaning that if a point $\mathbf{p}$ was supposed to be colored, say, red, we colored $R(\mathbf{p})$ red instead. This is a rather straightforward example, but many of the most importation transformations of images in computer graphics—from rotations and reflections, to fitting an image to a screen or window—are acheived by applying an appropriate linear transformation.

3.1.1 Linear Transformations of the Plane: Revisited

Recall from Definition 1.2.1, that a linear transformation of the plane was a function $L(x,y) = (u,v)$ where:

$$u = ax + by,$$

$$v = cx + dy,$$

for constants a, b, c, and d. We noticed in Homework 1 (cf. Exercise 1.8) that a linear transformation appeared to play well with addition, suggesting that it might be worth thinking the inputs and outputs of the function as vectors rather than points.

Notation 3.1. *Let* $\mathbb{R}^2$ *denote the collection of 2-dimensional column vectors* $\begin{bmatrix} x \\ y \end{bmatrix}$.

With this in hand we can revisit Definitions 1.2.1 and 1.2.2, with its notational updates. We remark that the difference is purely cosmetic, the content is identical.

Definition 3.1.1 A linear transformation is a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

where u and v are computed as follows:

$$u = ax + by,$$

$$v = cx + dy,$$

for constants a, b, c, d. The *2×2 matrix* associated to L is:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The four coefficients a, b, c, d completely determine L, therefore so does the matrix M. As such, we often just denote L just using the matrix itself:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

What is on the right looks like we are multiplying two matrices, and that is no accident. In fact, we know that this should equal the column vector

$$\begin{bmatrix} u \\ v \end{bmatrix},$$

whose formula is given in Definition 3.1.1. In particular, the first first row of the output should be $ax + by$ and the second should be $cx + dy$. This establishes our first formula for matrix multiplication:

Definition 3.1.2 — Matrix-Vector Multiplication: The 2×2 Case. The product of a 2×2 matrix M and a column vector $\mathbf{w}$ can be computed as follows:

$$M\mathbf{w} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

This recovers the usual definition of Matrix-Vector multiplication. Rather than just memorizing this formula, it is useful to think about how the computation as *a process*. Indeed, this process is what will generalize to matrix multiplication in general. To compute the matrix product, we first think about the first row of M and pair its entries in order with the entries in $\mathbf{w}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We then multiply the paired elements together, and add them up to obtain the first row of $M\mathbf{w}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ \end{bmatrix}.$$

To obtain the second row of the output, we pair the second row of M with the entries of $\mathbf{w}$ the same way, multiplying the corresponding elements and adding them together.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

This is a process you may already be familiar with, but when I first learned how to multiply matrices, I found the rule to feel kind of opaque and arbitrary. Hopefully Definition 3.1.1 makes this process feel more reasonable. Indeed, the *product* $M\mathbf{v}$ should be thought of as the *function* M being applied to the *vector* $\mathbf{v}$.

■ **Example 3.1** Let's compute the matrix product:

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Starting with the first row:

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1*-1+3*2 \\ \end{bmatrix} = \begin{bmatrix} 5 \\ \end{bmatrix}$$

Filling in the second row:

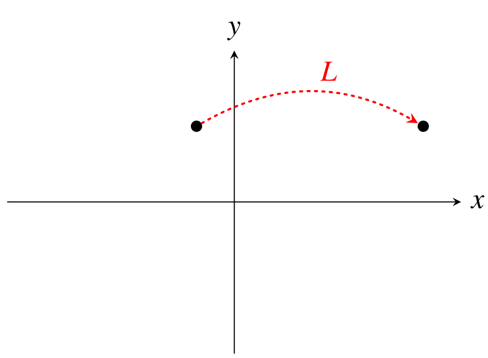
$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1*-1+3*2 \\ -2*-1+0*2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

To interpret this as a function, we see that the linear transformation L given by the rules:

$$u = x+3y$$

$$v = -2x,$$

takes $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Viewing these vectors as points:



■

■ **Question 3.1** Let M be the 2×2 matrix from Example 3.1, compute $M\hat{\mathbf{i}}$ and $M\hat{\mathbf{j}}$.

■ **Example 3.2** The linear transformation which reflected my picture into the correct orientation was given by:

$$u = -x,$$

$$v = y,$$

and therefore by the matrix:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying this to any point gives:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1*x+0*y \\ 0*x+1*y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

In particular, a computer would likely apply this function by using matrix multiplication on the coordinates of the pixels, rather than remember the functions and all the variables involved. ■

In Homework 1 (cf. Exercise 1.6) we asked the following question:

■ **Question 3.2** Let ℓ be a linear transformation of the plane. If I know $\ell\left(\hat{\mathbf{i}}\right)$ and $\ell\left(\hat{\mathbf{j}}\right)$, do I know the output of ℓ when applied to any element of $\mathbb{R}^2$?

Many folks already determined the answer. Let's see how this works in an example.

■ **Example 3.3** Suppose ℓ is a linear transformation, and that:

$$\ell\left(\hat{\mathbf{i}}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad \ell\left(\hat{\mathbf{j}}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

We also know ℓ corresponds to some matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore

$$M\hat{\mathbf{i}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a*1+b*0 \\ c*1+d*0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}.$$

In particular, $M\hat{\mathbf{i}}$ plucks out the first column of M, and since $M\hat{\mathbf{i}} = \ell\left(\hat{\mathbf{i}}\right)$ we have:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus $a = 1$ and $c = -2$. We can similarly compute that $M\hat{\mathbf{j}}$ is the second column of M:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a*0+b*1 \\ c*0+d*1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Since this is $\ell(\hat{\mathbf{j}})$ we have determined $b = 3$ and $d = 0$. In particular, we have determined:

$$M = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix},$$

which completely determines ℓ as the function:

$$u = x + 3y,$$

$$v = -2x.$$

This is in fact the function from Example 3.1. ■

To summarize, we have the following theorem.

Theorem 3.1.1 Let ℓ be a linear transformation of $\mathbb{R}^2$. Then ℓ is completely determined by the values $\ell(\hat{\mathbf{i}})$ and $\ell(\hat{\mathbf{j}})$. In particular, if:

$$\ell(\hat{\mathbf{i}}) = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \text{and} \qquad \ell(\hat{\mathbf{j}}) = \begin{bmatrix} b \\ d \end{bmatrix},$$

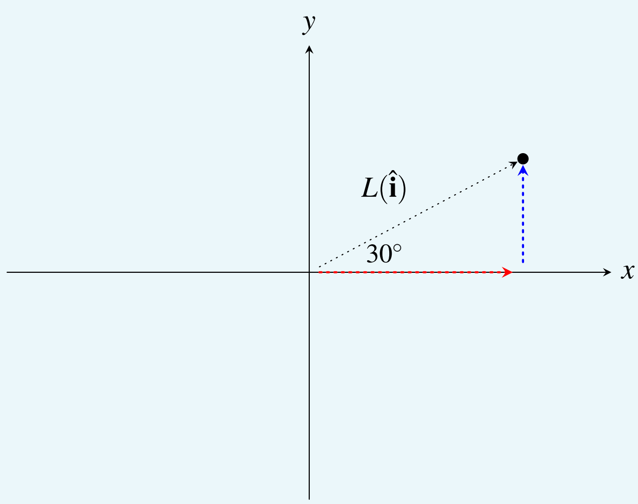
then the matrix associated to ℓ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

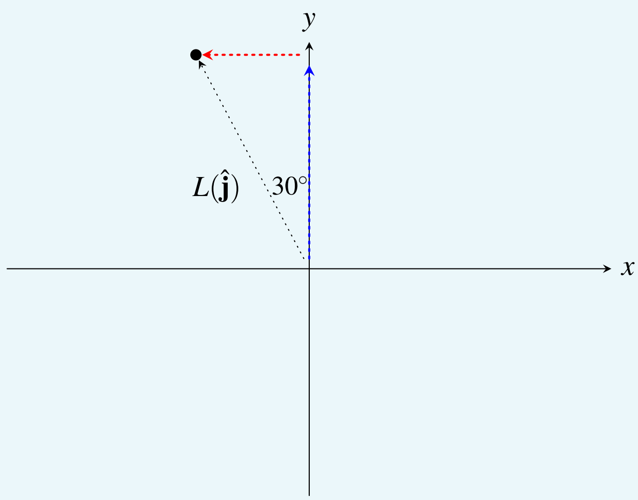
Exercise 3.1 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates the plane 30° counterclockwise. Let's determine the associated matrix. You may use the following facts:

$$\sin(30^\circ) = \frac{1}{2} = 0.5 \qquad \text{and} \qquad \cos(30^\circ) = \frac{\sqrt{3}}{2} \approx 0.866.$$

1. Find the lengths of the legs of the following triangle to determine $L(\hat{\mathbf{i}})$ as a column vector. (Note, since $\hat{\mathbf{i}}$ has length one, so does any rotation of $\hat{\mathbf{i}}$.)



2. Find the lengths of the legs of the following triangle to determine $L(\hat{\mathbf{j}})$ as a column vector. *Be careful with signs!* (Note, since $\hat{\mathbf{j}}$ has length one, so does any rotation of $\hat{\mathbf{j}}$.)



3. Let M be the matrix associated to the linear transformation L. Write down M. (Recall that $L(\hat{\mathbf{i}})$ and $L(\hat{\mathbf{j}})$ determine the columns of M).
4. Use matrix-vector multiplication to determine the image of the point $(-3, 1)$ after a $30°$ rotation. (You may leave your answer in exact form, or save one decimal point).

3.1.2 Linear Transformations and Linear Combinations

The entire *vector perspective* was motivated by Exercise 1.8, which suggested that a linear transformation plays well with addition. Thinking of the inputs and outputs as vectors, the property Exercise 1.8 suggested is that, for any linear transformation L and for any pair of 2d vectors $\mathbf{v}$ and $\mathbf{w}$, we have:

$$L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}).$$

If the matrix associated to L is M, then we can rewrite this in terms of matrix-vector multiplication:

$$M(\mathbf{v}+\mathbf{w}) = M\mathbf{v} + M\mathbf{w}.$$

But this is a familiar looking property: *the distributive property*! In particular, the fact that L commutes with addition is equivalent to the fact that matrix-vector multiplication satisfies the distributive property! We package this fact together with a related fact regarding scalar multiplication together in the following theorem.

Theorem 3.1.2 — Linearity of Linear Transformations: Planar Case. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ be vectors, and let c be a constant.
1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

Proof. We will prove the first statement. To do this, we introduce some notation. Let's represent L by the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and represent $\mathbf{v}$ and $\mathbf{w}$ by the column vectors:

$$\mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Then filling in the first row of $M(\mathbf{v}+\mathbf{w})$ gives:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix} = \begin{bmatrix} a(x_0+x_1) + b(y_0+y_1) \\ \dots \end{bmatrix}.$$

On the other hand, filling in the first row of $M\mathbf{v} + M\mathbf{w}$ gives:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 \\ \dots \end{bmatrix} + \begin{bmatrix} ax_1 + by_1 \\ \dots \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 + ax_1 + by_1 \\ \dots \end{bmatrix}.$$

The fact that these two first rows agree is simply the distributive property for usual addition. One can observe the second rows agree by an identical argument.[1]

We will leave the second part of the theorem for homework. ■

R One can prove Theorem 3.1.2 without comparing it to matrix multiplication, and just by plugging in generic values to the formula for a linear transformation. That being said, viewing this a matrix-vector multiplication, one can observe that the resemblance of Theorem 3.1.2.1 to the distributive property for matrix-vector multiplication is not purely cosmetic, it really boils down to the usual distributive property for numbers in each row.

■ **Question 3.3** Let L be a linear transformation, and suppose that $\mathbf{v}$ and $\mathbf{w}$ be two vectors, and suppose that $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Is it always true that $L(\mathbf{x})$ is a linear combination of $L(\mathbf{v})$ and $L(\mathbf{w})$?

In Question 3.3, we asked about what happens to linear combinations when we apply a linear transformation. Let's briefly revisit this to get a more geometric perspective on Theorem 3.1.1. If $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$, then (applying Theorem 3.1.2), we can see that:

$$L(\mathbf{x}) = L(a\mathbf{v} + b\mathbf{w}) = L(a\mathbf{v}) + L(b\mathbf{w}) = aL(\mathbf{v}) + bL(\mathbf{w}).$$

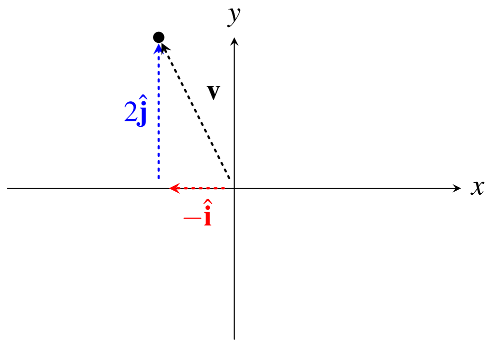
In particular, if $\mathbf{v} = \hat{\mathbf{i}}$ and $\mathbf{w} = \hat{\mathbf{j}}$, then a and b are the coordinates of $\mathbf{x}$, and we see that knowing these coordinates of together with the images of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ allows us to determin $L(\mathbf{x})$. Let's see this in the context of Example 3.3.

■ **Example 3.4** Adopt the setup of 3.1 and 3.3, and consider again the vector:

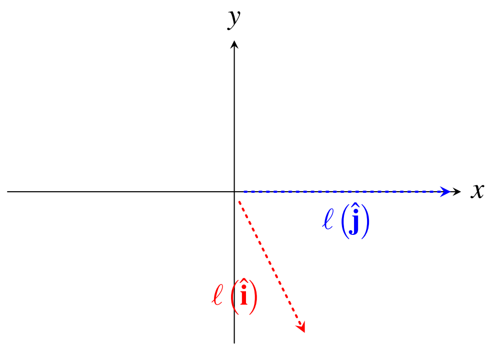
$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

In particular, $\mathbf{v}$ is acheived by doing $\hat{\mathbf{i}}$ backwards, and then doing $\hat{\mathbf{j}}$ twice.

[1] do it! do it! do it!



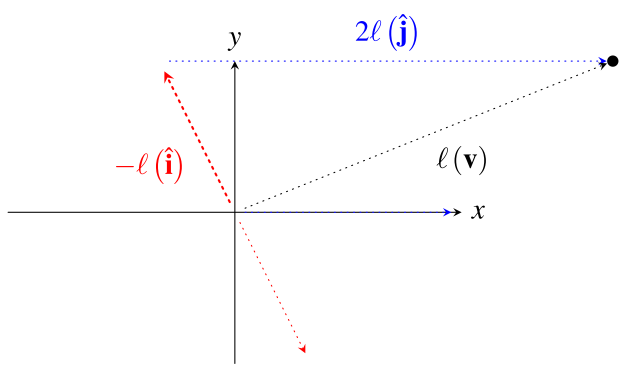
Now let's take a look at where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go after applying ℓ.



Theorem 3.1.1 suggests that knowing these values should be enough to know $\ell(\mathbf{v})$. And indeed, applying Theorem 3.1.2 we have:

$$\ell(\mathbf{v}) = \ell\left(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}}\right) = -\ell\left(\hat{\mathbf{i}}\right) + 2\ell\left(\hat{\mathbf{j}}\right).$$

So, because $\mathbf{v}$ does $\hat{\mathbf{i}}$ backwards and then $\hat{\mathbf{j}}$ twice, we know $\ell(\mathbf{v})$ does $\ell\left(\hat{\mathbf{i}}\right)$ backwards and then $\ell\left(\hat{\mathbf{j}}\right)$ twice. Let's throw that in the picture:



Throwing in some numbers: we can see that:

$$\ell(\mathbf{v}) = -\ell\left(\hat{\mathbf{i}}\right) + 2\ell\left(\hat{\mathbf{j}}\right) = -\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix},$$

which agrees with our output from Example 3.1 (as it must). The way to think about this is that when ℓ moves $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, it drags the entire grid along with it.[2] ■

3.1.3 Detecting Linearity

We've seen that linear transformations have really nice properties, but in order to utilize these properties, we have to make sure the function is linear. Our techniques for doing this so far haven't been particularly refined. For example, we can recognize that a function is linear if it's equations are linear with no constant terms...but to verify this you already have to know the entire function, making something like Theorem 3.1.1 somewhat less useful. In this section we will introduce an easy test for determining whether a function is linear. The starting point is Theorem 3.1.2.

In particular, suppose we are given a function:

$$F : \mathbb{R}^2 \to \mathbb{R}^2,$$

which satisfies the *conclusions* of Theorem 3.1.2. In particular, say we know that:

1. $F(\mathbf{x}+\mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$
2. $F(c\mathbf{x}) = cF(\mathbf{x})$.

What if I want to compute something like $F\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$. Well, using these two conditions I could observe that:

$$\begin{aligned} F\left(\begin{bmatrix}2\\3\end{bmatrix}\right) &= F(2\hat{\mathbf{i}}+3\hat{\mathbf{j}}) \\ &= F(2\hat{\mathbf{i}})+F(3\hat{\mathbf{j}}) \\ &= 2F(\hat{\mathbf{i}})+3F(\hat{\mathbf{j}}), \end{aligned}$$

In particular, if I know $F(\hat{\mathbf{i}})$ and $F(\hat{\mathbf{j}})$, that's enough information to figure out $F\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$. This is just like a linear function (cf. Theorem 3.1.1). And, of course, there was nothing special about the vector we chose. We could evaluate F at any vector in $\mathbb{R}^2$ the same way!

■ **Example 3.5** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfy the two conditions above, and let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function. Suppose also that:

$$F(\hat{\mathbf{i}}) = \begin{bmatrix}1\\7\end{bmatrix} \quad \text{and} \quad F(\hat{\mathbf{j}}) = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

$$L(\hat{\mathbf{i}}) = \begin{bmatrix}1\\7\end{bmatrix} \quad \text{and} \quad L(\hat{\mathbf{j}}) = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

Let's compare

$$F\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) \quad \text{and} \quad L\left(\begin{bmatrix}2\\-1\end{bmatrix}\right).$$

For F, we use the trick we used above:

$$F\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = F(2\hat{\mathbf{i}}-\hat{\mathbf{j}}) = 2F(\hat{\mathbf{i}})-F(\hat{\mathbf{j}}) = 2\begin{bmatrix}1\\7\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}3\\15\end{bmatrix}.$$

[2]This is animated brilliantly by Grant Sanderson: `https://3b1b-posts.us-east-1.linodeobjects.com//content/lessons/2016/linear-transformations/basis_example2.mp4`

For L, we start by finding the matrix for L using Theorem 3.1.1.

$$L = \begin{bmatrix} L(\hat{\mathbf{i}}) & L(\hat{\mathbf{j}}) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 7 & -1 \end{bmatrix}.$$

Then we can use matrix multiplication:

$$F\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 7 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}.$$

Of course, there was nothing special about $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ here. Using Theorem 3.1.2, we can see that this is true for any point.

$$\begin{aligned} L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= L(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \\ &= xL(\hat{\mathbf{i}}) + yL(\hat{\mathbf{j}}) \\ &= xF(\hat{\mathbf{i}}) + yF(\hat{\mathbf{j}}) \\ &= F(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \\ &= F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right). \end{aligned}$$

In particular, F and L are the same function! Therefore, since L has a matrix, so does F, and F is in fact a linear transformation. ■

In this example, we saw that because F satisfied the 2 conditions in the concusion of Theorem 3.1.2, F was in fact linear and has a matrix! The proof in the example didn't really rely on the specifics of F, just that it satisfied these conditions. This means that the converse of Theorem 3.1.2 is true as well. To summarize:

Theorem 3.1.3 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, and suppose that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and constant c, F satisfies the following conditions.

1. $F(\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$
2. $F(c\mathbf{x}) = cF(\mathbf{x})$

Then F is a linear transformation, and the matrix for F is:

$$\begin{bmatrix} F(\hat{\mathbf{i}}) & F(\hat{\mathbf{j}}) \end{bmatrix}.$$

3.2 September 19, 2024

3.2.1 Meaningfully interpreting mathematics in context

Before diving into the content, a bit of reflection. In Exercise 2.20.2 we are presented with 3 gas mixtures, gasses X,Y, and Z, which are combinations of Nitrogen, Oxygen, Carbon Dioxidee, and Argon. Each one is represented by a vector $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$, which records the components of the gas mixture (by weight) as its entries. We are then asked to describe *in plain english* the meaning of $\mathrm{span}\{\mathbf{x},\mathbf{y},\mathbf{z}\}$. Many folks said something along the lines of:

> $\mathrm{span}\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ is the collection of linear combinations of $\mathbf{x}, \mathbf{y},$ and $\mathbf{z}$.

Of course, this is a correct definition of the span, and this sentence would be correct no matter what the context, and no matter what the vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ represent. But I don't think I would call this *plain english*, and it doesn't really have anything to do with the problem at hand (mixing gasses in a greenhouse in space). When working on a problem like this, we prefer to think about things *in context*.

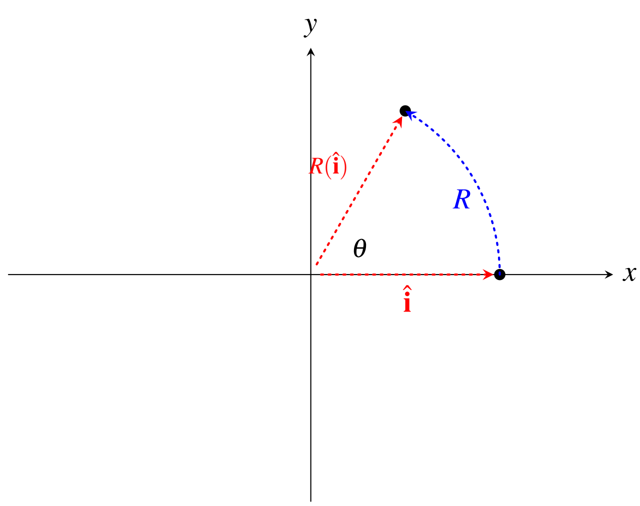
Rather than give the answer let's take a moment to think about this in the context of Exercise 2.16, where we study two vectors $\mathbf{x}$ and $\mathbf{y}$ representing the effect of adding an ounce of pigment X and pigment Y respectively. Then we could be asked to describe $\mathrm{span}\{\mathbf{x},\mathbf{y}\}$ in plain english. One could give the definition of span: *all vectors that are linear combinations of* $\mathbf{x}$ *and* $\mathbf{y}$ but this would leave out all the context in the problem. Instead, in context, one could say something like:

> $\mathrm{span}\{\mathbf{x},\mathbf{y}\}$ represents all colors that can be mixed from pigment X and pigment Y.

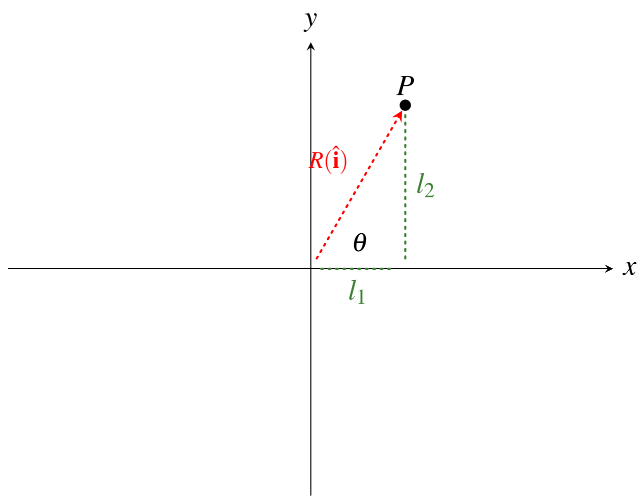
This is an accurate description of the span, *and provides context*. It is a description that keeps hold of the fact that, while doing all this vector math, *we are talking about mixing colors*. The purely mathematical description of the span erases this context.

3.2.2 An Example: Rotation Matrices

We can do something similar to what we did in Exercise 3.1 to compute matrices which can capture *any rotation of the plane*! Let's denote by R the linear function which rotates the plane counterclockwise by an angle of θ. To compute the matrix associated to R, it suffices to trace what the rotation does to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Let's start with $\hat{\mathbf{i}}$.



Here $R\left(\hat{\mathbf{i}}\right)$ starts at the origin, so to find it's representation as a column vector we can simply compute the coordinates of the point P where it ends. To do this we can do this by using the triangle below, together with some trig.



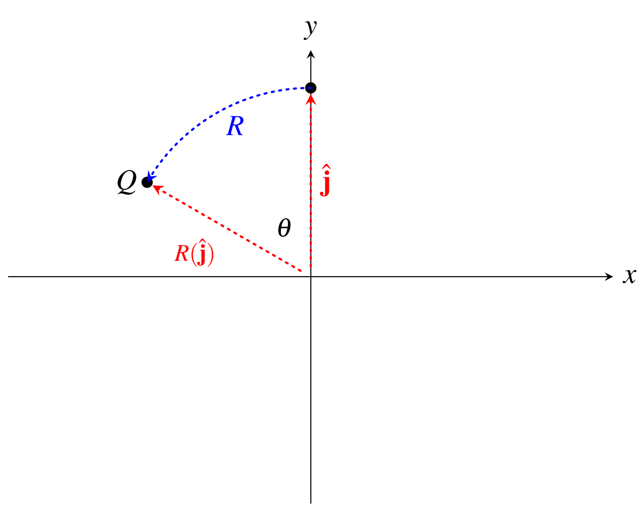
Once we compute the lengths l_1 and l_2, we will know that $P = (l_1, l_2)$. The length of $\hat{\mathbf{i}}$ is 1, and this remains true after rotation. As such, we can solve:

$$\cos\theta = \frac{l_1}{1}, \qquad \text{and} \qquad \sin\theta = \frac{l_2}{1}$$

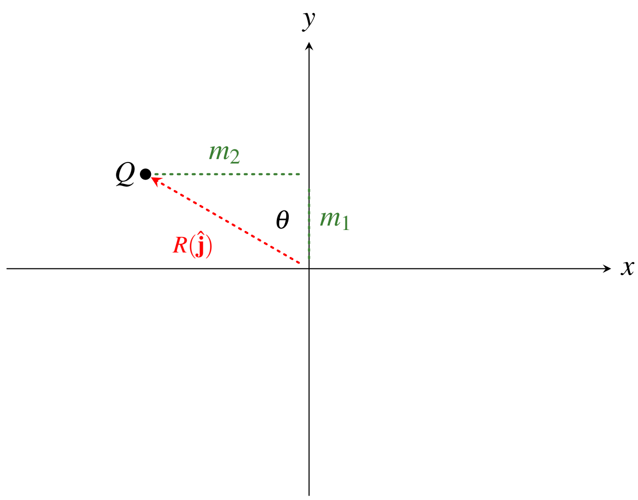
for $l_1 = \cos\theta$ and $l_2 = \sin\theta$. Therefore, $R(\hat{\mathbf{i}})$ is the vector from $(0,0)$ to $(\cos\theta, \sin\theta)$, so that we have:

$$R(\hat{\mathbf{i}}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

We can find $R(\hat{\mathbf{j}})$ similarly, looking for the coordinates of the point Q below



Again, we can can find the coordinates for Q by computing the lengths of the legs of the triangle below.



As above, because the length of $\hat{\mathbf{j}}$ is 1, length of $R(\hat{\mathbf{j}})$ is too, so that the trigonometric ratios tell us $m_1 = \cos\theta$ and $m_2 = \sin\theta$. Keeping signs in mind, we can conclude that $Q = (-m_2, m_1)$ so that:

$$R(\hat{\mathbf{j}}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

By Theorem 3.1.1, we can now conclude that the transformation R is given by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In summary, we have deduced the following result:

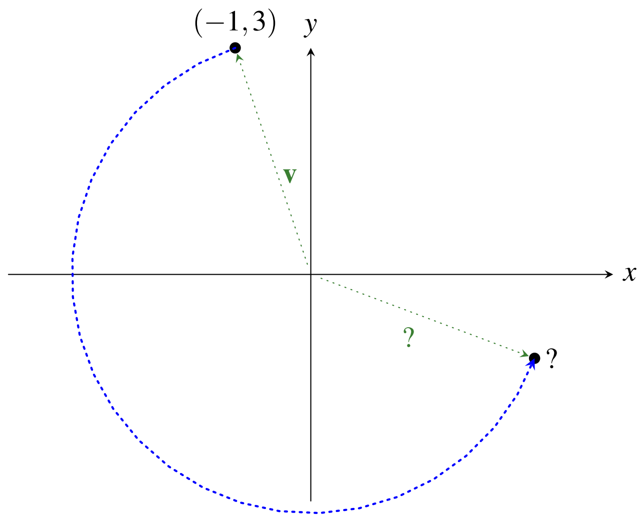
Proposition 3.2.1 Let R be the linear transformation which rotates the plane counterclockwise by an angle of θ, the R is given by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

R The careful reader may notice that our arguments above seemed to rely on the fact that θ was an acute angle. In fact, it is always true that the point unit circle making an angle of θ with the x-axis has coordinates $(\cos\theta, \sin\theta)$. This is actually *the definition* of the trigonometric functions for angles which are not acute. I encourage you to carefully work out the details!

Proposition 3.2.1 is a extremely powerful result, allowing for the rapid computation of any rotation using just matrix multiplication. This is very useful for rotating images on a screen, as doing trig in real time can be slow, but multiplying by matrices is quite fast!

■ **Example 3.6** If I rotate the plane 231°, where does the point $(-1, 3)$ end up?



Proposition 3.2.1 tells us that the function R which rotates the plane $231°$ is given by the matrix:

$$M = \begin{bmatrix} \cos(231°) & -\sin(231°) \\ \sin(231°) & \cos(231°) \end{bmatrix} = \begin{bmatrix} -0.629 & 0.777 \\ -0.777 & -0.629 \end{bmatrix}.$$

Letting $\mathbf{v}$ be the vector from $(0,0)$ to $(-1,3)$, we can compute:

$$M\mathbf{v} = \begin{bmatrix} -.629 & .777 \\ -.777 & -.629 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -.629*(-1)+.777*3 \\ -.777*(-1)-.629*3 \end{bmatrix} = \begin{bmatrix} 2.96 \\ -1.11 \end{bmatrix}.$$

Therefore we can conclude that after rotating $231°$, the point $(-1,3)$ moves to the point $(2.96,-1.11)$.
■

3.2.3 Linear Transformations of 3-space

So far we've been pretty focused on transformations of the plane, but in is also quite important in practice move beyond the plane. Let's begin by considering transformations of 3-dimensional space. For this it will be useful to give a definition of $\mathbb{R}^3$ which is analogous to that of $\mathbb{R}^2$.

Definition 3.2.1 The set $\mathbb{R}^3$ is the collection of 3-dimensional column vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

As before, we will sometimes conflate the idea of a 3d column vector with that of a 3d point (x,y,z) if our perspective is spacial. We actually already talked a bit about transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ in Homework 1 (Exercise 1.9). In particular, a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ can be defined by the rule:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

and we can define u, v, and w in terms of x,y,z:

$$u = u(x,y,z),$$
$$v = v(x,y,z),$$
$$w = w(x,y,z).$$

Example 3.7 Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the rule

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where

$$u = xyz,$$

$$v = x + y + z,$$

$$w = 1 + 2x.$$

Then, for example, we can compute:

$$T\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2*3*4 \\ 2+3+4 \\ 1+2*2 \end{bmatrix} = \begin{bmatrix} 24 \\ 9 \\ 5 \end{bmatrix}.$$

■

As with transformations of the plane, linear algebra focuses on functoins which are *purely linear*. That is, we want the equations of u, v, and w to be polynomials of degree 1, with no constant terms.

Definition 3.2.2 A function $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a *linear transformation* if it can be defined by the rule:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where:

$$u = ax + by + cz,$$

$$v = dx + ey + kz,$$

$$w = lx + my + nz,$$

for constants $a, b, c, d, e, k, l, m, n$.

Notice that the entire function is determined by the cofficients of x, y, and z in the equations for u, v, and w. Therefore, it is enough to remember just these coefficients, which we can arrange in a matrix.

Definition 3.2.3 The matrix associated to the linear transformation from Definition 3.2.2 is:

$$\begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

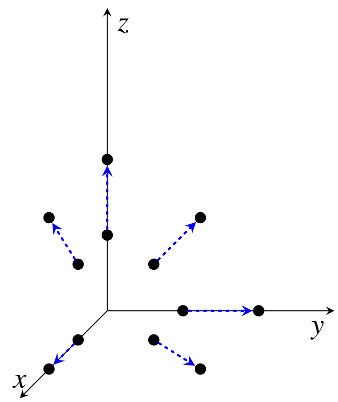
■ **Example 3.8** Consider the function $L : \mathbb{R}^3 \to \mathbb{R}^3$ which is given by the rules:

$$u = 2x \qquad v = 2y \qquad w = 2z.$$

The matrix associated to this function is:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let's visualize this as a transformation of space, like we did back at the beginning of the semester. Below we have a series of points. The points before and after applying T are connected by arrows.



It looks like points are being pushed away from the origin, and indeed:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so T takes a point and doubles its distance from the origin. ■

The fact that we can replace a function with a matrix tells us that we can get a formula for matrix-vector multiplication in 3d, analogous to Definition 3.1.2. Indeed, adopting the notation of Definitions 3.2.2 and 3.2.3, if T is the transformation associated to:

$$M = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

Then we can write:

$$\begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Plugging in for the formulas of u, v, w gives:

Definition 3.2.4 — Matix-Vector Multiplication: The 3×3 Case. The product of a 3×3 matrix and a column vector $\mathbf{v}$ can be computed as follows.

$$M\mathbf{v} = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+kz \\ lx+my+nz \end{bmatrix}.$$

This can follow a process just like in the 2×2 case, by going row by row in the matrix, and pairing each entry in the row with the appropriate entry in the vector.

$$\textbf{Row 1:} \quad \begin{bmatrix} \color{red}{a} & \color{blue}{b} & \color{green}{c} \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} \color{red}{x} \\ \color{blue}{y} \\ \color{green}{z} \end{bmatrix} = \begin{bmatrix} \color{red}{ax}+\color{blue}{by}+\color{green}{cz} \\ \\ \\ \end{bmatrix}$$

$$\textbf{Row 2:} \quad \begin{bmatrix} a & b & c \\ \color{red}{d} & \color{blue}{e} & \color{green}{k} \\ l & m & n \end{bmatrix} \begin{bmatrix} \color{red}{x} \\ \color{blue}{y} \\ \color{green}{z} \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ \color{red}{dx}+\color{blue}{ey}+\color{green}{kz} \\ \\ \end{bmatrix}$$

$$\textbf{Row 3:} \quad \begin{bmatrix} a & b & c \\ d & e & k \\ \color{red}{l} & \color{blue}{m} & \color{green}{n} \end{bmatrix} \begin{bmatrix} \color{red}{x} \\ \color{blue}{y} \\ \color{green}{z} \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+kz \\ \color{red}{lx}+\color{blue}{my}+\color{green}{nz} \end{bmatrix}$$

■ **Question 3.4** Compute the matrix product:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

As we get to bigger matrices, the process for matrix multiplication starts to get more tedious, and we will start to use computers to do it for us. But we will keep doing it by hand, just a little bit longer, until we can identify the general pattern and its meaning.

■ **Question 3.5** Let M be the 3×3 matrix from Question3.4. Compute:

$$M\hat{\mathbf{i}}, \quad M\hat{\mathbf{j}}, \quad M\hat{\mathbf{k}}.$$

Do you notice anything?

As you probably observed, we recovered the columns of M. In particular, a version of Theorem 3.1.1 holds true for 3×3 matrices as well.

Theorem 3.2.2 Let T be a linear transformation of $\mathbb{R}^3$. Then T is completely determined by the values $T\left(\hat{\mathbf{i}}\right)$, $T\left(\hat{\mathbf{j}}\right)$, and $T\left(\hat{\mathbf{k}}\right)$. In particular, if:

$$T\left(\hat{\mathbf{i}}\right) = \begin{bmatrix} a \\ d \\ l \end{bmatrix} \quad \text{and} \quad T\left(\hat{\mathbf{j}}\right) = \begin{bmatrix} b \\ e \\ m \end{bmatrix} \quad \text{and} \quad T\left(\hat{\mathbf{k}}\right) = \begin{bmatrix} c \\ k \\ n \end{bmatrix},$$

then the matrix associated to T is:

$$M = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

This theorem is very important in 3D modelling. Indeed, a rotation in 3 space is a complicated manuver to pin down, as there are 3-axes about which to rotate. That said, once you know where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ go, you can write down a matrix which completely captures the rotation! We will do this explicitly next week! Before moving on, we'd like to record that Theorem 3.1.2 holds true here as well, and can be computed directly using matrix multiplicaiton.

Theorem 3.2.3 — Linearity of Linear Transformations: 3D Case. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be vectors, and let c be a constant.

1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

3.2.4 Linear transformations between dimensions

We've talked for the time being about transformations from $\mathbb{R}^2$ to itself and from $\mathbb{R}^3$ to itself, but it is also sometimes important to think about transformations between different spaces. We've also been mainly thinking about $\mathbb{R}^2$ and $\mathbb{R}^3$ spacially, but sometimes in context we encounter linear transformations have sources and targets which aren't obviously spacial, but instead have some other concrete interpretations. In fact, *we have already done both of these things!* To see this, let's return to the paint mixing examples from Homework 2 (Exercises 2.16-2.19), and see how there was a linear transformation lying at the heart of it, even if it didn't quite look like that at the time.

To begin let's assign some specific meanings to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ which fit this context.

$$\mathbb{R}^2 = \left\{ \text{vectors } \begin{bmatrix} x \\ y \end{bmatrix} \text{ representing } x \text{ oz Pigment X and } y \text{ oz Pigment Y} \right\}.$$

$$\mathbb{R}^3 = \left\{ \text{vectors } \begin{bmatrix} r \\ g \\ b \end{bmatrix} \text{ representing } r \text{ units red, } g \text{ units green, and } b \text{ units blue} \right\}.$$

Then we can define a function:

$$F : \mathbb{R}^2 \to \mathbb{R}^3,$$

by the rule:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{the vector } \begin{bmatrix} r \\ g \\ b \end{bmatrix} \text{ representing the color mixed from } x \text{ oz Pigment X and } y \text{ oz Pigment Y}.$$

In fact, we can find equations for r, g, and b in terms of x and y. Indeed, we are given what the overall effects of our two pigments are.

Pigment X: Adding 1 ounce of Pigment X to paint adds:

1 unit of red, 2 units of green, 3 units of blue

Pigment Y: Adding 1 ounce of Pigment Y to paint adds:

7 units of red, 5 units of green, 2 units of blue

Therefore, we can compute:

$$r = 1 * x + 7 * y,$$

$$g = 2 * x + 5 * y,$$

$$b = 3 * x + 2 * y.$$

So we have computed equations for the function $F : \mathbb{R}^2 \to \mathbb{R}^3$. Even better, we can notice that r, g, and b are *linear functions* in x and y with no constant terms. So this looks like a linear function overall. This is completely determined by its coefficients, so following what we've done before, we can see that we know the whole function by just remember its coefficient matrix:

$$M = \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix}.$$

What happens if we try to extend the process of Matrix-Vector Multipication to this setting? Let's try with 3 ounces of say, 3 ounces of Pigment X and 3 ounces of Pigment Y. Well, going row by row:

$$\text{Row 1:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ \\ \\ \end{bmatrix},$$

$$\text{Row 2:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ 2*3+5*3 \\ \\ \end{bmatrix},$$

$$\text{Row 3:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ 2*3+5*3 \\ 3*3+2*3 \end{bmatrix} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix}.$$

Looks like the output is the vector representing the color containing

24 units of red, 21 units of green, 15 units of blue,

which is fancy gold! We learned in Exercise 2.17 tha this is exactly what we should get when mixing 3 ounces of Pigment X and 3 ounces of Pigment Y. That is:

$$M \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix} = F\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} \right),$$

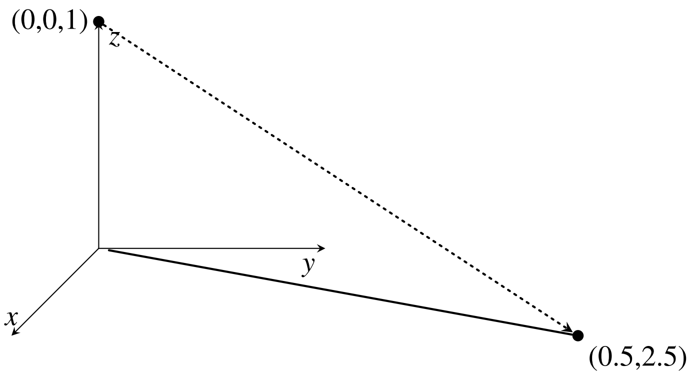
so as before, *matrix multiplication corresponds to applying the linear function!*.

■ **Question 3.6** Let $\hat{\mathbf{i}}$ be the vector in $\mathbb{R}^2$ representing one ounce of Pigment X, and let $\hat{\mathbf{j}}$ represent one ounce of Pigment Y. Compute:

$$M\hat{\mathbf{i}} \qquad \text{and} \qquad M\hat{\mathbf{j}}.$$

As you might observe, you yet again obtain the columns of M, so as before, so these values determine all of F and Theorem 3.1.1 holds! Of course, here it is no surprise that knowing the effect of one ounce of Pigment X and one ounce of Pigment Y is enough to tell you the effect of any mixture of them. It seems like adding this type context makes certain results easier or expect than it is initially in the purely geometric setting.

Let's see if we can run this philosophy in reverse. Let's choose coordinates (in meters), and place a meterstick vertically at the origin, just outside the library at ODY. It casts a shadow to a point on the ground 0.5 meters east and 2.5 meters north of the stick.



We then defined a linear function $S : \mathbb{R}^3 \to \mathbb{R}^2$ defined as follows:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

if a point x meters east and y meters north of the origin and z-meters off the ground casts its shadow on the ground at a point u-meters east and v-meters north of the origin. So for example, our picture tells us that:

$$S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 \\ 2.5 \end{bmatrix}.$$

■ **Question 3.7** What are $S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ and $S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$?

Every linear map we've seen so far comes with a matrix, and the columns of that matrix are precisely the values of the linear map applied to the standard basis. Let's see what that philosophy can tell us in this situation. Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be the standard basis of $\mathbb{R}^3$. Then S should have some matrix whose columns are $S(\hat{\mathbf{i}}), S(\hat{\mathbf{j}}), S(\hat{\mathbf{k}})$.

$$N = \begin{bmatrix} S(\hat{\mathbf{i}}) & S(\hat{\mathbf{j}}) & S(\hat{\mathbf{k}}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix}.$$

The entries of this matrix have always arisen as the coefficients of the linear equations defining our linear function. Assuming that holds here too, what would the equations for S be? We know S takes 3 inputs (x,y,z) and has 2 outputs (u,v), so we are looking for:

$$u = u(x,y,z),$$

$$v = v(x,y,z).$$

Let's go row by row:

$$\textbf{Row 1}: \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \longrightarrow u = 1x + 0y + 0.5z.$$

$$\textbf{Row 2:} \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \longrightarrow v = 0x + 1y + 2.5z.$$

We have now deduced what our function should be:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

where

$$u = x + 0.5z,$$

$$v = y + 2.5z.$$

So, for example, Gunnison Chappel is about 65 meters east from where we placed our meterstick, and stands about 30 meters tall. This gives the tip of the chapel the coordinates $(65,0,30)$. If we want to know where it casts its shadow, we can compute:

$$u = 65 + 0.5 * 30 = 80,$$

$$v = 0 + 2.5 * 30 = 75.$$

So it casts a shadow 80 meters east, and 75 meters north, of our meterstick. *What is remarkable about this, is we measured the point of a single shadow and we were able to determine another!* Finally, let's recall that we discovered the process for matrix-vector multiplication by thinking about multiplying a matrix as the same as applying the associated linear function. This works here too: if we apply the process of matrix multiplication, we get the same answer as applying S.

$$\textbf{Row 1}: \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \begin{bmatrix} 65 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 1*65+0*0+0.5*30 \\ \end{bmatrix} = \begin{bmatrix} 80 \\ \end{bmatrix},$$

$$\textbf{Row 2:} \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \begin{bmatrix} 65 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 1*65+0*0+0.5*30 \\ 0*65+1*0+2.5*30 \end{bmatrix} = \begin{bmatrix} 80 \\ 75 \end{bmatrix}.$$

3.3 Exercises

Exercise 3.2 Compute the following Matrix-Vector products.

1. $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. $\begin{bmatrix} 8 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$
4. $\begin{bmatrix} 8 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

■

Exercise 3.3 A linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ has the following effects on $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{j}}$:

$$T(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T(\hat{\mathbf{j}}) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \qquad T(\hat{\mathbf{k}}) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

What is $T(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 17\hat{\mathbf{k}})$? ■

Exercise 3.4 A linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has the following effects on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

$$L(\hat{\mathbf{i}}) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \qquad \text{and} \qquad L(\hat{\mathbf{j}}) = \begin{bmatrix} -7 \\ 13 \end{bmatrix}.$$

What has a larger magnitude:

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \qquad \text{or} \qquad L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)?$$

■

Exercise 3.5 Consider the following 2 vectors in $\mathbb{R}^2$.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1. Do you think $\text{span}\{\mathbf{v}, \mathbf{w}\} = \mathbb{R}^2$? Why or why not?
2. Let $\mathbf{x} = 7\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (*Hint:* Extract a system of equations from the expression $a\mathbf{v} + b\mathbf{w} = \mathbf{x}$, and then solve for the constants a and b).
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose that:

$$T(\mathbf{v}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{w}) = \begin{bmatrix} -6 \\ -5 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ by writing it as a linear combination of $T(\mathbf{v})$ and $T(\mathbf{w})$.

Exercise 3.6 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^2$ whose span is all of $\mathbb{R}^2$. Do you agree or disagree with the following statement? Explain your reasoning. (Use the intuition gained from Question 3.5.)

The values $L(\mathbf{v})$ and $L(\mathbf{w})$ determine $L(\mathbf{x})$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$.

Exercise 3.7 Let's prove Theorem 3.1.2.2. It states the following: If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $\mathbf{v}$ is a vector in $\mathbb{R}^2$, and n is a constant. Then

$$nL(\mathbf{v}) = L(n\mathbf{v}).$$

To do this, we'll introduce some notation. We denote the matrix associated to L by:

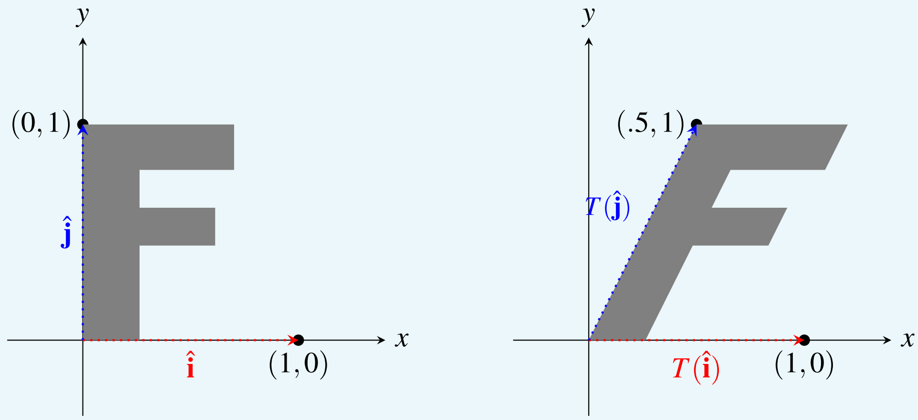
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We also give coordinates to $\mathbf{v}$:

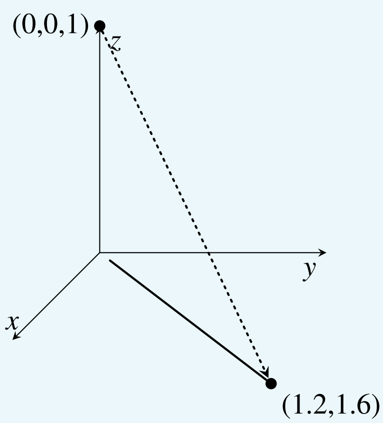
$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

1. Write a column vector for $n(M\mathbf{v})$ in terms of the variables a, b, c, d, n, x, and y. (First do the Matrix-Vector product $M\mathbf{v}$, then scale the result by n).
2. Write a column vector for $M(n\mathbf{v})$ in terms of the variables a, b, c, d, n, x, and y. (First write $n\mathbf{v}$ as a column vector in terms of n, x, and y, and then multiply this column by M).
3. Compare your answers to (a) and (b) to explain why the Theorem is true.

Exercise 3.8 A computer translates images from blockstyle fonts to *italics* by applying a linear transformation called a *shear*. Below is an image of the letter F before and after applying the shear. Use this image to determine the matrix associated to the shearing transformation.



Exercise 3.9 We can use linear maps to calculate how shadows are cast. Choose some coordinates in meters, put a meterstick vertically at the origin, and measure that it casts its shadow on the point 1.2 meters east and 1.6 meters north of the stick.



We define a linear function $S : \mathbb{R}^3 \to \mathbb{R}^2$ which takes a vector point in space (denoted by a vector in $\mathbb{R}^3$) to the point at which it casts its shadow (denoted by a vector in $\mathbb{R}^2$). **Notice: the input of this function is 3-dimensional, and the output is 2-dimensional. In particular, the general setup looks something like:**

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}.$$

1. What are $S(\hat{\mathbf{i}}), S(\hat{\mathbf{j}}), S(\hat{\mathbf{k}})$? Your answers should be 2D vectors. (*Hint*: You should be able to extract $S(\hat{\mathbf{k}})$ from the picture above. For the other two...where does a point on the ground cast its shadow?)
2. We've seen that many linear maps can be captured by matrices, and that the columns of

these matrices can be recovered by where the standard basis vectors are sent. Use this philosophy to write down a matrix which could represent S.

3. Extract some equations which could represent $S : \mathbb{R}^3 \to \mathbb{R}^2$ from the matrix in part (b).
4. The Eiffel Tower stands 330 meters tall. It also happens to be 500m west and 350m north from where you planted your meter stick at the start of this exercise. Use the work you've done to compute the coordinates of where the tip of its shadow should land (you may assume that Paris is completely flat). Try this with the equations from question 3, *and* with matrix vector multiplication.
5. Do you agree or disagree with the following statement? Explain your reasoning.

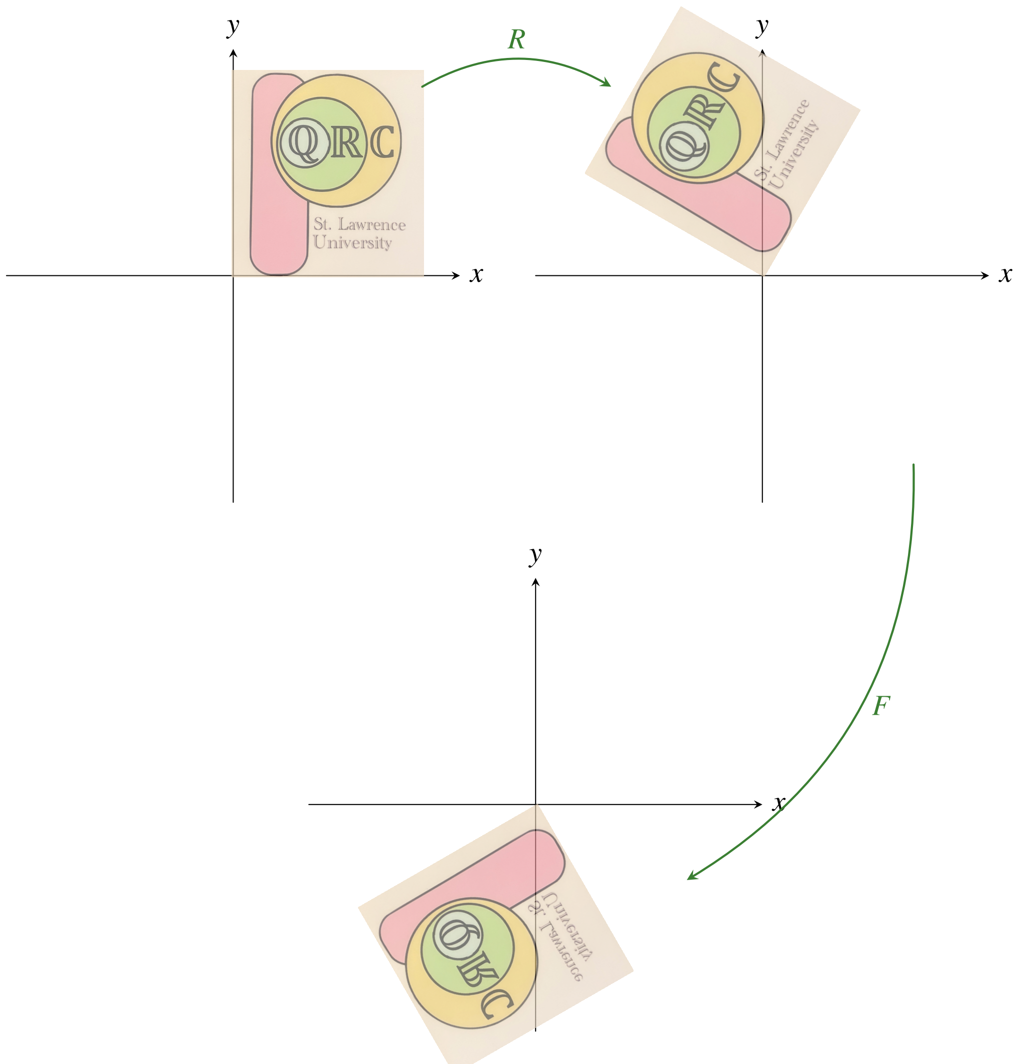
Once I know where a single point *above the ground* casts its shadow, I can compute where any point casts its shadow.

■

3.4 September 24, 2024

3.4.1 Composition of Linear Transformations

Suppose I start with an image that I'd like to manipulate an image of the PQRC logo in order to place it on a T-Shirt. There are two things I'd like to do. First, I want to rotate it 60°, and then I'd like to reflect it vertically over the x-axis.

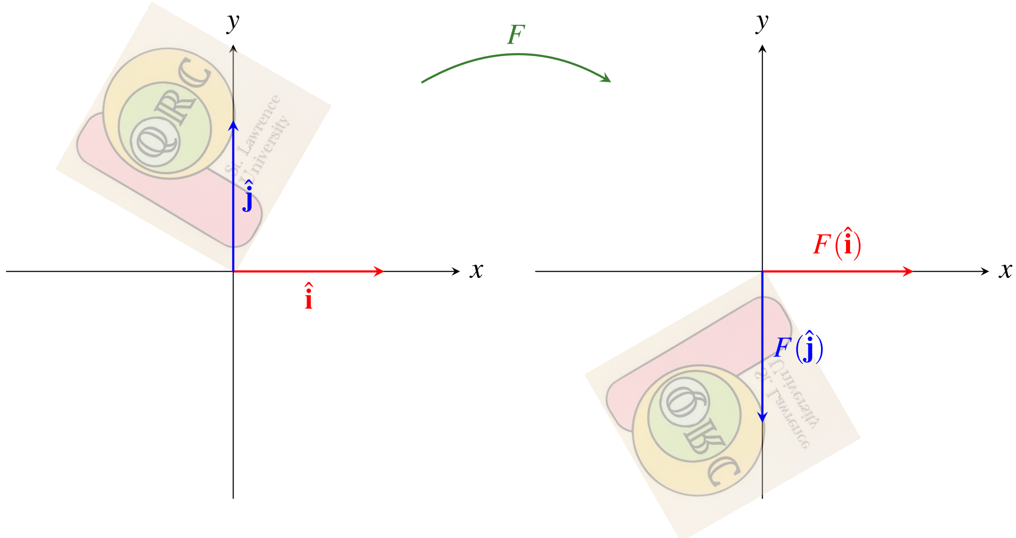


The transformatoins R (for *rotate*) and F (for *flip*) are both linear, and therefore each have an associated matrix. Let's find the matrix for the transformation with *rotates* and then *flips*. First let's find the matrix for the rotation R. In fact, in Section 3.2.2 we deduced that R is given by the rotation matrix:

$$N = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 0.5 & -.866 \\ .866 & 0.5 \end{bmatrix}.$$

What about F? To find the matrix for F, it is enough to compute $F(\hat{\mathbf{i}})$ and $F(\hat{\mathbf{j}})$.

■ **Question 3.8** Can you compute the matrix for F?



In particular, we have:

$$F(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad F(\hat{\mathbf{j}}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore the matrix for F is:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

To find the matrix for *rotate then flip*, we will follow a similar philosphy, by trying to track where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go. That is, we'd like to know the values $F(R(\hat{\mathbf{i}}))$ and $F(R(\hat{\mathbf{j}}))$. Before computing this, let's briefly unpack the notation. Given a vector $\mathbf{v}$ in $\mathbb{R}^2$, the value $R(\mathbf{v})$ represents where the vector $\mathbf{v}$ is after rotating. This is another vector in $\mathbb{R}^2$, and therefore it can be fed to the function F to be flipped. This value is $F(R(\mathbf{v}))$.

To compute $F(R(\hat{\mathbf{i}}))$, we first compute $R(\hat{\mathbf{i}})$, and we feed whatever the output is to F. But we can easily determine $R(\hat{\mathbf{i}})$: it is the first column of the matrix for R:

$$R(\hat{\mathbf{i}}) = \begin{bmatrix} .5 & -.866 \\ .866 & .5 \end{bmatrix} \begin{bmatrix} .5 \\ .866 \end{bmatrix}.$$

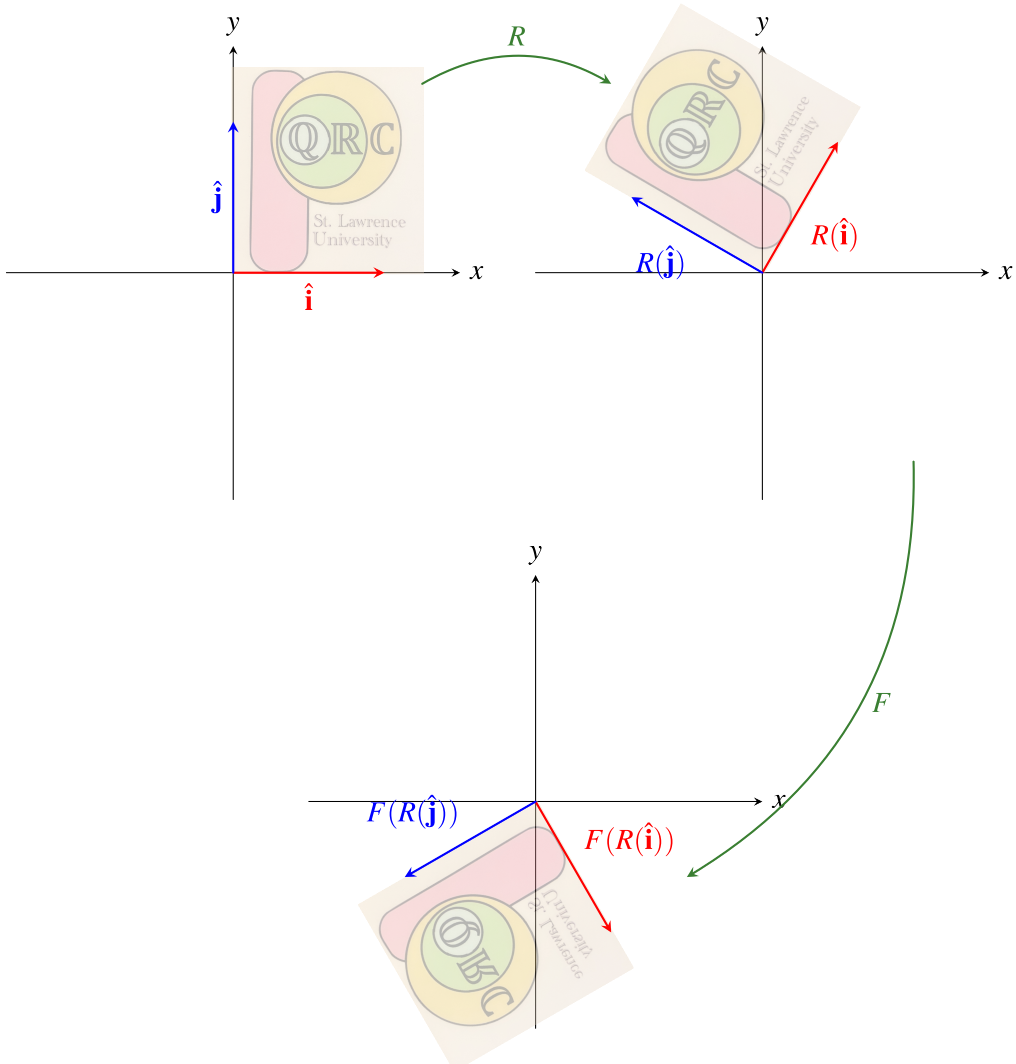
Therefore to compute $F(R(\hat{\mathbf{i}}))$, we can feed this output to F (i.e., multiply it by N).

$$F(R(\hat{\mathbf{i}})) = F\left(\begin{bmatrix} .5 \\ .866 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} .5 \\ .866 \end{bmatrix} = \begin{bmatrix} 1*.5+0*.866 \\ 0*.5+(-1)*.866 \end{bmatrix} = \begin{bmatrix} .5 \\ -.866 \end{bmatrix}.$$

Our strategy to compute $F(R(\hat{\mathbf{j}}))$ is similar, first noticing that $R(\hat{\mathbf{j}})$ is just the second column of the matrix for R, and then applying F to this column.

$$F(R(\hat{\mathbf{j}})) = F\left(\begin{bmatrix} -0.866 \\ .5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -.866 \\ .5 \end{bmatrix} = \begin{bmatrix} 1*(-.866)+0*.5 \\ 0*(-.866)+(-1)*.5 \end{bmatrix} = \begin{bmatrix} -.866 \\ -.5 \end{bmatrix}.$$

This process is illutrating in the following diagram.



We can now write down the matrix for for *rotate, then flip*, since its columns are the values on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ respectively. Let's call it P.

$$P = \begin{bmatrix} F(R(\hat{\mathbf{i}})) & F(R(\hat{\mathbf{j}})) \end{bmatrix} = \begin{bmatrix} .5 & -.866 \\ -.866 & -.5 \end{bmatrix}.$$

In particular, to *rotate, then flip* point in the image corresponding to a vector $\mathbf{v}$, we can just multiply by this matrix!

$$F(R(\mathbf{v})) = P\mathbf{v}.$$

Since *rotating* (applying R) is the same as multiplying by N, and *flipping* (applying F) is the same as multiplying by its matrix M, we can substitute this in:

$$MN\mathbf{v} = P\mathbf{v}.$$

It seems reasonable, then, to call this matrix P the *product* of M and N:

$$MN = P.$$

Exercise 3.10 Let's experiment with composition using functions whose domain might not be $\mathbb{R}^2$.

1. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates the *counterclockwise* by $90°$. Find the matrix for R. (Recall that $\sin 90° = 1$ and $\cos 90° = 0$.)
2. A matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ sends a vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ to $T(\mathbf{v}) = \begin{bmatrix} u \\ v \end{bmatrix}$ in $\mathbb{R}^2$ where:
$$u = 2x - z \qquad \text{and} \qquad v = -x - y + 5z.$$
Find the matrix associated to T.
3. Compute $R(T(\hat{\mathbf{i}})), R(T(\hat{\mathbf{j}}))$, and $R(T(\hat{\mathbf{k}}))$. (These should be 2-dimensional column vectors!)
4. To find the matrix for a linear transformation, we see what happens to the standard basis and use these outputs as our columns. Use this philosophy to find a matrix for the composition $R \circ T$ (this is the function which first applies T, and then applies R to the result).

■

3.4.2 Matrix Multiplication

Let's set this up more generally. First we recall the definition of the composition of two functions

Definition 3.4.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $L : \mathbb{R}^2 \to \mathbb{R}^2$ be transformations. The *composition* of L with T is the transformation: $L \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ that is obtained by first doing T, and then applying L to the result:
$$(L \circ T)(\mathbf{v}) = L(T(\mathbf{v})).$$

This is nice to visualize as follows:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T"] \arrow[rr, bend right, "L\circ T"'] & \mathbb{R}^2 \arrow[r, "L"] & \mathbb{R}^2 \end{tikzcd}$$

If T and L are linear, then they each come with a matrix.

Definition 3.4.2 — Matrix Multiplication: 2×2 case. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $L : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations, with associated matrices N and M respectively. Then the matrix product MN is the matrix associated to the composition $L \circ T$.

Let's find a formula for the matrix product, following the example of manipulating the PQRC sticker. Suppose:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then, since MN is associated to the composition $L \circ T$, it's columns are $L(T(\hat{\mathbf{i}}))$ and $L(T(\hat{\mathbf{j}}))$ respectively. Since $T(\hat{\mathbf{i}})$ and $T(\hat{\mathbf{j}})$ are the columns of N, we can compute these directly:

$$L(T(\hat{\mathbf{i}})) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} ap+br \\ cp+dr \end{bmatrix}$$

$$L(T(\hat{\mathbf{i}})) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q \\ s \end{bmatrix} = \begin{bmatrix} aq+bs \\ cq+ds \end{bmatrix}$$

Therefore we have established the usual formula for matrix multiplication.

Theorem 3.4.1 — A formula for 2×2 matrix multiplication. If

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

then the product:

$$MN = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

Much like matrix-vector multiplication, this is often best remembered as a *process*, where each column is a matrix-vector multiplication. In particular, to know the ij-entry of MN, you can pair the i'th row of M with the j'th column of N, multiplying the associated entries of each and adding them up.

$$\begin{bmatrix} \color{red}{a} & \color{blue}{b} \\ c & d \end{bmatrix} \begin{bmatrix} \color{red}{p} & q \\ \color{blue}{r} & s \end{bmatrix} = \begin{bmatrix} \color{red}{ap}+\color{blue}{br} & \\ & \end{bmatrix} \qquad \begin{bmatrix} \color{red}{a} & \color{blue}{b} \\ c & d \end{bmatrix} \begin{bmatrix} p & \color{red}{q} \\ r & \color{blue}{s} \end{bmatrix} = \begin{bmatrix} ap+br & \color{red}{aq}+\color{blue}{bs} \\ & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ \color{red}{c} & \color{blue}{d} \end{bmatrix} \begin{bmatrix} \color{red}{p} & q \\ \color{blue}{r} & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ \color{red}{cp}+\color{blue}{dr} & \end{bmatrix} \qquad \begin{bmatrix} a & b \\ \color{red}{c} & \color{blue}{d} \end{bmatrix} \begin{bmatrix} p & \color{red}{q} \\ r & \color{blue}{s} \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & \color{red}{cq}+\color{blue}{ds} \end{bmatrix}$$

■ **Question 3.9** Compute the matrix product:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 10 \end{bmatrix}$$

We've now got a general formula for how to multiply two matrices. Now, when we multiply two numbers, it doesn't really matter what order we do it in. For example:[3]

$$2*3 = 3*2$$

Is this true for matrices as well? Let' investigate.

■ **Question 3.10** Does order matter in matrix multiplication?

It's perhaps easiest to just look at an example and do some computations.

[3]This feels automatic because of how comfortable we are with it, but depending on how you define multiplication, it is slightly nontrivial. For example, we might be comparing *2 boxes with 3 things each* and *3 boxes with 2 things each*.

■ **Question 3.11** Consider the matrices:

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute the products MN and NM. Are they the same?

If we compute we get:

$$MN = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

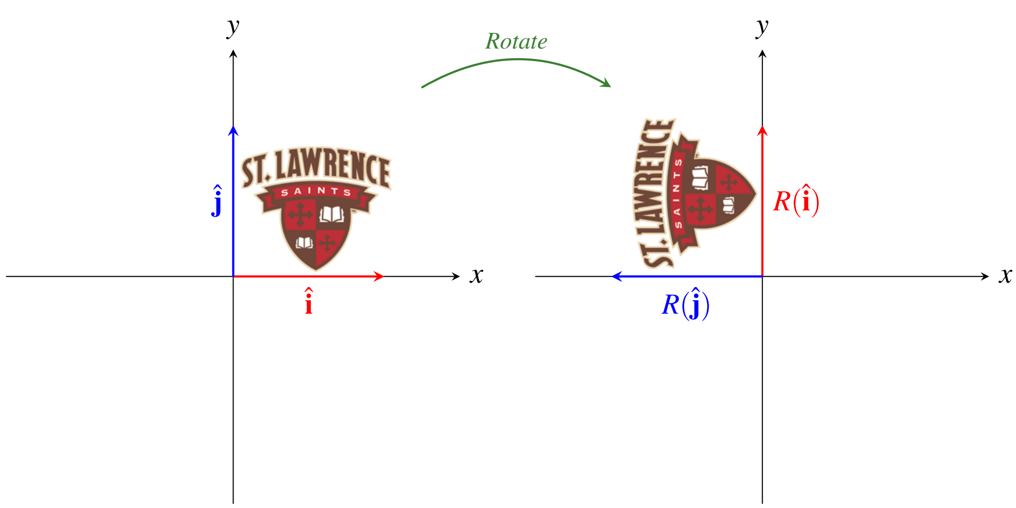
$$NM = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

So $MN \neq NM$. This gives an answer to Question 3.10: *yes, order does matter.* But, I don't think it's a very satisfying one. We can look at MN and NM and say: *see? They're different!* But it doesn't really tell us *why* they are different in any concrete way. To investigate this question a bit further, let's remember our guiding philosophy with matrices: *a matrix is a function!*. So what are the functions associated to M and N?

Let's start with M. Let's call the associated funtion $R : \mathbb{R}^2 \to \mathbb{R}^2$. To determine what R does, let's start by asking what it does to the standard basis vectors $\hat{\mathbf{i}}$ nad $\hat{\mathbf{j}}$. We can read this information off of the columns of the matrix M.

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad R(\hat{\mathbf{i}}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R(\hat{\mathbf{j}}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

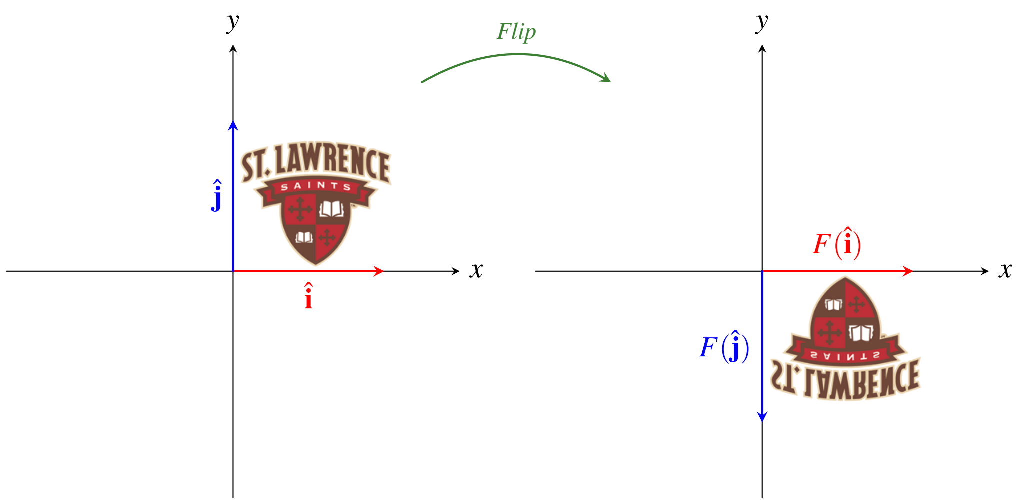
In particular, both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are rotated 90°. Since this determines the entire map, R must be the 90° rotation of the plane.



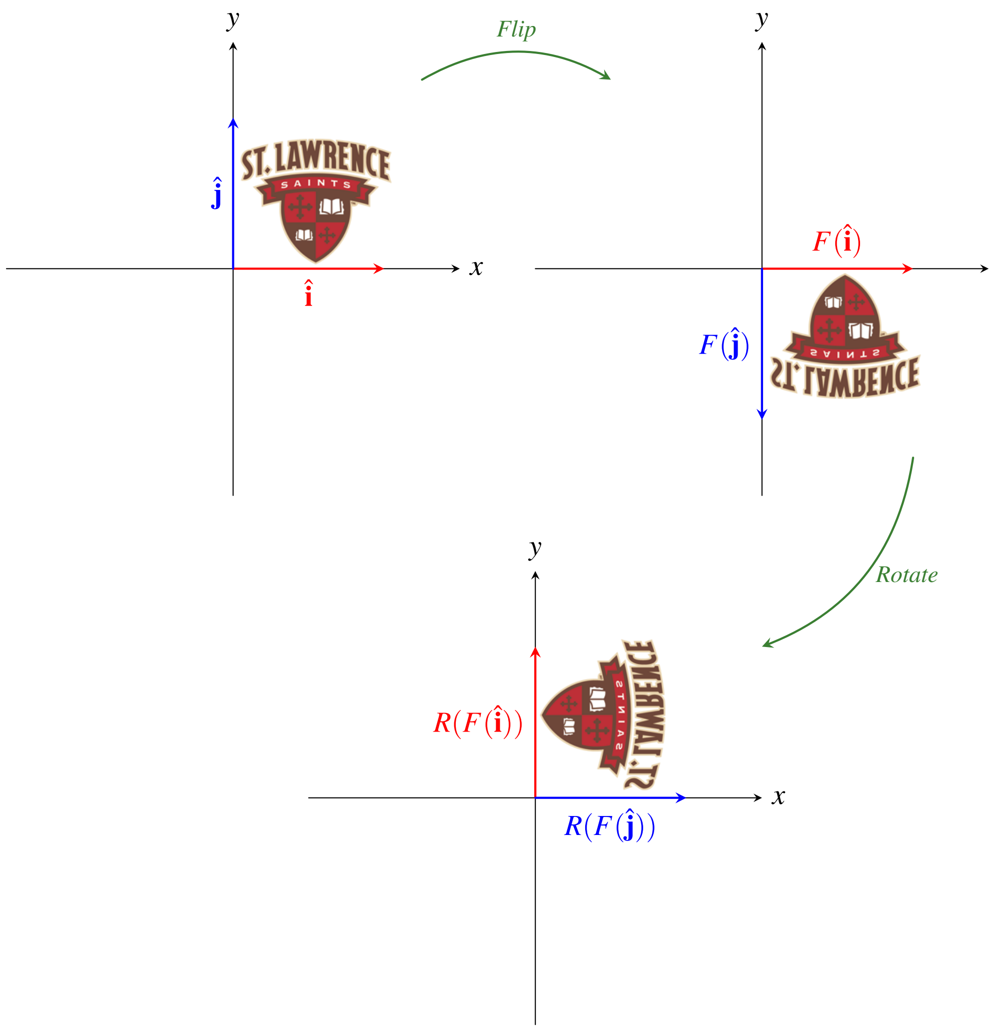
Let's do the same with N. Let's call the associated funtion $F : \mathbb{R}^2 \to \mathbb{R}^2$. To determine what F does, let's again ask what it does to the standard basis vectors $\hat{\mathbf{i}}$ nad $\hat{\mathbf{j}}$. We can read this information off of the columns of the matrix N.

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad F(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad F(\hat{\mathbf{j}}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We see that $\hat{\mathbf{i}}$ remains fixed and $\hat{\mathbf{j}}$ is flipped upside down, in particular both are reflected over the x-axis, so F is the reflection map over the x-axis (we saw this map in Section 3.4.1),

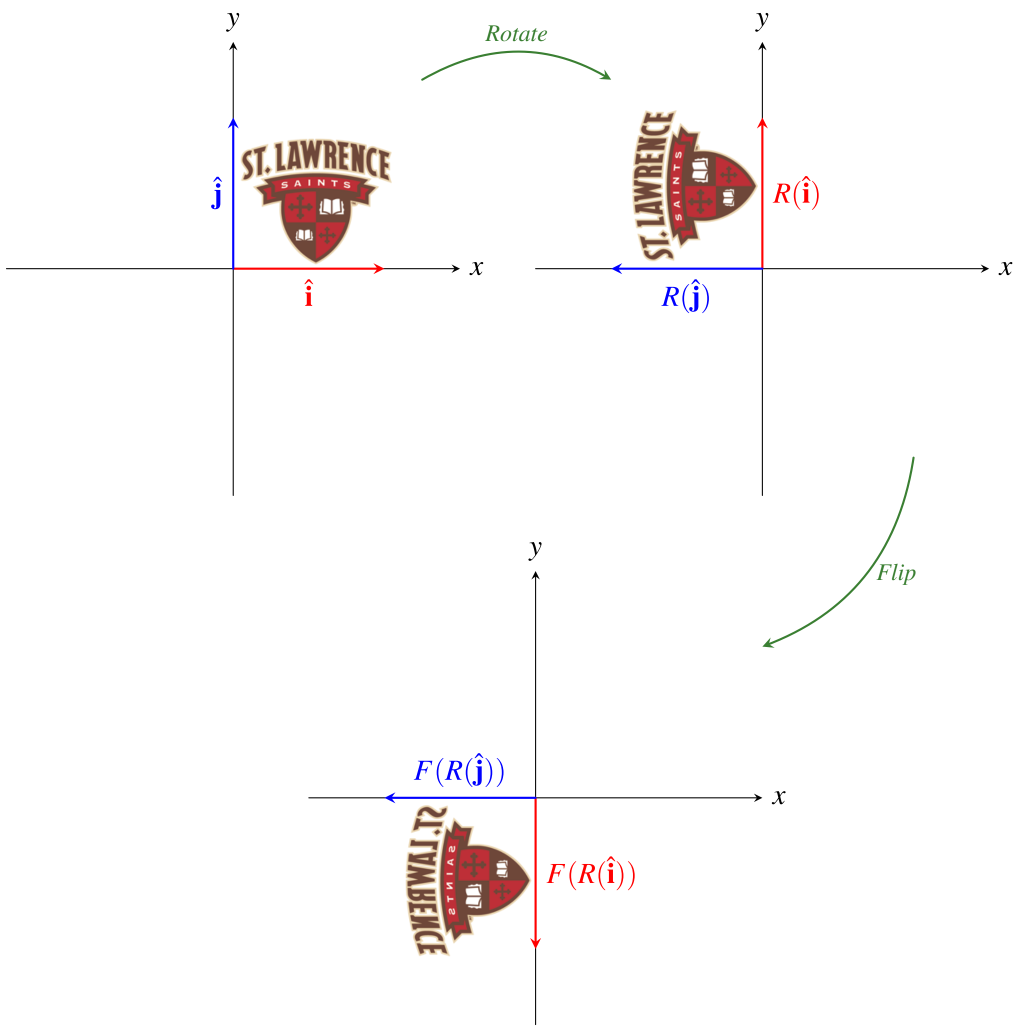


The product MN takes a vector $\mathbf{v}$ to $MN\mathbf{v} = R(F(\mathbf{v}))$, so its corresponding function is $R \circ F$: *first flip vertically, then rotate 90 degrees*. On the other hand, NM takes a vector $\mathbf{v}$ to $NM\mathbf{v} = F(R(\mathbf{v}))$, so its corresponding function is $F \circ R$: *first rotate 90 degrees, then flip vertically*. Should we expect *flip then rotate* to be the same as *rotate then flip*? Let's Look at a couple of images. First let's look at the effect of MN (or $R \circ F$): which flips first, and then rotates.



■ **Question 3.12** The matrix for $R \circ F$ is MN. Confirm that the coordinates for $R(F(\hat{\mathbf{i}}))$ and $R(F(\hat{\mathbf{j}}))$ are consistent with the columns of the matrix MN we computed above.

Now let's do the same experiment, but for the effect of NM, which corresponds to the function $F \circ R$: or *rotate, then flip*.



■ **Question 3.13** The matrix for $F \circ R$ is NM. Confirm that the coordinates for $F(R(\hat{\mathbf{i}}))$ and $F(R(\hat{\mathbf{j}}))$ are consistent with the columns of the matrix MN we computed above.

A visual inspection shows that *rotate, then flip* and *flip, then rotate* do different things. This, to me, gives a far more satisfying reason for why order matters in matrix multiplication: *order matters when you compose functions!*. This is a huge advantage of having both an algebraic perspective and a functional one. The algebraic perspective allows you to compute things, but the functional perspective *means something!* I think the most important takeaway from this section so far is the following:

■ **Slogan 3.1** A matrix is a function. Multiplying matrices is the same as composing functions.

R Above we had F for *flipping* and R for *rotating*. It may be a bit perplexing to see $F \circ R$ read that as *rotate, then flip*. In particular, we usually read from left to right, so why is it the case that in this instance we read right to left? The reason has to do with functional notation. In particular, when we have a function (say f), and we want to evaluate it at a value (say x), we put that value to the right of the function (so $f(x)$). Back to rotating and then flipping: if we start by rotating a vector $\mathbf{v}$, we feed it to the function, resulting in $R(\mathbf{v})$. If we want to flip the resulting value, we feed the whole thing to F (again on the right), resulting in $F(R(\mathbf{v}))$. The convention of the function *eating* the value to the right of it leads to us having to read from right to left. As a result, our function interpretation of matrix multiplication has us reading right to left as well. That is, the product MN is the function that takes a vector, first multiplies it by N, and then multiplies the result by M.

3.5 September 26, 2024

3.5.1 The Identity Matrix

In Homework 1 (Exercise 1.4) we were presented with the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

■ **Question 3.14** Let $\mathbf{v} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Compute $I\mathbf{v}$.

If you got $\mathbf{v}$ back, great job! In fact, in Exercise 1.4 was associated to the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose equations were:

$$u = x,$$

$$v = y.$$

In particuar, for plugging any vector:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we try to qualitatively describe what T does to the plane, we can deduce that it *does nothing!* Nothing get moved around by T, everything just stays put. This function is often called the identity function, and is denoted id.

Definition 3.5.1 The function $id : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $id(\mathbf{v}) = \mathbf{v}$ is called the *identity function*. The matrix associated to the identity function is called the *identity matrix*, and has the following form:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

■ **Question 3.15** Let I be the identity matrix. Is it possible to find a vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $I\mathbf{v} \neq \mathbf{v}$?

There are two ways to see that the answer to this question is no. One way is to use that the identity function corresponds with the identity matrix, so:

$$I\mathbf{v} = id(\mathbf{v}) = \mathbf{v}.$$

One can also choose variable coordinates for $\mathbf{v}$, and do matrix multiplication:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1*x+0*y \\ 0*x+1*y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Either way, we see that nothing happens! In particular, for matrix-vector multiplication, we see that multiplying by the identity matrix does nothing. What about for matrix multiplication in general?

■ **Question 3.16** Let I be the identity matrix and let M be any 2×2 matrix. Can you say anything about IM? What about MI?

Let's do an example: say:

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix}.$$

$$IM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1*1+0*-1 & 1*2+0*7 \\ 0*1+1*-1 & 0*2+1*7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix} = M.$$

So it looks like nothing happens! And as above, we could give M some variable coordinates and see that this is always the case.

$$IM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1*a+0*c & 1*b+0*d \\ 0*a+1*c & 0*b+1*d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = M.$$

That being said, this makes it look almost like an accident, a lucky and clever choice of numbers for I so that multiplication does nothing. Insead, let's take the approach from Slogan 3.1. Then I corresponds to the identity function id and M is associated with some other linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$. What happens if we compose them? The identity functoin *does nothing*, so for any $\mathbf{v}$ we choose:

$$id(L(\mathbf{v})) = L(\mathbf{v}).$$

Since the compositoin $id \circ L$ is the same as just doing L, the product $IM = M$. In the other direction:

$$L(id(x)) = L(x),$$

so that $L \circ id = L$, and therefore $MI = M$ as well.

Theorem 3.5.1 Let I be the identity matrix, and M any other 2×2 matrix. Then:

$$IM = MI = M,$$

In particular, the identity matrix behaves for matrix multiplication, much like the number 1 behaves for traditional multiplication.

3.5.2 Inverse Transforms and Inverse Matrices

When studying a function, it is often very useful to have an *undo button*: another function which reverses what the first one did. For example, consider the function $f(x) = x^3$. To *undo* this function, we take the cube root! To be more precise, there is a function $g(y) = \sqrt[3]{y}$, and if we compose f and g, we get back where we started. Let's try this out on a few numbers:

$$g(f(2)) = g(8) = \sqrt[3]{8} = 2.$$

This works both ways: cubing *undoes* the cuberoot.

$$f(g(12)) = f(\sqrt[3]{12}) = f(2.2894...) = (2.2894...)^3 = 12.$$

Plugging in variables:

$$g \circ f(x) = g(x^3) = \sqrt[3]{x^3} = x, \qquad \text{and} \qquad f \circ g(y) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y.$$

In particular, the composition $g \circ f$ is the *do nothing* function (also known as the identity function), and the same can be said for $f \circ g$. If $g \circ f = id$ and $f \circ g = id$, we call g the *inverse* of f, and denote it by f^{-1}.

R It's not true that every function has an inverse. For example, let $h(x) = x^2$. If I plug in 2 I get $h(2) = 4$, so this tells me that whatever the *undo* function is, it better take 4 to 2. On the other hand, $h(-2) = 4$ as well, so this undo function must also take 4 to -2. It can't do both! So h cannot have an inverse. In fact, we've stumbled upon something: for a function F to have an inverse, it must satisfy the following property: whenever I have $a \neq b$, we need $F(a) \neq F(b)$. If they were equal, we wouldn't know how to undo their value. This property is called being *one-to-one*, and we will revisit it further down the line.

■ **Question 3.17** Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates the plane $30°$. Does R have an inverse? Can you describe it?

Definition 3.5.2 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. The *inverse* to L (if it exists), is a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ so that when we compose in both directions we obtain the identity function:

$$L \circ T = id \qquad \text{and} \qquad T \circ L = id.$$

We often denote the inverse T by the symbol L^{-1} (pronounced L *inverse*).

■ **Example 3.9** To undo the function R which rotates the plane $30°$, we merely rotate the plane $-30°$, and get back to where we started.[4] Let's look at the matrices for R and R^{-1}, which we will call M and N respectively.

$$M = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix},$$

and

$$N = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} = \begin{bmatrix} .866 & .5 \\ -.5 & .866 \end{bmatrix}.$$

Can you guess what the product NM is? It should correspond to the function which first rotates $30°$, and then rotates $-30°$, that is, it corresponds to the identity function. So hopefully, it is the identity matrix. Let's check:

$$\begin{aligned} NM &= \begin{bmatrix} .866 & .5 \\ -.5 & .866 \end{bmatrix} \begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix} \\ &= \begin{bmatrix} .866 * .866 + .5 * .5 & .866 * (-.5) + .5 * (.866) \\ (-.5) * (.866) + .866 * .5 & (-.5) * (-.5) + .866 * .866 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Notice that seems a lot more surprising that it works out when you compute it directly, than when you compare it to the composition of rotating $30°$ forward and then back. Nonethelss, $NM = I$ and one can similarly see that $MN = I$ as well. ■

When multiplying numbers, the *inverse* (or multiplicative inverse) of a number (say 7), is the number we multiply to get 1 (in this case, $\frac{1}{7}$ or 0.14285...). In fact, we will often denote $\frac{1}{7}$ just by writing 7^{-1}. In matrix multiplication, the number 1 is replaced by the identity matrix I, so the natural way to define an inverse is as follows.

[4] we could also rotate $330°$.

Definition 3.5.3 Let M be a 2×2 matrix. The *inverse* of M (if it exists) is a matrix N such that:

$$MN = I \qquad \text{and} \qquad NM = I.$$

If an inverse to M exists, we will denote it by the symbol M^{-1}.

Question 3.18 Let

$$M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Is $N = M^{-1}$?

Let's use our philosophy that *a matrix is a function*, to connect matrix inverses and function inverses. In particular, let $L, T : \mathbb{R}^2 \to \mathbb{R}^2$ be functions, whose associated matrices are M and N respectively.

Question 3.19 If $T = L^{-1}$, does this mean $N = M^{-1}$?

The answer had better be yes, and indeed, the product MN is the matrix associated to the composition $L \circ T = id$, so $MN = I$. We can say the same for NM.

Example 3.10 — Inverses of Rotation Matrices. The inverse of the rotation matrix:

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

is the matrix which rotates the plane the same amount, but in the opposite direction:

$$M^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

■

Example 3.11 A matrix can be its own inverse! Indeed, let:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then it isn't hard to compute that $MM = I$. This can be elucidated by recognizing that M corresponds to the *flip* function $F : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects the plane over the x-axis. Then MM coresponds to $F \circ F$, which flips, and then flips again. But if you flip twice, you're back where you started! Since $F \circ F = id$, we know $MM = I$. ■

Rather than write MM, for multiplying a matrix by itself, we can write M^2.

Definition 3.5.4 Let M be a a 2×2 matrix, and let n be a positive integer. Then:

$$M^n = \underbrace{M \cdot M \cdots M}_{n\text{-times}}.$$

If M has an inverse M^{-1}, then we can write:

$$M^{-n} = \underbrace{M^{-1} \cdot M^{-1} \cdots M^{-1}}_{n\text{-times}}.$$

Finally, $M^0 = I$.

3.5.3 A Technique: Solving a System of Equations with Inverse Matrices

One application of matrix inversion (and linear algebra in general) is it gives a broad framework to solve systems of equations. Let's see an example of this, in the case of 2 equations and 2 unknowns. Suppose we want to solve the following system:

$$2x + 5y = 11,$$

$$x + 3y = -4.$$

We can make this a single (vector) equation by putting brackets around each side.

$$\begin{bmatrix} 2x+5y \\ x+3y \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

The left hand side can be factored into the product of a 2×2 matrix and a single (vector) variable.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

The 2×2 matrix is the matrix M from Question 3.18. Letting:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ -4 \end{bmatrix},$$

the system of equations boils down to the rather simple equation:

$$M\mathbf{x} = \mathbf{v}. \tag{3.1}$$

Now our intuition tells of the following: *if we want to solve for* $\mathbf{x}$*, we should divide both sides by* M. But we can't really divide by a matrix...can we? For numbers, if I wanted to divide by 7, we could instead multiply by $\frac{1}{7}$, or to more reflect our current setup, we could multiply by 7^{-1}. Let's model our next step on this, and try to multiply both sides by M^{-1}, which we found in Question 3.18 to be:

$$N = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

So let's try multiplying both sides of Equation (3.1) by N. *Careful! As we saw above, it matters whether we multiply on the left or the right. If we want to cancel out the M, we shouled probably multiply on the left.*

$$NM\mathbf{x} = N\mathbf{v}. \tag{3.2}$$

Zooming in on the left-hand-side, we can use that $NM = I$ to see:

$$NM\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

So multiplying by N worked just like dividing by M should! Plugging this back into Equation (3.2), we get:

$$\mathbf{x} = N\mathbf{v}.$$

That is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -4 \end{bmatrix} = \begin{bmatrix} 33+20 \\ -11-4 \end{bmatrix} = \begin{bmatrix} 53 \\ -19 \end{bmatrix}.$$

Let's check that this works:

$$2*53+5*(-19) = 11,$$

$$53+3(-19) = -4.$$

Magic! This is, of course, not magic. In fact, viewing this from the *functional* perspective clarifies the picture somewhat. The function L associated to the matrix M is given by equations:

$$u = 2x+5y,$$

$$v = x+3y,$$

which look very much like the system of equations we started with. Then solving the system of equations is looking for a vector that, when I apply L, results in a u-coordinate of 11 and a v-coordinate of -4. That is, we want $L(\mathbf{x}) = \mathbf{v}$. Multiplying by the inverse then correponds to applying L^{-1} to both sides, which gives:

$$L^{-1}(L(\mathbf{x})) = L^{-1}(\mathbf{v}),$$

which in turn simplifies to $\mathbf{x} = L^{-1}(\mathbf{v})$. Since we know L^{-1} (why?), we have now found $\mathbf{x}$ and therefore solved our system.

3.6 Exercises

Exercise 3.11 Compute the following matrix products.

1. $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 11 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

■

Exercise 3.12 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the equations:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \textit{where} \qquad \begin{aligned} u &= 2x-3y \\ v &= x+y \end{aligned}.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be another linear transformation given by the equations:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \textit{where} \qquad \begin{aligned} u &= -y \\ v &= 2x+2y \end{aligned}.$$

What are the equations for the composition $L \circ T$? (*Hint:* Can you translate this problem to computing a single matrix product?) ■

Exercise 3.13 Let $id : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity function on $\mathbb{R}^3$. That is, $id(\mathbf{v}) = \mathbf{v}$ for any 3D vector $\mathbf{v}$. What is the matrix for id? ■

Exercise 3.14 Let θ and φ be two angles, and consider the rotation matrices:

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

Do you think $MN = NM$? Why or why not? (Trying to study this by multiplying things out results in some hard trig. It's probably easier to think about how the functions associated to M and N compose.) ■

Exercise 3.15 When working between dimensions, sometimes matrix multiplication makes sense, and sometimes it doesn't. Let's think about this a bit more carefully. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the functions associated to the matrices:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \textit{and} \qquad N = \begin{bmatrix} -1 & 0 \\ -2 & 5 \end{bmatrix}.$$

1. Try to compute the following values, or explain why they cannot be computed.

$$F\left(G\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) \qquad \textit{and} \qquad G\left(F\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)\right).$$

2. One of the following 2 functionm compositions makes sense, and the other doesn't. Determine which is which, and explain why.

$$F \circ G \qquad \textit{or} \qquad G \circ F.$$

3. One of the following 2 matrix products makes sense, and the other doesn't. Determine which is which, and explain why.

$$MN \qquad \textit{or} \qquad NM.$$

   For the one that does make sense: what is the product? Your answer should be one matrix. (Recall: *a matrix is a function, and a matrix multiplication should reflect compositoin of functions*).

■

Exercise 3.16 Below are 4 matrices. Match each one with its inverse. Justify your reasoning.

$$M = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} \qquad N = \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} -0.5 & -2 \\ 0.5 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

■

Exercise 3.17 Below are 4 functions. Match each one with its inverse. Justify your reasoning.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -0.5x - 0.5y \\ v &= 0.5x - 0.5y \end{aligned}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= 2x + 4y \\ v &= -x - y \end{aligned}$$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -0.5x - 2y \\ v &= 0.5x + y \end{aligned}$$

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -x + y \\ v &= -x - y \end{aligned}$$

■

Exercise 3.18 Consider the following system of equations:

$$2x - 5y = 1$$

$$x + 2y = 3.$$

Write the system of equations as a single matrix equation:

$$M\mathbf{x} = \mathbf{v},$$

where $\mathbf{x}$ is a variable vector, and $\mathbf{v}$ is an actual vector. Then use the formula for matrix inversion (Theorem 3.7.1 on Page 98 of the notes) to solve for $\mathbf{x}$ and determine x and y. ■

Exercise 3.19 Let a, b, c, d, k, ℓ be constants, and consider the system of equations:

$$ax + by = k$$

$$cx + dy = \ell.$$

Do you agree or disagree with the following statement? Justify your reasoning:

If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse, then the system of equations has exactly one solution.

■

Exercise 3.20 Suppose that M and N are matrices with inverses. Does the product MN have an inverse as well? Justify your reasoning. (This might be easier to think of from the *functional* perspective than from the algebraic one). ■

Exercise 3.21 Some functions have inverses while others don't. The same is true for matrices. For each of the following matrices, determine if there is or is not an inverse. If there is an inverse, write it down. If there is no inverse, explain why. (*Hint: Remember that $h(x) = x^2$ has no inverse because $h(2) = h(-2) = 4$. Try to give a similar argument for the ones you think have no inverse.*)

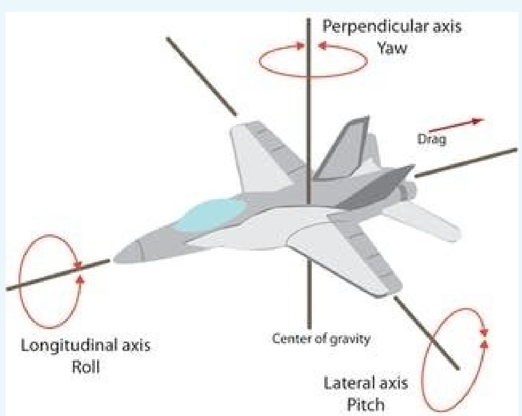
1. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.
2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
3. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$.
4. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

■

Exercise 3.22 We've defined inverses for 2×2 matrices. What do you think the definition for the inverse of a 3×3 matrix should be? ■

Exercise 3.23 Rotations in 3D space are controlled by 3 axes of rotation:
- *Roll*: Rotation about the x-axis.
- *Pitch*: Rotation about the y-axis.
- *Yaw*: Rotation about the z-axis.

These 3 axes of rotation are demonstrated by the fighter jet in the image below.



In terms of a plane then the axes can be described (from the point of view of the pilot) as follows:
- *Roll*: The wings of the plane rotating around while nose stays pointed forward.
- *Pitch*: The nose of the plane angling up or down.
- *Yaw*: The nose of the plane angling left or right.

A general rotation can be expressed in terms of its *roll angle* γ, its *pitch angle* β, and yaw angle

α. Rotating around an axis is a matrix transformation. Since a general rotation is a composition of rotations around the 3 axes, it is as well! Let's figure out which matrix it is.

2. Find the standard matrix associated to a roll of γ—that is, a rotation of angle γ about the x-axis which would appear counterclockwise when looking from the positive x-direction. (This should like a bit like a rotation matrix in 2D!)
3. Find the standard matrix associated to a pitch of β—that is, a rotation of angle β about the y-axis, which would appear counterclockwise when looking from the positive y-direction.
4. Find the standard matrix associated to a yaw of α—that is, a rotation of angle α about the z-axis, would appear counterclockwise when looking from the positive z direction.
5. A general rotation can be expressed in terms of its *roll angle* γ, its *pitch angle* β, and yaw angle α. Therefore, it is a composition of the 3 rotation matrices you've found so far. Compute the standard matrix for such a rotation. Remember that compositing matrix transformations corresponds to multiplying matrices.
6. Let M be the matrix for a roll of $30°$, a pitch of $-15°$, and a yaw of $90°$. Do you think M has an inverse? Why or why not?
7. Set the origin at the center of gravity of a jet, and suppose a bug is sitting on the nose of a jet, at the coordinates $(5,0,1)$. The jet rotates with a roll angle of $30°$, a pitch angle of $-15°$, and a yaw angle of $90°$. What are the coordinates of the bug after this rotation.

■

3.7 October 1, 2024

3.7.1 General Linear Transformations

We've now seen some examples of linear maps between spaces of the same dimension, as well spaces of different dimension. Let's tally up some of our observations. The color example was a map from $F : \mathbb{R}^2 \to \mathbb{R}^3$. As equations, it consisted of 3 equations in terms of 2 variables each. Translating this to a matrix, we had 3 rows (one for each equation), and 2 columns (one for each variable).

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \longrightarrow \qquad \begin{matrix} u = ax + by \\ v = cx + dy \\ w = ex + fy \end{matrix} \qquad \longrightarrow \qquad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

The shadow example gave us a map $S : \mathbb{R}^3 \to \mathbb{R}^2$. As equations, it consisted of 2 equations in 3 variables each, and translating this to a matrix we had 2 rows (one for each equatoin), and 2 columns (one for each variable).

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \longrightarrow \qquad \begin{matrix} u = ax + by + cz \\ v = dx + ey + fz \end{matrix} \qquad \longrightarrow \qquad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

We further observed that in each case, the columns of the matrix could be determined by simply evaluating the function on the standard basis vectors. Let us take this as a jumping off point for extending the theory to linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$ for general n and m. Let's first define what this $\mathbb{R}^n$ should be, recalling the definition of *the computer scientists approach* to n-dimensional vectors (cf. Definition 2.1.10.

Definition 3.7.1 — Higher Dimensional Vector Spaces. Let n be a positive integer. A *n-dimensional column vector* is an array of n numbers, arranged vertically:

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The collection of all n-dimensional column vectors is denoted $\mathbb{R}^n$.

We've seen a couple of examples of higher dimensional vectors, including Example 2.4 which discussed the 5D-vectors controlling a 5-axis CNC mill. Fix 2 positive integers, m and n, and lets consider a function $L : \mathbb{R}^m \to \mathbb{R}^n$. Such a function could be denoted by a rule:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

We have to define what each coordinate, u_i, of the target is, so such a function would need n-equations in terms of the m input variables.

$$u_1 = u_1(x_1, x_2, \cdots, x_m),$$

$$u_2 = u_2(x_1, x_2, \cdots, x_m),$$
$$\vdots$$
$$u_n = u_n(x_1, x_2, \cdots, x_m).$$

There are many many equations to choose from, which could be outrageously complicated. Linear algebra focuses on the *linear ones*, which for us means, *purely linear with no constant terms*.

Definition 3.7.2 — Linear Transformations in General. A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if it is given by the the equations

$$u_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m,$$
$$u_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m,$$
$$\vdots$$
$$u_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m,$$

for constants a_{ij}.

As usual, the data of this such a function is completely captured by *all* of the coefficients a_{ij} (there are now mn of them), so to remember this function, we need only remember these mn constants (in the correct order). We can do this by putting them in an array (or $n \times m$ matrix).

Definition 3.7.3 — $n \times m$ matrices. The $n \times m$ matrix associated to the linear transformation from Definition 3.7.2 is

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix}$$

Notice that if $L : \mathbb{R}^m \to \mathbb{R}^n$, the associated matrix has n rows and m columns (is an $n \times m$ matrix). This is because we get 1 row for each equation (i.e., each output, of which there are n), and one column for each variable (i.e., each input, of which there are m).

■ **Question 3.20** Let $L : \mathbb{R}^6 \to \mathbb{R}^8$ be a linear function. What is the shape of the matrix associated to L (that is, how many rows does it have, and how many columns?).

Again, M and L are interchangeable, so for any vector $\mathbf{v}$ in $\mathbb{R}^m$, it is reasonable to write:

$$L(\mathbf{v}) = M\mathbf{v}.$$

We know what $L(\mathbf{v})$ should be (using the equations from Definitoin 3.7.2), so that we can obtain a general formula for matrix-vector multiplication.

Definition 3.7.4 Given an $n \times m$ matrix and a vector in $\mathbb{R}^m$, we can define their product as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix}$$

As before, this can be expressed as a process where to give the ith row of Mv one pairs the entries of the ith row of M with the entries of v one by one, multiplying them together and adding them up.

$$\textbf{Row 1:} \quad \begin{bmatrix} \color{red}{a_{11}} & \color{blue}{a_{12}} & \cdots & \color{green}{a_{1m}} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} \color{red}{x_1} \\ \color{blue}{x_2} \\ \cdots \\ \color{green}{x_m} \end{bmatrix} = \begin{bmatrix} \color{red}{a_{11}x_1} + \color{blue}{a_{12}x_2} + \cdots + \color{green}{a_{1m}x_m} \\ \\ \vdots \\ \\ \end{bmatrix}$$

$$\textbf{Row 2:} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \color{red}{a_{21}} & \color{blue}{a_{22}} & \cdots & \color{green}{a_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} \color{red}{x_1} \\ \color{blue}{x_2} \\ \cdots \\ \color{green}{x_m} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ \color{red}{a_{21}x_1} + \color{blue}{a_{22}x_2} + \cdots + \color{green}{a_{2m}x_m} \\ \vdots \\ \\ \end{bmatrix}$$

$$\textbf{Row n:} \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \color{red}{a_{n1}} & \color{blue}{a_{n2}} & \cdots & \color{green}{a_{nm}}. \end{bmatrix} = \begin{bmatrix} \color{red}{x_1} \\ \color{blue}{x_2} \\ \cdots \\ \color{green}{x_m} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ \color{red}{a_{n1}x_1} + \color{blue}{a_{n2}x_2} + \cdots + \color{green}{a_{nm}x_m} \end{bmatrix}$$

■ **Example 3.12** The position of a 5-axis CNC rotor is given by a vector:

$$\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}$$

where x, y, z give the location in 3-space, and θ and ϕ measure its orientation (as rotations around the z and x axes). It is moved in space by 3 perpindicular arms, and rotated by a mechanism attached directly to the drill. In particular, when the computer sends the information to the arms, it doesn't need to send θ and ϕ, just the x, y, z-coordinates. This can be expressed by a function $F : \mathbb{R}^5 \to \mathbb{R}^3$, given by the rules:

$$F\left(\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where

$$u = x = 1x + 0y + 0z + 0\theta + 0\phi,$$

$$v = y = 0x + 1y + 0z + 0\theta + 0\phi,$$

$$w = z = 0x + 0y + 1z + 0\theta + 0\phi.$$

The matrix for F is therefore:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

■

■ **Question 3.21** The rotation mechanism attached to the drill head only needs to remember θ and ϕ. Define a function $G : \mathbb{R}^5 \to \mathbb{R}^2$ which only plucks out the rotation coordinates. Determine if it is linear, if it is, give the associated matrix.

An important property of linear maps we saw so far was that they could be determined by their values on a few chosen vectors. This happens here too. For example, if L is the linear transformation in Definition 3.7.2, then:

$$L\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} * 1 + a_{12} * 0 + \cdots + a_{1m} * 0 \\ a_{21} * 1 + a_{22} * 0 + \cdots + a_{2m} * 0 \\ \vdots \\ a_{n1} * 1 + a_{n2} * 0 + \cdots + a_{nm} * 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix},$$

which recovers the first column of the matrix for L. We can do similarly with the remaining columns. Before stating the general result, we will need to introduce some notation. For $\mathbb{R}^2$ we cared about $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and for $\mathbb{R}^3$ we cared about $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. For general $\mathbb{R}^m$. Since there are only a finite number of letters in the alphabet, and we want to work with general m, we need to slightly switch up our notation.

Definition 3.7.5 — The Standard Basis for $\mathbb{R}^n$. The *standard basis* for $\mathbb{R}^m$ is the collection of vectors:

$$\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \hat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \hat{\mathbf{e}}_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

In particular, $\hat{\mathbf{e}}_i$ is the m-dimensional vector which has a 1 in the ith entry, and zeroes everywhere else.

With this notation in hand, we can record the general result about determining linear functions.

Theorem 3.7.1 A linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ is determined by its values on the standard basis for $\mathbb{R}^m$. In particular, if:

$$L(\hat{\mathbf{e}}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \quad L(\hat{\mathbf{e}}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \quad \cdots \quad L(\hat{\mathbf{e}}_m) = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix},$$

then the matrix for L is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix}$$

The other thing we noticed for linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ is that they played well with addition (on HW1) and scalar multiplication (on this week's HW4). Before moving on, we'd like to record that this holds here as well.

Theorem 3.7.2 — Linearity of Linear Transformations. Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, let $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^m$ be vectors, and let c be a constant.

1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

The general framework of the proof for identical to the computation in the 2×2 case (Theorem 3.1.2 for Part 1 and Exercise 3.7 for part 2), except with more symbols to keep track of. We will omit it for now.

3.7.2 General Matrix Multiplication

In Exercise 3.10 we considered two functions. $R : \mathbb{R}^2 \to \mathbb{R}^2$ rotated the plane 90°, and was associated with the matrix:

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We also looked at a function $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix

$$N = \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

We were interested in the composition $R \circ T$, which first applies T to a vector in $\mathbb{R}^3$, and then rotates the result 90°. In particular, $T \circ R : \mathbb{R}^3 \to \mathbb{R}^2$, so it is represented by a 2×3 matrix-2 rows, one for each output, and 3 columns, one for each input. As usual find the matrix for $T \circ R$, it is enough to see what it does to $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

$$P = \begin{bmatrix} R(T(\hat{\mathbf{i}})) & R(T(\hat{\mathbf{j}})) & R(T(\hat{\mathbf{k}})) \end{bmatrix}.$$

But we can extract $T(\hat{\mathbf{i}}), T(\hat{\mathbf{j}}), T(\hat{\mathbf{k}})$ from the columns for N, and to apply R we just multiply by M, so:

$$P = \begin{bmatrix} M\begin{bmatrix} 2 \\ -1 \end{bmatrix} & M\begin{bmatrix} 0 \\ -1 \end{bmatrix} & M\begin{bmatrix} -1 \\ 5 \end{bmatrix} \end{bmatrix}.$$

Applying M column by column then gives our result:

$$P = \begin{bmatrix} 1 & 1 & -5 \\ 2 & 0 & -1 \end{bmatrix}.$$

Since matrix multiplication reflects the composition of the associated functions, we write:

$$MN = P,$$

or

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -5 \\ 2 & 0 & -1 \end{bmatrix}.$$

■ **Question 3.22** Let $\mathbf{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Compute $R(T(\mathbf{v}))$ by applying T and the R in succession. Compare this to $P\mathbf{v}$ computed by matrix multiplication.

We can follow a *process* to do this matrix multiplication, similar to how we did this above. In particular, to find the ij-entry of MN, we pair the elements of the ith row of M, with the jth row of N, multiply them together, and add them up. In this case, we have to do this 6 times. So for example, if you're interested in the entry in the first row and third column of MN, you'd pair the first row of M with the third column of N:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

You multiply the paired elements and add up the result:

$$0 \cdot -1 + -1 \cdot 5 = 5.$$

And indeed, the entry in the first row and third column of MN computed by composing the functions was indeed a 5.

■ **Question 3.23** Compute the second row, second column of MN this way, confirming you get the same result.

So the product MN makes sense, because it corresponds to the composition $R \circ T$. What if we wanted a product NM? This should correspond to composing $T \circ R$: that is *rotate the vector* 90°, *then apply T to the result.*

■ **Question 3.24** Does this make sense?

Let's investigate this further. Let $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and lets try to compute $T \circ R(\mathbf{v}) = T(R(\mathbf{v}))$. We first must compute $R(\mathbf{v})$, that is, we must rotate $\mathbf{v}$ by 90°.

$$R(\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now we'd like to apply this to T.

■ **Question 3.25** What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

If you said that this doesn't make sense, you're correct! $T:\mathbb{R}^3\to\mathbb{R}^2$, so it takes 3D vectors as inputs. It doesn't know what to do when we input a 2D vector, because 2D vectors are not in its domain! In summary, $R\circ T$ makes sense, but $T\circ R$ does not. Since matrix products must reflect composing their associated functions, this means that the matrix product MN makes sense, but the matrix product NM does not.

■ **Slogan 3.2** Matrix multiplication doesn't always make sense. This is because matrices are functions, and matrix multiplication is functoin composition. Matrix multiplication only works when the associated functions can be composed!

You do a similar exploration on Homework 5 (Exercise 3.15).

Working more generally, suppose we have two linear transformation $L:\mathbb{R}^n\to\mathbb{R}^m$ and $T:\mathbb{R}^p\to\mathbb{R}^q$. The composition $L\circ T$ should be the function which takes a vector $\mathbf{v}$ in $\mathbb{R}^p$, applies T to get $T(\mathbf{v})$ in $\mathbb{R}^q$, and then applies L to the result.

■ **Question 3.26** For $L\circ T$ to make sense, what needs to be true about q and n?

We are trying to applying L to a vector in $\mathbb{R}^q$, but L only knows what to do with vectors in $\mathbb{R}^n$, so in order for this to be possible, we *must* have $n=q$.

■ **Question 3.27** Let $L:\mathbb{R}^6\to\mathbb{R}^4$ and $T:\mathbb{R}^{11}\to\mathbb{R}^6$. What makes sense, $L\circ T$ or $T\circ L$?

What if we translate this to matrix multiplication. Associated to L is the matrix M, which has m rows and n columns (so it's an $n\times m$ matrix), and associated to T is the matrix N, which as q rows and p columns (so is $q\times p$). The produce MN should be the matrix associated to the composition $L\circ T$, which only makes sense if $n=q$.

■ **Slogan 3.3** The matrix product MN exists precisely when the number of columns of M is equal to the number of rows of N.

■ **Question 3.28** Consider the matrices:

$$M=\begin{bmatrix}1&2&3&4\\5&6&7&9\\9&10&11&12\\13&14&15&16\\17&18&19&20\\21&22&23&24\end{bmatrix}\qquad\text{and}\qquad N=\begin{bmatrix}3&1&4&1&5&9\\2&6&5&3&5&8\end{bmatrix}.$$

Which of the matrix products make sense (if any)?

$$MN \qquad \text{or} \qquad NM.$$

Since M has 4 columns, and N has 2 rows, MN doesn't make sense. On the other hand, N has 6 columns, and M has 6 rows, so NM does make sense.

■ **Question 3.29** Building on the same example: How many rows does NM have? How many columns does NM have?

To figure this out, we think about N as a function $T:\mathbb{R}^6\to\mathbb{R}^2$, and think about M as a functtion $L:\mathbb{R}^4\to\mathbb{R}^6$. Then the product NM corresponds to the compositoin $T\circ L$ which takes a vector in $\mathbb{R}^4$ as input, applies L to get a vector in $\mathbb{R}^6$, and applies T to the result to get a vector in $\mathbb{R}^2$.

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "L"] \arrow[rr, bend left, "T\circ L"] & \mathbb{R}^6 \arrow[r, "T"] & \mathbb{R}^2 \end{tikzcd}$$

In particular, $T \circ L : \mathbb{R}^4 \to \mathbb{R}^2$, so the associated matrix (which is NM) should have 2 rows and 4 columns.

Definition 3.7.6 Let $L : \mathbb{R}^p \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^p$. The function $L \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is the function which first does applies T to a vector in $\mathbb{R}^n$, and then applies L to the result.

If M is the $m \times p$ matrix associated with L, and N is the $p \times n$ matrix associated to T, then MN is the $m \times n$ matrix associated to $L \circ T$.

■ **Slogan 3.4** Matix multiplication *cancels the middle*. That is:

$$(m \text{ by } p) \times (p \text{ by } n) = m \text{ by n}.$$

How do we compute a matrix product? Adopting the notation of Definition 3.7.6, we are interested in the matrix that represents the composition $L \circ T$, which first applies T to a vector in $\mathbb{R}^n$, and then applies L to the result. To find this matrix, it is eanough to see what it does to the standard basis elements $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_n$:

$$MN = \begin{bmatrix} L(T(\hat{\mathbf{e}}_1)) & L(T(\hat{\mathbf{e}}_2)) & \cdots & L(T(\hat{\mathbf{e}}_n)) \end{bmatrix}.$$

Intoduce some notation so that:

$$N = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn}. \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}.$$

Here $\mathbf{c}_j$ is the jth column of N. We know by Theorem 3.7.1 is $T(\hat{\mathbf{e}}_j)$. Therefore

$$L(T(\hat{\mathbf{e}}_j)) = M\mathbf{c}_j = M \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix},$$

which is a usual matrix vector multiplication, and:

$$MN = \begin{bmatrix} M\mathbf{c}_1 & M\mathbf{c}_2 & \cdots & M\mathbf{c}_n \end{bmatrix}.$$

Let's introduce a bit more notation:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix}$$

Then:

$$M\mathbf{c}_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} = \begin{bmatrix} a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1p}b_{pj} \\ a_{21}b_{1j} + a_{22}b_{2j} + \cdots + a_{2p}b_{pj} \\ \vdots \\ a_{m1}b_{1j} + a_{m2}b_{2j} + \cdots + a_{mp}b_{pj} \end{bmatrix}.$$

To summarize: we can follow a *process* to do this matrix multiplication, similar to how we did above. In particular, to find the ij-entry of MN, we pair the elements of the ith row of M, with the jth row of N, multiply them together, and add them up.

Theorem 3.7.3 The matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn}. \end{bmatrix}$$

is the matrix with m rows and n columns, whose entry in row i and column j is:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj},$$

■ **Example 3.13** Let's find the entry in the first row and third column of the matrix product from Question 3.28. Do do this, we pair the 1st row and 3rd columns:

$$\begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}.$$

Multiplying associated elements and adding them up gives:

$$3*3 + 1*7 + 4*11 + 1*15 + 5*19 + 9*23 = 377.$$

So we have:

$$\begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix} = \begin{bmatrix} ? & ? & 377 & ? \\ ? & ? & ? & ? \end{bmatrix}.$$

Do this 7 more times, and we have the matrix product! ■

■ **Question 3.30** What is the entry in the second row and second column?

As you can probably tell, this process can be rather tedious, and doing the arithmetic by hand is not very enlightening in general. It also opens you up to small mistakes which are hard to notice, but may vastly change the outcome of whatever your working on. On the other hand, a computer can quickly do a series of additions and multiplications with data given in a table. For small matrices (2×2 or 3×3 at most), we will sometimes do things by hand, but once we start playing with larger matrices it is best to use technology for the arithmetic for use.

3.8 October 3, 2024

3.8.1 General Identity Matrices

In Section 3.5.1 we introduced the *2×2 identity matrix*:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which behaved for matrix multiplication the way the number 1 does for usual multiplication, and whose associated function—*the identity function*—does nothing to the place. In Homework 5 (Exercise 3.13), we looked at the identity function $id : \mathbb{R}^3 \to \mathbb{R}^3$ which does nothing to 3-space: $id(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v}$. To find equations we introduce coordinates:

$$id\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

and since this must equal the inputed vector, we have:

$$u = x = 1x + 0y + 0z,$$

$$v = y = 0x + 1y + 0z,$$

$$w = z = 0x + 0y + 1z.$$

Therefore, the matrix for id is:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

■ **Question 3.31** Let $\mathbf{v}$ be any vector in $\mathbb{R}^3$. Is it true that $I_3\mathbf{v} = \mathbf{v}$?

■ **Question 3.32** Compute the matrix product:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Can you explain why your answer makes sense in terms of composing the associated functions?

Indeed, this always works:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{bmatrix} = \begin{bmatrix} 1a+0d+0g & 1b+0e+0h & 1c+0f+0\ell \\ 0a+1d+0g & 0b+1e+0h & 0c+1f+0\ell \\ 0a+0d+1g & 0b+0e+1h & 0c+0f+1\ell \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{bmatrix}.$$

You try the other way:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

■ **Question 3.33** Compute the matrix products:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Can you explain why your answer makes sense in terms of composing the associated functions?

Let's do this more generally, in higher dimensions. The identity matrix should be associated to the identity function, so let's first figure out what that should be.

■ **Question 3.34** Is it possible to have an identity map $id : \mathbb{R}^3 \to \mathbb{R}^2$?

■ **Question 3.35** Suppose $m \neq n$. Is it possible to have an identity map $id : \mathbb{R}^n \to \mathbb{R}^m$?

If $id(\mathbf{v}) = \mathbf{v}$, then if $\mathbf{v}$ is in $\mathbb{R}^n$, we must have $id(\mathbf{v})$ is in $\mathbb{R}^n$ as well.

Definition 3.8.1 Fix a positive integer n. The identity map on $\mathbb{R}^n$ is the function $id : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $id(\mathbf{v}) = \mathbf{v}$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$.

■ **Question 3.36** Consider $id : \mathbb{R}^7 \to \mathbb{R}^7$. What is:

$$id\left(\begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \\ 3 \\ 0 \\ 9 \end{bmatrix}\right)?$$

The identity matrix should be associated to this function. Following what we've done above, we can extract equations for id. Here:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = id\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right).$$

Since id does nothing: we can extract equations for id:

$$u_1 = x_1 = 1x_1 + 0x_2 + \cdots + 0x_n,$$
$$u_2 = x_2 = 0x_1 + 1x_2 + \cdots + 0x_n,$$
$$\vdots$$
$$u_n = x_n = 0x_1 + 0x_2 + \cdots + 1x_n.$$

Putting the coefficients in a matrix gives us the identity matrix.

Definition 3.8.2 The $n \times n$ *identity matrix* is the $n \times n$ matrix which has 1s in the diagonal, and

0s everywhere else.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

■ **Question 3.37** Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Does $I_n\mathbf{v} = \mathbf{v}$?

3.8.2 Inverse Matrices in General

In Homework 6 (cf. Exercise 3.22) we asked what the definition is for the inverse of a 3×3 matrix. Let's reflect on that here. As usual, we should begin with the functional perspective.

Definition 3.8.3 Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. An inverse to L is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$L \circ T = id \qquad \text{and} \qquad T \circ L = id.$$

If such a T exists, we often write it as L^{-1}.

The coefficient matrix for such an L would be an $n \times n$ matrix M, and the coefficient matrix for T would be another $n \times n$ matrix N. The fact that $L \circ T = id$ means $MN = I_n$, and for similar reasons, $NM = I_n$. This suggests the following definition for the inverse of a matrix.

Definition 3.8.4 Let M be an $n \times n$ matrix. Another $n \times n$ matrix N is called an inversefor M if:

$$MN = I_n \qquad \text{and} \qquad NM = I_n.$$

If N exists, we will often write it as M^{-1}, and we will call M an *invertible matrix*.

We use the term *the inverse*, suggesting that there is only one possible inverse. Let's justify this.

Theorem 3.8.1 Let M be an invertible matrix, and let N_1 and N_2 be inverses for M. Then $N_1 = N_2$.

Proof. We give 2 proofs. The first is a direct application of matrix algebra.

$$N_1 = N_1 I_n = N_1(MN_2) = (N_1 M)N_2 = I_n N_2 = N_2.$$

So $N_1 = N_2$ as desired.

For a more conceptual approach using the function perspective, let's suppose that M corresponds to a function L, and N_1 and N_2 to T_1 and T_2 respectively. Fix some vector $\mathbf{x}$ in $\mathbb{R}^n$. Then $\mathbf{x} = L(\mathbf{y})$ for some vector $\mathbf{y}$ (in fact, we could take $\mathbf{y} = T_1(\mathbf{x})$). Then, since T_1 and T_2 both undo L, we know that

$$T_1(\mathbf{x}) = \mathbf{y} = T_2(\mathbf{x}).$$

Since T_1 and T_2 do the same thing to every point, they are the same function, and therefore have the same coefficient matrices. ■

We have only introduced inverses for functions $L : \mathbb{R}^n \to \mathbb{R}^n$, that is, where the domain and co-domain have the same dimension n. In time we will fully prove why, but let's start conceptually.

■ **Question 3.38** Let $L : \mathbb{R}^2 \to \mathbb{R}^3$. Can L have an inverse? Similarly, let $T : \mathbb{R}^3 \to \mathbb{R}^2$. Can T have an inverse?

Let's translate this to the matrix perspective. We've only defined inverses with the same number of rows and columns. Let's give that a name.

Definition 3.8.5 A matrix with the same number of columns as rows is called a *square matrix*.

■ **Question 3.39** Can a non-square matrix be invertible?

How can we find inverses?

It seems like it would be very useful to be able to invert linear functions and matrices. Inverting a number (like 7) is easy, you can just divide 1 by 7. For matrices, it seems much less clear. Some matrices, like rotations or reflections, are easy to invert (as we saw in Examples 3.10 and 3.11). On the other hand, the inverse for the matrix in Question 3.18 seemed to have come out of nowhere. For 2×2 matrices, it turns out there is a formula, which we will record here.

Theorem 3.8.2 — Inverses of 2×2 Matrices. Consider the matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then M has an inverse, which is given by the formula:

$$M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

It isn't too hard to check that this formula does indeed give an inverse,[5] but it is a little mysterious where this comes from. We will delay the discussion of where it comes from until a little later, but we record the formula sooner because it comes in quite handy.

■ **Question 3.40** Plug the matrix M from Question 3.18 into the inverse formula, and confirm that you get N.

3.9 Exercises

Exercise 3.24 Below are 4 matrices. Determine which (if any) can be multiplied together, and calculate the results. You may use a computer to do the computations.

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 5 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{bmatrix}.$$

■

[5] Do it do it do it!

Exercise 3.25 A linear map $L : \mathbb{R}^5 \to \mathbb{R}^3$ satisfies:

$$\begin{aligned}
L(\hat{\mathbf{e}}_1) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} \\
L(\hat{\mathbf{e}}_2) &= \hat{\mathbf{i}} - \hat{\mathbf{k}} \\
L(\hat{\mathbf{e}}_3) &= 2\hat{\mathbf{i}} - 2\hat{\mathbf{k}} \\
L(\hat{\mathbf{e}}_4) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \\
L(\hat{\mathbf{e}}_5) &= -5\hat{\mathbf{j}}
\end{aligned}$$

1. Let M be the matrix for the function L. What is M?
2. One of the following 2 values exists, and the other is undefined. Determine which whan exists, and compute it.

$$L\left(\begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}\right) \qquad or \qquad L\left(\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}\right).$$

3. Consider the matrix:

$$N = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 2 & 3 \\ -5 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

One of MN or NM is well defined. Determine which, and compute the matrix product. (You may use a computer to compute the product).

4. The matrix you found in part (c) corresponds to a function. Determine the domain and co-domain of this function.

■

Exercise 3.26

1. Compute the matrix product:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

by hand. What observations can you make about the result?

2. Let I_n be the $n \times n$ identity matrix, and let M be any $p \times n$ matrix. Determine the matrix product MI_n, and *fully explain your reasoning*. (*Hint:* It may be tedious to describe this by hand, try using the functional perspective instead).
3. Let I_n be the $n \times n$ identity matrix, and let M be any $n \times p$ matrix. Determine the matrix product I_nM, and *fully explain your reasoning*. (*Hint:* It may be tedious to describe this by hand, try using the functional perspective instead).

■

Exercise 3.27 1. Suppose $n \neq m$. Can there be an identity function $id : \mathbb{R}^n \to \mathbb{R}^m$? Why or why not?
2. Why does every identity matrix have to be square? (*Hint: Justify this using your answer to part (a).*)

■

Exercise 3.28 1. Let $\mathbf{c}$ be a n-dimensional column vector. If we instead think about it as an $n \times 1$ matrix, it determines a function. What is the domain and codomain of this function?
2. A n-dimensional row vector is a $1 \times n$ matrix. What is the domain and co-domain of the associated function?

For the rest of the problem we let:

$$\mathbf{c} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{r} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}.$$

1. Compute the product $\mathbf{cr}$ and informally interpret its meaning.
2. Compute the product $\mathbf{rc}$ and informally interpret its meaning.

■

Exercise 3.29 Here's an example using matrix methods to solve a system with 7 equations and 7 variables. To do this with traditional methods would be extremely tedious. But with matrix inversion (and the help of some technology), it's actually not so bad.

$$\begin{array}{rrrrrrrl}
x_1 & & +3x_3 & & & -2x_6 & & =3 \\
& +x_2 & & +\ x_4 & & +2x_6 & -3x_7 & =1 \\
& & -4x_3 & +7x_4 & & +\ x_6 & +\ x_7 & =4 \\
& +x_2 & & & +2x_5 & -\ x_6 & & =1 \\
2x_1 & & +3x_3 & & -3x_5 & & +\ x_7 & =5 \\
x_1 & & +4x_3 & & -\ x_5 & & -\ x_7 & =9 \\
& -x_2 & -\ x_3 & -\ x_4 & -\ x_5 & & +4x_7 & =2
\end{array}$$

1. The system of equations can be written as a single matrix-vector equation $M\mathbf{x} = \mathbf{v}$ where M is a 7×7 matrix, $\mathbf{v}$ is a 7-dimension column vector, and:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix},$$

is a column vector of your variables. Write down M and $\mathbf{v}$.
2. Now solve $M\mathbf{x} = \mathbf{v}$ for $\mathbf{x}$ using matrix multiplication, thereby solving the system of

equations. (You may do this by hand, or with a matrix multiplication calculator. There is one of these at `matrix.reshish.com` as well.)

3. Check that your solution works with 2 of the equations you started with.

4. Determinants

4.1 October 10, 2024

4.1.1 Determinants of 2×2 matrices

Consider a generic square 2×2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Theorem 3.8.2 tells us that if the value $ad - bc \neq 0$, then M has an inverse which is given by the formula:[1]

$$M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

It isn't too hard to see that this formula works (as we did during the groupwork on March 2nd). For example:

$$MM^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad}{ad-bc}+\frac{b(-c)}{ad-bc} & \frac{a(-b)}{ad-bc}+\frac{ba}{ad-bc} \\ \frac{cd}{ad-bc}+\frac{d(-c)}{ad-bc} & \frac{c(-b)}{ad-bc}+\frac{ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the fact that $M^{-1}M = I_2$ is very similar. This calculation does confirm that this formula correctly produces the inverse of M, but it might seem a bit like a lucky accident, and it raises a few questions. For example:

■ **Question 4.1** Where does this value $ad - bc$ come from? Does it have any further meaning?

Furthermore, Theorem 3.8.2 tells us that if $ad - bc \neq 0$, we have an inverse. If $ad - bc = 0$, then our formula for M^{-1} doesn't make any sense (because it would require dividing by zero). This raises another question.

[1] In the formula we multiplied a matrix by a scalar. Like vectors, when we scale a matrix we scale each entry.

■ **Question 4.2** If $ad-bc=0$, does that mean that M doesn't have an inverse? Or do we need a different formula to cover this case?

Let's start studying these questions by giving this mysterious $ad-bc$ value a name.

Definition 4.1.1 Let M be a 2×2 matrix.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The *determinant of M* is the value:

$$\det M = ad - bc.$$

In what follows we will answer questions 4.1 and 4.2. Let's start with the latter one.

What happens if the determinant is 0?

As usual, the way to really think about these questions is to use our running philosophy that *a matrix is a function*. Therefore, in this section we will try to understand what a determinant of 0 says about the associated function. Let's explore some examples of matrices whose determinant is 0.

■ **Example 4.1** The *zero matrix*:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The determinant is:

$$00 - 00 = 0.$$

The associated function is the *zero function*, which we will also denote by $\mathbf{0} : \mathbb{R}^2 \to \mathbb{R}^2$. This function takes any vector $\mathbf{v}$ in $\mathbb{R}^2$ to the zero vector:[2]

$$\mathbf{0}(\mathbf{v}) = \mathbf{0}.$$

Can this function have an inverse? Well, such an inverse should *undo* the function $\mathbf{0}$.

$$\text{Since } \mathbf{0}(\hat{\mathbf{i}}) = \mathbf{0} \text{ then need } \mathbf{0}^{-1}(\mathbf{0}) = \hat{\mathbf{i}}.$$

$$\text{Since } \mathbf{0}(\hat{\mathbf{j}}) = \mathbf{0} \text{ then need } \mathbf{0}^{-1}(\mathbf{0}) = \hat{\mathbf{j}}.$$

It can't be both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, so the inverse can't possibly exist. ■

We were able to determine that this function had no inverse because it sent different vectors ($\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$) to the same place (the zero vector). Let's give this property (or rather, its complement) a name.

Definition 4.1.2 A function f is said to be *injective* if it satisfies the following property:

Whenever $x \neq y$, are two distinct elements in the domain of f, $f(x) \neq f(y)$.

This property is also sometimes called being *one-to-one*.

[2]You can find an animation of this function here:`www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

If a function isn't injective, then it can't possibly have an inverse. Indeed, if $x \neq y$ but $f(x) = f(y)$, do we *undo* this output to x or to y?

Theorem 4.1.1 If a function isn't injective, it cannot have an inverse.

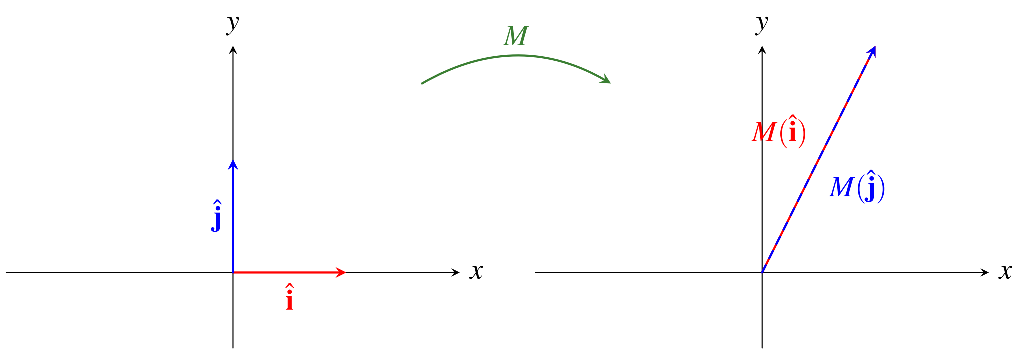
■ **Example 4.2** Let's consider the function associated to the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then the determinant of M is:

$$\det M = 1 \cdot 2 - 1 \cdot 2 = 0.$$

Does M have an inverse? Viewing M as a function $M : \mathbb{R}^2 \to \mathbb{R}^2$, it is equivalent to ask if the function M has an inverse.



We know that $\hat{\mathbf{i}} \neq \hat{\mathbf{j}}$, but

$$M(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = M(\hat{\mathbf{j}}).$$

So M is not injective, so Theorem 4.1.1 tells us it can't possibly have an inverse. ■

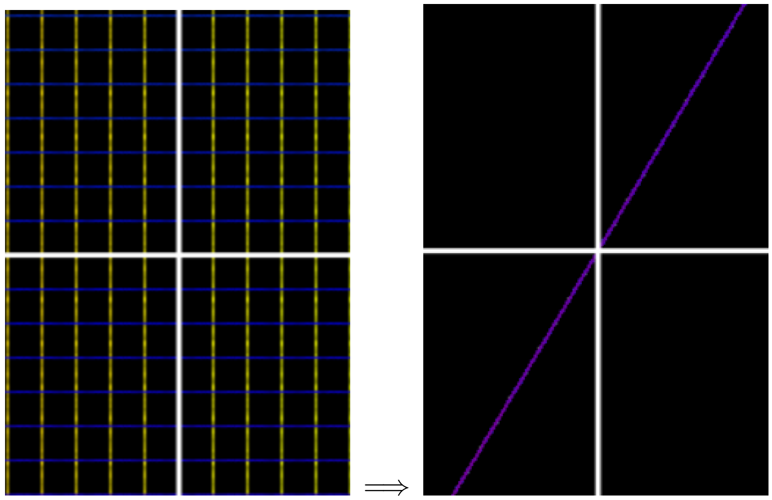
■ **Question 4.3** Consider any matrix whose columns match:

$$M = \begin{bmatrix} a & a \\ c & c \end{bmatrix}.$$

What is the determinant of M? Does M have an inverse?

Let's look a little bit more closely at the unction from Example 4.2. We can animate what the function does to the plane on a screen[3] and here are the before and after images.

[3]Click button Example 4.2 here: www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html



It looks like the entire grid collapsed to a line. And indeed, this is exactly what happens, as any vector $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}$ gets sent to:

$$M(a\hat{\mathbf{i}}+b\hat{\mathbf{j}}) = aM(\hat{\mathbf{i}})+bM(\hat{\mathbf{j}}) = a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}1\\2\end{bmatrix} = (a+b)\begin{bmatrix}1\\2\end{bmatrix}.$$

This shows that the image of the function M consists only of multiples of a single vector!

$$\operatorname{im}(M) = \text{ multiples of } \begin{bmatrix}1\\2\end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}.$$

As we have already seen, the span of a single vector is a line (can you find the equation of the line in this case?).

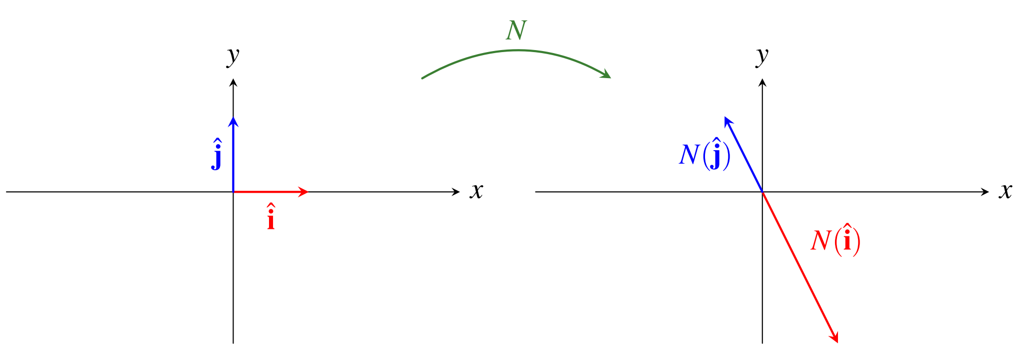
■ **Example 4.3** Let's consider the matrix:

$$N = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

The determinant can be computed to be:

$$\det N = 2\cdot 2-(-1)(-4) = 0.$$

Does N have an inverse? As above, let's start by viewing N as a function $N:\mathbb{R}^2\to\mathbb{R}^2$. We can where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go reading off the columns of N.



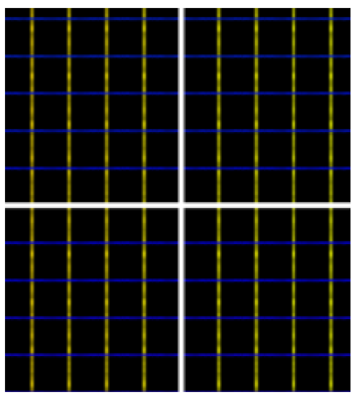
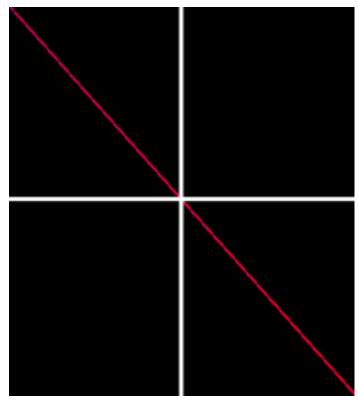
Although we don't have $N(\hat{\mathbf{i}}) = N(\hat{\mathbf{j}})$ this time, it does appear that they are parallel. Indeed, we can observe that:

$$-2N(\hat{\mathbf{j}}) = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = N(\hat{\mathbf{i}}).$$

We proved in Homework 4 (cf Exercise 3.7) that N commutes with scaling so that that $-2N(\hat{\mathbf{j}}) = N(-2\hat{\mathbf{j}})$. In summary:

$$-2\hat{\mathbf{j}} \neq \hat{\mathbf{i}} \quad \text{but} \quad N(-2\hat{\mathbf{j}}) = N(\hat{\mathbf{i}}).$$

So N is not injective, and therefore cannot have an inverse (again using Theorem 4.1.1). Let's look at an animation for N, and consider also the before and after images of the grid.[4]

 $\Longrightarrow$ 

Again, it seems to take the entire plane to a single line. Can you explain why? (*Hint:* Write $M(a\hat{\mathbf{i}} + b\hat{\mathbf{j}})$ in terms of $M(\hat{\mathbf{j}})$? Conclude that the image is the span of a single vector.) ■

■ **Question 4.4** Consider a 2×2 matrix where one column is a multiple of the other. What is the determinant of this matrix? Can it have an inverse?

We've seen a few examples of matrices M with determinant zero, and made the following observations in each case:

1. M collapses the plane down to a line (or in the case of the $\mathbf{0}$ matrix, a point).
2. M is not injective, and therefore cannot have an inverse.

In each case, it came down to the fact that the columns were multiples of eachother. This meant that $M(\hat{\mathbf{i}})$ and $M(\hat{\mathbf{j}})$ were multiples of eachother (for example $M(\hat{\mathbf{i}}) = cM(\hat{\mathbf{j}})$). Then by linearity, this tells us that $M(\hat{\mathbf{i}}) = M(c\hat{\mathbf{j}})$, so that M cannot be injective. It turns out this really covers all our bases.

Proposition 4.1.2 If M is a 2×2 matrix of determinant zero, then one column of M is a multiple of the other.

Proof. We can introduce some variables:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[4] Click button Example 4.3 here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

Then $\det M = ad - bc = 0$ means that $ad = bc$. There are a few cases to cover:

Case 1: One of the columns is all 0. Then it is the multiple of the other column by 0. For example, if:

$$M = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix},$$

the second column can be obtained by scaling the first by 0.

Case 2: One of the rows is all 0. For example, if

$$M = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

then the first column can be obtained by the second by scaling by $\frac{a}{b}$ (as long as $b \neq 0$, but if $b = 0$ then we're back in case 1).

Notice that if we don't fall in case 1 or 2, then all entries must be nonzero. Indeed, suppose one of the entries were 0, for example a. Then $ad = bc$ means $bc = 0$, so that either $b = 0$ (and therefore the first row is 0), or $c = 0$ (and therefore the first column is 0). Therefore all that remains is:

Case 3: All entries are nonzero. Then $ad = bc$ means that $\frac{a}{c} = \frac{b}{d}$. But two fractions agree precisely when you can scale the numerator and denomator of one to obtain the other, so this tells us that the columns are multiples of eachother. ■

As we saw before, once one column is a multiple of the other, the function can no longer be injective, and therefore can't have an inverse. We therefore have answered Question 4.2.

Theorem 4.1.3 A 2×2 matrix M has an inverse if and only if $\det M \neq 0$. Furthermore, if $\det M = 0$ then either:

1. M is the zero matrix, whence M collapses the entire plane to the origin, or,
2. M is nonzero. Then M collapses the entire plane to the line given by the span of one of its nonzero columns.

So we seem to understand the determinant zero case, and we can learn a lot of about the function associated with M by figuring out whether or not $\det M = 0$. But if $\det M \neq 0$, what can we say about its actual value?

What does the determinant mean?

To see what the determinant means when it's nonzero, let's experiment by looking at a couple of examples to get a bit of a baseline.

■ **Question 4.5** What is the determinant of the identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}?$$

Let's work with a more nontrivial example.

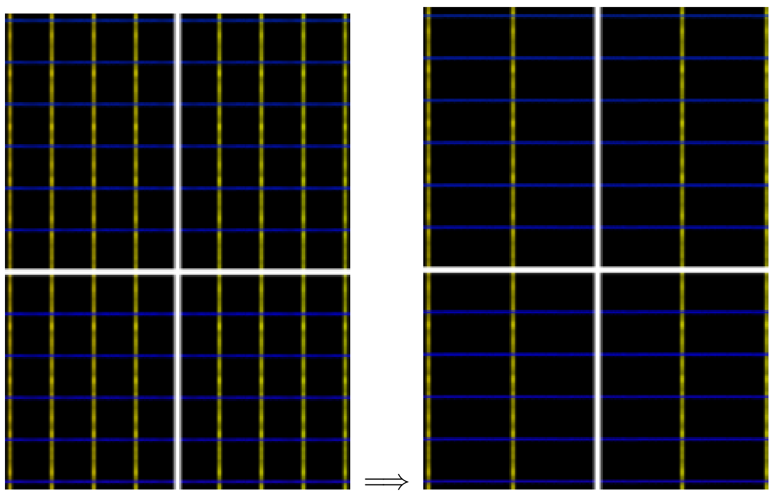
■ **Example 4.4** Consider the matrix:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

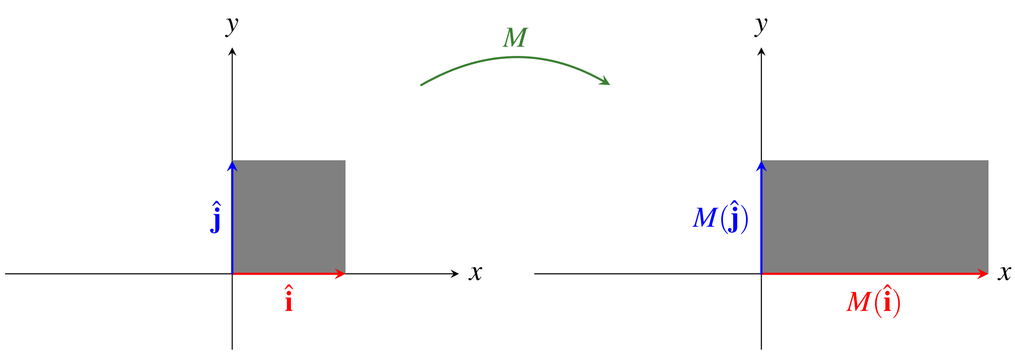
We can compute the determinant to be

$$\det M = 2 \cdot 1 - 0 \cdot 0 = 2.$$

We can look at an animation of the associated function,[5] and consider some before and after pictures of the grid.



It looks like the plane is *spreading out* in the horizontal direction. Let's try to figure out how much. Consider the square whose corners are the points $(0,0),(0,1),(1,1),(1,0)$. We will call this the *unit square*, which has area 1. We'd like to compare this to the area of the image of the unit square under the map M.



To compute the area of the new recangle, we can multiply base times height, and obtain an area of 2. This matches with the value of the determinant of M. ■

[5]Click example 4.4: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

■ **Question 4.6** Let $a, d > 0$ and consider the matrix

$$N = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

What is the determinant of N? What is the area of the image of the unit square after applying N.[6]

It looks like for *diagonal matrices*, the determinant is equal to the area of image of the unit square. In particular, if the determinant is bigger than 1, the unit square grows, and if the determinant is less than 1, the unit square shrinks. So the initial feeling is that the determinant calculates how much the linear map expands or contracts the plane. Let's do a few more examples and see if this pattern holds.

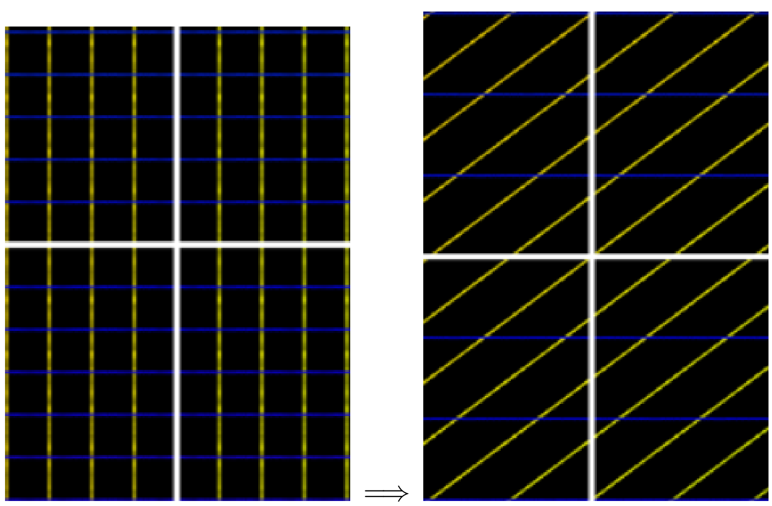
■ **Example 4.5** Let's consider the matrix:

$$P = \begin{bmatrix} 1.5 & 2 \\ 0 & 2 \end{bmatrix}.$$

The determinant is:

$$\det P = 1.5 \cdot 2 - 2 \cdot 0 = 3.$$

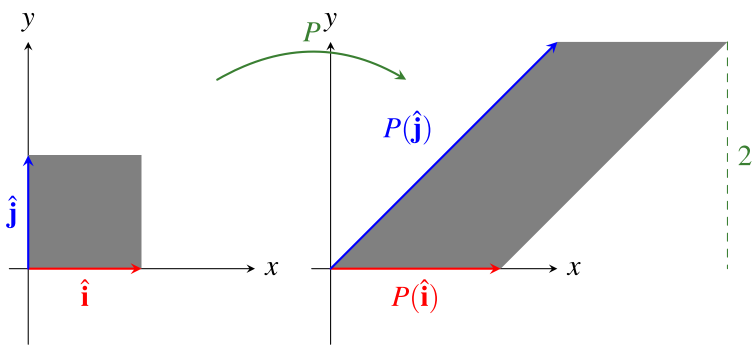
We can look at an animation of the associated function,[7] and consider some before and after pictures of the grid.



It again looks like we get some expansion, one would hope by a factor of 3.

[6]An example of this with $a = .75$ and $b = 2$ can be found by clicking Question 4.6 at the link `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`.

[7]Click Example 4.5: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

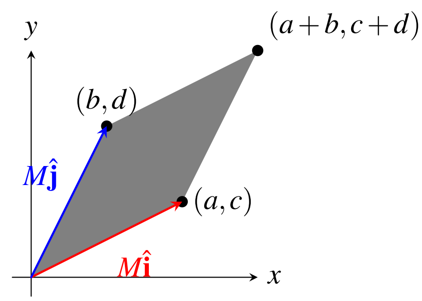


The area of a parallelogram is still base times height, which is $1.5 * 2 = 3!$ ■

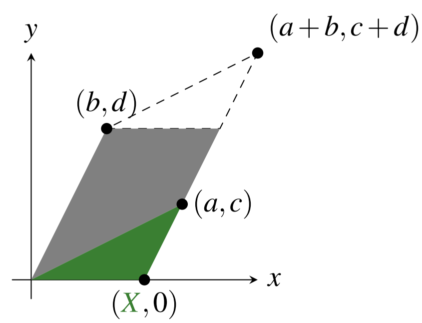
It's starting to look like the determinant exactly captures the area of the image of the unit square. Indeed, consider the matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we can trace where M take $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and determine where the unit square goes.



To compute the area, we can slice off the top of this parallelogram, and stick it at the bottom.



The new shaded region has the same area, but now we can more easily determine base times height. The height is d, and the base is the unknown quantity X. But X is the x-intercept of the line connecting (a, c) and $(a+b, c+d)$. This line has slope:

$$\frac{\Delta y}{\Delta x} = \frac{b}{d}.$$

Plugging this into point slope form with the point (a,c) gives an equation of the line:

$$y = \frac{d}{b}(x-a)+c.$$

And plugging in $y=0$ and solving for x gives us our x-intercept of:

$$X = a - \frac{bc}{d}.$$

As this is the base of the parallelogram, we need only to multiply by the height d to obtain:

$$d \cdot \left(a - \frac{bc}{d}\right) = ad - bc = \det M.$$

There is one issue with this proof: it relies on a picture where $M\hat{\mathbf{j}}$ is above $M\hat{\mathbf{i}}$. The following example demonstrates that this is rather problematic.

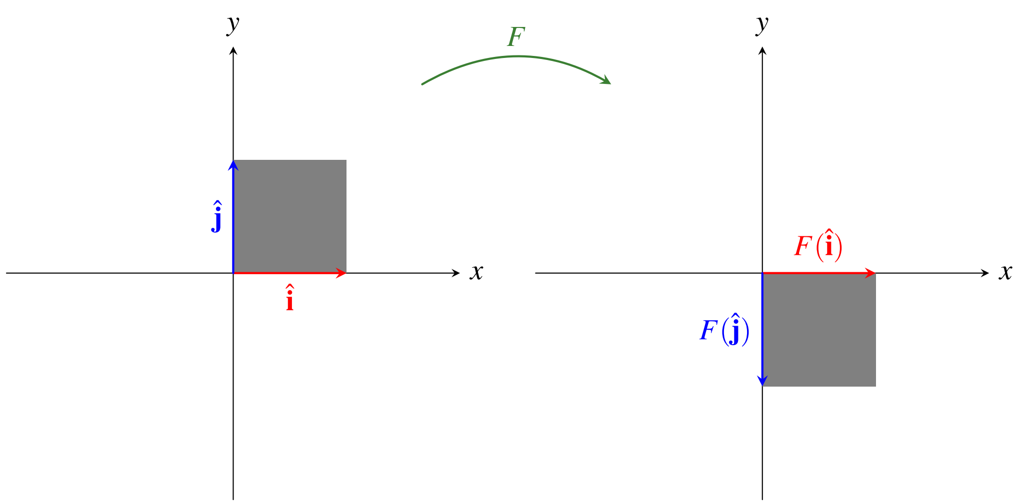
■ **Example 4.6** Consider the *flip* matrix:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then we can compute the determinant:

$$\det F = 1(-1) - 0 \cdot 0 = -1.$$

This seems immediately to be an issue. If the determinant measures an area, how can it be negative? Analyzing the image of the unit rectangle we see that its area is still 1.

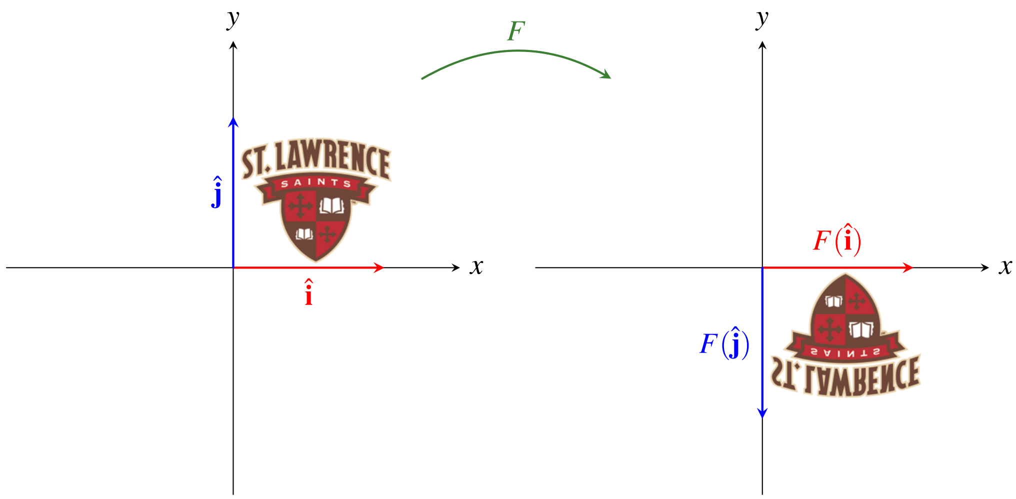


So the *absolute value* of the determinant seems to still capture the right area. So what does the negativity capture? One way to gain understanding is to look at the animation of this function.[8] In the animation, it looked like the whole grid needed to cross over itself in order to get to its final position. This corresponds to a change in *orientation*. We can describe this orientation in a couple of ways:

[8] Click Example 4.6 HERE. We won't post before and after screenshots because the grid ends up looking the same.

1. If you were to be looking in the direction of $\hat{\mathbf{i}}$, then $\hat{\mathbf{j}}$ is to the left. On the other hand, if you were to be looking in the direction of $F(\hat{\mathbf{i}})$, then $F(\hat{\mathbf{j}})$ is to the right. This corresponds to a change of orientation.
2. The angle from $F(\hat{\mathbf{i}})$ to $F(\hat{\mathbf{j}})$ (counting counterclockwise as usual) is greater than 180°.
3. If you wrote a word on the unit square, then it becomes reflected after applying F and now appears backwards.

It is clear that the first and second cases mean the same thing. For the third, let's put some words on the unit square and then apply the *flip* map.



■

All in all, we have established the following interpretation of the determinant of a 2×2 matrix.

Theorem 4.1.4 Let M be a 2×2 matrix. Then the absolute value of the determinant $|\det M|$ computes the area of the unit square after applying the function M. In particular:

1. If $|\det M| < 1$ then M contracts areas.
2. If $|\det M| = 1$ then M preserves areas.
3. If $|\det M| > 1$ then M expands areas.

Furthermore, the sign of $\det M$ determines whether M preserves orientation or not. In particular, if $\det M < 0$ then the function M is in part a reflection. More precisely:

1. If $\det M > 0$ then the angle from $M\hat{\mathbf{i}}$ to $M\hat{\mathbf{j}}$ is between 0° and 180°.
2. If $\det M < 0$ then the angle from $M\hat{\mathbf{i}}$ to $M\hat{\mathbf{j}}$ is between 180° and 360°.

■ **Question 4.7** Does the interpretation of determinants as areas hold up when $\det M = 0$?

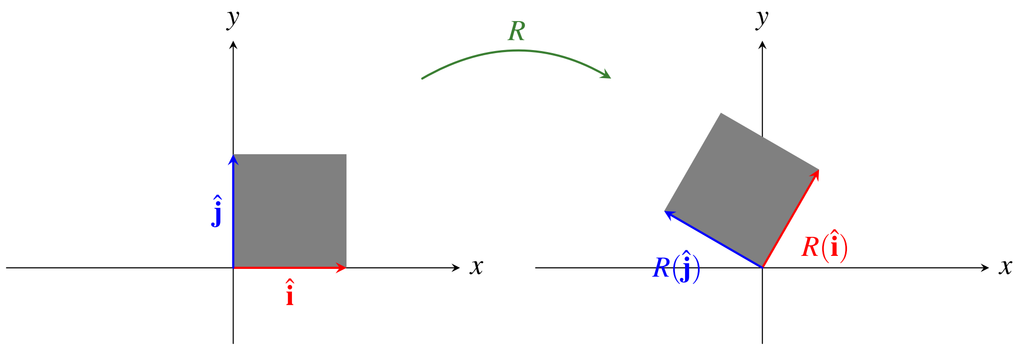
You can find a really cool interactive tool to play with determinants in real time here: `https://www.khanacademy.org/computer-programming/linear-transformation-playground-determinant-edition/6721406349426688`.

Exercise 4.1 Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates the plane by an angle of θ. What is the determinant of the matrix associated to R? Fully justify your reasoning. ■

4.2 October 15, 2024

Let's start today by going over Exercise 4.1, and in doing so, warm back up to the notion of determinants.

■ **Example 4.7** We are asked to compute the determinant of a 2×2 rotation matrix R. We will use that the determinant measures how much a transformation scales area. In particular, the absolute value of $\det R$ is the area of the image of the unit square after applying R. Below we depict what happens to R under this rotation.



As we can see, the area of the unit square doesn't change when rotating the plane, so that the absolute value of $\det R$ must be equal to 1. Furthermore, orientation is positive, as $R\hat{\mathbf{j}}$ remains to the left of $R\hat{\mathbf{i}}$, so $\det R$ is positive as well. Therefore:

$$\det R = 1.$$

Notice that we determined this without doing any computations, just by analyzing the geometry of the situation. An alternative approach could be to set:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then we could directly compute:

$$\det R = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta.$$

The fact that this is equal to 1 follows from the Pythagorean theorem. ■

4.2.1 Determinants of 3×3 matrices.

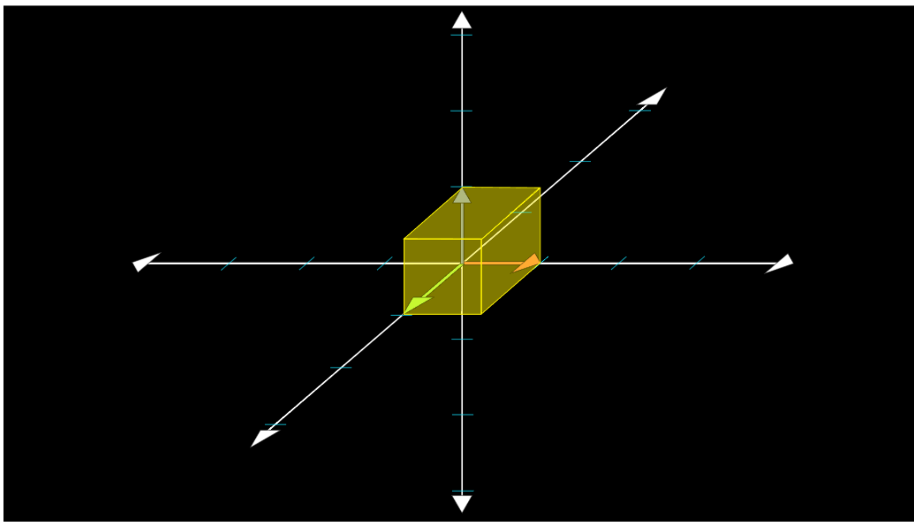
Now that we've unpacked the meaning of a 2×2 determinant, we'd like to see if we can arrive at a similar notion for 3×3 matrics, 4×4, and so on. A 2×2 matrix corresponds to a function $\mathbb{R}^2 \to \mathbb{R}^2$, and the determinant measures how much the matrix scales area in $\mathbb{R}^2$. Size in $\mathbb{R}^3$ isn't measured with areas, but instead with volumes, so this could give us a first idea.

Definition 4.2.1 — 3×3 determinants: a first attempt. Let M be a 3×3 matrix, associated to a function $\mathbb{R}^3 \to \mathbb{R}^3$ which we will also denote by M. The determinant of M is a constant which measures (up to a sign) how much volumes in $\mathbb{R}^3$ scale after applying M. That is, if V is a region in $\mathbb{R}^3$, then:

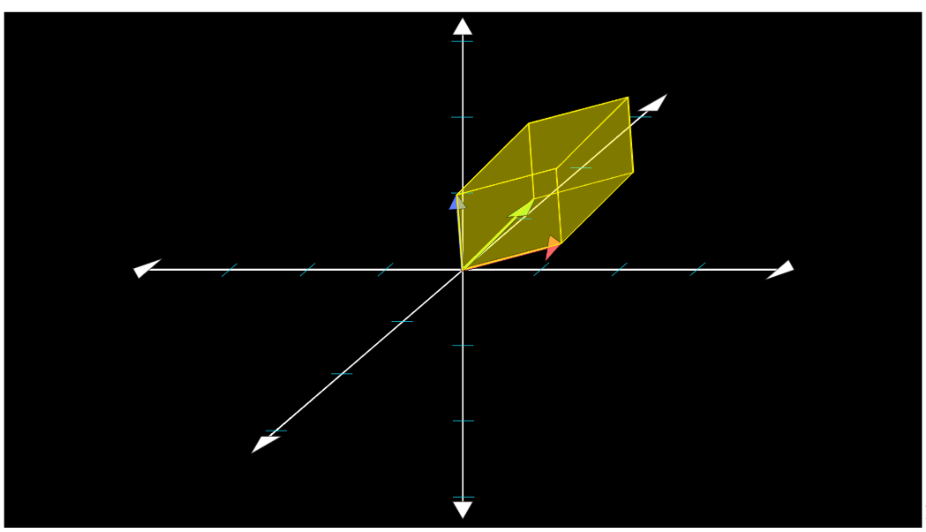
$$\text{Volume}(M(V)) = |\det M| \cdot \text{Volume}(V).$$

This definition gives us the determinant (or at least its absolute value). In order to arrive at a formula, we may want to follow what we did for 2×2 matrices. Here we started with the unit square which has area 1. Applying the transformation transformed it into a parallelogram, the area of which was precisely the absolute value of the determinant.

The analogous object to the unit square in 3-space is the *unit cube* C, formed by taking the vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ as legs of a cube.


[9]

After applying M, this gets transformed into the 3 dimensional version of a paralellogram: a *parallelepiped*, whose legs are formed by $M\hat{\mathbf{i}}, M\hat{\mathbf{j}}$ and $M\hat{\mathbf{k}}$.


[10]

[9]Image by Grant Sanderson/3Blue1Brown
[10]Image by Grant Sanderson/3Blue1Brown

If we call the unit cube C, Definition 4.2.1 says:

$$|\det(M)| \cdot \text{Volume}(C) = \text{Volume}(M(C)).$$

Subbing in that the volume of the unit cube is 1 tells us:

$$|\det(M)| = \text{Volume}(M(C)).$$

We now have pinned down a definition that captures the determinant up to a sign: just measure the volume of the image of the unit cube. This allows us now to study the following question:

■ **Question 4.8** Let M be a 3×3 matrix. Is is still true that M has an inverse if and only if $\det M = 0$?

Let's begin by collecting some evidence.

■ **Example 4.8** Let I_3 be the 3×3 identity matrix. This is an invertible matrix (what is the inverse?). Further, I_3 takes the unit cube to itself, so the determinant of I_3 is 1 (or possibly -1). ■

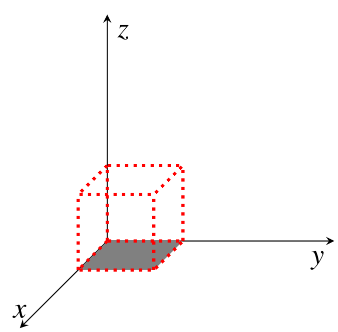
■ **Example 4.9** Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that $M(\hat{\mathbf{k}}) = \mathbf{0} = M(\mathbf{0})$. Therefore M is not injective, and cannot have an inverse. Can we work out $\det M$? Well:

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix},$$

so everything projects down vertically to the xy-plane.



As we can see, the unit cube (outlined in red) projects down to the unit square in the xy-plane (shaded in gray). The volume of the unit square is 0—being confined to 2 dimensions, it doesn't enclose any volume at all. This tells us that $\det M = 0$. ■

■ **Question 4.9** Let M be a 3×3 matrix and suppose one of its columns is zero. Is M injective? What is the determinant?

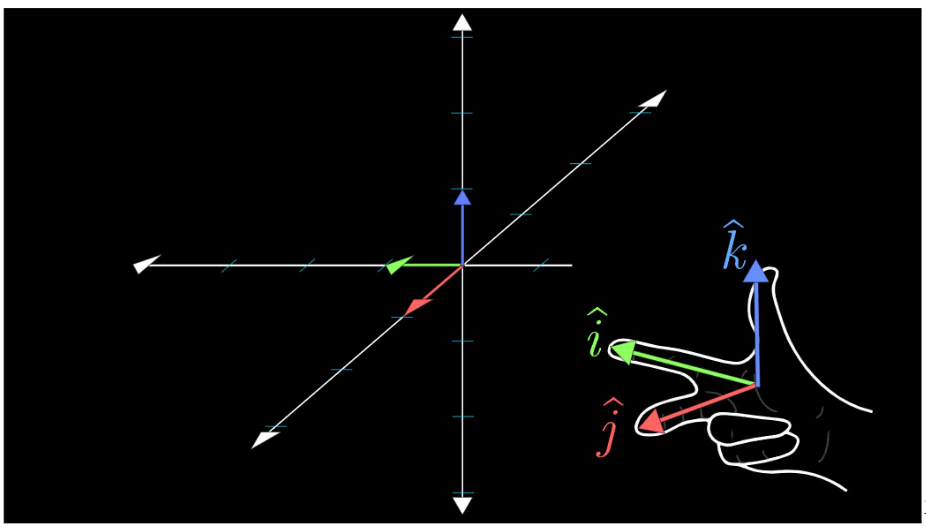
If the determinant is 0, this means that the unit cube gets projected down onto something which encloses no volume. In general, this means it goes down a dimension (like in the example we just saw). One can deduce from this that all of $\mathbb{R}^3$ gets projected onto something of smaller dimension (either a plane, a line, or a single point), from which you can deduce that the function is not injective, and therefore not invertible. We will explore this a bit more carefully on today's groupwork, answering the following question.

■ **Question 4.10** Let M be a 3×3 matrix, and suppose one of the columns is in the span of the other two. Is M injective? What is $\det M$?

Conversely, there is a formula for the inverse of a 3×3 matrix that involves dividing by the determinant (we will see this below), so that the following theorem holds.

Theorem 4.2.1 A 3×3 matrix is invertible if and only if its determinant is nonzero.

Notice we have only defined the absolute value of the determinant so far, but that's enough to determine whether or not it is zero, so the theorem above is not controversial. But it is worth taking a moment to analyze the sign of the determinant. It turns out, just like before, that it boils down to a subtle notion of orientation. One way to describe orientation in 3D is with the right hand rule: if you point your index finger of right hand in the direction of $\hat{\mathbf{i}}$, and curl your middle finger in the direction of $\hat{\mathbf{j}}$, then your thumb is pointing in the direction of $\hat{\mathbf{k}}$. This denotes the positive orientation.

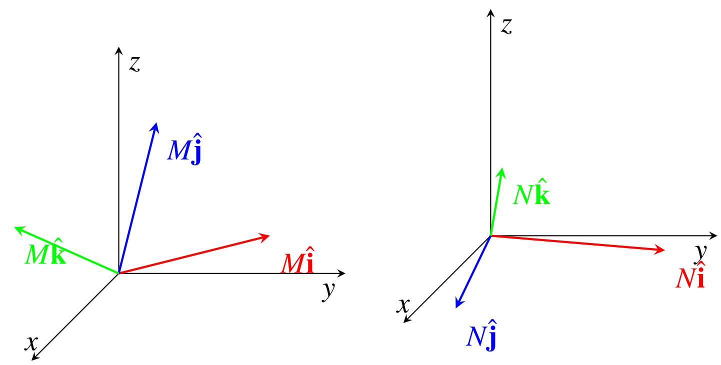


[11]

To determine whether a matrix preserves orientation or not, you apply the right-hand-rule to the image of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. In particular, point your index finger of right hand in the direction of $M\hat{\mathbf{i}}$, and curl your middle finger in the direction of $M\hat{\mathbf{j}}$. If $M\hat{\mathbf{k}}$ is going in the direction of your thumb, then orientation has remained positive. Otherwise, orientation has become negative. If M preserves orientation, then it will have positive determinant, otherwise the determinant will be negative.

■ **Example 4.10** The graphs below show where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ applying a matrix M (on left) or N (on right). Applying the right hand rule, we can observe that $\det M > 0$ and $\det N < 0$.

[11] Image by Grant Sanderson/3Blue1Brown



■

We now have a definition of a 3×3 determinant.

Definition 4.2.2 Let M be a 3×3 matrix. The determinant of M is a scalar values such that:

1. The absolute value $|\det M|$ is the volume of the image of the unit cube.
2. $\det M$ is positive if it preserves orientation (as defined by the right hand rule), and is negative if it reverses orientation.

■ **Question 4.11** We saw above that $\det I_3 = \pm 1$. Use the right hand rule to determine whether it is 1 or -1.

We have now completely defined the 3×3 determinant purely geometrically, without any difficult formulas. Of course, we would also like to compute it from time to time.

A formula for the 3×3 determinant

So how do we compute the 3×3 determinant of a matrix M? It turns out, we can use the following formula.

Proposition 4.2.2 — Cofactor Expansion along the First Row. Let:

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix}$$

Then the determinant of M can be computed via the following formula:

$$\det M = a\det\left(\begin{bmatrix} b & f \\ \ell & m \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ h & m \end{bmatrix}\right) + c\det\left(\begin{bmatrix} d & e \\ h & \ell \end{bmatrix}\right).$$

To compute this, we traverse the top row of the matrix, and for each entry in the top row of the matrix, we cross out the corresponding column along with the top row, so that a 2×2 matrix remains (often called the *minor* associated to that enty). We then multiply the determinant of this minor together with that entry. To finish, take the *alternating sum* of all the terms. That is, we add the first, subtract the second, and add the third.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix}$$

We will not prove that this formula works, but it is generally related to the *cross product* which you might see in multivariable calculus (Math 205).

R The formula we defined is often called *cofactor expansion along the first row*. One can do a similar computation along the second or third rows, and even along some columns. We will delay this generalization for the time being.

■ **Question 4.12** Compute $\det I_3$ using the formula, and compare to your answer to question 4.11.

■ **Question 4.13** Compute the determinant of:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Is M invertible?

■ **Question 4.14** Suppose M is a matrix which has a column of zeroes. Use the formula to confirm your answer to Question 4.9.

Inverting a 3×3 matrix using determinants

The 2×2 determinant was useful for computing inverse matrices. It turns out that there is a similar formula for 3×3 matrices, although it is quite a bit more complicated. First, notice that in computing a 3×3 determinant, it was useful to look at determinants of 2×2 matrices obtained by blocking out a row and a column. This turns out to be a useful idea.

Definition 4.2.3 Let M be a 3×3 matrix. The ij-minor of M is the 2×2 matrix M_{ij} obtained by removing row i and column j.

■ **Example 4.11** Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Let's compute the minor M_{23}. To do this we block out the second row and third column:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We put what remains in a matrix:

$$M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}.$$

■

■ **Question 4.15** Let M be the same matrix as in Example 4.11. What are M_{21} and M_{33}?

R This allow us to simplify the formula for the determinant a bit. Namely, if we call the ij-entry of M by the notation a_{ij}, then:

$$\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} \det M_{13}.$$

With this in hand, we can define the *adjoint matrix* for M.

Definition 4.2.4 Let M be a 3×3 matrix. Then the *adjoint* of M is:

$$\operatorname{Adj}(M) = \begin{bmatrix} \det M_{11} & -\det M_{21} & \det M_{31} \\ -\det M_{12} & \det M_{22} & -\det M_{32} \\ \det M_{13} & -\det M_{23} & \det M_{33} \end{bmatrix}.$$

■ **Warning 4.1** Notice in the definion: $\det M_{ij}$ appears in the ji-position of the adjoint matrix. This isn't a typo, this is on purpose. There is also something called the *cofactor matrix* which has $\det M_{ij}$ in the ij-position. The adjoint and cofactor matrices are related by an operation called the *transpose*, which swaps the ij and ji entries. We will cover transposes in the *matrix methods* section of this course.

With this in hand, we can now give a formula for inverting a 3×3 matrix!

Theorem 4.2.3 Let M be a 3×3 matrix whose determinant is nonzero. The inverse of M is given by the formula:

$$M^{-1} = \frac{1}{\det M}\operatorname{Adj}(M)$$

■ **Example 4.12** Let's consider the matrix

$$M = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix}.$$

We can compute the determinants of all the minors.

$$\det M_{11} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} -1 & 5 \\ 4 & -7 \end{bmatrix} = -1\cdot -7 - 4\cdot 5 = -13.$$

$$\det M_{21} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} -2 & 3 \\ 4 & -7 \end{bmatrix} = -2\cdot -7 - 4\cdot 3 = 2.$$

$$\det M_{31} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} -2 & 3 \\ -1 & 5 \end{bmatrix} = -2\cdot 5 - -1\cdot 3 = -7.$$

$$\det M_{12} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} 0 & 5 \\ -2 & -7 \end{bmatrix} = 0\cdot -7 = -2\cdot 5 = 10.$$

$$\det M_{22} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} 1 & -7 \\ -2 & 3 \end{bmatrix} = 1\cdot -7 - -2\cdot 3 = -1.$$

$$\det M_{32} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} = 1\cdot 5 - 0\cdot 3 = 5.$$

$$\det M_{13} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 0 & -1 \\ -2 & 4 \end{bmatrix} = 0 \cdot 4 - -2 \cdot -1 = -2.$$

$$\det M_{23} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = 1 \cdot 4 - -2 \cdot -2 = 0.$$

$$\det M_{33} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot -2 = 1.$$

We can plug these values in to the formula of the formula adjoint matrix, keeping track of signs, and see:

$$\mathrm{Adj}(M) = \begin{bmatrix} -13 & -2 & -7 \\ -10 & -1 & -5 \\ -2 & 0 & -1 \end{bmatrix}.$$

To finish computing M^{-1}, we also need the determinant of M. We can use the work we've already done:

$$\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} \det M_{13} = 1 \cdot (-13) - (-2) \cdot 10_3 \cdot (-2) = -13 + 20 - 6 = 1.$$

Therefore:

$$M^{-1} = \frac{1}{\det M} \mathrm{Adj}(M) = \begin{bmatrix} -13 & -2 & -7 \\ -10 & -1 & -5 \\ -2 & 0 & -1 \end{bmatrix}.$$

As we saw in Groupwork 6 Problem 2, this is exactly M^{-1}. ■

4.2.2 Determinants of $n \times n$ matrices

Everything we have done so far can be extended to 4×4 matrices and beyond (at the expense of a massive increase in computational complexity). At this point, the geometetric grounding starts to become a bit more abstract, so we will just introduce the relevant definitions. For the remainder of this section, we fix a generic square $n \times n$ matrix:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Definition 4.2.5 For integers i and j between 1 and n, we defined the ij-minor of M to be the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j. For example, M_{12} removes row 1 and column 2, resulting in:

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

We can then define the determinant of a matrix M in a similar way to how we defined it for 3×3 matrices.

Definition 4.2.6 The determinant of M is the value:

$$\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} M_{13} - \cdots + (-1)^{n+1} a_{1n} \det M_{1n} = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det M_{1j}.$$

R This is generally called cofactor expansion along the first row. We can, in fact, do so along any row. In particular, for any i, the determinant is also:

$$\det M = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij}.$$

We can also do cofactor expansion along any column.

$$\det M = \sum_{i=1}^{n} (-1)^{i+1} a_{ij} \det M_{ij}.$$

We will see later that these all give the same value.

R The determinant does have a geometric interpretation, telling us how n-dimensional volumes scale under the associated matrix transfomation. In particular, it can be a useful way to determine whether a matrix transformation generally *spreads stuff out* or *brings them together*.

To generalize Theorem 4.2.3, we must define a version of the adjoint.

Definition 4.2.7 The *adjoint* of M is the matrix $\mathrm{Adj}(M)$ whose ij-entry is $(-1)^{i+j} \det M_{ji}$. That is:

$$\mathrm{Adj}(M) = \begin{bmatrix} \det M_{11} & -\det M_{21} & \cdots & (-1)^{1+n} \det M_{n1} \\ -\det M_{12} & \det M_{22} & \cdots & (-1)^{2+n} \det M_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} \det M_{1n} & (-1)^{n+2} \det M_{2n} & \cdots & \det M_{nn} \end{bmatrix}$$

Observe the reversal in the i and the j in the subscript of M. As in Warning 4.1, this is no mistake.

We can now see exactly how the determinant helps us fully understand the inverse.

Theorem 4.2.4 Let M be an $n \times n$ matrix. Then M has an inverse if and only if $\det M \neq 0$. In this case the inverse can be computed as:

$$M^{-1} = \frac{1}{\det M} \operatorname{Adj}(M).$$

5. Systems of Linear Equations

5.1 October 22, 2024

5.1.1 Linear Systems and Matrix Equations

We have seen a couple of times (cf. Section 3.5.3 or Exercise 3.29) that if we write a system of linear equations as a single matrix equation, we can use matrix methods to solve the system more efficiently. Let's review this by unpacking Exercise 3.29, where we solve a *very large* system of linear equations, with seven equuations and seven unknowns, something that would have taken a long time with traditional methods.

■ **Example 5.1** Let's use matrix inversion to solve the following system of equations.

$$\begin{aligned}
x_1 \quad + 3x_3 \quad\quad\quad\quad - 2x_6 \quad &= 3\\
x_2 \quad + x_4 \quad + 2x_6 - 3x_7 &= 1\\
- 4x_3 + 7x_4 \quad + x_6 + x_7 &= 4\\
x_2 \quad\quad + 2x_5 - x_6 \quad &= 1\\
2x_1 \quad + 3x_3 \quad - 3x_5 \quad + x_7 &= 5\\
x_1 \quad + 4x_3 \quad - x_5 \quad - x_7 &= 9\\
- x_2 - x_3 - x_4 - x_5 \quad + 4x_7 &= 2
\end{aligned}$$

Indeed, we saw that this linear system could be rewritten as a *single matrix equation*

$$M\mathbf{x} = \mathbf{v}, \tag{5.1}$$

where:

$$M = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & -4 & 7 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 \\ 2 & 0 & 3 & 0 & -3 & 0 & 1 \\ 1 & 0 & 4 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \\ 2 \end{bmatrix}$$

If we multiply both sides of equationn 5.1 by M^{-1} *on the left* we obtaion:

$$M^{-1}M\mathbf{x} = M^{-1}\mathbf{v},$$

which simpifies to:

$$\mathbf{x} = M^{-1}\mathbf{v}.$$

Using a calculator, we can compute:

$$M^{-1} = \begin{bmatrix} 13 & 11.5 & --1 & -3.5 & -3 & -6 & 8 \\ -9.1 & -7.7 & 0.3 & 3.1 & 2.7 & 3.7 & -5.6 \\ -2.2 & -1.9 & 0.1 & 0.7 & 0.4 & 1.4 & -1.2 \\ -1.4 & -1.3 & 0.2 & 0.4 & 0.3 & 0.8 & -0.9 \\ 5.9 & 5.3 & -0.2 & -1.4 & -1.8 & -2.3 & 3.9 \\ 2.7 & 2.9 & -0.1 & -0.7 & -0.9 & -0.9 & 2.2 \\ -1.7 & -1.4 & 0.1 & 0.7 & 0.4 & 0.9 & -0.7 \end{bmatrix}$$

We now know every term in $\mathbf{x} = M^{-1}\mathbf{v}$ explicitly, so we can sub in and compute:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 13 & 11.5 & --1 & -3.5 & -3 & -6 & 8 \\ -9.1 & -7.7 & 0.3 & 3.1 & 2.7 & 3.7 & -5.6 \\ -2.2 & -1.9 & 0.1 & 0.7 & 0.4 & 1.4 & -1.2 \\ -1.4 & -1.3 & 0.2 & 0.4 & 0.3 & 0.8 & -0.9 \\ 5.9 & 5.3 & -0.2 & -1.4 & -1.8 & -2.3 & 3.9 \\ 2.7 & 2.9 & -0.1 & -0.7 & -0.9 & -0.9 & 2.2 \\ -1.7 & -1.4 & 0.1 & 0.7 & 0.4 & 0.9 & -0.7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4.9 \\ 4.8 \\ 2.6 \\ -1.1 \\ 1.7 \\ 3.3 \end{bmatrix}$$

In particular, we have solved the system of equations:

$$x_1 = -8$$

$$x_2 = 4.9$$

$$x_3 = 4.8$$

$$x_4 = 2.6$$

$$x_5 = -1.1$$

$$x_6 = 1.7$$

$$x_7 = 3.3$$

■

As this example shows, writing a system of equations as a single matrix equation provide us with a very efficient technique for solving a system of equations. Notice, though, that it was very important in this previous example that the matrix M was invertible. What happens if this is not the case?

■ **Example 5.2** Consider the system:

$$\begin{aligned} 2x - y &= -1 \\ -4x + 2y &= 2 \end{aligned}$$

We can write this system as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Unfortunately, we saw in Example 4.3 that M isn't invertible. We can check this, for example, by computing:

$$\det M = 2 * 2 - (-1)(-4) = 0.$$

What does this tell us about the system of equations? We cannot use the same technique as in Example 5.1, so where do we go from here? In this section we will answer these questions and more. ■

All of the examples we have seen so far correspond to systems with the same number equations and variables. In this case, the matrix we extract will be square. But this isn't always the case, as the following examples show.

■ **Example 5.3** Consider the system of equations:

$$x + 3y - 11z - 2w = 1,$$

$$2x + 7y - 3z + w = 5.$$

This can be written as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 2 & 7 & -3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

■

■ **Example 5.4** Consider the system of equations:

$$\begin{aligned} x + y &= 0 \\ 2x + 3y &= 5 \\ -x + 11y &= 13 \end{aligned}$$

This can be written as $N\mathbf{y} = \mathbf{w}$ where:

$$N = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 11 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 5 \\ 13 \end{bmatrix}.$$

■

The matrices M and N from Examples 5.3 and 5.4 are not square, which (we will soon see) means that they have no inverses. Where do we go from here? In this section we will expand the techniques used in Example 5.1, so that we can also use it in situations like those in Examples 5.2, 5.3, and 5.4. We should first observe that it was no coincidence that all 4 of the examples listed so far have been able to be translated into a matrix equation. This will always happen. Let's record this fact, and the central definition of this section.

Definition 5.1.1 A *linear system* is a system of linear equations in a fixed number of variables.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}
$$

Notice that any linear system can be written as a *matrix equation* which contains exactly the same data.

Definition 5.1.2 The *matrix equation* of the linear system from Definition 5.1.1 is the matrix equation:

$$M\mathbf{x} = \mathbf{v},$$

where:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Notice that the linear system in Definition 5.1.1 and the matrix equation in Definition 5.1.2 contain *exactly the same information*. They are interchangeable, and just differ notationally. In fact, we will refer to both just simply as a *linear system*. To record this one more time, we present another—equivalent but much more concise—definition of a linear system.

Definition 5.1.3 A linear system is a matrix equation $M\mathbf{x} = \mathbf{v}$ where M is a matrix, $\mathbf{x}$ is a column vector of variables, and $\mathbf{v}$ is a column vector of constants.

A Note About the Functional Perspective

So far during this course we have emphasized the perspective that *a matrix is a function*. This perspective is valuable in the context of linear systems as well. In particular, if M is an $n \times m$ matrix, then we often think about M as a function $M : \mathbb{R}^m \to \mathbb{R}^n$. Asking for a solution to the linear system:

$$M\mathbf{x} = \mathbf{v},$$

comes down to asking if there is some $\mathbf{x}$ in the domain $\mathbb{R}^m$, whose image after applying the function M is $\mathbf{v}$. Let's reframe a few of the previous examples with this perspective in mind.

■ **Example 5.5** Adopt the notation of Example 5.1. Here $M : \mathbb{R}^7 \to \mathbb{R}^7$, and we are asking for a vector whose image is

$$\mathbf{v} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3 + 1\hat{\mathbf{e}}4 + 5\hat{\mathbf{e}}_5 + 9\hat{\mathbf{e}}_6 + 2\hat{\mathbf{e}}_7.$$

Since M is invertible, we know that such a vector exists, and by applying the *undo function* M^{-1} to $\mathbf{v}$ we can even find it! In fact, this tells us that there *exactly one* vector $\mathbf{x}$ in the domain mapping to $\mathbf{v}$ under M (cf. Exercise 3.19). ■

■ **Example 5.6** Adopt the notation of Example 5.3. Here $M : \mathbb{R}^4 \to \mathbb{R}^2$, and asking for solutions $M\mathbf{x} = \mathbf{v}$ is asking for vectors $\mathbf{x}$ in the domain $\mathbb{R}^4$ whose image $M(\mathbf{x})$ in $\mathbb{R}^2$ is $\mathbf{v} = \hat{\mathbf{i}} + 5\hat{\mathbf{j}}$. Notice this perspective makes it seem like there could be no solutions (if $\mathbf{v}$ isn't in the image of the function M), or even many different solutions—if the function M sends different vectors to $\mathbf{v}$. Notice functions do this all the time, for example the *squaring* function sends 2 and -2 to the same place. ■

As the last example suggests, this function perspective tells us the following: a linear system could have no solutions, one solution, or many solutions. This suggests we should take a second to answer the following question:

5.1.2 What Does it Mean to Solve a Linear System?

Consider a linear system, written as a system of equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}$$

A *solution* to this linear system is an assignment of the values $x_1, x_2, \cdots, x_m$ so that each equation in the system is true *simultaneously*. When we rewrite this system as a matrix equation (cf Definition 5.1.2)

$$M\mathbf{x} = \mathbf{v},$$

a *solution* is a specific $\mathbf{x}$ in $\mathbb{R}^m$ which correctly solves this equation.

■ **Example 5.7** A solution for the system from Example 5.1 has can be given by the assignments:

$$x_1 = -8, \quad x_2 = 4.9, \quad x_3 = 4.8, \quad x_4 = 2.6, \quad x_5 = -1.1, \quad x_6 = 1.7, \quad x_7 = 3.3.$$

When we write the system as a matrix equation $M\mathbf{x} = \mathbf{v}$, the same solution can be given by the 7-dimensional column vector

$$\mathbf{x} = \begin{bmatrix} -8 \\ 4.9 \\ 4.8 \\ 2.6 \\ -1.1 \\ 1.7 \\ 3.3 \end{bmatrix}.$$

■

This is what it means to give *a solution* to a linear system. To *solve* a linear system, we'd like to write down *all possible* solutions to a given linear system (if they exist). In Example 5.1, we know there is only one solution, so we have already given every possible solution. This linear system has been *solved*. But what if there are multiple solutions?

■ **Example 5.8** Consider again the linear system from Example 5.2.

$$\begin{aligned} 2x - y &= -1 \\ -4x + 2y &= 2 \end{aligned}$$

Let's solve this the old fashioned way, first solving for y in the first equation:

$$y = 2x + 1.$$

Now we can plug this in to the second equation, and then simplifying and expanding the righthand-side.

$$\begin{aligned} -4x + 2(2x+1) &= 2 \\ -4x + 4x + 2 &= 2 \\ 2 &= 2 \end{aligned}$$

It simplified to the equation $2 = 2$, *which is always true!* To interpret this, we can observe that whenever the first equation is true, the second equation is also automatically true. This gives us many solutions. For example, if:

$$x = 0,$$

then

$$y = 2x + 1 = 2 \cdot 0 + 1 = 1.$$

Plugging this into the each term of the system of equations gives:

$$2 \cdot 0 - 1 = -1,$$

$$-4 \cdot 0 + 2 \cdot 1 = 2.$$

Each one holds! So $(x, y) = (0, 1)$ is a solution to the system. To see one more we can let:

$$x = -3.$$

Then,

$$y = 2x + 1 = 2(-3) + 1 = -5.$$

And we can quickly check that:

$$2(-3) - (-5) = -1,$$

$$-4(-3) + 2(-5) = 2.$$

This gives us another solution to the system: say $(x, y) = (-3, -5)$. In fact, we see that given any value we'd like to choose for x, (say some constant t), if we set y equal to one more than twice that value (that is $2t + 1$) we will get a solution to the system of equations. Since we can choose any x value we want, we will call x a *free variable*, because it is free to be whatever we'd like it to be. Once this value is chosen, though, the value for y has been set in stone as one more than twice the value of x. Since y depends on x, we will call y a *dependant variable*. We can now write a *general* solution to the system of equations in terms of a *parameter* t:

$$\begin{aligned} x &= t \\ y &= 2t+1 \end{aligned}$$

In this way we have written down *all the solutions* to the system of equations, parametrized by our *parameter* t. That is, given any choice of t, if we plug this choice into the two equations above we get a solution to the linear system. For example, if $t = 0$ we get:

$$x = 0, \quad \textit{and} \quad y = -1,$$

and if $t = -3$ we plug in to get:

$$x = -3, \quad \textit{and} \quad y = -5.$$

So one way we can *solve* a system of equations, is to write down all the solutions in terms of some *free parameters*, so that any choice of these free parameters give a solution to the equation. We can give this solution as a vector too:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t+1 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This again expresses all the solutions to the system (now as a matrix equation $M\mathbf{x} = \mathbf{v}$) in terms of a parameter t.

We can also give a more geometric flavor to the set of solutions to the system of equations. In particular, we know that if x and y satisfy the equation

$$y = 2x+1,$$

then they will solve the system. Notice that this is the equation of a line in $\mathbb{R}^2$, of slope 2 and y-intercept 1. So the solution to this system of equations could be expressed geometrically as a line. ■

■ **Example 5.9** A slight modification to the above example lands us in a squarely different situation. Let's instead consider the system of equations:

$$\begin{aligned} 2x - y &= {-1} \\ -4x + 2y &= -17 \end{aligned}$$

Again the first equation gives:

$$y = 2x+1$$

Plugging into the second equation we get:

$$\begin{aligned} -4x+2(2x+1) &= -17 \\ -4x+4x+2 &= -17 \\ 2 &= -17 \end{aligned}$$

But $2 \neq -17$, so whenver the first equation holds, the second equation fails to hold. This tells us that there are no solutions to the system of equations. ■

■ **Example 5.10** Let's consider another example.

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ \ 5w &= \ \ 3 \end{aligned}$$

This system of equations is just begging for us to solve for x and y in terms of z and w. In fact, the following system of equations is identical, albeit rearranged.

$$x = 68z + 17w - 8$$

$$y = -19z - 5w + 3$$

Notice that once we pick any values of z and w, the values for x and y are set. For example, if $z = -2$ and $w = 7$ then we have:

$$x = 68(-2) + 17 \cdot 7 - 8 = -25,$$

$$y = -19(-2) - 5 \cdot 7 + 3 = 6.$$

This gives us a solution to our system of equations via the assignments:

$$x = -25, \quad y = 6, \quad z = -2, \quad w = 7.$$

Indeed:

$$-25 - 68(-2) - 17 \cdot 7 = -8,$$

$$6 + 19(-2) + 5 \cdot 7 = 3.$$

This lands us to a situation similar to the one we encountered in Example 5.8. Once we fix *any* values for z and w (say, setting them equal to constants s and t), then we can write down values for x and y in terms of these values using the equations above to get a solution to the system of equations. Here z and w will be the *fre variables*, since they are free to be any number we'd like, and x and y are the dependent variables, whose values depends on the choice for z and w. As above, we can write down a *general solution* to the system, now in terms of 2 *parameters*, s and t.

$$x = 68s + 17t - 8,$$

$$y = -19s - 5t + 3,$$

$$z = s,$$

$$w = t.$$

Again, we have written down *all the solutions* to the system of equations, parametrized by our two parameters s and t. To generate individual solutions from this data, we need only choose values for s and t, for example, letting $s = -2$ and $t = 7$ will generate the same solution we gave above.

As in Example 5.8, we can also give this solution as a vector:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 68s + 17t - 8 \\ -19s - 5t + 3 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 68s \\ -19s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 17t \\ -5t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 68 \\ -19 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 19 \\ -5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Again, we have expressed all the solutions to the system (now as a matrix equation $M\mathbf{x} = \mathbf{v}$) in terms of our two parameters s and t.

It is a little trickier to express this solution set geometrically, given that it is contained in $\mathbb{R}^4$, which is difficult to visualize. That being said, we can see that the solution set to this system of equations is determined by 2 parameters, and therefore gives us 2 degrees of freedom when choosing solutions. This suggests that it may be reasonable to call the general solution to this system of equations *2 dimensional*. ■

Exercise 5.1 In this exercise we will be studying the linear system $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

1. Write the linear system $M\mathbf{x} = \mathbf{v}$ as a system of linear equations.
2. Write a general solution to the system of linear equations in terms of parameters s and t and r (*Note: you may not need all 3 parameters*).
3. Explicitly identify the free variables and dependent variables.
4. How many solutions are there to the linear system? Explain your reasoning.
5. By specifying parameters, give 2 explicit solutions to the linear system.

■

5.2 October 24, 2024

5.2.1 Augmented Matrices and Elementary Row Operations

In Example 5.10 it was very easy to solve for the dependant variables in terms of the free ones. Compare this to Example 5.3.

■ **Example 5.11**

$$x+3y-11z-2w=1,$$

$$2x+7y-3z+w=5.$$

If we try to follow the same outline here, we'd get:

$$x=-3y+11z+2w+1,$$

$$y=-\frac{2}{7}x+\frac{3}{7}z-\frac{1}{7}w+5.$$

It looks like perhaps we can set z and w freely, but x and y appear to be both dependent and free. Fixing y in the first equation would also set it in the second, where it should depend on the righthandside. We seem to have gotten nowhere. ■

The reason this worked better in Example 5.10 is because the dependent variables were already isolated, and only appeared in a single equation (better yet, with a coefficient of 1 so they were easy to solve for). The technique of *Gauss-Jordan Elimination* is a way to turn any linear system into one which is as easily solved as Example 5.10. This is generally done using matrix methods, but it adapted from techniques of solving a system of equations, so it may be best to consider the system of equations first.

■ **Example 5.12** We return to the following system.

$$x+3y-11z-2w=1.$$

$$2x+7y-3z+w=5$$

We'd like to try to isolate as many variables as possible, so that they only appear in a single equation. One thing we can always do is add multiples of one equation to another without changing the solution set. Let's begin but subtracting twice the first equation from the second.

$$\begin{array}{r} 2x+7y-3z+w=5 \\ \underline{-2\,(x+3y-11z-2w=1)} \\ 0x+1y+19z+5w=3 \end{array}$$

Therefore the following system of equations is equivalent to our last one.

$$\begin{aligned} x+3y-11z-2w&=1 \\ y+19z+5w&=3 \end{aligned}$$

Now x only appears in one equation. But every variable in the second equation appears in both equations, so we cannot solve for one without that variable looking free in the first equation, and dependant in the second. We can do a similar trick, subtracting twice 3 times the second equation from the first to get rid of y. Notice that because x is already eliminated in the second equation, it won't mess up our isolated x in the first. Doing this gives us an equivalent system:

$$\begin{aligned} x \quad -68z - 17w &= -8 \\ y + 19z + 5w &= 3 \end{aligned}$$

This looks familiar! We can now solve it as we did in Example 5.10. ■

Let's retrace this example with the matrix perspective in mind.

■ **Example 5.13** The linear system from 5.12 can be written as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 2 & 7 & -3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

We first subtracted twice the first equation of the system from the second. On the level of matrices, this has the effect *subtracting twice the first row from the second row of both M and* $\mathbf{v}$, resulting in:

$$M' = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 0 & 1 & 19 & 5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Our second step was to then subtract 3 times the second equation from the first equation. On the level of matrices, thi has the effect of *subtracting twice the second row from the first row of both M and* $\mathbf{v}$, resulting in:

$$M'' = \begin{bmatrix} 1 & 0 & -68 & -17 \\ 0 & 1 & 19 & 5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}' = \begin{bmatrix} -8 \\ 3 \end{bmatrix}.$$

These two moves give us an equivalent linear system $M''\mathbf{x} = \mathbf{v}''$ which corresponds to the easily solved linear system of Example 5.10.

Since we do the same things to both M and $\mathbf{v}$, it is often common to combine them into a single *augmented matrix*:

$$\left[M \mid \mathbf{v} \right].$$

The vertical bar tells us where our coeffficent matrix M ends, and our target vector $\mathbf{v}$ begins. Following the same 3 steps with the augmented matrix associated to our linear system looks as follows:

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 1 \\ 2 & 7 & -3 & 1 & 5 \end{array}\right]$$

Subtract 2*(Row 1) from (Row 2):

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 1 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right]$$

Subtract 3*(Row 2) from (Row 1):

$$\left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & -8 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right] = \left[M'' \mid \mathbf{v}'' \right]$$

We can now extract M'' and $\mathbf{v}''$ from either side of the vertical bar, and continue just as before. ■

The previous example introduced a new definition, called an augmented matrix. Let's record the general definition.

Definition 5.2.1 The *augmented matrix* associated to the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}$$

is the matrix

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm} & v_n.
\end{array}\right]$$

This can be expressed much more concisely when a linear system is already expressed as a matrix equation.

Definition 5.2.2 The *augmented matrix* associated to the linear system $M\mathbf{x} = \mathbf{v}$ is the matrix:

$$\left[M \mid \mathbf{v}\right].$$

In Example 5.12 we applied operations to the system of equations to make it easier to solve, and in Example 5.13 we traced what these operations did to the augmented matrix. In fact, there will be 3 different operations that we would like to apply to a linear system in order to solve it. Let's spell them out, and interpret them as operations on the associated augmented matrix.

1. Interchange any 2 equations.
2. Scale an equation by a *nonzero* constant.
3. Add a multiple of one equation to another equation.

It is the third operation that used twice in Example 5.12.

Interchange any 2 equations.

Swapping the i'th and j'th equations of a linear system has no effect on the solutions to the linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{red} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m} &{\color{red} = v_i}, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{blue} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m} &{\color{blue} = v_j}, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n. \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{blue} a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m} &{\color{blue} = v_j}, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{red} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m} &{\color{red} = v_i}, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n. \end{aligned}$$

In fact, it feels like this operation has on the linear system at all! On the augmented matrix, the effect is less trivial, as this corresponds to interchanging rows i and j.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {\color{red} a_{i1}} & {\color{red} a_{i2}} & \cdots & {\color{red} a_{im}} & {\color{red} v_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {\color{blue} a_{j1}} & {\color{blue} a_{j2}} & \cdots & {\color{blue} a_{jm}} & {\color{blue} v_j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right] \quad \Longrightarrow \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {\color{blue} a_{j1}} & {\color{blue} a_{j2}} & \cdots & {\color{blue} a_{jm}} & {\color{blue} v_j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {\color{red} a_{i1}} & {\color{red} a_{i2}} & \cdots & {\color{red} a_{im}} & {\color{red} v_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right]$$

Scale an equation by a nonzero constant

A common trick in solving an equation or system of equations is to multiply both sides of an equation by a nonzero constant. In particular, if we call this constant ${\color{blue} c} \neq 0$ we can multiply both sides of equation i by ${\color{blue} c}$ without changing the solution set. In particular, the following two systems will have the same set of solutions.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{red} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m} &{\color{red} = v_i}, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n. \end{aligned} \quad \Longrightarrow \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\ \vdots \qquad\qquad & \quad \vdots \\ {\color{blue} c}{\color{red} a_{i1}}x_1 + {\color{blue} c}{\color{red} a_{i2}}x_2 + \cdots + {\color{blue} c}{\color{red} a_{im}}x_m &= {\color{blue} c}{\color{red} v_i}, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n. \end{aligned}$$

On the augmented matrix, this would correspond to scaling row i by the same constant $c \neq 0$.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right] \implies \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ca_{i1} & ca_{i2} & \cdots & ca_{im} & cv_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right]$$

Adding a multiple of one row to another row

The trick we used twice in Examples 5.12 and 5.13 involved adding a multiple of one equation to another equation. Say we scale equation i by a constant c, and then add it to equation j. Doing this, and combining all like terms, has the following effect on the system of equations, without changing the solution set.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = v_1, & & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = v_2, & & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = v_2, \\ \vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = v_i, & & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = v_i, \\ \vdots \qquad\qquad \vdots & \implies & \vdots \qquad\qquad \vdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m = v_j, & & (a_{j1} + ca_{i1})x_1 + (a_{j2} + ca_{i2})x_2 + \cdots + (a_{jm} + ca_{im})x_m = v_j + cv_i, \\ \vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = v_n. & & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = v_n. \end{array}$$

On the level of augmented matrices, this has the effect of adding a multiple of row i (scaled by the same constant c) to row j.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jm} & v_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right] \implies \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jm} + ca_{im} & v_j + cv_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right]$$

Note: unlike when just scaling a row, one doesn't have to assume that $c \neq 0$. Of course, if $c = 0$, then the overall effect will be the same as doing absolutely nothing.

These three operations on matrices are inspired by the associated oprations on a system of equations, and become very useful for algorithmically solving any system of equations. They also turn out to have interesting interpretations for matrices arising in all sorts of contexts, so let's give them a name.

Definition 5.2.3 — Elementary Row Operations. Let M be any matrix. The three *elementary row operations* one can apply to M are:

1. Interchange any 2 rows of M.
2. Scale an entire row of M by a nonzero constant.
3. Add a scalar multiple of one row of M to another row of M.

■ **Example 5.14** Consider the matrix:

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

We can get a related matrix by applying the first elementary row operation, for example, interchanging the first and third rows.

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad M_1 = \begin{bmatrix} 6 & 2 & -1 \\ 0 & 1 & 0 \\ 2 & 3 & 5 \\ 3 & 0 & 0 \end{bmatrix}$$

We can get another related matrix by applying the second elementary row operations, say, scaling the third row by -3

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad M_2 = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ -3\cdot 6 & -3\cdot 2 & -3\cdot -1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ -18 & -6 & 3 \\ 3 & 0 & 0 \end{bmatrix}$$

As an example of the third elementary row operation, we can add 2 times row 1 to row 4.

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad M_3 = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3+2\cdot 2 & 0+2\cdot 3 & 0+2\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 7 & 6 & 10 \end{bmatrix}$$

■

In the previous example, we called two matrices *related* when we could get from one to the next via an elementary row operation, and indeed, if we can get from one matrix to another by a row operation (or a sequence of row operations), they will share many important properties. Let's give this relationship a name.

Definition 5.2.4 2 matrices M and N are called *row equivalent* if N can be obtained from N by a sequence of row operations.

■ **Example 5.15** The matrices M_1, M_2, M_3 in Example 5.14 are all row equivalent to M (and to eachother!). In fact, so is the matrix,

$$N = \begin{bmatrix} -4 & -6 & -10 \\ 6 & 2 & -1 \\ 0 & 1 & 0 \\ 3 & -5 & 0 \end{bmatrix}$$

as it can be obtained by N by 3 elementary row operations: first scaling Row 1 by -2, then adding -5*(Row 2) to Row 3, and finally, interchanging Rows 2 and 3.

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 6 & 2 & -1 \\ 0 & 1 & 0 \\ 3 & -5 & 0 \end{bmatrix}$$

■

If we apply an elementary row operation to an augmented matrix, we don't change the solutions to the associated linear system. After all, the row operations were based on 3 moves which didn't change the solutions of a linear system. Therefore we have the following fact.

Theorem 5.2.1 Let $M\mathbf{x} = \mathbf{v}$ and $N\mathbf{x} = \mathbf{w}$ be two linear systems. If the augmented matrices

$$\left[M \mid \mathbf{v}\right] \qquad \textit{and} \qquad \left[N \mid \mathbf{w}\right]$$

are row equivalent, then the two linear systems have the same solutions.

Theorem 5.2.1 is more or less what we used in Example 5.13 to turn the *difficult to solve* system of equations

$$x + 3y - 11z - 2w = 1.$$

$$2x + 7y - 3z + w = 5$$

into the *easy to solve* system of equations

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ 5w &= \ 3. \end{aligned}$$

In fact, Theorem 5.2.1 will be central our general strategy toward solving system of equations. Let's briefly outline it:

1. Start with linear system $M\mathbf{x} = \mathbf{v}$
2. Construct the augmented matrix $\left[M \mid \mathbf{v}\right]$
3. Via a sequence of elementary row operations, obtain row equivalent matrix $\left[N \mid \mathbf{w}\right]$ whose linear system is easier to solve
4. Solve $N\mathbf{x} = \mathbf{w}$ as in Example 5.10 potentially using parametric equations

At this point, Step 3 is the most mysterous, and for this strategy to be viable we have to answer the following 2 questions.

■ **Question 5.1** How do we know when $\left[N \mid \mathbf{w}\right]$ is easy to solve?

■ **Question 5.2** What sequence of elementary row operations should we follow?

5.2.2 Reduced Row Echelon Form

In Example 5.10, what made the system easy to solve? Let's look at it:

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ 5w &= \ 3. \end{aligned}$$

What made this so easy to solve, was the the first variable that appears in each equation doesn't appear in any other equation. Furthermore, it's coefficient is 1. Let's call this first variable that appears the *leading variable*. This made it very easy to solve for the leading variables: we could keep leave x in the first equation alone, and push everything else to the other side, thereby solving for the dependant variable x in one step. Since x doesn't appear in any other equation, this is the only dependancy it has, so we have completely solved for x. We could similarly solve for y in the second equation. In particular, we exploited that the system of equation had the following properties:

1. The leading variable in each equation has a coefficient of 1.
2. The leading variable of each equation, appears in that equation alone.

Let's translate this to the associated augmented matrix.

$$\begin{aligned} 1x + 0y - 68z - 17w &= -8 \\ 0x + 1y + 19z + \ \ 5w &= \ \ 3. \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & -8 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right]$$

Notice how the first entry of each row is 1 (highlighted in blue). Let's call these *leading 1*'s. The leading 1 in the first row corresponds to the coefficient of the leading variable x in the first equation, and the leading 1 in the second row appears in the second column, corresponding to the coefficient of the leading variable y in the second equation. To summarize, the first property of the system of equations we exploited is equivalent to the following condition on the augmented matrix:

The first nonzero value of each row is a 1.

Notice that above and below each leading 1, we only have zeros appearing. The red 0 appearing in the first entry of the second row corresponds to a coefficient of 0 next to the variable x in the second equation, and similar, the green 0 in the second entry of the first row corresponds to a coefficient of 0 for y in the second equation. In particular, the zeroes above and below the leading ones record exactly that the leading variables don't appear in any other equation. To summarize, the second property of the system of equations we exploited is equivalent to the following condition on the augmented matrix:

A leading 1 is the only nonzero entry in its column.

In particular, an augmented matrix satisfying the two properties we displayed above will correspond to a linear system that is easy to solve. Let's call such a matrix *nice*. An important fact is that any matrix is row equivalent to a nice matrix.

■ **Fact 5.1** Let M be any matrix. There exists a row equivalent matrix M^{red} which is *nice*. Furthermore, this matrix can be obtained algorithmically.

We will call this process *putting a matrix in reduced row echelon form* (cf. Definition 5.2.5). Though we will learn how to do this by hand, in practice, we generally let a computer implement this algorithm.

■ **Example 5.16** Consider the following system of equations.

$$\begin{aligned} 2x_1 - 4x_2 \qquad\quad + 6x_4 \qquad\quad &= 2 \\ 3x_2 - \ \ x_3 \qquad\quad + 3x_5 &= 1 \\ 4x_1 \qquad\quad + 2x_3 \qquad\quad - \ \ x_5 &= 1 \\ -2x_1 - \ \ x_2 + \ \ x_3 + \ \ x_4 + \ \ x_5 &= 0 \end{aligned}$$

The associated augmented matrix is:

$$\left[\begin{array}{ccccc|c} 2 & -4 & 0 & 6 & 0 & 2 \\ 0 & 3 & -1 & 0 & 3 & 1 \\ 4 & 0 & 2 & 0 & -1 & 1 \\ -2 & -1 & 1 & 1 & 1 & 0 \end{array}\right]$$

Using a calculator or computer, we can put this matrix into reduced row echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -\frac{16}{37} & \frac{6}{37} \\ 0 & 1 & 0 & 0 & \frac{83}{74} & \frac{29}{74} \\ 0 & 0 & 1 & 0 & \frac{27}{74} & \frac{13}{74} \\ 0 & 0 & 0 & 1 & \frac{33}{37} & \frac{20}{37} \end{array}\right]$$

Now we can turn this back into a system of equations (*unaugmenting*, so to speak).

$$\begin{array}{ccccccc} x_1 & & & & - & \frac{16}{37}x_5 & = & \frac{6}{37} \\ & x_2 & & & + & \frac{83}{74}x_5 & = & \frac{29}{74} \\ & & x_3 & & + & \frac{27}{74}x_5 & = & \frac{13}{74} \\ & & & x_4 & + & \frac{33}{37}x_5 & = & \frac{20}{37} \end{array}$$

Now it is clear that the leading (or dependant) variables are x_1, x_2, x_3, x_4, and the free variable will be x_5. So our general solution becomes:

$$x_1 = \frac{16}{37}t + \frac{6}{37},$$

$$x_2 = -\frac{83}{74}t + \frac{29}{74},$$

$$x_3 = -\frac{27}{74}t + \frac{13}{74},$$

$$x_4 = -\frac{33}{37}t + \frac{20}{37},$$

$$x_5 = t.$$

■

Let's give a name a matrix satisfying these properties (together which two more we haven't seen yet).

Definition 5.2.5 A matrix M is said to be in *reduced row echelon form* if

1. The first nonzero value of each row is 1. Call this value a *leading 1*.
2. Each leading 1 is the only nonzero entry in its column.
3. The leading 1 of each successive row appears further to the right.
4. All rows consisting only of zeros appear at the bottom of the matrix.

R The first 2 conditions constitute *niceness*, and are the ones that make solving equations easy. The second two can always be acheived from a matrix satisfying the first 2 by simply swapping rows.

R When asking whether an augmented matrix is in reduced row echelon form, there is nothing special about the final column, and you can freely ignore the vertical line.

We have now have another answer to question 5.1. It is easy to solve $N\mathbf{x} = \mathbf{w}$ when $\left[N \mid \mathbf{w}\right]$ is in reduced row echelon from.

■ **Example 5.17** In Exercise 5.1 we studied the linear system $M\mathbf{x} = \mathbf{v}$ where

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

The augmented matrix associated to this system is:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \end{array}\right]$$

This matrix *is* in reduced row echelon form. Indeed, the first nonzero entry of each row is a 1 (highlighted in red), and the entries sharing a column with a leading 1 are all 0 (highlighted in blue). The third condition holds as the first leading 1 is in the column 1, the second is in column 3, and the third is in column 4, so they are moving to the right as we descend rows. ■

■ **Example 5.18** The following matrix are not in reduced row echelon form.

$$M_1 = \begin{bmatrix} 0 & 3 & 9 \\ 0 & 1 & 4 \end{bmatrix}$$

This is because the first nonzero entry in row 1 is not a 1. The next matrix is still not in reduced row echelon form.

$$M_2 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4. \end{bmatrix}$$

This is because the leading 1 in the first row is not the only nonzero entry in its column. The next matrix is still not in reduced row echelon form.

$$M_3 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

This is because the leading 1 in the second row is not the only nonzero entry in its column. What about the next matrix?

$$M_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

Notice each successive matrix in Example 5.18 was obtained by an elemenatry matrix operation.

- M_2 is obtained from M_1 by multiplying the first row by $\frac{1}{3}$.
- M_3 is obtained from M_2 by subracting the first row from the second.
- M_4 is obtained from M_3 by subtracting 3 times the second row from the first.

Indeed, we were able to start with a *nonreduced* matrix, and via a sequence of elementary row operations, put in in reduced row echelon form. This starts to hint at how to answer to Question 5.2

5.3 Exercises

Exercise 5.2 In each of the following cases, compute $\det(M), \det(N)$, $\det(MN)$, and $\det(NM)$.

1. $M = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, and $N = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix}$.
2. $M = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$, and $N = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$.
3. Using the evidence gathered in parts (a) and (b), propose a relationship between $\det(M)$, $\det(N)$, and $\det(MN)$. Give a geometric explanation as to why you think this relationship is true.

■

Exercise 5.3 Let $\mathbf{v}, \mathbf{w}$ and $\mathbf{r}$ be 3 dimensional vectors, and suppose $\mathbf{r}$ is in the $\operatorname{span}\{\mathbf{v}, \mathbf{w}\}$. Consider the 3×3 matrix whose columns are formed by $\mathbf{v}, \mathbf{r}$, and $\mathbf{w}$:

$$M = \begin{bmatrix} \mathbf{v} & \mathbf{w} & \mathbf{r} \end{bmatrix}$$

. Compute $\det M$ and explain your reasoning. *Hint: Do you think M is invertible?* ■

Exercise 5.4 In Homework 6 Problem 8(d) you found a general 3×3 matrix R whose transformation corresponded to a 3d rotation with *roll angle* γ, its *pitch angle* β, and yaw angle α. Compute $\det R$? (*Hint: This will be much less messy if you interpret the determinant geometrically.*) ■

Exercise 5.5 In this exercise we will be experimenting with elementary row operations. In particular, we will look at these 3 operations:

- Operation 1: Swap rows 1 and 2.
- Operation 2: Multiply row 3 by -5.
- Operation 3: Subtract row 1 from row 3.

1. Let E_1, E_2, and E_3 be the matrices obtained by performing operations 1, 2, and 3 to the identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Find E_1, E_2, and E_3.
2. Let:

$$M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 6 & 2 \\ 6 & 5 & 3 & 5 \end{bmatrix}$$

   and let M_1, M_2, and M_3 be the matrices obtained by performing operations 1, 2, and 3 to M. Find M_1, M_2, and M_3.
3. Compute E_1M, E_2M, and E_3M (you may use a computer or do it by hand).

4. Did you observe any relations between (b) and (c)? Explain.

In Exercise 5.6, 5.7, and 5.8, solve the given system of equations in the following steps:

- Write the augmented matrix for the linear system.
- Use a calculator or computer to put the matrix from step (a) in reduced row echelon form.
- *Unagment* the matrix from step (b) to get a new system of equations with the same set of solutions.
- Solve the system of equations from step (c).

Exercise 5.6

$$\begin{aligned} -\ 2x_3 \qquad + 7x_55 &= 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 &= 28 \\ 2x_1 + 4x_2 - 5x_3 + 6x_4 - 5x_5 &= -5 \end{aligned}$$

Exercise 5.7

$$\begin{aligned} x \qquad + 2z &= 6 \\ -2x + y - z &= 0 \\ y + 2z &= -3 \\ 3x - 2y \qquad &= 1 \end{aligned}$$

Exercise 5.8

$$\begin{aligned} 2x + y - 2z &= 1 \\ 3y - z &= 1 \\ x + y + z &= 5 \end{aligned}$$

Exercise 5.9 Write the system of equations from Exercise 5.8 7 as a single matrix equation $M\mathbf{x} = \mathbf{v}$. Use a matrix inversion calculator to compute M^{-1} and solve for $\mathbf{x}$. Did you get the same asnwer this way?

5.4 Tuesday, October 29

5.4.1 Gauss-Jordan Elimination

To fill in the last gap in our strategy toward solving a linear system, we'd like to discuss the algorith alluded to in Fact 5.1. In particular, we'd like to outline a process which tells us a sequence of elementary row operations we can apply to any matrix to obtain one in reduced row echelon form. This process is called Gauss-Jordan Elimination:

Theorem 5.4.1 — Gauss-Jordan Elimination Theorem. Let M be any matrix. There exists a *unique* row equivalent matrix M^{red} which is in reduced row echelon form. Furthermore, this matrix can be obtained algorithmically.

Let's outline the algorithm, and then do a number of examples. There are two main phases in the algorithm, the *working down phase* and the *working up phase*.

	Working Down Phase
Step 1:	Locate and mark the leftmost column that isn't all zeroes.
Step 2:	Interchange the top row with one that has a nonzero entry in the marked columns.
Step 3:	Now the top entry in the marked column is $a \neq 0$. Scale the top row by $\frac{1}{a}$.
Step 4:	Add suitable multiples of the top row to the remaining rows to make the rest of the marked column 0.
Step 5:	Cover the top row, and return to step 1 with what is left.
	Working Up Phase
Step 6:	Uncover all the covered rows. Cover the bottom rows that are all zeroes.
Step 7:	Locate the bottom row and mark the column with a leading 1.
Step 8:	Add suitable multiples of the bottom row to the remaining rows to make the rest of the marked column 0.
Step 9:	Cover the bottom row, return to step 7 with what is left.
Step 10:	Uncover all the covered rows. You're done!

Let's do this in a simple example.

■ **Example 5.19** Consider the matrix:

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 0 \end{bmatrix}.$$

Let's start with the ***Working down phase***. **Step 1** asks us to locate and mark the leftmost column that isn't all zeroes.

$$\begin{bmatrix} \color{red}{2} & 4 \\ \color{blue}{-1} & -2 \\ \color{blue}{3} & 0 \end{bmatrix}.$$

Step 2 asks to make sure the top entry of the marked column is nonzero. It already is! Let's move onto **Step 3**. Since the top entry of the marked column is $\color{red}{2}$, we scale the top row by $\frac{1}{\color{red}{2}}$ to turn this red entry into a leading 1.

$$\begin{bmatrix} \color{red}{1} & \color{green}{2} \\ \color{blue}{-1} & -2 \\ \color{blue}{3} & 0 \end{bmatrix}.$$

Step 4 asks us to add suitable entries of the top row to the remaining rows, to kill the blue entries below our leading 1. To get rid of -1 in the second row, we can simply add the first row.

$$\begin{bmatrix} 1 & 2 \\ -1+1 & -2+2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}.$$

To get rid of 3 in the third row, we subtract triple the first row.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 3+(-3)*1 & 0+(-3)*2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -6 \end{bmatrix}.$$

Step 5 asks us to cover the top row, and go back to **Step 1** with what remains. Instead of covering it, we will gray it out.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -6 \end{bmatrix}$$

Now we return to **Step 1**. (Maybe make this a clicker question?). We want to mark the leftmost nonzero column. This is now the second column.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & -6 \end{bmatrix}$$

Step 2 asks us to make sure the top entry of the marked column is nonzero. It is not! To acheive this, we must swap rows 2 and 3.

$$\begin{bmatrix} 1 & 2 \\ 0 & -6 \\ 0 & 0 \end{bmatrix}$$

Step 3 tells us to turn the -6 into a leading 1, which we do by scaling all of row 2 by $\frac{1}{-6}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step 4 asks now to kill all remaining nonzero terms in the marked column, which is already done (note, the top row is all currently covered, so the 2 doesn't count yet until we work upwards). **Step 5** then commands us to cover the top row (or in our case, grey it out), and start over again.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Returning to **Step 1**, we see there are no nonzero columns, therefore no leading ones or clearing necessary. **Step 5** then covers the final row, and we have finished the ***working down phase***. We now start the ***Working up phase***, in **Step 6** by uncovering all the covered rows:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and then covering the rows that are all zero.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step 7 asks us to mark the column containing a leading 1 in the bottom row.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step 8 asks us to kill all the entries in the row with this marked leading 1. In particular, we'd like to get rid of the 2, which we can do by subtracting twice row 2 from row 1.

$$\begin{bmatrix} 1-2*0 & 2-2*1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Step 9 asks us to cover the bottom row.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Returning to **Step 7**, we mark the leading 1 in the bottom (and now only) row.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There is nothing to kill in **Step 8**, and **Step 9** leaves us with nothing left, so we can move on to **Step 10**, uncovering everything.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We now have a matrix in reduced row echelon form, which is row equivalent to the one we started with. ■

The following image[1] captures the process very well.

[1]From Interactive Linear Algebra by Dan Margalit, Joseph Rabinoff

Get a 1 here
$$\begin{pmatrix} \boxed{\star} & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} \boxed{1} & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Get a 1 here
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \boxed{1} & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \boxed{1} & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

(maybe these are already zero)
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & \boxed{0} & \star \\ 0 & 0 & \boxed{0} & \star \end{pmatrix}$$

Get a 1 here
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{\star} \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Clear down
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Matrix is in REF
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up
$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up
$$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & \boxed{1} & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is in RREF
$$\begin{pmatrix} 1 & 0 & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Watching this performed is not extremely enlightening. Instead, here are 2 for you to work out.

Exercise 5.10 Use Gauss-Jordan Elimination to put the following two matrices in Reduced Row Echelon Form.

1.

$$\begin{bmatrix} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{bmatrix}$$ [a]

2.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

3. Use Gauss-Jordan Elimination to Solve the following system of equations.

$$x+2y=3$$

$$4x + 5y = 6$$
$$7x + 8y = 9$$

[a]This is worked out in an animation here: `https://textbooks.math.gatech.edu/ila/row-reduction.html`

We first introduced elementary row operations and the Gauss-Jordan Elimination algorithm as a technique to solve linear systems. The technique proceeded as follows: given a linear system $M\mathbf{x} = \mathbf{v}$, we form the augmented matrix $\left[M \mid \mathbf{v}\right]$. We can then perform Gauss-Jordan Elimination on this matrix to obtain a matrix $\left[N \mid \mathbf{w}\right]$ which is in reduced row echelon form, and solve the (now quite simple) linear system $N\mathbf{x} = \mathbf{w}$ instead.

The elementary row operations were perfectly suited for this technique, as they corresponded precisely to operations we can perform on a system of linear equations without changing the solution set. Nevertheless, we can also perform elementary row operations—and by extension: Gauss-Jordan Elimination—on any matrix we please. It turns out that there are many good reasons to do this. The following example demonstrates one of them.

Example 5.20 Let's perform Gauss-Jordan elimination on the matrix:

$$\begin{bmatrix} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

The leading term in the first row is a 2, so to make a leading 1, let's divide the first row by 2.

$$\begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

Now to kill the leading 1 in the second row, we can subtract the first row from the second.

$$\begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

Now to make the leading $\frac{1}{2}$ in the second row into a leading 1, let's multiply the whole second row by 2.

$$\begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

Now we move to the *working up phase*, to kill the $\frac{5}{2}$ we'll subtract $\frac{5}{2}$ times the second row from the first:

$$\begin{bmatrix} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

The matrix is now in reduced row echelon form. Let's make the following observation. The first two columns of the matrix we started with is:

$$M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix},$$

so that we can write the matrix we started with as:

$$\left[M \mid I_2\right].$$

Since $\det M = 6 = 5 = 1$, M turns out to be an invertible matrix, and we can compute its inverse using Theorem 3.8.2.

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

But this is precisely the rightmost two columns of the row reduced matrix. In summary:

$$\left[M \mid I_2\right] \quad \rightsquigarrow \quad \text{Gauss-Jordan Elimination} \quad \rightsquigarrow \quad \left[I_2 \mid M^{-1}\right].$$

■

In this section, we'd like to understand this phenomenon. Does this always happen, and if so why? As a consequence, we will see that one can use Gauss-Jordan Elimination to find the inverse of a matrix (if it exists).

Do we need a new matrix inversion algorithm?

The reader may recall that we already encountered an algorithm to compute the inverse of a matrix in Theorem 4.2.4, indeed, that is how we noticed the inverse arose in Example 5.20. The problem is, Theorem 4.2.4 uses determinants to find the inverse, and to compute the determinant of an $n \times n$ matrix (using Definition 4.2.6), we need to compute $n!$ different 2×2 determinants.

For example, to invert a 20×20 matrix M using Theorem 4.2.4, we would have to compute:

$$20! = 2432902008176640000,$$

different 2×2 determinants, which even for a powerful computer is quite a task. On the other hand, we could row reduce:

$$\left[M \mid I_{20}\right],$$

which has us working down (row by row over 20 rows), and then working back up (row by row over another 20 rows), leading to about 40 column clearing steps, which is much *much* faster in practice (especially with a computer). If the reduced matrix is indeed:

$$\left[I_{20} \mid M^{-1}\right],$$

as in Example 5.20, then we can much more quickly recovered M^{-1} as the 20 rightmost columns of the matrix we were reducing.

5.5 October 31, 2024

5.5.1 Elementary Matrices

In Homework 8 (cf. Exercise 5.5) we considered the matrices associated to applying an elementary row operation the the identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For example, we could swap the first and second rows of the identity matrix to get:

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider what happens when we multiply E by the matrix:

$$M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 6 & 2 \\ 6 & 5 & 3 & 5 \end{bmatrix}.$$

Wheobtain:

$$EM = \begin{bmatrix} 5 & 9 & 6 & 2 \\ 3 & 1 & 4 & 1 \\ 6 & 5 & 3 & 5 \end{bmatrix},$$

which is precisely M, but with the first and second rows swapped. That is, multiplying by E has the same effect as applying the elementary row operation that was performed to create E. Exercise 5.5 shows that this happens with other elementary row operations as well. This suggests the following may be true.

■ **Slogan 5.1** One can perform elementary row operations using matrix multiplication.

This suggests that it is worth looking at the matrices which perform these operations.

Definition 5.5.1 An *elementary matrix* is a $n \times n$ matrix which can be obtained by performing a single elementary row operation on the identity matrix I_n.

■ **Example 5.21** Consider the matrix:

$$E = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is an elementary matrix, as it can be obtained from I_4 by subtracting twice row 3 from row 1. ■

■ **Example 5.22** Let:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

If we compute EM we see that

$$EM = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-2*5 & 2-2*6 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -9 & -10 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

This is precisely the matrix obtained from M by subtractince twice row 3 from row 1. ■

■ **Example 5.23** Consider the elementary matrix:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

which is obtained from I_2 by doubling the second row. Then multiplying by E (on the left) has exactly the effect of doubling the second row. Indeed:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{bmatrix} &= \begin{bmatrix} a_1 + 0b_1 & a_2 + 0b2 & \cdots & a_m + 0b_m \\ 0a_1 + 2b_1 & 0a_2 + 2b_2 & \cdots & 0a_m + 2b_m \end{bmatrix} \\ &= \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ 2b_1 & 2b_2 & \cdots & 2b_m \end{bmatrix}. \end{aligned}$$

■

Exercise 5.5 and Examples 5.21 and 5.22 suggest the following proposition.

Proposition 5.5.1 Let E be an $n \times n$ elementary matrix which is obtained by performing an elementary row operation to I_n, and let M be an $n \times m$ matrix. Then EM is the matrix obtained by performing the same elementary row operatoin to M.

We will omit the proof[2], but hope the reader notices while solving Exercise 5.5 and working through Examples 5.21 and 5.22 that it feels almost tautological.

We are trying to build a relationship between invertible matrices and elementary row operations. Seeing that elementary row operations can be performed via matrix multiplication puts these two concepts on the same playing court. To get closer, we'd like to observe that elementary matrices are invertible. The key observation is the following:

Proposition 5.5.2 Every elementary row operation is reversable.

Proof. If we swap two rows of a matrix, we can easily swap them back. If we scale a row by a nonzero c, we can undo this by scaling the same row by $\frac{1}{c}$. Finally, if we add a multiple of row i to row j, we can undo this by subtracting the same multiple of row i from row j. ■

As a consequence, we can prove the invertibility of elementary matrices.

Corollary 5.5.3 Every elementary matrix is invertible.

Proof. Let E be any elementary matrix. It is obtained by perfomring some row operation to I_n. Let F be the elementary matrix which performs the reverse operation. Then FE is the matrix obtained by this reverse operation to E, that is, undoing the operation by which we obtained E in the first place. In summary, $FE = I_n$. An identical argument tells us that $EF = I_n$, so that $F = E^{-1}$. ■

5.5.2 Gauss-Jordan Elimination and Invertible Matrices

Theorem 4.2.4 allowed us to recognize whether a matrix is invertible or not by computing its determinant. In particular, we saw that a square matrix M is invertible if and only if $\det M \neq 0$. Unfortunately, as we saw above, computing determinants using Definition 4.2.6 is very slow (even for a powerful computer). For examle, computing the determinant of a 20×20 matrix would involve computing $20! = 2432902008176640000$ 2×2 determinants. Gauss-Jordan elimination gives us another way to test if a matrix M is invertible: put it in reduced row echelon form!

[2] at least in this draft of the notes

Theorem 5.5.4 A matrix M is invertible precisely when M^{red} is the identity matrix.[a]

[a]Recall that M^{red} is the output when M is put in reduced row echelon form.

Proof. First let's assume M is invertible. Consider the linear system $M\mathbf{x} = \mathbf{0}$. Since M is invertible we know $\mathbf{x} = M^{-1}\mathbf{0} = \mathbf{0}$, solving the linear system. That is, the solution to the system of equations is:

$$\begin{array}{cccc} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \vdots \\ & & x_n & = 0 \end{array}$$

On the other hand, we can solve $M\mathbf{x} = \mathbf{0}$ using Gauss-Jordan Elimination. To do this, we reduce the augmented matrix $\left[M \mid \mathbf{0}\right]$, which reduces to $\left[M^{red} \mid \mathbf{0}\right]$. But this must be the same as the augmented matrix for the solved linear system, so that:

$$\left[M^{red} \mid \mathbf{0}\right] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right] = \left[I_n \mid \mathbf{0}\right]$$

Therefore we can see that $M^{red} = I_n$.

Conversely, let's assume that $M^{red} = I_n$. In particular, there are a sequence of elementary row operations we can perform to M to obtain I_n. Let $E_1, E_2, \cdots, E_t$ be the elementary matrices obtained by performing these operations to I_n. Then we see that:

$$(E_1 E_2 \cdots E_t)M = I_n.$$

Since each E_i is invertible, we have that:

$$M = E_t^{-1} \cdots E_2^{-1} E_1^{-1},$$

so that it is a product of invertible matrices, and is therefore invertible. ■

■ **Example 5.24** Consider the matrix:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

and computed that:

$$M^{red} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore we can conclude that M has no inverse. ■

■ **Example 5.25** Consider the matrix:

$$M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \\ 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 \end{bmatrix}$$

One can compute (by hand or with a calculator) that:

$$M^{red} = I_4,$$

so that M is invertible. ■

In Example 5.20 we saw something slightly stronger: that if M is invertible, then $\left[M \mid I\right]$ reduces to $\left[I \mid M^{-1}\right]$. Let's try to see why this might be true.

Consider an invertible matrix M. By Theorem 5.5.4 we know that $M^{red} = I_n$. Let's continue as in the proof of Theorem 5.5.4, fixing a sequence of elementary row operations we can perform to M to obtain I_n, and letting $E_1, E_2, \cdots, E_t$ be the elementary matrices obtained by performing these operations to I_n. Then:

$$E_1 E_2 \cdots E_t M = I_n.$$

Multiplying both sides on the left by M^{-1} we obtain:

$$M^{-1} = E_1 E_2 \cdots E_t.$$

Now let's row reduce $\left[M \mid I_n\right]$. To do this, we will perform the same row operations, i.e., we will multiply by $E_1 E_2 \cdots E_t$.

$$E_1 E_2 \cdots E_t \left[M \mid I_n\right] = \left[E_1 E_2 \cdots E_t M \mid E_1 E_2 \cdots E_t I_n\right] = \left[I_n \mid M^{-1}\right].$$

Furthermore, the matrix we end with is in reduced row echelon form. Indeed:

$$\left[I_n \mid M^{-1}\right] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & M^{-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \end{array}\right],$$

and we can verify directly that this satisfies the conditions of Definition 5.2.5. We have therefore proved the following fact.

Theorem 5.5.5 Let M be an invertible $n \times n$ matrix. Then:

$$\left[M \mid I_n\right]^{red} = \left[I_n \mid M^{-1}\right].$$

■ **Example 5.26** We saw in Example 5.25 that

$$M = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 5 & 9 & 2 & 6 \\ 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 \end{bmatrix}$$

is invertible. Let's use Theorem 5.5.5 and a calculator to compute its inverse.

$$[M \mid I] = \left[\begin{array}{cccc|cccc} 3 & 1 & 4 & 1 & 1 & 0 & 0 & 0 \\ 5 & 9 & 2 & 6 & 0 & 1 & 0 & 0 \\ 5 & 3 & 5 & 8 & 0 & 0 & 1 & 0 \\ 9 & 7 & 9 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

We use a calculator to put this in reduced row echelon form:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{-467}{98} & \frac{-123}{98} & \frac{79}{98} & \frac{191}{98} \\ 0 & 1 & 0 & 0 & \frac{171}{98} & \frac{57}{98} & \frac{-39}{98} & \frac{-67}{98} \\ 0 & 0 & 1 & 0 & \frac{163}{49} & \frac{38}{49} & \frac{-26}{49} & \frac{-61}{49} \\ 0 & 0 & 0 & 1 & \frac{12}{49} & \frac{4}{49} & \frac{5}{49} & \frac{-9}{49} \end{array}\right]$$

Therefore we can extract the last four columns as M^{-1}.

$$M^{-1} = \begin{bmatrix} \frac{-467}{98} & \frac{-123}{98} & \frac{79}{98} & \frac{191}{98} \\ \frac{171}{98} & \frac{57}{98} & \frac{-39}{98} & \frac{-67}{98} \\ \frac{163}{49} & \frac{38}{49} & \frac{-26}{49} & \frac{-61}{49} \\ \frac{12}{49} & \frac{4}{49} & \frac{5}{49} & \frac{-9}{49} \end{bmatrix}$$

■

5.6 Exercises

Exercise 5.11 Consider the matrix:

$$M = \begin{bmatrix} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

1. Use Gauss-Jordan Elimination to put M in reduced row echelon form. (Do this by hand, and label every elementary row operation you do).
2. Use your answer to part (a) to solve the linear system:

$$\begin{aligned} -7y - 4z &= 2 \\ 2x + 4y + 6z &= 12 \\ 3x + y - z &= 2 \end{aligned}$$

■

Exercise 5.12 Use the row reduction technique to invert the following 6×6 matrix. You may use a calculator to row reduce the matrix: but write down what the matrix it is you are reducing, and what the output of the reduction is.

$$\begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 & 8 \\ 9 & 7 & 9 & 3 & 2 & 3 \\ 8 & 4 & 6 & 2 & 6 & 4 \\ 3 & 3 & 8 & 3 & 2 & 7 \\ 9 & 5 & 0 & 2 & 8 & 8 \end{bmatrix}$$

■

Exercise 5.13 Use Theorem 5.5.4 to answer the following question. Explain your reasoning.

Can a matrix which is not square be invertible?

■

Exercise 5.14 Consider a linear system $M\mathbf{x} = \mathbf{v}$ with augmented matrix $\left[M \mid \mathbf{v}\right]$.

1. Suppose every column of the reduced matrix $\left[M \mid \mathbf{v}\right]^{red}$ except the rightmost column has a leading 1. How many solutions does the linear system have? Justify your answer.
2. Suppose the rightmost column of the row reduced matrix $\left[M \mid \mathbf{v}\right]^{red}$ has a leading 1. How many solutions does the linear system have? Justify your answer.
3. Suppose the rightmost column of the row reduced matrix $\left[M \mid \mathbf{v}\right]^{red}$ **does not** have a leading 1. Suppose that there is at least one other column without a leading 1. How many solutions does the linear system have? Justify your answer.

■

Exercise 5.15 Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

1. Compute the determinant of M by hand. Compute the determinant on the calculator you will use for the exam. Compare the results.
2. Based on 1, do you think M is invertible?
3. Compute the *adjoint matrix* for M by computing the determinants of all the 2×2 minors of M. (See Definition 4.2.7 on page 438 of the course notes).
4. Compute M^{-1} using the formula given in Theorem 2.4.2 (on page 139 of the course notes).

■

Exercise 5.16 Let M be the same matrix as in the previous question.

1. Use your calculator to row reduce the matrix:

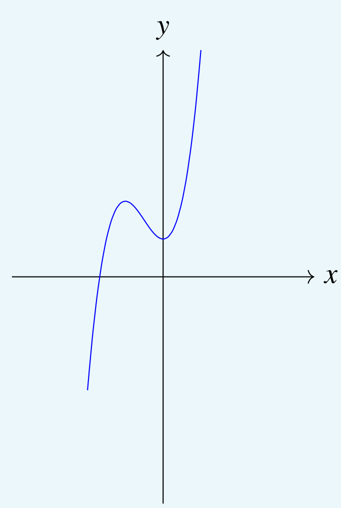
$$\begin{bmatrix} M \mid I_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   Extract M^{-1} from your result.
2. Use the row reduced matrix from part (a) to solve the following system of equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 \quad\quad &= 0 \\ x_2 + 4x_3 \quad + x_5 &= 0 \\ 5x_2 + 6x_2 \quad\quad\quad &= 1 \end{aligned}$$

3. Row reduce the matrix from part (a) by hand.

Exercise 5.17 The following is the graph of a cubic equation $y = ax^3 + bx^2 + cx + d$.



The following the following 4 points lie on the curve: $(-2,-3),(-1,2),(0,1),(1,6)$. Find the equation for the curve.

Hint: Let a,b,c,d be variables. Then each point on the curve will give you one linear equation, so you can construct a system of 4 linear equations in the variables a,b,c,d. You can use Gaussian Elimination (and a row reduction calculator) to solve this system, and deduce the values of a,b,c,d.

6. Bases and Dimension

6.1 Day 19

6.1.1 Linear Combinations and Spans Revisited

In Section 2.3.2 we introduced the notion of *linear combinations of vectors* and *spans of vectors*. The technique of Gaussian Elimination gives us new computational tools to study these notions. n this section, we will see that row reducing a matrix whose columns are built from a collection of vectors gives us lots of information about their span. We will see this by diving deeply back into our color mixing example. To begin, though, let's review the definitions.

Definition 6.1.1 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$. A *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a vector $\mathbf{w}$ in $\mathbb{R}^n$ which can be written in terms of the $\mathbf{v}_i$. That is, there exist constants $a_1, a_2, \cdots, a_r$ such that:

$$w = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r.$$

■ **Example 6.1** The vector:

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

is a linear combination of the two vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}.$$

Indeed, if I consider:

$$3\mathbf{v}_1 + 2\mathbf{v}_2 = 3\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 12 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \mathbf{w}.$$

■

■ **Example 6.2** In Homework 2 (cf Exercises 2.17, 2.18, 2.19) we discussed measuring colors is in RGB, where:

$$\begin{array}{cc} & \text{r units of red} \\ & \text{g units of green} \\ + & \text{b units of blue} \\ \hline & \text{a rich spectrum of colors} \end{array}$$

We considered an artist with two pigments, which they are hoping to mix together into paint to try and get the color they want.

Pigment X: Adding 1 ounce of Pigment X to paint adds:

1 unit of red, 2 units of green, 3 units of blue

Pigment Y: Adding 1 ounce of Pigment Y to paint adds:

7 units of red, 5 units of green, 2 units of blue

In this case we represented the effects of adding one ounce of Pigment X or Pigment Y to paint as colum vectors:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

Then, a linear combination of $\mathbf{x}$ and $\mathbf{y}$ is a color represented by the vector:

$$a\mathbf{x} + b\mathbf{y},$$

which is mixed from a ounces of Pigment X and b ounces of Pigment Y. Concretely, a linear combination of $\mathbf{x}$ and $\mathbf{Y}$ is a color that can be mixed from Pigment X and Pigment Y. For example, in Exercise 2.17 we observed that:

$$3\mathbf{x} + 3\mathbf{y} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix} = \mathbf{f}.$$

Concretely, the color Fancy Gold can be from Pigment X and Pigment Y. Equivalently, the vector $\mathbf{f}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.

On the other hand, in Exercise 2.18 we introduced a color super green, corresponding to the vector:

$$\mathbf{s} = \begin{bmatrix} 3 \\ 90 \\ 3 \end{bmatrix},$$

and saw that there is *no values* a and b such that:

$$\mathbf{s} = a\mathbf{x} + b\mathbf{y}.$$

That is, we cannot mix super green from Pigment X and Pigment Y, or equivalently, $\mathbf{s}$ is *not* a linear combination of $\mathbf{x}$ and $\mathbf{y}$. ■

As we saw in Example 6.2, sometimes very concrete questions come down to following question.

■ **Question 6.1** *Is some vector a linear combination of another collection of vectors?*

Sometimes the answer is *yes*, and sometimes the answer is *no*, and depending on your context, this can have some concrete meanings. To package this together, we introduced the notion of *a span*

Definition 6.1.2 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$. The *span* of these vectors, dentoted:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\},$$

is the collection of vectors that can be written as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$.

■ **Example 6.3** Example 6.1 tells us that:

$$\begin{bmatrix}1\\2\\4\end{bmatrix} \text{ is in span} \left\{ \begin{bmatrix}-1\\0\\4\end{bmatrix}, \begin{bmatrix}2\\1\\-4\end{bmatrix} \right\}.$$

■

■ **Question 6.2** Is there anything not in the span described in Example 6.3?

■ **Example 6.4** Example 6.2 tells us that $\mathbf{f}$ is in $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ and $\mathbf{s}$ is not. ■

As we saw in Exercises 2.17-2.19, the fundamental question of whether a certain vector is in the span of other vectors (Question 6.1) boils down to solving a linear system. Let's see this concretely.

■ **Example 6.5** If we are trying to mix Fancy Gold from Pigment X and Pigment Y, we are solving the equation

$$a\mathbf{x} + b\mathbf{y} = \mathbf{f}. \tag{6.1}$$

which plugging in column vectors is:

$$a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}7\\5\\2\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}.$$

This gives us the following matrix equation:

$$\begin{bmatrix}1a+7b\\2a+5b\\3a+2b\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}.$$

This factors into:

$$\begin{bmatrix}1 & 7\\2 & 5\\3 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}.$$

This can be written more concisely as:

$$\begin{bmatrix}\mathbf{x} & \mathbf{y}\end{bmatrix}\mathbf{z} = \mathbf{f},$$

where $\mathbf{z}$ is a column of indeterminates.[1] We can now use Gauss-Jordan Elimination, building an augmented matrix $\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{f}\right]$ which reduces to

$$\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{f}\right]^{red} = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

Similarly, if we are trying to mix super green from Pigment X and Pigment Y, we are solving:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{s},$$

for a and b. This boils down to solving the matrix equation:

$$\left[\mathbf{x} \quad \mathbf{y}\right]\mathbf{z} = \mathbf{s}.$$

We again build the augmented matrix $\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{s}\right]$ which reduces to

$$\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{g}\right]^{red} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

The third row unaugments to the equation:

$$0 = 1,$$

so we know there is no solution.

This exact argument works for any color. If we want to know if some color $\mathbf{c}$ can be mixed from our pigments, we can row reduce the matrix $\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{c}\right]$. Playing a particularly important role in each case was the matrix whose columns were formed by our pigments:

$$M = \left[\mathbf{x} \quad \mathbf{y}\right].$$

In fact, we know what this matrix reduces to:

$$M^{red} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This bottom row of zeros predicted the failure of mixing super green, or at least the failure some color. Indeed, whenever we try to mix the color $\mathbf{c}$, we reduce $\left[M \mid \mathbf{c}\right]$, and obtain:

$$\left[M \mid \mathbf{c}\right]^{red} = \left[M^{red} \mid \mathbf{c}'\right] = \left[\begin{array}{cc|c} 1 & 0 & c_1' \\ 0 & 1 & c_2' \\ 0 & 0 & c_3' \end{array}\right].$$

So if we ever end up with $c_3' \neq 0$, we will fail to make c'. ■

We will record what we learned from the last example into the following slogan, though we will delay proving the general case until further down the line.

[1] We use $\mathbf{z}$ because $\mathbf{x}$ has been taken to play the roll of Pigment X in this example.

■ **Slogan 6.1** We can get a lot of information about $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ by looking at the matrix whose columns are the v_i:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

Our analysis in Example 6.5 suggest we study the following question.

■ **Question 6.3** Let M be the matrix of the matrix from Slogan 6.1. If the bottom row of M^{red} is all zeros, does this mean there is some vector $\mathbf{w}$ which is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$?

Let's now continue Example 6.5 by throwing some Greenifier into the mix (cf. Exercise 2.19).

■ **Example 6.6** We've seen that we cannot mix super green from Pigment X and Pigment Y alone (cf. Exercise 2.18 or Example 6.5), or equivalently, that $\mathbf{s}$ is not in $\text{span}\{\mathbf{x}, \mathbf{y}\}$. In Exercise 2.19, we introduce a new pigment: Greenifier, which has the effect of subtracting red and blue pigment. Written as a column vector, Greenifier is represented by:

$$\mathbf{g} = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix}.$$

Exercise 2.19 went on to show that you could mix super green from Pigment X,Pigment Y, and Greenfier. Let's review this with matrix methods in mind. We'd like to solve:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} = \mathbf{s},$$

for values a, b, c. As in Example 6.5, we can factor this into a matrix equation:

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{s}.$$

To solve this, we can reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} \mathbf{x} & \mathbf{y} & \mathbf{g} & \mathbf{s} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 7 & -5 & 3 \\ 2 & 5 & 0 & 90 \\ 3 & 2 & -5 & 3 \end{array}\right]$$

Applying Gauss-Jordan to the matrix above results in the matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 22.5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 16.5 \end{array}\right]$$

Therefore we can un-augment and obtain:

$$a = 22.5$$

$$b = 9$$

$$c = 16.5,$$

and end up with a recipe for creating super green:

> 22.5 Ounces of Pigment X. 9 Ounces of Pigment Y. 16.5 Ounces of Greenifier.

As in Examplre 6.5, it seems like Slogan 6.1 rings true: if we want to understand span$\{\mathbf{x},\mathbf{y},\mathbf{g}\}$, it seems like a good idea to consider the matrix:

$$M = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix}.$$

In fact, we just computed that this matrix reduces to the identity: $M^{red} = I_3$. This turns out to be a crucial observation. It will help us answer the following question.

■ **Question 6.4** Can I mix any color I like from Pigment X,Pigment Y, and Greenifier?

Let's answer Question 6.4, by fixing a random color:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

To mix it from our pigments, we are asking if it is in span$\{\mathbf{x},\mathbf{y},\mathbf{g}\}$, i.e, to solve the equation:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} = \mathbf{v} \qquad or \qquad M\mathbf{z} = \mathbf{v},$$

where $M = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix}$ as above and

$$\mathbf{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

is our vector of indeterminates. To solve this we can reduce the augmented matrix:

$$\left[M \mid \mathbf{v} \right]^{red} = \left[M^{red} \mid \mathbf{w} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & w_1 \\ 0 & 1 & 0 & w_2 \\ 0 & 0 & 1 & w_3 \end{array}\right].$$

This unaugments to

$$a = w_1,$$

$$b = w_2,$$

$$c = w_3.$$

So we obtain a (unique) solution, and can indeed mix our random color from the pigments in stock! Notice that we could answer this without resorting to Gauss-Jordan Elimination. Mixing the color $\mathbf{v}$ came down to solving $M\mathbf{z} = \mathbf{v}$. Since $M^{red} = I_3$, we know M^{-1} exists, so we can solve for $\mathbf{z} = M^{-1}\mathbf{v}$.

To summarize: *we can mix any color from Pigment X,Pigment Y, and Greenifier.* To translate to a more general language, we can model our *rgb* color space by $\mathbb{R}^3$:

$$\mathbb{R}^3 = \left\{ \text{All colors} \begin{bmatrix} r \\ g \\ b \end{bmatrix} \right\}.$$

Then, the fact that we can mix any color from our pigments says that the vector $\mathbf{x}, \mathbf{y}$ and $\mathbf{g}$ span all of $\mathbb{R}^3$:

$$\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{g}\} = \mathbb{R}^3.$$

■

Let's summarize what we just observed into a general statement.

Proposition 6.1.1 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$, and consider the matrix whose columns are these vectors:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

If M is invertible, then the $\mathbf{v}_i$ span all of $\mathbb{R}^n$:

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\} = \mathbb{R}^n.$$

Proposition 6.1.1 has a restriction built in to its setup. In order for the matrix M to be invertible, it must at least be square, so in particular, $r = n$. This means that Proposition 6.1.1 really can only tell us about the situation if the number of vectors agrees with the dimension of the space. Does this feel too restricitve? That is:

■ **Question 6.5** If we have r vectors in $\mathbb{R}^n$, and $r \neq n$, is it possible for these vectors to span all of $\mathbb{R}^n$?

The key case to look at is where $r > n$. Let's rephrase this in terms of our color mixing examples.

■ **Question 6.6** Suppose I have more than 3 pigments. Is it possible to mix any color?

The answer to this question is obviously yes, but to try and get a sense of what to look for in the matrix algebra, let's carefully work out an example anyways.

■ **Example 6.7** We know that $\mathbf{x}, \mathbf{y}$ and $\mathbf{g}$ already span our whole color space. Suppose we get our hands on a fourth pigment: magenta. This is represented by the vector:

$$\mathbf{m} = \begin{bmatrix} 10 \\ 0 \\ 6 \end{bmatrix}.$$

Let's answer the question: *can I mix any color from Pigment X,Pigment Y,Greenifier, and magenta*? We already know the answer is yes, since the first 3 pigments will do, and we can just use 0 ounces of magenta. But let's continue anyway and see what we can learn about the matrix algebra. We let $\mathbf{v}$ be a random color in $\mathbb{R}^3$ and try to solve:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} + d\mathbf{m} = \mathbf{v} \qquad or \qquad M\mathbf{z} = \mathbf{v},$$

where $M = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} & \mathbf{m} \end{bmatrix}$ and:

$$\mathbf{z} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

To solve this:, we reduce the augmented matrix:

$$\left[M \mid \mathbf{v}\right]^{red} = \left[M^{red} \mid \mathbf{w}\right].$$

In particular, the reduction of the matrix whose columns are our pigments becomes relevant again (cf. Slogan 6.1). This reduced matrix is:

$$M^{red} = \begin{bmatrix} 1 & 7 & -5 & 10 \\ 2 & 5 & 0 & 0 \\ 3 & 2 & -5 & 6 \end{bmatrix}^{red} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & .4 \\ 0 & 0 & 1 & -1.64 \end{bmatrix}$$

Returning to our equation $M\mathbf{z} = \mathbf{v}$, we now can see our reduced augmented matrix to be:

$$\left[M \mid \mathbf{v}\right]^{red} = \left[M^{red} \mid \mathbf{w}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & w_1 \\ 0 & 1 & 0 & .4 & w_2 \\ 0 & 0 & 1 & -1.64 & w_3 \end{array}\right]$$

In particular, this unaugments to:

$$a - d = w_1,$$

$$b + .4d = w_2,$$

$$c - 1.64d = w_3.$$

We have 3 dependant variables (a, b, c) and a free variable d, to which we assign general parameter t: giving a general solution:

$$a = w_1 + t,$$

$$b = w_2 - .4t,$$

$$c = w_3 + 1.64t,$$

$$d = t.$$

Letting $t = 0$, we recover the way to mix $\mathbf{v}$ in terms of Pigment X,Pigment Y, and Greenifier, but observe that with magenta, we have many more ways to mix our general color $\mathbf{v}$. Infinitely many, in fact. ■

The important fact about the matrix M^{red} that guaranteed a solution in Example 6.7 was that *each row had a leading 1*. This meant that putting it as the *coefficient side* of any augmented matrix would generate a solution we could explicitly solve for. Equivalently, the fact that M^{red} has no rows that were all 0s, meaning it was impossible to put it into an augmented matrix which had a row

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right],$$

corresponding to an equation $0 = 1$ (this in direct contrast to what happened in Example 6.5). As a result, we can give the following strengthening to Proposition 6.1.1.

Theorem 6.1.2 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$, and consider the matrix whose columns are these vectors:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

If every row of M^{red} as a leading 1,[a] then

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\} = \mathbb{R}^n.$$

[a] equivalently, if M^{red} has no rows which are all 0

Proof. If we let $\mathbf{w}$ be a random vector in $\mathbb{R}^n$, we'd like to observe that there is a solution to:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_r\mathbf{v}_r = \mathbf{w}.$$

This can be written as the matrix equation $M\mathbf{x} = \mathbf{w}$, which can be solved by reducing the augmented matrix:

$$\begin{bmatrix} M \mid \mathbf{w} \end{bmatrix}^{red} = \begin{bmatrix} M^{red} \mid \mathbf{w}' \end{bmatrix}.$$

Since every row of M^{red} already has a leading one, then the final column of $\begin{bmatrix} M \mid \mathbf{w} \end{bmatrix}^{red}$ doesn't have a leading one. Therefore, as we argued in Exercise 5.14 (HW9 Problem 4), there must be a solution to the linear system. ■

R In fact, we can do better! If every column of M has a leading 1, then Exercise 5.14 tells us there is exactly one solution. Otherwise, there are infinitely many solutions. We can upack this as saying that if $r = n$ there is a unique way to write every vector in $\mathbb{R}^n$ in terms of the $\mathbf{v}_i$, and if $r > n$ there are infinitely many ways to do so. This will turn out to be a cruacial observation, which we will expand on with the notion of a *basis* below.

6.2 Exercises

Exercise 6.1 Consider the following four vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 0 \end{bmatrix}$$

1. Write down 3 vectors that are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
2. Is $\mathbf{v}_5$ a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$? Why or why not?
3. Is $\mathbf{v}_5$ in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$? Why or why not?
4. Do the 5 vectors above span all of $\mathbb{R}^4$? Why or why not?
5. Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ span all of $\mathbb{R}^4$? Why or why not?
6. Do the vectors $\mathbf{v}_1, \mathbf{v}_3$, and $\mathbf{v}_5$ span all of $\mathbb{R}^4$? Why or why not?

■

Exercise 6.2 Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be 3 vectors in $\mathbb{R}^4$, and consider the 4×3 matrix whose columns are the $\mathbf{w}_i$.

$$M = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix}.$$

1. What is the maximum number leading 1's that M^{red} can have?
2. What is the bottom row of M^{red}? Explain your reasoning.
3. Can 3 vectors ever span all of $\mathbb{R}^4$? Why or why not? *Hint: First write down some augmented matrix $\left[M^{red} \mid \mathbf{x}\right]$ that could cause trouble. Then use the fact that elementary row operations are reversable.*

■

Exercise 6.3 Do you agree or disagree with the following statement? Why or why not?

If $r < n$ then r vectors can never span all of $\mathbb{R}^n$

■

Exercise 6.4 Prove Theorem 6.1.2. (Hint: It should be an immediate consequence of Exercise 5.14). ■

Exercise 6.5 One way to measure colors is in RGB, where:

r units of red
g units of green
\+ b units of blue
a rich spectrum of colors

With this as a model, we can model our entire *color space* with $\mathbb{R}^3$, letting:

$$\mathbb{R}^3 = \left\{ \text{All colors } \begin{bmatrix} r \\ g \\ b \end{bmatrix} \right\}.$$

Another model of colors (generally used by printers, both at-home and industrial) uses as its building blocks: Cyan, Magenta, and Yellow. In the RGB spectrum, these are represented by the vectors:

$$\mathbf{c} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}.$$

1. We previously studied Pigment X and Pigment Y, which were given (in RGB) by the

vectors:

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

If we wanted to print these pigments, we'd need to express these colors as mixtures of Cyan, Magenta, and Yellow. How much of each do we need?

2. Explain why every color in the RGB spectrum can printed *uniquely* as a mixture of Cyan, Magenta, and Yellow.
3. Colors on a screen are generally stored in RGB, but when sent to a printer must be translated to a mixture of Cyan, Magenta and Yellow. Explain a process that can take a general RGB color vector:

$$\mathbf{v} = \begin{bmatrix} r \\ g \\ b \end{bmatrix},$$

and extract the values of Cyan, Magenta, and Yellow needed to print it. (We can think of this as a *change of coordinates*, where we are taking the RBG coordinates for $\mathbf{v}$, and tranlating them to CMY coordinates. These are two different ways to represent the same color as a 3D vector!)

4. Let's turn the process from part (c) into a function:

$$F : \mathbb{R}^3 \to \mathbb{R}^3.$$

The input is a vector

$$\mathbf{v} = \begin{bmatrix} r \\ g \\ b \end{bmatrix},$$

which represents the color in the RGB spectrum. The output is a vector:

$$F(\mathbf{v}) = \begin{bmatrix} c \\ m \\ y \end{bmatrix},$$

which represents *the same color* but now in the CMY spectrum. Write down the matrix for the function F. That is, find a 3×3 matrix N such that:

$$F(\mathbf{v}) = N\mathbf{v}.$$

(This should be a *specific matrix*).

5. Use the matrix from part (d) to find the CMY values necessary to print red, green, and blue.

6.3 November 7, 2024

6.3.1 Linear Independence

Last week we described a procedure to answer the following question:

Question 6.7 Can a vector $\mathbf{w}$ could be expressed as a linear combination of another collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$.

Question 6.7 came down to solving the linear system

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix} \mathbf{x} = \mathbf{w},$$

which in turn boiled down to studying the reduction of the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdot & \mathbf{v}_r & \mathbf{w} \end{array}\right].$$

These techniques also allowed us to answer the following related question:

Question 6.8 When does a collection of n-dimensional vectors $\mathbf{v}_1, \cdots, \mathbf{v}_r$ span all of $\mathbb{R}^n$?

This question was answered by Theorem 6.1.2 and boiled down to studying the reduction of the matrix:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

Today, we'd like to look at a more refined version of Question 6.7. To motivate this, observe that it can have a few different types of answers.

Not a linear combination: Sometimes the answer to question 6.7 is just *no*. In Example 6.5 we saw that we cannot mix super green from Pigment X and Pigment Y. That is: there is no way to write $\mathbf{s}$ as a linear combination of $\mathbf{x}$ and $\mathbf{y}$. This corresponded to the linear system

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \mathbf{z} = \mathbf{s},$$

having no solution.

A unique linear combination: Sometimes $\mathbf{w}$ can be written in terms of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ in *exactly one way*. In Example **??** we saw that to mix super green from Pigment X, Pigment Y, and Greenifier, we needed exactly

22.5 Ounces of Pigment X. 9 Ounces of Pigment Y. 16.5 Ounces of Greenifier.

That is, not only does

$$22.5\mathbf{x} + 9\mathbf{y} + 16.5\mathbf{g} = \mathbf{s},$$

but *no other coefficients would do!*. This boils down to the linear system

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix} \mathbf{z} = \mathbf{s},$$

having a unique solution.

Infinitely many linear combinations: Sometimes $\mathbf{w}$ can be written in terms of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ in *many different ways*. In Example 6.7 we saw that we could mix super green from Pigment X, Pigment Y, Greenifier and Magenta in many different ways. In fact, we saw that:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} + d\mathbf{m} = \mathbf{s},$$

holds whenever:

$$a = 22.5 + t,$$

$$b = 9 - .4t,$$

$$c = 16.5 + 1.64t,$$

$$d = t,$$

for some value t. This means, for example, we could mix

22.5 oz of Pigment X. 9 oz of Pigment Y. 16.5 oz of Greenifier. 0 oz of Magenta.

Or we could mix

32.5 oz of Pigment X. 5 oz of Pigment Y. 32.9 oz of Greenifier. 10 oz of Magenta.

This is letting $t = 0$ or $t = 10$, but we could let t be any number we want! This corresponded to the linear system

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} & \mathbf{m} \end{bmatrix} \mathbf{z} = \mathbf{s}$$

having infinitely many solutions.

Our main goal for today is to develop tools to tell the difference between these three situations, and we will introduce some useful terminology along the way. In particular, we will work on answering a more refined version of Question **??**

■ **Question 6.9** In how many ways can a vector $\mathbf{w}$ could be expressed as a linear combination of another collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$.

We've actually already developed all the tools we need to answer this question. *Let's see if we can start to brainstorm a list of relevant questions we can ask.*

- Does $r = n$?
- Is $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}$ is invertible?
- Does $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix} \mathbf{x} = \mathbf{w}$ have free variables?

As we observed last week (cf. Slogan 6.1), it seems relevant to consider the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

In fact, if this matrix is invertible, we can immmediately answer question **??**, arguing exactly as we did in Proposition 6.1.1.

Proposition 6.3.1 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of n dimensional vectors, and suppose the matrix:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix},$$

is invertible. Then any n-dimensional vector $\mathbf{w}$ can be written uniquely as a linear combination of the $\mathbf{v}_i$.

Proof. We are trying to solve for constants $a_1, a_2, \cdots, a_r$ such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r = \mathbf{w}.$$

This factors as

$$M\mathbf{x} = \mathbf{w} \qquad \textit{where} \qquad \mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}$$

Since M is invertible we can solve for

$$\mathbf{x} = M^{-1}\mathbf{w},$$

giving our unique solution. ■

Exactly like in Proposition 6.1.1, this Proposition 6.3.1 has a restriction built into its setup. In order for M to be invertible, it must at least be square, so again we must have $r = n$. This means that Propositoin 6.3.1 can only tell us about the situation if the number of vectors agrees with the dimension of the space, which is too restrive. Question 6.9 still feels relevant if $r \neq n$.

We already know how to tell the difference between when something is or isn't in the span. We should expect the *infinteness* to come from the existence of free variables, and this will turn out to be true, but let's take a slightly different approach first. To start to get a sense about where exactly the *infiniteness* comes from, let's look at a simpler example.

■ **Example 6.8** Let's work in $\mathbb{R}^2$, and let's let:

$$\mathbf{v}_1 = \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

How can we write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$? For example, we could write:

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix} = 3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} = 3\mathbf{v}_1 + 7\mathbf{v}_2 + 0\mathbf{v}_3.$$

But we could also write:

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\mathbf{v}_1 + 6\mathbf{v}_2 + 1\mathbf{v}_3.$$

Or even:

$$\mathbf{w} = 1\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3 = 0\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3 = \cdots.$$

There are infinitely many different ways that we could do this! Where does this redundancy come from? Well:

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \hat{\mathbf{i}} + \hat{\mathbf{j}} = \mathbf{v}_1 + \mathbf{v}_2.$$

That is, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. This means that anywhere we have $\mathbf{v}_1 + \mathbf{v}_2$, we could instead write $\mathbf{v}_3$. This means that nothing that we can write in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ can be written uniquely.

$$\begin{aligned} a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 &= (a-1)\mathbf{v}_1 + (b-1)\mathbf{v}_2 + (\mathbf{v}_1 + \mathbf{v}_2) + c\mathbf{v}_3 \\ &= (a-1)\mathbf{v}_1 + (b-1)\mathbf{v}_2 + (v_3) + c\mathbf{v}_3 \\ &= (a-1)\mathbf{v}_1 + (b-1)\mathbf{v}_2 + (c+1)\mathbf{v}_3 \end{aligned}$$

■

In Example 6.8, the infiniteness arose because $\mathbf{v}_3$ could be written in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, these 3 vectors didn't exist independently, they shared an extra relationship expressed by the equation

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$$

In order to have any hope of uniqueness, we would need to eliminate this property. This motivates the following definition.

Definition 6.3.1 A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are called *linearly independent* if none of the vectors can be written as a linear combination of the rest. That is, for each $i = 1, 2, \cdots, r$,

$$\mathbf{v}_i \text{ is not in } \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_r\}.$$

A collection of vectors which is **not** linearly independent is called *linearly dependent*.

■ **Example 6.9** The vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ in $\mathbb{R}^3$ are linearly independent. Notice that that $\hat{\mathbf{k}}$ is not in $\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$. Indeed, try to write

$$\hat{\mathbf{k}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}.$$

As column vectors, this is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix},$$

which can never hold because the third coordinates don't agree. Similarly, $\hat{\mathbf{j}}$ is not in $\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}\}$ because we can never have:

$$\hat{\mathbf{j}} = a\hat{\mathbf{i}} + b\hat{\mathbf{k}} \qquad \text{or} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}.$$

One can argue that $\hat{\mathbf{i}}$ is not in $\operatorname{span}\{\hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ very similarly. ■

■ **Question 6.10** Are $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in $\mathbb{R}^2$ linearly independent?

■ **Question 6.11** Are $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ linearly independent?

A Trick to Determine Linear Independance

The definition of linear independence is somewhat intuitive: a collection of vectors is linearly independent if they don't satisfy any extra relationships, as evidenced by one vector being written in terms of the others. But as we started to see in Example 6.9, this definition can be tedious to verfiy. A priori, to verify if r vectors are linearly independent, we have to check that r different equations can't be satisfied. It turns out we can reduce this to one simple check. To do this, let's consider return to Question 6.11

■ **Example 6.10** We can see that $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are linearly dependant because:

$$-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} = \mathbf{v}.$$

Pushing everything to one side of the equation, we have that:

$$-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \mathbf{v} = \mathbf{0}.$$

That is, we can write the zero vector as a *nontrivial* linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\mathbf{v}$. ■

Let's introduce some notation:

Definition 6.3.2 A *nontrivial* linear combination of a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ is a linear combination:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r,$$

where the a_i are not all zero.

Notice that we can do what we did in Example 6.10 for any linearly dependant set. Suppose I have a collection of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_r$ which are *not* linearly independent. This means I can write one vector in terms of the others. Perhaps reording the list, we can assume this is $\mathbf{v}_r$.

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{r-1}\mathbf{v}_{r_1} = \mathbf{v}_r.$$

Following Example 6.10, we can move everything to one side:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{r-1}\mathbf{v}_{r_1} - \mathbf{v}_r = \mathbf{0}.$$

The coefficient of $\mathbf{v}_r$ is -1, so we have written $\mathbf{0}$ as a *nontrivial* linear combination of $\mathbf{v}_1$ through $\mathbf{v}_r$.

The converse is true as well. If I can write $\mathbf{0}$ as a nontrivial linear combination of some vectors, I can solve for one vector in terms of the rest, showing that the set must be linearly dependent. Therefore, we have the following characterization of linear dependence.

Theorem 6.3.2 A collection $\mathbf{v}_1, \cdots, \mathbf{v}_r$ of vectors is linearly independent if and only if whenever:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r = \mathbf{0},$$

we have,

$$a_1 = 0 \quad a_2 = 0 \quad \cdots \quad a_r = 0.$$

■ **Example 6.11** The standard basis for $\mathbb{R}^n$

$$\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \hat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \hat{\mathbf{e}}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

is linearly independent. Indeed, suppose

$$a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n = \mathbf{0}.$$

This can be rewritten as:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

■

Theorem 6.3.2 boils the question of linear independence down to studying the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r = \mathbf{0}.$$

which factors as the linear system:

$$M\mathbf{x} = \mathbf{w} \qquad \textit{where} \qquad M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix} \quad \textit{and} \quad \mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}$$

Therefore, the reduction type of M seems relevant again, and Slogan 6.1 re-enters the picture. Let's consider a few examples.

■ **Example 6.12** Consider the vectors corresponding to Pigment X and Pigment Y.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

To determine if these vectors are linearly independent we consider:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{0}, \qquad \text{or equivalently} \qquad \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0}.$$

The augmented matrix is:

$$\left[\begin{array}{cc|c} 1 & 7 & 0 \\ 2 & 5 & 0 \\ 3 & 2 & 0 \end{array}\right],$$

which reduces to:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

so the only solution is $a = b = 0$. Therefore $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. ■

■ **Example 6.13** Let's throw in the vector for Greenifier.

$$\mathbf{g} = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix}.$$

To determine independence, we'd like to solve the system:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} = \mathbf{0}, \qquad \text{or equivalently} \qquad \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}.$$

To do this, we reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 7 & -5 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 2 & -5 & 0 \end{array}\right] \overset{red}{=} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right],$$

so that $a = 0, b = 0, c = 0$, giving independence of these 3 vectors. ■

■ **Example 6.14** Let's throw in the vector for Magenta.

$$\mathbf{m} = \begin{bmatrix} 10 \\ 0 \\ 6 \end{bmatrix}.$$

To determine independence, we'd like to solve the system:

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{g} + d\mathbf{m} = \mathbf{0}, \qquad orequivalently \qquad \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} & m \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0}.$$

To do this, we reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 7 & -5 & 10 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 3 & 2 & -56 & & 0 \end{array}\right] \overset{red}{=} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 \\ 0 & 0 & 1 & -1.64 & 0 \end{array}\right].$$

This un-augments to the system:

$$a - d = 0,$$

$$b + .4d = 0,$$

$$c - 1.64d = 0.$$

We have 3 dependant variables (a,b,c) and a free variable d, to which we assign general parameter t: giving a general solution:

$$a = t,$$

$$b = -.4t,$$

$$c = 1.64t,$$

$$d = t.$$

For example, letting $t = 1$, we have:

$$\mathbf{x} - 0.4\mathbf{y} + 1.64\mathbf{g} + \mathbf{m} = \mathbf{0} \tag{6.2}$$

Therefore, these four vectors are **not** linearly independent. This checks out, because in Example 6.6 we determined that any color in the RGB spectrum could be written in terms of Pigment X, Pigment Y, and Greenify. This spectrum certainly includes Magenta. In fact, Equation 6.2 gives us a way to do this, by solving for $\mathbf{m}$[2]

$$\mathbf{m} = -\mathbf{x} + 0.4\mathbf{y} - 1.64\mathbf{g}.$$

■

Notice that is precisely the existence of free variables in the linear system:

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix} \mathbf{x} = \mathbf{0},$$

that allowed us to determine that there was a nontrivial solution. We can summarize this with the following strengthening of Proposition 6.3.1.

Theorem 6.3.3 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$, and consider the matrix:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

The vectors are linearly independent precisely when every column of M^{red} has a leading 1.

Proof. We are studying solutions to the equation:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r = \mathbf{0}.$$

This factors as:

$$M\mathbf{x} = \mathbf{0} \qquad \textit{where} \qquad \mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}.$$

[2] this reflects the proof of Theorem 6.3.2, which tells us that if $\mathbf{0}$ is a nontrivial linear combination, we can solve for one of the vectors in terms of the rest

Notice that the *trivial solution* where all the $a_i = 0$ (that is: $\mathbf{x} = \mathbf{0}$) is always a solution. If we row reduce the augmented matrix we obtain:

$$\left[M \mid \mathbf{0}\right]^{red} = \left[M^{red} \mid \mathbf{0}\right].$$

If every column of M^{red} has a leading 1, then this is the only solution (cf. Exercise 5.14.2). Otherwise, we have a free variable, and infinitely many solution (cf. Exercise 5.14.3), which must include a nontrivial one. ■

6.4 November 12, 2024

Last time we discussed what it meant for a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ to be *linearly independent*. This roughly meant that there were no notrivial equations that the vectors satisfied. More precisely, no one of the vectors could be written in terms of the others:

$$\mathbf{v}_i \neq a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_{i+}\mathbf{v}_{i+1} + \cdots + a_r\mathbf{v}_r.$$

With some effort, we were able see translate this to a more technical statement about the matrix whose columns consist of the $\mathbf{v}_i$,

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

In particular, we established in Theorem 6.3.3 that:

$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are linearly independent precisely when each column of M^{red} has a leading 1

Let's take a moment to think about the structure of a matrix in reduced row echelon form where every column has a leading 1. (This is essentially a simplified Sudoku puzzle).

■ **Example 6.15** Let N be a 5×3 matrix in reduced row echelon form where every column has a leading 1.

$$N = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

The first row must be nonzero (since there is at least one nonzero row), so it must have a leading 1. Since the first column has a leading 1, and leading 1s move to the right, this tells us the first entry is a 1. The entries below must therefore be zeroes.

$$N = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

We need a leading 1 in the second column as well, so this must go in the second row (again because leading ones move the right as you descend). Above and below this leading 1 are all zeroes.

$$N = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

Finally, we observe that there is a leading one in the third column, therefore one more nonzero row, which must be the third one. Above and below it, we have all zeroes.

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c} I_3 \\ \hline \mathbf{0_{2\times 3}} \end{array}\right]$$

In particular, the first 3 rows form the identity matrix, with all zeroes below. ■

Notation 6.1. *The $m \times n$ zero matrix is denoted $\mathbf{0}_{m\times n}$. We will sometimes abuse notation and just denote it by $\mathbf{0}$.*

There was nothing special about the matrix being 5×3. The same argument could go through for any reduced matrix which as a leading 1 in any column. In particular, if N is a $m \times n$ matrix in reduced row echelon form, having a leading 1 in every column, then

$$N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \left[\frac{I_m}{\mathbf{0}_{m-n\times n}}\right]$$

To summarize, we have the following more precise version of Theorem 6.3.3.

Theorem 6.4.1 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$, and consider the matrix:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

The vectors are linearly independent precisely when

$$M^{red} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \left[\frac{I_r}{\mathbf{0}}\right]$$

Recall that our goal when we introduce linear independence was to answer Question 6.9: how many ways can a vector $\mathbf{w}$ be written in terms of the vectors $\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_r$. We first saw in Example 6.8 that these ideas were connected. Let's analyze example 6.12 to hash this out a bit further.

■ **Example 6.16** Suppose we're trying to mix some color $\mathbf{v}$ from Pigment X and Pigment Y. We'd like to reduce the augmented matrix:

$$\left[\mathbf{x} \quad \mathbf{y} \mid \mathbf{v}\right] = \left[\begin{array}{cc|c} 1 & 0 & w_1 \\ 0 & 1 & w_2 \\ 0 & 0 & w_3 \end{array}\right]$$

for some values w_1, w_2, w_3. Now, if $w_3 \neq 0$ then there is no way to mix this color. Otherwise, we can solve for w_1 ounces of Pigment X and w_2 ounces of Pigment Y. In particular, in this case, we can either mix $\mathbf{v}$ in 0 ways, or in one unique way. The *uniqueness* of the solution (if it existed), came from the fact that every column in $\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}$ had a leading 1 (so every variable was dependent). This is the same reason that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent!. ■

What happened in Example 6.16 happens in general, allowing us to answer Question **??** completely.

Theorem 6.4.2 Let $\mathbf{w}$ be in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$.

1. If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$, are linearly independent, then $\mathbf{w}$ can be written uniquely as a linear combination of them.
2. If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are linearly dependent, then $\mathbf{w}$ can be written as a linear combination of them in infinitely many ways.

Proof. Letting $M = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r\end{bmatrix}$, we are counting solutions to $M\mathbf{x} = \mathbf{w}$. To do this, we look at the reduction of the augmented matrix

$$\left[M \mid \mathbf{w}\right]^{red} = \left[M^{red} \mid \mathbf{w}'\right].$$

If the vectors are linearly independent, each entry of M^{red} has a leading 1, so every variable is dependent and the solution must be unique (cf. Exercise 5.14.2). If they are independent, there is a free variable, so there are infinitely many solution (cf. Exercise 5.14.3). ■

We can summarize what this means for our running examples.

Example 6.17 Since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent (Example 6.12), we know that any color mixed from Pigment X and Pigment Y can only be mixed in a unique way. This isn't to say that every color can be mixed from them (we have already seen that super green cannot), just that the ones that can can only be mixed in one way. This agrees with our computatoin in Example 6.16. ■

Example 6.18 Since $\mathbf{x}$ and $\mathbf{y}$ and $\mathbf{g}$ are linearly independent (Example 6.13), we know that any color mixed from Pigment X and Pigment Y and Greenifier can only be mixed in a unique way. We saw last week (Example 6.6) that any color can be mixed from these 3 pigments. We now know that any color can be mixed from them *in a unique way*. ■

Example 6.19 Since $\mathbf{x}$ and $\mathbf{y}$ and $\mathbf{g}$ and $\mathbf{m}$ are linearly *dependent* (Example 6.14), we know that any color that can be mixed from Pigment X and Pigment Y and Greenifier and Magenta can be done so in *infinitely many different ways*. We also saw last week (Example 6.7) that any color can be mixed from these four pigments. We know know that any color can be mixed from them in *infnitely many different ways*. ■

Before moving on, let's reflect on Slogan 6.1: for a collection $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ of n-dimesnional vectors, what does the reduction of the matrix:

$$M = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r\end{bmatrix},$$

tell us?

If every row has a leading 1: Then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\} = \mathbb{R}^n$. So every vector in $\mathbb{R}^n$ can be written in terms of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$.

If every column has a leading 1: Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ are linearly independent. Then any vector in $\mathbb{R}^n$ which can be written in terms of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ can be written that way uniquely.

Question 6.12 What can you say about when both every row and every column have a leading 1?

Question 6.13 Can more than n vectors in $\mathbb{R}^n$ be linearly independent?

Question 6.14 Can fewer than n vectors in $\mathbb{R}^n$ span all of $\mathbb{R}^n$?

6.4.1 Bases

Let's analyze Question 6.12 by returning to our running color example.

Example 6.20 Our painter has 3 pigments, Pigment X, Pigment Y, and Greenifier. These correspond to vectors:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix}.$$

If we row reduce:

$$M = \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{g} \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ 2 & 5 & 0 \\ 3 & 2 & -5 \end{bmatrix},$$

we obtain

$$M^{red} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Every row has a leading 1, so this tells us that the vectors span all of $\mathbb{R}^3$. Equivalently, we can make any color from the 3 pigments. Also, every column also has a leading 1, so the 3 vectors are linearly independent. Theorem 6.4.2 then tells us that any vector in $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{g}\}$ can be written as a *unique* linear combination of the 3 vectors. Putting these two observations together in plain english:

Every color can be mixed be mixed from Pigment X, Pigment Y, and Greenifier *in a unique way!*.

This means, that to the artist, they have a dictionary between a color, and a recipe to make it. The color represented by the vector $\mathbf{v}$ is:

$$\mathbf{v} = a\mathbf{x} + b\mathbf{y} + c\mathbf{g},$$

is completely captured by the coefficients a, b, c, meaning we may as well represent it by just those digits:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \text{The color mixed from } a \text{ oz. of Pigment X, b oz. of Pigment Y, and c oz. of Greenifier.}$$

This gives a *new coordinate system* to the color specturm, where instead of describing a color with a vector describing the amounts of red, green, and blue, we describe a vector with describing the amounts of each pigment. In fact, this coordinate system is more likely to be useful to the artist. Instead of describing super green as:

$$\begin{bmatrix} 3 \\ 90 \\ 3 \end{bmatrix},$$

the artist could describe it as:

$$\begin{pmatrix} 22.5 \\ 9 \\ 16.5 \end{pmatrix},$$

and extract directly from this vector a recipe to make the intended color. Both coordinate systems are described similarly: the coordinates of the vector describe how much of some basic vector to add. The RGB coordinates, describe how many units of red, green, and blue. But this was somewhat arbitrary, and the artist may find this second coordinate system more useful.

$$\begin{bmatrix} r \\ g \\ b \end{bmatrix} = r\hat{\mathbf{i}} + g\hat{\mathbf{j}} + b\hat{\mathbf{k}} \qquad \textit{but} \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{x} + b\mathbf{y} + c\mathbf{g}.$$

■

This works in general: If I have a linear independent set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_r$ that span all of $\mathbb{R}^n$, this tells me that any vector in $\mathbb{R}^n$ can be written as

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r,$$

in a unique way. In this sense, once the $\mathbf{v}_i$ are fixed, the vector is determined exactly by the coefficients:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix}.$$

This gives us a coordinate system for $\mathbb{R}^n$. In fact, this is how we defined coordinates in the first place: extracting the coefficients from an initial set of vectors which uniquely span $\mathbb{R}^n$.

$$\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + \cdots + b_n\hat{\mathbf{e}}_n.$$

This suggests that a spanning collection of linearly independent vectors might be important, so let's give it a name.

Definition 6.4.1 A collection $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ of vectors in $\mathbb{R}^n$ is called a *basis* for $\mathbb{R}^n$ if they are linearly indepndent, and span all of $\mathbb{R}^n$.

Our discussion immediately preceeding this definition suggests the following slogan for bases.

■ **Slogan 6.2** A basis gives rise to a coordinate system.

■ **Example 6.21** The standard basis $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ for $\mathbb{R}^3$ is a basis. Indeed:

$$\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

This is already in reduced row echelon form, and has a leading 1 in every column and row. Therefore they form a basis. Their associated coordinate system is a standard coordinate system. ■

Example 6.22 The standard basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_n$ for $\mathbb{R}^n$ is a basis. Indeed:

$$\begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \cdots & \hat{\mathbf{e}}_n \end{bmatrix} = I_n.$$

■

Example 6.23 In last week's groupwork (cf. Exercise 6.5) we considered the colors cyan, magenta, and yellow

$$\mathbf{c} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}.$$

Notice that this is a basis. Indeed, we the inverse of

$$\begin{bmatrix} \mathbf{c} & \mathbf{m} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 & 10 & 5 \\ 5 & 0 & 10 \\ 10 & 5 & 0 \end{bmatrix}$$

which implies that it reduces to I_3. We now have 3 different bases for the color spectrum.

The Computer's Basis: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. A general color in this coordinate system is represented by a vector:

$$\begin{aligned} \begin{bmatrix} r \\ g \\ b \end{bmatrix} &= r\hat{\mathbf{i}} + g\hat{\mathbf{j}} + b\hat{\mathbf{k}} \\ &= \text{r units of red, g units of green, b units of blue.} \end{aligned}$$

The Artist's Basis: $\mathbf{x}, \mathbf{y}, \mathbf{g}$. A general color in this coordinate system is represented by a vector:

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= a\mathbf{x} + b\mathbf{y} + c\mathbf{g} \\ &= a \text{ oz. of Pigment X, b oz. of Pigment Y, c oz. of Greenifier.} \end{aligned}$$

The Printer's Basis: $\mathbf{c}, \mathbf{m}, \mathbf{w}$. A general color in this coordinate system is represented by a vector:

$$\begin{aligned} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} &= p\mathbf{c} + q\mathbf{m} + r\mathbf{w} \\ &= p \text{ units of Cyan, } q \text{ units of Magenta, } r \text{ units of yellow.} \end{aligned}$$

Each basis gives a recipe to uniquely describe a color, and it isn't clear that one is *better* than another, it really just depends on context. A computer screen would likely prefer the first basis, since it colors its pixels according to amounts of red, green, and blue. On the other hand, the artist may prefer the second basis, since they will be mixing colors from the pigments they have on hand, and if they use that basis they could avoid doing any calculations before mixing a color. An industrial printer, on the other hand, generally prints in CMY, and so it would generally want a vector using the third basis. ■

In the previous example, we used that the matrix associated to CMY was invertible to prove that it was a basis. This is generally true.

Theorem 6.4.3 A collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$ precisely when the matrix

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}$$

is invertible.

Proof. We know it forms a basis precisely when M^{red} has a leading 1 in every column *and* in every row. This can only happen if M^{red} is the identity matrix, which means that M is invertible (by Theorem 5.5.4). ■

■ **Question 6.15** How many vectors must be in a basis for $\mathbb{R}^n$?

6.5 Exercises

Exercise 6.6 Consider the following four vectors in $\mathbb{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 5 \\ 3 \\ 6 \\ 3 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 0 \end{bmatrix}$$

1. Are the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ linearly independent? Why or why not?
2. Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ linearly independent? Why or why not?
3. Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_5$ linearly independent? Why or why not?
4. Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5$ and $\mathbf{v}_6$ linearly indepndent? Why or why not?
5. Is the collection of all 6 vectors above linearly independent? Why or why not?
6. Choose a subcollection of the 6 vectors above which form a basis for $\mathbb{R}^4$.

■

Exercise 6.7 Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ and $\mathbf{w}_5$ be 5 vectors in $\mathbb{R}^4$, and consider the 4×5 whith these vectors as columns.

$$M = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{w}_4 & \mathbf{w}_5 \end{bmatrix}.$$

1. What is the maximum number leading 1's that M^{red} can have?
2. Can 5 vectors in $\mathbb{R}^4$ ever be linearly independent? Why or why not?

■

Exercise 6.8 Do you agree or disagree with the following statements? Why or why not?
1. If $r > n$, it is possible for r vectors in $\mathbb{R}^n$ to be linearly independent.
2. If $r \leq n$, it is possible for r vectors in $\mathbb{R}^n$ to be linearly independent.
3. If $r \leq n$, then r vectors in $\mathbb{R}^n$ *must be* linearly independent.

■

Exercise 6.9 Do you agree or disagree with the following statements? Why or why not?

1. If $r < n$, it is possible for r vectors to span $\mathbb{R}^n$.
2. If $r \geq n$, it is possible for r vectors to span $\mathbb{R}^n$.
3. If $r \geq n$, then r vectors *must* span $\mathbb{R}^n$.

■

Exercise 6.10 Fix a pair of 2-dimensional column vectors (say $\mathbf{v}_1$ and $\mathbf{v}_2$), and explore the relationship between the following 3 questions:

- Are $\mathbf{v}_1$ and $\mathbf{v}_2$ linearly independent?
- Do $\mathbf{v}_1$ and $\mathbf{v}_2$ span $\mathbb{R}^2$?
- What is $\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$?

It appears that the answer to these questions are related. Let's explore this a bit futher.

1. Suppose that $\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \neq 0$. What is $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{red}$?
2. Suppose that $\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \neq 0$. Do $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$? Why or why not? (*Hint:* Use part (a) to look at where the leading 1s of the reduction of the matrix.)
3. Explain why 2 vectors in $\mathbb{R}^2$ are linearly independent exactly when they span $\mathbb{R}^2$.

■

Exercise 6.11 Let's generalize question 5, letting $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n$ be n vectors in $\mathbb{R}^n$. Let M be the matrix:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Are the answer's to the following questions the same? Why are why not?

- Are the vectors linearly independent?
- Do the vectors span $\mathbb{R}^n$?
- Do the vectors form a basis?
- Does M have an inverse?

■

Exercise 6.12 We have seen 3 bases for the color spectrum. **The Computer's Basis:** $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. A general color in this coordinate system is represented by a vector:

$$\begin{aligned} \begin{bmatrix} r \\ g \\ b \end{bmatrix} &= r\hat{\mathbf{i}} + g\hat{\mathbf{j}} + b\hat{\mathbf{k}} \\ &= \text{r units of red, g units of green, b units of blue.} \end{aligned}$$

The Artist's Basis: $\mathbf{x}, \mathbf{y}, \mathbf{g}$ corresponding to the vectors (in standard coordinates):

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} -5 \\ 0 \\ -5 \end{bmatrix}.$$

A general color in this coordinate system is represented by a vector:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{x} + b\mathbf{y} + c\mathbf{g}$$

$$= a \text{ oz. of Pigment X, b oz. of Pigment Y, c oz. of Greenifier.}$$

The Printer's Basis: $\mathbf{c}, \mathbf{m}, \mathbf{w}$ corresponding to the vecotrs (in standard coordinates)

$$\mathbf{c} = \begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}.$$

A general color in this coordinate system is represented by a vector:

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = p\mathbf{c} + q\mathbf{m} + r\mathbf{w}$$

$$= p \text{ units of Cyan, } q \text{ units of Magenta, } r \text{ units of yellow.}$$

Let's practice translating between them.

1. The color **fancy gold** corresons (in RGB) to the vectors

$$\mathbf{f} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix}.$$

   Write fancy gold in both the Artist's Coordinates and the Printer's Coordinates.

2. The painter mixes a color from

   10 oz. of Pigment X, 15 oz. of Pigment Y, 5 oz. of Greenifier.

   Write this color in all 3 coordinate systems.

3. A printer uses equal parts of Cyan, Magenta, and Yellow (10 units each). What RGB coordinates does a computer need to display this on the screen? How much of each pigment does the artist need mix this color?

Exercise 6.13 A map of a city is modelled by $\mathbb{R}^2$, and a reasonable set of coordinates for this city can be given by the by vectors representing movements in the cardinal directions:

$$\hat{\mathbf{i}} = 1 \text{ meter North} \qquad \hat{\mathbf{j}} = 1 \text{ meter East.}$$

The inhabitants of this city tend to describe how to walk around slightly differently. The basis for their directions are:

$$\mathbf{b}_1 = 1 \text{ block uptown} \qquad \mathbf{b}_2 = 1 \text{ block towards the east side.}$$

The blocks that move to the east side don't go exactly east. Instead, they go 30° north of due East. These blocks are 40 meters long. On the other hand, uptown is *due south* of downtown. The blocks between uptown and downttown are longer, 100 meters long..

1. Compute $\mathbf{b}_1$ and $\mathbf{b}_2$ in the coordinates induced by $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.
2. Do $\mathbf{b}_1$ and $\mathbf{b}_2$ form a basis for the city? Why or why not?
3. You want to go 200 meters southwest. Express the vector charting this path in both coordinate systems (the cardinal direction system, and the city block system).
4. You'd like to direct an inhabitant of this city to go 150 meters north, and 255 meters east. How would you give them directions in city block coordinates?
5. An inhabitant tells you to go 20 blocks downtown, and 10 blocks west. Write down the vector charting this path in the cardinal direction coordinate system.

■

6.6 November 19, 2024

6.6.1 Change of Basis Theorems–or–Translating Between Coordinate Systems

Having various different coordinate systems can be very useful, but it is also useful to have a dictionary between them. The computer screen, which displays colors in RGB, needs to translate to CMY before sending an image to a color printer. We started studying this in last week's groupwork (cf. 6.5), let's flesh this out more carefully.

■ **Example 6.24** Suppose we wanted to print, say, supergreen in CMY. This boils down to trying to solve the equation:

$$p\mathbf{c}+q\mathbf{m}+r\mathbf{w}=\mathbf{s}.$$

This is $M\mathbf{z}=\mathbf{s}$ where:

$$M=\begin{bmatrix}0 & 10 & 5\\ 5 & 0 & 10\\ 10 & 5 & 0\end{bmatrix} \quad ,\mathbf{z}=\begin{bmatrix}p\\ q\\ r\end{bmatrix}, \quad \textit{and} \quad \mathbf{s}=\begin{bmatrix}3\\ 90\\ 3\end{bmatrix}$$

Because $\mathbf{c},\mathbf{m},\mathbf{w}$ form a basis, we can invert M, and compute:

$$\mathbf{z}=M^{-1}\mathbf{s}.=\begin{bmatrix}-2/45 & 1/45 & 4/45\\ 4/45 & -2/45 & 1/45\\ 1/45 & 4/45 & -2/45\end{bmatrix}\begin{bmatrix}3\\ 90\\ 3\end{bmatrix}.$$

Therefore the CMY coordinates we send to the printer are:

$$\begin{Bmatrix}p\\ q\\ r\end{Bmatrix}=\begin{Bmatrix}32/15\\ -11/3\\ 119/15\end{Bmatrix}.$$

This process had really nothing to do with the color super green. To convert any color in RGB,

$$\mathbf{v}=\begin{bmatrix}r\\ g\\ b\end{bmatrix}$$

into CMY, we solve:

$$p\mathbf{c}+q\mathbf{m}+r\mathbf{w}=\mathbf{v} \qquad \textit{or} \qquad M\mathbf{z}=\mathbf{v},$$

and obtain the CMY coordinates of $\mathbf{z}=M^{-1}\mathbf{v}$. In summary:

To convert from RGB to CMY, multiply by M^{-1}.

■

This result is very general.

Theorem 6.6.1 Let $\mathbf{v}_1,\cdots,\mathbf{v}_n$ be a basis for $\mathbb{R}^n$, and let:

$$M=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}.$$

To convert from the standard coordinates to those induced by our basis, it suffices to multiply by

M^{-1}.

Proof. Fix a vector $\mathbf{w}$. To find the new coordinates (in terms of our new basis), we must solve for constants a_i such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{w}.$$

This factors as

$$M\mathbf{x} = \mathbf{w} \qquad \textit{where} \qquad \mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We conclude by multiplying both sides by M^{-1} on the left. ■

Last week we introduced the notion of a *basis* for $\mathbb{R}^n$. This was a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ which are linearly independent, and span all of $\mathbb{R}^n$. We observed that given a basis for $\mathbb{R}^n$, we could obtain *coordinates relative to that basis*. To do this, we observe that a random vector $\mathbf{w}$ in $\mathbb{R}^n$ could be written as:

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n,$$

(because the $\mathbf{v}_i$ span) *in a unique way* (because they are linearly independent). In particular, once we know the basis, to remember $\mathbf{w}$ we really only need to know the cooefficients $a_1, a_2, \cdots, a_n$. Therefore we can think of this as a system of coordinates, where we pluck out the a_1 as the first coordinate, a_2 as the second coordinate, and so on.

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Last time we started to explore the idea of *changing coordinates*. Let's suppose we know a vector $\mathbf{w}$ in terms of its standard coordinates:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + w_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = w_1\hat{\mathbf{e}}_1 + w_2\hat{\mathbf{e}}_2 + \cdots + w_n\hat{\mathbf{e}}_n.$$

We'd like to translate this to the nonstandard coordinates in terms of the new basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. For this, we need to solve:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{w},$$

for $x_1, x_2, \cdots, x_n$. This can be rewritten as:

$$M\mathbf{x} = \mathbf{w} \qquad \textit{where} \qquad M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \quad \textit{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Since the $\mathbf{v}_i$ form a basis, M is invertible, so we can solve for

$$\mathbf{x} = M^{-1}\mathbf{w}.$$

Therefore: *To convert from standard coordinates to the nonstandard coordinates defined by the* $\mathbf{v}_i$, *we can multiply by* M^{-1}. This was Theorem 6.6.1. What about going the other direction?

■ **Question 6.16** To change from the coordinates from the $\mathbf{v}_i$ back to standard coordinates do I:

- Multiply by M^{-1}?
- Multiply by M?
- Multiply by $Adj(M)$?
- Something else altogether?

Let's explore this more carefully with a concrete example.

■ **Example 6.25** Returning to the setup of Example 6.24, it is worth trying to understand the effect of multiplying by

$$M = \begin{bmatrix} \mathbf{c} & \mathbf{m} & \mathbf{w} \end{bmatrix}.$$

Notice that:

$$M\begin{bmatrix}1\\0\\0\end{bmatrix} = \mathbf{c} = \begin{bmatrix}0\\5\\10\end{bmatrix}.$$

So multiplying by $\hat{\mathbf{i}}$ gives the RGB coordinates for Cyan. Similarly:

$$M\begin{bmatrix}0\\1\\0\end{bmatrix} = \mathbf{m} = \begin{bmatrix}10\\0\\5\end{bmatrix} \qquad \textit{and} \qquad M\begin{bmatrix}0\\0\\1\end{bmatrix} = \mathbf{2} = \begin{bmatrix}5\\10\\0\end{bmatrix}.$$

In particular, if we multiply by some general vector:

$$\mathbf{v} = \begin{bmatrix}p\\q\\r\end{bmatrix}.$$

we get:

$$M\mathbf{v} = p\mathbf{c} + q\mathbf{m} + r\mathbf{w},$$

that is, we obtain the RGB coordinates for the color consisting of p units of Cyan, q units of Magenta, and r units of Yellow. To summarize, if a color

$$\begin{Bmatrix}p\\q\\r\end{Bmatrix},$$

represents a color in CMY coordinates, then the product:

$$M\begin{bmatrix}p\\q\\r\end{bmatrix},$$

outputs the entries for the same color in RGB coordinates. In summary:

To convert from CMY to RGB, multiply by M.

■

This result is also very general.

Theorem 6.6.2 Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$, and let:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

To convert from the the coordinates induced by our basis back to standard coordinates, it suffices to multiply by M.

Proof. This is immediate, as a vector in our basis whose nonstandard coordinates are:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

are by definition:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = M \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

■

6.6.2 Change of Basis for Linear Maps

■ **Example 6.26** Suppose we wanted to rotate the image on a screen 90° (call this function R). We saw (way back in Chapter 3) that to do so, we could multiply the coordinates of a pixel on the screen by:

$$M = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is important to emphasize that, for this to work, we have decided that our coordinates correspond to the *standard basis*, where $\hat{\mathbf{i}}$ represents moving 1 pixel to the right, and $\hat{\mathbf{j}}$ represents moving 1 pixel up. It turns out that not every computer uses these coordinates on its screen.[3] Suppose instead, that the computer stores a location on the screen according to the following basis:

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \textit{and} \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

■ **Question 6.17** Do $\mathbf{b}_1$ and $\mathbf{b}_2$ form a basis for the screen (that is, for $\mathbb{R}^2$). Why?

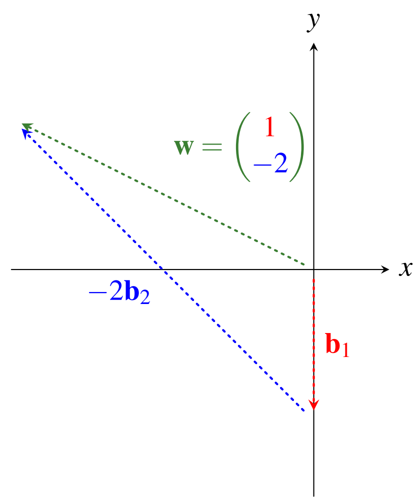
[3] In fact, this is very rarely the coordinate system a screen uses. Usually a screen will start counting from the top, so $\hat{\mathbf{i}}$ represents moving one pixel down!

Let's look at coordinates relative to this basis. A general vector:

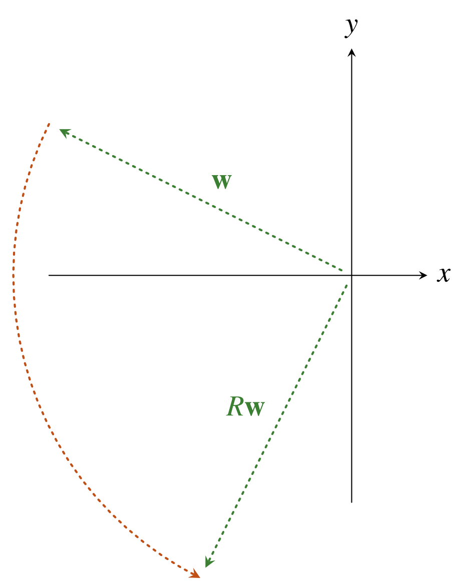
$$\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a\mathbf{b}_1 + b\mathbf{b}_2.$$

So for example, we can consider:

$$\mathbf{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$



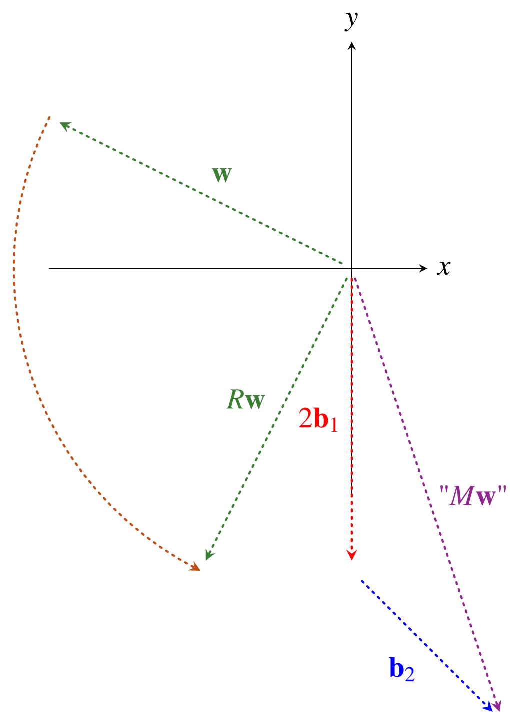
Let's rotate $\mathbf{w}$ 90° on the picture.



Now, to compute this in standard coordinates, we multiplied by M. What do we get if we multiply the non-standard coordinates of $\mathbf{w}$ by this matrix?

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{b}_1 + \mathbf{b}_2.$$

Let's add this vector to the map as well.



One might hope that "$M\mathbf{w}$" would recover $R\mathbf{w}$, but this seems not to be the case. In fact, this is perhaps an unsurprising pathology, since defining M relied in an important way on the coordinates we started with:

$$M = \begin{bmatrix} M\hat{\mathbf{i}} & M\hat{\mathbf{j}} \end{bmatrix}.$$

That being said, the computer which displays pixels on the screen would likely want to understand what the function *Rotate 90 Degrees* does in terms of its coordinate system. How do we figure this out.

To start, we should reflect that, if $\mathbf{w}$ were expressed in the standard coordinates, then we could rotate it 90° by multiplying by M. So perhaps we can start by figuring out what M is in terms of the standard coordinates. We can do this by hand, by computing:

$$\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{b}_1 - 2\mathbf{b}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

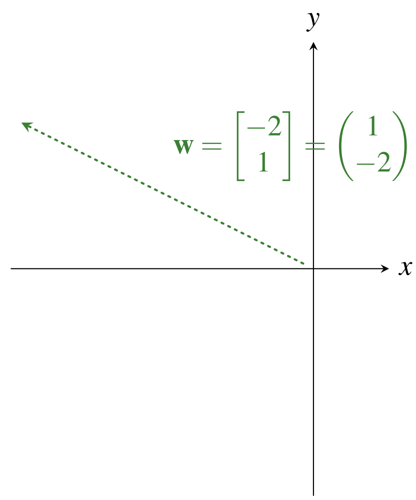
Note, according to Theorem 6.6.2, we could also do this by multiplying our nonstandard coordinates by the *Change of Basis Matrix* whose columns are the vectors of our basis:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Indeed:

$$B \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

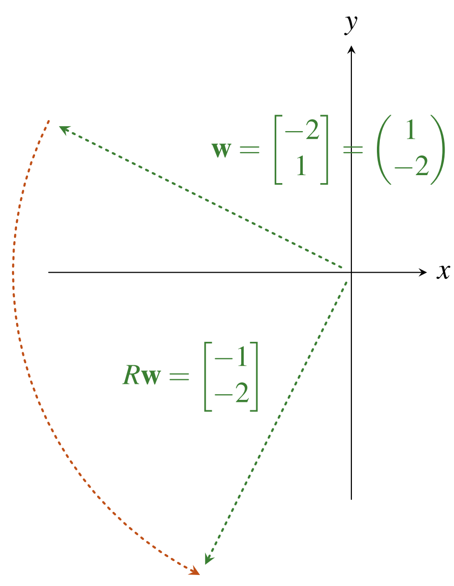
Either way, we get the same answer (thank goodness!). Let's add this to the drawing.



In standard coordinates, we can rotate by multiplying by M:

$$M\mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

If we add this to our drawing, it does look like it's in the right place!



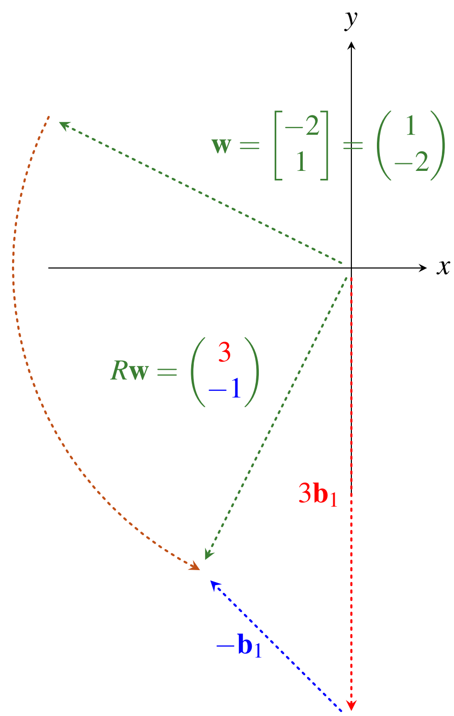
Of course, we are not quite done. The computer screen doesn't want $R\mathbf{w}$ in standard coordinates, it wants $R\mathbf{w}$ in its own (albeit bizzare) coordinate system. To do this, we can leverage Theorem 6.6.1, noting that to convert from standard to our nonstandard coordinates we can multiply by B^{-1}. We first have to compute B^{-1}.

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{-1} = \frac{1}{\det B} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

We can now compute:

$$B^{-1}\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{pmatrix}-3\\1\end{pmatrix} = 3\mathbf{b}_1 - \mathbf{b}_2.$$

And indeed, if we add $3\mathbf{b}_1$ and $-\mathbf{b}_2$ on our diagram, it looks like we get to the right place.



To summarize, we first multiplied the (nonstandard) coordinates by B, to change to standard coordinates. Then we multiplied by M to do the rotation in standard coordinates. And finally we multiplied by B^{-1} to convert back to our nonstandard coordinates.

$$R\mathbf{w} = (B^{-1}MB)\mathbf{w} = \underbrace{B^{-1}}_{\text{Convert back to nonstandard basis}} \underbrace{M}_{\text{Rotate using standard matrix}} \underbrace{B}_{\text{Convert to standard basis}} \underbrace{\begin{pmatrix}1\\-2\end{pmatrix}}_{\text{Our vector in nonstandard coordinates}}.$$

Of course, there was nothing important about rotating $\mathbf{w}$ in particular, what mattered was the process:

1. Convert to standard coordinates (Multiply by B).
2. Rotate via matrix multipcation (Multiply by M).
3. Convert back to nonstandard coordinates (Multiply by B^{-1}).

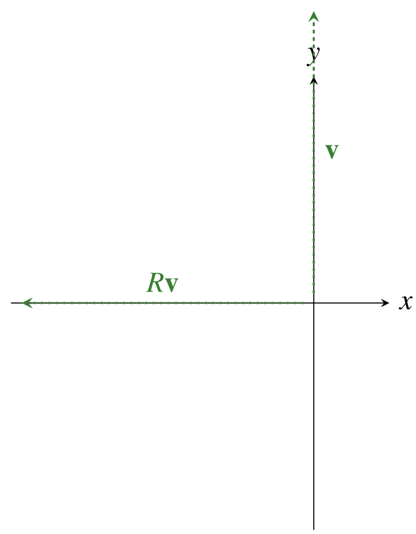
We have therefore deduced the following fact: to rotate a vector whose coordinates are given in terms of $\mathbf{b}_1$ and $\mathbf{b}_2$ $90°$, we can simply multiply by:

$$B^{-1}MB = \begin{bmatrix}-1 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & -1\end{bmatrix} = \begin{pmatrix}-1 & -2\\1 & 1\end{pmatrix}.$$

So, for example, suppose we wanted to rotate the vector:

$$\mathbf{v} = \begin{bmatrix}-2\\0\end{bmatrix}.$$

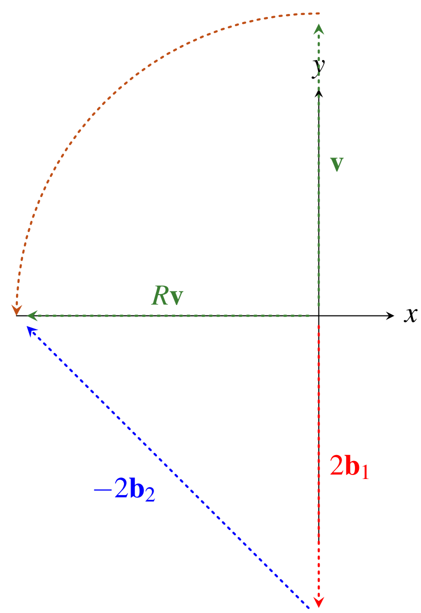
This correspoinds to $-2\mathbf{b}_1$ which points 2 units straight up, so we know that the output will have to point two units to the left.



Let's try to compute this in terms of matrix multiplication (with respect to these nonstandard coordinates).

$$R\mathbf{v} = B^{-1}MB\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}-1 & -2\\1 & 1\end{pmatrix}\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}.$$

So it looks like $R\mathbf{v} = 2\mathbf{b}_1 - 2\mathbf{b}_2$. Let's put this on the image.



This looks just right! ∎

What happened in Example 6.26 serves as a guide to answer the following question.

■ **Question 6.18** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function which is represented by the matrix M using the standard basis, and let $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$ be another basis. What matrix represents F with respect this new basis?

It turns out, we've already worked out the answer. Indeed, if we let:

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix},$$

then we have to follow the same three steps:

1. Convert to standard coordinates (Multiply by B).
2. Rotate via matrix multipcation (Multiply by M).
3. Convert back to nonstandard coordinates (Multiply by B^{-1}).

Then we can follow the same formula. If

$$\mathbf{w} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n,$$

is a vector given in the nonstandard $\mathbf{b}_i$ coordinates, we can compute $F(\mathbf{v})$ as

$$F(\mathbf{v}) = \underbrace{B^{-1}}_{\substack{\text{Convert back to} \\ \text{nonstandard basis}}} \underbrace{M}_{\substack{\text{Rotate using} \\ \text{standard matrix}}} \underbrace{B}_{\substack{\text{Convert to} \\ \text{standard basis}}} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\substack{\text{Our vector in} \\ \text{nonstandard coordinates}}} = (B^{-1}MB)\mathbf{v}.$$

In summary, we have proved the following *change of basis theorem*.

Theorem 6.6.3 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function which is represented by the matrix M using the standard basis, and let $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_n$ be another basis. Let

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix}.$$

Then the matrix $B^{-1}MB$ represents the function F in terms of the coordinates associated with the new basis.

■ **Question 6.19** The nonstandard coordiates for pixels on a computer screen that we introduced in Example 6.26 were somewhat contrived. More common is the basis:

$$\mathbf{v}_1 = \text{ 1 pixel to the right,}$$

$$\mathbf{v}_2 = \text{ 1 pixel down.}$$

In particular, a pixel on the screen is represented by a vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \text{ pixels to the right and } b \text{ pixels down.}$$

Find the standard matrix for a $90°$ rotation in terms of this basis.

Another common basis reverses this order:

$$\mathbf{w}_1 = \text{ 1 pixel down,}$$

$$\mathbf{w}_2 = \text{ 1 pixel to the right.}$$

Find the standard matrix for a 90° rotation in terms of this basis as well.

■ **Example 6.27** Placing a green filter on the screen can be thought of as a linear transformation of our color spectrum: $G : \mathbb{R}^3 \to \mathbb{R}^3$. In RGB coordinates, this function is:

$$G \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}.$$

A certain printer has a setting which applies a green filter as well. To compute this, the printer needs to input the color of each point in CMY coordinates, and compute the output of applying the green filter in CMY coordinates as well. This is a linear transformation, which can be computed via a single matrix multiplication. But which matrix do we multiply by? Well, if we let:

$$B = \begin{bmatrix} \mathbf{c} & \mathbf{m} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 & 10 & 5 \\ 5 & 0 & 10 \\ 10 & 5 & 0 \end{bmatrix},$$

then Theorem 6.6.3 tells us we need to multiply by:

$$B^{-1}MB,$$

where M is the standard matrix for the transformation in RGB coordinates. We can start by finding M, noticing that:

$$M = \begin{bmatrix} F(\hat{\mathbf{i}}) & F(\hat{\mathbf{j}}) & F(\hat{\mathbf{k}}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can also compute B^{-1} (using a determinants, row reduction, or a calculator).

$$B^{-1} = \begin{bmatrix} -2/45 & 1/45 & 4/45 \\ 4/45 & -2/45 & 1/45 \\ 1/45 & 4/45 & -2/45 \end{bmatrix}.$$

Therefore, the matrix we'd like is:

$$B^{-1}MB = \begin{bmatrix} -2/45 & 1/45 & 4/45 \\ 4/45 & -2/45 & 1/45 \\ 1/45 & 4/45 & -2/45 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 10 & 5 \\ 5 & 0 & 10 \\ 10 & 5 & 0 \end{bmatrix} = \begin{pmatrix} 1/9 & 0 & 2/9 \\ -2/9 & 0 & -4/9 \\ 4/9 & 0 & 8/9 \end{pmatrix}.$$

Let's interpret this. Once we apply a red filter, Cyan turns into:

$$F(\mathbf{c}) = \begin{pmatrix} 1/9 \\ -2/9 \\ 4/9 \end{pmatrix} = \frac{1}{9}\mathbf{c} - \frac{2}{9}\mathbf{m} + \frac{4}{9}\mathbf{w}.$$

■ **Question 6.20** What happens to Magenta under the green filter? Does this make sense?

■ **Question 6.21** The artist works in XYG coordinates with respect to amounts of Pigment X, Pigment Y, and Greenifier. Can you compute a matrix will calculate how they need to modify the colors they are mixing when applying a green filter?

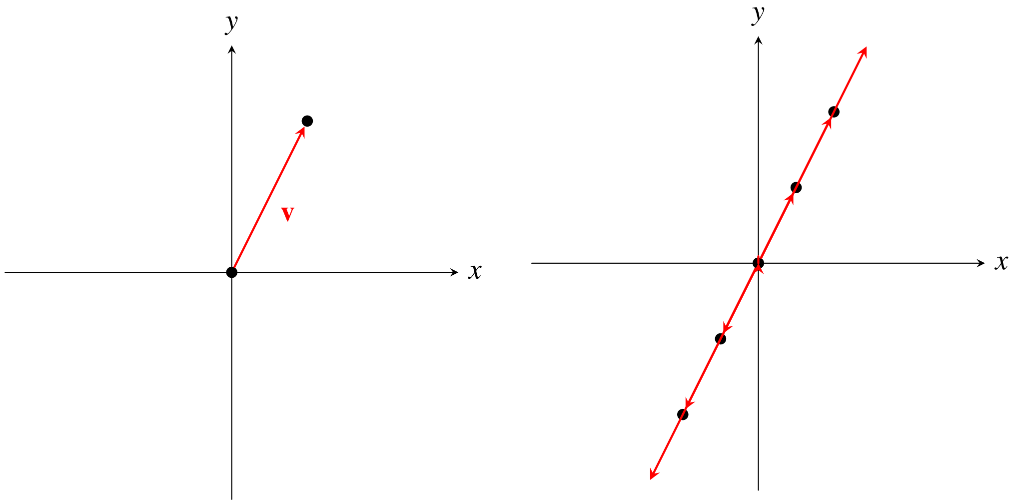
■

6.7 November 21, 2024

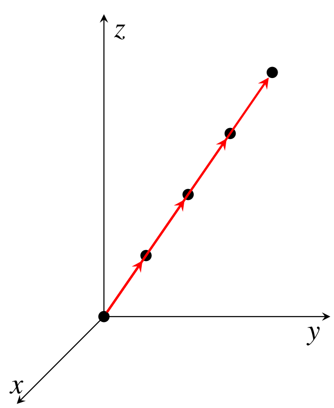
6.7.1 Dimension Theory

The notion of dimension has popped up in a couple of different contexts in this course so far. Most clearly, we have encountered the dimension of vectors, or more generally, the dimension of the ambient space $\mathbb{R}^n$. This, for obvious reasons, we have often described as n-dimensional. But this isn't the only place where the notion of dimension has popped up.

When we first encountered the notion of a span (back in Chapter 2), we started by considering the span of a single *nonzero* vector $\mathbf{v}$ in $\mathbb{R}^2$. This span was the collection of multiples of $\mathbf{v}$.



As we can see, this span traces out a line in $\mathbb{R}^2$. Similarly, we can consider the span of a single nonzero vector in $\mathbb{R}^3$.



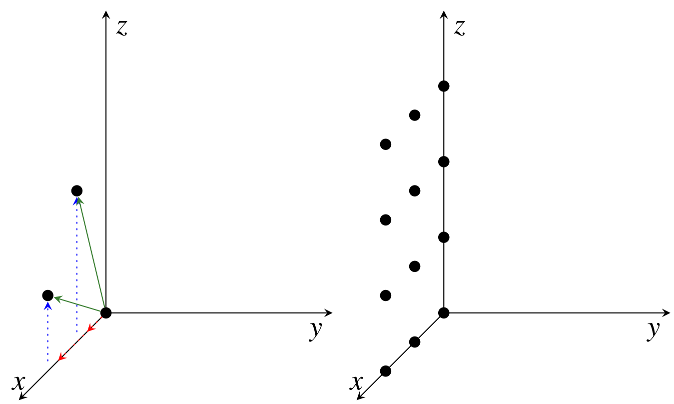
Again, this span sweeps out a line (now living in $\mathbb{R}^3$). In each case, it looked like we got something that could reasonably be called 1-dimensional, living inside our ambient space (either $\mathbb{R}^2$ or $\mathbb{R}^3$). This suggests the following question.

■ **Question 6.22** Is the span of a single nonzero vector in $\mathbb{R}^n$ always 1-dimensional.

Right now, it's unclear how to answer this question. This is because we don't have a precise definition of what the dimension of something living inside $\mathbb{R}^n$ should be. In this section we'd like to develop a definition of dimension. Whatever the dimension may be, we'd better hope that it will give an affirmative answer to Question **??**.

Let's now think about the dimension of the span of 2-vectors. Let's start in $\mathbb{R}^2$, letting $\mathbf{v}$ and $\mathbf{w}$ are a pair of *non-parallel* vectors in $\mathbb{R}^2$. Then the span all of $\mathbb{R}^2$, which is certainly 2-dimensional. What about 2 vectors in $\mathbb{R}^3$? Let's revisit an example from February (Example 2.17).

■ **Example 6.28** Consider the span of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$. As we start to plot out a few values:



As we can see, we plot out the entire xz-plane. ■

Indeed, the span of 2 *nonparallel* vectors in $\mathbb{R}^3$ will always trace out a plane in $\mathbb{R}^3$. This suggests the answer to the following question may also be affirmative.

■ **Question 6.23** Is the span of 2 (nonparallel) vectors in $\mathbb{R}^n$ always 2-dimensional?

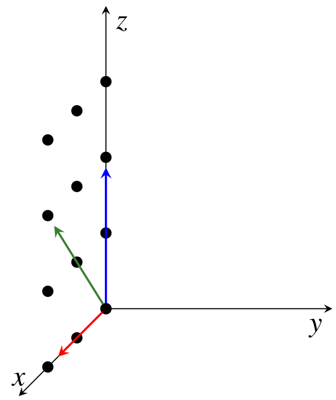
These examples may suggest investigating the following question as well:

■ **Question 6.24** Is the span of r vectors in $\mathbb{R}^n$ always r-dimensional?

The answer to this question should probably be no. For example, the span of 17 vectors in $\mathbb{R}^2$ can never be bigger than $\mathbb{R}^2$, so it's unlikely that it should be any more than 2-dimensional. So we will likely need to be a bit more careful. To tease out what goes wrong, let's look again at Example 6.28

■ **Example 6.29** We saw that $\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}\}$ in $\mathbb{R}^3$ appeared to be 2-dimensional. Let's compare this to $\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}, \mathbf{v}\}$ where:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \hat{\mathbf{i}} + \hat{\mathbf{k}}.$$



It doesn't look like we can really escape the xz plane, and indeed, considering a general vector in this span we have:

$$a\hat{\mathbf{i}} + b\hat{\mathbf{k}} + c\mathbf{v} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} a+c \\ 0 \\ b+c \end{bmatrix}.$$

As we can see, we are still contained in the xz plane no matter what we do. As a result:

$$\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}\} = xz\text{-plane} = \operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}, \mathbf{v}\}.$$

This gives us more evidence that the answer to Question 6.24 is no. If it were yes, then the xz-plane would have to be both 2-dimensional and 3-dimensional, which would suggest a pretty useless definition of dimension. Of course, we already believed that Question **??** had a negative ansewr, but this example gives us a bit more information. Of course we wouldn't enlarge $\operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{k}}\}$ by adding in $\mathbf{v}$, because $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$ *was already in the span!* ■

The previous example suggests that the dimension of r vectors will be less than r, *whenever one of the vectors is already in the span of the rest*. We already have a name for this: linear dependence. This suggests that the span of r *linearly dependent* vectors will likely be smaller than r. In fact, we already included this in our earlier analysis. When we claimed the span of a single vector was 1-dimensional, we required that vector to be *nonzero*, which for a single vector, is the same as linear independence.[4] When we claimed that the span of 2 vectors was 2-dimensional, we required them to be *nonparallel*, which for 2 vecytors is the same as linear indepndence.[5]. And indeed, when a collection of linear indepndent vectors span $\mathbb{R}^n$, there are exactly n of them. So perhaps the right definition of dimension will give an affirmative answer to the following question.

■ **Question 6.25** Is the span of r *linearly indepndent* vectors in $\mathbb{R}^n$ always r-dimensional?

6.7.2 Subspaces and Dimension

When trying to define dimension, we need to know what kind of *things* we are trying to define the dimension of. It would be too restrictive to just consider $\mathbb{R}^n$ for various n, since we already know that this is n-dimensional. We would like to also allow for certain things that live *inside of* $\mathbb{R}^n$, like spans. Any subset of $\mathbb{R}^n$ is perhaps a bit too broad to study using the tools of linear algebra alone. The Goldilocks medium[6] is the notion of a *subspace*.

[4] Can you prove this?
[5] Can you prove this too?
[6] just right!

Definition 6.7.1 Let V be a collection of vectors in $\mathbb{R}^n$ (possibly, and in fact usually, infinite). We call V a *subspace of* $\mathbb{R}^n$ if:

- Whenever $\mathbf{v}$ and $\mathbf{w}$ are in V, so is $\mathbf{v}+\mathbf{w}$.
- Whenever $\mathbf{v}$ is in V, so is $c\mathbf{v}$ for every constant c.

■ **Example 6.30** The xz-plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. Indeed, take any two vectors in the xz-plane:

$$\mathbf{v} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} x \\ 0 \\ y \end{bmatrix}.$$

Then their sum

$$\mathbf{v}+\mathbf{w} = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix}$$

is still in the xz-plane. Similarly, if we scale $\mathbf{v}$ by c, we get:

$$c\mathbf{v} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix},$$

which is still in the xz-plane. ■

Back on Homework 2 (cf. Exercise 2.22) we proved that the span of 2 vectors in $\mathbb{R}^3$ was a subspace (without yet having these words). Indeed, given $\mathbf{x}$ and $\mathbf{y}$ in the span$\{\mathbf{v},\mathbf{w}\}$, we can write:

$$\mathbf{x} = a_1\mathbf{v}+b_1\mathbf{w} \qquad \textit{and} \qquad \mathbf{y} = a_2\mathbf{v}+b_2\mathbf{w}.$$

Then:

$$\mathbf{x}+\mathbf{y} = (a_1+a_2)\mathbf{v}+(b_1+b_2)\mathbf{w},$$

$$c\mathbf{x} = ca_1\mathbf{v}+cb_1\mathbf{w}.$$

Both $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are able to be written in terms of $\mathbf{v}$ and $\mathbf{w}$, and are therefore still in the span. The argument for the span of an arbitrary number of vectors is identical (but with more $\cdots$). Therefore:

Proposition 6.7.1 The span of a collection of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

R The converse is in fact true as well. Any subspace of $\mathbb{R}^n$ is the span of all the vectors it contains (which could be an infinite number of vectors). In this sense, introducing the notion of a subspace doesn't really introduce anything new that wasn't already covered by spans. That said, it is a useful name to have on hand.

We saw in (WHERE?) that a basis for $\mathbb{R}^n$ always consists of exactly n vectors. As $\mathbb{R}^n$ is n-dimensional (being our model for dimension), this suggests the following approach to defining dimension:

The dimension should correspond to the size of a basis.

This gives us an avenue to extract a definition of dimension for spans and subspaces. We first need bases, which we can model off of the definition for $\mathbb{R}^n$.

Definition 6.7.2 Let V be a subspace of $\mathbb{R}^n$. A collection of vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$ in V is called a *basis* for V if

- $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r$ are linearly independent and
- $\text{span}\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r\} = V$.

As before, having a basis for V gives us a set of coordinates for V. The fact that the $\mathbf{b}_i$ span V tells us we can write any vector in V in terms of the $\mathbf{b}_i$, and linear independence tells us we can do so in a unique way. Therefore, just remembering the coefficinets of the $\mathbf{b}_i$ give us coordinates, and any vector in V can be identified by these coordinates:

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_r\mathbf{b}_r = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix}.$$

This coordinate system is starting to make V look a lot like $\mathbb{R}^r$, even though it had no coordinate system to begin with! It also reinforces the idea that the size of a basis measures dimension. Let's record that as our definition.

Definition 6.7.3 Let V be a subpspace of $\mathbb{R}^n$ and let $\mathbf{b}_1, \cdots, \mathbf{b}_r$ be a basis for V. Then we say that V is r-dimensional.

In particular, this gives us an affirmative answer to Question 6.25

Theorem 6.7.2 The span of r linearly indepenedent vectors in $\mathbb{R}^n$ in r-dimensional.

Proof. The definitions are set up to make this almost tautological. If $\mathbf{b}_1, \cdots, \mathbf{b}_r$ are linearly independent, and V is their span, they they form a basis for V (they span by definition, and they were linearly indepndent by assumption). Therefore the dimension of V is exactly r. ■

Therefore, given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, its forms a single element basis for its span $V = \text{span}\{\mathbf{v}\}$, which makes it 1-dimensional, giving an affirmative answer to Question 6.22. Furthermore, we can extract a coordinate system for V by just taking the coefficient of $\mathbf{v}$. That is, every element of V is:

$$c\mathbf{v} = \begin{pmatrix} c \end{pmatrix}.$$

This gives us a nice correspondence between V and $\mathbb{R}$, associating to a real number c the vector $\begin{pmatrix} c \end{pmatrix} = c\mathbf{v}$ in V.

Similarly, given any two *nonparallel* vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, we see that $W = \text{span}\{\mathbf{v}, \mathbf{w}\}$ is 2-dimensional, giving an affirmative answer to Question 6.23. And indeed, we can extract a coordinate system for W from its basis $\mathbf{v}$ and $\mathbf{w}$. Indeed, any vector $\mathbf{u}$ in this span can be written uniquely as:

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

This makes W, which *a priori* was some span living in $\mathbb{R}^n$, look a lot like $\mathbb{R}^2$, and gives us a precise way to talk about planes living in higher dimensional spaces like $\mathbb{R}^{17}$.

■ **Example 6.31 — Maybe Skip?.** Consider the span of $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{w} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ in $\mathbb{R}^3$. If we consider $W = \text{span}\{\mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$, we trace out a plane, and the linear indepndence of $\mathbf{v}$ and $\mathbf{w}$ tell us that this is indeed 2-dimensional. Better yet, the fact that $\mathbf{v}$ and $\mathbf{w}$ form a basis tells us that this embedded plane in $\mathbb{R}^3$ can be studied like $\mathbb{R}^2$ in a linear algebraic standpoint. That is, every vector in W can be given 2D-coordinates in terms of $\mathbf{v}$ and $\mathbf{w}$. Take, for example:

$$\begin{bmatrix}1\\2\\1\end{bmatrix} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} = \mathbf{v} + \mathbf{w} = \begin{pmatrix}1\\1\end{pmatrix}.$$

This dictionary can be made even more precise by thinking about the map $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by $F(\hat{\mathbf{i}}) = \mathbf{v}$ and $F(\hat{\mathbf{j}}) = \mathbf{w}$. This linear function embeds $\mathbb{R}^2$ into $\mathbb{R}^3$ exactly as the span of $\mathbf{v}$ and $\mathbf{w}$ with these exact coordinates! ■

6.7.3 A technique for computing the dimension of a span

We now know how to define the the definition of a span, and Theorem 6.7.2 even gives us a way to compute it in some cases. Nevertheless, suppose we are given a collection of vectors which are not linearly independent—for example, the $\hat{\mathbf{i}}, \hat{\mathbf{k}}$, and $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ from Example 6.29 whose span V is the xz-plane. In this example, we can see by inspection that $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ form a basis for this span, and so it is 2-dimensional. But how can we do this in general?

To extract a technique, it might be useful to consider this example more closely. Let's take $\hat{\mathbf{i}}, \hat{\mathbf{k}}$ and $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$ and put them a matrix. Our goal is to find a linearly independent subcollection of $\{\hat{\mathbf{i}}, \hat{\mathbf{k}}, \mathbf{v}\}$ whose span is the same. Let's start by using the same technique we used before to determine if a group of vectors is linearly independent in the first place: throw them into the columns of a matrix and row reduce.

$$M = \begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{k}} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & 0 & 0\\0 & 1 & 1\end{bmatrix}.$$

If we reduce this matrix, we obtain:

$$M^{red} = \begin{bmatrix}{\color{red}1} & 0 & 1\\0 & {\color{red}1} & 1\\0 & 0 & 0\end{bmatrix}.$$

Notice that there are leading 1's in the first and second columns (highlighted in red). This tells us that they weren't linearly independent in the first place. But in fact, it tells us more than this. To elucidate why, let's suggestively grey out the third column (which didn't have a leading 1 in the end).

$$\begin{bmatrix}1 & 0 & {\color{gray}1}\\0 & 0 & {\color{gray}0}\\0 & 1 & {\color{gray}1}\end{bmatrix} \longrightarrow RREF \longrightarrow \begin{bmatrix}1 & 0 & {\color{gray}1}\\0 & 1 & {\color{gray}1}\\0 & 0 & {\color{gray}0}\end{bmatrix}.$$

The various columns don't interfere with eachother as we row reduce, so this tells us that if we had row reduced just the first two columns (which consisted of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$), we would have ended up with exactly the non-grayed out part of the reduced matrix. Since each column has a leading 1, this tells us that the vectors we started with in these columns ($\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$) are linearly independent . Furthermore, throwing in the gray column breaks independence, so the vector in the third column must be in the span of the first 2. In particular, it looks like:

- The number of leading 1's in the matrix is the dimension of the span, and
- The vectors corresponding to columns with leading 1's form a basis of the span.

Let's try this with another more complicated example.

■ **Example 6.32** Consider the following vectors in $\mathbb{R}^4$.[7]

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ -5 \\ -2 \\ -6 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 10 \\ 18 \\ 5 \\ 15 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 6 \\ 9 \\ 3 \\ 9 \end{bmatrix}.$$

Let's compute the dimension and a basis for $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. We know it can't be 5 dimensional (since it is living in $\mathbb{R}^4$), so we want to find some sub-collection of the $\mathbf{v}_i$ which form a basis for V. To do this, we want to put the matrix:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 10 & 0 & 6 \\ 3 & -5 & 18 & 0 & 9 \\ 1 & -2 & 5 & 1 & 3 \\ 3 & -6 & 15 & 1 & 9 \end{bmatrix}.$$

When we row reduce this we obtain:

$$M^{red} = \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again, selectively graying out the columns that don't end up with leading 1's, we have:

$$\begin{bmatrix} 2 & -4 & 10 & 0 & 6 \\ 3 & -5 & 18 & 0 & 9 \\ 1 & -2 & 5 & 1 & 3 \\ 3 & -6 & 15 & 1 & 9 \end{bmatrix} \longrightarrow RREF \longrightarrow \begin{bmatrix} 1 & 0 & 11 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In particular, zoning in on columns 1,2, and 4, we see that

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -5 & 0 \\ 1 & -2 & 1 \\ 3 & -6 & 1 \end{bmatrix} \longrightarrow RREF \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Every column has a leading 1, so $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_4$ are linearly independent. Furthermore, if we throw back in one more vector (say, $\mathbf{v}_3$), we obtain:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 10 & 0 \\ 3 & -5 & 18 & 0 \\ 1 & -2 & 5 & 1 \\ 3 & -6 & 15 & 1 \end{bmatrix} \longrightarrow RREF \longrightarrow \begin{bmatrix} 1 & 0 & 11 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

[7] This example is adapted from Anton and Rorres

In particular, we are no longer linearly independent, so we must have had $\mathbf{v}_3$ in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ to begin with. A similar argument tells us that $\mathbf{v}_5$ is in this span, so we can conclude that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_4$ must have already spanned V to begin with! Since they are linearly independent, they in fact form a *basis* for V, so that V is 3-dimensional! ■

This gives us a technique, not only to compute the dimension of a span, but to in fact extract a a basis!

Theorem 6.7.3 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r$ be a collection of vectors in $\mathbb{R}^n$, and let V be their span. Build a matrix whose columns are the $\mathbf{v}_i$:

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}.$$

Then the dimenion of V is equal to the number of leading ones in M^{red}. Furthermore, if we select the $\mathbf{v}_i$ associated to the columns with leading 1's, we obtain a basis for V.

6.8 Tuesday, December 3, 2024

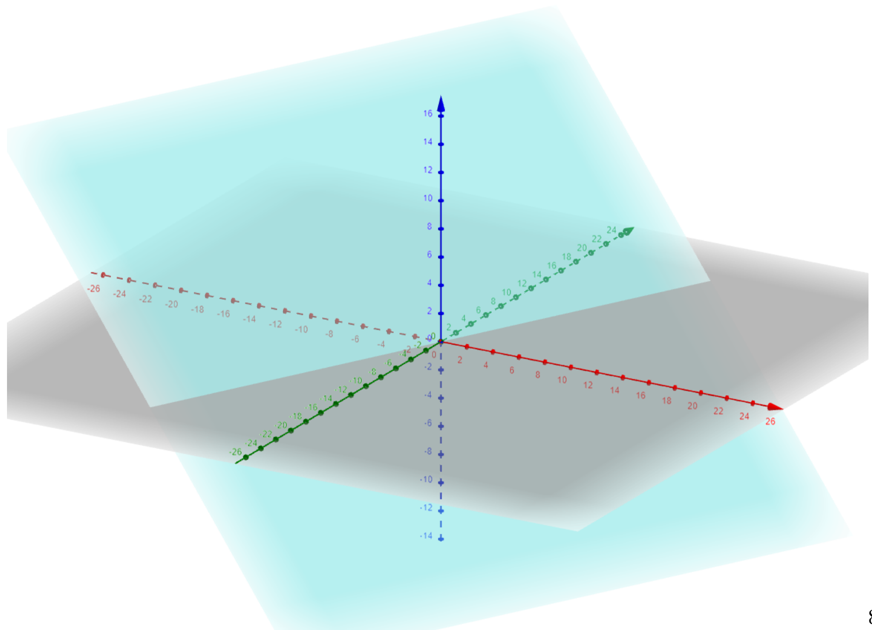
6.8.1 Null Spaces and the Dimension of the Solutions to a Linear System

The span of a collection of vectors gives us a subspace of $\mathbb{R}^n$ whose dimension we may want to measure, but this is not the only way to obtain such a thing. We have also seen interesting subcollections of vectors in $\mathbb{R}^n$ but looking at solutions to linear systems.

■ **Example 6.33** Consider the vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

in $\mathbb{R}^3$ which satisfy $2x - y + 3z = 0$. This is in fact the equation of a plane:



[8]

This is certainly 2-dimensional. It is also the solution space to a linear system. Namely,

$$M\mathbf{x} = \mathbf{0},$$

where M is the 1×3 matrix:

$$M = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

If we solve this linear system, we see that there are 2 free variables (y and z), and that we'd obtain:

$$x = \frac{1}{2}s - \frac{3}{2}t,$$

$$y = s,$$

$$z = t.$$

Each parameter feels like a degree of freedom, or perhaps even a coordinate. In fact, we seem to get a coordinate system in for this set of solution in terms of s and s:

$$\begin{pmatrix} s \\ t \end{pmatrix}.$$

To make this precise, we'd like to translate these coordinates into a coordinate space for some basis of this solution space. To do this, let's write a general element of this solution space:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} .5s - 1.5t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} .5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1.5 \\ 0 \\ 1 \end{bmatrix}.$$

[8] Plotted in GeoGebra

This tells us that any member of this plane can be written as a linear combination of

$$\mathbf{v} = \begin{bmatrix} .5 \\ 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{w} = \begin{bmatrix} -1.5 \\ 0 \\ 1 \end{bmatrix},$$

and the coordinates are precisely that s and t! So we have obtained a basis for the plane, $\mathbf{v}$ and $\mathbf{w}$, and it is (as we expected), 2-dimensional in a linear algebraic sense (which is good, because we'd like it to match the calculus sense of the word!). ■

Example 6.33 gave us a technique we can try to replicate find the basis for the space of solutions spaces to linear systems, and therefore to learn its dimension. To really find a basis though, the solution space must be a subspace. So (for now), we can only do this to certain special linear systems, where the right side of the equations are all 0. We call this a *null space*.

Definition 6.8.1 Let M be a matrix. The *null space* of M is the set of solutions to the linear system $M\mathbf{x} = \mathbf{0}$.

Proposition 6.8.1 Let M be a $m \times n$ matrix. Then the null space of M is a subspace of $\mathbb{R}^n$.

Proof. If $\mathbf{x}$ and $\mathbf{y}$ are in the null space of M, then:

$$M(\mathbf{x}+\mathbf{y}) = M\mathbf{x} + M\mathbf{y} = \mathbf{0} + \mathbf{0},$$

so that $\mathbf{x}+\mathbf{y}$ is also in the null space. Similarly, for any constant c,

$$M(c\mathbf{x}) = cM\mathbf{x} = c \cdot \mathbf{0} = \mathbf{0}.$$

■

■ **Example 6.34** Accordint to the previous proposition, the set of vectors:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix},$$

which satisfy the equations

$$x + 3y - 11z - 2w = 0,$$

$$2x + 7y - 3z + w = 0,$$

form a subspace of $\mathbb{R}^4$. We can find its dimension and a basis by solving the linear system. To do this, we can row reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 0 \\ 2 & 7 & -3 & 1 & 0 \end{array}\right] \longrightarrow RREF \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & 0 \\ 0 & 1 & 19 & 5 & 0 \end{array}\right].$$

This un-augments to:

$$x - 68z - 17w = 0,$$

$$y + 19z + 5w = 0,$$

so that we have 2 free variables (z and w). Our general solution is therefore:

$$x = 68s + 17t,$$

$$y = -19s - 5t,$$

$$z = s,$$

$$w = t.$$

As in Example 6.33, it feels like each free variable gives us a dimension of freedom. 2 free variables looks like 2 degrees of freedom, which in turn suggests a 2-dimensional null space. And indeed, a general vector looks like:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 68s + 17t \\ -19s - 5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 68 \\ -19 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 17 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

And indeed, the s and t do give us a coordinate system and we obtain a basis for the solutions to this linear system:

$$\left\{ \begin{bmatrix} 68 \\ -19 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 17 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

■

Defining dimension correctly, and finding bases in this way, allows us to prove a theorem we probably already assumed was true.

Theorem 6.8.2 The dimension of the null space $M\mathbf{x} = 0$ is the number of free variables in the linear system.

The solutions to a linear system $M\mathbf{x} = \mathbf{v}$ are not a subspace, and therefore we cannot compute their dimension by counting a basis. That being said, the solutions to $M\mathbf{x} = \mathbf{v}$ (if there are any) can be obtained as a *translation* of the solutions to $M\mathbf{x} = \mathbf{0}$ by a single vector in the solution space of $M\mathbf{x} = \mathbf{v}$. Therefore, the dimension should be exactly the same. Let's see how this works in an example.

■ **Example 6.35** Let's consider a system closely related to the one from Example 6.34.

$$x + 3y - 11z - 2w = 1,$$

$$2x + 7y - 3z + w = 5.$$

We can row reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 1 \\ 2 & 7 & -3 & 1 & 5 \end{array}\right] \longrightarrow RREF \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & -8 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right],$$

which un-augments to

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ \ 5w &= \ \ 3 \end{aligned}$$

We have the same two free variables (z and w) giving us the following general solutionThis system of equations is just begging for us to solve for x and y in terms of z and w. IN fact, the following system of equations is identical, albeit rearranged.

$$x = 68s + 17t - 8,$$

$$y = -19s - 5t + 3,$$

$$z = s,$$

$$w = t.$$

As in Example 6.34, the two free variables suggest that we have 2 degrees of freedom. Unfortunately, the solution space to this linear system is not a subspace. Consider, for example

$$\mathbf{v} = \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

If we scale by, say, 0, do we still have a solution? Therefore, we cannot expect to be able to find a basis for this linear system. What happens if we write down a general element as in Example 6.34?

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 68s + 17t \\ -19s - 5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 68 \\ -19 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 17 \\ -5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

As we can see, the solutions to this linear system, are exactly the subspace from Example 6.34, simply translated by $\mathbf{v}$ (that is, shifted -8 units in the x-direction, and 3 units in the y-direction, with 0 units in the z or w directions). Shifting a space shouldn't change its dimension, so this solution space should be 2 dimensional as well. ■

The previous example illustrates a general principle.

■ **Slogan 6.3** The solutions to $M\mathbf{x} = \mathbf{v}$ (if there are any) are merely a translation of the solutions to $M\mathbf{x} = \mathbf{0}$.

Indeed, let $\mathbf{w}$ is any solution to the first system. Then for any $\mathbf{x}$ that solves the second system, we can obtain one that solves the first by adding $\mathbf{w}$:

$$M(\mathbf{x} + \mathbf{w}) = M\mathbf{x} + M\mathbf{w} = M\mathbf{w} = \mathbf{v}.$$

Conversely, given $\mathbf{y}$ that solves the first sytem, we get one that sovles the second by subtracting $\mathbf{w}$:

$$M(\mathbf{y} - \mathbf{w}) = M\mathbf{y} - M\mathbf{w} = \mathbf{v} - \mathbf{v} = 0.$$

It is reasonable then to say that the dimension of the set of solutions to $M\mathbf{x} = \mathbf{v}$ is the number of free (since it is just a translation to the null space of soltutions to $M\mathbf{x} = \mathbf{0}$). We can therefore extend Theorem 6.8.2:

Corollary 6.8.3 If there are any solutions to $M\mathbf{x} = \mathbf{v}$, then the dimension of the set of solutions is equal to the number of free variables.

6.8.2 The Rank-Nullity Theorem

Let M be an $m \times n$ matrix, and let N be its null space (that is, the solutions to $M\mathbf{x} = \mathbf{0}$). Then the dimension of N is the number of free variables, or equivalently, the number of columns of M^{red} which **do not have a leading 1**. Let V be the span of the columns of M. That is, write M as a concatonation of columns of vectors in $\mathbb{R}^m$

$$M = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix},$$

and let $V = \mathrm{span}\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\}$ (this is sometimes called the *column space* of n). Then the dimension of V is the number of columns of M^{red} which **do have a leading 1**. In particular:

$$\dim N + \dim V = \#\text{ columns of } M = n.$$

This is often called the *Rank-Nullity Theorem*. It is work pointing out that if we think about $M : \mathbb{R}^n \to \mathbb{R}^m$, then the column space of M is:

$$\mathrm{span}\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n\} = \mathrm{span}\{M\hat{\mathbf{e}}_1, M\hat{\mathbf{e}}_2, \cdots, M\hat{\mathbf{e}}_n\}.$$

This is precisely the image (or range) of the function M. Indeed, since everything in $\mathbb{R}^n$ can be written in terms of the $\hat{\mathbf{e}}_i$ we have:

$$M\mathbf{v} = M(a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + \cdots + a_n\hat{\mathbf{e}}_n) = a_1 M\hat{\mathbf{e}}_1 + a_2 M\hat{\mathbf{e}}_2 + \cdots + a_n M\hat{\mathbf{e}}_n = a_1\mathbf{c}_1 + a_2\mathbf{c}_2 + \cdots + a_n\mathbf{c}_n.$$

Therefore, we can write the rank nullity theorem as follows.

Theorem 6.8.4 Let $M : \mathbb{R}^n \to \mathbb{R}^m$. Them:

$$\dim(\mathrm{Null}(M)) + \dim(\mathrm{Range}(M)) = n.$$

R This result is generalized in group theory as the *First Isomorphism Theorem*.

The rank nullity theorem is a very powerful numerical result. It can be used, for example, to guarantee that certain linear systems have many solutions.

■ **Example 6.36** Let $M : \mathbb{R}^{1000} \to \mathbb{R}$ be a linear map. Then the collection of vectors in $\mathbb{R}^{1000}$ which map to zero is at least 999 dimensional. Indeed, we area looking for the dimension of the null space N. Call the columns space V. Then:

$$\dim N + \dim V = 1000.$$

Since V lives in $\mathbb{R}$, so cannot be more than 1-dimensional, so N needs to make up the remaining 999 dimensions. ■

■ **Question 6.26** What is the minimum dimension of the solution set to the following linear system?

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0,$$

$$b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 = 0,$$

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 = 0.$$

6.9 Exercises

Exercise 6.14 Consider the following two vectors in $\mathbb{R}^2$:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

1. Explain why $\mathbf{b}_1, \mathbf{b}_2$ form a basis for $\mathbb{R}^2$.
2. Draw the the vectors $\mathbf{v} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ and $\mathbf{w} = 10\hat{\mathbf{j}}$ in in $\mathbb{R}^2$, and then express them as sums of $\mathbf{b}_1$ and $\mathbf{b}_2$ in your drawing.
3. Write $\mathbf{v}$ and $\mathbf{w}$ in the coordinates giving by our new basis $\mathbf{b}_1$ and $\mathbf{b}_2$.
4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the *flip* map which reflects over the x-axis. Find the standard matrix for F in terms of the new basis $\mathbf{b}_1$ and $\mathbf{b}_2$.
5. Compute the coordinates $F(\mathbf{v})$ and $F(\mathbf{w})$ in terms of our new basis. Sketch where they en up in on $\mathbb{R}^2$, and confirm that your output makes sense.

■

Exercise 6.15 Placing a blue filter on the screen can be thought of as a linear transformation of our color spectrum: $B : \mathbb{R}^3 \to \mathbb{R}^3$. In RGB coordinates, this function is:

$$B \begin{bmatrix} r \\ g \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}.$$

1. Find the stardard matrix (with respect to RGB coordinates) for the blue filter function B.
2. A certain printer has a setting which applies a blue filter as well. To compute this, the printer needs to input the color of each point in CMY (cf. Homework 11 Problem 7) coordinates, and compute the output of applying the blue filter in CMY coordinates. This is a linear transformation, which can be computed via a single matrix multiplication. Which matrix do we multiply by?
3. A printer is given equal parts of Cyan, Magenta, and Yellow (10 units each). If we ask the printer to print this color after a blue shift, what color will it print? (Express your answer in CMY coordinates).
4. Our painter wants to compute how to apply a blue filter to the colors they mix (from Pigment X, Pigment Y, and Greenifier cf. HW11 Problem 7). To compute this, they start with the color XYG coordiantes (the artist's basis from Homework 11 Problem 7) coordinates, and compute the output of applying the blue filter in XYG coordinates as well. This is a linear transformation, which can be computed via a single matrix multiplication. Which matrix does the artist multiply by?
5. The painter wants to apply a blue filter to the color they mix from:

   10 oz. of Pigment X, 15 oz. of Pigment Y, 5 oz. of Greenifier.

   How much of each color should the end up using?

■

Exercise 6.16 Consider the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

1. Are the $\mathbf{v}_i$ linearly independent?
2. Do the $\mathbf{v}_i$ span $\mathbb{R}^4$?
3. What is the dimension of span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

■

Exercise 6.17 Let P_3 be the collection of polynomials of degree ≤ 3. That is, all polynomials of the form:

$$a + bx + cx^2 + dx^3,$$

for some constants a, b, c, d.

1. There are 4 special elements of P_3, $1, x, x^2$ and x^3. Can everything in P_3 be written in terms of these four elements? Why or why not?
2. Do you think $1, x, x^2$ and x^3 are linearly independent? Why or why not? (Hint, what happens if a nontrivial linear combination of them was 0).
3. Use your answers from problems 1 and 2 to determine the dimension of P_3.
4. It seems like $1, x, x^2$ and x^3 form a basis for P_3. Recall that a basis gives rise to coordinates. Write:

   $$2 + x + 3x^2 - x^3,$$

   as a column vector

   $$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

   in terms of the coordinates induced by this basis.
5. Write the column vector:

   $$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix},$$

   as a polynomial.
6. We can think of the derivative as a functio $D : P_3 \to P_3$. It is also linear (for example $(f+g)' = f' + g'$). Therefore, our philosophy tells us that it should be representable by a

matrix. Find:

$$D\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad D\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad D\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad D\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Use these values to find a matrix M for D.

7. Compute the derivative of $2+x+3x^2-x^3$ the usual way. Compare this to what happens when you multiply the column vector you found in problem 4 by M (from problem 6), and convert the output to a polynomial.
8. Multiply the column vector

$$\begin{pmatrix}1\\0\\2\\4\end{pmatrix}$$

by M. Compare this to what happens when you take the derivative of the polynomial you found in problem 5, and convert that derivative to a column vector.

******Wait to attempt problems 5 and 6 until after class on Tuesday, May 2nd.******

Exercise 6.18 Consider the following 5 vectors in $\mathbb{R}^5$.

$$\mathbf{w}_1=\begin{bmatrix}3\\1\\4\\1\\5\end{bmatrix}\quad \mathbf{w}_2=\begin{bmatrix}2\\7\\1\\8\\7\end{bmatrix}\quad \mathbf{w}_3=\begin{bmatrix}-1\\6\\-3\\7\\2\end{bmatrix}\quad \mathbf{w}_4=\begin{bmatrix}1\\6\\1\\8\\0\end{bmatrix}\quad \mathbf{w}_5=\begin{bmatrix}-8\\-55\\-5\\-71\\-9\end{bmatrix}.$$

Let $W=\operatorname{span}\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_5\}$. Compute the dimension of W and give a basis for W.

Exercise 6.19 Consider the following linear system.

$$\begin{aligned}
3x_1+2x_2-x_3+x_4-8x_5&=0\\
x_1+7x_2+6x_3+6x_4-55x_5&=0\\
4x_1+x_2-3x_3+x_4-5x_5&=0\\
x_1+8x_2+7x_3+8x_4-71x_5&=0\\
5x_1+7x_2+2x_3-9x_5&=0
\end{aligned}$$

Let N be the set of solutions to this linear system. Compute the dimension of N, and give a basis for N.

7. Eigenvectors and Eigenvalues

7.1 Day 25

One of the most universally applicable topics in Linear Algebra is the theory of Eigenvalues and Eigenvectors, which has found itself extremely important in computer grapics, data science, network theory, and even search engine design (indeed, the google search engine essentially workes by solving finding eigenvectors in a very large dimensional dataset). On the last day here we really only have time to scratch the surface of the theory, but we will try to give a brief sketch of what kind of things are going on.

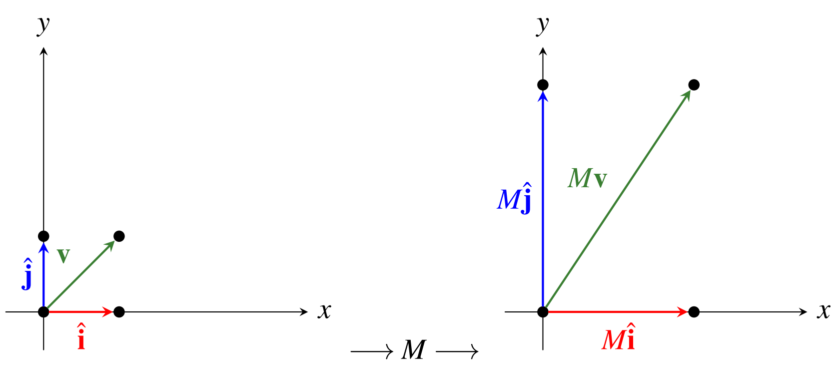
7.1.1 What is an Eigenvector?

■ **Example 7.1** Consider the matrix $M : \mathbb{R}^2 \to \mathbb{R}^2$, given by the matrix:

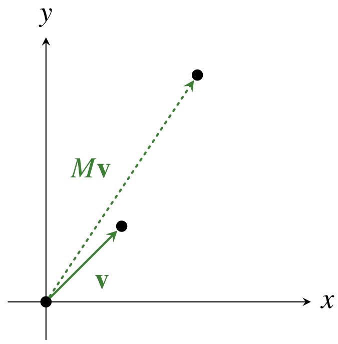
$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Let's visualize M with what it does to the plane. For example, let's see how it acts on the vectors:

$$\hat{\mathbf{i}}, \quad \hat{\mathbf{j}}, \quad \mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$



Observe that M just stretches $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, without changing their direction at all. Indeed, $M\hat{\mathbf{i}} = 2\hat{\mathbf{i}}$ and $M\hat{\mathbf{j}} = 3\hat{\mathbf{j}}$. Is the same true for $\mathbf{v}$? It doesn't quite look like it. Indeed, let's plot $\mathbf{v}$ and $M\mathbf{v}$ on the same coordinate plane.



It appears that M does not change the direction of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, but does change the direction of $\mathbf{v}$. ■

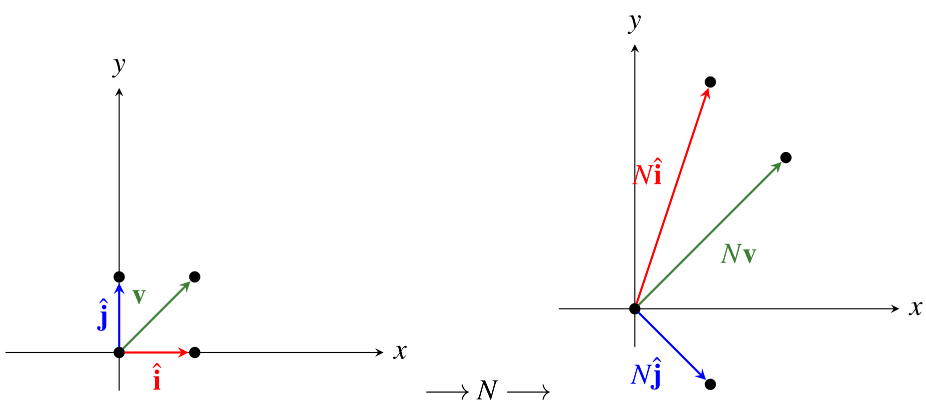
Let's compare this to another function.

■ **Example 7.2** Consider $N : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix:

$$N = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.$$

Let's see what N does to the same 3 vectors. We can deduce $N\hat{\mathbf{i}}$ and $N\hat{\mathbf{j}}$ from as the first and second columns of N respectively, and:

$$N\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$



Something different happened than what happened in Example 7.1: N changes the direction of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, but $N\mathbf{v} = 2\mathbf{v}$, so it's direction is unchanged! ■

In the 2 linear transformations of $\mathbb{R}^2$ from Examples 7.1 and 7.2, we saw that some vectors point the same direction after applying the transformation, while others may change direction. This is the notion of an *eigenvector*.

Definition 7.1.1 Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. An *eigenvector for M* is a vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $M\mathbf{v}$ points in the same (or an opposite) direction as $\mathbf{v}$. In particular, it is a vector $\mathbf{v}$ such that $M\mathbf{v}$ is *a nonzero multiple* of $\mathbf{v}$.

$$M\mathbf{v} = \lambda\mathbf{v} \text{ for some nonzero } \lambda.$$

7.1.2 What is an Eigenvalue?

In Example 7.1, we saw that $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ were both eigenvectors for the linear transformation:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

In particular, we saw that M preserves the $\hat{\mathbf{i}}$ directions and $\hat{\mathbf{j}}$ directions. But it does something different in each direction. For example, $M\hat{\mathbf{i}} = 2\hat{\mathbf{i}}$, so the length of $\hat{\mathbf{i}}$ is doubled by M. In particular, it preserves the $\hat{\mathbf{i}}$ direction, and expands in this direction by a factor of 2. On the other hand, $M\hat{\mathbf{j}} = 3\hat{\mathbf{j}}$, so the length of $\hat{\mathbf{j}}$ is tripled by M. In particular, it preserves the $\hat{\mathbf{j}}$ direction, and expands in that direction by a factor of 3. This scaling value is called an *eigenvalue for M*.

Definition 7.1.2 Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $\mathbf{w}$ be an eigenvector for M. Then:

$$M\mathbf{w} = \lambda\mathbf{w},$$

for some constant λ. This constant value λ is called the *eignevalue* for $\mathbf{w}$ under M.

In particular, under the matrix M from Example 7.1, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are eigenvectors, with eigenvalues 2 and 3 resepctively.

■ **Question 7.1** Consider the setup of Example 7.2, where:

$$N = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \quad \textit{and} \quad \mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

We've seen $\mathbf{v}$ is an eigenvector for N. What is the associated eigenvalue?

R An eigenvector for a linear transformation can be thought of as direction which the linear transformation doesn't change. In particular, the line pointing this direction can only be stretched or shrunk (or reverse direction). The eigenvalue for this eigenvectors measures this stretching, shrinking, or direction reversal.

7.1.3 How do we find eigenvalues and eigenvectors?

■ **Example 7.3** Suppose we'd like to find the (nonzero) eigenvectors for the matrix:

$$M = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

That is, what directions in $\mathbb{R}^2$ does this matrix fix? In particular, we are looking for vectors $\mathbf{v}$ such that:

$$M\mathbf{v} = \lambda\mathbf{v}, \tag{7.1}$$

for some constant λ. Notice that $\lambda \mathbf{v} = \lambda I \mathbf{v}$ where:

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

Pushing everything in Equation 7.1 to the left side and factoring out $\mathbf{v}$ we get:

$$(M - \lambda I)\mathbf{v} = 0.$$

In particular, $\mathbf{v}$ is a nontrivial vector in the null space of the matrix:

$$M - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix}.$$

Since $M - \lambda I$ has sends a nonzero vector $\mathbf{v}$ to zero, it cannot be invertible (would you undo $\mathbf{0}$ to $\mathbf{0}$ or to $\mathbf{v}$?). Therefore its determinant must be zero!

$$\det(M - \lambda I) = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix} = (3-\lambda)(-1-\lambda) = 0.$$

Therefore, we must have $\lambda = 3$ or $\lambda = -1$. *Perhaps surprisingly, it was easier to find the eigenvalues first, and with that in hand, and furthermore, there are only 2.* We have now found the eigenvalues, to continue, we have to find the eigenvectors. Let's break it up into each case:

The $\lambda = 3$ *case:* Now we are looking for a vector $\mathbf{v}$ in the null space of:

$$M - 3I = \begin{bmatrix} 3-3 & 0 \\ 8 & -1-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix}.$$

In particular, we are looking for a solution to the system of equations:

$$0x + 0y = 0$$

$$8x - 4y = 0.$$

Normally, we could use Gauss-Jordan elimination, but this is simple enough letting $y = t$ be the free variable, so:

$$x = \frac{1}{2}t$$

$$y = t.$$

This gives us *infinitely many eigenvectors with eigenvalue* 3*,* and they are parametrized by t. For example, when $t = 2$ we have:

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And indeed, if we compute $M\mathbf{v}$ we have:

$$M\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{v}.$$

The $\lambda = -1$ *case:* Now we are looking for a vector $\mathbf{w}$ in the null space of:

$$M - (-1)I = M + I = \begin{bmatrix} 3+1 & 0 \\ 8 & -1+1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix}.$$

In particular, we are looking for a solution to the system of equations:

$$4x + 0y = 0,$$

$$8x + 0y = 0.$$

In particuar, $x = 0$ and $y =$anything! For example, $\mathbf{w} = \hat{\mathbf{j}}$ would work. And indeed:

$$M\hat{\mathbf{j}} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\hat{\mathbf{j}}.$$

■

R Notice that $\mathbf{v}$ and $\hat{\mathbf{j}}$ form a basis for $\mathbb{R}^2$. It is easy to find the matrix for the transformation with respect to this basis, since it merely triples the first element, and negates the second:

$$\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

In particular, we end up with a *diagonal* whose entries are the eigenvalues (like in Example 7.1). This basis is often called an *eigenbasis*, and the new matrix is called the *diagonalization of* M.

This technique is often useful in computer graphics, where we would like to distort a screen by stretching or shrinking in a few perscribed directions. This amounts to fixing a few eigenvectors, whose eigenvalues come from the factors by which we stretch or shrink.

Example 7.3 gives us a perscribed technique for finding the eigenvalues and eigenvectors for a given square matrix M.

- **Step 1:** Compute the matrix $M - \lambda I$ treating λ as an variable.
- **Step 2:** Compute the determinant $p(\lambda) = \det(M - \lambda I)$, which will be a polynomial in the variable λ.
- **Step 3:** Set $P(\lambda) = 0$ and solve for the roots $\lambda_1, \lambda_2, \cdots, \lambda_n$. *These are your eigenvalues!*.
- **Step 4:** Compute the null space of $M - \lambda_i I$ for each eigenvalue i. That is, solve the linear system

  $$(M - \lambda_i I)\mathbf{x} = 0.$$

  The solutions to this linear system will be eigenvectors whose eigenvalues is λ_i.

R The polynomial $P(\lambda)$ is called the *characteristic polynomial* of M, and is a really interesting thing to study in its own right.

7.1.4 Why do we care about eigenvalues and eigenvectors?

We could spend a month (or semester...or career) discussing this topic. Unfortunately, we don't have that much time left. I'll give a few brief examples.

In Computer Graphics:

As we alluded to above, eigenvectors are useful in 2D (and 3D) computer animations, as they tell you the general directions of movement of a given linear transformation, and the eigenvalues tell you how movement scales in those directions. This means that if we want to prescribe movement or distortion of some specific type, it is enough to prescribe eigenvalues and eigenvectors.

In Theoretical Mathematics:

Finding the matrix of a linear transformation depends on the choice of a basis, which can lead to some ambiguity and confusion. But the eigenvectors and eigenvalues of a linear transformation don't depend on the basis. In fact, as we saw in Example 7.3 and the following Remark, the eigenvectors can sometimes give a *eigenbasis* with respect to which the matrix is diagonal! Diagonal matrices are much easier to study (and invert, and compute determinats of, and...), so choosing an eigenbasis is often an important technique when studying a linear transformation. This process is often known as *diagonalizetion*.

In Data Science and Machine Learning:

If you a linear transformation of an enormous dimensional data set, it can be often quite difficult extract meaningful data. That being said, the eigenvectors with the largest eigenvalues will point in the direction of the where where the linear transformation is *having the largest impact*. These are (roughly speaking) the *principle components* of this linear transformation, and restricting your attention to a selection of these most impactful eigenvalues allows you to reduce the dimension of your data set, by focussing attention on the most important directions. This technique (roughly speaking) is called *principle components analysis*.

In Sports Analytics:

The W-L record of a sports team is a rather *crude* metric of how good it is. To get deeper understanding, one would want to weight wins against better teams more (for example). How can we do this? One way is to make a matrix whose ij entry is how many times team i beat team j. The eigenvector with the largest eigenvalue will then point in the direction of the *best teams* (although it may not be specifically in the direction of one team). This gives some continuous and geometric data which you can use to compare performance.

In Internet Search Enginges:

In fact, the technique used in sports analytics is essentially how google's search engine works! We'd like to rank pages which are linked to often, but we'd like to rank links from important websites higher. One way to do this is to make a matrix whose ij entry is a 1 if page i links to page j, and a 0 otherwise. As in the sports analytics example, the eigenvectors with largest eigenvalues tend to point in the direction where information on the web generally flows!