Homework Assignment 9

Due Friday, April 8

1. Let G be a finite group, p a prime, and $P \in Syl_p(G)$ a Sylow p-subgroup. Use **(Sylow 2)** to show that $P \trianglelefteq G$ if and only if $n_p = \#Syl_p(G) = 1$.

2. Use Sylow's theorems to prove that a group of order 200 can never be simple.

3. Generalizing question 2, let G be a group of order p^2q for primes p and q. We will show that G always has a nontrivial *normal* Sylow subgroup.

   (a) Suppose $p > q$. Show that G has a normal subgroup of order p^2.

   (b) Suppose $q > p$. Show that either G has a normal subgroup of order q, or else $G \cong A_4$. (You may use the result from the April 5 lecture that if $|G| = 12$ and $n_3 \neq 1$ then $G \cong A_4$).

   (c) Explain why a group of order p^2q can never be simple. (You may need to treat the cases where $G = A_4$ or $p = q$ separately).

4. Let G be a group of order $99 = 3^2 * 11$. Let's show that G is abelian.

   (a) Let $P \leq G$ be a Sylow 3-subgroup. Show that $P \trianglelefteq G$.

   (b) Construct an injective homomorphism $G/C_G(P) \hookrightarrow Aut(P)$. (Can you use group actions and part (a)?)

   (c) Deduce from part (b) and Lagrange's theorem that $G = C_G(P)$. Leverage this fact to prove that G is abelian.