Takehome Assigment 3

Due Monday, April 26

In this assignment we establish some basic facts about prime and maximal ideals in *commutative unital* rings. **In this assignment all rings are commutative unital rings, and all ring homomorphisms are unital, meaning that they send 1 to 1**,

1. Let $\varphi : R \to S$ be a homomorphism between commutative unital rings with $\varphi(1_R) = 1_S$.
   (a) Let $\mathfrak{q} \subseteq S$ be a prime ideal. Show that $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R.
   (b) Suppose φ is surjective, and $\mathfrak{m} \subseteq S$ is a maximal ideal. Show that $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal of R.
   (c) Give a counterexample to part (b) if φ is not surjective.

2. In class we defined the ring of fractions for a *good multiplicative subset* of a ring, i.e., a subset of R which contains no zero divisors and is closed under multiplication. Let's generalize this. We define a subset $S \subseteq R$ to a be *multiplicative subset* if it is closed under multiplication and contains 1. In this exercise we will describe the ring of fractions $S^{-1}R$.
   (a) Consider the subset $\{(a, b) : a \in R, b \in S\} \subseteq R \times R$. Prove that:
   $$(a_1, b_1) \sim (a_2, b_2) \text{ if there exits } t \in S \text{ such that } t(a_1 b_2 - b_1 a_2) = 0,$$
   is an equivalence relation on R. The equivalence class of (a, b) will be denoted $\frac{a}{b}$. Explain why if S contains no zero divisors (or zero), this is the same equivalence relation as the one defined in class.
   (b) Let $S^{-1}R = \{\frac{a}{b} : a \in R, b \in S\}$ be the set of equivalence classes of the relation described above. Define addition and multiplication on $S^{-1}R$ by the rules:
   $$\begin{aligned} \frac{a_1}{b_1} + \frac{a_2}{b_2} &= \frac{a_1 b_2 + a_2 b_1}{b_1 b_2} \\ \frac{a_1}{b_1} \times \frac{a_2}{b_2} &= \frac{a_1 a_2}{b_1 b_2}. \end{aligned}$$
   Show that these rules make $S^{-1}R$ into a commutative ring with identity. (You must first show that they are well defined. Then show that the ring axioms are satisfied)
   (c) Define $\iota : R \to S^{-1}R$ by the rule $\iota(r) = \frac{r}{1}$. Show that ι is a ring homomorphism, that $\iota(1_R) = 1_{S^{-1}R}$ and that if $s \in S \subseteq R$, the $\iota(s)$ is a unit in $S^{-1}R$. Prove also that ι is injective if and only if S contains no zero divisors (or zero),
   (d) Show that $S^{-1}R$ satisfies the following *universal property*. For any commutative unital ring A, and ring homomorphisms $\varphi : R \to A$ such that $\varphi(s) \in A^\times$ for every $s \in S$, there is a unique homomorphism $\tilde{\varphi} : S^{-1}R \to A$ such that $\tilde{\varphi} \circ \iota = \varphi$.
   $$\begin{tikzcd} S^{-1}R \arrow[dr, dotted, "\tilde{\varphi}"] & \\ R \arrow[u, "\iota"] \arrow[r, "\varphi"'] & A. \end{tikzcd}$$
   Deduce that there is a bijection:
   $$\{\text{Homomorphisms } \varphi : R \to A \text{ such that elements of } S \text{ map to } A^\times\}$$
   $$\Updownarrow$$
   $$\{\text{(Unital) homomorphisms } \tilde{\varphi} : S^{-1}R \to A\}.$$

(e) Let $r \in R$ be nonzero and consider the multiplicative set $S = \{1, r, r^2, r^3, \cdots\}$. Define $R[1/r] := S^{-1}R$. Show that $R[1/r] = 0$ if and only if r is nilpotent.

3. In this exercise we calculate the intersection of all the prime ideals in a commutative unital ring R.

   (a) Show that the element 0 is contained in every ideal of R.

   (b) Let r be a nilpotent element of R. Show that r is contained in every prime ideal of R.

   (c) Conversely, suppose r is not nilpotent. Show that there is some prime ideal not containing r. Deduce that:
$$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}.$$
   (Hint: To find such a prime ideal, try applying 1(a) and 2(e) to the map $\iota : R \to R[1/r]$.)

   (d) Deduce that the intersection of all the prime ideals in an integral domain is the 0 ideal.

   (e) Suppose that r is in the intersection of all the prime ideals of R. Show that $1 - ry \in R^\times$ for every $y \in R$. (We will see below that the converse is not true in general, but that we can characterize all elements satisfying this property).

4. In this exercise we calculate the intersection of all the maximal ideals in a commutative unital ring R. Given a ring R, we define the *Jacobson radical* of R to be:
$$\mathfrak{J}(R) = \bigcap_{\mathfrak{m} \subseteq R \text{ maximal}} \mathfrak{m}.$$

   (a) Show that $\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$.

   (b) Show that an element $r \in R$ is a unit if and only if it is not contained in any maximal ideal.

   (c) Suppose $\mathfrak{m}$ is a maximal ideal and $r \in R \setminus \mathfrak{m}$. Compute the ideal $(\mathfrak{m}, r)$ generated by $\mathfrak{m}$ and r.

   (d) Prove that the condition from 3(e) actually characterizes elements in the *Jacobson Radical*! That is, prove that $r \in \mathfrak{J}(R)$ if and only if $1 - ry \in R^\times$ for every $y \in R$. (Parts (b) and (c) might help!)