Takehome Assignment 2

Due Monday, April 4 at 5pm

In this assignment, we complete the proof of Sylow's Theorems. Let's recall the relevant definitions and statements.

Definition 1. *Let p be a prime number. A group H is called a **p-group** if $|H| = p^r$ for some r. If G is a group and $H \leq G$ is a subgroup which is a p-group, we call it a **p-subgroup** of G.*

Definition 2. *Let G be a finite group of order $|G| = p^\alpha m$ for p a prime not dividing m. A subgroup $P \leq G$ of order p^α is called a **Sylow p-subgroup** of G. The collection of all Sylow p-subgroups of G is denoted $Syl_p(G)$ and the number of Sylow p-subgroups is often denoted $n_p = \#Syl_p(G)$.*

Theorem 3 (Sylow's Theorems)**.** *Adopt the notation from Definition 2.*

- ***(Sylow 1)*** *There exists a Sylow p-subgroup of G.*
- ***(Sylow 2)*** *Let $P \in Syl_p(G)$ and let $Q \leq G$ any p-subgroup of G. Then there exists some $g \in G$ with $gQg^{-1} \leq P$.*
- ***(Sylow 3)*** *Let $P \in Syl_p(G)$.*
  - *(a)* $n_p \equiv 1 \mod p$.
  - *(b)* $n_p = [G : N_G(P)]$. *In particular* $n_p | m$.

We already proved **(Sylow 1)** in class, **(Sylow 2)** and **(Sylow 3)** remain. As is often the case, group actions will be a useful tool! To help us along the way, we introduce one more definition.

Definition 4. *Let G be a group acting on a set A. The fixed points of the action are:*

$$A^G = \{a \in A : g \cdot a = a \text{ for all } g \in G\}.$$

1. Let's establish a few facts about the fixed points.
   (a) Let G be a group. Compute the fixed points of the following actions.
      i. G acting on G by left multiplication.
      ii. G acting on G by conjugation.

   (b) Let G be a p-group acting on a finite set A. Show that $|A^G| \equiv |A| \mod p$. (*Hint:* One could model this off of the proof of the class equation. Use the orbit-stabilizer theorem to see what happens when reducing mod p).

   (c) Let G be a p-group acting on a nonempty set A, and suppose that p does not divide $|A|$. Show that the action of G on A has at least one fixed point.

(Sylow 2) now follows from a clever application of 1(c). All we have to do is look at the right group action!

2. Let G be as in Definition 2, and P a Sylow p-subgroup of G. Let $Q \leq G$ be a p-subgroup.
   (a) Use 1(c) to deduce that the action of Q on G/P by left multiplication has a fixed point. (There are 2 cardinality conditions to apply 1(c), explain why they both hold.)

   (b) Use the fixed point of this action to show that a conjugate of Q is contained in P, thereby proving **(Sylow 2)**.

(c) Deduce that all Sylow p-subgroups of G are conjugate and isomorphic.

The two parts of **(Sylow 3)** follow from the orbit-stabilizer theorem and clever application of 1(b), keeping careful track of the numerics!

3. Let G be as in Definition 2, and P a Sylow p-subgroup of G.

   (a) Show that G acts on the set $Syl_p(G)$ by conjugation. What is the stabilizer of P?

   (b) Use the orbit-stabilizer theorem of the action from part (a) to prove **(Sylow 3)(b)**. (You can use 2(c) to compute the orbit $G * P$).

   (c) Restrict the action from part (a) to an action of P on $Syl_p(G)$. Show that the action of P on $Syl_p(G)$ has a single fixed point: P itself!

   (d) Deduce **(Sylow 3)(a)** from 1(b) and 3(c).

Good job! You did it! We will explore many consequences of these results in the coming week!