Homework Assignment 6

Due Friday, March 5

1. Let G be a group, and $M, N \trianglelefteq G$ normal subgroups such that $MN = G$.
   (a) Show $G/(M \cap N) \cong (G/M) \times (G/N)$
   (b) Suppose further that $M \cap N = \{1\}$. Show that $G \cong M \times N$.

2. Let G be a group and $Z(G)$ its center.
   (a) Suppose $H \leq Z(G)$. Show that H is a normal subgroup. (In particular, $Z(G)$ is normal).
   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   (c) Let p and q be prime numbers (not necessarily distinct), and G a group of order pq. Show that if G is not abelian, than $Z(G) = \{1\}$.

3. Let's classify all groups of order 6. To begin, let G be a nonabelian group of order 6. We will show $G \cong S_3$.
   (a) Show that there is an element $x \in G$ of order 2. (Once we have Cauchy's theorem for nonabelian groups this part becomes easy, but since G has 6 elements, one can do this by inspection using Lagrange's theorem).
   (b) Let $x \in G$ have order 2, and let $H = \langle x \rangle$. Show that H is not normal in G. (*Hint:* Show that if H is normal than $H \leq Z(G)$, then apply 2(c) to find a contradiction.)
   (c) Consider the action of G on $A = G/H$ by left multiplication. Show that the associated permutation representation is injective. Conclude that $G \cong S_3$.
   (d) Complete the classification of all groups of order 6 by showing that if Z is an abelian group of order 6 then $Z \cong Z_6$. (*Hint:* We do have Cauchy's theorem for abelian groups.) *We've now classified groups of order* ≤ 7.

4. Let G be a group. Let $[G, G] = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$.
   (a) Show that $[G, G]$ is a normal subgroup of G.
   (b) Show that $G/[G, G]$ is abelian.

   $[G, G]$ is called the *commutator subgroup* of G, and $G/[G, G]$ is called the *abelianization* of G, denoted G^{ab}. The rest of this exercise explains why.

   (c) Let $\varphi : G \to H$ be a homomorhism with H abelian. Show $[G, G] \subseteq \ker \varphi$.
   (d) Conclude that for H an abelian group there is a bijection:

   $$\left\{ \text{ Homomorphisms } \varphi : G \to H \ \right\} \Longleftrightarrow \left\{ \text{ Homomorphisms } \tilde{\varphi} : G^{\text{ab}} \to H \ \right\}$$

   Hint. *Recall the technique of passing to the quotient described at the beginning of the 2/23 lecture*

5. Let's now compute D_{2n}^{ab}. We should begin computing $xyx^{-1}y^{-1}$. There are 3 cases.
   (a) Compute $x^{-1}y^{-1}xy$ in each of the following 3 cases. (*Hint:* HW2#9(e) gives the inverse for a reflection.)
      (i) x, y both reflections. So $x = sr^i$ and $y = sr^j$.

(ii) x a reflection and y not a reflection. So $x = sr^i$ and $y = r^j$.

(iii) Neither x nor y are reflections. So $x = r^i$ and $y = r^j$.

(b) Prove that $[D_{2n}, D_{2n}] = \langle r^2 \rangle$. If n is odd one could choose another generator. What is it?

(c) Now prove that D_{2n}^{ab} is either V_4 or Z_2 depending on whether n is odd or even. Note that since this is so small we should interpret this as suggesting that D_{2n} is far from abelian.

For the remainder we will study the quaternion group Q_8. It is a nonabelian group with very interesting properties.

Definition 1. *The quaternion group of order 8, denoted Q_8 is the group of the following 8 elements:*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

subject to the relations:

$$(-1)^2 = 1$$
$$i^2 = j^2 = k^2 = -1,$$
$$(-1)x = -x = x(-1) \text{ for all } x,$$

$$\begin{aligned} ij &= k, & ji &= -k, \\ jk &= i, & kj &= -i, \\ ki &= j, & ik &= -j. \end{aligned}$$

6. Let's start with a few simple facts. Much of this is worked out in the book.
   (a) Write the entire multiplication table for Q_8.
   (b) Find 2 elements which generate all of Q_8. (*Bonus:* Can you give a presentation of Q_8?)
   (c) Prove that Q_8 is not isomorphic to D_8.
   (d) Find all the subgroups of Q_8, and draw its lattice. (*Hint*: there are 6 total subgroups).
   (e) Prove that every subgroup of Q_8 is normal.
   (f) Prove that every subgroup and quotient group of Q_8 is abelian (*Hint*: recall TH1#4).
   (g) Compute $Z(Q_8)$ and $Q_8/Z(Q_8)$ (*Hint for the second part*: you can do this by hand, but it might be slicker to apply 2(b)).

7. Now let's follow the proof of Cayley's theorem to exhibit Q_8 as a subgroup of S_8.
   (a) Label $\{1, -1, i, -i, j, -j, k, -k\}$ as the numbers $\{1, 2, \cdots, 8\}$. Then the action of Q_8 on itself by left multiplication gives an injective map $Q_8 \to S_8$. Write the permutation representations for -1 and i as elements $\sigma_{-1}, \sigma_i \in S_8$, and verify that $\sigma_i^2 = \sigma_{-1}$. (Using the multiplication table from question 1 will make this easier).
   (b) Use the generators from question 1(b) to give two elements of S_8 which generate a subgroup $H \leq S_8$ isomorphic to Q_8.
   (c) Is σ_i even or odd?
   (d) $A_8 \cap H$ is isomorphic to a subgroup of Q_8. Which one?

8. Cayley's theorem says that if $|G| = n$ then G embeds at S_n. One could ask if this n is *sharp*, or if perhaps G can embed in some smaller symmetric group. For example, D_8 embeds in S_4 (thinking about symmetries of the square as permutations of the vertices, cf HW3#5). Nevertheless, for Q_8 the symmetric group given by Cayley's theorem is the smallest.

   (a) Let Q_8 act an a set A with $|A| \leq 7$. Let $a \in A$. Show that the stabilizer of a, $(Q_8)_a \leq Q_8$ must contain the subgroup $\{\pm 1\}$. (*Hint:* The orbit stabilizer theorem might help.)

   (b) Deduce that the kernel of the action of Q_8 on A contains $\{\pm 1\}$.

   (c) Conclude that Q_8 cannot embed into S_n for $n \leq 7$.