Homework Assignment 3

Due Friday, February 12

1. We begin by establishing important basic facts about group homomorphisms that we will use repeatedly throughout the course. Let G, H, K be groups, and let $\varphi : G \to H$ and $\psi : H \to K$ a homomorphisms.
   (a) Show that $\varphi(1_G) = 1_H$.
   (b) Show that $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.
   (c) Show that if $g \in G$ has finite order, then $|\varphi(g)|$ divides $|g|$.
   (d) Show that if φ is an isomorphism, then so is φ^{-1}.
   (e) Show that if φ is an isomorphism, $|\varphi(g)| = |g|$.
   (f) Show that the composition $\psi \circ \varphi : G \to K$ is a homomorphism.
   (g) Suppose φ and ψ are both isomorphisms. Show that the composition $\psi \circ \varphi$ is as well.
   (h) Conclude that the relation *is isomorphic to* is an equivalence relation on the set of all groups.

2. Given a homomorphism $\varphi : G \to H$, we obtain 2 important subgroups, one of G and one of H. They are called the *kernel of* φ and *image of* φ and are defined by the following rules:

$$\begin{aligned} \ker \varphi &= \{g \in G : \varphi(g) = 1_H\}, \\ \operatorname{im} \varphi &= \{h \in H : h = \varphi(g) \text{ for some } g \in G\}. \end{aligned}$$

   (a) Show that $\ker \varphi$ is a subgroup of G.
   (b) Show that $\operatorname{im} \varphi$ is a subgroup of H.
   (c) *Important:* Show that φ is injective if and only if $\ker \varphi = \{1_G\}$. (This is an incredibly useful fact!)

3. The kernel has the following important generalization. For $h \in H$ define the *fiber over* h as

$$\varphi^{-1}(h) = \{g \in G : \varphi(g) = h\}.$$

   This is sometimes also called the *preimage of* h. Observe that by definition, the kernel of φ is the fiber over 1.
   (a) Show that the fiber over h is a subgroup if and only if $h = 1_H$.
   (b) Show that the *nonempty* fibers of φ form a partition of G. (In particular, if φ is surjective its fibers partition G.)
   (c) Show that all nonempty fibers have the same cardinality. (Hint: if $\varphi^{-1}(h)$ is nonempty, build a bijection between it and $\ker \varphi$.) Observe that this generalizes 2(c).

4. Recall that we defined the kernel of a group action in class. Let's justify our terminology. Let $G \times A \to A$ be an action of G on a set A and let $\varphi : G \to S_A$ be the associated permutation representation.
   (a) Show that the kernel of the group action is equal to $\ker \varphi$.
   (b) Show that the action is faithful if and only if the φ is injective. (Hint: Use 2(c).)

5. We've seen that there is a relationship between the dihedral and symmetric groups. Let's explore this a bit.

   (a) Describe an injective homomorphism from $\varphi : D_{2n} \to S_n$ (you may describe this in words, but make sure to justify injectivity).

   (b) In the map you described, what is the cycle decomposition of $\varphi(r)$ (where as usual r is the generator corresponding to clockwise rotation of the n-gon by $2\pi/n$)?

   (c) Prove that $D_6 \cong S_3$.

6. In this exercise we show that you can compute the order of a permutation from its cycle decomposition.

   (a) Let G be a group. Two elements $x, y \in G$ are called *commuting elements* if $xy = yx$. Show that if x and y are commuting elements, then $(xy)^n = x^n y^n$.

   (b) Give a counterexample to part (a) if the chosen elements do not commute.

   (c) Let $\sigma = (a_1, a_2, \cdots, a_r) \in S_n$ be an r-cycle. Show that $|\sigma| = r$.

   (d) Prove that the order of a permutation is the least common multiple of the lengths of the cycles in its cycle decomposition. (Hint: You may freely use that disjoint cycles are commuting elements. You may find it useful to establish that the product of nontrivial disjoint cycles is never 1).

7. We hinted in class that if A and B are sets of the same cardinality, then their permutation groups S_A and S_B (defined in HW2#5) are isomorphic. Let's prove it. To begin, fix a bijective function $\theta : A \to B$.

   (a) Let $f : A \to A$ be bijective. Show that $\theta \circ f \circ \theta^{-1} : B \to B$ is bijective. (Hint: what is its inverse?)

   (b) Part (a) allows us to construct the following function:

$$\begin{array}{ccc} S_A & \xrightarrow{\varphi} & S_B \\ f & \longmapsto & \theta \circ f \circ \theta^{-1}. \end{array}$$

   Show that φ is an isomorphism, thereby proving the result. (Note: There are two parts to this. You must show that φ is bijctive, and that it is a homomorphism.)

8. The set S_3 has 6 elements. Compute the order and cycle decomposition of each element.