Homework Assignment 4

Due Friday, February 21

1. Let G be a group. Let $H, K \leq G$ be two subgroups.
   (a) Show that the intersection $H \cap K$ is a subgroup of G.
   (b) Give an example to show that the union $H \cup K$ need not be a subgroup of G.
   (c) Show that $H \cup K$ is a subgroup of G if and only if $H \subset K$ or $K \subset H$.

2. Let A be an *abelian* group.
   (a) Let $A^n = \{a^n | a \in A\}$ be the collection of nth powers of elements in A. Show that this is a subgroup of A.
   (b) Let $A[n] = \{a \in A | a^n = 1\}$. Show that $A[n]$ is a subgroup of A. This is often called the *n-torsion* subgroup of A.
   (c) Let $A^{\text{tors}} = \{a \in A | \ |a| < \infty\}$. Show that A^{tors} is a subgroup of A. This is often called the *torsion* subgroup of A.
   (d) Give an example of a nonabelian group G where G^{tors} is not a subgroup of G. (Note that G must be infinite, as if G were finite every element would have finite order so that we would have $G^{\text{tors}} = G$).

3. Compute the center of the dihedral group. Explicitly, let n be an integer ≥ 3. Compute $Z(D_{2n})$. (Note: you will need to split into the two cases, where n is even or n is odd).

4. Let G be a group.
   (a) Show that if H is a subgroup of G, then $H \leq N_G(H)$.
   (b) Give an example where $A \subset G$ is a a subset (not necessarily a subgroup), and $A \not\subset N_G(A)$.
   (c) Show that $H \leq C_G(H)$ if and only if H is abelian.

5. In class we classified all finite cyclic groups and their generators. In this exercise you take care of the infinite case. Let $H = \langle x \rangle$ be a cyclic group of infinite order.
   (a) Show that the map $\varphi : \mathbb{Z} \to H$ defined by the rule $\varphi(a) = x^a$ is an isomorphism.
   (b) Since H is cyclic every element of H is of the form x^a for some a. Show that x^a generates H if and only if $a = \pm 1$.

6. In this exercise we study products of finite cyclic groups. Recall that we denote by Z_n the cyclic group of order n (written multiplicatively).
   (a) Prove that $Z_2 \times Z_2$ is not a cyclic group.
   (b) Prove that $Z_2 \times Z_3 \cong Z_6$. Conclude that $Z_2 \times Z_3$ is a cyclic group.

   Those two examples really cover all the bases. Use the intuition you gained from them to prove the following classification result.

   (c) Show that $Z_n \times Z_m$ is cyclic if and only if $\gcd(n, m) = 1$. (Hint: recall that up to isomorphism there is only one cyclic group of order N for every positive integer N).

7. Let $G = S_n$ be the symmetric group equipped with it's natural action on $\Omega_n = \{1, 2, \cdots, n\}$ by permutations. For $i \in \Omega_n$, let $G_i = \{\sigma \in G | \sigma(i) = i\}$ be the stabilizer of i. What is $|G_i|$?