Takehome Assigment 3 Solutions

Due Monday, April 26

In this assignment we establish some basic facts about prime and maximal ideals in *commutative unital* rings. **In this assignment all rings are commutative unital rings, and all ring homomorphisms are unital, meaning that they send 1 to 1**,

1. Let $\varphi : R \to S$ be a homomorphism between commutative unital rings with $\varphi(1_R) = 1_S$.

   (a) Let $\mathfrak{q} \subseteq S$ be a prime ideal. Show that $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R.

   Proof. We present two proofs. The first is direct. Let $x, y \in R$ with $xy \in \varphi^{-1}(\mathfrak{q})$. Then $\varphi(xy) = \varphi(x)\varphi(y) \in \mathfrak{q}$, so that $\varphi(x) \in \mathfrak{q}$ or else $\varphi(y) \in \mathfrak{q}$ (by the primality of $\mathfrak{q}$). This in turn implies that either x or y are in $\varphi^{-1}(\mathfrak{q})$, which implies primality of $\varphi^{-1}(\mathfrak{q})$.

   Here's another proof I consider cuter. Consider the composition $R \to S \to S/\mathfrak{q}$. The kernel of this map is plainly $\varphi^{-1}(\mathfrak{q})$, so that by the first isomorphism theorem we obtain an injective unital ring homomorphisms $R/\varphi^{-1}(\mathfrak{q}) \hookrightarrow S/\mathfrak{q}$. Since $\mathfrak{q}$ is prime, $S/\mathfrak{q}$ is an integral domain. Therefore $R/\varphi^{-1}(\mathfrak{q})$ is isomorphic to a subring of an integral domain, so it cannot have any zero divisors (else $S/\mathfrak{q}$ would). This in turn implies $\varphi^{-1}(\mathfrak{q})$ is prime. □

   (b) Suppose φ is surjective, and $\mathfrak{m} \subseteq S$ is a maximal ideal. Show that $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal of R.

   Proof. Consider the composition $R \to S \to S/\mathfrak{m}$. The composition is surjective, and the kernel is $\varphi^{-1}(\mathfrak{m})$. Therefore by the first isomorphism theorem, $R/\varphi^{-1}(\mathfrak{m}) \cong S/\mathfrak{m}$. Since $\mathfrak{m}$ is maximal, $S/\mathfrak{m}$ is a field, and therefore so is $R/\varphi^{-1}(\mathfrak{m})$, so that $\varphi^{-1}(\mathfrak{m})$ is maximal in R.

   Here's another proof using the fourth isomorphism theorem (HW11 Problem 2(a)). Since $S \cong R/\ker\varphi$, there is an inclusion preserving bijection between ideals of S and those of R containing $\ker\varphi$. In particular, an ideal I with $\varphi^{-1}(\mathfrak{m}) \subseteq I \subsetneq R$ corresponds to an ideal $\mathfrak{m} \subseteq \overline{I} \subsetneq S$. The maximality of $\mathfrak{m}$ shows that $\overline{I} = \mathfrak{m}$, so that $I = \varphi^{-1}(\mathfrak{m})$. □

   (c) Give a counterexample to part (b) if φ is not surjective.

   Proof. Consider $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Then the 0 ideal is maximal in $\mathbb{Q}$, but its preimage in $\mathbb{Z}$ is also the zero ideal, which is not maximal, since for any $n \geq 2$ we have $0 \subsetneq n\mathbb{Z} \subsetneq \mathbb{Z}$. □

2. In class we defined the ring of fractions for a *good multiplicative subset* of a ring, i.e., a subset of R which contains no zero divisors and is closed under multiplication. Let's generalize this. We define a subset $S \subseteq R$ to a be *multiplicative subset* if it is closed under multiplication and contains 1. In this exercise we will describe the ring of fractions $S^{-1}R$.

   (a) Consider the subset $\{(a, b) : a \in R, b \in S\} \subseteq R \times R$. Prove that:

   $$(a_1, b_1) \sim (a_2, b_2) \text{ if there exits } t \in S \text{ such that } t(a_1b_2 - b_1a_2) = 0,$$

   is an equivalence relation on R. The equivalence class of (a, b) will be denoted $\frac{a}{b}$. Explain why if S contains no zero divisors (or zero), this is the same equivalence relation as the one defined in class.

Proof. We must show it is symmetric, reflexive, and transitive.

Symmetric: This follows because the ring is commutative. Explicitly: suppose $(a_1, b_1) \sim (a_2, b_2)$. Then there is some $t \in S$ so that $t(a_1b_2 - b_1a_2) = t(a_2b_1 - b_2a_1) = 0$ which is what it means for $(a_2, b_2) \sim (a_1, b_1)$.

Reflexive: To see $(a_1, b_1) \sim (a_1b_1)$ we observe that $t(a_1b_1 - b_1a_1) = 0$, for any t, so in particular for any $t \in S$.

Transitive: Suppose $(a_1, b_1) \sim (a_2, b_2)$ and $(a_2, b_2) \sim (a_3, b_3)$. Therefore there are $t, s \in S$ such that:

$$t(a_1b_2 - b_1a_2) = 0 \text{ and } s(a_2b_3 - b_2a_3) = 0. \tag{1}$$

In particular, we can multiply the first equation by s and the second by t to obtain:

$$sta_1b_2 = sta_2b_1 \text{ and } sta_2b_3 = stb_2a_3. \tag{2}$$

Multiply the first by b_3 and the second by b_1, then add them together to obtain:

$$sta_1b_2b_3 + stb_1a_2b_3 = stb_1a_2b_3 + stb_1b_2a_3. \tag{3}$$

We observe that $stb_1a_2b_3 = stb_1a_2b_3$, by Equation (2), so that we can cancel this from equation (3) and rearrange it to obtain:

$$stb_2(a_1b_3 - b_1a_3) = 0 \tag{4}$$

Since $s, t, b_2 \in S$, and S is multiplicatively closed, then $stb_2 \in S$, so that Equation (4) witnesses the relation $(a_1, b_1) \sim (a_3, b_3)$, completing the proof of transitivity.

Finally, if S contains no zero divisors (or 0), then if $t(a_1b_2 - b_1a_2) = 0$ for some $t \in S$ if and only if $a_1b_2 - b_1a_2 = 0$, which is precisely the condition we described in class. □

(b) Let $S^{-1}R = \{\frac{a}{b} : a \in R, b \in S\}$ be the set of equivalence classes of the relation described above. Define addition and multiplication on $S^{-1}R$ by the rules:

$$\begin{aligned} \frac{a_1}{b_1} + \frac{a_2}{b_2} &= \frac{a_1b_2 + a_2b_1}{b_1b_2} \\ \frac{a_1}{b_1} \times \frac{a_2}{b_2} &= \frac{a_1a_2}{b_1b_2}. \end{aligned}$$

Show that these rules make $S^{-1}R$ into a commutative ring with identity. (You must first show that they are well defined. Then show that the ring axioms are satisfied)

Proof. There is a whole laundry list of things to check.

+ is well defined: Suppose $\frac{a_1}{b_1} = \frac{\hat{a}_1}{\hat{b}_1}$ and $\frac{a_2}{b_2} = \frac{\hat{a}_2}{\hat{b}_2}$. These are witnessed by $s, t \in S$ such that:

$$t(a_1\hat{b}_1 - b_1\hat{a}_1) = 0 \text{ and } s(a_2\hat{b}_2 - b_2\hat{a}_2) = 0. \tag{5}$$

Therefore:

$$\begin{aligned} & st((a_1b_2 + a_2b_1)\hat{b}_1\hat{b}_2) - b_1b_2(\hat{a}_1\hat{b}_2 + \hat{a}_2\hat{b}_1)) \\ = \;& (ta_1\hat{b}_1)(sb_2\hat{b}_2) + (sa_2\hat{b}_2)(tb_1\hat{b}_1) - (tb_1\hat{a}_1)(sb_2\hat{b}_2) - (tb_1\hat{b}_1)(sb_2\hat{a}_2) \\ = \;& t(a_1\hat{b}_1 - b_1\hat{a}_1)(sb_2\hat{b}_2) + s(a_2\hat{b}_2 - b_2\hat{a}_2)(tb_1\hat{b}_1) \\ = \;& 0 \end{aligned}$$

Where the last step comes from substituting en Equation (5). We may therefore conclude that:

$$\frac{a_1b_2 + a_2b_1}{b_1b_2} = \frac{\hat{a}_1\hat{b}_2 + \hat{a}_2\hat{b}_1}{\hat{b}_1\hat{b}_2}$$

+ is associative: We directly check that:

$$\begin{aligned}
\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) + \frac{a_3}{b_3} &= \frac{a_1b_2 + a_2b_1}{b_1b_2} + \frac{a_3}{b_3} \\
&= \frac{(a_1b_2 + a_2b_1)b_3 + (b_1b_2)a_3}{(b_1b_2)b_3} \\
&= \frac{a_1(b_2b_3) + b_1(a_2b_3 + b_2a_3)}{b_1(b_2b3)} \\
&= \frac{a_1}{b_1} + \frac{a_2b_3 + b_2a_3}{b_2b_3} \\
&= \frac{a_1}{b_1} + \left(\frac{a_2}{b_2} + \frac{a_3}{b_3}\right).
\end{aligned}$$

+ has identity: We notice that $\frac{0}{t}$ is an additive identity for any $t \in S$ (uniqueness doesn't need to be checked, it follows from the group axioms). Indeed, for any $\frac{a}{b}$ we have:

$$\frac{a}{b} = \frac{0}{t} = \frac{at + b0}{bt} = \frac{at}{bt}.$$

But since $abt - bat = t(ab - ba) = 0$, we see that $\frac{a}{b} = \frac{at}{bt}$.

+ has inverses: We see that $-\frac{a}{b} = \frac{-a}{b}$. Indeed:

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab + -ab}{bb} = \frac{0}{b^2},$$

as desired.

+ is commutative: This is immediate as + and $\times$ both are in R. Indeed:

$$\frac{a}{b} + \frac{a'}{b'} = \frac{ab' + a'b}{bb'} = \frac{a'b + ab'}{b'b} = \frac{a'}{b'} + \frac{a}{b}.$$

This completes the proof that $(S^{-1}R, +)$ is an abelian group under +. We now move on to proving $\times$ as the properties we'd like.

$\times$ is well defined: We begin exactly as in showing + is well defined with $\frac{a_1}{b_1} = \frac{\hat{a}_1}{\hat{b}_1}$ and $\frac{a_2}{b_2} = \frac{\hat{a_2}}{\hat{b_2}}$, witnessed by Equation (5) above. Rarranging we see that:

$$ta_1\hat{b}_1 = tb_1\hat{a}_1 \text{ and } sa_2\hat{b}_2 = sb_2\hat{a}_2.$$

Therefore:

$$sta_1a_2\hat{b}_1\hat{b}_2 = stb_1b_2\hat{a}_1\hat{a}_2,$$

so that

$$st(a_1a_2\hat{b}_1\hat{b}_2 - b_1b_2\hat{a}_1\hat{a}_2).$$

Since $st \in S$ this witnesses that:
$$\frac{a_1 a_2}{b_1 b_2} = \frac{\hat{a}_1 \hat{a}_2}{\hat{b}_1 \hat{b}_2},$$
as desired.

$\times$ is associative and commutative: This is an easy check, following immediately from the associativity and commutativity of R. Indeed:
$$\left(\frac{a_1}{b_1}\frac{a_2}{b_2}\right)\frac{a_3}{b_3} = \frac{(a_1a_2)a_3}{(b_1b_2)b_3} = \frac{a_1(a_2a_3)}{b_1(b_2b_3)}.$$
$$\frac{a_1}{b_1}\frac{a_2}{b_2} = \frac{a_1a_2}{b_1b_2} = \frac{a_2a_1}{b_2b_1} = \frac{a_2}{b_2}\frac{a_1}{b_1}.$$

The distributive law holds: Since we checked commutativity first, we need only the distributive law on one side. Let's do it:
$$\begin{aligned}\frac{c}{d}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) &= \frac{c}{d}\frac{a_1b_2 + a_2b_1}{b_1b_2} \\ &= \frac{c(a_1b_2 + a_2b_1)}{db_1b_2}.\end{aligned}$$

On the other hand:
$$\begin{aligned}\frac{c}{d}\frac{a_1}{b_1} + \frac{c}{d}\frac{a_2}{b_2} &= \frac{ca_1}{db_1} + \frac{ca_2}{db_2} \\ &= \frac{cda_1b_2 + cda_2b_1}{d^2b_1b_2}.\end{aligned}$$

To see that these are the same we cross multiply and observe that:
$$c(a_1b_2+a_2b_1)d^2b_1b_2-(cda_2b_2+cda_2b_1)db_1b_2 = cd^2b_1b_2((a_1b_2+a_2b_1)-(a_1b_2-a_2b_1)) = 0.$$

Therefore these are equal and distributivity holds. We have now shown that $S^{-1}R$ is a commutative ring. One thing remains.

$\times$ is unital: We finish by observing that $\frac{1}{1}$ is a unit (here we are using that $1 \in S$, but in fact $\frac{t}{t}$ would work for any $t \in S$). Indeed:
$$\frac{1}{1}\frac{a}{b} = \frac{1a}{1b} = \frac{a}{b}.$$

The other side immediately holds by commutativity. □

(c) Define $\iota : R \to S^{-1}R$ by the rule $\iota(r) = \frac{r}{1}$. Show that ι is a ring homomorphism, that $\iota(1_R) = 1_{S^{-1}R}$ and that if $s \in S \subseteq R$, the $\iota(s)$ is a unit in $S^{-1}R$. Prove also that ι is injective if and only if S contains no zero divisors (or zero),

Proof. **ι is a unital homomorphism:** Fix $r, s \in R$. Then:
$$\iota(r) + \iota(s) = \frac{r}{1} + \frac{s}{1} = \frac{r1 + 1s}{11} = \frac{r+s}{1} = \iota r + s,$$

$$\iota(r)\iota(s) = \frac{r}{1}\frac{s}{1} = \frac{rs}{1} = \iota(rs),$$

so ι is a homomorphism. Furthermore, $\iota(1) = \frac{1}{1}$ which we verified in part (b) was the identity of $S^{-1}R$.

ι takes elements of S to units: Let $s \in S$. Then $\iota(s) = \frac{s}{1}$. Notice that $\frac{1}{s} \in S^{-1}R$, and that $\frac{1}{s}\frac{s}{1} = \frac{s}{s}$ But $\frac{s}{s} = \frac{1}{1}$. Indeed, if we cross multiply this is $s1 - 1s = 0$. Therefore $\iota(s)$ has a left inverse, which is also a right inverse by commutativity. Thus $\iota(s)$ is a unit.

Characterizing injectivity: Notice that $\iota(r) = 0$ if and only if $\frac{r}{1} = \frac{0}{1}$. This holds if and only if there is some $t \in S$ such that $t(r1 - 1 \cdot 0) = tr = 0$. Therefore, there is some nonzero $r \in \ker \iota$ if and only if there is some $t \in S$ such that $tr = 0$, which holds if and only if $t = 0$ or t is a zero divisor. So ι is not injective if and only if S contains a zero or zero divisor. $\square$

(d) Show that $S^{-1}R$ satisfies the following *universal property*. For any commutative unital ring A, and ring homomorphisms $\varphi : R \to A$ such that $\varphi(s) \in A^\times$ for every $s \in S$, there is a unique homomorphism $\tilde{\varphi} : S^{-1}R \to A$ such that $\tilde{\varphi} \circ \iota = \varphi$.

$$\begin{tikzcd}
S^{-1}R \arrow[dr, dotted, "\tilde{\varphi}"] & \\
R \arrow[u, "\iota"] \arrow[r, "\varphi"'] & A.
\end{tikzcd}$$

Deduce that there is a bijection:

$$\{\text{Homomorphisms } \varphi : R \to A \text{ such that elements of } S \text{ map to } A^\times\}$$

$$\Updownarrow$$

$$\{\text{(Unital) homomorphisms } \tilde{\varphi} : S^{-1}R \to A\}.$$

Proof. We first verify the following fact.

Lemma 1. *Let $\psi : T_1 \to T_2$ be a unital ring homomorphism (i.e. that takes 1 to 1). Then if $t \in T_1^\times$, $\varphi(t) \in T_2^\times$ and $\varphi(t^{-1}) = \varphi(t)^{-1}$.*

Proof. This is clear because $\varphi(t)\varphi(t^{-1}) = \varphi(tt^{-1}) = \varphi(1) = 1$, and varifying that it is also a left inverse is identical. $\square$

Fix some $\varphi : R \to A$ such that $\varphi(s) \in A^\times$ for every $s \in S$. Notice that then φ is unital, since it must take 1 to a unit, and therefore a nonzero divisor, which by HW11 1(d) means it must take 1 to 1. We build a homomorphism $\tilde{\varphi} : S^{-1}R \to A$ by the rule $\tilde{\varphi}\left(\frac{a}{b}\right) = \varphi(a)\varphi(b)^{-1}$ (where $\varphi(b)$ has an inverse in A since $b \in S$).

$\tilde{\varphi}$ is well defined: Suppose $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, so that there is some $t \in S$ so that $t(a_1b_2 - a_2b_1) = -$. Applying φ gives

$$\varphi(t)(\varphi(a_1)\varphi(b_2) - \varphi(a_2)\varphi(b_1)) = 0.$$

As $\varphi(t)$ is a unit, we can multiply by itse inverse and deduce that $\varphi(a_1)\varphi(b_2) = \varphi(a_2)\varphi(b_1)$. Multiplying on both sides by the inverses of $\varphi(b_1)$ and $\varphi(b_2)$ gives:

$$\tilde{\varphi}\left(\frac{a_1}{b_1}\right) = \varphi(a_1)\varphi(b_1)^{-1} = \varphi(a_2)\varphi(b_2)^{-1}\tilde{\varphi}\left(\frac{a_2}{b_2}\right).$$

$\tilde{\varphi}$ is a homomorphism: We check directly:

$$\begin{aligned}
\tilde{\varphi}\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}\right) &= \tilde{\varphi}\left(\frac{a_1b_2+a_2b_1}{b_1b_2}\right)\\
&= \varphi(a_1b_2+a_2b_1)\varphi(b_1b_2)^{-1}\\
&= (\varphi(a_1)\varphi(b_2)+\varphi(a_2)\varphi(b_1))\varphi(b_1)^{-1}\varphi(b_2)^{-1}\\
&= \varphi(a_1)\varphi(b_1)^{-1}+\varphi(a_2)\varphi(b_2)^{-1}\\
&= \tilde{\varphi}\left(\frac{a_1}{b_1}\right)+\tilde{\varphi}\left(\frac{a_2}{b_2}\right).
\end{aligned}$$

$$\begin{aligned}
\tilde{\varphi}\left(\frac{a_1}{b_1}\frac{a_2}{b_2}\right) &= \tilde{\varphi}\left(\frac{a_1a_2}{b_1b_2}\right)\\
&= \varphi(a_1a_2)\varphi(b_1b_2)^{-1}\\
&= \varphi(a_1)\varphi(b_1)^{-1}\varphi(a_2)\varphi(b_2)^{-1}\\
&= \tilde{\varphi}\left(\frac{a_1}{b_1}\right)\tilde{\varphi}\left(\frac{a_2}{b_2}\right).
\end{aligned}$$

Now we prove the bijection. Indeed, given some $\varphi : R \to A$ we get a unique $\tilde{\varphi}$ applying the construction we just discussed. Conversely, given some $\tilde{\varphi}$ we let φ be the compositon:

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[rr, bend left, "\varphi"] & S^{-1}R \arrow[r, "\tilde{\varphi}"] & A. \end{tikzcd}$$

We must show that φ takes elements of S to units in A. Fix $t \in S$. Then $\iota(t)$ is a unit in $S^{-1}R$. Since $\tilde{\varphi}$ is unital, Lemma 1 implies that $\tilde{\varphi}\iota(t)$ is a unit in A, but this is just $\varphi(t)$. These two constructions are clearly inverses two eachother. Indeed, starting at the top we have $\varphi \mapsto \tilde{\varphi} \mapsto \iota \circ \tilde{\varphi} = \varphi$, so that going down and back up is the identity. Conversely, we have $\tilde{\varphi} \mapsto \tilde{\varphi} \circ \iota \mapsto \widetilde{\tilde{\varphi} \circ \iota}$, but the latter has to be $\tilde{\varphi}$ due to the uniqueness of the contructino of $\tilde{\varphi}$. ☐

(e) Let $r \in R$ be nonzero and consider the multiplicative set $S = \{1, r, r^2, r^3, \cdots\}$. Define $R[1/r] := S^{-1}R$. Show that $R[1/r] = 0$ if and only if r is nilpotent.

Proof. Suppose $R[1/r] = 0$. Then for all $a \in R$, we have $\frac{a}{1} = \frac{0}{1}$. In particular there is some $t \in S$ such that $t(a \cdot 1 - 1 \cdot 0) = ta = 0$. Since each $t = r^m$ for some m, this means that for all a there is some naturan number n such that $r^n a = 0$. In particular, letting $a = 0$, we see that there is an n such that $r^n 1 = r^n = 0$, so that r is nilotent. Conversely, suppose r is nilpotent. Then $r^n = 0$ for some n. Therefore any $\frac{a}{b} = \frac{0}{1}$. Indeed, $r^n(a \cdot 1 - b \cdot 0) = 0$ because $r^n = 0$ to begin with. ☐

3. In this exercise we calculate the intersection of all the prime ideals in a commutative unital ring R.

   (a) Show that the element 0 is contained in every ideal of R.

   Proof. Let I be an ideal. It must be nonempty, so fix some $r \in I$. Then $0 = 0 \cdot r \in I$ by the ideal property. □

   (b) Let r be a nilpotent element of R. Show that r is contained in every prime ideal of R.

   Proof. We first prove the following claim: if $\mathfrak{p}$ is a prime ideal and $f^n \in \mathfrak{p}$, then $f \in p$. We proceed by induction on n. The base case is $n = 1$ and is trivial. For the general case, suppose $f^n = f \cdot f^{n-1} \in \mathfrak{p}$. Then by primality either $f \in \mathfrak{p}$ (in which case we win), or else $f^{n-1} \in \mathfrak{p}$. But then by induction, $f^{n-1} \in \mathfrak{p}$ implies $f \in \mathfrak{p}$, so we win. Now suppose $r \in R$ is nilpotent. Then for any prime ideal $\mathfrak{p}$, $r^n = 0 \in \mathfrak{p}$ by part (a). But now primality implies $r \in \mathfrak{p}$ as desired. □

   (c) Conversely, suppose r is not nilpotent. Show that there is some prime ideal not containing r. Deduce that:

   $$\mathfrak{N}(R) = \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}.$$

   (Hint: To find such a prime ideal, try applying 1(a) and 2(e) to the map $\iota : R \to R[1/r]$.)

   Proof. We consider the map $\iota : R \to R[1/r]$. Since $R[1/r]$ is a unital ring, and is nonzero by 2(e), there is some maximal ideal $\mathfrak{m} \in R[1/r]$. Since maximal ideals are prime, 1(a) shows that $\mathfrak{p} := \iota^{-1}(\mathfrak{m})$ is a prime ideal of R. To conclude we must show that $r \notin \mathfrak{p}$. But if it were, then $\iota(r) \in \mathfrak{m}$. Then $\mathfrak{m}$ contains a unit, so that by HW11 Problem 5(c), we have that $\mathfrak{m} = R[1/r]$, contradicting that $\mathfrak{m}$ is maximal (since maximal ideals are proper). □

   (d) Deduce that the intersection of all the prime ideals in an integral domain is the 0 ideal.

   Proof. HW10 Problem 3(a) implies that nonzero nilpotents are zero divisors. Therefore in an integral domain R we have $\mathfrak{N}(R) = \{0\}$. Then the result is an immediate consequence of part (c). □

   (e) Suppose that r is in the intersection of all the prime ideals of R. Show that $1 - ry \in R^\times$ for every $y \in R$. (We will see below that the converse is not true in general, but that we can characterize all elements satisfying this property).

   Proof. By part (c) we see that r is nilpotent, and so $-ry$ is nilpotent as well (by HW10 Problem 3(d)). But then $1 + (-ry)$ is a unit by HW10 Problem 3(e). □

4. In this exercise we calculate the intersection of all the maximal ideals in a commutative unital ring R. Given a ring R, we define the *Jacobson radical* of R to be:

   $$\mathfrak{J}(R) = \bigcap_{\mathfrak{m} \subseteq R \text{ maximal}} \mathfrak{m}.$$

(a) Show that $\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$.

Proof. A nilpotent element x is in every prime ideal (by 3(b)). But every maximal is prime, so in particular x is in every maximal ideal. $\square$

(b) Show that an element $r \in R$ is a unit if and only if it is not contained in any maximal ideal.

Proof. Let r be a unit. If r is contained in some maximal ideal $\mathfrak{m}$, that ideal must be all of R by HW11 Problem 5(c). But maximal ideals are proper, so this cannot be. Therefore units aren't contained in maximal ideals. Conversely, suppose r is not in any maximal ideal. If (r) were a proper ideal, it would have to be contined in some maximal ideal, so this says that (r) isn't proper, that is, $(r) = R$. By HW11 Problem 5(a) we may conclude that r is a unit. $\square$

(c) Suppose $\mathfrak{m}$ is a maximal ideal and $r \in R \setminus \mathfrak{m}$. Compute the ideal $(\mathfrak{m}, r)$ generated by $\mathfrak{m}$ and r.

Proof. Notice that $\mathfrak{m} \subseteq (\mathfrak{m}, r) \subseteq R$. Since $r \in (\mathfrak{m}, r) \setminus \mathfrak{m}$, we see that $\mathfrak{m} \neq (\mathfrak{m}, r)$. By the maximality of $\mathfrak{m}$, we conclude that $(\mathfrak{m}, r) = R$. $\square$

(d) Prove that the condition from 3(e) actually characterizes elements in the *Jacobson Radical*! That is, prove that $r \in \mathfrak{J}(R)$ if and only if $1 - ry \in R^\times$ for every $y \in R$. (Parts (b) and (c) might help!)

Proof. Suppose $r \in \mathfrak{J}(R)$, then so is ry. If $1 - ry$ is not a unit, then $1 - ry \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, but so is ry (since it is in every maximal ideal), so that $1 - ry + ry = 1 \in \mathfrak{m}$, implying that $\mathfrak{m} = R$, contradicting that it is a maximal ideal. Therefore $1 - ry$ must be a unit.

Conversely, if $r \notin \mathfrak{J}(R)$ then $r \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. But then $(r, \mathfrak{m}) = R$, Therefore, there is some $y \in Y$ and $m \in \mathfrak{m}$ so that $ry + m = 1$. In particular, $1 - ry \in \mathfrak{m}$ and therefore is not a unit. $\square$