

Homework Assignment 10

Due Friday, April 16

1. Let R be a ring. Recall that for $a \in R$ we denote the *additive* identity of a by $-a$. Establish the following identities.
 - (a) $(-a)b = a(-b) = -ab$
 - (b) $(-a)(-b) = ab$
 - (c) If $1 \in R$ then $(-1)a = -a$.
 - (d) Suppose R is an integral domain. Show that if $a^2 = 1$ then $a = \pm 1$.
2. Let R be a ring with $1 \neq 0$.
 - (a) Let $R^\times \subseteq R$ be the set of units of R . Show that R^\times is a group under the multiplication operation of R .
 - (b) Suppose that $a \in R$ is a zero divisor. Show that $a \notin R^\times$.
 - (c) Suppose R is a subring of some ring S . Show that if $a \in R^\times$ then $a \in S^\times$. Give an example to show the converse is false.
3. Let R be a commutative ring. An element $r \in R$ is called *nilpotent* if there exists a positive n such that $r^n = 0$. A commutative ring is called *reduced* if it has no nilpotent elements.
 - (a) Show that a nilpotent element of a ring is either 0 or a zero divisor.
 - (b) Give an example of a ring with a nonzero nilpotent element.
 - (c) Show that the sum of nilpotent elements is nilpotent.
 - (d) Suppose r is nilpotent. Show that rx is nilpotent for all $x \in R$. (Note, in future terminology, (c) and (d) prove that the set of nilpotent elements is an *ideal* of R , which we will call the *nilradical*).
 - (e) Suppose R is a commutative ring with $1 \neq 0$, and suppose $r \in R$ is nilpotent. Show that $1 + r \in R^\times$.
4.
 - (a) Let $\{S_i \subseteq R\}$ be a nonempty collection of subrings of R . Show that $\bigcap_i S_i$ is a subring of R .
 - (b) Suppose S is a subring of R , and R is a subring of T . Show that S is a subring of T .
5. For a ring R , define the *center* of R to be:

$$Z(R) = \{r \in R \mid ra = ar \text{ for all } a \in R\}.$$

- (a) Show that $Z(R)$ is a subring of R .
- (b) Suppose R has $1 \neq 0$. Show that $R^\times \cap Z(R) \subseteq Z(R^\times)$. (The converse is *not true* in general, but I don't consider this to be obvious. Perhaps we will see an example later).
- (c) Show that the center of a division ring is a field.
- (d) Let \mathbb{H} be Hamilton's quaternions (defined in Lecture 21 or [DF] Example 5 on Page 224). Compute $Z(\mathbb{H})$. (Notice that \mathbb{H} contains a copy of \mathbb{C} , is this the center?)

6. Let R be ring, and X any set. Define

$$\text{Maps}(X, R) = \{f : X \rightarrow R \mid f \text{ is a function}\}.$$

Define binary operations $+$ and \times as follows.

$$(f + g)(x) = f(x) + g(x) \quad (f \times g)(x) = f(x)g(x).$$

- (a) Show that $\text{Maps}(X, R)$ is a ring.
 - (b) Suppose R is commutative, show that $\text{Maps}(X, R)$ is too.
 - (c) Suppose R is unital, show that $\text{Maps}(X, R)$ is too.
 - (d) Suppose R is reduced (defined in Problem 3), show that $\text{Maps}(X, R)$ is too.
 - (e) Give an example to show that even if R is a field, $\text{Maps}(X, R)$ need not be.
 - (f) Give an example to show that even if R is an integral domain, $\text{Maps}(X, R)$ need not be.
7. We now develop an example of rings that appear along the intersection of the algebraic and analytic theory (for example in *functional analysis*). You may use without proof the following facts from elementary calculus: **(1)** If f, g are continuous so are their sum and product. **(2)** If f, g are differentiable then they are continuous and:

$$(f + g)' = f' + g' \quad (fg)' = f'g + fg'$$

- (a) Let \mathcal{P} be a property of maps from $X \rightarrow R$, and let

$$\text{Maps}_{\mathcal{P}}(X, R) = \{f : X \rightarrow R \mid f \text{ has property } \mathcal{P}\}.$$

Suppose that if f and g have property \mathcal{P} , then so do $f - g$ and $f \times g$. Show that $\text{Maps}_{\mathcal{P}}(X, R)$ is a subring of $\text{Maps}(X, R)$.

- (b) Let $X = R = \mathbb{R}$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ have property \mathcal{C}^0 if f is continuous, and define $C^0(\mathbb{R}) = \text{Maps}_{\mathcal{C}^0}(\mathbb{R}, \mathbb{R})$ to be the set of continuous functions from \mathbb{R} to \mathbb{R} . Use part (a) to show that $C^0(\mathbb{R})$ is a subring of $\text{Maps}(\mathbb{R}, \mathbb{R})$.
- (c) For each $n > 0$ let $f : \mathbb{R} \rightarrow \mathbb{R}$ have property \mathcal{C}^n if f has a derivative everywhere, and df/dx has property \mathcal{C}^{n-1} . (So for example, f is \mathcal{C}^1 if it is differentiable and its derivative is continuous). Show by induction on n that $C^n(\mathbb{R}) = \text{Maps}_{\mathcal{C}^n}(\mathbb{R}, \mathbb{R})$ is a subring of $C^{n-1}(\mathbb{R})$.
- (d) A function $f : \mathbb{R} \rightarrow \mathbb{R}$ has property \mathcal{C}^∞ if for each positive n the n 'th derivative of f exists and is continuous. (Such a function is also often called *smooth*). Show that $C^\infty(\mathbb{R}) = \text{Maps}_{\mathcal{C}^\infty}(\mathbb{R}, \mathbb{R})$ is a subring of $C^n(\mathbb{R})$ for each \mathbb{R} . (Hint: rather than prove this directly, you could use (4)).

8. Let A be an abelian group (written additively). Define the *endomorphism ring* of A as follows:

$$\text{End}(A) = \{f : A \rightarrow A \mid f \text{ is a homomorphism}\}.$$

Give $\text{End}(A)$ 2 binary operations $+$ and \times as follows:

$$(f + g)(a) = f(a) + g(a) \quad (f \times g)(a) = f(g(a)).$$

- (a) Prove that $\text{End}(A)$ is a ring.
- (b) Prove that $(\text{End}(A))^\times \cong \text{Aut}(A)$.
- (c) Let E be an elementary abelian p -group of order p^n . Show that $\text{End}(E) \cong M_n(\mathbb{F}_p)$ (You may use that $n \times n$ matrices over a field F correspond to linear maps $F^n \rightarrow F^n$. Compare to HW7 Problem 5).