Homework Assignment 11

Due Friday, April 22

This assignment will fill in many details from lecture, and do a few hands on classifications. To begin we will confirm that the semidirect product is indeed a group. First recall the definition.

Definition 1. *Let H, K be groups, and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism. Denote the induced action of K on H by:*

$$k \cdot h = \varphi(k)(h).$$

The semidirect product of H and K with respect to φ is the set $H \rtimes K = \{(h, k) : h \in H, k \in K\}$, where multiplication is defined by the rule:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1k_2).$$

1. Let's make sure that $H \rtimes K$ is a group.
   (a) Show that $(1, 1) \in H \rtimes K$ is the identity. (Remember you have to check both sides).
   (b) Show that $(h, k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$. (As above, you have to check both sides).
   (c) Prove that multiplication is associative.
2. Consider again the setup of Definition 1. Let's prove some basic properties about $G = H \rtimes K$.
   (a) Show that the subset $\{(h, 1) : h \in H\} \subseteq G$ is a subgroup isomorphic to H. Similarly, show that $\{(1, k) : k \in K\} \subseteq G$ is a subgroup isomorphic to K. In what follows we identify H and K with these subgroups, and write $H, K \leq G$.
   (b) Prove that $H \cap K = \{1_G\}$.
   (c) Show that $H \trianglelefteq G$ and $G/H \cong K$.

In HW 10 Problem 5 we proved the *Recognition Theorem for Direct Products* (HW10 Theorem 3). There is an analogous result for semidirect products, and in fact you already did most of the work. Let's state the result.

Theorem 2 (Recognition Theorem for Semidirect Products)**.** *Suppose G is a group and $H, K \leq G$ are subgroups. Suppose that $H \trianglelefteq G$ is normal, and that $H \cap K = 1$. Then*

$$HK \cong H \rtimes_\varphi K,$$

where $\varphi : K \to \mathrm{Aut}(H)$ corresponds to the action of K on H by conjugation (in G). In particular, if $HK = G$ then $G \cong H \rtimes_\varphi K$.

3. Prove Theorem 2 by showing that function $H \rtimes_\varphi K \to HK$ defined by the rule $(h, k) \mapsto hk$ is an isomorphim. (*Note:* Bijectivity should follow from HW10 Problem 5(a), so the main verification is that it is a homomorphism).
4. A lot of studying semidirect products comes down to enumerating and classifying homomorphisms.
   (a) Show that giving a homomorphism $Z_n \to G$ is the same as selecting an element $g \in G$ with $|g|$ dividing n. That is, give a bijection between the following sets:

$$\left\{ \begin{array}{c} \text{Homomorphisms} \\ Z_n \to G \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Elements } g \in G \\ \text{where } |g| \text{ divide } n \end{array} \right\}$$

(b) If p is prime show that giving a *nontrivial* map $Z_p \to G$ is the same as choosing an element of order p in G. (Note: the trivial map is the one that sends every element to the identity of G).

(c) Show that giving a homomorphism $Z_{n_1} \times \cdots \times Z_{n_r} \to G$ is the same as chosing elements $g_1, \cdots, g_r \in G$ such that all the g_i commute with eachother and each $|g_i|$ divides n_i.

(d) Suppose G is abelian and p is prime. Describe the set of homomorphisms $Z_p \times Z_p \to G$ as a subset of $G \times G$.

Any homomorphism $\varphi : K \to \operatorname{Aut} H$ allows us to build a semidirect product $H \rtimes_\varphi K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used a special case of the following lemma.

Lemma 3. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \operatorname{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:*

$$\begin{tikzcd} K \arrow[d, "\gamma"'] \arrow[dr, "\varphi"] & \\ K \arrow[r, "\psi"'] & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 4. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitely, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_\alpha : \beta \mapsto \alpha\beta\alpha^{-1}$, and suppose that $\psi = \sigma_\alpha \circ \varphi$:*

$$\begin{tikzcd} & \operatorname{Aut} H \arrow[d, "\sigma_\alpha"] \\ K \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

5. Lemmas 3 and 4 say that if we alter φ by an automorphism of K, or an inner automorphism of $\operatorname{Aut} H$, (or both), we don't change the semidirect products. Let's prove this.

   (a) Consider the setup of Lemma 3. Show that the map:

$$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h, k) & \mapsto & (h, \gamma(k)) \end{array}$$

   is an isomorphism, thereby proving the lemma.

(b) Consider the setup of Lemma 4. Show that the map:

$$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (\alpha(h), k) \end{array}$$

is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_α is an automorphism of $\operatorname{Aut} H$, given by conjugation by α. In unweildy notation, this says $\sigma_\alpha \in \operatorname{Aut}(\operatorname{Aut} H)$.)

(c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\ K \arrow[r, "\psi"] & \operatorname{Aut} H. \end{tikzcd}$$

That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$. (Hint: This should follow formally from Lemmas 3 and 4, so you shouldn't have to do any lengthy computations).

6. We've seen 5 groups of order 12: $Z_{12}, Z_6 \times Z_2, D_{12}, A_4$, and a nontrivial semidirect product $Z_3 \rtimes Z_4$ where the generator of Z_4 acts on Z_3 by inverting elements. Let's prove this is all of them!

(a) Let G be a group of order 12. Show that if $G \ncong A_4$, then $G \cong Q \rtimes P$ where P is a Sylow 2-subgroup and Q is a Sylow 3-subgroup. (*Hint*: **(Sylow 3)** and the Theorem 2 should help).

(b) Show that there is only one abelian and one nonabelian semidirect product $Z_3 \rtimes Z_4$ up to isomorphism.

(c) Show that there is only one abelian and one nonabelian semidirect prodcut $Z_3 \rtimes (Z_2 \times Z_2)$ up to isomorphism. (You might need Lemma 3).

(d) Put together parts (a)-(c) to deduce that there are exactly 5 groups of order 12 up to isomorphism. Of the semidirect products classified in parts (b) and (c), which one corresponds to D_{12}?

7. In this problem we classify all groups of order 75 up to isomorphism. (There should be 3 total).

(a) List all the abelian groups of order 75 using the fundamental theorem of finite abelian groups.

(b) Prove that a group of order 75 is isomorphic to $P \rtimes Q$ where P is a Sylow 5-subgroup and Q is a Sylow 3-subgroup.

(c) Prove that if a group of order 75 has a *cyclic* Sylow 5-subgroup, then it is abelian.

(d) Show that there is a unique nonabelian group of order 75 up to isomorphism. (*Hint:* Show that 3 is a maximal 3-divisor of $|GL_2(\mathbb{F}_5)|$. Then use Sylow's theorems and 5(c).)