Homework Assignment 5

Due Friday, February 25

In this assignment we answer the following question:

Question 1. *Let G be a group, and $H \leq G$ a subgroup. When is G/H a group?*

More specifically we are asking when the set of a cosets a group under the mutliplication rule given by $(g_1H)(g_2H) = g_1g_2H$. We want a way to answer the question *intrinsically* to G and H. For this we recall the following definition from the 2/22 lecture.

Definition 2. *Let G be a group and $H \leq G$ a subgroup. For $g \in G$ the* ***conjugate of** H **by** g* *is the set:*

$$gHg^{-1} = \{ghg^{-1} \ : \ h \in H\}.$$

We say that H is a ***normal subgroup*** *if for every $g \in G$ we have $gHg^{-1} = H$. If $H \leq G$ is a normal subgroup, we write $H \trianglelefteq G$.*

An intrinsic answer to Question 1 is given by the following theorem.

Theorem 3. *Let G be a group and $H \leq G$ a subgroup. The following are equivalent.*

(i) $H \trianglelefteq G$

(ii) G/H *is a group under the rule* $(g_1H)(g_2H) = g_1g_2H$.

(iii) H *is the kernel of a group homomorphism with domain* G.

1. There is only one goal in this assignment: to prove Theorem 3. To achieve this goal, we will prove $(i) \implies (ii) \implies (iii) \implies (i)$.

   (a) Suppose $H \trianglelefteq G$. Show that G/H is a group under the rule $(g_1H)(g_2H) = g_1g_2H$. This shows that $(i) \implies (ii)$.

   Proof. The first step is to show that this multiplication law is even well defined. That is, suppose $g_1H = \hat{g}_1H$ and $g_2H = \hat{g}_2H$. We must show that the cosets g_1g_2H and $\hat{g}_1\hat{g}_2H$ agree. Let $i = 1$ or 2 and notice that $g_iH = \hat{g}_iH$ means in particular that $\hat{g}_i \in g_iH$, which implies that $\hat{g}_i = g_ih_i$ for some $h_i \in H$. Therefore $\hat{g}_1\hat{g}_2 = g_1h_1g_2h_2$. Since H is normal, $g_2^{-1}h_1g_2 = h \in H$, so that in particular $h_1g_2 = g_2h$. Therefore we can substitute:

   $$\hat{g}_1\hat{g}_2 = g_1h_1g_2h_2 = g_1g_2(hh_2) \in g_1g_2H.$$

   Since $\hat{g}_1\hat{g}_2$ is in the coset generated by g_1g_2, we may conclude that they generate the same equivalence class (or coset), so that $g_1g_2H = \hat{g}_1\hat{g}_2H$ as desired.

   Now that multiplication is well defined, the group axioms follow easily. Indeed, 1_GH is the identity element since multiplication is ultimately happening in G, and associativity is inherited from G for the same reason, as is the fact that $(gH)^{-1} = g^{-1}H$. $\square$

   (b) Suppose that G/H is a group under the rule $(g_1H)(g_2H) = g_1g_2H$. Produce a group homomorphism from G to some group whose kernel is H. This shows that $(ii) \implies (iii)$. (Note: we essentially gave this argument in the 2/22 lecture, but do include a full proof here as well).

Proof. We define a function $\pi : G \to G/H$ by the rule $\pi(g) = gH$. (This is often called the *natural projection* or the *reduction mod H map.*) It is easy to see it is a homomorphism, since:

$$\pi(x)\pi(y) = (xH)(yH) = xyH = \pi(xy).$$

The kernel of π consists of $g \in G$ such that the coset $gH = 1H$, that is, such that $g \in 1H = H$. Therefore $\ker \pi = H$, as desired. $\square$

(c) Let $\varphi : G \to G'$ be a group homomorphism with kernel H. Show that $H \trianglelefteq G$. This proves $(iii) \implies (i)$, thereby completing the proof of Theorem 3.

Proof. We showed in class that to show a group is normal, it is enough to show that $gHg^{-1} \subseteq H$ for every $g \in G$. Therefore, fixing $h \in H$ and $g \in G$, we must show that $ghg^{-1} \in H$. Applying φ and using that $h \in H = \ker \varphi$ we compute:

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = 1.$$

Therefore $ghg^{-1} \in \ker \varphi = H$. $\square$