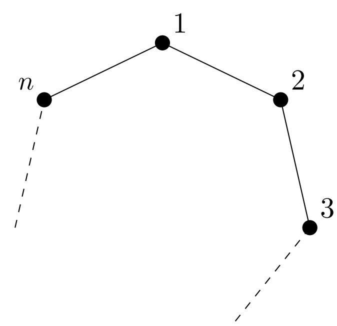
Homework Assignment 3

Due Friday, February 11

1. In class we developed the theory of the group D_{12} of rigid symmetries of the regular hexagon (on bCourses: 'Lecture 4' from 55:00 until the end). In fact, everything we developed should go through almost exactly the same way for D_{2n}: the rigid symmetries of regular n-sided polygon, pictured below:



   (a) Explain why D_{2n} is a group under composition of symmetries.

   (b) Show that there are exactly $2n$ rigid symmetries of the regular n-gon.

   (c) Let r be the rotation by $2\pi/n$ in the clockwise direction, and s be the reflection along the vertical line going through the vertex labelled '1'. Compute the elements of D_{2n} in terms of r and s in the following steps:

      i. Compute the order of r and s (justifying your answers).

      ii. Let $i_1, i_2 \in \{0, 1\}$ and $j_1, j_2 \in \{0, 1, \cdots, n-1\}$. Show that:
$$s^{i_1} r^{j_1} = s^{i_2} r^{j_2} \text{ if and only if } i_1 = i_2 \text{ and } j_1 = j_2.$$
      (Hint: You could first show $s \neq r^i$ for any i using geometry. The rest of the cases should follow from this and part (i) by using cancellation and 8(b).)

      iii. Conclude that $D_{2n} = \{s^i r^j | i = 0, 1 \text{ and } j = 0, 1, \cdots, n-1\}$. In particular, r and s generate D_{2n}.

   (d) Show that $rs = sr^{-1}$. Deduce inductively from this that $r^n s = sr^{-n}$ for all n.

   We now completely understand the algebraic structure of D_{2n}. In particular, we know what every element looks like (in terms of r and s) by (c), and we know how to multiply any two elements using the relation in part (d). We summarize this by saying that D_{2n} has the following presentation:
$$D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

   (e) Use this presentation to give an algebraic proof that every element which is not a power of r has order 2.

2. The set S_3 has 6 elements. Compute the order and cycle decomposition of each element.

3. Some of the arguments in problem 1 used a connection between symmetries of polygons and permutations of the vertices. Let's make this explicit!

   (a) Describe an injective homomorphism from $\varphi : D_{2n} \to S_n$ (you may describe this in words, but make sure to justify injectivity).

   (b) In the map you described, what is the cycle decomposition of $\varphi(r)$ (where r is the generator corresponding to clockwise rotation of the n-gon by $2\pi/n$)?

   (c) Prove that $D_6 \cong S_3$.

4. No we important basic facts about group homomorphisms that we will use repeatedly throughout the course. Let G, H, K be groups, and let $\varphi : G \to H$ and $\psi : H \to K$ a homomorphisms.

   (a) Show that $\varphi(1_G) = 1_H$.

   (b) Show that $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.

   (c) Show that if $g \in G$ has finite order, then $|\varphi(g)|$ divides $|g|$.

   (d) Show that if φ is an isomorphism, then so is φ^{-1}.

   (e) Show that if φ is an isomorphism, $|\varphi(g)| = |g|$.

   (f) Show that the composition $\psi \circ \varphi : G \to K$ is a homomorphism.

   (g) Suppose φ and ψ are both isomorphisms. Show that the composition $\psi \circ \varphi$ is as well.

   (h) Conclude that the relation *is isomorphic to* is an equivalence relation on the set of all groups.

5. In this exercise we show that you can compute the order of a permutation from its cycle decomposition.

   (a) Let G be a group. Two elements $x, y \in G$ are called *commuting elements* if $xy = yx$. Show that if x and y are commuting elements, then $(xy)^n = x^n y^n$.

   (b) Give a counterexample to part (a) if the chosen elements do not commute.

   (c) Let $\sigma = (a_1, a_2, \cdots, a_r) \in S_n$ be an r-cycle. Show that $|\sigma| = r$.

   (d) Prove that the order of a permutation is the least common multiple of the lengths of the cycles in its cycle decomposition. (Hint: You may freely use that disjoint cycles are commuting elements. You may find it useful to establish that the product of nontrivial disjoint cycles is never 1).

6. We suggested in class that if A and B are sets of the same cardinality, then their permutation groups S_A and S_B (defined in HW2#5) are isomorphic. Let's prove it. To begin, fix a bijective function $\theta : A \to B$.

   (a) Let $f : A \to A$ be bijective. Show that $\theta \circ f \circ \theta^{-1} : B \to B$ is bijective. (Hint: what is its inverse?)

   (b) Part (a) allows us to construct the following function:

$$\begin{array}{ccc} S_A & \xrightarrow{\varphi} & S_B \\ f & \longmapsto & \theta \circ f \circ \theta^{-1}. \end{array}$$

   Show that φ is an isomorphism, thereby proving the result. (Note: There are two parts to this. You must show that φ is bijective, and that it is a homomorphism.)

   (c) Use (a) and (b) to conclude that if A be a finite set with n elements, then $S_A \cong S_n$.