Homework Assignment 3

Due Friday, February 14

In problems 1-4 we establish some important basic results about group homomorphisms. For all four problems we fix a homomorphism $\varphi : G \to H$.

1. (a) Show that $\varphi(1_G) = 1_H$.
   (b) Show that $\varphi(x^{-1}) = \varphi(x)^{-1}$ for all $x \in G$.
   (c) Show that if $g \in G$ has finite order, then $|\varphi(g)|$ divides $|g|$.
   (d) Show that if φ is an isomorphism, then so is φ^{-1}.
   (e) Conclude that if φ is an isomorphism, $|\varphi(g)| = |g|$.

2. Define the *kernel* of φ to be

$$\ker \varphi = \{g \in G : \varphi(g) = 1_H\}$$

   (a) Show that $\ker \varphi$ is a subgroup of G.
   (b) Show that φ is injective if and only if $\ker \varphi = \{1_G\}$.

3. More generally, for $h \in H$ define the *fiber over* h to be

$$\varphi^{-1}(h) = \{g \in G : \varphi(g) = h\}.$$

   (a) Show that $\ker \varphi = \varphi^{-1}(1)$
   (b) Show that the fiber over h is a subgroup if and only if $h = 1_H$.
   (c) Show that the *nonempty* fibers of φ form a partition of G. (In particular, if φ is surjective its fibers partition G.)
   (d) Show that all nonempty fibers have the same cardinality. (Hint: if $\varphi^{-1}(h)$ is nonempty, build a bijection between it and $\ker \varphi$)

4. Define the *image* of φ to be

$$\operatorname{im} \varphi = \{h \in H : h = \varphi(g) \text{ for some } g \in G\}.$$

   Show that $\operatorname{im} \varphi$ is a subgroup of H.

Recall that we defined the kernel of a group action in class. The following exercise shows that the kernel of a homomorphism and the kernel of a group action are related, justifying our terminology.

5. Let $G \times A \to A$ be an action of G on a set A. Let $\varphi : G \to \operatorname{Aut}(A)$ be the associated permutation representation. Show that the kernel of the group action is equal to $\ker \varphi$.

6. Describe an injective homomorphism from $\varphi : D_{2n} \to S_n$ (you may describe this in words). In the map you described, what is the cycle decomposition of $\varphi(r)$ (where as usual r is the generator corresponding to rotation of the n-gon by $2\pi/n$)?

7. The set S_3 has 6 elements. Compute the order and cycle decomposition of each element.