Linear Algebra

Math 217: Saint Lawrence University

Gabriel Dorfsman-Hopkins

GABRIEL DORFSMAN-HOPKINS
SAINT LAWRENCE UNIVERSITY
DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE

These are growing course notes developed for a first course in Linear Algebra at Saint Lawrence University in Canton, NY. They are written and maintained by Professor Gabriel Dorfsman-Hopkins. The notes have been typeset in LaTeXwith a template that was adapted from the Legrand Orange Book Template by Mathias Legrand (`legrand.mathias@gmail.com`) with modifications by Vel (`vel@latextemplates.com`).

First release, Spring 2023

Contents

1. Introduction

1.1 January 24, 2023

1.1.1 Functions

Almost any course in mathematics is centered around studying types of *functions*. For example, in *Calculus* we study the behavior of functions of a single variable, that is, functions whose input is a single real number and whose output is a single real number, looking especially closely at functions which are *continous* or *differentiable*.

■ Example 1.1 — Functions of a single variable. Consider the function

$$f(x) = 3x.$$

Its input is a real number, x, and the output is computed by multiplying the input by 3. To see what this function does to a real number, say, 11, we can compute:

$$f(11) = 3 \times 11 = 33.$$

Explicitly, f takes an input of eleven an *transforms it* into an output of 33. ■

■ Example 1.2 Consider the function:

$$g(x) = x^2 - 2x + 1.$$

What does this function do to the number 2? ■

The study of calculus looks closely at these functions of a single variable, establishing concepts like *derivatives* and *integrals*, and connecting them to many real world questions and situations. A shorthand that we will adopt to describe a function f of a single variable is the following

$$f : \mathbb{R} \to \mathbb{R}$$

This can be read aloud as *f is a function from $\mathbb{R}$ to $\mathbb{R}$*. It signifies that f takes a real number (on the left of the arrow), and runs it through the arrow to produce another real number (on the right of the arrow). *Note: The set before the arrow is called the* ***domain*** *of the function. It is also sometimes called the* ***source****. The set after the arrow is called the* ***co-domain****. It is sometimes also called the* ***target****.*

In *Mulltivariable Calculus* we develop similar ideas, **but the types of functions we study are different**. In particular, we allow for functions which take more than one real number as an input. Allowing for mutli-variable inputs allows calculus to be applied to our multi-dimensional world, and vastly expands the applications of derivatives, integrals, and related ideas.

■ Example 1.3 — Functions of 2 variables. In multivariable calculus you may encounter a function like:

$$f(x,y) = x - y.$$

It takes as input a *pair* of real numbers (x,y), and outputs their difference. For example, to see what the function does to the pair of number $(5,2)$ we can compute:

$$f(5,2) = 5 - 2 = 3.$$

In partiuclar, f will *transform* the pair of numbers $(5,2)$ into the single number 3. ■

■ Example 1.4 — Functions of 3 variables. Consider the function of 3 variables:

$$f(x,y,z) = xyz + 1.$$

What does this function do to the triple $(1,2,3)$? ■

The *arrow notation* of a function introduced above carries over here as well. For example, if f is a function of two variables, (whose input is 2 real numbers) we may write:

$$f : \mathbb{R}^2 \to \mathbb{R},$$

which we read as *f is a function from $\mathbb{R}^2$ to $\mathbb{R}$*. Here $\mathbb{R}^2$ denotes the collection of *pairs of real numbers*. Similarly, if g is a function of 3 variables (like in Example 1.4), we may write

$$g : \mathbb{R}^3 \to \mathbb{R}.$$

Notice that for each function we've describe so far, the output is *1-dimensional*. That is, we may have a function into which takes multiple real numbers as an input, but in each case the output is *a single real number*.[1] But just as allowing a multi-dimensional input massively expanded the scope of calculus, allowing functions to have a multidimensional output can be very useful as well.

■ Example 1.5 — Analyzing Ocean Currents. A group of oceanographers are measuring the movement of the water in the Atlantic, by studying where a collection of sensors start and end over the course of two weeks. They compile their data into a function C whose input is the GPS coordinates of a location in the Atlantic, and whose output of where the water at that location ends up 2 weeks later. For example,

$$C(40.47, -68.73) = (41.71, -64.07),$$

[1] You may recall that $\mathbb{R}$ can be thought of as a line, $\mathbb{R}^2$ as a plane, and $\mathbb{R}^3$ as 3-dimensional space. We will eventually adopt this notion of dimensionality, and explore it more carefully.

means that a drop of water whose GPS Coordinates are 40.47N 68.73W will move over the course of two weeks to the location 41.71N 64.07W. Observe that this is a function that takes as input two real numbers, and outputs *2 real numbers* as well! That is, both the input and the output are *2-dimensional*. In our arrow notation, we would write:

$$C : \mathbb{R}^2 \to \mathbb{R}^2.$$

TODO: put an image here!

■

■ **Example 1.6 — Casting Shadows.** Shadows are cast when a body in space blocks the sun from hitting the ground. If we'd like to study the shape of shadows mathematically, it is worth modelling shadows with a function, say S. Here:

$$S(\text{A point in space}) = \text{The spot on the ground where it casts a shadow.}$$

Modelling 3-dimensional space with $\mathbb{R}^3$ and the 2-dimensional ground with $\mathbb{R}^2$, this gives a function:

$$S : \mathbb{R}^3 \to \mathbb{R}^2.$$

In fact, this will be a *projection function*, a certain kind of *linear transformation* that we will study in TBA.

TODO: put an image here!

■

As we can see, functions with multivariable outputs are not hard to come up with, and model many different situations we would hope to study with mathematics. Let us begin by looking at a very special case:

1.1.2 Functions $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose you wanted to describe a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. How would you go about it? Both the input and output of f consist of pairs of numbers, so to be explicit with our notation, let's give the first $\mathbb{R}^2$ the coordinates (x, y), and the second $\mathbb{R}^2$ the coordinates (u, v). In particular, our function will look something like

$$f(x, y) = (u, v).$$

The function should be a rule so that, given a pair (x, y) of real numbers, we return with another pair of numbers, (u, v). In particular, we have to say what u is, and what v is. But each of these coordinates depend on both x and y, so in essense this is just *two functions* whose output is a real number:

$$u = u(x, y)$$

$$v = v(x, y).$$

■ **Slogan 1.1** To describe a function whose output is two real numbers, you can give 2 functions which output a single real number each.

Let's see how this works with an example.

■ **Example 1.7** Lets define a function $f:\mathbb{R}^2\to\mathbb{R}^2$ via the rule $f(x,y)=(u,v)$ where:

$$u = u(x,y) = xy+1,$$
$$v = v(x,y) = x+2y^2$$

The input of this function is a pair of numbers (x,y), and the output is *another* pair of number (u,v). So, for example, if we feed the function the pair $(-1,3)$, we can compute:

$$u = u(-1,3) = -1\times 3+1 = -3+1 = -2$$
$$v = v(-1,3) = -1+2\times 3^2 = -1+18 = 17.$$

Therefore, this function transforms the pair $(-1,3)$ to the pair $(-2,17)$:

$$f(-1,3) = (-2,17).$$

■

■ **Example 1.8** Define a function $g:\mathbb{R}^2\to\mathbb{R}^2$ which takes (x,y) to (u,v) via the rule

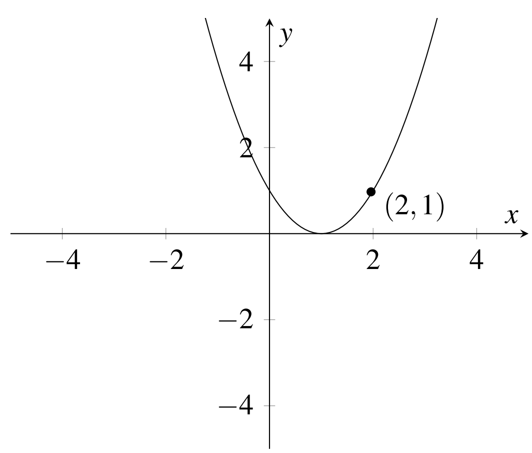
$$u = u(x,y) = 2x-2y,$$
$$v = v(x,y) = \frac{1}{2}x+y.$$

Where does g take the point $(1,1)$? ■

It is often useful to think about a function as something that *moves* the point (x,y) to the point (u,v), and to emphasize this intuition, we will often refer a function $\mathbb{R}^2\to\mathbb{R}^2$ as a *trnasformation of the plane*.

1.1.3 Visualizing Transformations of the Plane

How do we visualize these types of functions? Since these will be central objects of study, let's start by spending some time developing techniques for how to think about and imagine a function from $\mathbb{R}^2$ to itself. Recall that in calculus you often visualize functions $g:\mathbb{R}\to\mathbb{R}$ using their graphs in the xy-plane. Here the x axis plays the role of the domain, and the y-axis the role of the co-domain, and the graph is generally a curve consisting of the points $(x,g(x))$. For example, the graph of the function $g(x)=x^2-2x+1$ from Example 1.2 is below.



The fact that $f(2) = 1$ is captured by the fact that $(2,1)$ lies on the curve. A similar approach is used in multivariable functions, where now the domain is the entire xy-plane, and the co-domain is the z-axis. Then a function $f : \mathbb{R}^2 \to \mathbb{R}$ can be graphed in 3-dimensional space, coloring in the points $(x,y,f(x,y))$, generally giving rise to a surface in 3-dimensional space.

■ **Question 1.1** Can we take a similar approach to a function $f : \mathbb{R}^2 \to \mathbb{R}^2$? Why or why not?

Given the dimensional constraints, we have to come up with another way to represent a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. One way to do this is to get to the heart of what a function really does: *it transforms a point in $\mathbb{R}^2$ to another point in $\mathbb{R}^2$*. In particular, we can think about such a function as *something that transforms the plane*, moving the points of the plane around.

■ **Slogan 1.2** Think about a function $\mathbb{R}^2 \to \mathbb{R}^2$ as something that moves around the points on a single plane. The input (x,y) is where the point starts, and the output $(u,v) = f(x,y)$ is where the point ends.

In fact, this is exactly what the function from Example 1.5 does, it keeps track of of where a drop of water in the Atlantic moves over the course of two weeks!

■ **Example 1.9** Let's visualizae the function from Example 1.7, which was function $f(x,y) = (u,v)$ where:

$$u = u(x,y) = xy + 1,$$
$$v = v(x,y) = x + 2y^2$$

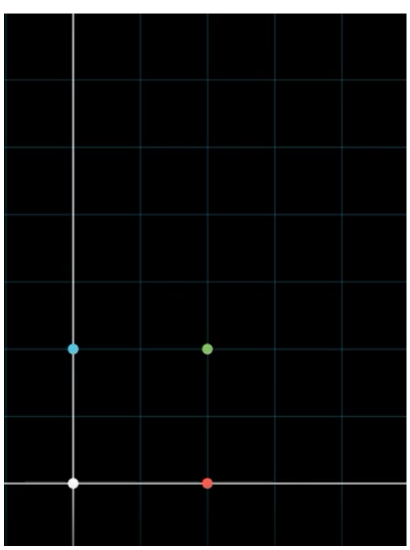
To get a sense of what kind of movement, let's keep track of what happens to a few points:

$$(0,0), (1,0), (0,1), (1,1).$$

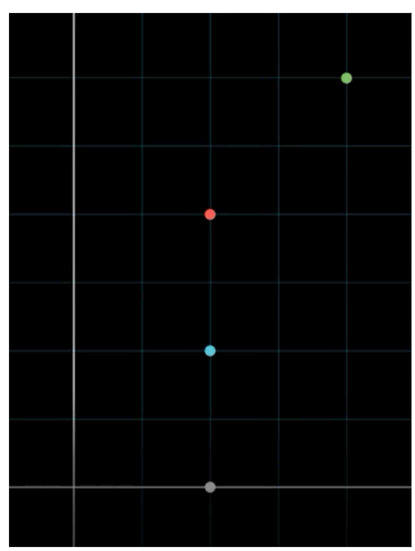
Using the formulas we can compute where f takes these points, just like in Example 1.7.

$$f(0,0) = (1,0), \quad f(1,0) = (1,2),$$
$$f(0,1) = (1,1), \quad f(1,1) = (2,3).$$

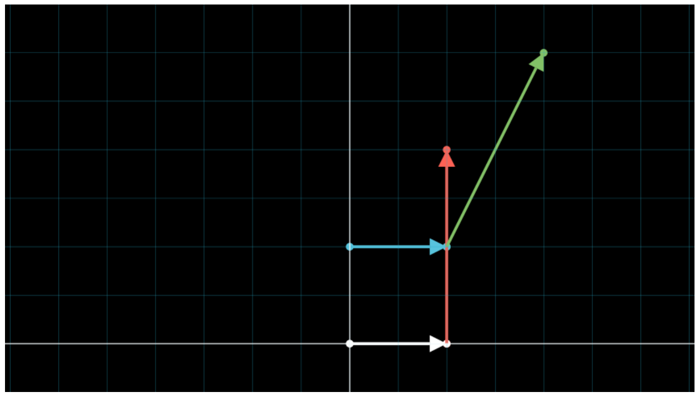
Instead of a single graph of the function, we can represent what f does with two pictures of the plane, a *before* shot and an *after* shot. On the left, we see the 4 points before applying f, and on the right, we see them after.



$\Longrightarrow$



The *movement* of the situation can be captured nicely by an animation linked below.[2] You can also emphasize that it is movement on a single page by using arrows that point from the start to the finish of the various points:



■

Exercise 1.1 — January 24th Checkin. Consider the transformation $L(x,y) = (u,v)$ of the plane $\mathbb{R}^2$, given by the following two equations:

$$u = u(x,y) = y$$

$$v = v(x,y) = -x.$$

On a single coordinate plane, draw what the function does to a number of points. Do this by plotting a point (x,y), its image $L(x,y)$, and connecting them with an arrow. Use a few sentences to describe what the transformation L is doing to the plane. This can be a *qualitative* description. *What does it look like is happening?* ■

[2] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan24_Quad.mp4`

1.2 January 26, 2023

Let's begin by recalling some of the techniques discussed last time to visualize a function $\mathbb{R}^2 \to \mathbb{R}^2$ as a *transformation* of the plane.

■ **Example 1.10** Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 1.8. In particular, it is given by the rule $g(x,y) = (u,v)$ where:

$$u = 2x - 2y \text{ and,}$$

$$v = \frac{1}{2}x + y.$$

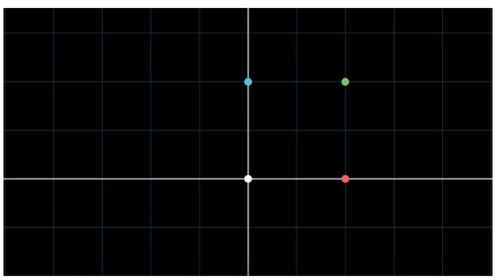
We compute compute where g takes the four points: $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. For example, $g(0,0)$, we may compute the u coordinate to be $2 \times 0 - 2 \times 0 = 0$ and the v coordinate to be $\frac{1}{2} \times 0 + 0 = 0$, so that $g(0,0) = 0$. Similar computations show that:

$$g(1,0) = (2,0.5)$$

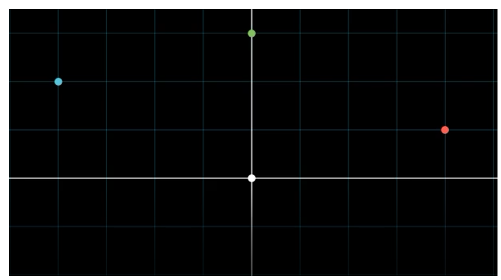
$$g(0,1) = (-2,1)$$

$$g(1,1) = (0,1.5).$$

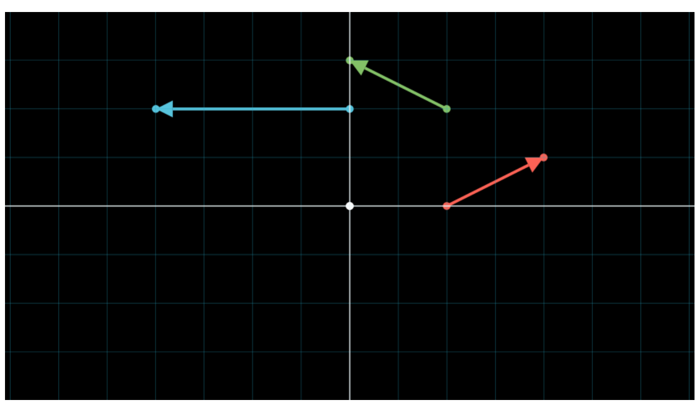
Plotting the points before and after applying g gives:



$\Longrightarrow$



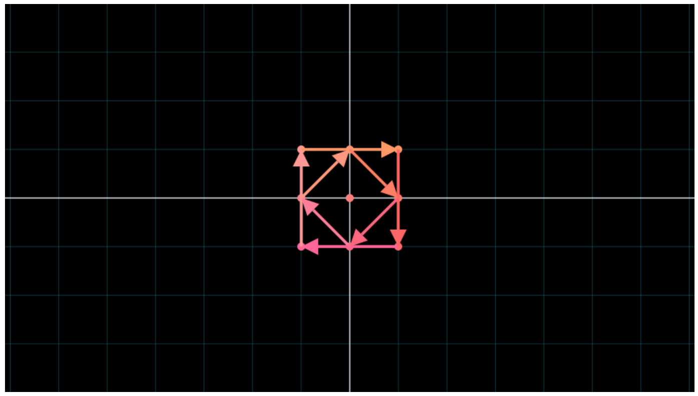
Plotting the before and after on the same plane, connecting (x,y) with $g(x,y)$ using arrows gives the following picture.



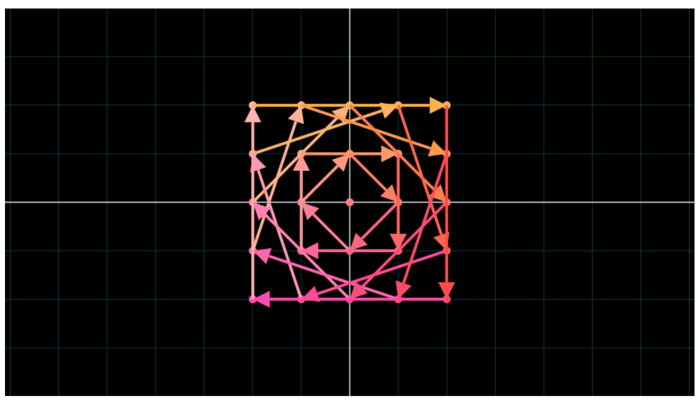
We should imagine this function something *moving around the points on the plane*, a perspective that is emphasized when animating the function. You find an animation of the moving points below.[3]. Try to give a qualitative description of what this function is doing to the plane. Plotting moer points may give a better picture. ■

[3] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26_Linear.mp4`

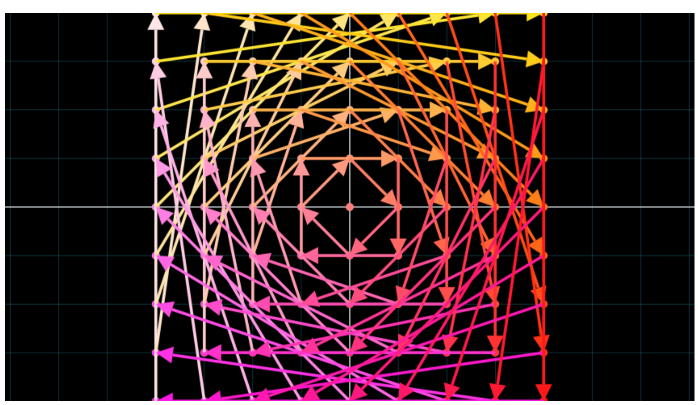
■ **Example 1.11** At the end of our last class, we did a similar exercise using the function $L(x,y) = (y,-x)$ (cf. Exercise 1.1). Let's draw a few pictures and see if we can arrive at a description of what is happening to the plane. First, we plot all the points whose x and y coordinate's are between -1 and 1, connecting the points before and after applying L with an arrow.



Can you begin to describe what L is doing to the plane? Let's throw in a few more points, now letting the coordinates range between -2 and 2.



As you can see, it appears that L is *rotating the plane clockwise!*. We include one more image now with coordinates ranging between -5 and 5.

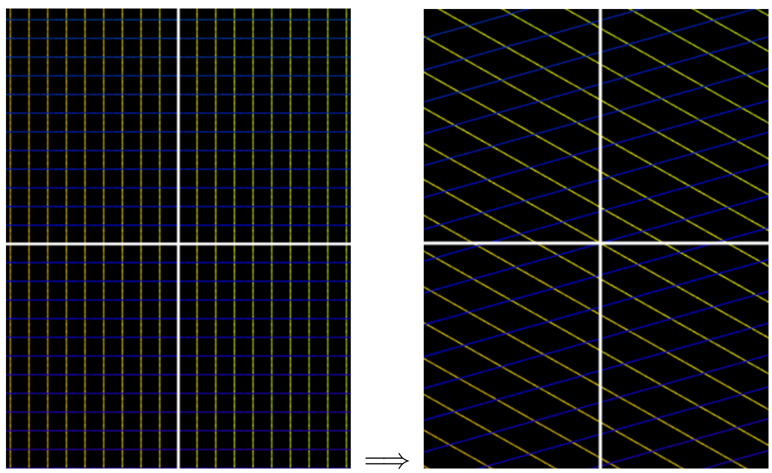


Althought the image is starting to get cluttered, this definitely appears to be a rotation, and indeed, replacing the arrows with an animation makes this clear (see the animation below[4]). ■

[4] `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26_Rotate.mp4`

To summarize, plotting where points go under a function can give a sense qualitative sense of how a function moves the plane. That said, Examples 1.9 and 1.10 suggest that only drawing where a few points go gives an incomplete picture. On the other hand, as we saw at the end of Example 1.11, if we to fill in more and more points, the image can start to get cluttered and it may become difficult to infer much from the picture. [5]. That being said, if you carefully pic which points to keep track of, you can get a nice sense of the *geometric* properties of a function. One way to do this, is by keeping track of what the function does to the *gridlines* of the plane.

■ **Example 1.12** To get a better picture of the function g from Examples 1.8 and 1.10, let's analyze what it does does to the gridlines of the plane.

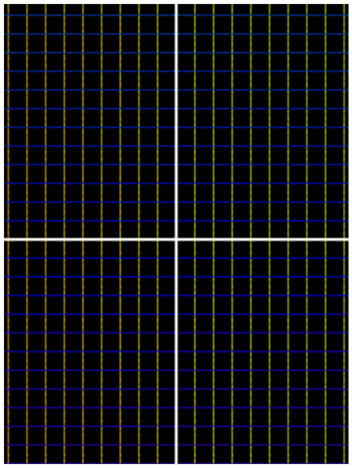
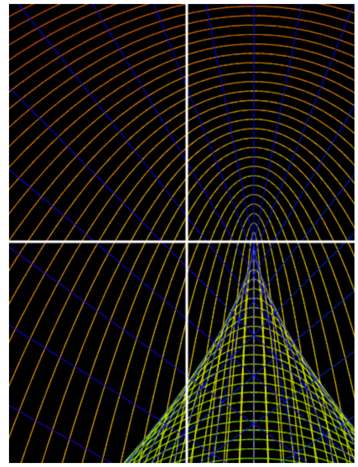


One can really get a sense for how g *moves* the plane by playing around with the tool linked below[6]. In particular, we see that it sort of *stretches* and *rotates* the plane, distorting it slightly but not too much. In this course we will develop a vocabulary to mathematically describe terms like *stretching the plane*, and ways to extract that information from the equations given in Exampel 1.8, but for now we're trying to get a qualitative sense of what's going on. ■

■ **Example 1.13** Let's also look at what the function f from Example 1.7 does to the gridlines of the plane.

[5] *Try this! For some functions you can actually get a nice picture! In fact, the situation in Example 1.11 is a particularly nice one. In general it will be much more complicated*

[6] Click the *linear* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26Gridlines/index.html`

 $\Longrightarrow$ 

The animation is actually quite nice to look at[7]. ■

It is fair to say that the function in Example 1.13 appears far more complicated than the one in Example 1.12. In fact, in some sense it is complicated in a way that puts it beyond the purview of *linear algebra*[8] For the context of linear algebra, we will have to restrict ourselves to functions more like that of Example 1.12, functions that we will call *linear transformations*. Before describing exactly what these are, it might be worth while to ponder the following question. Qualitative answers are always welcome!

■ **Question 1.2** What are some differences between what happens to the gridlines in the two examples on the previous page?

1.2.1 Linear Transformations of the Plane

One answer to Question 1.2 could be: *In example 1.8 the gridlines remain as lines after applying g, but in Example 1.7 the gridlines become curvy*. This is a good observation. Recall that lines played a special role in calculus. Not only where they the simplest functions, we used them to model more complicated functions locally, by taking *tangent lines*. We do something similar in multivariable calculus, modelling more complicated functions with linear ones by taking the *tangent plane*. Not only were these functions simple *geomterically* (being lines and planes), but they were also simple *algebracially*. For example, a line usually has the following equation:

$$f(x) = mx + b.$$

Above we highlighted the *linear term* in red, and the *constant term* in blue. Similarly, a plane had a simple equation as well:

$$h(x,y) = mx + ny + b,$$

where again the linear terms are highlighted in red, and the constant term in blue. Looking at the function $g(x,y) = (u,v)$ from Example 1.8, we see that the equations for both u and v have only linear terms (and no constant terms).

$$u = u(x,y) = 2x - 2y,$$

[7] Click the *Quadratic* button here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Jan26Gridlines/index.html`

[8] This is the kind of function studied in *algebraic geometry*.

$$v = v(x,y) = \frac{1}{2}x + y.$$

This will turn out to be a good definition for a linear function.

Definition 1.2.1 — Linear Transformations of the Plane. A *linear transformation of the plane*, also called a *linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$, $L(x,y) = (u,v)$, where u and v are given by linear equations with no constant term:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

where a,b,c, and d are real numbers.

■ **Warning 1.1** A linear transformation is not quite the same as a linear function from Calculus, because a linear function from calculus can have a constant term, and a linear transformation cannot. This is an unfortunate inconsistency in terminology, but perhaps you can think about a linear transformation as being more *purely linear* since the only terms it has are linear terms, and no constant terms.

■ **Warning 1.2** In light of Question 1.2, you may want a geometric definition of a linear transformation of the plane to be something like: *it takes gridlines to lines*. This isn't quite the case (we will see some examples of this). To be completely precise, we also need the gridlines to remain parallel and evenly spaced, and we need $L(0,0) = (0,0)$. We will discuss this geometric reformulation more later, but for now I just wanted to mention that a this first guess is not quite enough.

You might be getting this far and thinking *wait...I thought linear algebra was about matrices? Where do those fit in?* This is a good question, so let's give a preliminary answer. Take a linear tranformation $L(x,y) = (u,v)$ where:

$$u = u(x,y) = ax + by,$$

$$v = v(x,y) = cx + dy,$$

This function is completely determined by the coefficients of x, and the coefficeints of y. That is, to know L, it is enough to know a, b, c, and d. So, we can completely capture all the data for L in the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For now we should just think of a matrix as a rectangular array of numbers, so that a linear transformation of the plane corresponds to a 2×2 matrix.

Definition 1.2.2 The matrix associated to the linear transformation in Definition 1.2.1 is the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■ **Example 1.14** Consider the function $g(x,y) = (u,v)$ from example 1.8. Observe that the coefficient of y in the first equation is -2, because adding $-2y$ is the same as subtracting $2y$. Also, the coefficient of y in the second equation is a 1 becuase $y = 1 \times y$.

$$u = u(x,y) = 2x + -2y,$$

$$v = v(x,y) = \frac{1}{2}x + 1y.$$

The matrix associated to this fuction is therefore:

$$\begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1. \end{bmatrix}$$

■

This correspondance goes in both directions. That is, given a matrix, you can extract a linear function.

Definition 1.2.3 Consider a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The *linear function associated to this matrix* is the function $L(x,y) = (u,v)$ where:

$$u = ax + by \text{ and,}$$

$$v = cx + dy.$$

Let's run through an example of applying a function, given only a matrix.

■ **Example 1.15** We compute $T(1,-2)$ where $T(x,y)$ is the function associated to the matrix

$$\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

Applying the definition we see that $T(x,y) = (u,v)$ where:

$$u = 3x + 1y = 3x + y,$$

$$v = -1x + 0y = -x.$$

Plugging in $(x,y) = (1,-2)$ gives:

$$u = 3 \times 1 + (-2) = 1, \qquad \text{and} \qquad v = -1$$

Therefore $T(1,-2) = (1,-1)$. ■

R Later we will streamline this process using *matrix multiplication*.

Exercise 1.2 Consider the matrix:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.

2. Evaluate L at $(0,0),(0,1),(1,0)$ and $(1,1)$.
3. Plot the four points of part (b), before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

So far we've only seen how a correspondance between linear transformations of the plane and 2×2 matrices. We will work out in the coming weeks how this fits in to notions of matrix multiplication, determinants, and other matrix operations. For now, we the main take away should be the following.

■ **Slogan 1.3** A matrix is a function.

1.3 Homework 1

Exercise 1.3 Consider the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $L(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate L at all the points with integer coordinates are between -1 and 1. (There should be nine such points).
3. Plot the 9 points from part (b) before and after applying L, and connect them with arrows.
4. Give a qualitative description of what you think L is doing to the plane.

■

Exercise 1.4 Consider the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $T(x,y) = (u,v)$ be the associated linear transformation.

1. Write down the formulas for u and v in terms of x and y.
2. Evaluate T at 5 points of your choice.
3. Plot the 5 points from part (b) before and after applying T, and connect them with arrows.
4. Give a qualitative description of what you think T is doing to the plane.

■

Exercise 1.5 For this problem, adopt the notation of Exercises 1.3 and 1.4. Also consider the matrix N associated to the function $g(x,y)$ from Example **??**:

$$N = \begin{bmatrix} 2 & -2 \\ \frac{1}{2} & 1 \end{bmatrix}$$

1. Can you identify any relationships between the outputs $L(1,0), L(0,1)$ and the matrix M?
2. Can you identify any relationships between the outputs $T(1,0), T(0,1)$ and the matrix I?
3. Can you identify any relationships between the outputs $g(1,0), g(0,1)$ and the matrix N?

4. Now let's treat the general case: pet $\ell(x,y)$ be a linear transformation associated to a general matrix:

$$P = \begin{bmatrix} a & b \\ c & d. \end{bmatrix}$$

Describe the relationship between $\ell(1,0)$, $\ell(0,1)$ and the matrix P. Give reasoning for your answer.

■

Exercise 1.6 Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Do you agree or disagree with the following statement?

Once I know $\ell(1,0)$ and $\ell(0,1)$, I can determine $\ell(x,y)$ for any pair (x,y).

Explain your reasoning. ■

Exercise 1.7 Consider a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, and suppose that $F(0,0) = (1,1)$. Is it possible for F to be a linear transformation? Why or why not? ■

Exercise 1.8 Adopt the notation of Problem 1.5. Define a rule for adding two points as follows:

$$(x_0,y_0)+(x_1,y_1) = (x_0+x_1, y_0+y_1).$$

Let $P = (1,2)$ and $Q = (3,-2)$.

1. Can you identify any relationship between $L(P), L(Q)$, and $L(P+Q)$?
2. Can you identify any relationship between $g(P), g(Q)$, and $g(P+Q)$?
3. To see that it's not a fluke, do parts (a) and (b) again, but with new points P and Q of your choice.
4. Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a general linear transformation, and let P and Q be two random points in $\mathbb{R}^2$. Make a conjecture for the relationship between $\ell(P)$, $\ell(Q)$, and $\ell(P+Q)$. (There is no need to prove this yet, but you can extrapolate from the evidence collected in (a) through (c)).

■

Exercise 1.9 To give a function $\mathbb{R}^2 \to \mathbb{R}^2$ we needed to give 2 functions of 2 variables which output a single number each (for more detail, see Section 1.1.2 in the course notes). Let's see if we can work out what to do in higher dimensions. In particular, adapt Section 1.1.2 in the course notes to describe a function $F : \mathbb{R}^3 \to \mathbb{R}^3$. You can make up any funcion you like, just make sure that you describe it fully. Evaluate this function at the points $(0,0,0), (1,0,0)$, and $(1,2,3)$. ■

2. Vectors

Acknowledgment: I'd like to attribute this approach to vectors in part to Grant Sanderson, author of the delightful youtube channel `3Blue1Brown`. In particular, I borrow heavily his description of the three perspectives of vectors presented below as the *physicist's perspective*, the *computer scientist's perspective*, and the *mathematician's perspective*.

2.1 January 31, 2023

We suggested in the Introduction that the field of Linear Algebra is centered around the study of *linear transformations*. Furthermore, Exercise 1.8 suggests that a linear transformation L satisfies the following equation:[1]

$$L(P+Q) = L(P) + L(Q).$$

It is worth taking some time to unpack what $+$ is doing here. If P and Q are points, what is their sum? In Exercise 1.8, we defined the sum of 2 points to be a third point, whose coordinates correspond to adding the coordinates of P and Q. In $\mathbb{R}^2$ this is written as follows:

$$(x_0, y_0) + (x_1, y_1) = (x_0 + x_1, y_0 + y_1).$$

This is a perfectly valid formula, but it may also seem a bit strange. In particular, we should unpack a concrete interpretation of this algebraic operation to answer the following question:

■ **Question 2.1** What exactly is the meaning of adding coordinates of points in $\mathbb{R}^2$?

By trying to answer this question, we naturally encounter the notion of a *vector*. In fact, a first definition of a vector is pretty much exactly the idea of *points you can add*.

---

[1] At least when $L : \mathbb{R}^2 \to \mathbb{R}^2$

Definition 2.1.1 — 2-dimensional vectors: the computer scientist's perspective. A two dimensional vector is an array of two numbers aligned vertically:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

These can be added coordinatewise:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 \\ y_0 + y_1 \end{bmatrix}.$$

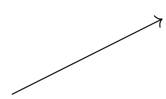
Notation 2.1. *Aligning vectors* vertically *allows them to fit into the matrix theory we will develop in the coming weeks. We will sometimes use the term* column vector *when writing it this way.*

We call this *the computer scientist's perspective*, because it remembers a vector as a light-weight data type stored in a way that easily allows for vector operations (like addition and applying linear maps) to be computed efficiently by a computer in almost any programming language. This perspective also has the advantage of generalizing very easily to higher dimensions, *can you see how?* That being said, it doesn't really get us any closer to answering Question 2.1. In order to do this, we give another perspective on vectors you may have seen in a physics course.

Definition 2.1.2 — 2-dimensional vectors: the physicist's perspective. A two dimensional vector is a quantity specifying a *magnitude* together with a *direction* in the two dimensional plane. This can be represented by an arrow, pointing in the given direction, whose length is the given magnitude.

The physicist's vectors can be added too, essentially by *concatenating the two arrows*. We define this more carefully in Definition **??** below. In particular, we have encountered two different perspectives on the notion of a vector. Below to the left is an example of a computer scientist's vector, and to the right is an example of a physicist's vector.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

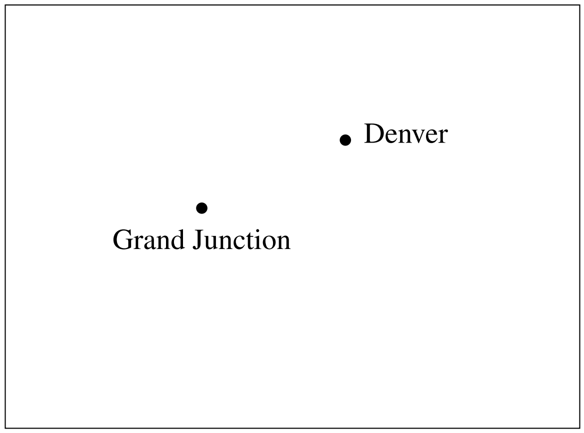


In fact, one could say that these are two perspectives on the same vector, *can you explain why?*. An important part of linear algebra involves learning how to pass seemlessly between these two perspectives, as one lends itself better to computations, while the other lends itself better to interpretations. In this section, we will explore these two perspectives, and start developing a dictionary between them, keeping track of what information can get lost in translation. Along the way we will extract algebraic properties of vectors, and have a first encounter the notions of *linear combinations* and *spans*, which are among the most important in this course.

R Another thing we can do from both perspectives is *scale* vectors by numbers. In fact, there is a third perspective on vectors, which we can call *the mathematician's perspective*, which essentially defines vectors as: *theoretical objects which can be added together and scaled*. We will postpone discussion of this third, more abstract, perspective until the end of the semester. The attentive reader may want to pay attention to how most properties of vectors can be expressed in terms of of these two operations (addition and scaling).

2.1.1 The Physicist's Perspective

A vector is a natural quantity to describe relationships between objects in the physical world. For example, suppose that a pilot is hoping to fly from Denver, Colorado to Grand Junction, Colorado, and asks you for directions.[2]

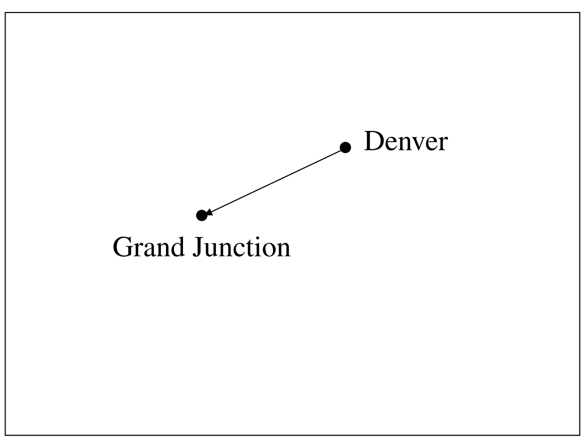


The distance between Denver and Grand Junction is 220 miles. That be but it is *not enough* to tell the pilot to fly 220 miles. If they don't want to end up in Wyoming, they must also know which direction to fly. So for example, you may tell the pilot to fly 220 miles *west-southwest*. These two peices of information constitute a quantity which we call a *vector*:

Magnitude: 220 miles,

Direction: West-Southwest.

All of the defining this quantity information can be exhibited on a map, by drawing an arrow from the start to the finish. One can recover the direction from the direction of the arrow, and the magnitude from the length of the arrow.



This gives us another way to think about Definition 2.1.2.

[2]Map not to scale.

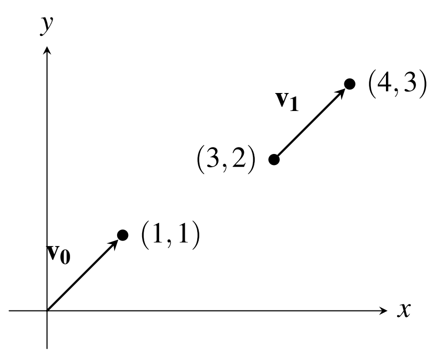
Definition 2.1.3 [Vectors as Displacement] Given two points P and Q, the *displacement vector from P to Q* is the arrow whose tip is at the point Q whenever its tail is placed on P.

Notation 2.2. *Given a vector* $\mathbf{v}$*, we denote its magnitude by* $||\mathbf{v}||$.

With this definition we think of a *displacement vector* as something you can apply to a point. To apply a vector to a point P, we put its tail at P, and the output is wherever its tip points. Importantly, the same vector can be placed in different locations. Let us take this perspective to our vector which takes us from Denver to Grand Junction. This vector is completely determined by the fact that it goes 220 miles west-southwest. It isn't necessary for its to lie on Grand Junction. For example, we could apply the *same vector* starting from Canton, and we would end up somewhere near Toronto. We would still be using the same magnitude (220 miles) and direction (west-southwest), and therefore following the same vector. This is an important point: *a vector is determined by magnitude and direction*. 2 vectors of the same length and pointing in the same direction are the same vector, even if they are drawn at different places.

■ **Slogan 2.1** The vector is the arrow, not where the arrow is.

■ **Example 2.1** Consider following vectors. $\mathbf{v_0}$ connects $(0,0)$ and $(1,1)$, while $\mathbf{v_1}$ connects $(3,2)$ and $(4,3)$. Does $\mathbf{v_0} = \mathbf{v_1}$?



■

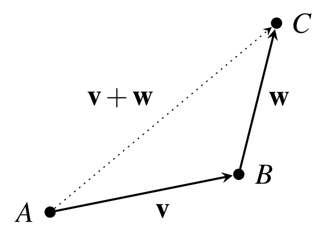
It is common in linear algebra to distinguish between *vectors*, which have a direction and a magnitude, numbers without any direction. The latter is just a number, but we will also often refer to it as a *scalar*.

Adding and Scaling Arrow Vectors

Recall that Question 2.1 asked what adding vectors meant. By thinking of vectors as measuring displacement, we can get a geometrically and physically meaningful understanding how they add, subtract, and scale. We will explore this with the following thought experiment in mind:

You are programming autonomous vehicles. To command a vehicle to move, you give it a vector. The vehicle will then move along the vector: in the given direction for the given magnitude.

Addition: Suppose you give your vehicle a vector $\mathbf{v}$ to follow, and it moves from point A to point B. Once it has arrived at point B, you give it another vector $\mathbf{w}$, and it moves from point B to point C. At this point, the net displacement that the vehicle has travelled is from point A to point C. Define $\mathbf{v}+\mathbf{w}$ to be this displacement vector from A to C.



We can summarize in the following definitoin.

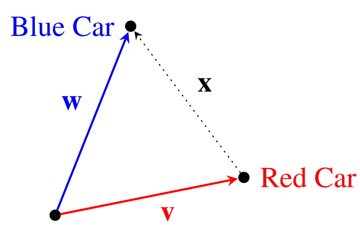
Definition 2.1.4 The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$, is the combined displacement resulting from first applying $\mathbf{v}$, and then applying $\mathbf{w}$ to the result.

■ **Question 2.2** Given 2 vectors $\mathbf{v}$ and $\mathbf{w}$, is it always true that

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}?$$

With addition now defined, we can move on to:

Subtraction: Suppose you control two vehicles, a red vehicle and a blue one, both starting at the same point. You send the red one along vector $\mathbf{v}$, and send the blue one along vector $\mathbf{w}$. After they arrive, the blue vehicle breaks down, so you must send the red vehicle to rescue it. What vector $\mathbf{x}$ must you command the red car to follow?



In particular, if the red car first does $\mathbf{v}$, and then does $\mathbf{x}$, it should overall be following $\mathbf{w}$, and therefore should end up alongside the blue car. We translate this by Definition 2.1.4 to the statement,

$$\mathbf{w}=\mathbf{v}+\mathbf{x}.$$

If subtraction of vectors were to make any sense, then we could subtract $\mathbf{v}$ from both sides and discover that $\mathbf{x}$ really should be the difference of $\mathbf{w}$ and $\mathbf{v}$:

$$\mathbf{w}-\mathbf{v}=\mathbf{x}.$$

Therefore, that is the definition we will make.

Definition 2.1.5 The difference $\mathbf{w}-\mathbf{v}$ of two vectors $\mathbf{w}$ and $\mathbf{v}$, is the vector which, when added to $\mathbf{v}$, gives $\mathbf{w}$.

We can now add and subtract vectors in a geometrically meaningful way. Bringing us closer to getting meaningful answer to Question 2.1. Before moving on, though, we'd like to introduce a special vector.

The Zero Vector: If your vehicle doesn't move at all, what vector does it follow? Since magnitude is the net distance covered, the magnitude is 0. As for direction, this isn't really well defined, since if you move 0 units in any direction, you've stayed put. We will call the vector from a point to itself the *zero vector*.

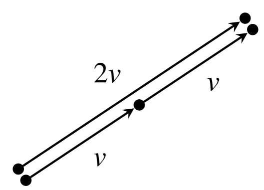
Definition 2.1.6 The vector whose magnitude is zero is called the *zero vector*, and is denoted $\mathbf{0}$.

R The zero vector is the only vector whose direction is unspecified.

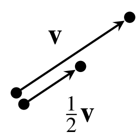
Exercise 2.1 Let $\mathbf{v}$ be any vector. What is $\mathbf{v}+\mathbf{0}$? What about $\mathbf{v}-\mathbf{v}$? ■

Let's introduce one more important operation that vectors allow.

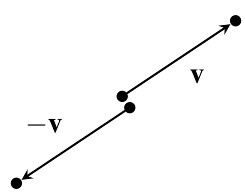
Scalar Multiplication: Suppose you'd like to send your car in the same direction as a vector $\mathbf{v}$, but twice as far as the vector $\mathbf{v}$ allows. You could acheive this by applying $\mathbf{v}$, and then doing so again. With Definition 2.1.4 in mind, you send your car along $\mathbf{v}+\mathbf{v}$, which we can write as $2\mathbf{v}$.



Similarly, if you wanted your car to go in the same direction as $\mathbf{v}$, but half as far, you could follow a vector $\mathbf{w}$ which satisfied $\mathbf{v}=\mathbf{w}+\mathbf{w}$. Since $2\mathbf{w}=\mathbf{v}$ we could reasonably say that $\mathbf{w}=\frac{1}{2}\mathbf{v}$.



Alternatively, suppose you wanted to go the same distance as $\mathbf{v}$, but in the opposite direction. You could follow a vector $\mathbf{x}$. Notice that if the car first does $\mathbf{v}$ and then does $\mathbf{x}$, it will travel along $\mathbf{v}$, and move the same distance in the opposite direction until it gets back to where it started. In particular, we have that $\mathbf{v}+\mathbf{x}=\mathbf{0}$, so it is reasonable to write $\mathbf{x}=-\mathbf{v}$.



Following this logic we can deduce that to scale a vector by a positive number, you scale its magnitude. The negative of a vector reverses direction. What about scaling by a negative number? The following formula should shed some light.

$$-2\mathbf{v}=-(2\mathbf{v}).$$

It appears that to scale by negative 2, you can first scale by 2, and then reverse direction.

Exercise 2.2 With $\mathbf{v}$ as in the figures above, sketch $-2\mathbf{v}$ ■

We can put all this together into the following definition.

Definition 2.1.7 Let $\mathbf{v}$ be a vector and c any scalar. Then the vector $c\mathbf{v}$ is defined by the following data.

- If c is positive, the direction of $c\mathbf{v}$ is the same as $\mathbf{v}$. Otherwise, the direction of $c\mathbf{v}$ is opposite to that of $\mathbf{v}$.
- The magnitude of $c\mathbf{v}$ is:

$$||c\mathbf{v}|| = |c| \cdot ||\mathbf{v}||,$$

where $|c|$ denotes the absolute value of the scalar c.

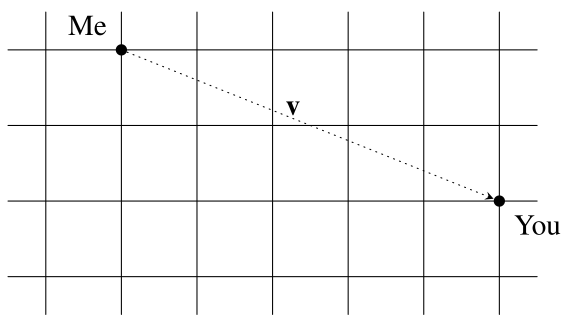
Exercise 2.3 Let $\mathbf{v}$ be any vector. What is $0\mathbf{v}$? ■

A nice output of this geometric approach is that we can give geometric names to certain algebraic operations. For example, we should call 2 vectors parallel, if when we draw their arrows are parallel as lines segments. Our definition of scalar multiplication tells us that this is equivalent to one being a scalar multiple of the other, giving us a algebraic notion of parallel-ness (that can and will extend to higher dimensions).

Definition 2.1.8 Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if there is some constant c such that $\mathbf{v} = c\mathbf{w}$.

2.1.2 Decomposing a Vector into Components

We'd like to connect the ideas described in the *physicist's perspective* on vectors, to the ordered pair numbers which we called the *computer scientist's perspective*. To do this, we'll take the example of giving directions in a city.



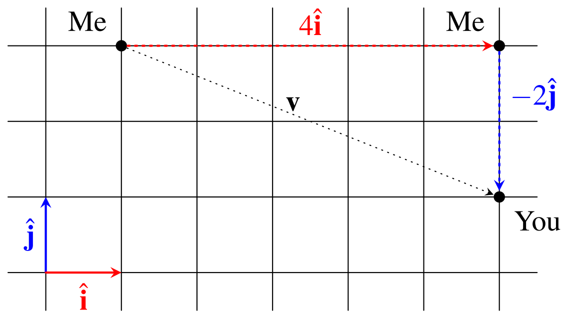
Above is a map depiction of a city, with the thick vertical and horizontal lines roads. You'd like to tell me how to find you. In particular, you need to tell what vector $\mathbf{v}$ I need to follow, in order to get to your location. One way to describe this to me is to give a magnitude and direction. But it is unlikely that you'll tell me something like *go about 530 meters in a direction that is mostly east but somewhat south*. In fact, even if you were more precise with the angles and distance, it is unlikely that I would be able to follow the directions (without walking through buildings).

Instead, you'd probably say something like *walk 5 blocks east, and then 2 blocks south*. Indeed, the regular gridlines of the city give us two natural vectors which we can all agree on:

$$\hat{\mathbf{i}} = \text{ one block east},$$

$$\hat{\mathbf{j}} = \text{ one block north}.$$

With this in hand, everyone can agree on a set of navigational rules. Let's put them on our plot.



So we can see that in order to get to you, I first have to follow $4\hat{\mathbf{i}}$ and then $-2\hat{\mathbf{j}}$. In particular:

$$\mathbf{v} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}}.$$

Once we all agree on a definition of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, we can represent the vector $\mathbf{v}$, just in terms of numbers 4 and 2. This is what the computer scientist would call:

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

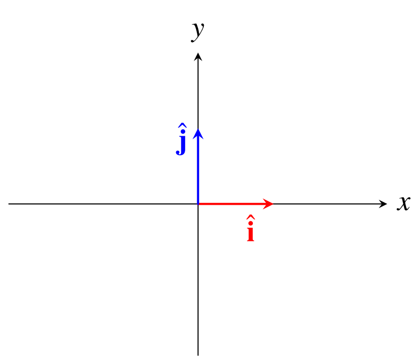
R It is important here that we all agree on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. One can imagine a city which is not grided parallel to north an south, and instead has *updown* and *downtown* directions. Then you may represent a vector using the coefficient of $\hat{\mathbf{j}}$ to describe how many units it goes uptown or downtown, but someone else might represent the vector using the coefficient of $\hat{\mathbf{j}}$ to represent how many units it goes north or south. The coordinates the computer scientist would write down would be different in each case. This is an important subtlety, but one that we will table until we are discussing *bases* and *change of bases*. For now, just remember that the coordinates that you might right down for a vector depend on a *choice* of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. This is one advantage of the physicist's perspective over the computer scientist's perspective.

The general setup (in 2-dimensions) is essentially the same

Definition 2.1.9 Given a cartesian coordinate system (that is, an *xy*-plane with coordinates), we can define the *standard basis vectors* $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to be the vectors:

$$\hat{\mathbf{i}} = \text{ one unit in the positive } x \text{ direction},$$

$$\hat{\mathbf{j}} = \text{ one unit in the positive } y \text{ direction}.$$



Given any vector $\mathbf{v}$ in the coordinate plane, we can *resolve* it into its components:

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}}.$$

Here v_1 is a *scalar*, representing how far $\mathbf{v}$ goes in the x-direction, and v_2 is a *scalar* representing how far $\mathbf{v}$ goes in the y-direction. They are unique.

Notation 2.3. *Given a vector in component form:*

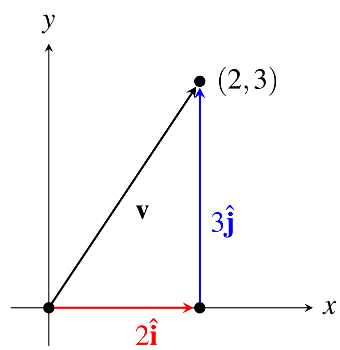
$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}},$$

we call the coefficients v_1 and v_2 the components *of* $\mathbf{v}$. *We can take these components and write* $\mathbf{v}$ *as a* column vector*:*

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

It is important to note that the coordinates of the column vector depend on $\hat{\mathbf{i}}$ *and* $\hat{\mathbf{j}}$. *In particular, we should think of this column vector as meaning,* first do $v_1\hat{\mathbf{i}}$, then do $v_2\hat{\mathbf{j}}$.

■ **Example 2.2** Consider the displacement vector $\mathbf{v}$ from $(0,0)$ to $(2,3)$.



Therefore we see that:

$$\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

■

Notice that the displacement vector from $(0,0)$ to (x,y) can always be written

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 2.4 Consider the vectors $\mathbf{v_0}$ and $\mathbf{v_1}$ from Example 2.1. Write them both in component form and as column vectors. Use your result to decide whether they are equivalent. ■

Exercise 2.5 Let $P = (x_0, y_0)$, $Q = (x_1, y_1)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ in component form and as a column vector, in terms of the coordinates of P and Q. ■

Exercise 2.6 — Checkin 1. For problems 1 through 3, fix the following vectors:

$$\mathbf{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} \qquad \mathbf{w} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}.$$

1. Suppose you start at the origin, first travel along the displacement vector $\mathbf{v}$, and then travel along the displacement vector $\mathbf{w}$. Sketch your overall path to determine your endpoint. Use this to write down the vector $\mathbf{v} + \mathbf{w}$ in terms of its components.
2. Suppose you start at the origin and travel in the direction of $\mathbf{v}$, but twice as far. Sketch the path you travel to determine your endpoint. Use this to write down the vector $2\mathbf{v}$ in terms of its components.
3. Suppose you start at the origin and travel along the displacement vector $\mathbf{v}$, and then along the displacement vector $-\mathbf{w}$. Sketch the path you travel to determine your endpoint. Use this to write down the vector $\mathbf{v} - \mathbf{w}$ in terms of its components.
4. Now fix generic vectors

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} \quad \text{and} \quad \mathbf{w} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}}$$

and let c be a scalar. Use the intuition developed on the previous page to write down formulas for $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$, and $c\mathbf{v}$.

■

2.2 February 2, 2023

Let's start with a couple of warmup questions.

Exercise 2.7 Let $P = (-1,2)$ and $Q = (3,4)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ as a column vector. ■

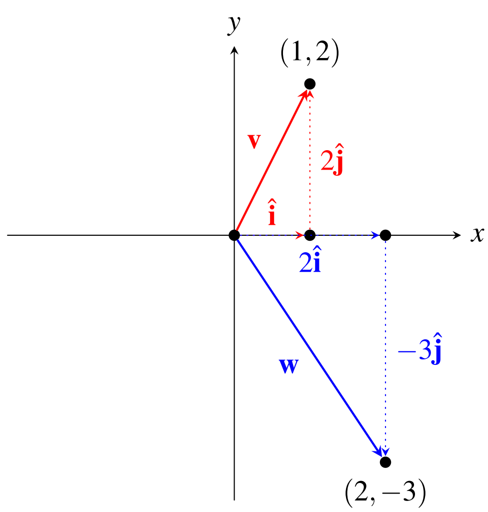
Exercise 2.8 Now let $P = (x_0, y_0)$ and $Q = (x_1, y_1)$, and let $\mathbf{v}$ be the vector from P to Q. Write $\mathbf{v}$ as a column vector (in terms of x_0, x_1, y_0, and y_1). ■

In Exercise 2.6, we explored what addition, subtraction, and scalar multiplication looked like when we put vectors in component form. Let's start by working through an example summarizing our observations.

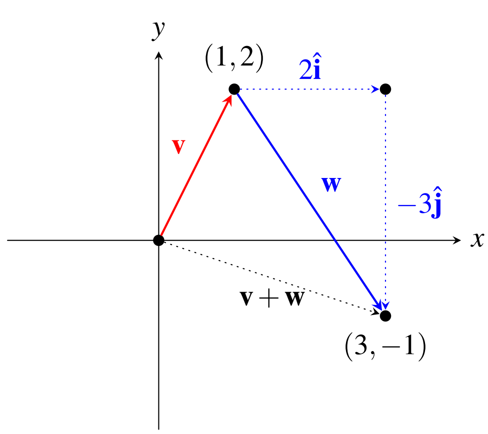
■ **Example 2.3** Consider the following two vectors given in component form:

$$\mathbf{v} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} \qquad \mathbf{w} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}.$$

Let's plot them both starting at the origin.



Let's add them together! To do this, we move the tail of $\mathbf{w}$ to the tip of $\mathbf{v}$, so that we can see what happens when you iterate them.



In particular, we have computed that: $\mathbf{v}+\mathbf{w} = 3\hat{\mathbf{i}}-\hat{\mathbf{j}}$. Let's compare this to adding together these two vectors *as computer scientists*. The column vector forms of $\mathbf{v}$ and $\mathbf{w}$ are (as in Notation 2.3):

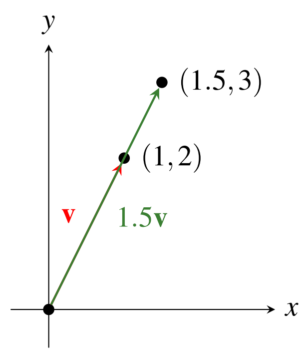
$$\mathbf{v} = \hat{\mathbf{i}}+2\hat{\mathbf{j}} = \begin{bmatrix}1\\2\end{bmatrix} \qquad \mathbf{w} = 2\hat{\mathbf{i}}-3\hat{\mathbf{j}} = \begin{bmatrix}2\\-3\end{bmatrix}.$$

Then if we add coordinatewise we get:

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}1+2\\2-3\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix},$$

which translates to a the vector $3\hat{\mathbf{i}}-\hat{\mathbf{j}}$ in component form. We get the same answer! Let's say a word as to why this makes sense. Since $\mathbf{v}$ moves 1 unit in the x direction, and $\mathbf{w}$ moves 2 units in the x direction, then doing both moves $1+2=3$ units in the x direction. This is exactly what the computer scientist's approach does, adding together the x-coordinates. An identical argument explains why the $\hat{\mathbf{j}}$ components agree with both perspectives!

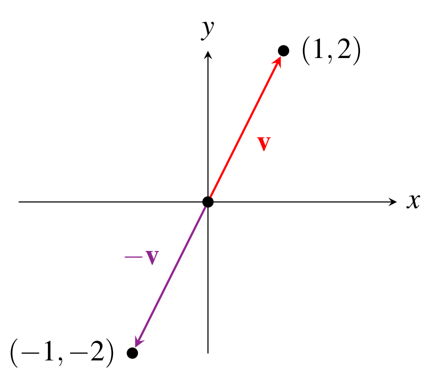
Now let's look at scalar multiplication, comparing $\mathbf{v}$ and $1.5\mathbf{v}$.



Since $1.5\mathbf{v}$ is 1.5 times longer, it (in particular), goes 1.5 times further in the x-direction, and 1.5 times further in the y-direction, so that:

$$1.5\mathbf{v} = 1.5(\hat{\mathbf{i}}+2\hat{\mathbf{j}}) = 1.5\hat{\mathbf{i}}+1.5*2\hat{\mathbf{j}} = \begin{bmatrix}1.5\\3\end{bmatrix}.$$

Let's also compare $\mathbf{v}$ and $-\mathbf{v}$.



Notice that $-\mathbf{v}$ goes the same distance along the x-axis $\mathbf{v}$ but in the opposite direction, and similarly along the y-axis. Therefore:

$$-\mathbf{v} = -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} \qquad \text{or} \qquad -\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

■

This example suggests that to add a pair of vectors, we can merely add the $\hat{\mathbf{i}}$ components and the $\hat{\mathbf{j}}$ components, and to scale a vector, we can just scale the components. The second statement in fact gives us a formula for scaling column vectors as well: scaling a column vector can be acheived by scaling each entry. Let's record this:

Theorem 2.2.1 Fix two general vectors in component form (and as column vectors).

$$\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \mathbf{w} = w_1\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

1. We can compute the sum using the formula:

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1)\hat{\mathbf{i}} + (v_2 + w_2)\hat{\mathbf{j}}.$$

In particular, the physicist's perspective on vector addition agrees with the computer scientist's formula:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

2. Let c be any constant. We can compute the scalar multiple $c\mathbf{v}$ using the formula:

$$c\mathbf{v} = (cv_1)\hat{\mathbf{i}} + (cv_2)\hat{\mathbf{j}}.$$

From this we can derive a formula for scaling a column vector which agrees with the physicist's perspective on vector scaling:

$$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}.$$

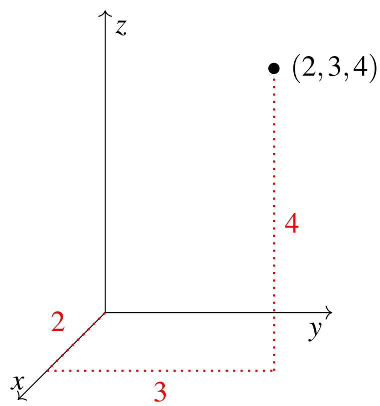
Proof. The argument for why Theorem 2.2.1 is true is essentially identical to the arguments appearing in Example 2.3 For example, $\mathbf{v}$ goes v_1 units in the x-direction, and $\mathbf{w}$ goes w_1 units in the x-direction, so doing both in succession results in an overall movement of $v_1 + w_1$ units in the x-direction. See if you can adapt the remaining arguments from Example 2.3 to the general setup. ■

This gives a complete answer to Question 2.1. To concretely interpret what *adding points* means, we should think about the points as vectors. Then the physicist's perspective on vector addition gives us a concrete way to think about the addition geometrically.

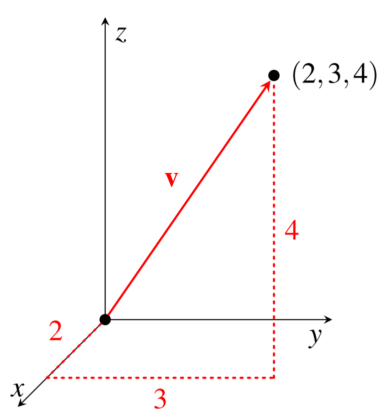
Exercise 2.9 Adopt the notation of Exercise 2.7. Suppose the tail for $-2.5\mathbf{v}$ is placed at P, what are the coordinates of its tip? ■

2.2.1 Extending this to 3 dimensions...and beyond

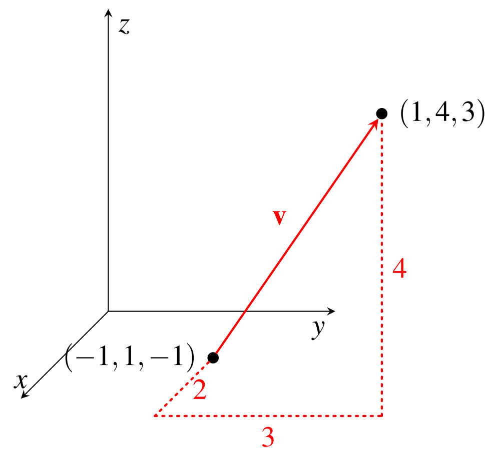
Everything described so far also works in 3 dimensions with a few cosmetic adjusments. Much like before, we can take the physicist's perspective or the computer scientist's perspective, and use coordinates to connect the two. The main difference is that we pass from our 2 dimensional xy-plane, to 3 dimensional space, whose points are now specified by 3 coordinates (x,y,z). We will usually adopt the convention of viewing the x axis as coming forward out of the page, the y axis as going to the right, and the vertical z-axis as going up. Here, for example, is how you plot the point $(2,3,4)$.



The Physicist's Perspective in 3d: As in 2d, the physicist's vector is a quantity $\mathbf{v}$ specifying a *magnitude* and *direction*, though the direction is now in 3d. We can again represent a vector by an arrow in 3-dimensional space: it should point in the *direction* of $\mathbf{v}$ and its length should be the *magnitude* of $\mathbf{v}$. Given two points P and Q in 3-space, we can obtain the vector from P to Q as an arrow connecting P and Q. For example, here we draw the vector $\mathbf{v}$ connecting $(0,0,0)$ and $(2,3,4)$.



As before *the vector is the arrow*, not the location. If we put the tail of $\mathbf{v}$ somewhere else, the vector remains the same.



In this way, we can think about a vector as something we can *apply* to a point. For example $\mathbf{v}$ takes $(0,0,0)$ to $(2,3,4)$, and also takes $(-1,1,-1)$ to $(1,4,3)$.

With this perspective, the definitions of addition, subtraction, and scalar multiplicate defined in Definitions 2.1.4, 2.1.5, and 2.1.7 go through unchanged. For example, to add two vectors $\mathbf{v}$ and $\mathbf{w}$, we take the vector obtained by *first applying* $\mathbf{v}$*, and then applying* $\mathbf{w}$.

The computer scientist's perspective in 3d: For a computer scientist, a 3-dimensional vector is an array of 3 numbers, aligned vertically:

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The can be added and scaled coordinatewise:

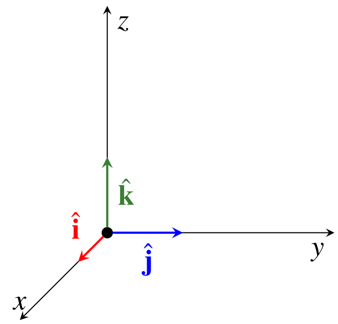
$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0+x_1 \\ y_0+y_1 \\ z_0+z_1 \end{bmatrix} \qquad \text{and} \qquad c\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}.$$

Combining the two perspectives in 3d: To go between these two perspectives in 3 dimensions, we follow a similar approach to what we did in 2d. To start, we define the standard unit vectors by their directions and magnitudes:

$$\hat{\mathbf{i}} = \text{one unit in the positive } x \text{ direction}$$

$$\hat{\mathbf{j}} = \text{one unit in the positive } y \text{ direction}$$

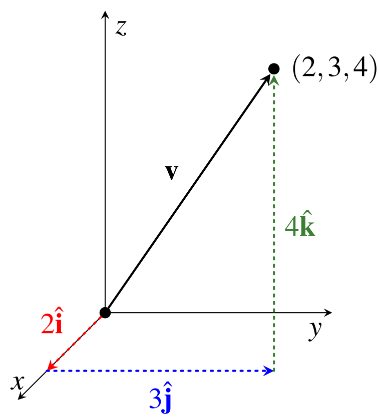
$$\hat{\mathbf{k}} = \text{one unit in the positive } z \text{ direction}$$



We can now express any vector in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. For example, the vector from $(0,0,0)$ to $(2,3,4)$ can be expressed as

$$\mathbf{v} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}},$$

because it goes 2 units in the x direction (achieved by applying $\hat{\mathbf{i}}$ twice), 3 units in the y direction (acheived by applying $\hat{\mathbf{j}}$ 3 times), and 4 units in the z direction (achieved by applying $\hat{\mathbf{k}}$ four times).



To get this in the form of a column vector, just arrange the coefficients of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vertically.

$$2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Exercise 2.10 Let $\mathbf{w}$ be the vector from $(5,-3,9)$ to $(1,1,2)$. Write $\mathbf{w}$ as a column vector. ■

We can generalize 2.8 to 3d as well.

Proposition 2.2.2 Let $P = (x_0, y_0, z_0)$, $Q = (x_1, y_1, z_1)$, and let $\mathbf{w}$ be the vector from P to Q. Then $\mathbf{w}$ can be written as a column vector in the following way:

$$\mathbf{w} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{bmatrix}.$$

Exercise 2.11 Explain why Proposition 2.2.2 is true. ■

Higher dimensions: The computer scientist's approach to vectors generalizes to higher dimensions. To the computer scientist, a 2d vector was an array of 2 numbers, and a 3d vector was an array of 3 numbers. Following the pattern, an n-dimensional vector should be an array of n-numbers.

Definition 2.2.1 — Higher Dimensional Vectors: The Computer Scientist's Approach. Let n be a positive integer. A *n-dimensional column vector* is an array of n numbers, arranged vertically:

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The can be added and scaled coordinatewise:

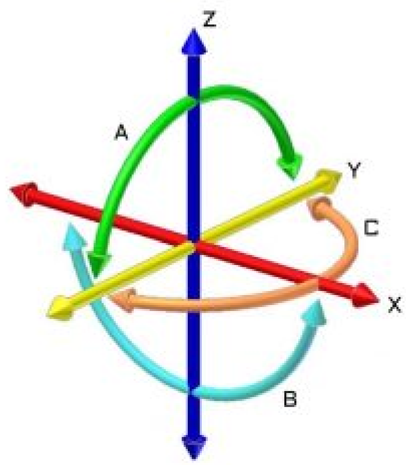
$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_0 + y_0 \\ x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \qquad \text{and} \qquad c \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_0 \\ cx_1 \\ \vdots \\ cx_n \end{bmatrix}.$$

At first we might be wary of straying beyond 3-dimensions, especially our interest is limited to applications within the physical world. Nevertheless, there are many reasons one might be interested in having more than 3 axes of data in a vector, even for things that get modelled in the physical world. Let's see a couple of examples.

■ **Example 2.4 — 5-axis CNC drilling.** Where a drilling machine drills a hole depends not only on the location of the tip of the drill bit, but also on its orientation in space (what direction is the drill bit pointing?). For these reasons, programmable drills control the movement of the drilling head using a five dimensional vector:

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}.$$

The first three entries, x, y, z, control the location of the machine-head in space, while θ and ϕ control rotation, in order to angle the drill to the necessary position. In particular, θ controls rotation in the xy-plane (about the z-axis), while ϕ controls rotatoin in the yz-plane (about the x-axis).



[4]

The drill can be controlled much like the autonomous vehicles of Section **??**, by being sent vectors. For example, if you would like to send the drill 1 cm in the x-direction and 2 cm down, without changing rotation, you may send it the vector:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

If the drill is pointing down and you would like to point it up, and lift it 5 cm, you could then send it:

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 180 \end{bmatrix}.$$

To do these in succession, we would first do $\mathbf{v}$ (say to drill a hole), and then do $\mathbf{w}$ (say, to lift it and drill another hole from below), which a physicist may say should add the vectors. And indeed, tracing the overall movement of the printhead would have it end up following the vector

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 180 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 180 \end{bmatrix} = \mathbf{v} + \mathbf{w}.$$

In particular, the computer scientist's definition of vector addition has a physical interpretation as well! ■

This example shows that adding even 5-dimensional vectors can concretely understood once we specify the data that the entries store. The following exercise shows you can do something similar for scalar multiplication.

[3] From Autodesk: `https://blogs.autodesk.com/inventor/understanding-process-5-axis-machining/`
[4] TODO: Make a better graphic

Exercise 2.12 Suppose you have a 5-axis drill, at its *home position*. You program it to move along the 5-dimensional $\mathbf{v}$ to drill a hole. If you'd like it to return to home position, what vector should you ask it to follow? ■

■ **Example 2.5 — A 171,000 dimensional data set.** The study of *stylometry* studies variations in literary style using statistical analysis, and part of this work involves measuring how frequently certain words appear. There are approximately 171,000 words in the english language, so if you would like to record how many times each word appears in a given book, you could do so in a vector:

$$\mathbf{v} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{171,000} \end{bmatrix},$$

where w_1 counts the number of times the first word in the dictionary appears, and w_2 counts how many times the second word in the dictionary appears, and so on. Adding the vectors associated to a 2 different books will count how many times each word appears in both. Having a mathematical model grounded in geometry allows one to use linear algebraic techniques to study questions about variations in word frequency. One could also use scalar multiples to weight the importance or prominence of certain sources over others. ■

2.3 Homework 2

Exercise 2.13 Let $P = (1,2), Q = (2,1)$ and $R = (-3,-1)$. Let $\mathbf{v}$ be the vector from P to Q, and let $\mathbf{w}$ be the vector from Q to R.

1. Do you agree or disagree with the following statement:

   $$\mathbf{v}+\mathbf{w} \text{ is the vector from } P \text{ to } R.$$

   Justify your answer by drawing a picture.
2. Write $\mathbf{v}, \mathbf{w}$ and $2\mathbf{v}-\mathbf{w}$ in component form, and as column vectors.
3. Draw $2\mathbf{v}-\mathbf{w}$ on the plane, with its tail starting at R. What are the coordinates of where the tip lands?

■

Exercise 2.14 Write the standard 3-dimensional unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ as column vectors. ■

Exercise 2.15 To a physicist, a vector was a quantity with *magnitude* and *direction*. We saw how to turn such a quantity into a *column vector*. Let's start going in the other direction.

1. Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the standard unit vectors in 2d. What are their magitudes: $||\hat{\mathbf{i}}||$ and $||\hat{\mathbf{j}}||$? (*Recall:* When representing a vector with an arrow magnitude, is the length of the arrow.)
2. Let $\mathbf{v}$ and $\mathbf{w}$ be the same vectors from Problem 1. Use the Pythagorean theorem to compute their magnitudes: $||\mathbf{v}||$ and $||\mathbf{w}||$.

3. Consider a general column vector:

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Write formula for the magnitude $||\mathbf{u}||$ in terms of x and y. Explain your reasoning.

An artist is trying to mix the perfect color. One way to measure colors is in RGB, where:

$$\begin{array}{rc} & \text{r units of red} \\ & \text{g units of green} \\ + & \text{b units of blue} \\ \hline & \text{a rich spectrum of colors} \end{array}$$

The artist has two pigments, which they are hoping to mix together into paint to try and get the color they want.

Pigment X: Adding 1 ounce of Pigment X to paint adds:

1 unit of red, 2 units of green, 3 units of blue

Pigment Y: Adding 1 ounce of Pigment Y to paint adds:

7 units of red, 5 units of green, 2 units of blue

Exercise 2.16 The effect of adding an ounce **pigment X** can be represented by a vector $\mathbf{x}$ and the effect of adding **pigment Y** by a vector $\mathbf{y}$.
1. Give a concrete interpretation of what the vector $\mathbf{x}+\mathbf{y}$ represents.
2. Express $\mathbf{x}$ and $\mathbf{y}$ as column vectors.

Exercise 2.17 The artist wants to create a color the called **fancy gold**, consisting of:

24 units of red, 21 units of green, 15 units of blue

1. Write **fancy gold** as a column vector $\mathbf{f}$.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X and b units of Pigment Y produce fancy gold.

3. Translate the single vector equation from part 2 into a system of linear equations. Solve this system of equations to see how many ounces of **Pigment X** and how many ounces of **Pigment Y** the artist needs to mix to get **fancy gold**.

Exercise 2.18 The artist also whats to create **super green**, which consists of:

3 units of red, 90 units of green, 3 units of blue.

1. Write **super green** as a column vector **s**.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X and b units of Pigment Y produce super green.

3. Translate the single vector equation from part 2 into a system of linear equations. Try to solve this system of equations to determine if it is possible to mix **super green** from **Pigment X** and **Pigment Y**.

■

Exercise 2.19 The artist got their hands on a fancy new pigment called **Greenifier**, which, perhaps suprisingly, doesn't actually contain any green. Instead, it works by absorbing red and blue light. The net effect of adding 1 ounce of **Greenifier** is

subtract 5 units of red, subtract 5 units of blue

1. Write **Greenifier** as a column vector **g**.
2. Write an equation using only vector addition and scalar multiplication which can be interpreted as saying:

   a units of Pigment X, b units of Pigment Y, and c units of Greenifier produce super green.

3. Translate the single vector equation from part 2 into a system of linear equations. Solve this system of equations to see how many ounces of **Pigment X**, how many ounces of **Pigment Y**, and how many ounces of **Greenifier** the artist needs to mix to get **fancy gold**.

■

2.4 February 7, 2023

Let's start by unpacking Exercises 2.17 and 2.18, and in doing so, get a quick review of how we solve systems of linear equations. Our goal in Exercise 2.17 is to mix fancy gold from the two pigments, Pigment X and Pigment Y. How much of each pigment do we add?

To introduce mathematical notation, we let the vector $\mathbf{x}$ represent the effect of adding an ounce of Pigment X and $\mathbf{y}$ represent adding an ounce of Pigment Y. In particular, if we add a ounces of Pigment X and b ounces of Pigment Y, the overall effect on color can be represented by the vector:

$$a\mathbf{x} + b\mathbf{y}.$$

If we let $\mathbf{f}$ represent the color fancy gold, we are therefore looking for integers a and b so that:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{f}. \tag{2.1}$$

Plugging in the column vectors from Exercises 2.16.2 and 2.17.1 turns this into:

$$a\begin{bmatrix}1\\2\\3\end{bmatrix} + b\begin{bmatrix}7\\5\\2\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}.$$

But we know how to scale and add column vectors. So this becomes:

$$\begin{bmatrix}a+7b\\2a+5b\\3a+2b\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix}$$

Now we can just remove the matrix brackets and obtain:

$$a + 7b = 24 \tag{2.2}$$
$$2a + 5b = 21 \tag{2.3}$$
$$3a + 2b = 15, \tag{2.4}$$

which is a system of linear equations with 3 equations and 2 unknowns. We want to find values a and b so that *all 3 equations hold*! Many readers have probably solved systems of linear equations before, but it may have been a long time, so let's briefly review how one might solve this. We will establish a completely systematic way of doing this in Section **??**, but for now, let's just follow our noses. A first step can be to solve for a in Equation (2.2):

$$a = 24 - 7b. \tag{2.5}$$

We can now plug this value of a into Equation (2.3). The left hand side is:

$$2a + 5b = 2(24 - 7b) + 5b = 48 - 9b.$$

So Equation (2.3) turns into:

$$48 - 9b = 21.$$

We can therefore solve for $b = 3$. Plugging this value into Equation (2.5) gives:

$$a = 24 - 7(3) = 3.$$

So in order for the first Equations (2.2) and (2.3) to hold, we need $a = 3$ and $b = 3$. What about Equation (2.4)? Plugging in $a = 3$ and $b = 3$ gives:

$$3a + 2b = 3 * 3 + 2 * 3 = 15.$$

So we have determined that $a = 3$ and $b = 3$ solves all three equations in our system. Translating back into our vector equations we have:

$$3\mathbf{x} + 3\mathbf{y} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}7\\5\\2\end{bmatrix} = \begin{bmatrix}3*1+3*7\\3*2+3*5\\3*3+3*2\end{bmatrix} = \begin{bmatrix}24\\21\\15\end{bmatrix} = \mathbf{f}.$$

So $a = 3$ and $b = 3$ also form solutions to the vector equation (2.1)! Translating back into plain english gives:

3 ounces of Pigment X and 3 ounces of Pigment Y results in Fancy Gold

One main takeaway from this example is that, the *single* linear equation of vectors (2.1), corresponds exactly to the *system* of linear equation*s* (2.2), (2.3), and (2.3).

■ **Slogan 2.2** Solving *one* linear equation of vectors is the same as solving *a system* of linear equations.

What about Super Green? The question becomes one of solving the single vector equation:

$$a\mathbf{x} + b\mathbf{y} = \mathbf{s},$$

which we can unpack to the system of equations:[5]

$$\begin{aligned} a + 7b &= 3 \\ 2a + 5b &= 90 \\ 3a + 2b &= 3 \end{aligned}$$

Following the same script as before, we can use the first two equations to solve for $a = \frac{199}{3}$ and $b = \frac{-28}{3}$ (the reader fill should in the missing steps!). This already presents a problem, how can we add a negative amount of Pigment Y? Let's ignore this for a moment, and pretend the artist had some tool to remove pigment. Would this solve it? Well, we see that to get 3 units of red and 90 units of green we are forced to add $\frac{199}{3}$ ounces of Pigment X and to (somehow) subtract $\frac{28}{3}$ ounces of Pigment Y. But we haven't even checked if we have the correct amount of blue. This is measured in the third equation, let's see if our values of a and b work:

$$3a + 2b = 3 \times \left(\frac{199}{3}\right) + 2\left(\frac{-28}{3}\right) = \frac{541}{3}.$$

That is way more than 3 units of blue! So our attempt at making Super Green is going to come out more blue than anything else, and we will fail. What we've encountered here is the following fact:

[5] the reader should fill in the details here!

sometimes a system of linear equations has no solutions. Indeed, if there are more equations than unknowns, this is rather common: notice we were able to find an a and b that satisfied the first two equations, but the third equation that caused us trouble. When we translate Exercise 2.19 into a system of equations, we have a third unknown, giving us added flexibility we can exploit. For now, let's just record the following observation:

■ **Slogan 2.3** Sometimes a linear equation of vectors can have no solution.

For the rest of this section, we will put these observations into a more general context, introducing some language with which we can more concisely describe these observations.

2.4.1 Linear Combinations and Spans: A First Pass

In our color mixing example, we were studying which colors we could mix from Pigment X and Pigment Y. Assigning variables a and b to the amount of each pigment added, we were able to translate this question into one which studies whether we can write a vector in the form:[6]

$$a\mathbf{x} + b\mathbf{y},$$

In a more general language, we are asking to know which vectors can be written *in terms of* $\mathbf{x}$ *and* $\mathbf{y}$ with the operations of scalar multiplication and addition. A vector that can be written in that form is called a *linear combination* of $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.4.1 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. A *linear combination* of $\mathbf{u}$ and $\mathbf{v}$ is a vector $\mathbf{w}$ which can be written:

$$\mathbf{w} = c\mathbf{u} + d\mathbf{v},$$

for constants c and d.

■ **Example 2.6** The vector $\mathbf{f}$ from Exercise 2.17 is a linear combination of $\mathbf{x}$ and $\mathbf{y}$ because:

$$\mathbf{f} = 3\mathbf{x} + 3\mathbf{y}.$$

■

■ **Question 2.3** Is the vector $\mathbf{s}$ a linear combination of $\mathbf{x}$ and $\mathbf{y}$.

We already encountered linear combinations of vectors before this color example. Indeed, it was the tool we used to translate between the *physicist's perspective* on vectors to the *computer scientist's perspective*.

■ **Example 2.7** The column vector:

$$\mathbf{v} = \begin{bmatrix} -11 \\ 9 \end{bmatrix},$$

is a linear combination of the standard unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ because

$$\mathbf{v} = -11\hat{\mathbf{i}} + 9\hat{\mathbf{j}}.$$

■

[6] In fact, one could interpret the example as having us restrict to positive values of a and b, but we will not make that restriction.

■ **Question 2.4** Let $\mathbf{w}$ be a 2-dimension vector. Is $\mathbf{w}$ a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$?

Indeed, this is what the column vector *means*. The column vector:

$$\mathbf{w} = \begin{bmatrix} c \\ d \end{bmatrix}$$

exactly says that $\mathbf{w}$ can be acheived by scaling $\hat{\mathbf{i}}$ by c, and scaling $\hat{\mathbf{j}}$ by d. That is $\mathbf{w} = c\hat{\mathbf{i}} + d\hat{\mathbf{j}}$.

We can also consider linear combinations of more than 2 vectors. We have already done so to represent 3-dimensional vectors as column matrices, and also in Exercise 2.19.

Definition 2.4.2 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be a collection of n-vectors (of the same dimension). A linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a vector $\mathbf{w}$ which can be written in the form:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

for constants $c_1, c_2, \cdots, c_n$.

■ **Example 2.8** Exercise 2.19 asks whether the vector $\mathbf{s}$ is a linear combination of $\mathbf{x}, \mathbf{y}$ and $\mathbf{g}$ where $\mathbf{g}$ represents the effect of adding one ounce of greenify. ■

■ **Example 2.9** Every 3d vector is a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. ■

■ **Question 2.5** Consider the column vector:

$$\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

If I know that $\mathbf{w}$ is a linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$, what can we say about its middle entry b?

■ **Example 2.10** Let $\mathbf{v}$ be a single vector. A linear combination of $\mathbf{v}$ is a vector of the form $c\mathbf{v}$ for some constant c. That is, a linear combination of $\mathbf{v}$ is the same as a multiple of $\mathbf{v}$. ■

As we are starting to see, many questions in linear algebra boil down to variants of the following type of question:

■ **Question 2.6** When is one vector a linear combination of another collection of vectors?

We will see many variations of this question, so let's introduce some terminology to simplify the exposition.

Definition 2.4.3 Let $\mathbf{u}$ and $\mathbf{v}$ be a pair of vectors. The *span* of $\mathbf{u}$ and $\mathbf{v}$ is the collection of vectors that can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \{\text{The collection } c\mathbf{u} + d\mathbf{v} \text{ for constants } c \text{ and } d\}.$$

■ **Example 2.11** Returning to our color mixing example: $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ is the collection of colors that can be mixed from Pigment X and Pigment Y. In particular, $\mathbf{f}$ is in $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$, while $\mathbf{s}$ is not. ■

■ **Example 2.12** Denote the entire collection of 2d vectors by $\mathbb{R}^2$. Then $\mathbb{R}^2 = \operatorname{span}\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\}$. ■

We can also consider the spans of more than 2 vectors.

Definition 2.4.4 Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be a collection of n-vectors (of the same dimension). The span of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is the collection of vectors that are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.

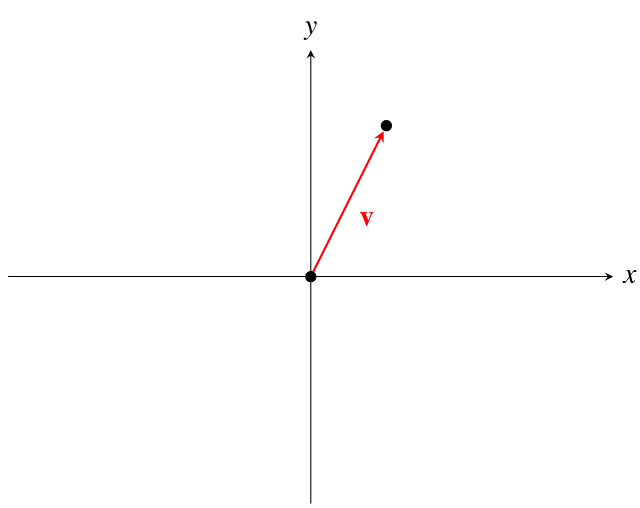
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \{\text{ the collection } c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \text{ for constants } c_1, c_2, \cdots, c_n\}.$$

■ **Example 2.13** Exercise 2.19 asks whether the vector $\mathbf{s}$ is in span$\{\mathbf{x}, \mathbf{y}, \mathbf{g}\}$. ■

Visualizing Spans

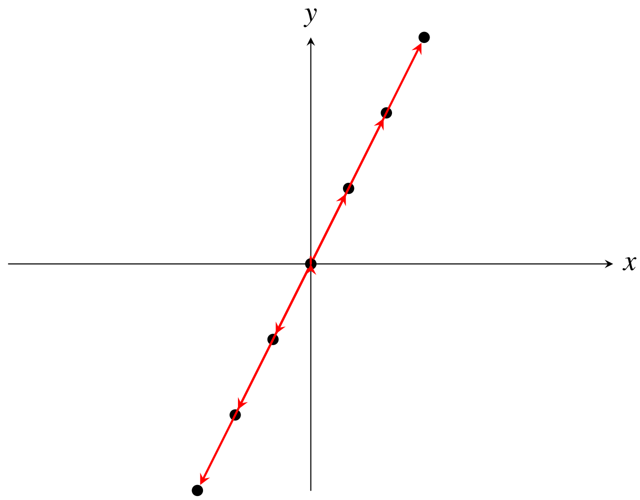
Spans can be a tricky thing to intuit on a first pass through linear algebra, but they are also an important and fundamental part of the theory. We will delay the general practice of explicitly computing spas until Section **??**, when we have developed a bit of matrix theory.[7] But if we can develop a visual intuition of what spans look like, we recognize where they arise and get some intuition about what they mean before doing any computations.

■ **Example 2.14 — The Span of a Single Vector.** Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$:

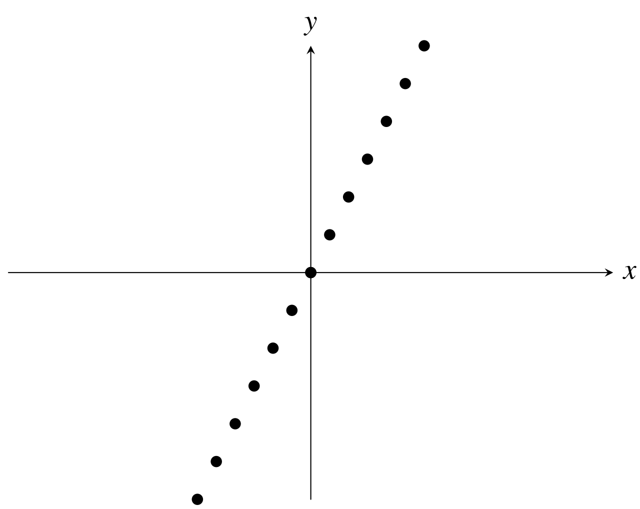


The span of $\mathbf{v}$ is the collection of linear combinations of $\mathbf{v}$. This is precisely the collectoin of multiples of $\mathbf{v}$ (cf. Example **??**). Let's plot a few of these. In particular, we'll plot: all the vectors $-1.5\mathbf{v}, -\mathbf{v}, -0.5\mathbf{v}, 0\mathbf{v}, .5\mathbf{v}, 1\mathbf{v}, 1.5\mathbf{v}$, with their tails starting at the origin.

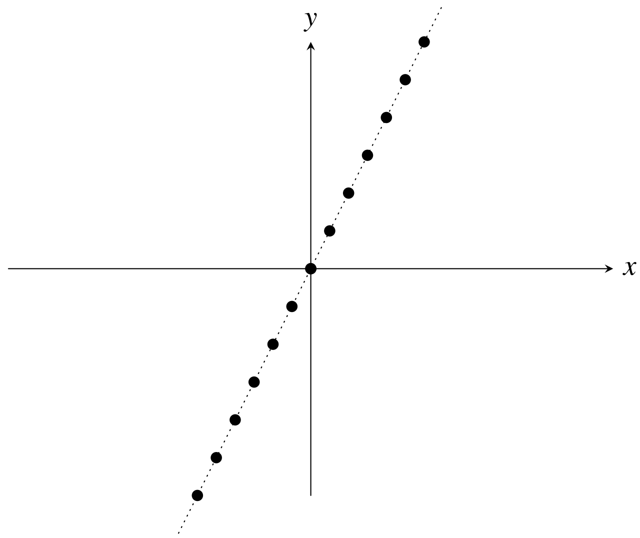
[7] though notice that Exercises 2.17 through 2.19 suggest a relationship with systems of linear equations



As we start to fill in more and more multiples, the arrows start to crowd the picture, so let's just draw the where the tips lie.



As the picture fills in, we start to see that the tips of all the multiples of **v** trace out a straight line.

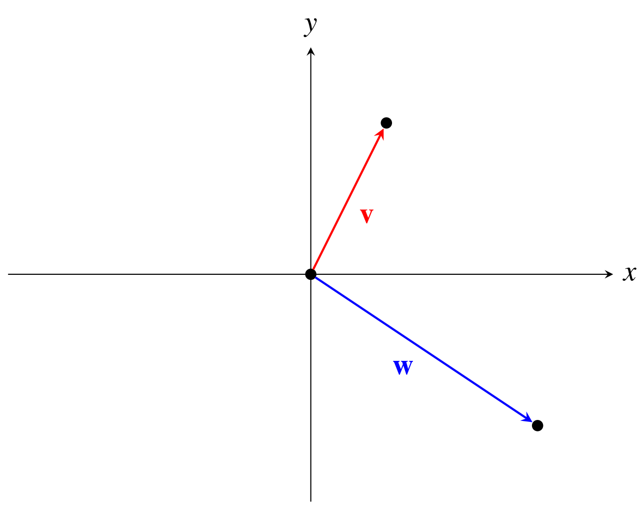


■

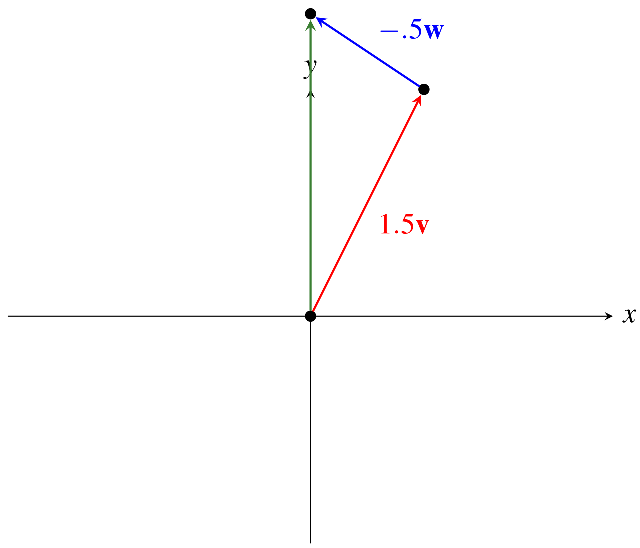
In the previous example, we deduced (without any explicit computations) some geometric facts about the span of a single vector!

■ **Slogan 2.4** The span of a single vector is a straight line through the origin.

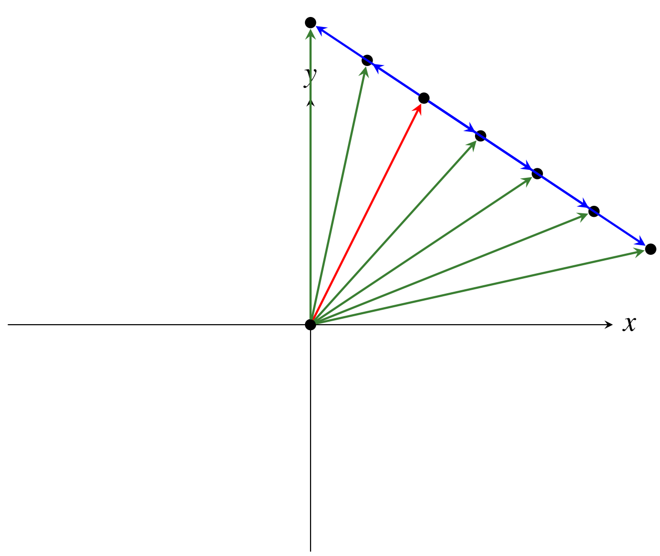
■ **Example 2.15 — The Span of 2 Vectors.** Let $\mathbf{v}$ and $\mathbf{w}$ be 2 nonzero vectors in $\mathbb{R}^2$. For now let's assume that they aren't parallel.



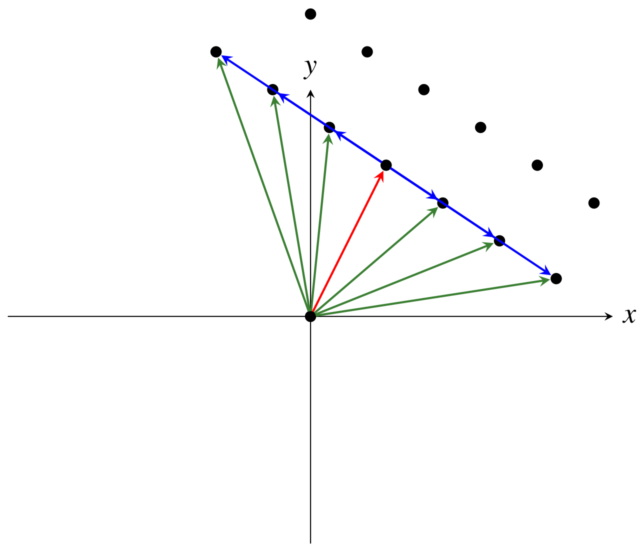
The span of $\mathbf{v}$ and $\mathbf{w}$ is the collection of vectors that can be written $c\mathbf{v}+d\mathbf{w}$. For example, $1.5\mathbf{v}-0.5\mathbf{w}$:



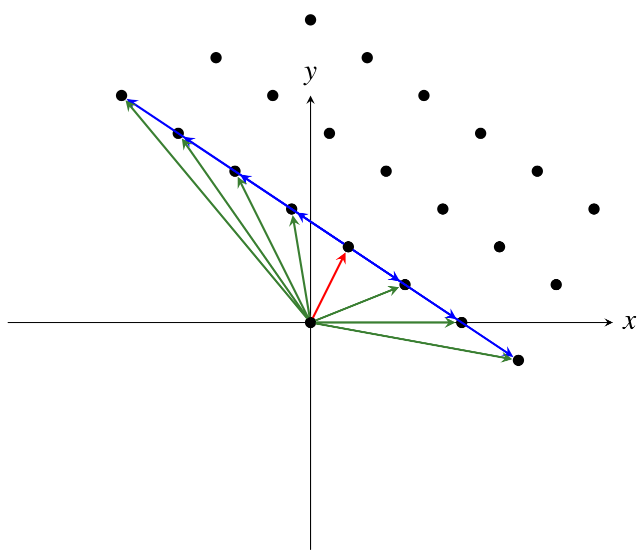
Let's plot a few more, fixing $c = 1.5$. That is, let's look at vectors of the form $1.5\mathbf{v} + d\mathbf{w}$ for various values of d.



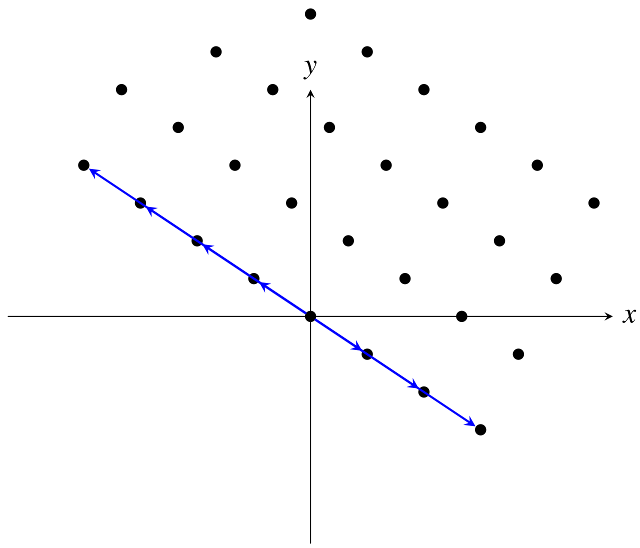
We can do the same, for $c = 1$.

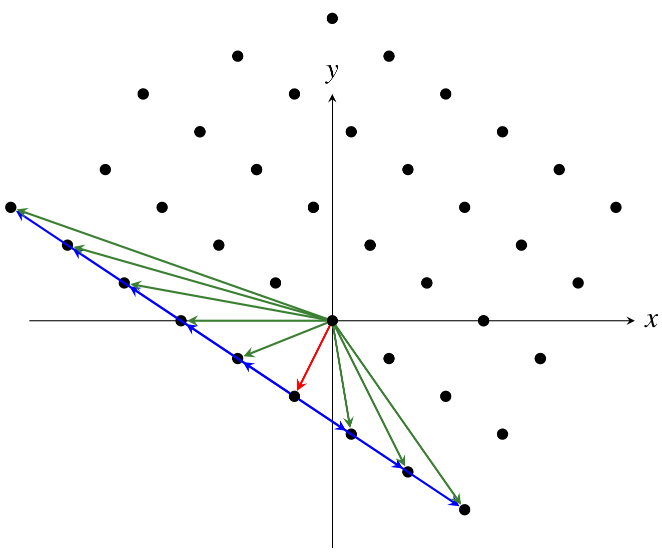


Keeping the points the tip has hit marked, let's also include $c = .5$



Before moving on, Let's throw in $c = 0$ (which are just multiples of $\mathbf{w}$) and $c = -.5$.





It starting to look like we can get anywhere in the plane. Indeed, imagine having two dials, one which modifies c and another which modifies d. Then c is changing the length of the red arrow, and d and changing the length of the blue arrow, tacked on to the tip of the red arrow. Then just by turning these dials, we should be able to get anywhere we want. This actually becomes even more clear when looking at an animation.[8] ■

In the previous example, we deduced (without any explicit computations) some geometric facts about the span of a pair of vectors!

■ **Slogan 2.5** The span of 2 nonzero and nonparallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$.

[8]For example `https://3b1b-posts.us-east-1.linodeobjects.com//content/lessons/2016/span/linear_combination.mp4`

Exercise 2.20 — Checkin 2. A group of exo-ecologists are experimenting with mixtures of gasses for a greenhouse in space. The gasses is a mix of Nitrogen, Oxygen, Carbon Dioxide, and Argon. They have 3 gas mixtures, whose compositions are given below (measured by mass).

- **Gas X:** 80% Nitrogen and 20% Oxygen.
- **Gas Y:** Pure Oxygen
- **Gas Z:** 60% Nitrogen, 30% Oxygen, 2% Carbon Dioxide, and 8% Argon

1. We represent a mixture containing a grams of Nitrogen, b grams of Oxygen, c grams of Carbon Dioxide, and d grams of Argon, by the column vector

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be the vectors representing the a mixture of *one gram* of gasses X, Y, and Z respectively. Write $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ as column vectors.
2. In plain english, describe what $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ represents.
3. The exo-ecologists would like a gas mixture $\mathbf{e}$ modelled on the concentration of gasses in earth's atmosphere, which they call *the eartlike mixture*. Fill in the blanks in the following sentence so that it means: *The earthlike mixture can be mixed from gasses X, Y, and Z.*

The vector(s) ___________ is/are a linear combination of the vector(s) ___________

4. Do you think the exo-biologists can mix pure Carbon Dioxide from their gas mixtures? Why or why not?

2.4.2 Homework 3

Exercise 2.21 We introduced vectors because in HW1 Problem 6 it looked like linear maps played well with adding points, suggesting that it might be more meaningful to think about the inputs and outputs as vectors rather than points. Let's try to make this more precise. To do this, we will shift our perspective slightly by letting $\mathbb{R}^2$ be the collection of 2-dimensional *vectors*:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ then is a rule:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

and can therefore be described by an equation for u and one for v. I want to emphasize that this is merely a shift of perspective (and notation), but the content is the same as in HW1.

1. Let $\mathbf{w} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ via the rule:

$$f(\mathbf{x}) = \mathbf{x} + \mathbf{w}.$$

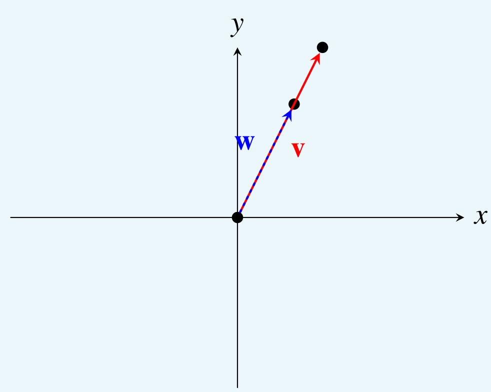
   (a) Write the two equations $u = u(x,y)$ and $v = v(x,y)$ representing f.
   (b) Is f a linear function? Why or why not?
   (c) If you determined f is a linear function, write down the 2×2 matrix associated to f. Otherwise skip this part.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the rule $g(\mathbf{x}) = 2\mathbf{x}$.
   (a) Write the two equations $u = u(x,y)$ and $v = v(x,y)$ representing g.
   (b) Is g is linear function? Why or why not?
   (c) If you determined g is a linear function, write down the 2×2 matrix associated to g. Otherwise skip this part.

Exercise 2.22 Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors in $\mathbb{R}^3$. Do you agree or disagree with the following statements? Explain your reasoning for each.

1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are in* $\operatorname{span}\{\mathbf{v},\mathbf{w}\}$, *the so is* $\mathbf{x}+\mathbf{y}$.
2. *If* $\mathbf{x}$ *is in* $\operatorname{span}\{\mathbf{v},\mathbf{w}\}$, *then so is* $c\mathbf{x}$ *for any constant* c.
3. *If* $\mathbf{x}$ *is in* $\operatorname{span}\{\mathbf{v},\mathbf{w}\}$, *then so is* $\mathbf{x}+\hat{\mathbf{i}}$.

Exercise 2.23 Let's think about a couple more spans in 2d.

1. Let $\mathbf{0}$ be the zero vector. Give a description of $\operatorname{span}\{\mathbf{0}\}$.
2. Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors which are parallel. What can you say about their span? In particular, would you say their span is all of $\mathbb{R}^2$? A line? A single point? Something else entirely? Explain your reasoning.



Exercise 2.24 Which of the following vectors in $\mathbb{R}^3$ are linear combinations of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$? Explain your reasoning.

1. The zero vector $\mathbf{0}$.
2. The column vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. The column vector $\begin{bmatrix} 0 \\ -5 \\ 11 \end{bmatrix}$
4. The vector $\hat{\mathbf{i}}$.

■

Exercise 2.25 We saw that the span of a single nonzero vector in $\mathbb{R}^2$ traces out a line. Let $\mathbf{v} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$, so that the tips of every vector in $\text{span}\{\mathbf{v}\}$ trace out a line. Find the equation of that line. (Recall, the equation of a line can be written $y = mx + b$ where m is the slope and b is the y-intercept. Can you find m and b?) ■

Exercise 2.26 Let's see if we can get some intution about the relationship between spans and dimension, without needing to do any explicit computation.

1. We saw that 2 vectors can span $\mathbb{R}^2$. Can fewer than 2 vectors do this? Why or why not?
2. Give an example of 3 vectors that can span $\mathbb{R}^3$. Do you think fewer that 3 vectors do this? Explain your answer (this explanation can be informal, it doesn't have to be a proof).
3. Let $\mathbb{R}^n$ be the collection of n-dimensional column vectors. How many vectors do you think are necessary to span all of $\mathbb{R}^n$?

■

2.5 February 14th, 2023

It's been a week of break. Let's begin by reminding ourselves of the definitions of linear combinations and spans, and do a few more examples. Recall that a *linear combination* of a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a vector $\mathbf{w}$ which can be written in terms of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. That is, there are constants $c_1, c_2, \cdots, c_n$ such that we can write:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The *span* of $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is the collection of *all vectors* $\mathbf{w}$ which are linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_n$. It is denoted:

$$\operatorname{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}.$$

2.5.1 Checkin 2 Comments: Meaningfully interpreting mathematics in context

Before moving on, I'd like to talk a bit about Checkin 2, in particular, part 2 (cf. Exercise 2.20.2). We are presented with 3 gas mixtures, gasses X,Y, and Z, which are combinations of Nitrogen, Oxygen, Carbon Dioxidee, and Argon. Each one is represented by a vector $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$, which records the components of the gas mixture (by weight) as its entries. We are then asked to describe *in plain english* the meaning of $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$. Many folks said something along the lines of:

> $\operatorname{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is the collection of linear combinations of $\mathbf{x}, \mathbf{y},$ and $\mathbf{z}$.

Of course, this is a correct definition of the span, and this sentence would be correct no matter what the context, and no matter what the vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ represent. But I don't think I would call this *plain english*, and it doesn't really have anything to do with the problem at hand (mixing gasses in a greenhouse in space). When working on a problem like this, we prefer to think about things *in context*.

Rather than give the answer,[9] let's take a moment to think about this in the context of Exercise 2.16, where we study two vectors $\mathbf{x}$ and $\mathbf{y}$ representing the effect of adding an ounce of pigment X and pigment Y respectively. Then we could be asked to describe $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ in plain english. One could give the definition of span: *all vectors that are linear combinations of* $\mathbf{x}$ *and* $\mathbf{y}$ but this would leave out all the context in the problem. Instead, in context, one could say something like:

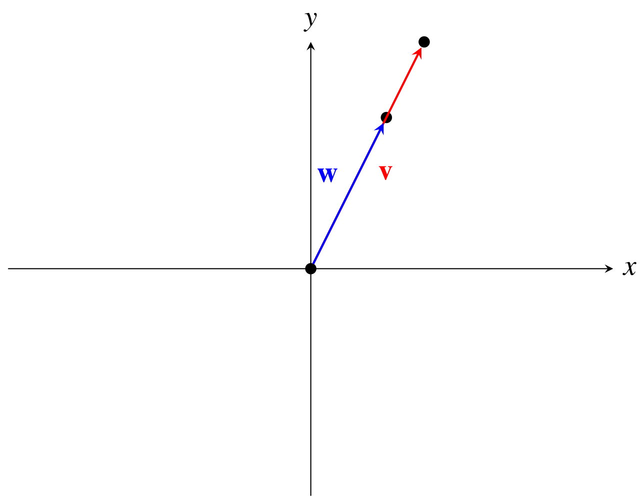
> $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$ represents all colors that can be mixed from pigment X and pigment Y.

This is an accurate description of the span, *and provides context*. It is a description that keeps hold of the fact that, while doing all this vector math, *we are talking about mixing colors*. The purely mathematical description of the span erases this context.

A few more examples of spans

■ **Question 2.7** Let $\mathbf{v}$ and $\mathbf{w}$ be nonzero vectors which are parallel. What can you say about their span?

[9] I will update the notes later, after checkin revisions have been collected



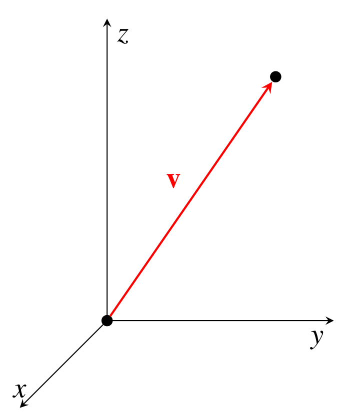
■ **Question 2.8** Can you say anything about the span of the zero vector $\mathbf{0}$?

To summarize, it looks like 3 things can happen:

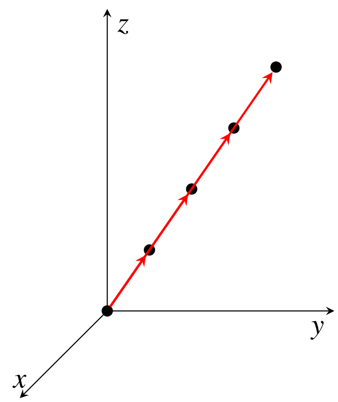
1. The span is just the zero vector. This happens if you are taking the span of the zero vector.
2. The span is a line. This happens if you are taking the span of a single nonzero vector, or of parallel vectors at least one of which is nonzero.
3. The span is all of $\mathbb{R}^2$. This happens if you take the span of at least two nonzero vectors which are not parallel.

What is remarkable is that that we were able to deduce all of this without doing any explicit computations. The story gets a bit more interesting if we move into 3-dimensions. As before, the span of the 0 vector will be just the origin, and with enough vectors it is possible to get all of $\mathbb{R}^3$ (for example, $\text{span}\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$). We can also also have again that the span of a single nonzero vector gives a straight line.

■ **Example 2.16** Let $\mathbf{v}$ be a nonzero vector.



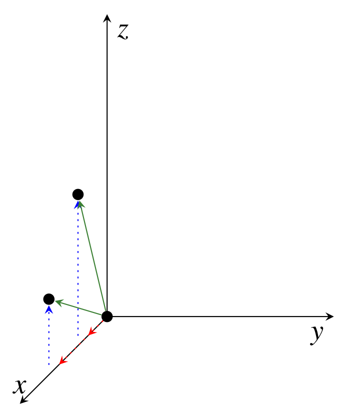
Taking a few multiples we have:



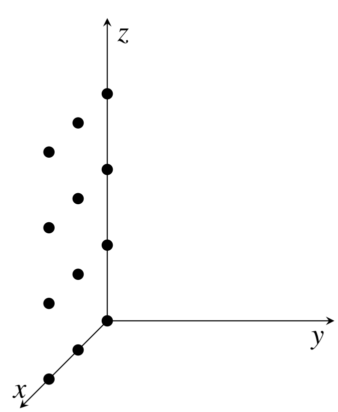
And we can observe that we are tracing out a line through the origin in $\mathbb{R}^3$. ∎

So we can get 0, everything, and a straight line. But something different can happen as well, when considering the span of two parallel vectors.

∎ **Example 2.17** Let's see if we can work out the span of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$. Below we plot a few different values.



It they lie in the xz-plane. Let's plot where the tips of $a\hat{\mathbf{i}}+b\hat{\mathbf{k}}$ for a few more values of a and b.



It's starting to appear that we can get anywhere in the xz-plane, but also that we can't escape from it.

This is actually not too hard to work out explicitly:

$$\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{k}} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\0\\b\end{bmatrix}.$$

So if we put the tail of $\mathbf{v}$ at the origin, then we can decide the x-coordinate of where its tip lands by choosing a, a similarly we can decide the z-coordinate of its tip by choosing b. The y-coordinate, on the other hand, must always stay at 0. ■

If you let $\mathbf{v}$ and $\mathbf{w}$ be two non-parallel vectors in $\mathbb{R}^3$, you can proceed as in Example 2.15 and think about where $c\mathbf{v} + d\mathbf{w}$. I encourage you to do this, and maybe you can convince yourself of the following fact:[10]

■ **Slogan 2.6** The span of 2 nonzero and nonparallel vectors in $\mathbb{R}^3$ trace out a plane in $\mathbb{R}^3$.

R Linear combinations are one of the most central and most important concepts in linear algebra. Even before doing any computations, we benefit greatly from thinking about how to visualize them and, and how to think about them in terms of concrete problems. This can be aided by well-made images and good animations. If you find 10 minutes to spare over break, I highly recommend Grant Sanderson's video on visualizing linear combinations: `https://www.3blue1brown.com/lessons/span`.

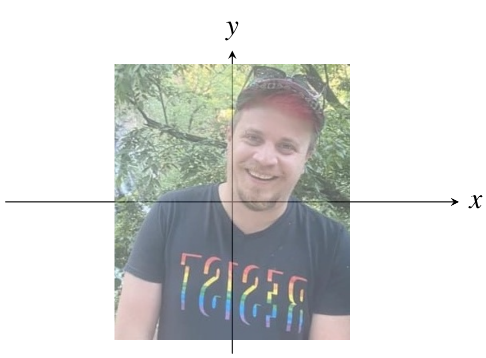
[10] TODO: Make an animation of this.

3. Linear Transformations and Matrices

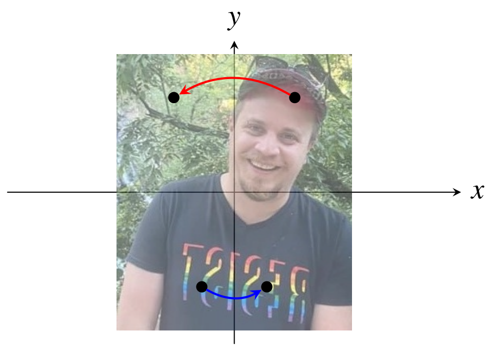
3.1 February 14th, 2023 (Continued...)

I was traveling over the past summer, and took the following selfie.

I didn't notice anything right away, but when I was reflecting back on my vacation photos, I realized that the logo on my shirt was backwards! *Oh no! I got a mirror image!* To fix this, I had to apply a horizontal reflection to the image. *But how?* First, let's introduce some coordinate axes.



With these coordinates chosen, the goal is to reflect the plane over the y-axis. For example, the point where my hair is colored pink is currently at $(1,2)$, but we should instead color my pink hair at the reflected point, $(-1,2)$. Similarly, the part of the letter S that is colored blue at $(-.5,-2)$, should instead be drawn blue at the reflected point $(.5,2)$.



We should do this for every pixel in the image. If the pixel at (x,y) is colored a certain way, we should instead color the pixel $(-x,y)$ that way. Doing this pixel by pixel, we recover the corrected image, with the text appearing legibly!

To summarize, we applied a certain *function* or *transformation* to the plane. In Chapter 1, we studied functions from the plane to itself, and this one certainly fits. Using that language, we defined a tranformation $R : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $R(x,y) = (u,v)$ where:

$$u = -x,$$

$$v = y.$$

Notice that this is even a *linear transformation* (cf. Defintion 1.2.1). We applied this linear transformation to the image, meaning that if a point $\mathbf{p}$ was supposed to be colored, say, red, we colored $R(\mathbf{p})$ red instead. This is a rather straightforward example, but many of the most importation transformations of images in computer graphics—from rotations and reflections, to fitting an image to a screen or window—are acheived by applying an appropriate linear transformation.

3.1.1 Linear Transformations of the Plane: Revisited

Recall from Definition 1.2.1, that a linear transformation of the plane was a function $L(x,y) = (u,v)$ where:

$$u = ax + by,$$

$$v = cx + dy,$$

for constants a, b, c, and d. We noticed in Homework 1 (cf. Exercise 1.8) that a linear transformation appeared to play well with addition, suggesting that it might be worth thinking the inputs and outputs of the function as vectors rather than points.

Notation 3.1. *Let $\mathbb{R}^2$ denote the collection of 2-dimensional column vectors* $\begin{bmatrix} x \\ y \end{bmatrix}$.

With this in hand we can revisit Definitions 1.2.1 and 1.2.2, with its notational updates. We remark that the difference is purely cosmetic, the content is identical.

Definition 3.1.1 A linear transformation is a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

where u and v are computed as follows:

$$u = ax + by,$$

$$v = cx + dy,$$

for constants a, b, c, d. The *2×2 matrix* associated to L is:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The four coefficients a, b, c, d completely determine L, therefore so does the matrix M. As such, we often just denote L just using the matrix itself:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

What is on the right looks like we are multiplying two matrices, and that is no accident. In fact, we know that this should equal the column vector

$$\begin{bmatrix} u \\ v \end{bmatrix},$$

whose formula is given in Definition 3.1.1. In particular, the first first row of the output should be $ax + by$ and the second should be $cx + dy$. This establishes our first formula for matrix multiplication:

Definition 3.1.2 — Matrix-Vector Multiplication: The 2×2 Case. The product of a 2×2 matrix M and a column vector $\mathbf{w}$ can be computed as follows:

$$M\mathbf{w} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

This recovers the usual definition of Matrix-Vector multiplication. Rather than just memorizing this formula, it is useful to think about how the computation as *a process*. Indeed, this process is what will generalize to matrix multiplication in general. To compute the matrix product, we first think about the first row of M and pair its entries in order with the entries in $\mathbf{w}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We then multiply the paired elements together, and add them up to obtain the first row of $M\mathbf{w}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ \end{bmatrix}.$$

To obtain the second row of the output, we pair the second row of M with the entries of $\mathbf{w}$ the same way, multiplying the corresponding elements and adding them together.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

This is a process you may already be familiar with, but when I first learned how to multiply matrices, I found the rule to feel kind of opaque and arbitrary. Hopefully Definition 3.1.1 makes this process feel more reasonable. Indeed, the *product* $M\mathbf{v}$ should be thought of as the *function* M being applied to the *vector* $\mathbf{v}$.

■ **Example 3.1** Let's compute the matrix product:

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Starting with the first row:

$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1*-1+3*2 \\ \end{bmatrix} = \begin{bmatrix} 5 \\ \end{bmatrix}$$

Filling in the second row:

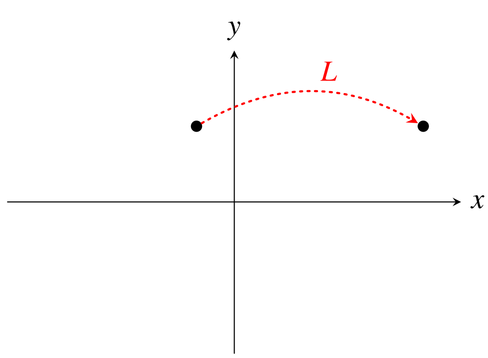
$$\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1*-1+3*2 \\ -2*-1+0*2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

To interpret this as a function, we see that the linear transformation L given by the rules:

$$u = x+3y$$

$$v = -2x,$$

takes $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Viewing these vectors as points:



■

■ **Question 3.1** Let M be the 2×2 matrix from Example 3.1, compute $M\hat{\mathbf{i}}$ and $M\hat{\mathbf{j}}$.

■ **Example 3.2** The linear transformation which reflected my picture into the correct orientation was given by:

$$u = -x,$$

$$v = y,$$

and therefore by the matrix:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Applying this to any point gives:

$$\begin{bmatrix} -1n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1*x+0*y \\ 0*x+1*y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

In particular, a computer would likely apply this function by using matrix multiplication on the coordinates of the pixels, rather than remember the functions and all the variables involved. ■

In Homework 1 (cf. Exercise 1.6) we asked the following question:

■ **Question 3.2** Let ℓ be a linear transformation of the plane. If I know $\ell\left(\hat{\mathbf{i}}\right)$ and $\ell\left(\hat{\mathbf{j}}\right)$, do I know the output of ℓ when applied to any element of $\mathbb{R}^2$?

Many folks already determined the answer. Let's see how this works in an example.

■ **Example 3.3** Suppose ℓ is a linear transformation, and that:

$$\ell\left(\hat{\mathbf{i}}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad \ell\left(\hat{\mathbf{j}}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

We also know ℓ corresponds to some matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Therefore

$$M\hat{\mathbf{i}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a*1+b*0 \\ c*1+d*0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}.$$

In particular, $M\hat{\mathbf{i}}$ plucks out the first column of M, and since $M\hat{\mathbf{i}} = \ell\left(\hat{\mathbf{i}}\right)$ we have:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus $a = 1$ and $c = -2$. We can similarly compute that $M\hat{\mathbf{j}}$ is the second column of M:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a*0+b*1 \\ c*0+d*1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Since this is $\ell\left(\hat{\mathbf{j}}\right)$ we have determined $b = 3$ and $d = 0$. In particular, we have determined:

$$M = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix},$$

which completely determines ℓ as the function:

$$u = x - 2y,$$

$$v = -2x.$$

This is in fact the function from Example 3.1. ■

To summarize, we have the following theorem.

Theorem 3.1.1 Let ℓ be a linear transformation of $\mathbb{R}^2$. Then ℓ is completely determined by the values $\ell\left(\hat{\mathbf{i}}\right)$ and $\ell\left(\hat{\mathbf{j}}\right)$. In particular, if:

$$\ell\left(\hat{\mathbf{i}}\right) = \begin{bmatrix} a \\ c \end{bmatrix} \qquad \text{and} \qquad \ell\left(\hat{\mathbf{j}}\right) = \begin{bmatrix} b \\ d \end{bmatrix},$$

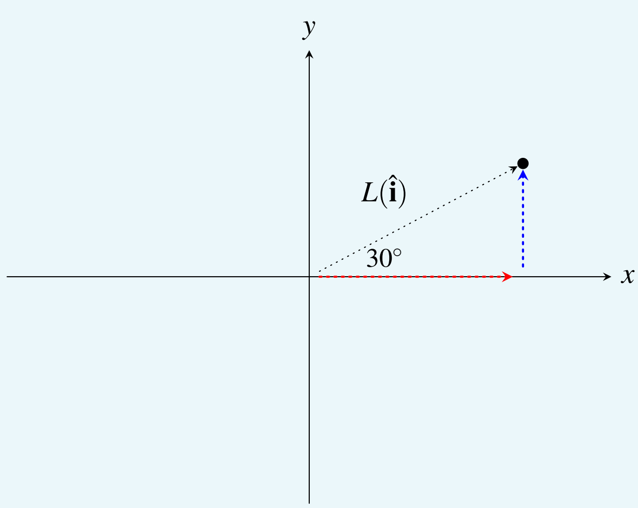
then the matrix associated to ℓ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

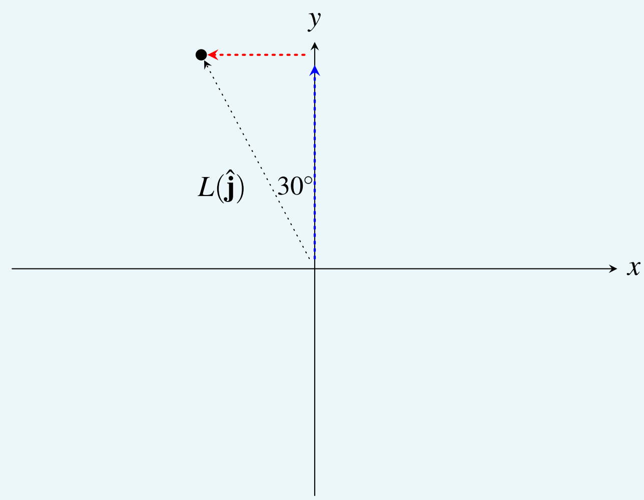
Exercise 3.1 — Checkin 3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates the plane 30° counterclockwise. Let's determine the associated matrix. You may use the following facts:

$$\sin(30^\circ) = \frac{1}{2} = 0.5 \qquad \text{and} \qquad \cos(30^\circ) = \frac{\sqrt{3}}{2} \approx 0.866.$$

1. Find the lengths of the legs of the following triangle to determine $L(\hat{\mathbf{i}})$ as a column vector. (Note, since $\hat{\mathbf{i}}$ has length one, so does any rotation of $\hat{\mathbf{i}}$.)



2. Find the lengths of the legs of the following triangle to determine $L(\hat{\mathbf{j}})$ as a column vector. *Be careful with signs!* (Note, since $\hat{\mathbf{j}}$ has length one, so does any rotation of $\hat{\mathbf{j}}$.)

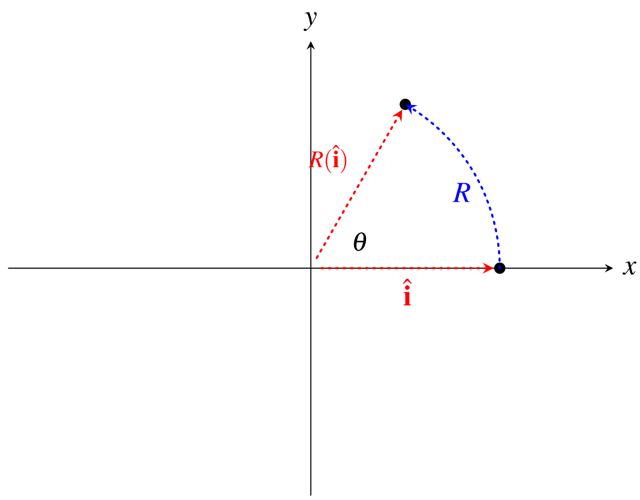


3. Let M be the matrix associated to the linear transformation L. Write down M. (Recall that $L(\hat{\mathbf{i}})$ and $L(\hat{\mathbf{j}})$ determine the columns of M).
4. Use matrix-vector multiplication to determine the image of the point $(-3,1)$ after a $30°$ rotation. (You may leave your answer in exact form, or save one decimal point).

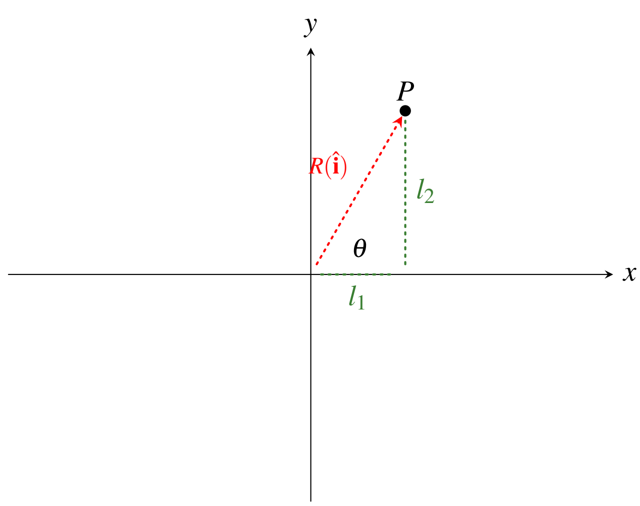
3.2 February, 16, 2023

3.2.1 An Example: Rotation Matrices

We can do something similar to what we did in Checkin 3 (Exercise 3.1) to compute matrices which can capture *any rotation of the plane*! Let's denote by R the linear function which rotates the plane counterclockwise by an angle of θ. To compute the matrix associated to R, it suffices to trace what the rotation does to $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Let's start with $\hat{\mathbf{i}}$.



Here $R\left(\hat{\mathbf{i}}\right)$ starts at the origin, so to find it's representation as a column vector we can simply compute the coordinates of the point P where it ends. To do this we can do this by using the triangle below, together with some trig.



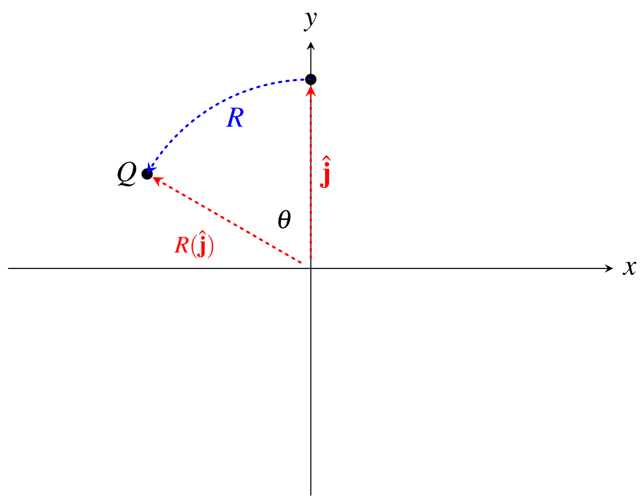
Once we compute the lengths l_1 and l_2, we will know that $P = (l_1, l_2)$. The length of $\hat{\mathbf{i}}$ is 1, and this remains true after rotation. As such, we can solve:

$$\cos\theta = \frac{l_1}{1}, \qquad \text{and} \qquad \sin\theta = \frac{l_2}{1}$$

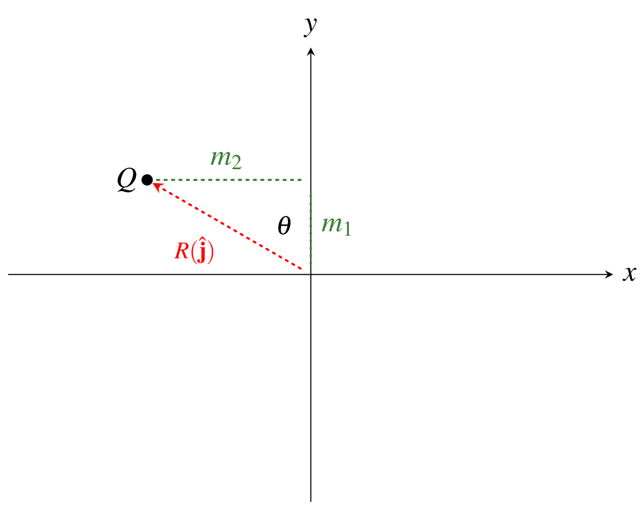
for $l_1 = \cos\theta$ and $l_2 = \sin\theta$. Therefore, $R(\hat{\mathbf{i}})$ is the vector from $(0,0)$ to $(\cos\theta, \sin\theta)$, so that we have:

$$R(\hat{\mathbf{i}}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

We can find $R(\hat{\mathbf{j}})$ similarly, looking for the coordinates of the point Q below



Again, we can can find the coordinates for Q by computing the lengths of the legs of the triangle below.



As above, because the length of $\hat{\mathbf{j}}$ is 1, length of $R(\hat{\mathbf{j}})$ is too, so that the trigonometric ratios tell us $m_1 = \cos\theta$ and $m_2 = \sin\theta$. Keeping signs in mind, we can conclude that $Q = (-m_2, m_1)$ so that:

$$R(\hat{\mathbf{j}}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

By Theorem 3.1.1, we can now conclude that the transformation R is given by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In summary, we have deduced the following result:

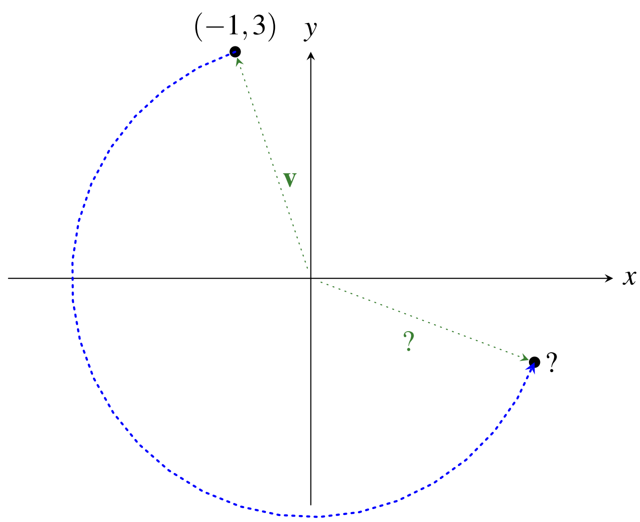
Proposition 3.2.1 Let R be the linear transformation which rotates the plane counterclockwise by an angle of θ, the R is given by the matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

> **R** The careful reader may notice that our arguments above seemed to rely on the fact that θ was an acute angle. In fact, it is always true that the point unit circle making an angle of θ with the x-axis has coordinates $(\cos\theta, \sin\theta)$. This is actually *the definition* of the trigonometic functions for angles which are not acute. I encourage you to carefully work out the details!

Proposition 3.2.1 is a extremely powerful result, allowing for the rapid computation of any rotation using just matrix multiplication. This is very useful for rotating images on a screen, as doing trig in real time can be slow, but multiplying by matrices is quite fast!

■ **Example 3.4** If I rotate the plane 231°, where does the point $(-1,3)$ end up?



Proposition 3.2.1 tells us that the function R which rotates the plane 231° is given by the matrix:

$$M = \begin{bmatrix} \cos(231°) & -\sin(231°) \\ \sin(231°) & \cos(231°) \end{bmatrix} = \begin{bmatrix} -0.629 & 0.777 \\ -0.777 & -0.629 \end{bmatrix}.$$

Letting $\mathbf{v}$ be the vector from $(0,0)$ to $(-1,3)$, we can compute:

$$M\mathbf{v} = \begin{bmatrix} -.629 & .777 \\ -.777 & -.629 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -.629*(-1)+.777*3 \\ -.777*(-1)-.629*3 \end{bmatrix} = \begin{bmatrix} 2.96 \\ -1.11 \end{bmatrix}.$$

Therefore we can conclude that after rotating 231°, the point $(-1,3)$ moves to the point $(2.96, -1.11)$.
■

3.2.2 Linear Transformations and Linear Combinations

The entire *vector perspective* was motivated by Exercise 1.8, which suggested that a linear transformation plays well with addition. Thinking of the inputs and outputs as vectors, the property Exercise 1.8 suggested is that, for any linear transformation L and for any pair of 2d vectors $\mathbf{v}$ and $\mathbf{w}$, we have:

$$L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v})+L(\mathbf{w}).$$

If the matrix associated to L is M, then we can rewrite this in terms of matrix-vector multiplication:

$$M(\mathbf{v}+\mathbf{w}) = M\mathbf{v}+M\mathbf{w}.$$

But this is a familiar looking property: *the distributive property*! In particular, the fact that L commutes with addition is equivalent to the fact that matrix-vector multiplication satisfies the distributive property! We package this fact together with a related fact regarding scalar multiplication together in the following theorem.

Theorem 3.2.2 — Linearity of Linear Transformations: Planar Case. Let $L:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, let $\mathbf{v},\mathbf{w} \in \mathbb{R}^2$ be vectors, and let c be a constant.

1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v})+L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

Proof. We will prove the first statement. To do this, we introduce some notation. Let's represent L by the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and represent $\mathbf{v}$ and $\mathbf{w}$ by the column vectors:

$$\mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Then filling in the first row of $M(\mathbf{v}+\mathbf{w})$ gives:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0+x_1 \\ y_0+y_1 \end{bmatrix} = \begin{bmatrix} a(x_0+x_1)+b(y_0+y_1) \\ \dots \end{bmatrix}.$$

On the other hand, filling in the first row of $M\mathbf{v}+M\mathbf{w}$ gives:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_0+by_0 \\ \dots \end{bmatrix} + \begin{bmatrix} ax_1+by_1 \\ \dots \end{bmatrix} = \begin{bmatrix} ax_0+by_0+ax_1+by_1 \\ \dots \end{bmatrix}.$$

The fact that these two first rows agree is simply the distributive property for usual addition. One can observe the second rows agree by an identical argument.[1]

We will leave the second part of the theorem for homework. ■

[1] do it! do it! do it!

R One can prove Theorem 3.2.2 without comparing it to matrix multiplication, and just by plugging in generic values to the formula for a linear transformation. That being said, viewing this a matrix-vector multiplication, one can observe that the resemblance of Theorem 3.2.2.1 to the distributive property for matrix-vector multiplication is not purely cosmetic, it really boils down to the usual distributive property for numbers in each row.

■ **Question 3.3** Let L be a linear transformation, and suppose that $\mathbf{v}$ and $\mathbf{w}$ be two vectors, and suppose that $\mathbf{x}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. Is it always true that $L(\mathbf{x})$ is a linear combination of $L(\mathbf{v})$ and $L(\mathbf{w})$?

In Question 3.3, we asked about what happens to linear combinations when we apply a linear transformation. Let's briefly revisit this to get a more geometric perspective on Theorem 3.1.1. If $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$, then (applying Theorem 3.2.2), we can see that:

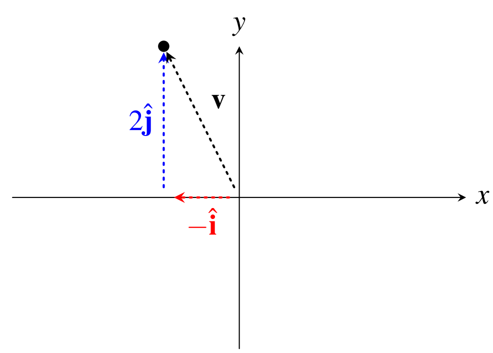
$$L(\mathbf{x}) = L(a\mathbf{v} + b\mathbf{w}) = L(a\mathbf{v}) + L(b\mathbf{w}) = aL(\mathbf{v}) + bL(\mathbf{w}).$$

In particular, if $\mathbf{v} = \hat{\mathbf{i}}$ and $\mathbf{w} = \hat{\mathbf{j}}$, then a and b are the coordinates of $\mathbf{x}$, and we see that knowing these coordinates of together with the images of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ allows us to determin $L(\mathbf{x})$. Let's see this in the context of Example 3.3.

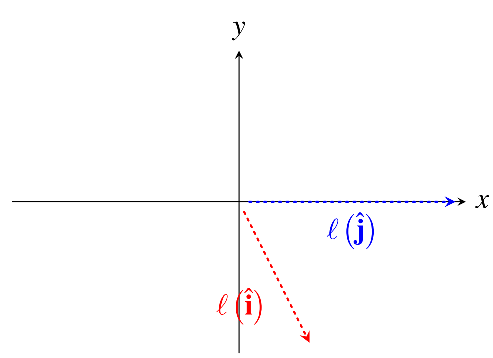
■ **Example 3.5** Adopt the setup of 3.1 and 3.3, and consider again the vector:

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}.$$

In particular, $\mathbf{v}$ is acheived by doing $\hat{\mathbf{i}}$ backwards, and then doing $\hat{\mathbf{j}}$ twice.



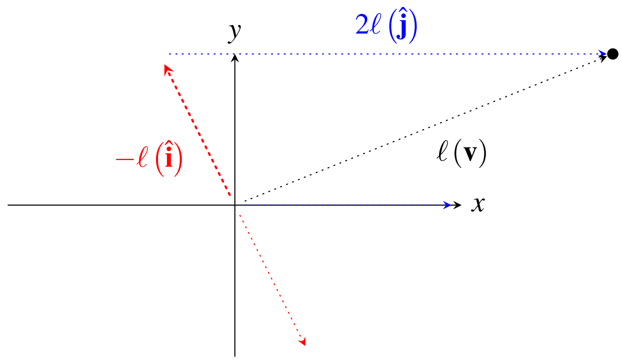
Now let's take a look at where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go after applying ℓ.



Theorem 3.1.1 suggests that knowing these values should be enough to know $\ell(\mathbf{v})$. And indeed, applying Theorem 3.2.2 we have:

$$\ell(\mathbf{v}) = \ell\left(-\hat{\mathbf{i}}+2\hat{\mathbf{j}}\right) = -\ell\left(\hat{\mathbf{i}}\right)+2\ell\left(\hat{\mathbf{j}}\right).$$

So, because $\mathbf{v}$ does $\hat{\mathbf{i}}$ backwards and then $\hat{\mathbf{j}}$ twice, we know $\ell(\mathbf{v})$ does $\ell\left(\hat{\mathbf{i}}\right)$ backwards and then $\ell\left(\hat{\mathbf{j}}\right)$ twice. Let's throw that in the picture:



Throwing in some numbers: we can see that:

$$\ell(\mathbf{v}) = -\ell\left(\hat{\mathbf{i}}\right)+2\ell\left(\hat{\mathbf{j}}\right) = -\begin{bmatrix}1\\-2\end{bmatrix}+\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix},$$

which agrees with our output from Example 3.1 (as it must). The way to think about this is that when ℓ moves $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, it drags the entire grid along with it.[2] ■

3.2.3 Linear Transformations of 3-space

So far we've been pretty focused on transformations of the plane, but in is also quite important in practice move beyond the plane. Let's begin by considering transformations of 3-dimensional space. For this it will be useful to give a definition of $\mathbb{R}^3$ which is analogous to that of $\mathbb{R}^2$.

Definition 3.2.1 The set $\mathbb{R}^3$ is the collection of 3-dimensional column vectors $\begin{bmatrix}x\\y\\z\end{bmatrix}$

As before, we will sometimes conflate the idea of a 3d column vector with that of a 3d point (x,y,z) if our perspective is spacial. We actually already talked a bit about transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ in Homework 1 (Exercise 1.9). In particular, a function $T:\mathbb{R}^3\to\mathbb{R}^3$ can be defined by the rule:

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}u\\v\\w\end{bmatrix},$$

and we can define u, v, and w in terms of x, y, z:

$$u = u(x,y,z),$$

[2]This is animated brilliantly by Grant Sanderson: `https://3b1b-posts.us-east-1.linodeobjects.com//content/lessons/2016/linear-transformations/basis_example2.mp4`

$$v = v(x,y,z),$$
$$w = w(x,y,z).$$

■ **Example 3.6** Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the rule

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where

$$u = xyz,$$
$$v = x + y + z,$$
$$w = 1 + 2x.$$

Then, for example, we can compute:

$$T\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2*3*4 \\ 2+3+4 \\ 1+2*2 \end{bmatrix} = \begin{bmatrix} 24 \\ 9 \\ 5 \end{bmatrix}.$$

■

As with transformations of the plane, linear algebra focuses on functoins which are *purely linear*. That is, we want the equations of u, v, and w to be polynomials of degree 1, with no constant terms.

Definition 3.2.2 A function $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a *linear transformation* if it can be defined by the rule:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where:

$$u = ax + by + cz,$$
$$v = dx + ey + kz,$$
$$w = lx + my + nz,$$

for constants $a, b, c, d, e, k, l, m, n$.

Notice that the entire function is determined by the cofficients of x, y, and z in the equations for u, v, and w. Therefore, it is enough to remember just these coefficients, which we can arrange in a matrix.

Definition 3.2.3 The matrix associated to the linear transformation from Definition 3.2.2 is:

$$\begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

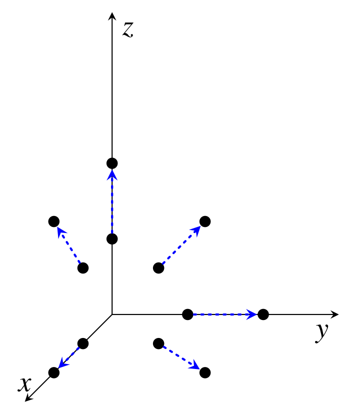
■ **Example 3.7** Consider the function $L : \mathbb{R}^3 \to \mathbb{R}^3$ which is given by the rules:

$$u = 2x \qquad v = 2y \qquad w = 2z.$$

The matrix associated to this function is:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let's visualize this as a transformation of space, like we did back at the beginning of the semester. Below we have a series of points. The points before and after applying T are connected by arrows.



It looks like points are being pushed away from the origin, and indeed:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so T takes a point and doubles its distance from the origin.[3] ■

The fact that we can replace a function with a matrix tells us that we can get a formula for matrix-vector multiplication in 3d, analogous to Definition 3.1.2. Indeed, adopting the notation of Definitions 3.2.2 and 3.2.3, if T is the transformation associated to:

$$M = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

Then we can write:

$$\begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Plugging in for the formulas of u, v, w gives:

[3] TODO: Make an animation for this example

Definition 3.2.4 — Matix-Vector Multiplication: The 3×3 Case. The product of a 3×3 matrix and a column vector $\mathbf{v}$ can be computed as follows.

$$M\mathbf{v} = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+kz \\ lx+my+nz \end{bmatrix}.$$

This can follow a process just like in the 2×2 case, by going row by row in the matrix, and pairing each entry in the row with the appropriate entry in the vector.

$$\textbf{Row 1:} \quad \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ \\ \\ \end{bmatrix}$$

$$\textbf{Row 2:} \quad \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+kz \\ \\ \end{bmatrix}$$

$$\textbf{Row 3:} \quad \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+by+cz \\ dx+ey+kz \\ lx+my+nz \end{bmatrix}$$

■ **Question 3.4** Compute the matrix product:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

As we get to bigger matrices, the process for matrix multiplication starts to get more tedious, and we will start to use computers to do it for us. But we will keep doing it by hand, just a little bit longer, until we can identify the general pattern and its meaning.

■ **Question 3.5** Let M be the 3×3 matrix from Question3.4. Compute:

$$M\hat{\mathbf{i}}, \quad M\hat{\mathbf{j}}, \quad M\hat{\mathbf{k}}.$$

Do you notice anything?

As you probably observed, we recovered the columns of M. In particular, a version of Theorem 3.1.1 holds true for 3×3 matrices as well.

Theorem 3.2.3 Let T be a linear transformation of $\mathbb{R}^3$. Then T is completely determined by the values $T\left(\hat{\mathbf{i}}\right)$, $T\left(\hat{\mathbf{j}}\right)$, and $T\left(\hat{\mathbf{k}}\right)$. In particular, if:

$$T\left(\hat{\mathbf{i}}\right) = \begin{bmatrix} a \\ d \\ l \end{bmatrix} \quad \text{and} \quad T\left(\hat{\mathbf{j}}\right) = \begin{bmatrix} b \\ e \\ m \end{bmatrix} \quad \text{and} \quad T\left(\hat{\mathbf{k}}\right) = \begin{bmatrix} c \\ k \\ n \end{bmatrix},$$

then the matrix associated to T is:

$$M = \begin{bmatrix} a & b & c \\ d & e & k \\ l & m & n \end{bmatrix}.$$

This theorem is very important in 3D modelling. Indeed, a rotation in 3 space is a complicated manuver to pin down, as there are 3-axes about which to rotate. That said, once you know where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ go, you can write down a matrix which completely captures the rotation! We will do this explicitly next week! Before moving on, we'd like to record that Theorem 3.2.2 holds true here as well, and can be computed directly using matrix multiplicaiton.

Theorem 3.2.4 — Linearity of Linear Transformations: 3D Case. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be vectors, and let c be a constant.

1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

3.3 Homework 4

Exercise 3.2 Compute the following Matrix-Vector products.

1. $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
3. $\begin{bmatrix} 8 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$
4. $\begin{bmatrix} 8 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

■

Exercise 3.3 A linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ has the following effects on $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{j}}$:

$$T(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T(\hat{\mathbf{j}}) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \qquad T(\hat{\mathbf{k}}) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

What is $T(3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 17\hat{\mathbf{k}})$? ■

Exercise 3.4 A linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ has the following effects on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

$$L(\hat{\mathbf{i}}) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \qquad \text{and} \qquad L(\hat{\mathbf{j}}) = \begin{bmatrix} -7 \\ 13 \end{bmatrix}.$$

What has a larger magnitude:

$$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) \quad \text{or} \quad L\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)?$$

■

Exercise 3.5 Consider the following 2 vectors in $\mathbb{R}^2$.

$$\mathbf{v} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix}1\\-1\end{bmatrix}.$$

1. Do you think $\operatorname{span}\{\mathbf{v},\mathbf{w}\} = \mathbb{R}^2$? Why or why not?
2. Let $\mathbf{x} = 7\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. (*Hint:* Extract a system of equations from the expression $a\mathbf{v} + b\mathbf{w} = \mathbf{x}$, and then solve for the constants a and b).
3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose that:

$$T(\mathbf{v}) = \begin{bmatrix}3\\2\end{bmatrix} \qquad \text{and} \qquad T(\mathbf{w}) = \begin{bmatrix}-6\\-5\end{bmatrix}.$$

Compute $T(\mathbf{x})$ by writing it as a linear combination of $T(\mathbf{v})$ and $T(\mathbf{w})$.

■

Exercise 3.6 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^2$ whose span is all of $\mathbb{R}^2$. Do you agree or disagree with the following statement? Explain your reasoning. (Use the intuition gained from Question 3.5.)

The values $L(\mathbf{v})$ and $L(\mathbf{w})$ determine $L(\mathbf{x})$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$.

■

Exercise 3.7 Let's prove Theorem 3.2.2.2. It states the following: If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $\mathbf{v}$ is a vector in $\mathbb{R}^2$, and n is a constant. Then

$$nL(\mathbf{v}) = L(n\mathbf{v}).$$

To do this, we'll introduce some notation. We denote the matrix associated to L by:

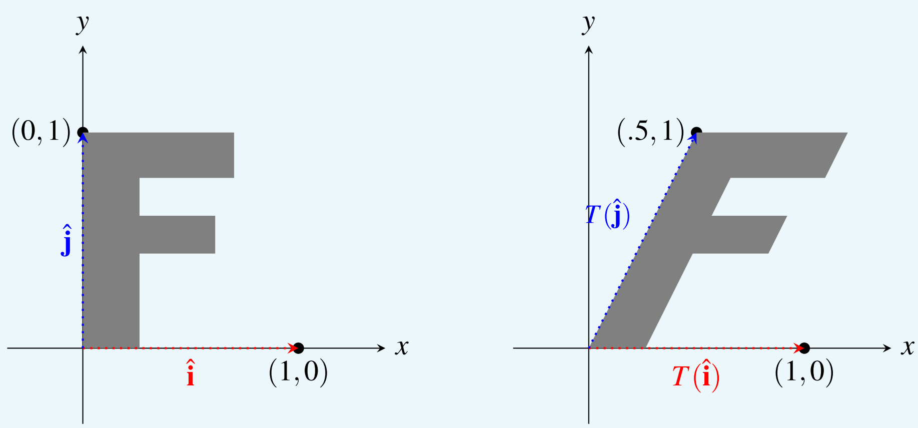
$$M = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

We also give coordinates to $\mathbf{v}$:

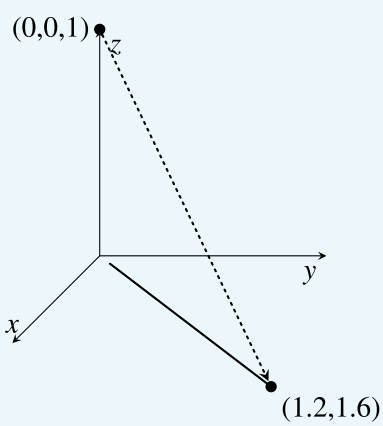
$$\mathbf{v} = \begin{bmatrix}x\\y\end{bmatrix}.$$

1. Write a column vector for $n(M\mathbf{v})$ in terms of the variables a,b,c,d,n,x, and y. (First do the Matrix-Vector product $M\mathbf{v}$, then scale the result by n).
2. Write a column vector for $M(n\mathbf{v})$ in terms of the variables a,b,c,d,n,x, and y. (First write $n\mathbf{v}$ as a column vector in terms of n,x, and y, and then multiply this column by M).
3. Compare your answers to (a) and (b) to explain why the Theorem is true.

Exercise 3.8 A computer translates images from blockstyle fonts to *italics* by applying a linear transformation called a *shear*. Below is an image of the letter F before and after applying the shear. Use this image to determine the matrix associated to the shearing transformation.



Exercise 3.9 We can use linear maps to calculate how shadows are cast. Choose some coordinates in meters, put a meterstick vertically at the origin, and measure that it casts its shadow on the point 1.2 meters east and 1.6 meters north of the stick.



We define a linear function $S : \mathbb{R}^3 \to \mathbb{R}^2$ which takes a vector point in space (denoted by a vector in $\mathbb{R}^3$) to the point at which it casts its shadow (denoted by a vector in $\mathbb{R}^2$). **Notice: the input of this function is 3-dimensional, and the output is 2-dimensional. In particular, the general**

setup looks something like:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix}.$$

1. What are $S(\hat{\mathbf{i}}), S(\hat{\mathbf{j}}), S(\hat{\mathbf{k}})$? Your answers should be 2D vectors. (*Hint*: You should be able to extract $S(\hat{\mathbf{k}})$ from the picture above. For the other two...where does a point on the ground cast its shadow?)
2. We've seen that many linear maps can be captured by matrices, and that the columns of these matrices can be recovered by where the standard basis vectors are sent. Use this philosophy to write down a matrix which could represent S.
3. Extract some equations which could represent $S : \mathbb{R}^3 \to \mathbb{R}^2$ from the matrix in part (b).
4. The Eiffel Tower stands 330 meters tall. It also happens to be 500m west and 350m north from where you planted your meter stick at the start of this exercise. Use the work you've done to compute the coordinates of where the tip of its shadow should land (you may assume that Paris is completely flat). Try this with the equations from question 3, *and* with matrix vector multiplication.
5. Do you agree or disagree with the following statement? Explain your reasoning.

   Once I know where a single point *above the ground* casts its shadow, I can compute where any point casts its shadow.

■

3.4 February 21, 2023

3.4.1 Linear transformations between dimensions

We've talked for the time being about transformations from $\mathbb{R}^2$ to itself and from $\mathbb{R}^3$ to itself, but it is also sometimes important to think about transformations between different spaces. We've also been mainly thinking about $\mathbb{R}^2$ and $\mathbb{R}^3$ spacially, but sometimes in context we encounter linear transformations have sources and targets which aren't obviously spacial, but instead have some other concrete interpretations. In fact, *we have already done both of these things!* To see this, let's return to the paint mixing examples from Homework 2 (Exercises 2.16-2.19), and see how there was a linear transformation lying at the heart of it, even if it didn't quite look like that at the time.

To begin let's assign some specific meanings to vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ which fit this context.

$$\mathbb{R}^2 = \left\{ \text{vectors } \begin{bmatrix} x \\ y \end{bmatrix} \text{ representing } x \text{ oz Pigment X and } y \text{ oz Pigment Y} \right\}.$$

$$\mathbb{R}^3 = \left\{ \text{vectors } \begin{bmatrix} r \\ g \\ b \end{bmatrix} \text{ representing } r \text{ units red, } g \text{ units green, and } b \text{ units blue} \right\}.$$

Then we can define a function:

$$F : \mathbb{R}^2 \to \mathbb{R}^3,$$

by the rule:

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{the vector } \begin{bmatrix} r \\ g \\ b \end{bmatrix} \text{ representing the color mixed from } x \text{ oz Pigment X and } y \text{ oz Pigment Y}.$$

In fact, we can find equations for r, g, and b in terms of x and y. Indeed, we are given what the overall effects of our two pigments are.

Pigment X: Adding 1 ounce of Pigment X to paint adds:

1 unit of red, 2 units of green, 3 units of blue

Pigment Y: Adding 1 ounce of Pigment Y to paint adds:

7 units of red, 5 units of green, 2 units of blue

Therefore, we can compute:

$$r = 1 * x + 7 * y,$$

$$g = 2 * x + 5 * y,$$

$$b = 3 * x + 2 * y.$$

So we have computed equations for the function $F : \mathbb{R}^2 \to \mathbb{R}^3$. Even better, we can notice that r, g, and b are *linear functions* in x and y with no constant terms. So this looks like a linear function overall. This is completely determined by its coefficients, so following what we've done before, we can see that we know the whole function by just remember its coefficient matrix:

$$M = \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix}.$$

What happens if we try to extend the process of Matrix-Vector Multipication to this setting? Let's try with 3 ounces of say, 3 ounces of Pigment X and 3 ounces of Pigment Y. Well, going row by row:

$$\text{Row 1:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ \\ \\ \end{bmatrix},$$

$$\text{Row 2:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ 2*3+5*3 \\ \\ \end{bmatrix},$$

$$\text{Row 3:} \quad \begin{bmatrix} 1 & 7 \\ 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1*3+7*3 \\ 2*3+5*3 \\ 3*3+2*3 \end{bmatrix} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix}.$$

Looks like the output is the vector representing the color containing

24 units of red, 21 units of green, 15 units of blue,

which is fancy gold! We learned in Exercise 2.17 tha this is exactly what we should get when mixing 3 ounces of Pigment X and 3 ounces of Pigment Y. That is:

$$M \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 24 \\ 21 \\ 15 \end{bmatrix} = F\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right),$$

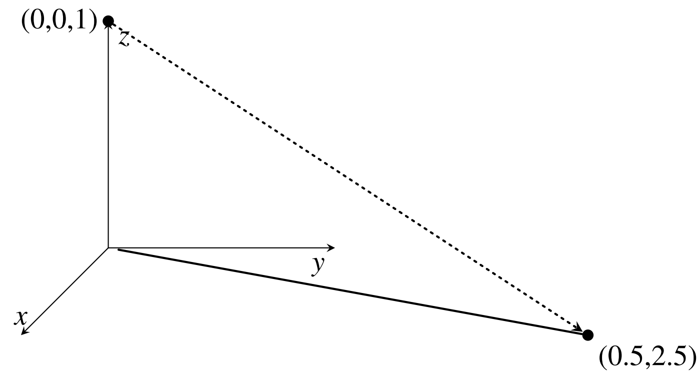
so as before, *matrix multiplication corresponds to applying the linear function!*.

■ **Question 3.6** Let $\hat{\mathbf{i}}$ be the vector in $\mathbb{R}^2$ representing one ounce of Pigment X, and let $\hat{\mathbf{j}}$ represent one ounce of Pigment Y. Compute:

$$M\hat{\mathbf{i}} \qquad \text{and} \qquad M\hat{\mathbf{j}}.$$

As you might observe, you yet again obtain the columns of M, so as before, so these values determine all of F and Theorem 3.1.1 holds! Of course, here it is no surprise that knowing the effect of one ounce of Pigment X and one ounce of Pigment Y is enough to tell you the effect of any mixture of them. It seems like adding this type context makes certain results easier or expect than it is initially in the purely geometric setting.

Let's see if we can run this philosophy in reverse. On last Thursday's groupwork (cf. Exercise 3.9) we played with another example of a linear map between dimensions, this time from $\mathbb{R}^3 \to \mathbb{R}^2$. Since we're finishing this on the homework, I'll use different numbers, but the idea will be exactly the same. We choose coordinates (in meters), and place a meterstick vertically at the origin, just outside the library at ODY. It casts a shadow to a point on the ground 0.5 meters east and 2.5 meters north of the stick.



We then defined a linear function $S : \mathbb{R}^3 \to \mathbb{R}^2$ defined as follows:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

if a point x meters east and y meters north of the origin and z-meters off the ground casts its shadow on the ground at a point u-meters east and v-meters north of the origin. So for example, our picture tells us that:

$$S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0.5 \\ 2.5 \end{bmatrix}.$$

■ **Question 3.7** What are $S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ and $S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$?

Every linear map we've seen so far comes with a matrix, and the columns of that matrix are precisely the values of the linear map applied to the standard basis. Let's see what that philosophy can tell us in this situation. Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be the standard basis of $\mathbb{R}^3$. Then S should have some matrix whose columns are $S(\hat{\mathbf{i}}), S(\hat{\mathbf{j}}), S(\hat{\mathbf{k}})$.

$$N = \begin{bmatrix} S(\hat{\mathbf{i}}) & S(\hat{\mathbf{j}}) & S(\hat{\mathbf{k}}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix}.$$

The entries of this matrix have always arisen as the coefficients of the linear equations defining our linear function. Assuming that holds here too, what would the equations for S be? We know S takes 3 inputs (x, y, z) and has 2 outputs (u, v), so we are looking for:

$$u = u(x, y, z),$$

$$v = v(x, y, z).$$

Let's go row by row:

$$\textbf{Row 1 :} \begin{bmatrix} \color{red}{1} & \color{blue}{0} & \color{green}{0.5} \\ 0 & 1 & 2.5 \end{bmatrix} \longrightarrow u = \color{red}{1}x + \color{blue}{0}y + \color{green}{0.5}z.$$

$$\textbf{Row 2:}\ \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \quad \longrightarrow \quad v = 0x + 1y + 2.5z.$$

We have now deduced what our function should be:

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix},$$

where

$$u = x + 0.5z,$$

$$v = y + 2.5z.$$

So, for example, Gunnison Chappel is about 65 meters east from where we placed our meterstick, and stands about 30 meters tall. This gives the tip of the chapel the coordinates $(65, 0, 30)$. If we want to know where it casts its shadow, we can compute:

$$u = 65 + 0.5 * 30 = 80,$$

$$v = 0 + 2.5 * 30 = 75.$$

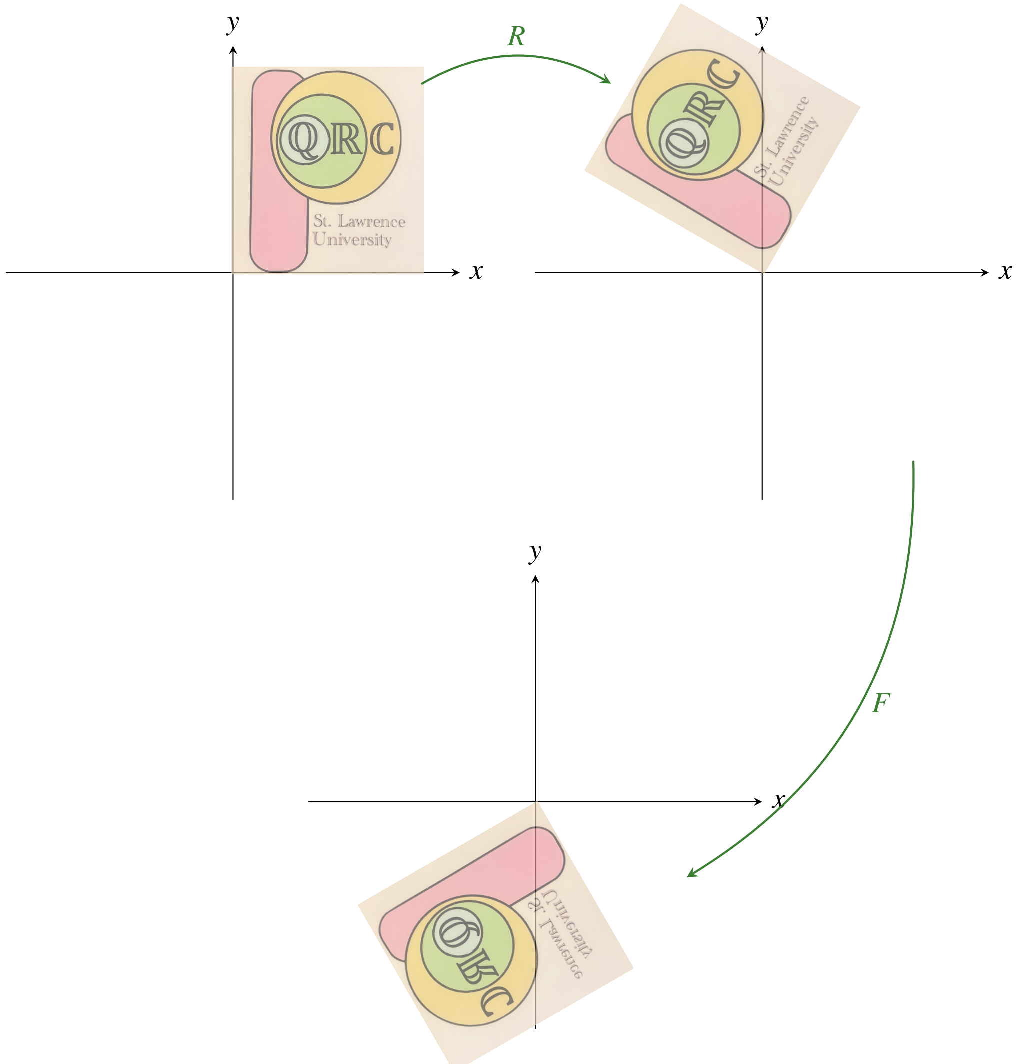
So it casts a shadow 80 meters east, and 75 meters north, of our meterstick. *What is remarkable about this, is we measured the point of a single shadow and we were able to determine another!* Finally, let's recall that we discovered the process for matrix-vector multiplication by thinking about multiplying a matrix as the same as applying the associated linear function. This works here too: if we apply the process of matrix multiplication, we get the same answer as applying S.

$$\textbf{Row 1:}\ \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \begin{bmatrix} 65 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 1*65 + 0*0 + 0.5*30 \\ \end{bmatrix} = \begin{bmatrix} 80 \\ \end{bmatrix},$$

$$\textbf{Row 2:}\ \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix} \begin{bmatrix} 65 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} 1*65 + 0*0 + 0.5*30 \\ 0*65 + 1*0 + 2.5*30 \end{bmatrix} = \begin{bmatrix} 80 \\ 75 \end{bmatrix}.$$

3.4.2 Composition of Linear Transformations

Suppose I start with an image that I'd like to manipulate an image of the PQRC logo in order to place it on a T-Shirt. There are two things I'd like to do. First, I want to rotate it 60°, and then I'd like to reflect it vertically over the x-axis.

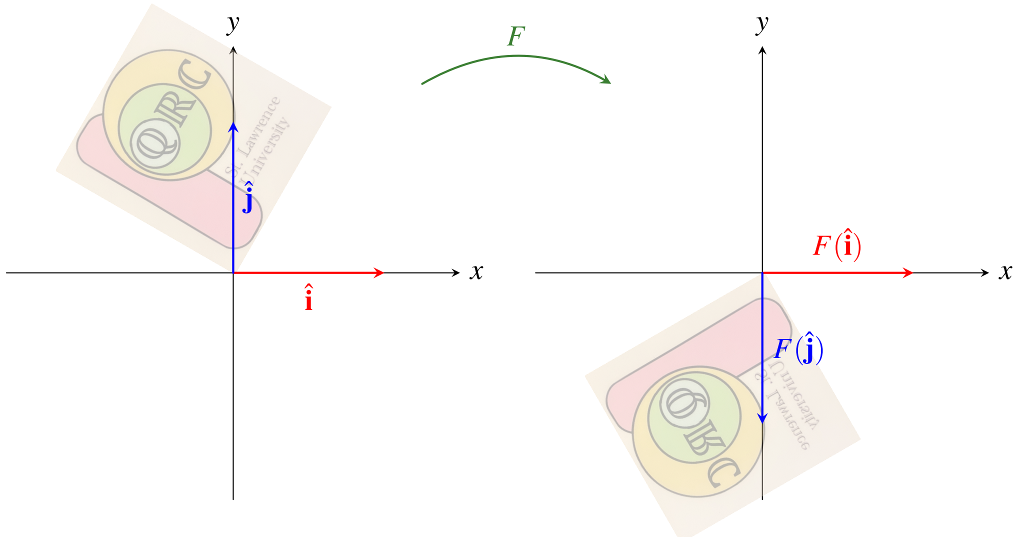


The transformatoins R (for *rotate*) and F (for *flip*) are both linear, and therefore each have an associated matrix. Let's find the matrix for the transformation with *rotates* and then *flips*. First let's find the matrix for the rotation R. In fact, in Section 3.2.1 we deduced that R is given by the rotation matrix:

$$N = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 0.5 & -.866 \\ .866 & 0.5 \end{bmatrix}.$$

What about F? To find the matrix for F, it is enough to compute $F(\hat{\mathbf{i}})$ and $F(\hat{\mathbf{j}})$.

■ **Question 3.8** Can you compute the matrix for F?



In particular, we have:

$$F(\hat{\mathbf{i}}) = \begin{bmatrix}1\\0\end{bmatrix} \qquad \text{and} \qquad F(\hat{\mathbf{j}}) = \begin{bmatrix}0\\-1\end{bmatrix}.$$

Therefore the matrix for F is:

$$M = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}.$$

To find the matrix for *rotate then flip*, we will follow a similar philosphy, by trying to track where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go. That is, we'd like to know the values $F(R(\hat{\mathbf{i}}))$ and $F(R(\hat{\mathbf{j}}))$. Before computing this, let's briefly unpack the notation. Given a vector $\mathbf{v}$ in $\mathbb{R}^2$, the value $R(\mathbf{v})$ represents where the vector $\mathbf{v}$ is after rotating. This is another vector in $\mathbb{R}^2$, and therefore it can be fed to the function F to be flipped. This value is $F(R(\mathbf{v}))$.

To compute $F(R(\hat{\mathbf{i}}))$, we first compute $R(\hat{\mathbf{i}})$, and we feed whatever the output is to F. But we can easily determine $R(\hat{\mathbf{i}})$: it is the first column of the matrix for R:

$$R(\hat{\mathbf{i}}) = \begin{bmatrix}.5 & -.866\\.866 & .5\end{bmatrix}\begin{bmatrix}.5\\.866\end{bmatrix}.$$

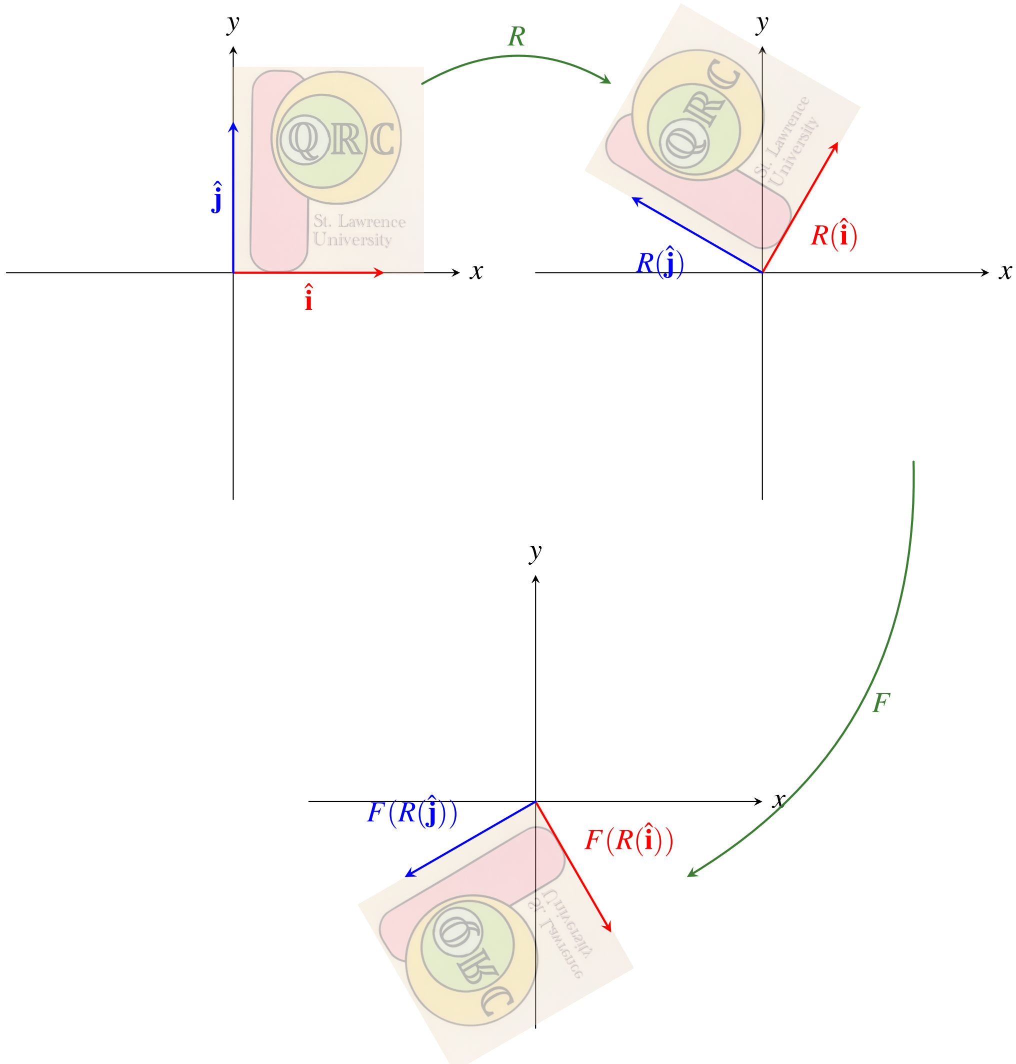
Therefore to compute $F(R(\hat{\mathbf{i}}))$, we can feed this output to F (i.e., multiply it by N).

$$F(R(\hat{\mathbf{i}})) = F\left(\begin{bmatrix}.5\\.866\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}.5\\.866\end{bmatrix} = \begin{bmatrix}1*.5+0*.866\\0*.5+(-1)*.866\end{bmatrix} = \begin{bmatrix}.5\\-.866\end{bmatrix}.$$

Our strategy to compute $F(R(\hat{\mathbf{j}}))$ is similar, first noticing that $R(\hat{\mathbf{j}})$ is just the second column of the matrix for R, and then applying F to this column.

$$F(R(\hat{\mathbf{j}})) = F\left(\begin{bmatrix}-0.866\\.5\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-.866\\.5\end{bmatrix} = \begin{bmatrix}1*(-.866)+0*.5\\0*(-.866)+(-1)*.5\end{bmatrix} = \begin{bmatrix}-.866\\-.5\end{bmatrix}.$$

This process is illutrating in the following diagram.



We can now write down the matrix for for *rotate, then flip*, since its columns are the values on $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ respectively. Let's call it P.

$$P = \begin{bmatrix} F(R(\hat{\mathbf{i}})) & F(R(\hat{\mathbf{j}})) \end{bmatrix} = \begin{bmatrix} .5 & -.866 \\ -.866 & -.5 \end{bmatrix}.$$

In particular, to *rotate, then flip* point in the image corresponding to a vector $\mathbf{v}$, we can just multiply by this matrix!

$$F(R(\mathbf{v})) = P\mathbf{v}.$$

Since *rotating* (applying R) is the same as multiplying by N, and *flipping* (applying F) is the same as multiplying by its matrix M, we can substitute this in:

$$MN\mathbf{v} = P\mathbf{v}.$$

It seems reasonable, then, to call this matrix P the *product* of M and N:

$$MN = P.$$

Exercise 3.10 — Checkin 4. Let's experiment with composition using functions whose domain might not be $\mathbb{R}^2$.

1. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates the *counterclockwise* by $90°$. Find the matrix for R. (Recall that $\sin 90° = 1$ and $\cos 90° = 0$.)
2. A matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ sends a vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ to $T(\mathbf{v}) = \begin{bmatrix} u \\ v \end{bmatrix}$ in $\mathbb{R}^2$ where:
$$u = 2x - z \qquad \text{and} \qquad v = -x - y + 5z.$$
Find the matrix associated to T.
3. Compute $R(T(\hat{\mathbf{i}})), R(T(\hat{\mathbf{j}}))$, and $R(T(\hat{\mathbf{k}}))$. (These should be 2-dimensional column vectors!)
4. To find the matrix for a linear transformation, we see what happens to the standard basis and use these outputs as our columns. Use this philosophy to find a matrix for the composition $R \circ T$ (this is the function which first applies T, and then applies R to the result).

■

3.5 February 23, 2023

3.5.1 Matrix Multiplication

Let's set this up more generally. First we recall the definition of the composition of two functions

Definition 3.5.1 Let $T:\mathbb{R}^2\to\mathbb{R}^2$ and $L:\mathbb{R}^2\to\mathbb{R}^2$ be transformations. The *composition* of L with T is the transformation: $L\circ T:\mathbb{R}^2\to\mathbb{R}^2$ that is obtained by first doing T, and then applying L to the result:

$$(L\circ T)(\mathbf{v})=L(T(\mathbf{v})).$$

This is nice to visualize as follows:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "T"] \arrow[rr, bend right, "L\circ T"'] & \mathbb{R}^2 \arrow[r, "L"] & \mathbb{R}^2
\end{tikzcd}$$

If T and L are linear, then they each come with a matrix.

Definition 3.5.2 — Matrix Multiplication: 2×2 case. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ and $L:\mathbb{R}^2\to\mathbb{R}^2$ be linear transformations, with associated matrices N and M respectively. Then the matrix product MN is the matrix associated to the composition $L\circ T$.

Let's find a formula for the matrix product, following the example of manipulating the PQRC sticker. Suppose:

$$M=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\qquad\text{and}\qquad N=\begin{bmatrix}p & q\\ r & s\end{bmatrix}.$$

Then, since MN is associated to the composition $L\circ T$, it's columns are $L(T(\hat{\mathbf{i}}))$ and $L(T(\hat{\mathbf{j}}))$ respectively. Since $T(\hat{\mathbf{i}})$ and $T(\hat{\mathbf{j}})$ are the columns of N, we can compute these directly:

$$L(T(\hat{\mathbf{i}}))=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}p\\ r\end{bmatrix}=\begin{bmatrix}ap+br\\ cp+dr\end{bmatrix}$$

$$L(T(\hat{\mathbf{i}}))=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}q\\ s\end{bmatrix}=\begin{bmatrix}aq+bs\\ cq+ds\end{bmatrix}$$

Therefore we have established the usual formula for matrix multiplication.

Theorem 3.5.1 — A formula for 2×2 matrix multiplication. If

$$M=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\qquad\text{and}\qquad N=\begin{bmatrix}p & q\\ r & s\end{bmatrix},$$

then the product:

$$MN=\begin{bmatrix}ap+br & aq+bs\\ cp+dr & cq+ds\end{bmatrix}$$

Much like matrix-vector multiplication, this is often best remembered as a *process*, where each column is a matrix-vector multiplication. In particular, to know the ij-entry of MN, you can pair the i'th row of M with the j'th column of N, multiplying the associated entries of each and adding them up.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & \\ & \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

■ **Question 3.9** Compute the matrix product:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 10 \end{bmatrix}$$

We've now got a general formula for how to multiply two matrices. Now, when we multiply two numbers, it doesn't really matter what order we do it in. For example:[4]

$$2*3 = 3*2$$

Is this true for matrices as well? Let' investigate.

■ **Question 3.10** Does order matter in matrix multiplication?

It's perhaps easiest to just look at an example and do some computations.

■ **Question 3.11** Consider the matrices:

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compute the products MN and NM. Are they the same?

If we compute we get:

$$MN = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$NM = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

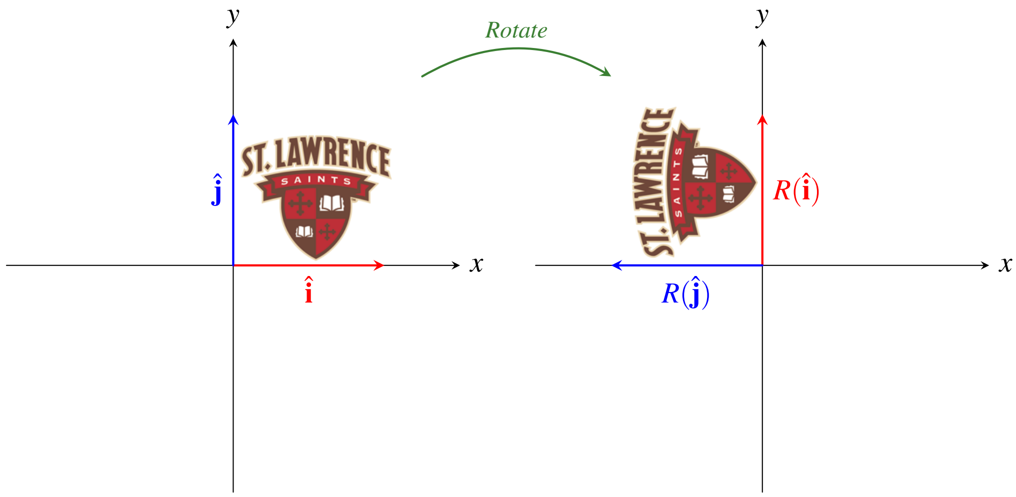
So $MN \neq NM$. This gives an answer to Question 3.10: *yes, order does matter.* But, I don't think it's a very satisfying one. We can look at MN and NM and say: *see? They're different!* But it doesn't really tell us *why* they are different in any concrete way. To investigate this question a bit further, let's remember our guiding philosophy with matrices: *a matrix is a function!*. So what are the functions associated to M and N?

[4]This feels automatic because of how comfortable we are with it, but depending on how you define multiplication, it is slightly nontrivial. For example, we might be comparing *2 boxes with 3 things each* and *3 boxes with 2 things each*.

Let's start with M. Let's call the associated funtion $R:\mathbb{R}^2\to\mathbb{R}^2$. To determine what R does, let's start by asking what it does to the standard basis vectors $\hat{\mathbf{i}}$ nad $\hat{\mathbf{j}}$. We can read this information off of the columns of the matrix M.

$$M=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \qquad R(\hat{\mathbf{i}})=\begin{bmatrix}0\\1\end{bmatrix} \qquad R(\hat{\mathbf{j}})=\begin{bmatrix}-1\\0\end{bmatrix}.$$

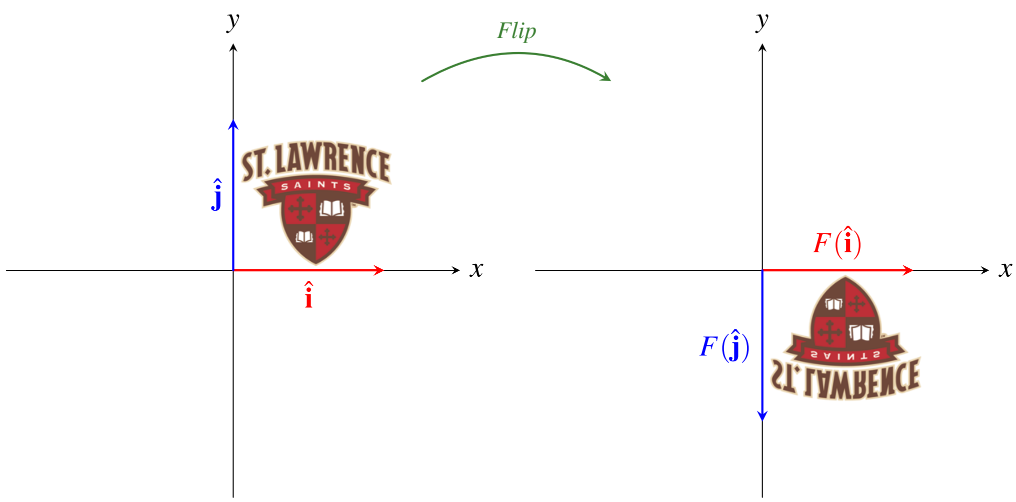
In particular, both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are rotated 90°. Since this determines the entire map, R must be the 90° rotation of the plane.



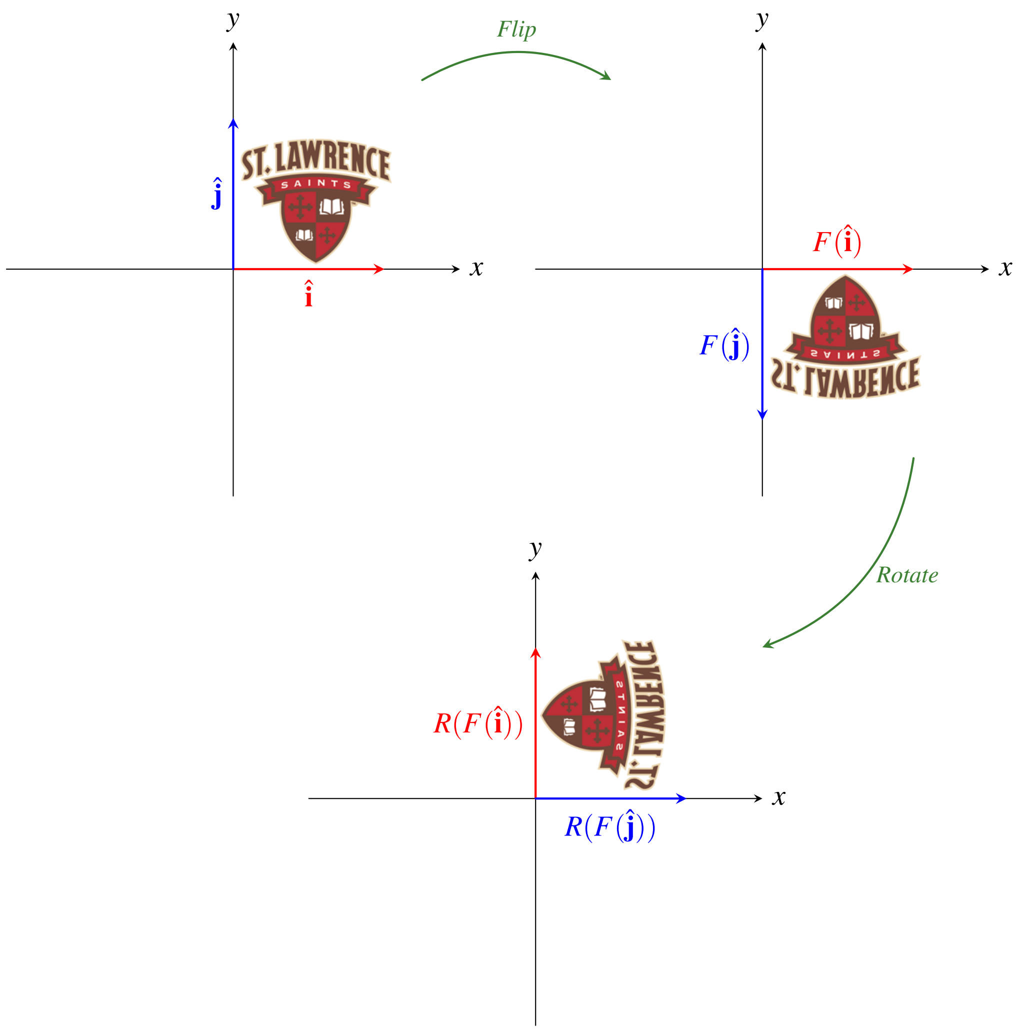
Let's do the same with N. Let's call the associated funtion $F:\mathbb{R}^2\to\mathbb{R}^2$. To determine what F does, let's again ask what it does to the standard basis vectors $\hat{\mathbf{i}}$ nad $\hat{\mathbf{j}}$. We can read this information off of the columns of the matrix N.

$$M=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad F(\hat{\mathbf{i}})=\begin{bmatrix}1\\0\end{bmatrix} \qquad F(\hat{\mathbf{j}})=\begin{bmatrix}0\\-1\end{bmatrix}.$$

We see that $\hat{\mathbf{i}}$ remains fixed and $\hat{\mathbf{j}}$ is flipped upside down, in particular both are reflected over the x-axis, so F is the reflection map over the x-axis (we saw this map in Section 3.4.2),

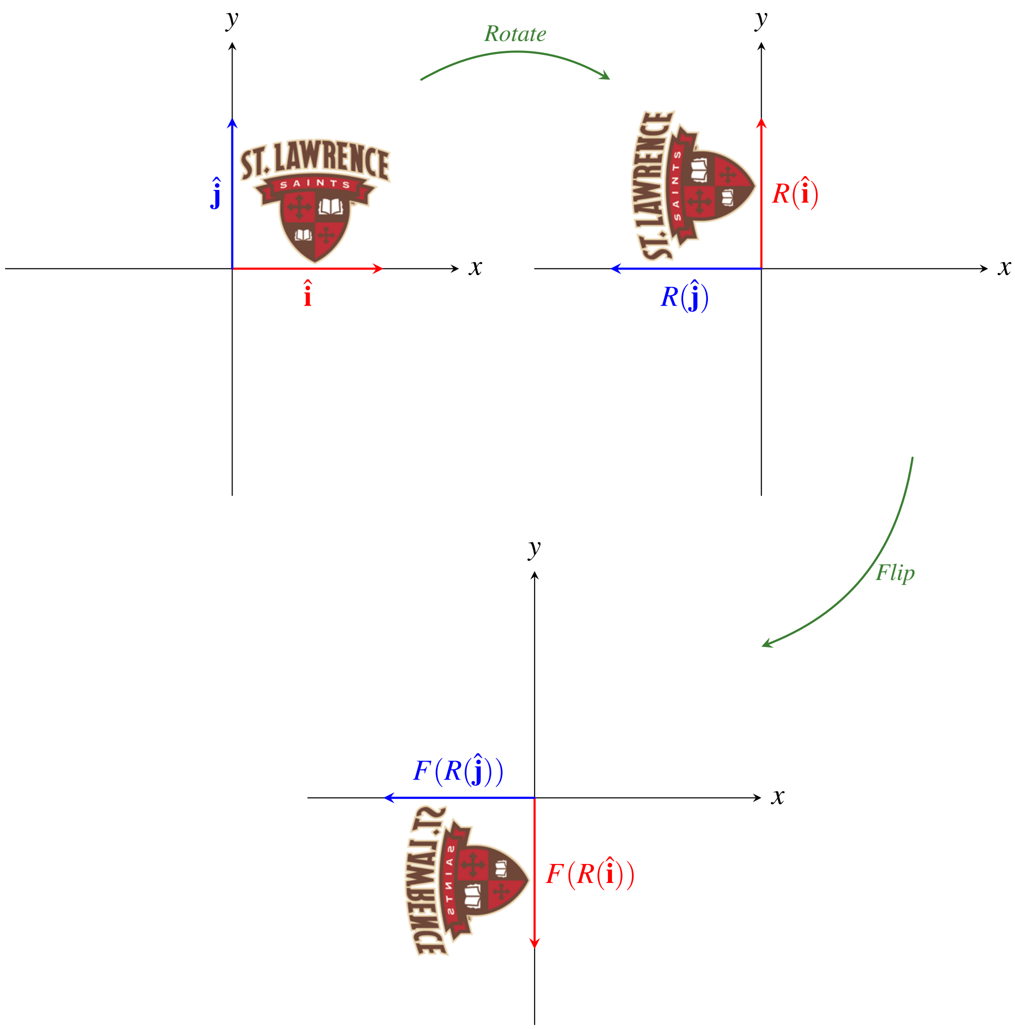


The product MN takes a vector $\mathbf{v}$ to $MN\mathbf{v} = R(F(\mathbf{v}))$, so its corresponding function is $R \circ F$: *first flip vertically, then rotate 90 degrees*. On the other hand, NM takes a vector $\mathbf{v}$ to $NM\mathbf{v} = F(R(\mathbf{v}))$, so its corresponding function is $F \circ R$: *first rotate 90 degrees, then flip vertically*. Should we expect *flip then rotate* to be the same as *rotate then flip*? Let's Look at a couple of images. First let's look at the effect of MN (or $R \circ F$): which flips first, and then rotates.



■ **Question 3.12** The matrix for $R \circ F$ is MN. Confirm that the coordinates for $R(F(\hat{\mathbf{i}}))$ and $R(F(\hat{\mathbf{j}}))$ are consistent with the columns of the matrix MN we computed above.

Now let's do the same experiment, but for the effect of NM, which corresponds to the function $F \circ R$: or *rotate, then flip*.



■ **Question 3.13** The matrix for $F \circ R$ is NM. Confirm that the coordinates for $F(R(\hat{\mathbf{i}}))$ and $F(R(\hat{\mathbf{j}}))$ are consistent with the columns of the matrix MN we computed above.

A visual inspection shows that *rotate, then flip* and *flip, then rotate* do different things. This, to me, gives a far more satisfying reason for why order matters in matrix multiplication: *order matters when you compose functions!*. This is a huge advantage of having both an algebraic perspective and a functional one. The algebraic perspective allows you to compute things, but the functional perspective *means something!* I think the most important takeaway from this section so far is the following:

■ **Slogan 3.1** A matrix is a function. Multiplying matrices is the same as composing functions.

R Above we had F for *flipping* and R for *rotating*. It may be a bit perplexing to see $F \circ R$ read that as *rotate, then flip*. In particular, we usually read from left to right, so why is it the case that in this instance we read right to left? The reason has to do with functional notation. In particular, when we have a function (say f), and we want to evaluate it at a value (say x), we put that value to the right of the function (so $f(x)$). Back to rotating and then flipping: if we start by rotating a vector $\mathbf{v}$, we feed it to the function, resulting in $R(\mathbf{v})$. If we want to flip the resulting value, we feed the whole thing to F (again on the right), resulting in $F(R(\mathbf{v}))$. The convention of the function *eating* the value to the right of it leads to us having to read from right to left. As a result, our function interpretation of matrix multiplication has us reading right to left as well. That is, the product MN is the function that takes a vector, first multiplies it by N, and then multiplies the result by M.

3.5.2 The Identity Matrix

In Homework 1 (Exercise 1.4) we were presented with the matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

■ **Question 3.14** Let $\mathbf{v} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Compute $I\mathbf{v}$.

If you got $\mathbf{v}$ back, great job! In fact, in Exercise 1.4 was associated to the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose equations were:

$$u = x,$$

$$v = y.$$

In particuar, for plugging any vector:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

If we try to qualitatively describe what T does to the plane, we can deduce that it *does nothing!* Nothing get moved around by T, everything just stays put. This function is often called the identity function, and is denoted id.

Definition 3.5.3 The function $id : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $id(\mathbf{v}) = \mathbf{v}$ is called the *identity function*. The matrix associated to the identity function is called the *identity matrix*, and has the following form:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

■ **Question 3.15** Let I be the identity matrix. Is it possible to find a vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $I\mathbf{v} \neq \mathbf{v}$?

There are two ways to see that the answer to this question is no. One way is to use that the identity function corresponds with the identity matrix, so:

$$I\mathbf{v} = id(\mathbf{v}) = \mathbf{v}.$$

One can also choose variable coordinates for $\mathbf{v}$, and do matrix multiplication:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 * x + 0 * y \\ 0 * x + 1 * y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Either way, we see that nothing happens! In particular, for matrix-vector multiplication, we see that multiplying by the identity matrix does nothing. What about for matrix multiplication in general?

■ **Question 3.16** Let I be the identity matrix and let M be any 2×2 matrix. Can you say anything about IM? What about MI?

Let's do an example: say:

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix}.$$

$$IM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1*1+0*-1 & 1*2+0*7 \\ 0*1+1*-1 & 0*2+1*7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 7 \end{bmatrix} = M.$$

So it looks like nothing happens! And as above, we could give M some variable coordinates and see that this is always the case.

$$IM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1*a+0*c & 1*b+0*d \\ 0*a+1*c & 0*b+1*d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = M.$$

That being said, this makes it look almost like an accident, a lucky and clever choice of numbers for I so that multiplication does nothing. Insead, let's take the approach from Slogan 3.1. Then I corresponds to the identity function id and M is associated with some other linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$. What happens if we compose them? The identity functoin *does nothing*, so for any $\mathbf{v}$ we choose:

$$id(L(\mathbf{v})) = L(\mathbf{v}).$$

Since the compositoin $id \circ L$ is the same as just doing L, the product $IM = M$. In the other direction:

$$L(id(x)) = L(x),$$

so that $L \circ id = L$, and therefore $MI = M$ as well.

Theorem 3.5.2 Let I be the identity matrix, and M any other 2×2 matrix. Then:

$$IM = MI = M,$$

In particular, the identity matrix behaves for matrix multiplication, much like the number 1 behaves for traditional multiplication.

3.6 Homework 5

Exercise 3.11 Compute the following matrix products.

1. $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 11 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

■

Exercise 3.12 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the equations:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \textit{where} \qquad \begin{aligned} u =& 2x - 3y \\ v =& x + y \end{aligned}.$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be another linear transformation given by the equations:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \textit{where} \qquad \begin{aligned} u =& -y \\ v =& 2x + 2y \end{aligned}.$$

What are the equations for the composition $L \circ T$? (*Hint:* Can you translate this problem to computing a single matrix product?) ■

Exercise 3.13 Let $id : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity function on $\mathbb{R}^3$. That is, $id(\mathbf{v}) = \mathbf{v}$ for any 3D vector $\mathbf{v}$. What is the matrix for id? ■

Exercise 3.14 Let θ and φ be two angles, and consider the rotation matrices:

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

Do you think $MN = NM$? Why or why not? (Trying to study this by multiplying things out results in some hard trig. It's probably easier to think about how the functions associated to M and N compose.) ■

Exercise 3.15 When working between dimensions, sometimes matrix multiplication makes sense, and sometimes it doesn't. Let's think about this a bit more carefully. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the functions associated to the matrices:

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \textit{and} \qquad N = \begin{bmatrix} -1 & 0 \\ -2 & 5 \end{bmatrix}.$$

1. Try to compute the following values, or explain why they cannot be computed.

$$F\left(G\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right) \qquad \textit{and} \qquad G\left(F\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)\right).$$

2. One of the following 2 functionm compositions makes sense, and the other doesn't. Determine which is which, and explain why.

$$F \circ G \qquad \textit{or} \qquad G \circ F.$$

3. One of the following 2 matrix products makes sense, and the other doesn't. Determine

which is which, and explain why.

$$MN \qquad or \qquad NM.$$

For the one that does make sense: what is the product? Your answer should be one matrix. (Recall: *a matrix is a function, and a matrix multiplication should reflect compositoin of functions*).

3.7 March 2, 2023

3.7.1 Inverse Transforms and Inverse Matrices

When studying a function, it is often very useful to have an *undo button*: another function which reverses what the first one did. For example, consider the function $f(x) = x^3$. To *undo* this function, we take the cube root! To be more precise, there is a function $g(y) = \sqrt[3]{y}$, and if we compose f and g, we get back where we started. Let's try this out on a few numbers:

$$g(f(2)) = g(8) = \sqrt[3]{8} = 2.$$

This works both ways: cubing *undoes* the cuberoot.

$$f(g(12)) = f(\sqrt[3]{12}) = f(2.2894...) = (2.2894...)^3 = 12.$$

Plugging in variables:

$$g \circ f(x) = g(x^3) = \sqrt[3]{x^3} = x, \qquad \text{and} \qquad f \circ g(y) = f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y.$$

In particular, the composition $g \circ f$ is the *do nothing* function (also known as the identity function), and the same can be said for $f \circ g$. If $g \circ f = id$ and $f \circ g = id$, we call g the *inverse* of f, and denote it by f^{-1}.

> **R** It's not true that every function has an inverse. For example, let $h(x) = x^2$. If I plug in 2 I get $h(2) = 4$, so this tells me that whatever the *undo* function is, it better take 4 to 2. On the other hand, $h(-2) = 4$ as well, so this undo function must also take 4 to -2. It can't do both! So h cannot have an inverse. In fact, we've stumbled upon something: for a function F to have an inverse, it must satisfy the following property: whenever I have $a \neq b$, we need $F(a) \neq F(b)$. If they were equal, we wouldn't know how to undo their value. This property is called being *one-to-one*, and we will revisit it further down the line.

■ **Question 3.17** Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates the plane 30°. Does R have an inverse? Can you describe it?

Definition 3.7.1 Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. The *inverse* to L (if it exists), is a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ so that when we compose in both directions we obtain the identity function:

$$L \circ T = id \qquad \text{and} \qquad T \circ L = id.$$

We often denote the inverse T by the symbol L^{-1} (pronounced L *inverse*).

■ **Example 3.8** To undo the function R which rotates the plane 30°, we merely rotate the plane $-30°$, and get back to where we started.[5] Let's look at the matrices for R and R^{-1}, which we will call M and N respectively.

$$M = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix},$$

and

$$N = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} = \begin{bmatrix} .866 & .5 \\ -.5 & .866 \end{bmatrix}.$$

[5] we could also rotate 330°.

Can you guess what the product NM is? It should correspond to the function which first rotates 30°, and then rotates -30°, that is, it corresponds to the identity function. So hopefully, it is the identity matrix. Let's check:

$$\begin{aligned} NM &= \begin{bmatrix} .866 & .5 \\ -.5 & .866 \end{bmatrix} \begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix} \\ &= \begin{bmatrix} .866 * .866 + .5 * .5 & .866 * (-.5) + .5 * (.866) \\ (-.5) * (.866) + .866 * .5 & (-.5) * (-.5) + .866 * .866 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Notice that seems a lot more surprising that it works out when you compute it directly, than when you compare it to the composition of rotating 30° forward and then back. Nonethelss, $NM = I$ and one can similarly see that $MN = I$ as well. ■

When multiplying numbers, the *inverse* (or multiplicative inverse) of a number (say 7), is the number we multiply to get 1 (in this case, $\frac{1}{7}$ or 0.14285...). In fact, we will often denote $\frac{1}{7}$ just by writing 7^{-1}. In matrix multiplication, the number 1 is replaced by the identity matrix I, so the natural way to define an inverse is as follows.

Definition 3.7.2 Let M be a 2×2 matrix. The *inverse* of M (if it exists) is a matrix N such that:

$$MN = I \qquad \text{and} \qquad NM = I.$$

If an inverse to M exists, we will denote it by the symbol M^{-1}.

■ **Question 3.18** Let

$$M = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

Is $N = M^{-1}$?

Let's use our philosophy that *a matrix is a function*, to connect matrix inverses and function inverses. In particular, let $L, T : \mathbb{R}^2 \to \mathbb{R}^2$ be functions, whose associated matrices are M and N respectively.

■ **Question 3.19** If $T = L^{-1}$, does this mean $N = M^{-1}$?

The answer had better be yes, and indeed, the product MN is the matrix associated to the composition $L \circ T = id$, so $MN = I$. We can say the same for NM.

■ **Example 3.9 — Inverses of Rotation Matrices.** The inverse of the rotation matrix:

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

is the matrix which rotates the plane the same amount, but in the opposite direction:

$$M^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

■

■ **Example 3.10** A matrix can be its own inverse! Indeed, let:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then it isn't hard to compute that $MM = I$. This can be elucidated by recognizing that M corresponds to the *flip* function $F : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects the plane over the x-axis. Then MM coresponds to $F \circ F$, which flips, and then flips again. But if you flip twice, you're back where you started! Since $F \circ F = id$, we know $MM = I$. ■

Rather than write MM, for multiplying a matrix by itself, we can write M^2.

Definition 3.7.3 Let M be a a 2×2 matrix, and let n be a positive integer. Then:

$$M^n = \underbrace{M \cdot M \ \cdots M}_{n\text{-times}}.$$

If M has an inverse M^{-1}, then we can write:

$$M^{-n} = \underbrace{M^{-1} \cdot M^{-1} \ \cdots M^{-1}}_{n\text{-times}}.$$

Finally, $M^0 = I$.

3.7.2 A Technique: Solving a System of Equations with Inverse Matrices

One application of matrix inversion (and linear algebra in general) is it gives a broad framework to solve systems of equations. Let's see an example of this, in the case of 2 equations and 2 unknowns. Suppose we want to solve the following system:

$$2x + 5y = 11,$$

$$x + 3y = -4.$$

We can make this a single (vector) equation by putting brackets around each side.

$$\begin{bmatrix} 2x + 5y \\ x + 3y \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

The left hand side can be factored into the product of a 2×2 matrix and a single (vector) variable.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.$$

The 2×2 matrix is the matrix M from Question 3.18. Letting:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ -4 \end{bmatrix},$$

the system of equations boils down to the rather simple equation:

$$M\mathbf{x} = \mathbf{v}. \tag{3.1}$$

Now our intuition tells of the following: *if we want to solve for* $\mathbf{x}$*, we should divide both sides by* M. But we can't really divide by a matrix...can we? For numbers, if I wanted to divide by 7, we could instead multiply by $\frac{1}{7}$, or to more reflect our current setup, we could multiply by 7^{-1}. Let's model our next step on this, and try to multiply both sides by M^{-1}, which we found in Question 3.18 to be:

$$N = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$

So let's try multiplying both sides of Equation (3.1) by N. *Careful! As we saw above, it matters whether we multiply on the left or the right. If we want to cancel out the M, we shouled probably multiply on the left.*

$$NM\mathbf{x} = N\mathbf{v}. \tag{3.2}$$

Zooming in on the left-hand-side, we can use that $NM = I$ to see:

$$NM\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

So multiplying by N worked just like dividing by M should! Plugging this back into Equation (3.2), we get:

$$\mathbf{x} = N\mathbf{v}.$$

That is:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -4 \end{bmatrix} = \begin{bmatrix} 33+20 \\ -11-4 \end{bmatrix} = \begin{bmatrix} 53 \\ -19 \end{bmatrix}.$$

Let's check that this works:

$$2*53+5*(-19) = 11,$$

$$53+3(-19) = -4.$$

Magic! This is, of course, not magic. In fact, viewing this from the *functional* perspective clarifies the picture somewhat. The function L associated to the matrix M is given by equations:

$$u = 2x+5y,$$

$$v = x+3y,$$

which look very much like the system of equations we started with. Then solving the system of equations is looking for a vector that, when I apply L, results in a u-coordinate of 11 and a v-coordinate of -4. That is, we want $L(\mathbf{x}) = \mathbf{v}$. Multiplying by the inverse then correponds to applying L^{-1} to both sides, which gives:

$$L^{-1}(L(\mathbf{x})) = L^{-1}(\mathbf{v}),$$

which in turn simplifies to $\mathbf{x} = L^{-1}(\mathbf{v})$. Since we know L^{-1} (why?), we have now found $\mathbf{x}$ and therefore solved our system.

How can we find inverses?

It seems like it would be very useful to be able to invert linear functions and matrices. Inverting a number (like 7) is easy, you can just divide 1 by 7. For matrices, it seems much less clear. Some matrices, like rotations or reflections, are easy to invert (as we saw in Examples 3.9 and 3.10). On the other hand, the inverse for the matrix in Question 3.18 seemed to have come out of nowhere. For 2×2 matrices, it turns out there is a formula, which we will record here.

Theorem 3.7.1 — Inverses of 2×2 Matrices. Consider the matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then M has an inverse, which is given by the formula:

$$M^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

It isn't too hard to check that this formula does indeed give an inverse,[6] but it is a little mysterious where this comes from. We will delay the discussion of where it comes from until a little later, but we record the formula sooner because it comes in quite handy.

■ **Question 3.20** Plug the matrix M from Question 3.18 into the inverse formula, and confirm that you get N.

3.8 Homework 6

Exercise 3.16 Below are 4 matrices. Match each one with its inverse. Justify your reasoning.

$$M = \begin{bmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} \qquad N = \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} -0.5 & -2 \\ 0.5 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

■

Exercise 3.17 Below are 4 functions. Match each one with its inverse. Justify your reasoning.

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -0.5x - 0.5y \\ v &= 0.5x - 0.5y \end{aligned}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= 2x + 4y \\ v &= -x - y \end{aligned}$$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -0.5x - 2y \\ v &= 0.5x + y \end{aligned}$$

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where} \quad \begin{aligned} u &= -x + y \\ v &= -x - y \end{aligned}$$

■

[6] Do it do it do it!

Exercise 3.18 Consider the following system of equations:

$$2x - 5y = 1$$

$$x + 2y = 3.$$

Write the system of equations as a single matrix equation:

$$M\mathbf{x} = \mathbf{v},$$

where $\mathbf{x}$ is a variable vector, and $\mathbf{v}$ is an actual vector. Then use the formula for matrix inversion (Theorem 3.7.1 on Page 98 of the notes) to solve for $\mathbf{x}$ and determine x and y. ■

Exercise 3.19 Let a, b, c, d, k, ℓ be constants, and consider the system of equations:

$$ax + by = k$$

$$cx + dy = \ell.$$

Do you agree or disagree with the following statement? Justify your reasoning:

If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse, then the system of equations has exactly one solution.

■

Exercise 3.20 Suppose that M and N are matrices with inverses. Does the product MN have an inverse as well? Justify your reasoning. (This might be easier to think of from the *functional* perspective than from the algebraic one). ■

Exercise 3.21 Some functions have inverses while others don't. The same is true for matrices. For each of the following matrices, determine if there is or is not an inverse. If there is an inverse, write it down. If there is no inverse, explain why. (*Hint: Remember that $h(x) = x^2$ has no inverse because $h(2) = h(-2) = 4$. Try to give a similar argument for the ones you think have no inverse.*)

1. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.
2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
3. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$.
4. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

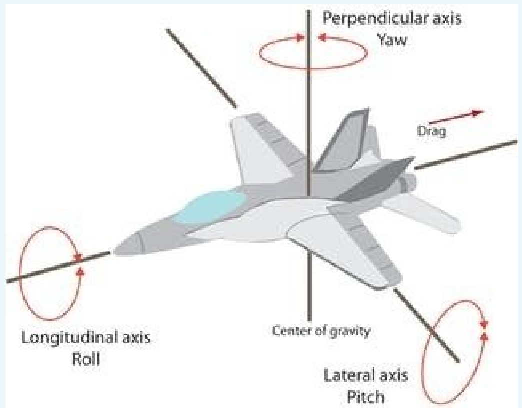
■

Exercise 3.22 We've defined inverses for 2×2 matrices. What do you think the definition for the inverse of a 3×3 matrix should be? ■

Exercise 3.23 Rotations in 3D space are controlled by 3 axes of rotation:

- *Roll*: Rotation about the x-axis.
- *Pitch*: Rotation about the y-axis.
- *Yaw*: Rotation about the z-axis.

These 3 axes of rotation are demonstrated by the fighter jet in the image below.



In terms of a plane then the axes can be described (from the point of view of the pilot) as follows:

- *Roll*: The wings of the plane rotating around while nose stays pointed forward.
- *Pitch*: The nose of the plane angling up or down.
- *Yaw*: The nose of the plane angling left or right.

A general rotation can be expressed in terms of its *roll angle* γ, its *pitch angle* β, and yaw angle α. Rotating around an axis is a matrix transformation. Since a general rotation is a composition of rotations around the 3 axes, it is as well! Let's figure out which matrix it is.

2. Find the standard matrix associated to a roll of γ—that is, a rotation of angle γ about the x-axis which would appear counterclockwise when looking from the positive x-direction. (This should like a bit like a rotation matrix in 2D!)
3. Find the standard matrix associated to a pitch of β—that is, a rotation of angle β about the y-axis, which would appear counterclockwise when looking from the positive y-direction.
4. Find the standard matrix associated to a yaw of α—that is, a rotation of angle α about the z-axis, would appear counterclockwise when looking from the positive z direction.
5. A general rotation can be expressed in terms of its *roll angle* γ, its *pitch angle* β, and yaw angle α. Therefore, it is a composition of the 3 rotation matrices you've found so far. Compute the standard matrix for such a rotation. Remember that compositing matrix transformations corresponds to multiplying matrices.
6. Let M be the matrix for a roll of $30°$, a pitch of $-15°$, and a yaw of $90°$. Do you think M has an inverse? Why or why not?
7. Set the origin at the center of gravity of a jet, and suppose a bug is sitting on the nose of a jet, at the coordinates $(5,0,1)$. The jet rotates with a roll angle of $30°$, a pitch angle of $-15°$, and a yaw angle of $90°$. What are the coordinates of the bug after this rotation.

■

3.9 March 7, 2023

3.9.1 General Linear Transformations

We've now seen some examples of linear maps between spaces of the same dimension, as well spaces of different dimension. Let's tally up some of our observations. The color example was a map from $F:\mathbb{R}^2\to\mathbb{R}^3$. As equations, it consisted of 3 equations in terms of 2 variables each. Translating this to a matrix, we had 3 rows (one for each equation), and 2 columns (one for each variable).

$$F\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}u\\v\\w\end{bmatrix}\qquad\longrightarrow\qquad\begin{aligned}u&=ax+by\\v&=cx+dy\\w&=ex+fy\end{aligned}\qquad\longrightarrow\qquad\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$$

The shadow example gave us a map $S:\mathbb{R}^3\to\mathbb{R}^2$. As equations, it consisted of 2 equations in 3 variables each, and translating this to a matrix we had 2 rows (one for each equatoin), and 2 columns (one for each variable).

$$S\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}u\\v\end{bmatrix}\qquad\longrightarrow\qquad\begin{aligned}u&=ax+by+cz\\v&=dx+ey+fz\end{aligned}\qquad\longrightarrow\qquad\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$$

We further observed that in each case, the columns of the matrix could be determined by simply evaluating the function on the standard basis vectors. Let us take this as a jumping off point for extending the theory to linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$ for general n and m. Let's first define what this $\mathbb{R}^n$ should be, recalling the definition of *the computer scientists approach* to n-dimensional vectors (cf. Definition 2.2.1.

Definition 3.9.1 — Higher Dimensional Vector Spaces. Let n be a positive integer. A *n-dimensional column vector* is an array of n numbers, arranged vertically:

$$\mathbf{v}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}.$$

The collection of all n-dimensional column vectors is denoted $\mathbb{R}^n$.

We've seen a couple of examples of higher dimensional vectors, including Example 2.4 which discussed the 5D-vectors controlling a 5-axis CNC mill. Fix 2 positive integers, m and n, and lets consider a function $L:\mathbb{R}^m\to\mathbb{R}^n$. Such a function could be denoted by a rule:

$$L\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix}\right)=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}.$$

We have to define what each coordinate, u_i, of the target is, so such a function would need n-equations in terms of the m input variables.

$$u_1=u_1(x_1,x_2,\cdots,x_m),$$

$$u_2 = u_2(x_1, x_2, \cdots, x_m),$$
$$\vdots$$
$$u_n = u_n(x_1, x_2, \cdots, x_m).$$

There are many many equations to choose from, which could be outrageously complicated. Linear algebra focuses on the *linear ones*, which for us means, *purely linear with no constant terms*.

Definition 3.9.2 — Linear Transformations in General. A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if it is given by the the equations

$$u_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m,$$
$$u_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m,$$
$$\vdots$$
$$u_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m,$$

for constants a_{ij}.

As usual, the data of this such a function is completely captured by *all* of the coefficients a_{ij} (there are now mn of them), so to remember this function, we need only remember these mn constants (in the correct order). We can do this by putting them in an array (or $n \times m$ matrix).

Definition 3.9.3 — $n \times m$ matrices. The $n \times m$ matrix associated to the linear transformation from Definition 3.9.2 is

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix}$$

Notice that if $L : \mathbb{R}^m \to \mathbb{R}^n$, the associated matrix has n rows and m columns (is an $n \times m$ matrix). This is because we get 1 row for each equation (i.e., each output, of which there are n), and one column for each variable (i.e., each input, of which there are m).

■ **Question 3.21** Let $L : \mathbb{R}^6 \to \mathbb{R}^8$ be a linear function. What is the shape of the matrix associated to L (that is, how many rows does it have, and how many columns?).

Again, M and L are interchangeable, so for any vector $\mathbf{v}$ in $\mathbb{R}^m$, it is reasonable to write:

$$L(\mathbf{v}) = M\mathbf{v}.$$

We know what $L(\mathbf{v})$ should be (using the equations from Definitoin 3.9.2), so that we can obtain a general formula for matrix-vector multiplication.

Definition 3.9.4 Given an $n \times m$ matrix and a vector in $\mathbb{R}^m$, we can define their product as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix}$$

As before, this can be expressed as a process where to give the ith row of Mv one pairs the entries of the ith row of M with the entries of v one by one, multiplying them together and adding them up.

Row 1: $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ \vdots \\ \end{bmatrix}$$

Row 2: $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ \end{bmatrix}$$

Row n: $$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix}$$

■ **Example 3.11** The position of a 5-axis CNC rotor is given by a vector:

$$\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}$$

where x, y, z give the location in 3-space, and θ and ϕ measure its orientation (as rotations around the z and x axes). It is moved in space by 3 perpindicular arms, and rotated by a mechanism attached directly to the drill. In particular, when the computer sends the information to the arms, it doesn't need to send θ and ϕ, just the x, y, z-coordinates. This can be expressed by a function $F : \mathbb{R}^5 \to \mathbb{R}^3$, given by the rules:

$$F\left(\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \end{bmatrix}\right) = \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where

$$u = x = 1x + 0y + 0z + 0\theta + 0\phi,$$

$$v = y = 0x + 1y + 0z + 0\theta + 0\phi,$$

$$w = z = 0x + 0y + 1z + 0\theta + 0\phi.$$

The matrix for F is therefore:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

■

■ **Question 3.22** The rotation mechanism attached to the drill head only needs to remember θ and ϕ. Define a function $G : \mathbb{R}^5 \to \mathbb{R}^2$ which only plucks out the rotation coordinates. Determine if it is linear, if it is, give the associated matrix.

An important property of linear maps we saw so far was that they could be determined by their values on a few chosen vectors. This happens here too. For example, if L is the linear transformation in Definition 3.9.2, then:

$$L\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} * 1 + a_{12} * 0 + \cdots + a_{1m} * 0 \\ a_{21} * 1 + a_{22} * 0 + \cdots + a_{2m} * 0 \\ \vdots \\ a_{n1} * 1 + a_{n2} * 0 + \cdots + a_{nm} * 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix},$$

which recovers the first column of the matrix for L. We can do similarly with the remaining columns. Before stating the general result, we will need to introduce some notation. For $\mathbb{R}^2$ we cared about $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, and for $\mathbb{R}^3$ we cared about $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. For general $\mathbb{R}^m$. Since there are only a finite number of letters in the alphabet, and we want to work with general m, we need to slightly switch up our notation.

Definition 3.9.5 — The Standard Basis for $\mathbb{R}^n$. The *standard basis* for $\mathbb{R}^m$ is the collection of vectors:

$$\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \hat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \hat{\mathbf{e}}_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

In particular, $\hat{\mathbf{e}}_i$ is the m-dimensional vector which has a 1 in the ith entry, and zeroes everywhere else.

With this notation in hand, we can record the general result about determining linear functions.

Theorem 3.9.1 A linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ is determined by its values on the standard basis for $\mathbb{R}^m$. In particular, if:

$$L(\hat{\mathbf{e}}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, \quad L(\hat{\mathbf{e}}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \quad \cdots \quad L(\hat{\mathbf{e}}_m) = \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{bmatrix},$$

then the matrix for L is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix}$$

The other thing we noticed for linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ is that they played well with addition (on HW1) and scalar multiplication (on this week's HW4). Before moving on, we'd like to record that this holds here as well.

Theorem 3.9.2 — Linearity of Linear Transformations. Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, let $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^m$ be vectors, and let c be a constant.

1. $L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$
2. $L(c\mathbf{v}) = cL(\mathbf{w})$.

The general framework of the proof for identical to the computation in the 2×2 case (Theorem 3.2.2 for Part 1 and Exercise 3.7 for part 2), except with more symbols to keep track of. We will omit it for now.

3.10 March 9, 2023

3.10.1 General Matrix Multiplication

In Checkin 4 (Exercise 3.10) we considered two functions. $R : \mathbb{R}^2 \to \mathbb{R}^2$ rotated the plane 90°, and was associated with the matrix:

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We also looked at a function $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the matrix

$$N = \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

We were interested in the composition $R \circ T$, which first applies T to a vector in $\mathbb{R}^3$, and then rotates the result 90°. In particular, $T \circ R : \mathbb{R}^3 \to \mathbb{R}^2$, so it is represented by a 2×3 matrix−2 rows, one for each output, and 3 columns, one for each input. As usual find the matrix for $T \circ R$, it is enough to see what it does to $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

$$P = \begin{bmatrix} R(T(\hat{\mathbf{i}})) & R(T(\hat{\mathbf{j}})) & R(T(\hat{\mathbf{k}})) \end{bmatrix}.$$

But we can extract $T(\hat{\mathbf{i}}), T(\hat{\mathbf{j}}), T(\hat{\mathbf{k}})$ from the columns for N, and to apply R we just multiply by M, so:

$$P = \begin{bmatrix} M\begin{bmatrix} 2 \\ -1 \end{bmatrix} & M\begin{bmatrix} 0 \\ -1 \end{bmatrix} & M\begin{bmatrix} -1 \\ 5 \end{bmatrix} \end{bmatrix}.$$

Applying M column by column then gives our result:

$$P = \begin{bmatrix} 1 & 1 & -5 \\ 2 & 0 & -1 \end{bmatrix}.$$

Since matrix multiplication reflects the composition of the associated functions, we write:

$$MN = P,$$

or

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -5 \\ 2 & 0 & -1 \end{bmatrix}.$$

■ **Question 3.23** Let $\mathbf{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$. Compute $R(T(\mathbf{v}))$ by applying T and the R in succession. Compare this to $P\mathbf{v}$ computed by matrix multiplication.

We can follow a *process* to do this matrix multiplication, similar to how we did this above. In particular, to find the ij-entry of MN, we pair the elements of the ith row of M, with the jth row of N, multiply them together, and add them up. In this case, we have to do this 6 times. So for example, if you're interested in the entry in the first row and third column of MN, you'd pair the first row of M with the third column of N:

$$\begin{bmatrix} \color{red}{0} & \color{blue}{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & \color{red}{-1} \\ -1 & -1 & \color{blue}{5} \end{bmatrix}.$$

You multiply the paired elements and add up the result:

$$0 \cdot -1 + -1 \cdot 5 = 5.$$

And indeed, the entry in the first row and third column of MN computed by composing the functions was indeed a 5.

■ **Question 3.24** Compute the second row, second column of MN this way, confirming you get the same result.

So the product MN makes sense, because it corresponds to the composition $R \circ T$. What if we wanted a product NM? This should correspond to composing $T \circ R$: that is *rotate the vector* $90°$, *then apply* T *to the result.*

■ **Question 3.25** Does this make sense?

Let's investigate this further. Let $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and lets try to compute $T \circ R(\mathbf{v}) = T(R(\mathbf{v}))$. We first must compute $R(\mathbf{v})$, that is, we must rotate $\mathbf{v}$ by $90°$.

$$R(\mathbf{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now we'd like to apply this to T.

■ **Question 3.26** What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

If you said that this doesn't make sense, you're correct! $T : \mathbb{R}^3 \to \mathbb{R}^2$, so it takes 3D vectors as inputs. It doesn't know what to do when we input a 2D vector, because 2D vectors are not in its domain! In summary, $R \circ T$ makes sense, but $T \circ R$ does not. Since matrix products must reflect composing their associated functions, this means that the matrix product MN makes sense, but the matrix product NM does not.

■ **Slogan 3.2** Matrix multiplication doesn't always make sense. This is because matrices are functions, and matrix multiplication is functoin composition. Matrix multiplication only works when the associated functions can be composed!

You do a similar exploration on Homework 5 (Exercise 3.15).

Working more generally, suppose we have two linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^p \to \mathbb{R}^q$. The composition $L \circ T$ should be the function which takes a vector $\mathbf{v}$ in $\mathbb{R}^p$, applies T to get $T(\mathbf{v})$ in $\mathbb{R}^q$, and then applies L to the result.

■ **Question 3.27** For $L \circ T$ to make sense, what needs to be true about q and n?

We are trying to applying L to a vector in $\mathbb{R}^q$, but L only knows what to do with vectors in $\mathbb{R}^n$, so in order for this to be possible, we *must* have $n = q$.

■ **Question 3.28** Let $L : \mathbb{R}^6 \to \mathbb{R}^4$ and $T : \mathbb{R}^{11} \to \mathbb{R}^6$. What makes sense, $L \circ T$ or $T \circ L$?

What if we translate this to matrix multiplication. Associated to L is the matrix M, which has m rows and n columns (so it's an $n \times m$ matrix), and associated to T is the matrix N, which as q rows and p columns (so is $q \times p$). The produce MN should be the matrix associated to the composition $L \circ T$, which only makes sense if $n = q$.

Slogan 3.3 The matrix product MN exists precisely when the number of columns of M is equal to the number of rows of N.

Question 3.29 Consider the matrices:

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix} \qquad \text{and} \qquad N = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 & 8 \end{bmatrix}.$$

Which of the matrix products make sense (if any)?

$$MN \qquad \text{or} \qquad NM.$$

Since M has 4 columns, and N has 2 rows, MN doesn't make sense. On the other hand, N has 6 columns, and M has 6 rows, so NM does make sense.

Question 3.30 Building on the same example: How many rows does NM have? How many columns does NM have?

To figure this out, we think about N as a function $T : \mathbb{R}^6 \to \mathbb{R}^2$, and think about M as a functtion $L : \mathbb{R}^4 \to \mathbb{R}^6$. Then the product NM corresponds to the compositoin $T \circ L$ which takes a vector in $\mathbb{R}^4$ as input, applies L to get a vector in $\mathbb{R}^6$, and applies T to the result to get a vector in $\mathbb{R}^2$.

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "L"] \arrow[rr, bend left, "T \circ L"] & \mathbb{R}^6 \arrow[r, "T"] & \mathbb{R}^2 \end{tikzcd}$$

In particular, $T \circ L : \mathbb{R}^4 \to \mathbb{R}^2$, so the associated matrix (which is NM) should have 2 rows and 4 columns.

Definition 3.10.1 Let $L : \mathbb{R}^p \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^p$. The function $L \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is the function which first does applies T to a vector in $\mathbb{R}^n$, and then applies L to the result.

If M is the $m \times p$ matrix associated with L, and N is the $p \times n$ matrix associated to T, then MN is the $m \times n$ matrix associated to $L \circ T$.

Slogan 3.4 Matix multiplication *cancels the middle*. That is:

$$(m \text{ by } p) \times (p \text{ by } n) = m \text{ by n}.$$

How do we compute a matrix product? Adopting the notation of Definition 3.10.1, we are interested in the matrix that represents the composition $L \circ T$, which first applies T to a vector in $\mathbb{R}^n$, and then applies L to the result. To find this matrix, it is eanough to see what it does to the standard basis elements $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \cdots, \hat{\mathbf{e}}_n$:

$$MN = \begin{bmatrix} L(T(\hat{\mathbf{e}}_1)) & L(T(\hat{\mathbf{e}}_2)) & \cdots & L(T(\hat{\mathbf{e}}_n)) \end{bmatrix}.$$

Intoduce some notation so that:

$$N = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn}. \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}.$$

Here $\mathbf{c}_j$ is the jth column of N. We know by Theorem 3.9.1 is $T(\hat{\mathbf{e}}_j)$. Therefore

$$L(T(\hat{\mathbf{e}}_j)) = M\mathbf{c}_j = M\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix},$$

which is a usual matrix vector multiplication, and:

$$MN = \begin{bmatrix} M\mathbf{c}_1 & M\mathbf{c}_2 & \cdots & M\mathbf{c}_n \end{bmatrix}.$$

Let's introduce a bit more notation:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix}$$

Then:

$$M\mathbf{c}_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix}\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{bmatrix} = \begin{bmatrix} a_{11}b_{1j} + a_{12}b_{2j} + \cdots + a_{1p}b_{pj} \\ a_{21}b_{1j} + a_{22}b_{2j} + \cdots + a_{2p}b_{pj} \\ \vdots \\ a_{m1}b_{1j} + a_{m2}b_{2j} + \cdots + a_{mp}b_{pj} \end{bmatrix}.$$

To summarize: we can follow a *process* to do this matrix multiplication, similar to how we did above. In particular, to find the ij-entry of MN, we pair the elements of the ith row of M, with the jth row of N, multiply them together, and add them up.

Theorem 3.10.1 The matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp}. \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn}. \end{bmatrix}$$

is the matrix with m rows and n columns, whose entry in row i and column j is:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj},$$

■ **Example 3.12** Let's find the entry in the first row and third column of the matrix product from Question 3.29. Do do this, we pair the 1st row and 3rd columns:

$$\begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 \\ 2 & 6 & 5 & 3 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}.$$

Multiplying associated elements and adding them up gives:

$$3*3+1*7+4*11+1*15+5*19+9*23=377.$$

So we have:

$$\begin{bmatrix}3 & 1 & 4 & 1 & 5 & 9\\2 & 6 & 5 & 3 & 5 & 8\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 9\\9 & 10 & 11 & 12\\13 & 14 & 15 & 16\\17 & 18 & 19 & 20\\21 & 22 & 23 & 24\end{bmatrix}=\begin{bmatrix}? & ? & 377 & ?\\? & ? & ? & ?\end{bmatrix}.$$

Do this 7 more times, and we have the matrix product! ■

■ **Question 3.31** What is the entry in the second row and second column?

As you can probably tell, this process can be rather tedious, and doing the arithmetic by hand is not very enlightening in general. It also opens you up to small mistakes which are hard to notice, but may vastly change the outcome of whatever your working on. On the other hand, a computer can quickly do a series of additions and multiplications with data given in a table. For small matrices (2×2 or 3×3 at most), we will sometimes do things by hand, but once we start playing with larger matrices it is best to use technology for the arithmetic for use.

3.10.2 General Identity Matrices

In Section 3.5.2 we introduced the *2×2 identity matrix*:

$$I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$

which behaved for matrix multiplication the way the number 1 does for usual multiplication, and whose associated function−*the identity function*−does nothing to the place. In Homework 5 (Exercise 3.13), we looked at the identity function $id:\mathbb{R}^3\to\mathbb{R}^3$ which does nothing to 3-space: $id(\mathbf{v})=\mathbf{v}$ for every $\mathbf{v}$. To find equations we introduce coordinates:

$$id\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}u\\v\\w\end{bmatrix},$$

and since this must equal the inputed vector, we have:

$$u=x=1x+0y+0z,$$

$$v=y=0x+1y+0z,$$

$$w=z=0x+0y+1z.$$

Therefore, the matrix for id is:

$$I_3=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

■ **Question 3.32** Let $\mathbf{v}$ be any vector in $\mathbb{R}^3$. Is it true that $I_3\mathbf{v} = \mathbf{v}$?

■ **Question 3.33** Compute the matrix product:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Can you explain why your answer makes sense in terms of composing the associated functions?

Indeed, this always works:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&\ell\end{bmatrix} = \begin{bmatrix}1a+0d+0g & 1b+0e+0h & 1c+0f+0\ell\\0a+1d+0g & 0b+1e+0h & 0c+1f+0\ell\\0a+0d+1g & 0b+0e+1h & 0c+0f+1\ell\end{bmatrix} = \begin{bmatrix}a&b&c\\d&e&f\\g&h&\ell\end{bmatrix}.$$

You try the other way:

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&\ell\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

■ **Question 3.34** Compute the matrix products:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Can you explain why your answer makes sense in terms of composing the associated functions?

Let's do this more generally, in higher dimensions. The identity matrix should be associated to the identity function, so let's first figure out what that should be.

■ **Question 3.35** Is it possible to have an identity map $id : \mathbb{R}^3 \to \mathbb{R}^2$?

■ **Question 3.36** Suppose $m \neq n$. Is it possible to have an identity map $id : \mathbb{R}^n \to \mathbb{R}^m$?

If $id(\mathbf{v}) = \mathbf{v}$, then if $\mathbf{v}$ is in $\mathbb{R}^n$, we must have $id(\mathbf{v})$ is in $\mathbb{R}^n$ as well.

Definition 3.10.2 Fix a positive integer n. The identity map on $\mathbb{R}^n$ is the function $id : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $id(\mathbf{v}) = \mathbf{v}$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$.

■ **Question 3.37** Consider $id : \mathbb{R}^7 \to \mathbb{R}^7$. What is:

$$id\left(\begin{bmatrix}8\\6\\7\\5\\3\\0\\9\end{bmatrix}\right)?$$

The identity matrix should be associated to this function. Following what we've done above, we can extract equations for id. Here:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = id\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right).$$

Since id does nothing: we can extract equations for id:

$$\begin{aligned} u_1 &= x_1 = 1x_1 + 0x_2 + \cdots + 0x_n, \\ u_2 &= x_2 = 0x_1 + 1x_2 + \cdots + 0x_n, \\ &\vdots \\ u_n &= x_n = 0x_1 + 0x_2 + \cdots + 1x_n. \end{aligned}$$

Putting the coefficients in a matrix gives us the identity matrix.

Definition 3.10.3 The $n \times n$ *identity matrix* is the $n \times n$ matrix which has 1s in the diagonal, and 0s everywhere else.

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

■ **Question 3.38** Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Does $I_n\mathbf{v} = \mathbf{v}$?

3.10.3 Inverse Matrices in General

In Homework 6 (cf. Exercise 3.22) we asked what the definition is for the inverse of a 3×3 matrix. Let's reflect on that here. As usual, we should begin with the functional perspective.

Definition 3.10.4 Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. An inverse to L is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that:

$$L \circ T = id \qquad \text{and} \qquad T \circ L = id.$$

If such a T exists, we often write it as L^{-1}.

The coefficient matrix for such an L would be an $n \times n$ matrix M, and the coefficient matrix for T would be another $n \times n$ matrix N. The fact that $L \circ T = id$ means $MN = I_n$, and for similar reasons, $NM = I_n$. This suggests the following definition for the inverse of a matrix.

Definition 3.10.5 Let M be an $n \times n$ matrix. Another $n \times n$ matrix N is called an inversefor M if:

$$MN = I_n \qquad \text{and} \qquad NM = I_n.$$

If N exists, we will often write it as M^{-1}, and we will call M an *invertible matrix*.

We use the term *the inverse*, suggesting that there is only one possible inverse. Let's justify this.

Theorem 3.10.2 Let M be an invertible matrix, and let N_1 and N_2 be inverses for M. Then $N_1 = N_2$.

Proof. We give 2 proofs. The first is a direct application of matrix algebra.

$$N_1 = N_1 I_n = N_1(MN_2) = (N_1 M)N_2 = I_n N_2 = N_2.$$

So $N_1 = N_2$ as desired.

For a more conceptual approach using the function perspective, let's suppose that M corresponds to a function L, and N_1 and N_2 to T_1 and T_2 respectively. Fix some vector $\mathbf{x}$ in $\mathbb{R}^n$. Then $\mathbf{x} = L(\mathbf{y})$ for some vector $\mathbf{y}$ (in fact, we could take $\mathbf{y} = T_1(\mathbf{x})$). Then, since T_1 and T_2 both undo L, we know that

$$T_1(\mathbf{x}) = \mathbf{y} = T_2(\mathbf{x}).$$

Since T_1 and T_2 do the same thing to every point, they are the same function, and therefore have the same coefficient matrices. ■

We have only introduced inverses for functions $L : \mathbb{R}^n \to \mathbb{R}^n$, that is, where the domain and co-domain have the same dimension n. In time we will fully prove why, but let's start conceptually.

■ **Question 3.39** Let $L : \mathbb{R}^2 \to \mathbb{R}^3$. Can L have an inverse? Similarly, let $T : \mathbb{R}^3 \to \mathbb{R}^2$. Can T have an inverse?

Let's translate this to the matrix perspective. We've only defined inverses with the same number of rows and columns. Let's give that a name.

Definition 3.10.6 A matrix with the same number of columns as rows is called a *square matrix*.

■ **Question 3.40** Can a non-square matrix be invertible?

3.11 Homework 7

Exercise 3.24 Below are 4 matrices. Determine which (if any) can be multiplied together, and calculate the results. You may use a computer to do the computations.

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 5 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \end{bmatrix}.$$

■

Exercise 3.25 A linear map $L : \mathbb{R}^5 \to \mathbb{R}^3$ satisfies:

$$\begin{aligned} L(\hat{\mathbf{e}}_1) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} \\ L(\hat{\mathbf{e}}_2) &= \hat{\mathbf{i}} - \hat{\mathbf{k}} \\ L(\hat{\mathbf{e}}_3) &= 2\hat{\mathbf{i}} - 2\hat{\mathbf{k}} \\ L(\hat{\mathbf{e}}_4) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} \\ L(\hat{\mathbf{e}}_5) &= -5\hat{\mathbf{j}} \end{aligned}$$

1. Let M be the matrix for the function L. What is M?
2. One of the following 2 values exists, and the other is undefined. Determine which whan exists, and compute it.

$$L\left(\begin{bmatrix}3\\1\\4\\1\\5\end{bmatrix}\right) \quad or \quad L\left(\begin{bmatrix}3\\1\\4\end{bmatrix}\right).$$

3. Consider the matrix:

$$N=\begin{bmatrix}6&0&2\\0&2&3\\-5&0&2\\0&1&2\end{bmatrix}.$$

One of MN or NM is well defined. Determine which, and compute the matrix product. (You may use a computer to compute the product).
4. The matrix you found in part (c) corresponds to a function. Determine the domain and co-domain of this function.

■

Exercise 3.26 1. Compute the matrix product:

$$\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

by hand. What observations can you make about the result?
2. Let I_n be the $n\times n$ identity matrix, and let M be any $p\times n$ matrix. Determine the matrix product MI_n, and *fully explain your reasoning*. (*Hint:* It may be tedious to describe this by hand, try using the functional perspective instead).
3. Let I_n be the $n\times n$ identity matrix, and let M be any $n\times p$ matrix. Determine the matrix product I_nM, and *fully explain your reasoning*. (*Hint:* It may be tedious to describe this by hand, try using the functional perspective instead).

■

Exercise 3.27 1. Suppose $n\neq m$. Can there be an identity function $id:\mathbb{R}^n\to\mathbb{R}^m$? Why or why not?
2. Why does every identity matrix have to be square? (*Hint: Justify this using your answer to part (a).*)

■

Exercise 3.28 1. Let $\mathbf{c}$ be a n-dimensional column vector. If we instead think about it as an $n\times 1$ matrix, it determines a function. What is the domain and codomain of this function?

2. A n-dimensional row vector is a $1 \times n$ matrix. What is the domain and co-domain of the associated function?

For the rest of the problem we let:

$$\mathbf{c} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{r} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}.$$

1. Compute the product **cr** and informally interpret its meaning.
2. Compute the product **rc** and informally interpret its meaning.

Exercise 3.29 Here's an example using matrix methods to solve a system with 7 equations and 7 variables. To do this with traditional methods would be extremely tedious. But with matrix inversion (and the help of some technology), it's actually not so bad.

$$\begin{aligned}
x_1 \quad + 3x_3 \qquad\qquad - 2x_6 \qquad &= 3 \\
+ x_2 \qquad + x_4 \qquad + 2x_6 - 3x_7 &= 1 \\
- 4x_3 + 7x_4 \qquad + x_6 + x_7 &= 4 \\
+ x_2 \qquad\qquad + 2x_5 - x_6 \qquad &= 1 \\
2x_1 \quad + 3x_3 \qquad - 3x_5 \qquad + x_7 &= 5 \\
x_1 \quad + 4x_3 \qquad - x_5 \qquad - x_7 &= 9 \\
- x_2 - x_3 - x_4 - x_5 \qquad + 4x_7 &= 2
\end{aligned}$$

1. The system of equations can be written as a single matrix-vector equation $M\mathbf{x} = \mathbf{v}$ where M is a 7×7 matrix, $\mathbf{v}$ is a 7-dimension column vector, and:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix},$$

is a column vector of your variables. Write down M and $\mathbf{v}$.
2. Now solve $M\mathbf{x} = \mathbf{v}$ for $\mathbf{x}$ using matrix multiplication, thereby solving the system of equations. (You may do this by hand, or with a matrix multiplication calculator. There is one of these at `matrix.reshish.com` as well.)
3. Check that your solution works with 2 of the equations you started with.

4. Determinants

4.1 March 14, 2023

4.1.1 Determinants of 2×2 matrices

Consider a generic square 2×2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Theorem 3.7.1 tells us that if the value $ad - bc \neq 0$, then M has an inverse which is given by the formula:[1]

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

It isn't too hard to see that this formula works (as we did during the groupwork on March 2nd). For example:

$$MM^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad}{ad-bc} + \frac{b(-c)}{ad-bc} & \frac{a(-b)}{ad-bc} + \frac{ba}{ad-bc} \\ \frac{cd}{ad-bc} + \frac{d(-c)}{ad-bc} & \frac{c(-b)}{ad-bc} + \frac{ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the fact that $M^{-1}M = I_2$ is very similar. This calculation does confirm that this formula correctly produces the inverse of M, but it might seem a bit like a lucky accident, and it raises a few questions. For example:

■ **Question 4.1** Where does this value $ad - bc$ come from? Does it have any further meaning?

Furthermore, Theorem 3.7.1 tells us that if $ad - bc \neq 0$, we have an inverse. If $ad - bc = 0$, then our formula for M^{-1} doesn't make any sense (because it would require dividing by zero). This raises another question.

[1] In the formula we multiplied a matrix by a scalar. Like vectors, when we scale a matrix we scale each entry.

■ **Question 4.2** If $ad - bc = 0$, does that mean that M doesn't have an inverse? Or do we need a different formula to cover this case?

Let's start studying these questions by giving this mysterious $ad - bc$ value a name.

Definition 4.1.1 Let M be a 2×2 matrix.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The *determinant of M* is the value:

$$\det M = ad - bc.$$

In what follows we will answer questions 4.1 and 4.2. Let's start with the latter one.

What happens if the determinant is 0?

As usual, the way to really think about these questions is to use our running philosophy that *a matrix is a function*. Therefore, in this section we will try to understand what a determinant of 0 says about the associated function. Let's explore some examples of matrices whose determinant is 0.

■ **Example 4.1** The *zero matrix*:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The determinant is:

$$00 - 00 = 0.$$

The associated function is the *zero function*, which we will also denote by $\mathbf{0} : \mathbb{R}^2 \to \mathbb{R}^2$. This function takes any vector $\mathbf{v}$ in $\mathbb{R}^2$ to the zero vector:[2]

$$\mathbf{0}(\mathbf{v}) = \mathbf{0}.$$

Can this function have an inverse? Well, such an inverse should *undo* the function $\mathbf{0}$.

$$\text{Since } \mathbf{0}(\hat{\mathbf{i}}) = \mathbf{0} \text{ then need } \mathbf{0}^{-1}(\mathbf{0}) = \hat{\mathbf{i}}.$$

$$\text{Since } \mathbf{0}(\hat{\mathbf{j}}) = \mathbf{0} \text{ then need } \mathbf{0}^{-1}(\mathbf{0}) = \hat{\mathbf{j}}.$$

It can't be both $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, so the inverse can't possibly exist. ■

We were able to determine that this function had no inverse because it sent different vectors ($\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$) to the same place (the zero vector). Let's give this property (or rather, its complement) a name.

Definition 4.1.2 A function f is said to be *injective* if it satisfies the following property:

Whenever $x \neq y$, are two distinct elements in the domain of f, $f(x) \neq f(y)$.

This property is also sometimes called being *one-to-one*.

[2]You can find an animation of this function here:`www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

If a function isn't injective, then it can't possibly have an inverse. Indeed, if $x \neq y$ but $f(x) = f(y)$, do we *undo* this output to x or to y?

Theorem 4.1.1 If a function isn't injective, it cannot have an inverse.

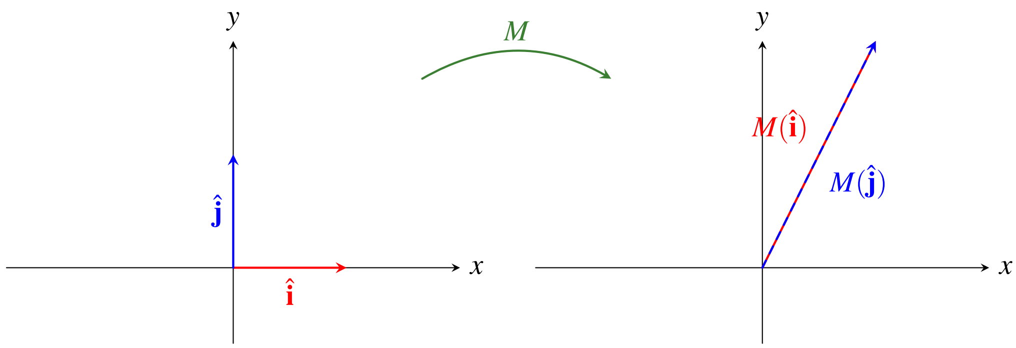
■ **Example 4.2** Let's consider the function associated to the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Then the determinant of M is:

$$\det M = 1 \cdot 2 - 1 \cdot 2 = 0.$$

Does M have an inverse? Viewing M as a function $M : \mathbb{R}^2 \to \mathbb{R}^2$, it is equivalent to ask if the function M has an inverse.



We know that $\hat{\mathbf{i}} \neq \hat{\mathbf{j}}$, but

$$M(\hat{\mathbf{i}}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = M(\hat{\mathbf{j}}).$$

So M is not injective, so Theorem 4.1.1 tells us it can't possibly have an inverse. ■

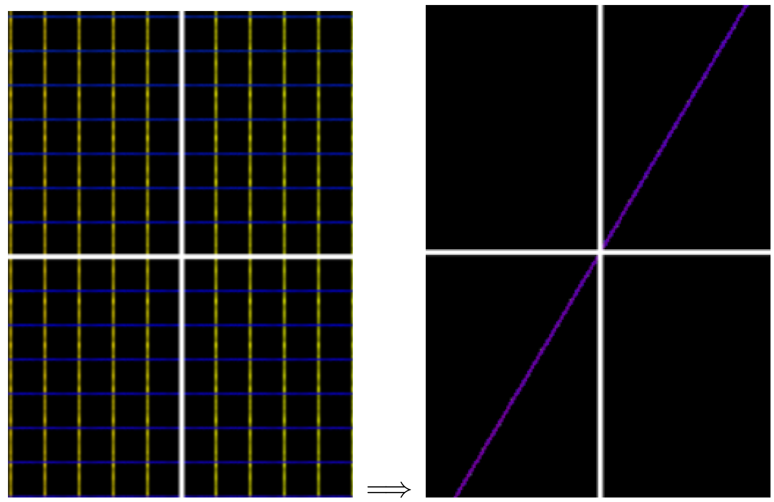
■ **Question 4.3** Consider any matrix whose columns match:

$$M = \begin{bmatrix} a & a \\ c & c \end{bmatrix}.$$

What is the determinant of M? Does M have an inverse?

Let's look a little bit more closely at the unction from Example 4.2. We can animate what the function does to the plane on a screen[3] and here are the before and after images.

[3] Click button Example 4.2 here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`



It looks like the entire grid collapsed to a line. And indeed, this is exactly what happens, as any vector $a\hat{\mathbf{i}}+b\hat{\mathbf{j}}$ gets sent to:

$$M(a\hat{\mathbf{i}}+b\hat{\mathbf{j}}) = aM(\hat{\mathbf{i}})+bM(\hat{\mathbf{j}}) = a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}1\\2\end{bmatrix} = (a+b)\begin{bmatrix}1\\2\end{bmatrix}.$$

This shows that the image of the function M consists only of multiples of a single vector!

$$\operatorname{im}(M) = \text{ multiples of } \begin{bmatrix}1\\2\end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}.$$

As we have already seen, the span of a single vector is a line (can you find the equation of the line in this case?).

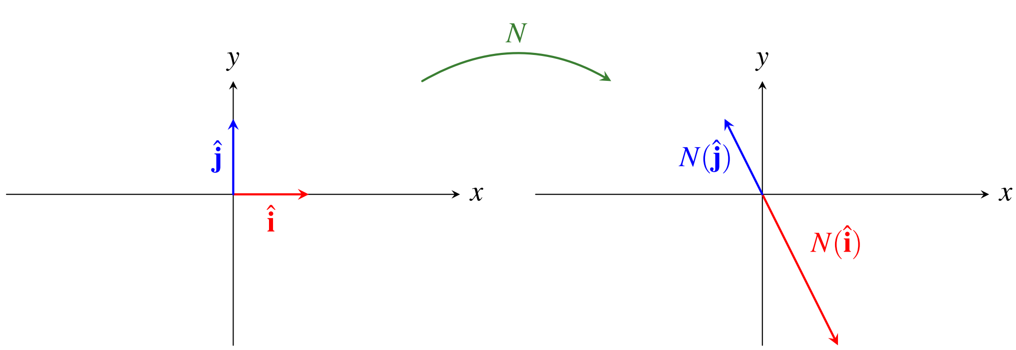
■ **Example 4.3** Let's consider the matrix:

$$N = \begin{bmatrix}2 & -1\\-4 & 2\end{bmatrix}.$$

The determinant can be computed to be:

$$\det N = 2\cdot 2-(-1)(-4) = 0.$$

Does N have an inverse? As above, let's start by viewing N as a function $N:\mathbb{R}^2\to\mathbb{R}^2$. We can where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ go reading off the columns of N.



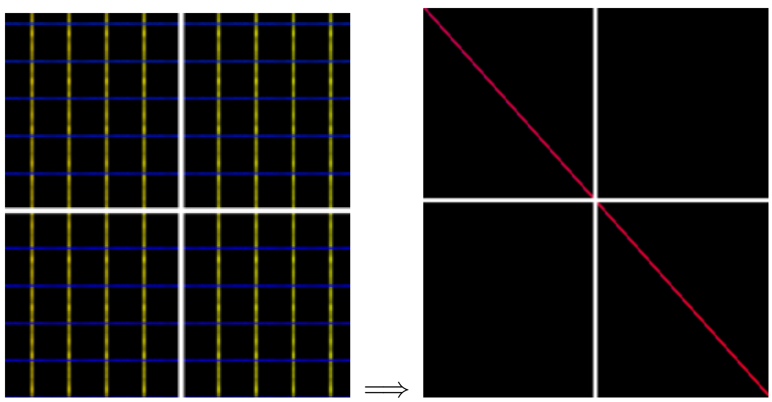
Although we don't have $N(\hat{\mathbf{i}}) = N(\hat{\mathbf{j}})$ this time, it does appear that they are parallel. Indeed, we can observe that:

$$-2N(\hat{\mathbf{j}}) = 2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} = N(\hat{\mathbf{i}}).$$

We proved in Homework 4 (cf Exercise 3.7) that N commutes with scaling so that that $-2N(\hat{\mathbf{j}}) = N(-2\hat{\mathbf{j}})$. In summary:

$$-2\hat{\mathbf{j}} \neq \hat{\mathbf{i}} \quad \text{but} \quad N(-2\hat{\mathbf{j}}) = N(\hat{\mathbf{i}}).$$

So N is not injective, and therefore cannot have an inverse (again using Theorem 4.1.1). Let's look at an animation for N, and consider also the before and after images of the grid.[4]



Again, it seems to take the entire plane to a single line. Can you explain why? (*Hint:* Write $M(a\hat{\mathbf{i}} + b\hat{\mathbf{j}})$ in terms of $M(\hat{\mathbf{j}})$? Conclude that the image is the span of a single vector.) ■

■ **Question 4.4** Consider a 2×2 matrix where one column is a multiple of the other. What is the determinant of this matrix? Can it have an inverse?

We've seen a few examples of matrices M with determinant zero, and made the following observations in each case:

1. M collapses the plane down to a line (or in the case of the $\mathbf{0}$ matrix, a point).
2. M is not injective, and therefore cannot have an inverse.

In each case, it came down to the fact that the columns were multiples of eachother. This meant that $M(\hat{\mathbf{i}})$ and $M(\hat{\mathbf{j}})$ were multiples of eachother (for example $M(\hat{\mathbf{i}}) = cM(\hat{\mathbf{j}})$). Then by linearity, this tells us that $M(\hat{\mathbf{i}}) = M(c\hat{\mathbf{j}})$, so that M cannot be injective. It turns out this really covers all our bases.

Proposition 4.1.2 If M is a 2×2 matrix of determinant zero, then one column of M is a multiple of the other.

Proof. We can introduce some variables:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[4] Click button Example 4.3 here: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

Then $\det M = ad - bc = 0$ means that $ad = bc$. There are a few cases to cover:

Case 1: One of the columns is all 0. Then it is the multiple of the other column by 0. For example, if:

$$M = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix},$$

the second column can be obtained by scaling the first by 0.

Case 2: One of the rows is all 0. For example, if

$$M = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

then the first column can be obtained by the second by scaling by $\frac{a}{b}$ (as long as $b \neq 0$, but if $b = 0$ then we're back in case 1).

Notice that if we don't fall in case 1 or 2, then all entries must be nonzero. Indeed, suppose one of the entries were 0, for example a. Then $ad = bc$ means $bc = 0$, so that either $b = 0$ (and therefore the first row is 0), or $c = 0$ (and therefore the first column is 0). Therefore all that remains is:

Case 3: All entries are nonzero. Then $ad = bc$ means that $\frac{a}{c} = \frac{b}{d}$. But two fractions agree precisely when you can scale the numerator and denomator of one to obtain the other, so this tells us that the columns are multiples of eachother. ■

As we saw before, once one column is a multiple of the other, the function can no longer be injective, and therefore can't have an inverse. We therefore have answered Question 4.2.

Theorem 4.1.3 A 2×2 matrix M has an inverse if and only if $\det M\ \emptyset$. Furthermore, if $\det M = 0$ then either:

1. M is the zero matrix, whence M collapses the entire plane to the origin, or,
2. M is nonzero. Then M collapses the entire plane to the line given by the span of one of its nonzero columns.

So we seem to understand the determinant zero case, and we can learn a lot of about the function associated with M by figuring out whether or not $\det M = 0$. But if $\det M \neq 0$, what can we say about its actual value?

What does the determinant mean?

To see what the determinant means when it's nonzero, let's experiment by looking at a couple of examples to get a bit of a baseline.

■ **Question 4.5** What is the determinant of the identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}?$$

Let's work with a more nontrivial example.

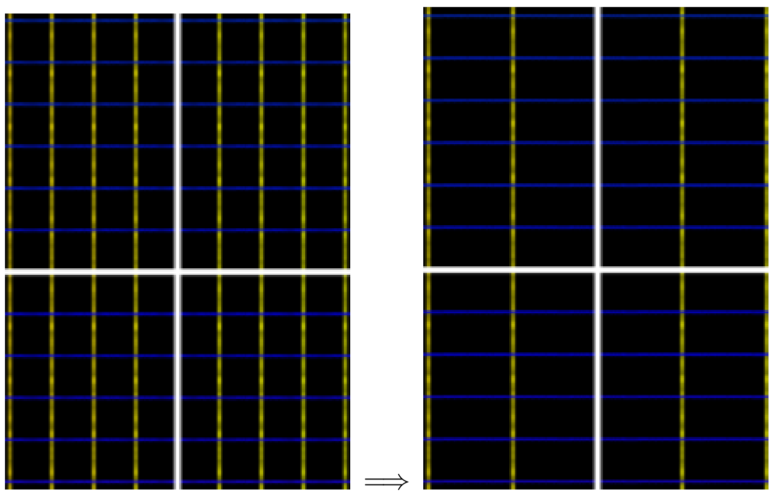
■ **Example 4.4** Consider the matrix:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

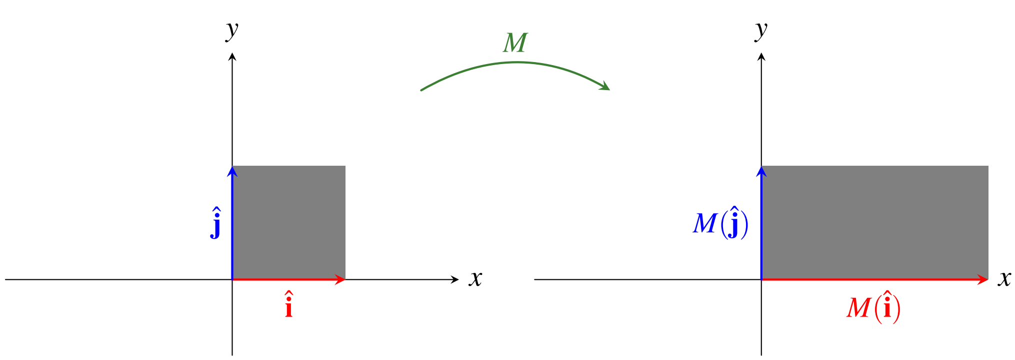
We can compute the determinant to be

$$\det M = 2 \cdot 1 - 0 \cdot 0 = 2.$$

We can look at an animation of the associated function,[5] and consider some before and after pictures of the grid.



It looks like the plane is *spreading out* in the horizontal direction. Let's try to figure out how much. Consider the square whose corners are the points $(0,0),(0,1),(1,1),(1,0)$. We will call this the *unit square*, which has area 1. We'd like to compare this to the area of the image of the unit square under the map M.



To compute the area of the new recangle, we can multiply base times height, and obtain an area of 2. This matches with the value of the determinant of M. ■

[5]Click example 4.4: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

■ **Question 4.6** Let $a, d > 0$ and consider the matrix

$$N = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

What is the determinant of N? What is the area of the image of the unit square after applying N.[6]

It looks like for *diagonal matrices*, the determinant is equal to the area of image of the unit square. In particular, if the determinant is bigger than 1, the unit square grows, and if the determinant is less than 1, the unit square shrinks. So the initial feeling is that the determinant calculates how much the linear map expands or contracts the plane. Let's do a few more examples and see if this pattern holds.

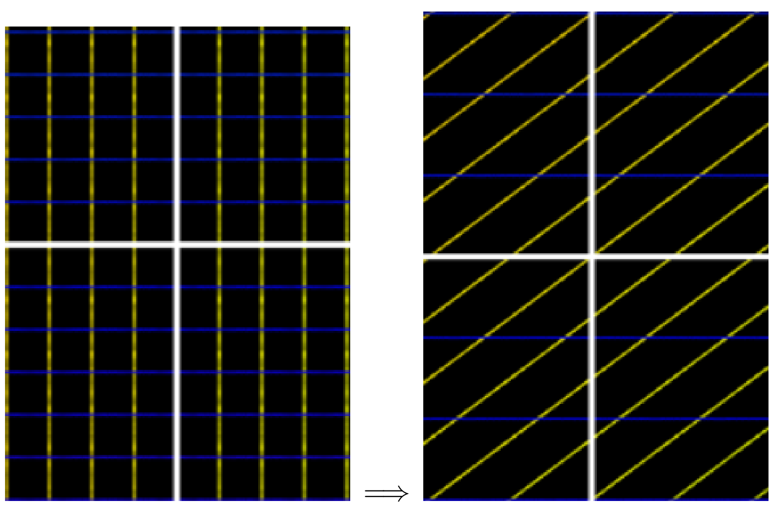
■ **Example 4.5** Let's consider the matrix:

$$P = \begin{bmatrix} 1.5 & 2 \\ 0 & 2 \end{bmatrix}.$$

The determinant is:

$$\det P = 1.5 \cdot 2 - 2 \cdot 0 = 3.$$

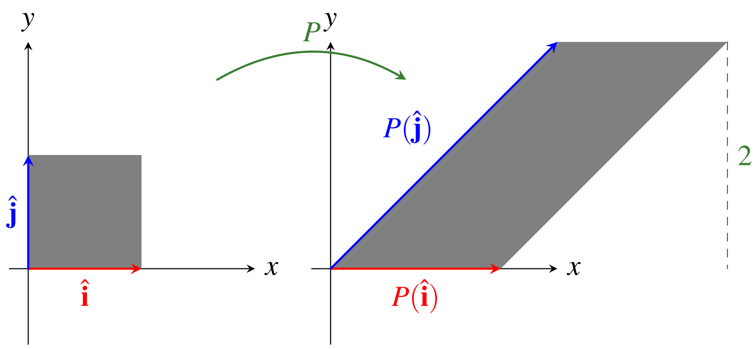
We can look at an animation of the associated function,[7] and consider some before and after pictures of the grid.



It again looks like we get some expansion, one would hope by a factor of 3.

[6]An example of this with $a = .75$ and $b = 2$ can be found by clicking Question 4.6 at the link `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`.

[7]Click Example 4.5: `www.gabrieldorfsmanhopkins.com/LinearAlgebraNotes/animationsAndTools/Mar14Examples/index.html`

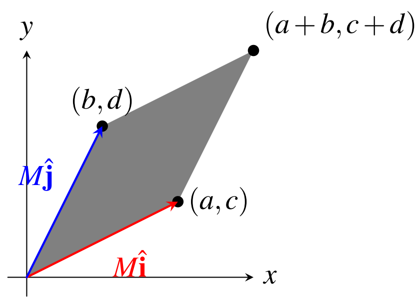


The area of a parallelogram is still base times height, which is $1.5 * 2 = 3$! ■

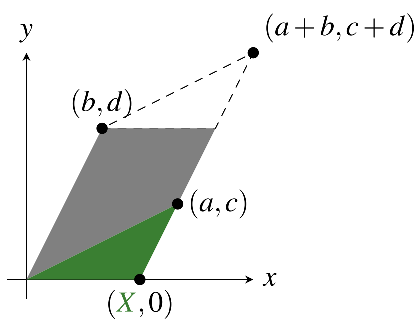
It's starting to look like the determinant exactly captures the area of the image of the unit square. Indeed, consider the matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we can trace where M take $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and determine where the unit square goes.



To compute the area, we can slice off the top of this parallelogram, and stick it at the bottom.



The new shaded region has the same area, but now we can more easily determine base times height. The height is d, and the base is the unknown quantity X. But X is the x-intercept of the line connecting (a,c) and $(a+b,c+d)$. This line has slope:

$$\frac{\Delta y}{\Delta x} = \frac{b}{d}.$$

Plugging this into point slope form with the point (a,c) gives an equation of the line:

$$y = \frac{d}{b}(x-a)+c.$$

And plugging in $y=0$ and solving for x gives us our x-intercept of:

$$X = a - \frac{bc}{d}.$$

As this is the base of the parallelogram, we need only to multiply by the height d to obtain:

$$d\cdot\left(a-\frac{bc}{d}\right) = ad - bc = \det M.$$

There is one issue with this proof: it relies on a picture where $M\hat{\mathbf{j}}$ is above $M\hat{\mathbf{i}}$. The following example demonstrates that this is rather problematic.

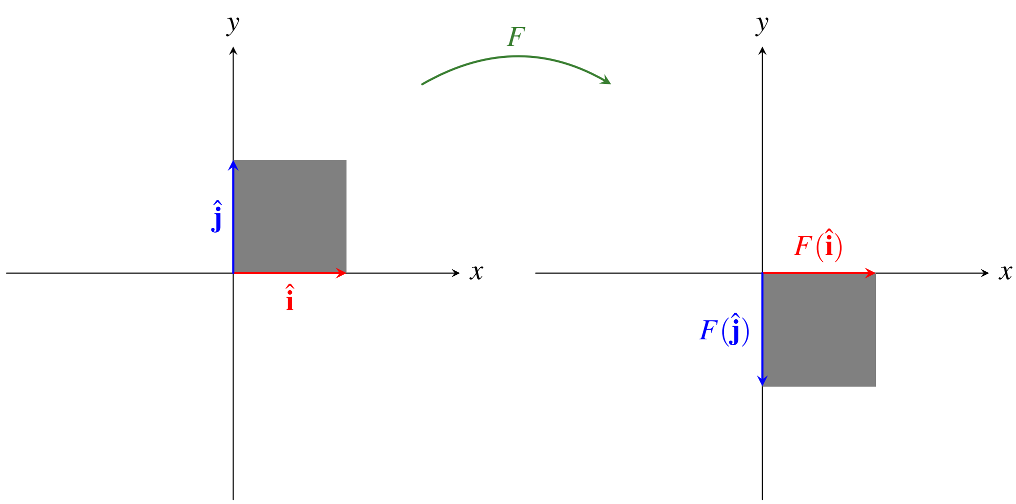
■ **Example 4.6** Consider the *flip* matrix:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then we can compute the determinant:

$$\det F = 1(-1) - 0\cdot 0 = -1.$$

This seems immediately to be an issue. If the determinant measures an area, how can it be negative? Analyzing the image of the unit rectangle we see that its area is still 1.

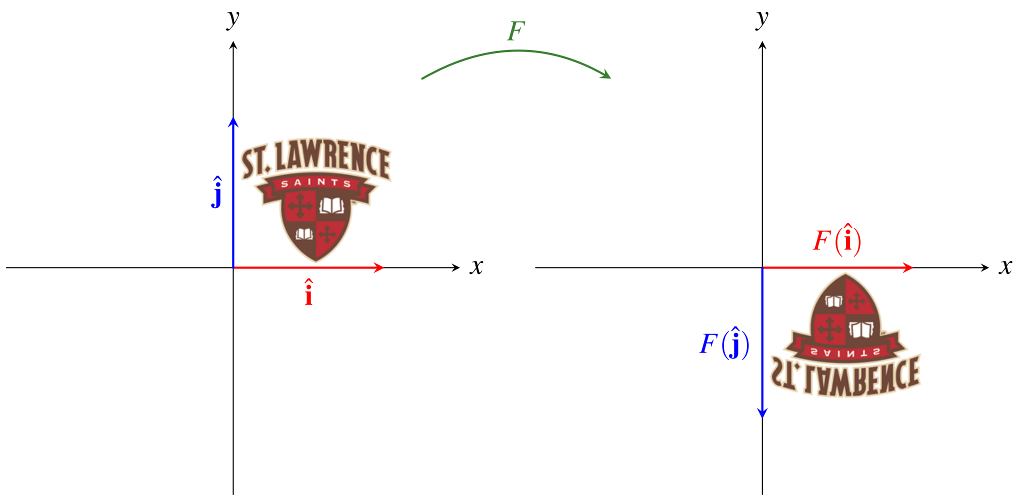


So the *absolute value* of the determinant seems to still capture the right area. So what does the negativity capture? One way to gain understanding is to look at the animation of this function.[8] In the animation, it looked like the whole grid needed to cross over itself in order to get to its final position. This corresponds to a change in *orientation*. We can describe this orientation in a couple of ways:

[8]Click Example 4.6 HERE. We won't post before and after screenshots because the grid ends up looking the same.

1. If you were to be looking in the direction of $\hat{\mathbf{i}}$, then $\hat{\mathbf{j}}$ is to the left. On the other hand, if you were to be looking in the direction of $F(\hat{\mathbf{i}})$, then $F(\hat{\mathbf{j}})$ is to the right. This corresponds to a change of orientation.
2. The angle from $F(\hat{\mathbf{i}})$ to $F(\hat{\mathbf{j}})$ (counting counterclockwise as usual) is greater than 180°.
3. If you wrote a word on the unit square, then it becomes reflected after applying F and now appears backwards.

It is clear that the first and second cases mean the same thing. For the third, let's put some words on the unit square and then apply the *flip* map.



■

All in all, we have established the following interpretation of the determinant of a 2×2 matrix.

Theorem 4.1.4 Let M be a 2×2 matrix. Then the absolute value of the determinant $|\det M|$ computes the area of the unit square after applying the function M. In particular:

1. If $|\det M| < 1$ then M contracts areas.
2. If $|\det M| = 1$ then M preserves areas.
3. If $|\det M| > 1$ then M expands areas.

Furthermore, the sign of $\det M$ determines whether M preserves orientation or not. In particular, if $\det M < 0$ then the function M is in part a reflection. More precisely:

1. If $\det M > 0$ then the angle from $M\hat{\mathbf{i}}$ to $M\hat{\mathbf{j}}$ is between 0° and 180°.
2. If $\det M < 0$ then the angle from $M\hat{\mathbf{i}}$ to $M\hat{\mathbf{j}}$ is between 180° and 360°.

■ **Question 4.7** Does the interpretation of determinants as areas hold up when $\det M = 0$?

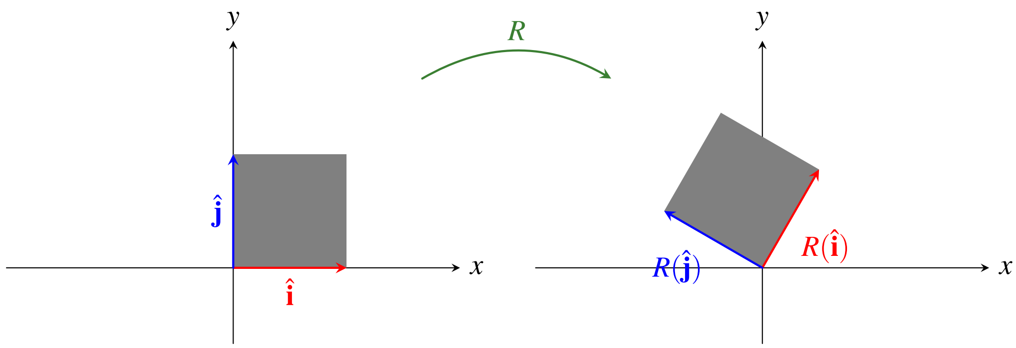
You can find a really cool interactive tool to play with determinants in real time here: `https://www.khanacademy.org/computer-programming/linear-transformation-playground-determinant-edition/6721406349426688`.

Exercise 4.1 — Checkin 6. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates the plane by an angle of θ. What is the determinant of the matrix associated to R? Fully justify your reasoning. ■

4.2 March 16, 2023

Let's start today by going over Checkin 6 (Exercise 4.1), and in doing so, warm back up to the notion of determinants.

■ **Example 4.7** We are asked to compute the determinant of a 2×2 rotation matrix R. We will use that the determinant measures how much a transformation scales area. In particular, the absolute value of $\det R$ is the area of the image of the unit square after applying R. Below we depict what happens to R under this rotation.



As we can see, the area of the unit square doesn't change when rotating the plane, so that the absolute value of $\det R$ must be equal to 1. Furthermore, orientation is positive, as $R\hat{\mathbf{j}}$ remains to the left of $R\hat{\mathbf{i}}$, so $\det R$ is positive as well. Therefore:

$$\det R = 1.$$

Notice that we determined this without doing any computations, just by analyzing the geometry of the situation. An alternative approach could be to set:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Then we could directly compute:

$$\det R = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta.$$

The fact that this is equal to 1 follows from the Pythagorean theorem. ■

4.2.1 Determinants of 3×3 matrices.

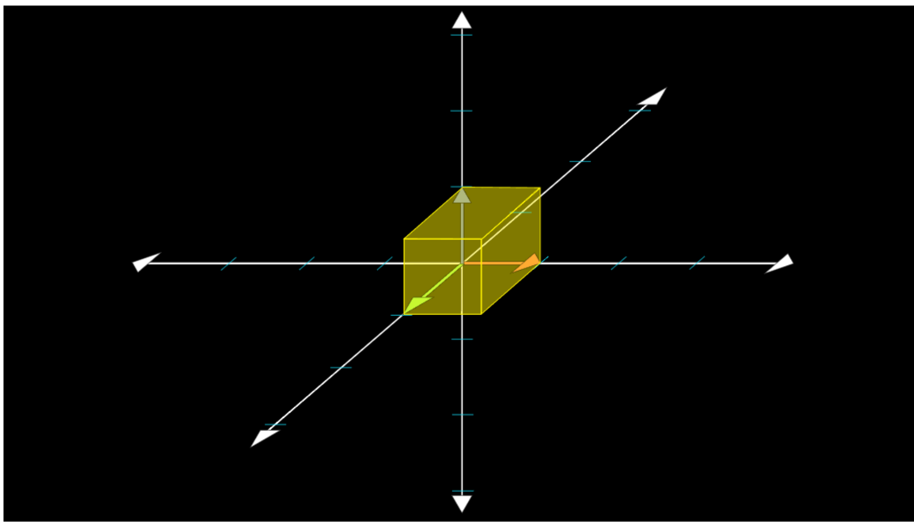
Now that we've unpacked the meaning of a 2×2 determinant, we'd like to see if we can arrive at a similar notion for 3×3 matrics, 4×4, and so on. A 2×2 matrix corresponds to a function $\mathbb{R}^2 \to \mathbb{R}^2$, and the determinant measures how much the matrix scales area in $\mathbb{R}^2$. Size in $\mathbb{R}^3$ isn't measured with areas, but instead with volumes, so this could give us a first idea.

Definition 4.2.1 — 3×3 determinants: a first attempt. Let M be a 3×3 matrix, associated to a function $\mathbb{R}^3 \to \mathbb{R}^3$ which we will also denote by M. The determinant of M is a constant which measures (up to a sign) how much volumes in $\mathbb{R}^3$ scale after applying M. That is, if V is a region in $\mathbb{R}^3$, then:

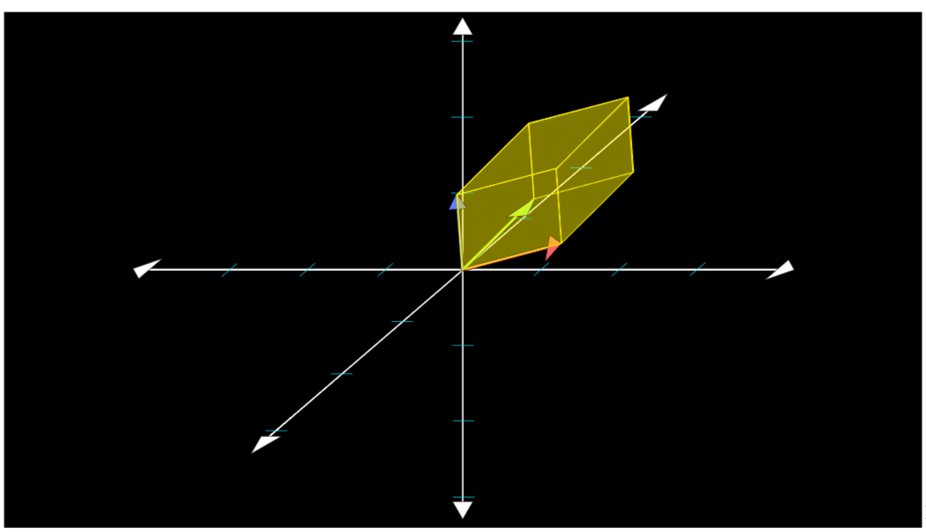
$$\text{Volume}(M(V)) = |\det M| \cdot \text{Volume}(V).$$

This definition gives us the determinant (or at least its absolute value). In order to arrive at a formula, we may want to follow what we did for 2×2 matrices. Here we started with the unit square which has area 1. Applying the transformation transformed it into a parallelogram, the area of which was precisely the absolute value of the determinant.

The analogous object to the unit square in 3-space is the *unit cube* C, formed by taking the vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ as legs of a cube.


[9]

After applying M, this gets transformed into the 3 dimensional version of a paralellogram: a *parallelepiped*, whose legs are formed by $M\hat{\mathbf{i}}, M\hat{\mathbf{j}}$ and $M\hat{\mathbf{k}}$.


[10]

[9]Image by Grant Sanderson/3Blue1Brown
[10]Image by Grant Sanderson/3Blue1Brown

If we call the unit cube C, Definition 4.2.1 says:

$$|\det(M)| \cdot \text{Volume}(C) = \text{Volume}(M(C)).$$

Subbing in that the volume of the unit cube is 1 tells us:

$$|\det(M)| = \text{Volume}(M(C)).$$

We now have pinned down a definition that captures the determinant up to a sign: just measure the volume of the image of the unit cube. This allows us now to study the following question:

■ **Question 4.8** Let M be a 3×3 matrix. Is is still true that M has an inverse if and only if $\det M = 0$?

Let's begin by collecting some evidence.

■ **Example 4.8** Let I_3 be the 3×3 identity matrix. This is an invertible matrix (what is the inverse?). Further, I_3 takes the unit cube to itself, so the determinant of I_3 is 1 (or possibly -1). ■

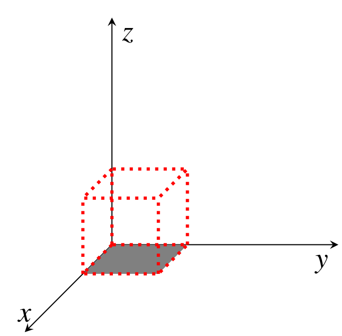
■ **Example 4.9** Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that $M(\hat{\mathbf{k}}) = \mathbf{0} = M(\mathbf{0})$. Therefore M is not injective, and cannot have an inverse. Can we work out $\det M$? Well:

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix},$$

so everything projects down vertically to the xy-plane.



As we can see, the unit cube (outlined in red) projects down to the unit square in the xy-plane (shaded in gray). The volume of the unit square is 0—being confined to 2 dimensions, it doesn't enclose any volume at all. This tells us that $\det M = 0$. ■

■ **Question 4.9** Let M be a 3×3 matrix and suppose one of its columns is zero. Is M injective? What is the determinant?

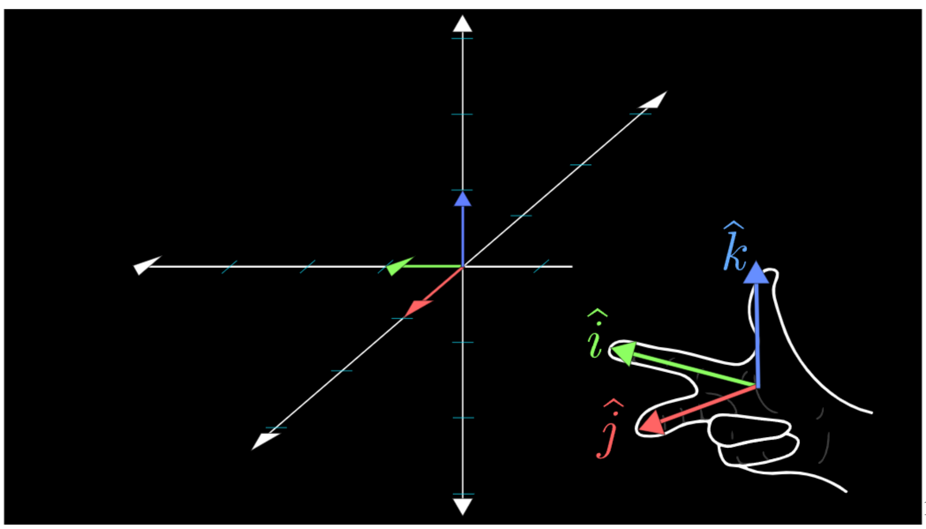
If the determinant is 0, this means that the unit cube gets projected down onto something which encloses no volume. In general, this means it goes down a dimension (like in the example we just saw). One can deduce from this that all of $\mathbb{R}^3$ gets projected onto something of smaller dimension (either a plane, a line, or a single point), from which you can deduce that the function is not injective, and therefore not invertible. We will explore this a bit more carefully on today's groupwork, answering the following question.

■ **Question 4.10** Let M be a 3×3 matrix, and suppose one of the columns is in the span of the other two. Is M injective? What is $\det M$?

Conversely, there is a formula for the inverse of a 3×3 matrix that involves dividing by the determinant (we will see this below), so that the following theorem holds.

Theorem 4.2.1 A 3×3 matrix is invertible if and only if its determinant is nonzero.

Notice we have only defined the absolute value of the determinant so far, but that's enough to determine whether or not it is zero, so the theorem above is not controversial. But it is worth taking a moment to analyze the sign of the determinant. It turns out, just like before, that it boils down to a subtle notion of orientation. One way to describe orientation in 3D is with the right hand rule: if you point your index finger of right hand in the direction of $\hat{\mathbf{i}}$, and curl your middle finger in the direction of $\hat{\mathbf{j}}$, then your thumb is pointing in the direction of $\hat{\mathbf{k}}$. This denotes the positive orientation.

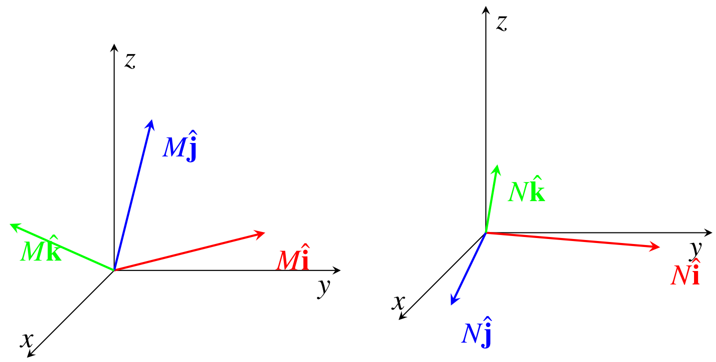


[11]

To determine whether a matrix preserves orientation or not, you apply the right-hand-rule to the image of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. In particular, point your index finger of right hand in the direction of $M\hat{\mathbf{i}}$, and curl your middle finger in the direction of $M\hat{\mathbf{j}}$. If $M\hat{\mathbf{k}}$ is going in the direction of your thumb, then orientation has remained positive. Otherwise, orientation has become negative. If M preserves orientation, then it will have positive determinant, otherwise the determinant will be negative.

■ **Example 4.10** The graphs below show where $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ applying a matrix M (on left) or N (on right). Applying the right hand rule, we can observe that $\det M > 0$ and $\det N < 0$.

[11] Image by Grant Sanderson/3Blue1Brown



■

We now have a definition of a 3×3 determinant.

Definition 4.2.2 Let M be a 3×3 matrix. The determinant of M is a scalar values such that:

1. The absolute value $|\det M|$ is the volume of the image of the unit cube.
2. $\det M$ is positive if it preserves orientation (as defined by the right hand rule), and is negative if it reverses orientation.

■ **Question 4.11** We saw above that $\det I_3 = \pm 1$. Use the right hand rule to determine whether it is 1 or -1.

We have now completely defined the 3×3 determinant purely geometrically, without any difficult formulas. Of course, we would also like to compute it from time to time.

A formula for the 3×3 determinant

So how do we compute the 3×3 determinant of a matrix M? It turns out, we can use the following formula.

Proposition 4.2.2 — Cofactor Expansion along the First Row. Let:

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix}$$

Then the determinant of M can be computed via the following formula:

$$\det M = a \det\left(\begin{bmatrix} b & f \\ \ell & m \end{bmatrix}\right) - b \det\left(\begin{bmatrix} d & f \\ h & m \end{bmatrix}\right) + c \det\left(\begin{bmatrix} d & e \\ h & \ell \end{bmatrix}\right).$$

To compute this, we traverse the top row of the matrix, and for each entry in the top row of the matrix, we cross out the corresponding column along with the top row, so that a 2×2 matrix remains (often called the *minor* associated to that enty). We then multiply the determinant of this minor together with that entry. To finish, take the *alternating sum* of all the terms. That is, we add the first, subtract the second, and add the third.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & m \end{bmatrix}$$

We will not prove that this formula works, but it is generally related to the *cross product* which you might see in multivariable calculus (Math 205).

R The formula we defined is often called *cofactor expansion along the first row*. One can do a similar computation along the second or third rows, and even along some columns. We will delay this generalization for the time being.

■ **Question 4.12** Compute $\det I_3$ using the formula, and compare to your answer to question 4.11.

■ **Question 4.13** Compute the determinant of:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Is M invertible?

■ **Question 4.14** Suppose M is a matrix which has a column of zeroes. Use the formula to confirm your answer to Question 4.9.

Inverting a 3×3 matrix using determinants

The 2×2 determinant was useful for computing inverse matrices. It turns out that there is a similar formula for 3×3 matrices, although it is quite a bit more complicated. First, notice that in computing a 3×3 determinant, it was useful to look at determinants of 2×2 matrices obtained by blocking out a row and a column. This turns out to be a useful idea.

Definition 4.2.3 Let M be a 3×3 matrix. The ij-minor of M is the 2×2 matrix M_{ij} obtained by removing row i and column j.

■ **Example 4.11** Let

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Let's compute the minor M_{23}. To do this we block out the second row and third column:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We put what remains in a matrix:

$$M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}.$$

■

■ **Question 4.15** Let M be the same matrix as in Example 4.11. What are M_{21} and M_{33}?

R This allow us to simplify the formula for the determinant a bit. Namely, if we call the ij-entry of M by the notation a_{ij}, then:

$$\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} \det M_{13}.$$

With this in hand, we can define the *adjoint matrix* for M.

Definition 4.2.4 Let M be a 3×3 matrix. Then the *adjoint* of M is:

$$\operatorname{Adj}(M) = \begin{bmatrix} \det M_{11} & -\det M_{21} & \det M_{31} \\ -\det M_{12} & \det M_{22} & -\det M_{32} \\ \det M_{13} & -\det M_{23} & \det M_{33} \end{bmatrix}.$$

■ **Warning 4.1** Notice in the definion: $\det M_{ij}$ appears in the ji-position of the adjoint matrix. This isn't a typo, this is on purpose. There is also something called the *cofactor matrix* which has $\det M_{ij}$ in the ij-position. The adjoint and cofactor matrices are related by an operation called the *transpose*, which swaps the ij and ji entries. We will cover transposes in the *matrix methods* section of this course.

With this in hand, we can now give a formula for inverting a 3×3 matrix!

Theorem 4.2.3 Let M be a 3×3 matrix whose determinant is nonzero. The inverse of M is given by the formula:

$$M^{-1} = \frac{1}{\det M} \operatorname{Adj}(M)$$

■ **Example 4.12** Let's consider the matrix

$$M = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix}.$$

We can compute the determinants of all the minors.

$$\det M_{11} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} -1 & 5 \\ 4 & -7 \end{bmatrix} = -1 \cdot -7 - 4 \cdot 5 = -13.$$

$$\det M_{21} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} -2 & 3 \\ 4 & -7 \end{bmatrix} = -2 \cdot -7 - 4 \cdot 3 = 2.$$

$$\det M_{31} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} -2 & 3 \\ -1 & 5 \end{bmatrix} = -2 \cdot 5 - -1 \cdot 3 = -7.$$

$$\det M_{12} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 0 & 5 \\ -2 & -7 \end{bmatrix} = 0 \cdot -7 = -2 \cdot 5 = 10.$$

$$\det M_{22} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & -7 \\ -2 & 3 \end{bmatrix} = 1 \cdot -7 - -2 \cdot 3 = -1.$$

$$\det M_{32} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} = 1 \cdot 5 - 0 \cdot 3 = 5.$$

$$\det M_{13} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 0 & -1 \\ -2 & 4 \end{bmatrix} = 0 \cdot 4 - -2 \cdot -1 = -2.$$

$$\det M_{23} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = 1 \cdot 4 - -2 \cdot -2 = 0.$$

$$\det M_{33} = \det \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 5 \\ -2 & 4 & -7 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot -2 = 1.$$

We can plug these values in to the formula of the formula adjoint matrix, keeping track of signs, and see:

$$\mathrm{Adj}(M) = \begin{bmatrix} -13 & -2 & -7 \\ -10 & -1 & -5 \\ -2 & 0 & -1 \end{bmatrix}.$$

To finish computing M^{-1}, we also need the determinant of M. We can use the work we've already done:

$$\det M = a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} \det M_{13} = 1 \cdot (-13) - (-2) \cdot 10_3 \cdot (-2) = -13 + 20 - 6 = 1.$$

Therefore:

$$M^{-1} = \frac{1}{\det M} \mathrm{Adj}(M) = \begin{bmatrix} -13 & -2 & -7 \\ -10 & -1 & -5 \\ -2 & 0 & -1 \end{bmatrix}.$$

As we saw in Groupwork 6 Problem 2, this is exactly M^{-1}. ■

4.2.2 Determinants of $n \times n$ matrices

Everything we have done so far can be extended to 4×4 matrices and beyond (at the expense of a massive increase in computational complexity). At this point, the geometetric grounding starts to become a bit more abstract, so we will just introduce the relevant definitions. For the remainder of this section, we fix a generic square $n \times n$ matrix:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Definition 4.2.5 For integers i and j between 1 and n, we defined the ij-minor of M to be the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j. For example, M_{12} removes row 1 and column 2, resulting in:

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix}.$$

We can then define the determinant of a matrix M in a similar way to how we defined it for 3×3 matrices.

Definition 4.2.6 The determinant of M is the value:

$$\det M = a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}M_{13} - \cdots + (-1)^{n+1}a_{1n}\det M_{1n} = \sum_{j=1}^{n} (-1)^{j+1} a_{1j}\det M_{1j}.$$

R This is generally called cofactor expansion along the first row. We can, in fact, do so along any row. In particular, for any i, the determinant is also:

$$\det M = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\det M_{ij}.$$

We can also do cofactor expansion along any column.

$$\det M = \sum_{i=1}^{n} (-1)^{i+1} a_{ij}\det M_{ij}.$$

We will see later that these all give the same value.

R The determinant does have a geometric interpretation, telling us how n-dimensional volumes scale under the associated matrix transfomation. In particular, it can be a useful way to determine whether a matrix transformation generally *spreads stuff out* or *brings them together*.

To generalize Theorem 4.2.3, we must define a version of the adjoint.

Definition 4.2.7 The *adjoint* of M is the matrix $\mathrm{Adj}(M)$ whose ij-entry is $(-1)^{i+j}\det M_{ji}$. That is:

$$\mathrm{Adj}(M) = \begin{bmatrix} \det M_{11} & -\det M_{21} & \cdots & (-1)^{1+n}\det M_{n1} \\ -\det M_{12} & \det M_{22} & \cdots & (-1)^{2+n}\det M_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}\det M_{1n} & (-1)^{n+2}\det M_{2n} & \cdots & \det M_{nn} \end{bmatrix}$$

Observe the reversal in the i and the j in the subscript of M. As in Warning 4.1, this is no mistake.

We can now see exactly how the determinant helps us fully understand the inverse.

Theorem 4.2.4 Let M be an $n \times n$ matrix. Then M has an inverse if and only if $\det M \neq 0$. In this case the inverse can be computed as:

$$M^{-1} = \frac{1}{\det M} \operatorname{Adj}(M).$$

5. Systems of Linear Equations

5.1 March 28, 2023

5.1.1 Linear Systems and Matrix Equations

We have seen a couple of times (cf. Section 3.7.2 or Exercise 3.29) that if we write a system of linear equations as a single matrix equation, we can use matrix methods to solve the system more efficiently. Let's review this by unpacking Exercise 3.29, where we solve a *very large* system of linear equations, with seven equuations and seven unknowns, something that would have taken a long time with traditional methods.

■ **Example 5.1** Let's use matrix inversion to solve the following system of equations.

$$\begin{array}{rrrrrrrl}
x_1 & & +3x_3 & & & -2x_6 & & =3 \\
& x_2 & & +\ x_4 & & +2x_6 & -3x_7 & =1 \\
& & -4x_3 & +7x_4 & & +\ x_6 & +\ x_7 & =4 \\
& x_2 & & & +2x_5 & -\ x_6 & & =1 \\
2x_1 & & +3x_3 & & -3x_5 & & +\ x_7 & =5 \\
x_1 & & +4x_3 & & -\ x_5 & & -\ x_7 & =9 \\
& -x_2 & -\ x_3 & -\ x_4 & -\ x_5 & & +4x_7 & =2
\end{array}$$

Indeed, we saw that this linear system could be rewritten as a *single matrix equation*

$$M\mathbf{x} = \mathbf{v}, \tag{5.1}$$

where:

$$M = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & -4 & 7 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 \\ 2 & 0 & 3 & 0 & -3 & 0 & 1 \\ 1 & 0 & 4 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 & 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \\ 2 \end{bmatrix}$$

If we multiply both sides of equationn 5.1 by M^{-1} *on the left* we obtaion:

$$M^{-1}M\mathbf{x} = M^{-1}\mathbf{v},$$

which simpifies to:

$$\mathbf{x} = M^{-1}\mathbf{v}.$$

Using a calculator, we can compute:

$$M^{-1} = \begin{bmatrix} 13 & 11.5 & --1 & -3.5 & -3 & -6 & 8 \\ -9.1 & -7.7 & 0.3 & 3.1 & 2.7 & 3.7 & -5.6 \\ -2.2 & -1.9 & 0.1 & 0.7 & 0.4 & 1.4 & -1.2 \\ -1.4 & -1.3 & 0.2 & 0.4 & 0.3 & 0.8 & -0.9 \\ 5.9 & 5.3 & -0.2 & -1.4 & -1.8 & -2.3 & 3.9 \\ 2.7 & 2.9 & -0.1 & -0.7 & -0.9 & -0.9 & 2.2 \\ -1.7 & -1.4 & 0.1 & 0.7 & 0.4 & 0.9 & -0.7 \end{bmatrix}$$

We now know every term in $\mathbf{x} = M^{-1}\mathbf{v}$ explicitly, so we can sub in and compute:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 13 & 11.5 & --1 & -3.5 & -3 & -6 & 8 \\ -9.1 & -7.7 & 0.3 & 3.1 & 2.7 & 3.7 & -5.6 \\ -2.2 & -1.9 & 0.1 & 0.7 & 0.4 & 1.4 & -1.2 \\ -1.4 & -1.3 & 0.2 & 0.4 & 0.3 & 0.8 & -0.9 \\ 5.9 & 5.3 & -0.2 & -1.4 & -1.8 & -2.3 & 3.9 \\ 2.7 & 2.9 & -0.1 & -0.7 & -0.9 & -0.9 & 2.2 \\ -1.7 & -1.4 & 0.1 & 0.7 & 0.4 & 0.9 & -0.7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4.9 \\ 4.8 \\ 2.6 \\ -1.1 \\ 1.7 \\ 3.3 \end{bmatrix}$$

In particular, we have solved the system of equations:

$$x_1 = -8$$

$$x_2 = 4.9$$

$$x_3 = 4.8$$

$$x_4 = 2.6$$

$$x_5 = -1.1$$

$$x_6 = 1.7$$

$$x_7 = 3.3$$

■

As this example shows, writing a system of equations as a single matrix equation provide us with a very efficient technique for solving a system of equations. Notice, though, that it was very important in this previous example that the matrix M was invertible. What happens if this is not the case?

■ **Example 5.2** Consider the system:

$$\begin{aligned} 2x - y &= -1 \\ -4x + 2y &= 2 \end{aligned}$$

We can write this system as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Unfortunately, we saw in Example 4.3 that M isn't invertible. We can check this, for example, by computing:

$$\det M = 2 * 2 - (-1)(-4) = 0.$$

What does this tell us about the system of equations? We cannot use the same technique as in Example 5.1, so where do we go from here? In this section we will answer these questions and more. ■

All of the examples we have seen so far correspond to systems with the same number equations and variables. In this case, the matrix we extract will be square. But this isn't always the case, as the following examples show.

■ **Example 5.3** Consider the system of equations:

$$x + 3y - 11z - 2w = 1,$$

$$2x + 7y - 3z + w = 5.$$

This can be written as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 2 & 7 & -3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

■

■ **Example 5.4** Consider the system of equations:

$$\begin{aligned} x + y &= 0 \\ 2x + 3y &= 5 \\ -x + 11y &= 13 \end{aligned}$$

This can be written as $N\mathbf{y} = \mathbf{w}$ where:

$$N = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 11 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 5 \\ 13 \end{bmatrix}.$$

■

The matrices M and N from Examples 5.3 and 5.4 are not square, which (we will soon see) means that they have no inverses. Where do we go from here? In this section we will expand the techniques used in Example 5.1, so that we can also use it in situations like those in Examples 5.2, 5.3, and 5.4. We should first observe that it was no coincidence that all 4 of the examples listed so far have been able to be translated into a matrix equation. This will always happen. Let's record this fact, and the central definition of this section.

Definition 5.1.1 A *linear system* is a system of linear equations in a fixed number of variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\ \vdots \qquad\qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n. \end{aligned}$$

Notice that any linear system can be written as a *matrix equation* which contains exactly the same data.

Definition 5.1.2 The *matrix equation* of the linear system from Definition 5.1.1 is the matrix equation:

$$M\mathbf{x} = \mathbf{v},$$

where:

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Notice that the linear system in Definition 5.1.1 and the matrix equation in Definition 5.1.2 contain *exactly the same information*. They are interchangeable, and just differ notationally. In fact, we will refer to both just simply as a *linear system*. To record this one more time, we present another—equivalent but much more concise—definition of a linear system.

Definition 5.1.3 A linear system is a matrix equation $M\mathbf{x} = \mathbf{v}$ where M is a matrix, $\mathbf{x}$ is a column vector of variables, and $\mathbf{v}$ is a column vector of constants.

A Note About the Functional Perspective

So far during this course we have emphasized the perspective that *a matrix is a function*. This perspective is valuable in the context of linear systems as well. In particular, if M is an $n \times m$ matrix, then we often think about M as a function $M : \mathbb{R}^m \to \mathbb{R}^n$. Asking for a solution to the linear system:

$$M\mathbf{x} = \mathbf{v},$$

comes down to asking if there is some $\mathbf{x}$ in the domain $\mathbb{R}^m$, whose image after applying the function M is $\mathbf{v}$. Let's reframe a few of the previous examples with this perspective in mind.

■ **Example 5.5** Adopt the notation of Example 5.1. Here $M : \mathbb{R}^7 \to \mathbb{R}^7$, and we are asking for a vector whose image is

$$\mathbf{v} = 3\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + 4\hat{\mathbf{e}}_3 + 1\hat{\mathbf{e}}4 + 5\hat{\mathbf{e}}_5 + 9\hat{\mathbf{e}}_6 + 2\hat{\mathbf{e}}_7.$$

Since M is invertible, we know that such a vector exists, and by applying the *undo function* M^{-1} to $\mathbf{v}$ we can even find it! In fact, this tells us that there *exactly one* vector $\mathbf{x}$ in the domain mapping to $\mathbf{v}$ under M (cf. Exercise 3.19). ■

■ **Example 5.6** Adopt the notation of Example 5.3. Here $M : \mathbb{R}^4 \to \mathbb{R}^2$, and asking for solutions $M\mathbf{x} = \mathbf{v}$ is asking for vectors $\mathbf{x}$ in the domain $\mathbb{R}^4$ whose image $M(\mathbf{x})$ in $\mathbb{R}^2$ is $\mathbf{v} = \hat{\mathbf{i}} + 5\hat{\mathbf{j}}$. Notice this perspective makes it seem like there could be no solutions (if $\mathbf{v}$ isn't in the image of the function M), or even many different solutions—if the function M sends different vectors to $\mathbf{v}$. Notice functions do this all the time, for example the *squaring* function sends 2 and -2 to the same place. ■

As the last example suggests, this function perspective tells us the following: a linear system could have no solutions, one solution, or many solutions. This suggests we should take a second to answer the following question:

5.1.2 What Does it Mean to Solve a Linear System?

Consider a linear system, written as a system of equations,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}
$$

A *solution* to this linear system is an assignment of the values $x_1, x_2, \cdots, x_m$ so that each equation in the system is true *simultaneously*. When we rewrite this system as a matrix equation (cf Definition 5.1.2)

$$M\mathbf{x} = \mathbf{v},$$

a *solution* is a specific $\mathbf{x}$ in $\mathbb{R}^m$ which correctly solves this equation.

■ **Example 5.7** A solution for the system from Example 5.1 has can be given by the assignments:

$$x_1 = -8, \quad x_2 = 4.9, \quad x_3 = 4.8, \quad x_4 = 2.6, \quad x_5 = -1.1, \quad x_6 = 1.7, \quad x_7 = 3.3.$$

When we write the system as a matrix equation $M\mathbf{x} = \mathbf{v}$, the same solution can be given by the 7-dimensional column vector

$$\mathbf{x} = \begin{bmatrix} -8 \\ 4.9 \\ 4.8 \\ 2.6 \\ -1.1 \\ 1.7 \\ 3.3 \end{bmatrix}.$$

■

This is what it means to give *a solution* to a linear system. To *solve* a linear system, we'd like to write down *all possible* solutions to a given linear system (if they exist). In Example 5.1, we know there is only one solution, so we have already given every possible solution. This linear system has been *solved*. But what if there are multiple solutions?

■ **Example 5.8** Consider again the linear system from Example 5.2.

$$\begin{aligned} 2x - \ \ y &= -1 \\ -4x + 2y &= \ \ 2 \end{aligned}$$

Let's solve this the old fashioned way, first solving for y in the first equation:

$$y = 2x + 1.$$

Now we can plug this in to the second equation, and then simplifying and expanding the righthand-side.

$$\begin{aligned} -4x + 2(2x+1) &= 2 \\ -4x + 4x + 2 &= 2 \\ 2 &= 2 \end{aligned}$$

It simplified to the equation $2 = 2$, *which is always true!* To interpret this, we can observe that whenever the first equation is true, the second equation is also automatically true. This gives us many solutions. For example, if:

$$x = 0,$$

then

$$y = 2x + 1 = 2 \cdot 0 + 1 = 1.$$

Plugging this into the each term of the system of equations gives:

$$2 \cdot 0 - 1 = -1,$$

$$-4 \cdot 0 + 2 \cdot 1 = 2.$$

Each one holds! So $(x, y) = (0, 1)$ is a solution to the system. To see one more we can let:

$$x = -3.$$

Then,

$$y = 2x + 1 = 2(-3) + 1 = -5.$$

And we can quickly check that:

$$2(-3) - (-5) = -1,$$

$$-4(-3) + 2(-5) = 2.$$

This gives us another solution to the system: say $(x, y) = (-3, -5)$. In fact, we see that given any value we'd like to choose for x, (say some constant t), if we set y equal to one more than twice that value (that is $2t + 1$) we will get a solution to the system of equations. Since we can choose any x value we want, we will call x a *free variable*, because it is free to be whatever we'd like it to be. Once this value is chosen, though, the value for y has been set in stone as one more than twice the value of x. Since y depends on x, we will call y a *dependant variable*. We can now write a *general* solution to the system of equations in terms of a *parameter* t:

$$\begin{aligned} x &= t \\ y &= 2t+1 \end{aligned}$$

In this way we have written down *all the solutions* to the system of equations, parametrized by our *parameter* t. That is, given any choice of t, if we plug this choice into the two equations above we get a solution to the linear system. For example, if $t = 0$ we get:

$$x = 0, \quad \textit{and} \quad y = -1,$$

and if $t = -3$ we plug in to get:

$$x = -3, \quad \textit{and} \quad y = -5.$$

So one way we can *solve* a system of equations, is to write down all the solutions in terms of some *free parameters*, so that any choice of these free parameters give a solution to the equation. We can give this solution as a vector too:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t+1 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This again expresses all the solutions to the system (now as a matrix equation $M\mathbf{x} = \mathbf{v}$) in terms of a parameter t.

We can also give a more geometric flavor to the set of solutions to the system of equations. In particular, we know that if x and y satisfy the equation

$$y = 2x+1,$$

then they will solve the system. Notice that this is the equation of a line in $\mathbb{R}^2$, of slope 2 and y-intercept 1. So the solution to this system of equations could be expressed geometrically as a line. ■

■ **Example 5.9** A slight modification to the above example lands us in a squarely different situation. Let's instead consider the system of equations:

$$\begin{aligned} 2x - y &= -1 \\ -4x + 2y &= -17 \end{aligned}$$

Again the first equation gives:

$$y = 2x+1$$

Plugging into the second equation we get:

$$\begin{aligned} -4x+2(2x+1) &= -17 \\ -4x+4x+2 &= -17 \\ 2 &= -17 \end{aligned}$$

But $2 \neq -17$, so whenver the first equation holds, the second equation fails to hold. This tells us that there are no solutions to the system of equations. ■

■ **Example 5.10** Let's consider another example.

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + 5w &= 3 \end{aligned}$$

This system of equations is just begging for us to solve for x and y in terms of z and w. IN fact, the following system of equations is identical, albeit rearranged.

$$x = 68z + 17w - 8$$

$$y = -19z - 5w + 3$$

Notice that once we pick any values of z and w, the values for x and y are set. For example, if $z = -2$ and $w = 7$ then we have:

$$x = 68(-2) + 17 \cdot 7 - 8 = -25,$$

$$y = -19(-2) - 5 \cdot 7 + 3 = 6.$$

This gives us a solution to our system of equations via the assignments:

$$x = -25, \quad y = 6, \quad z = -2, \quad w = 7.$$

Indeed:

$$-25 - 68(-2) - 17 \cdot 7 = -8,$$

$$6 + 19(-2) + 5 \cdot 7 = 3.$$

This lands us to a situation similar to the one we encountered in Example 5.8. Once we fix *any* values for z and w (say, setting them equal to constants s and t), then we can write down values for x and y in terms of these values using the equations above to get a solution to the system of equations. Here z and w will be the *fre variables*, since they are free to be any number we'd like, and x and y are the dependent variables, whose values depends one the choice for z and w. As above, we can write down a *general solution* to the system, now in terms of 2 *parameters*, s and t.

$$x = 68s + 17t - 8,$$

$$y = -19s - 5t + 3,$$

$$z = s,$$

$$w = t.$$

Again, we have written down *all the solutions* to the system of equations, parametrized by our two parameters s and t. To generate individual solutions from this data, we need only choose values for s and t, for example, letting $s = -2$ and $t = 7$ will generate the same solution we gave above.

As in Example 5.8, we can also give this solution as a vector:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 68s + 17t - 8 \\ -19s - 5t + 3 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 68s \\ -19s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 17t \\ -5t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 68 \\ -19 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 19 \\ -5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Again, we have expressed all the solutions to the system (now as a matrix equation $M\mathbf{x} = \mathbf{v}$) in terms of our two parameters s and t.

It is a little trickier to express this solution set geometrically, given that it is contained in $\mathbb{R}^4$, which is difficult to visualize. That being said, we can see that the solution set to this system of equations is determined by 2 parameters, and therefore gives us 2 degrees of freedom when choosing solutions. This suggests that it may be reasonable to call the general solution to this system of equations *2 dimensional*. ■

Exercise 5.1 — Checkin 7. In this checkin we will be studying the linear system $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

1. Write the linear system $M\mathbf{x} = \mathbf{v}$ as a system of linear equations.
2. Write a general solution to the system of linear equations in terms of parameters s and t and r (*Note: you may not need all 3 parameters)*.
3. Explicitly identify the free variables and dependent variables.
4. How many solutions are there to the linear system? Explain your reasoning.
5. By specifying parameters, a 2 explicit solutions to the linear system.

■

5.2 March 29, 2023

5.2.1 Augmented Matrices and Elementary Row Operations

In Example 5.10 it was very easy to solve for the dependant variables in terms of the free ones. Compare this to Example 5.3.

■ **Example 5.11**

$$x+3y-11z-2w=1,$$

$$2x+7y-3z+w=5.$$

If we try to follow the same outline here, we'd get:

$$x=-3y+11z+2w+1,$$

$$y=-\frac{2}{7}x+\frac{3}{7}z-\frac{1}{7}w+5.$$

It looks like perhaps we can set z and w freely, but x and y appear to be both dependent and free. Fixing y in the first equation would also set it in the second, where it should depend on the righthandside. We seem to have gotten nowhere. ■

The reason this worked better in Example 5.10 is because the dependent variables were already isolated, and only appeared in a single equation (better yet, with a coefficient of 1 so they were easy to solve for). The technique of *Gauss-Jordan Elimination* is a way to turn any linear system into one which is as easily solved as Example 5.10. This is generally done using matrix methods, but it adapted from techniques of solving a system of equations, so it may be best to consider the system of equations first.

■ **Example 5.12** We return to the following system.

$$x+3y-11z-2w=1.$$

$$2x+7y-3z+w=5$$

We'd like to try to isolate as many variables as possible, so that they only appear in a single equation. One thing we can always do is add multiples of one equation to another without changing the solution set. Let's begin but subtracting twice the first equation from the second.

$$\begin{array}{r} 2x+7y-3z+w=5 \\ \underline{-2\,(x+3y-11z-2w=1)} \\ 0x+1y+19z+5w=3 \end{array}$$

Therefore the following system of equations is equivalent to our last one.

$$\begin{aligned} x+3y-11z-2w&=1 \\ y+19z+5w&=3 \end{aligned}$$

Now x only appears in one equation. But every variable in the second equation appears in both equations, so we cannot solve for one without that variable looking free in the first equation, and dependant in the second. We can do a similar trick, subtracting twice 3 times the second equation from the first to get rid of y. Notice that because x is already eliminated in the second equation, it won't mess up our isolated x in the first. Doing this gives us an equivalent system:

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ \ 5w &= \ \ 3 \end{aligned}$$

This looks familiar! We can now solve it as we did in Example 5.10. ■

Let's retrace this example with the matrix perspective in mind.

■ **Example 5.13** The linear system from 5.12 can be written as $M\mathbf{x} = \mathbf{v}$ where:

$$M = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 2 & 7 & -3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

We first subtracted twice the first equation of the system from the second. On the level of matrices, this has the effect *subtracting twice the first row from the second row of both M and* **v**, resulting in:

$$M' = \begin{bmatrix} 1 & 3 & -11 & -2 \\ 0 & 1 & 19 & 5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Our second step was to then subtract 3 times the second equation from the first equation. On the level of matrices, thi has the effect of *subtracting twice the second row from the first row of both M and* **v**, resulting in:

$$M'' = \begin{bmatrix} 1 & 0 & -68 & -17 \\ 0 & 1 & 19 & 5 \end{bmatrix} \qquad \textit{and} \qquad \mathbf{v}' = \begin{bmatrix} -8 \\ 3 \end{bmatrix}.$$

These two moves give us an equivalent linear system $M''\mathbf{x} = \mathbf{v}''$ which corresponds to the easily solved linear system of Example 5.10.

Since we do the same things to both M and $\mathbf{v}$, it is often common to combine them into a single *augmented matrix*:

$$\left[M \mid \mathbf{v}\right].$$

The vertical bar tells us where our coeffficent matrix M ends, and our target vector $\mathbf{v}$ begins. Following the same 3 steps with the augmented matrix associated to our linear system looks as follows:

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 1 \\ 2 & 7 & -3 & 1 & 5 \end{array}\right]$$

Subtract 2*(Row 1) from (Row 2):

$$\left[\begin{array}{cccc|c} 1 & 3 & -11 & -2 & 1 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right]$$

Subtract 3*(Row 2) from (Row 1):

$$\left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & -8 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right] = \left[M'' \mid \mathbf{v}''\right]$$

We can now extract M'' and $\mathbf{v}''$ from either side of the vertical bar, and continue just as before. ■

The previous example introduced a new definition, called an augmented matrix. Let's record the general definition.

Definition 5.2.1 The *augmented matrix* associated to the linear system

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = v_1, \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = v_2, \\
&\vdots \qquad\qquad\qquad \vdots \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = v_n.
\end{aligned}$$

is the matrix

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm} & v_n.
\end{array}\right]$$

This can be expressed much more concisely when a linear system is already expressed as a matrix equation.

Definition 5.2.2 The *augmented matrix* associated to the linear system $M\mathbf{x} = \mathbf{v}$ is the matrix:

$$\left[M \mid \mathbf{v}\right].$$

In Example 5.12 we applied operations to the system of equations to make it easier to solve, and in Example 5.13 we traced what these operations did to the augmented matrix. In fact, there will be 3 different operations that we would like to apply to a linear system in order to solve it. Let's spell them out, and interpret them as operations on the associated augmented matrix.

1. Interchange any 2 equations.
2. Scale an equation by a *nonzero* constant.
3. Add a multiple of one equation to another equation.

It is the third operation that used twice in Example 5.12.

Interchange any 2 equations.

Swapping the i'th and j'th equations of a linear system has no effect on the solutions to the linear system:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m &= v_i, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m &= v_j, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m &= v_j, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m &= v_i, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}$$

In fact, it feels like this operation has on the linear system at all! On the augmented matrix, the effect is less trivial, as this corresponds to interchanging rows i and j.

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jm} & v_j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm} & v_n.
\end{array}\right]
\quad \Longrightarrow \quad
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{j1} & a_{j2} & \cdots & a_{jm} & v_j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm} & v_n.
\end{array}\right]$$

Scale an equation by a nonzero constant

A common trick in solving an equation or system of equations is to multiply both sides of an equation by a nonzero constant. In particular, if we call this constant $c \neq 0$ we can multiply both sides of equation i by c without changing the solution set. In particular, the following two systems will have the same set of solutions.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m &= v_i, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= v_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= v_2, \\
\vdots \qquad\qquad & \quad \vdots \\
ca_{i1}x_1 + ca_{i2}x_2 + \cdots + ca_{im}x_m &= cv_i, \\
\vdots \qquad\qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= v_n.
\end{aligned}$$

On the augmented matrix, this would correspond to scaling row i by the same constant $c \neq 0$.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right] \implies \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ca_{i1} & ca_{i2} & \cdots & ca_{im} & cv_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right]$$

Adding a multiple of one row to another row

The trick we used twice in Examples 5.12 and 5.13 involved adding a multiple of one equation to another equation. Say we scale equation i by a constant c, and then add it to equation j. Doing this, and combining all like terms, has the following effect on the system of equations, without changing the solution set.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = v_1, & & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = v_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = v_2, & & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = v_2, \\ \vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = v_i, & & a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = v_i, \\ \vdots \qquad\qquad \vdots & \implies & \vdots \qquad\qquad \vdots \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jm}x_m = v_j, & & (a_{j1} + ca_{i1})x_1 + (a_{j2} + ca_{i2})x_2 + \cdots + (a_{jm} + ca_{im})x_m = v_j + cv_i, \\ \vdots \qquad\qquad \vdots & & \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = v_n. & & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = v_n. \end{array}$$

On the level of augmented matrices, this has the effect of adding a multiple of row i (scaled by the same constant c) to row j.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jm} & v_j \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right] \implies \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{im} & v_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j1} + ca_{i1} & a_{j2} + ca_{i2} & \cdots & a_{jm} + ca_{im} & v_j + cv_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & v_n. \end{array}\right]$$

Note: unlike when just scaling a row, one doesn't have to assume that $c \neq 0$. Of course, if $c = 0$, then the overall effect will be the same as doing absolutely nothing.

These three operations on matrices are inspired by the associated oprations on a system of equations, and become very useful for algorithmically solving any system of equations. They also turn out to have interesting interpretations for matrices arising in all sorts of contexts, so let's give them a name.

Definition 5.2.3 — Elementary Row Operations. Let M be any matrix. The three *elementary row operations* one can apply to M are:

1. Interchange any 2 rows of M.
2. Scale an entire row of M by a nonzero constant.
3. Add a scalar multiple of one row of M to another row of M.

■ **Example 5.14** Consider the matrix:

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

We can get a related matrix by applying the first elementary row operation, for example, interchanging the first and third rows.

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad M_1 = \begin{bmatrix} 6 & 2 & -1 \\ 0 & 1 & 0 \\ 2 & 3 & 5 \\ 3 & 0 & 0 \end{bmatrix}$$

We can get another related matrix by applying the second elementary row operations, say, scaling the third row by -3

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad M_2 = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ -3\cdot 6 & -3\cdot 2 & -3\cdot -1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ -18 & -6 & 3 \\ 3 & 0 & 0 \end{bmatrix}$$

As an example of the third elementary row operation, we can 3 times row 1 to row 4.

$$M = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad M_3 = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3+2\cdot 2 & 0+2\cdot 3 & 0+2\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 7 & 6 & 10 \end{bmatrix}$$

■

In the previous example, we called two matrices *related* when we could get from one to the next via an elementary row operation, and indeed, if we can get from one matrix to another by a row operation (or a sequence of row operations), they will share many important properties. Let's give this relationship a name.

Definition 5.2.4 2 matrices M and N are called *row equivalent* if N can be obtained from N by a sequence of row operations.

■ **Example 5.15** The matrices M_1, M_2, M_3 in Example 5.14 are all row equivalent to M (and to eachother!). In fact, so is the matrix,

$$N = \begin{bmatrix} -4 & -6 & -10 \\ 6 & 2 & -1 \\ 0 & 1 & 0 \\ 3 & -5 & 0 \end{bmatrix}$$

as it can be obtained by N by 3 elementary row operations: first scaling Row 1 by -2, then adding -5*(Row 2) to Row 3, and finally, interchanging Rows 2 and 3.

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 0 & 1 & 0 \\ 6 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -4 & -6 & -10 \\ 6 & 2 & -1 \\ 0 & 1 & 0 \\ 3 & -5 & 0 \end{bmatrix}$$

■

If we apply an elementary row operation to an augmented matrix, we don't change the solutions to the associated linear system. After all, the row operations were based on 3 moves which didn't change the solutions of a linear system. Therefore we have the following fact.

Theorem 5.2.1 Let $M\mathbf{x} = \mathbf{v}$ and $N\mathbf{x} = \mathbf{w}$ be two linear systems. If the augmented matrices

$$\left[M \mid \mathbf{v}\right] \qquad \textit{and} \qquad \left[N \mid \mathbf{w}\right]$$

are row equivalent, then the two linear systems have the same solutions.

Theorem 5.2.1 is more or less what we used in Example 5.13 to turn the *difficult to solve* system of equations

$$x + 3y - 11z - 2w = 1.$$

$$2x + 7y - 3z + w = 5$$

into the *easy to solve* system of equations

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ 5w &= \ 3. \end{aligned}$$

In fact, Theorem 5.2.1 will be central our general strategy toward solving system of equations. Let's briefly outline it:

1. Start with linear system $M\mathbf{x} = \mathbf{v}$
2. Construct the augmented matrix $\left[M \mid \mathbf{v}\right]$
3. Via a sequence of elementary row operations, obtain row equivalent matrix $\left[N \mid \mathbf{w}\right]$ whose linear system is easier to solve
4. Solve $N\mathbf{x} = \mathbf{w}$ as in Example 5.10 potentially using parametric equations

At this point, Step 3 is the most mysterous, and for this strategy to be viable we have to answer the following 2 questions.

■ **Question 5.1** How do we know when $\left[N \mid \mathbf{w}\right]$ is easy to solve?

■ **Question 5.2** What sequence of elementary row operations should we follow?

5.2.2 Reduced Row Echelon Form

In Example 5.10, what made the system easy to solve? Let's look at it:

$$\begin{aligned} x \quad - 68z - 17w &= -8 \\ y + 19z + \ 5w &= \ 3. \end{aligned}$$

What made this so easy to solve, was the the first variable that appears in each equation doesn't appear in any other equation. Furthermore, it's coefficient is 1. Let's call this first variable that appears the *leading variable*. This made it very easy to solve for the leading variables: we could keep leave x in the first equation alone, and push everything else to the other side, thereby solving for the dependant variable x in one step. Since x doesn't appear in any other equation, this is the only dependancy it has, so we have completely solved for x. We could similarly solve for y in the second equation. In particular, we exploited that the system of equation had the following properties:

1. The leading variable in each equation has a coefficient of 1.
2. The leading variable of each equation, appears in that equation alone.

Let's translate this to the associated augmented matrix.

$$\begin{aligned} 1x + 0y - 68z - 17w &= -8 \\ 0x + 1y + 19z + \ \ 5w &= \ \ 3. \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -68 & -17 & -8 \\ 0 & 1 & 19 & 5 & 3 \end{array}\right]$$

Notice how the first entry of each row is 1 (highlighted in blue). Let's call these *leading 1*'s. The leading 1 in the first row corresponds to the coefficient of the leading variable x in the first equation, and the leading 1 in the second row appears in the second column, corresponding to the coefficient of the leading variable y in the second equation. To summarize, the first property of the system of equations we exploited is equivalent to the following condition on the augmented matrix:

> The first nonzero value of each row is a 1.

Notice that above and below each leading 1, we only have zeros appearing. The red 0 appearing in the first entry of the second row corresponds to a coefficient of 0 next to the variable x in the second equation, and similar, the green 0 in the second entry of the first row corresponds to a coefficient of 0 for y in the second equation. In particular, the zeroes above and below the leading ones record exactly that the leading variables don't appear in any other equation. To summarize, the second property of the system of equations we exploited is equivalent to the following condition on the augmented matrix:

> A leading 1 is the only nonzero entry in its column.

In particular, an augmented matrix satisfying the two properties we displayed above will correspond to a linear system that is easy to solve. Let's give a name a matrix satisfying these properties (together which two more we haven't seen yet).

Definition 5.2.5 A matrix M is said to be in *reduced row echelon form* if

1. The first nonzero value of each row is 1. Call this value a *leading 1*.
2. Each leading 1 is the only nonzero entry in its column.
3. The leading 1 of each successive row appears further to the right.
4. All rows consisting only of zeros appear at the bottom of the matrix.

R The first 2 conditions are the ones that make solving equations easy. The second two can always be acheived from a matrix satisfying the first 2 by simply swapping rows.

We have now answere question 5.1. It is easy to solve $N\mathbf{x} = \mathbf{w}$ when $\left[N \mid \mathbf{w}\right]$ is in reduced row echelon from.

■ **Example 5.16** In Checkin 7 (cf. Exercise 5.1) we studied the linear system $M\mathbf{x} = \mathbf{v}$ where

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

The augmented matrix associated to this system is:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \end{array}\right]$$

This matrix *is* in reduced row echelon form. Indeed, the first nonzero entry of each row is a 1 (highlighted in red), and the entries sharing a column with a leading 1 are all 0 (highlighted in blue). The third condition holds as the first leading 1 is in the column 1, the second is in column 3, and the third is in column 4, so they are moving to the right as we descend rows. ■

■ **Example 5.17** The following matrix are not in reduced row echelon form.

$$M_1 = \begin{bmatrix} 0 & 3 & 9 \\ 0 & 1 & 4 \end{bmatrix}$$

This is because the first nonzero entry in row 1 is not a 1. The next matrix is still not in reduced row echelon form.

$$M_2 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4. \end{bmatrix}$$

This is because the leading 1 in the first row is not the only nonzero entry in its column. The next matrix is still not in reduced row echelon form.

$$M_3 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

This is because the leading 1 in the second row is not the only nonzero entry in its column. What about the next matrix?

$$M_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

Notice each successive matrix in Example 5.17 was obtained by an elemenatry matrix operation.

- M_2 is obtained from M_1 by multiplying the first row by $\frac{1}{3}$.
- M_3 is obtained from M_2 by subracting the first row from the second.
- M_4 is obtainde3 from M_3 by subtracting 3 times the second row from the first.

Indeed, we were able to start with a *nonreduced* matrix, and via a sequence of elementary row operations, put in in reduced row echelon form. This suggests that there may be a good answer to Question 5.2

5.2.3 Gauss-Jordan Elimination

Our goal is to solve a linear system. To do this, we look at the augmented matrix. If an augmented matrix is in reduced row echelon form, we know the linear system will be easy to solve. Theorem 5.2.1 tells us also that we can apply elementary matrix operations without changing the solutions to the linear system. So to fill in the last gap, we need a way to apply a sequence of elementary row operations to obtain a row equivalent matrix in reduced row echelon form. The Gauss-Jordan elimination theorem does this for us.

Theorem 5.2.2 — Gauss-Jordan Elimination Theorem. Let M be any matrix. There exists a row equivalent matrix M^{red} which is in reduced row echelon form. Furthermore, this matrix can be obtained algorithmically.

We will discuss the algorithm in a little more detail next time. In practice, we generally let a computer implement the algorithm for us, rather than reducing matrices by hand. Let's see how powerful this theorem is with an example.

■ **Example 5.18** Consider the following system of equations.

$$\begin{aligned}
2x_1 - 4x_2 \qquad + 6x_4 \qquad &= 2 \\
3x_2 - x_3 \qquad + 3x_5 &= 1 \\
4x_1 \qquad + 2x_3 \qquad - x_5 &= 1 \\
-2x_1 - x_2 + x_3 + x_4 + x_5 &= 0
\end{aligned}$$

The associated augmented matrix is:

$$\left[\begin{array}{ccccc|c}
2 & -4 & 0 & 6 & 0 & 2 \\
0 & 3 & -1 & 0 & 3 & 1 \\
4 & 0 & 2 & 0 & -1 & 1 \\
-2 & -1 & 1 & 1 & 1 & 0
\end{array}\right]$$

Using a calculator or computer, we can put this matrix into reduced row echelon form.

$$\left[\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & -\frac{16}{37} & \frac{6}{37} \\
0 & 1 & 0 & 0 & \frac{83}{74} & \frac{29}{74} \\
0 & 0 & 1 & 0 & \frac{27}{74} & \frac{13}{74} \\
0 & 0 & 0 & 1 & \frac{33}{37} & \frac{20}{37}
\end{array}\right]$$

Now we can turn this back into a system of equations (*unaugmenting*, so to speak).

$$\begin{array}{ccccccc}
x_1 & & & & - & \frac{16}{37}x_5 & = & \frac{6}{37} \\
& x_2 & & & + & \frac{83}{74}x_5 & = & \frac{29}{74} \\
& & x_3 & & + & \frac{27}{74}x_5 & = & \frac{13}{74} \\
& & & x_4 & + & \frac{33}{37}x_5 & = & \frac{20}{37}
\end{array}$$

Now it is clear that the leading (or dependant) variables are x_1, x_2, x_3, x_4, and the free variable will be x_5. So our general solution becomes:

$$x_1 = \frac{16}{37}t + \frac{6}{37},$$

$$x_2 = -\frac{83}{74}t + \frac{29}{74},$$

$$
\begin{aligned}
x_3 &= -\frac{27}{74}t + \frac{13}{74}, \\
x_4 &= -\frac{33}{37}t + \frac{20}{37}, \\
x_5 &= t.
\end{aligned}
$$

■