

## Take Home Assignment 2

Due Monday, March 23

With everything going on right now the Monday deadline is flexible. That being said, if you are going to need extra time *please let me know*, I will be granting extensions no questions asked but I need to know when to expect your assignment so that nothing falls between the cracks. Good luck and stay safe.

In this set of problems we will study the quaternion group  $Q_8$ . It is a nonabelian group with very interesting properties.

**Definition 1.** *The quaternion group of order 8, denoted  $Q_8$  is the group of the following 8 elements:*

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

*subject to the relations:*

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \quad ji = -k,$$

$$jk = i, \quad kj = -i,$$

$$ki = j, \quad ik = -j.$$

1. Let's start with a few simple facts. Much of this is worked out in the book.
  - (a) Write the entire multiplication table for  $Q_8$ .
  - (b) Find a presentation for  $Q_8$  with 2 generators and 3 relations.
  - (c) Prove that  $Q_8$  is not isomorphic to  $D_8$ .
  - (d) Find all the subgroups of  $Q_8$ , and draw its lattice. (Hint: there are 6 total subgroups).
  - (e) Prove that every subgroup of  $Q_8$  is normal.
  - (f) Prove that every subgroup and quotient group of  $Q_8$  is abelian (Hint: use the classification of groups of order 4 and 2, as well as Lagrange's theorem).
  - (g) Compute  $Z(Q_8)$  and  $Q_8/Z(Q_8)$  (Hint for the second part: you can do this by hand, but it might be slicker to apply Homework 6 problem 5(b)).
  - (h) Write a composition series for  $Q_8$ .
2. Now let's follow the proof of Cayley's theorem to exhibit  $Q_8$  as a subgroup of  $S_8$ .
  - (a) Label  $\{1, -1, i, -i, j, -j, k, -k\}$  as the numbers  $\{1, 2, \dots, 8\}$ . Then the action of  $Q_8$  on itself by left multiplication gives an injective map  $Q_8 \rightarrow S_8$ . Write the permutation representations for  $-1$  and  $i$  as elements  $\sigma_{-1}, \sigma_i \in S_8$ , and verify that  $\sigma_i^2 = \sigma_{-1}$ . (Using the multiplication table from question 1 will make this easier).
  - (b) Use the generators from question 1(b) to give two elements of  $S_8$  which generate a subgroup  $H \leq S_8$  isomorphic to  $Q_8$ .
  - (c) Is  $\sigma_i$  even or odd?
  - (d)  $A_8 \cap H$  is isomorphic to a subgroup of  $Q_8$ . Which one?
3. Cayley's theorem says that if  $|G| = n$  then  $G$  embeds in  $S_n$ . But might not be the smallest symmetric group that  $G$  embeds in. For example,  $D_8$  embeds in  $S_4$  (thinking about symmetries of the square as permutations of the vertices). Nevertheless, for  $Q_8$  the symmetric group given by Cayley's theorem is the smallest.

- (a) Let  $Q_8$  act on a set  $A$  with  $|A| \leq 7$ . Let  $a \in A$ . Show that the stabilizer of  $a$ ,  $(Q_8)_a \leq Q_8$  must contain the subgroup  $\{\pm 1\}$ .
  - (b) Deduce that the kernel of the action of  $Q_8$  on  $A$  contains  $\{\pm 1\}$ .
  - (c) Conclude that  $Q_8$  cannot embed into  $S_n$  for  $n \leq 7$ .
4. Finally let's say a few things about the automorphism group of  $Q_8$ .
- (a) By counting possible places where the generators may go, show that  $|\text{Aut}(Q_8)| \leq 24$ .
  - (b) What is  $\text{Inn}(Q_8)$ ? (Hint: You already did this in question 1(g)!)
  - (c) Use parts (a) and (b) to conclude that  $|\text{Aut}(Q_8)|$  must be one of  $\{4, 8, 12, 16, 20, 24\}$ . (Note: it will turn out that it is 24, but the proof of this fact is more involved).