Homework Assignment 13

Due Saturday, May 7

1. In class we proved a cancellation law for integral domains. We can actually say something a bit stronger (and quite useful). Let R be a ring and $a, b, c \in R$. Suppose that a is not zero or a zero divisor, and that $ab = ac$. Prove $b = c$.

2. Let R and S be rings and $\varphi : R \to S$ a ring homomorphism.
   (a) Show that $\operatorname{im} \varphi$ is a subring of S.
   (b) Show that $\ker \varphi$ is a (two-sided) ideal of R.
   (c) Suppose $J \subseteq S$ is an ideal. Show that $\varphi^{-1}(J)$ is an ideal of R.
   (d) Suppose R and S are unital rings with *nonzero* identities 1_R and 1_S respectively. Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is either zero, or a zero divisor in S.
   (e) Deduce that if S is an integral domain and φ is nonzero then $\varphi(1_R) = 1_S$. (*Remark:* many authors require rings to be unital, and also require ring homomorphisms to take the identity to the identity.)

3. In this exercise we prove the third and fourth isomorphism theorems for rings.
   (a) We start with the fourth isomorphism theorem. Let R be a ring and $I \subseteq R$ an ideal. In particular (since R is abelian), I is a normal subgroup. Therefore, applying the fourth isomorphism theorem for groups (HW7 Problem 3), there is a bijection:

$$\left\{ \begin{array}{c} \text{Subgroups } A \leq R \\ \text{such that } I \leq A \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Subgroups} \\ \overline{A} \leq R/I \end{array} \right\}$$

   Prove the following ring theoretic enhancements hold:
      i. A is a subring of R if and only if $\overline{A}$ is a subring of R/I.
      ii. If A is a subring of R, then I is an ideal of A and that $A/I \cong \overline{A}$.
      iii. A is a left ideal of R if and only if $\overline{A}$ is a left ideal of R/I.
      iv. A is a right ideal of R if and only if $\overline{A}$ is a right ideal of R/I.
      v. A is an ideal of R if and only if $\overline{A}$ is an ideal of R/I.

   (b) We now prove the third isomorphism theorem for rings. Let $J \subseteq I \subseteq R$, with J, I ideals of a ring R. By part (a) we know that I/J is an ideal of R/J. Prove that:

$$\frac{R/J}{I/J} \cong \frac{R}{I}.$$

   (c) We finish with a ring theoretic analog of *passing to the quotient*. Suppose $\varphi : R \to S$ is a ring map, and suppose that $I \subseteq \ker \varphi$. Prove that there is a unique map $\overline{\varphi} : R/I \to S$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\pi"] & S \\ R/I \arrow[ur, dotted, "\overline{\varphi}"'] & \end{tikzcd}$$

   That is, $\overline{\varphi}$ is the unique map so that $\overline{\varphi} \circ \pi = \varphi$. (*Hint*: We already know from group theory that there is a unique such map on the level of group homomorphisms. What remains is to confirm that map is a ring homomorphism.)

4. Let R be a ring.

   (a) Suppose $\{I_j\}$ is a collection of left ideals of R. Show that the intersection $\cap I_j$ is a left ideal of R.

   (b) Show that part (a) also holds for right ideals and two-sided ideals.

   (c) Let R be a ring with $1 \neq 0$. Recall that:

$$RAR := \{r_1 a_1 s_1 + \cdots + r_n a_n s_n | r_i, s_i \in R \text{ and } a_i \in A.\}$$

$$(A) := \bigcap_{A \subset I \text{ an ideal}} I.$$

   Prove that RAR is an ideal of R, and that $RAR = (A)$.

   (d) State the analog for part (c) for left and right ideals. (The proof will be identical, so I won't make you repeat yourself.)

5. Let I and J be ideals of a ring R.

   (a) Prove that $I + J$ is the smallest ideal of R containing both I and J.

   (b) Recall that:

$$IJ = \{i_1 j_1 + \cdots + i_n j_n | i_k \in I \text{ and } j_k \in J.\}$$

   Show that IJ is an ideal contained in $I \cap J$

   (c) Give an example where $IJ \neq I \cap J$

   (d) Suppose R is commutative and unital, and that $I + J = R$. Show $IJ = I \cap J$.

6. Let R be a commutative ring with $1 \neq 0$.

   (a) Fix $a \in R$. Show that $(a) = R$ if and only if $a \in R^\times$.

   (b) Fix $a, b \in R$, and suppose that a is not a zero divisor. Show that $(a) = (b)$ if and only if $a = ub$ for some unit $u \in R^\times$.

   (c) Let I be any ideal. Show that $I = R$ if and only if I contains a unit $u \in R^\times$.

   (d) Prove that R is a field if and only if the only ideals in R are (0) and R itself.

   (e) Now suppose S is a (not necessarily commutative) ring with $1 \neq 0$. Show that S is a division ring if and only if the only all left, right, and 2-sided ideals are one of S or (0). (*Hint*: Start by proving a version of part (c) for noncommutative rings.)

7. Let R be a ring. The *nilradical* of R is $\mathfrak{N}(R) = \{r \in R : r \text{ is nilpotent}\}$. By HW12 Problem 3 we know that $\mathfrak{N}(R)$ is an ideal of R.

   (a) Show that $R/\mathfrak{N}(R)$ is reduced. This is often called the *reduction of R,* and is denoted R_{red}.

   (b) Compute $\mathfrak{N}(R)$ and R_{red} for the following two rings.

      i. $R = \mathbb{Z}[x]/(x)^n$ for $n \geq 2$.

      ii. $R = \mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 2$.

   (c) Let $\varphi : R \to S$ be any ring homomorphism. Show that $\varphi(\mathfrak{N}(R)) \subseteq \mathfrak{N}(S)$. Deduce that if S is reduced then $\mathfrak{N}(R)$ is contained in the kernel of φ.

(d) Let S be a reduced ring. Show that there is a bijection:

$$\{\text{Ring homomorphisms } \varphi : R \to S\} \Longleftrightarrow \{\text{ Ring homomorphisms } \tilde{\varphi} : R_{red} \to S\}.$$

Hint: Use passing to the quotient! *Remark:* This should feel reminicient of the *abelianization* from HW7 Problem 5. In fact, both are examples of something more general, called a *universal property.* Keep your eyes open for things like this, they appear all over mathematics!