

## Homework Assignment 10

Due Friday, April 16

1. Let  $R$  be a ring. Recall that for  $a \in R$  we denote the *additive* identity of  $a$  by  $-a$ . Establish the following identities.
  - (a)  $(-a)b = a(-b) = -ab$
  - (b)  $(-a)(-b) = ab$
  - (c) If  $1 \in R$  then  $(-1)a = -a$ .
  - (d) Suppose  $R$  is an integral domain. Show that if  $a^2 = 1$  then  $a = \pm 1$ .
2. Let  $R$  be a ring with  $1 \neq 0$ .
  - (a) Let  $R^\times \subseteq R$  be the set of units of  $R$ . Show that  $R^\times$  is a group under the multiplication operation of  $R$ .
  - (b) Suppose that  $a \in R$  is a zero divisor. Show that  $a \notin R^\times$ .
  - (c) Suppose  $R$  is a subring of some ring  $S$ . Show that if  $a \in R^\times$  then  $a \in S^\times$ . Give an example to show the converse is false.
3. Let  $R$  be a commutative ring. An element  $r \in R$  is called *nilpotent* if there exists a positive  $n$  such that  $r^n = 0$ . A commutative ring is called *reduced* if it has no nonzero nilpotent elements.
  - (a) Show that a nilpotent element of a ring is either 0 or a zero divisor.
  - (b) Give an example of a ring with a nonzero nilpotent element.
  - (c) Show that the sum of nilpotent elements is nilpotent.
  - (d) Suppose  $r$  is nilpotent. Show that  $rx$  is nilpotent for all  $x \in R$ . (*Note*, in future terminology, (c) and (d) prove that the set of nilpotent elements is an *ideal* of  $R$ , which we will call the *nilradical*).
  - (e) Suppose  $R$  is a commutative ring with  $1 \neq 0$ , and suppose  $r \in R$  is nilpotent. Show that  $1 + r \in R^\times$ .
4. (a) Let  $\{S_i \subseteq R\}$  be a nonempty collection of subrings of  $R$ . Show that  $\bigcap_i S_i$  is a subring of  $R$ .
  - (b) Suppose  $S$  is a subring of  $R$ , and  $R$  is a subring of  $T$ . Show that  $S$  is a subring of  $T$ .
5. For a ring  $R$ , define the *center* of  $R$  to be:

$$Z(R) = \{r \in R \mid ra = ar \text{ for all } a \in R\}.$$

- (a) Show that  $Z(R)$  is a subring of  $R$ .
- (b) Suppose  $R$  has  $1 \neq 0$ . Show that  $R^\times \cap Z(R) \subseteq Z(R^\times)$ . (The converse is *not true* in general, but I don't consider this to be obvious. Perhaps we will see an example later).
- (c) Show that the center of a division ring is a field.
- (d) Let  $\mathbb{H}$  be Hamilton's quaternions (defined in Lecture 21 or [DF] Example 5 on Page 224). Compute  $Z(\mathbb{H})$ . (Notice that  $\mathbb{H}$  contains a copy of  $\mathbb{C}$ , is this the center?)

6. Let  $R$  be ring, and  $X$  any set. Define

$$\text{Maps}(X, R) = \{f : X \rightarrow R \mid f \text{ is a function}\}.$$

Define binary operations  $+$  and  $\times$  as follows.

$$(f + g)(x) = f(x) + g(x) \quad (f \times g)(x) = f(x)g(x).$$

- Show that  $\text{Maps}(X, R)$  is a ring.
  - Suppose  $R$  is commutative, show that  $\text{Maps}(X, R)$  is too.
  - Suppose  $R$  is unital, show that  $\text{Maps}(X, R)$  is too.
  - Suppose  $R$  is reduced (defined in Problem 3), show that  $\text{Maps}(X, R)$  is too.
  - Give an example to show that even if  $R$  is a field,  $\text{Maps}(X, R)$  need not be.
  - Give an example to show that even if  $R$  is an integral domain,  $\text{Maps}(X, R)$  need not be.
7. We now develop an example of rings that appear along the intersection of the algebraic and analytic theory (for example in *functional analysis*). You may use without proof the following facts from elementary calculus: **(1)** If  $f, g$  are continuous so are their sum and product. **(2)** If  $f, g$  are differentiable then they are continuous and:

$$(f + g)' = f' + g' \quad (fg)' = f'g + fg'$$

- Let  $\mathcal{P}$  be a property of maps from  $X \rightarrow R$ , and let

$$\text{Maps}_{\mathcal{P}}(X, R) = \{f : X \rightarrow R \mid f \text{ has property } \mathcal{P}\}.$$

Suppose that the 0 map has property  $\mathcal{P}$ . Suppose also that if  $f$  and  $g$  have property  $\mathcal{P}$ , then so do  $f - g$  and  $f \times g$ . Show that  $\text{Maps}_{\mathcal{P}}(X, R)$  is a subring of  $\text{Maps}(X, R)$ .

- Let  $X = R = \mathbb{R}$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  have property  $\mathcal{C}^0$  if  $f$  is continuous, and define  $C^0(\mathbb{R}) = \text{Maps}_{\mathcal{C}^0}(\mathbb{R}, \mathbb{R})$  to be the set of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Use part (a) to show that  $C^0(\mathbb{R})$  is a subring of  $\text{Maps}(\mathbb{R}, \mathbb{R})$ .
- For each  $n > 0$  let  $f : \mathbb{R} \rightarrow \mathbb{R}$  have property  $\mathcal{C}^n$  if  $f$  has a derivative everywhere, and  $df/dx$  has property  $\mathcal{C}^{n-1}$ . (So for example,  $f$  is  $\mathcal{C}^1$  if it is differentiable and its derivative is continuous). Show by induction on  $n$  that  $C^n(\mathbb{R}) = \text{Maps}_{\mathcal{C}^n}(\mathbb{R}, \mathbb{R})$  is a subring of  $C^{n-1}(\mathbb{R})$ .
- A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has property  $\mathcal{C}^\infty$  if for each positive  $n$  the  $n$ 'th derivative of  $f$  exists and is continuous. (Such a function is also often called *smooth*). Show that  $C^\infty(\mathbb{R}) = \text{Maps}_{\mathcal{C}^\infty}(\mathbb{R}, \mathbb{R})$  is a subring of  $C^n(\mathbb{R})$  for each  $n$ . (Hint: rather than prove this directly, you could use (4)).

8. Let  $A$  be an abelian group (written additively). Define the *endomorphism ring* of  $A$  as follows:

$$\text{End}(A) = \{f : A \rightarrow A \mid f \text{ is a homomorphism}\}.$$

Give  $\text{End}(A)$  2 binary operations  $+$  and  $\times$  as follows:

$$(f + g)(a) = f(a) + g(a) \quad (f \times g)(a) = f(g(a)).$$

- Prove that  $\text{End}(A)$  is a ring.
- Prove that  $(\text{End}(A))^\times \cong \text{Aut}(A)$ .
- Let  $E$  be an elementary abelian  $p$ -group of order  $p^n$ . Show that  $\text{End}(E) \cong M_n(\mathbb{F}_p)$  (You may use that  $n \times n$  matrices over a field  $F$  correspond to linear maps  $F^n \rightarrow F^n$ . Compare to HW7 Problem 5).