Homework Assignment 13

Due Saturday, May 7

1. In class we proved a cancellation law for integral domains. We can actually say something a bit stronger (and quite useful). Let R be a ring and $a, b, c \in R$. Suppose that a is not zero or a zero divisor, and that $ab = ac$. Prove $b = c$.

2. Let R and S be rings and $\varphi : R \to S$ a ring homomorphism.

   (a) Show that $\operatorname{im}\varphi$ is a subring of S.

   (b) Show that $\ker\varphi$ is a (two-sided) ideal of R.

   (c) Suppose $J \subseteq S$ is an ideal. Show that $\varphi^{-1}(J)$ is an ideal of R.

   (d) Suppose R and S are unital rings with *nonzero* identities 1_R and 1_S respectively. Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is either zero, or a zero divisor in S.

   (e) Deduce that if S is an integral domain and φ is nonzero then $\varphi(1_R) = 1_S$. (*Remark:* many authors require rings to be unital, and also require ring homomorphisms to take the identity to the identity.)

6. Let R be a commutative ring with $1 \neq 0$.

   (a) Fix $a \in R$. Show that $(a) = R$ if and only if $a \in R^\times$.

   (b) Fix $a, b \in R$, and suppose that a is not a zero divisor. Show that $(a) = (b)$ if and only if $a = ub$ for some unit $u \in R^\times$.

   (c) Let I be any ideal. Show that $I = R$ if and only if I contains a unit $u \in R^\times$.

   (d) Prove that R is a field if and only if the only ideals in R are (0) and R itself.

7. Let R be a commutative ring. The *nilradical* of R is $\mathfrak{N}(R) = \{r \in R : r \text{ is nilpotent}\}$. By HW12 Problem 3 we know that $\mathfrak{N}(R)$ is an ideal of R.

   (a) Show that $R/\mathfrak{N}(R)$ is reduced. This is often called the *reduction of R,* and is denoted R_{red}.

   (b) Compute $\mathfrak{N}(R)$ and R_{red} for the following two rings.

      i. $R = \mathbb{Z}[x]/(x)^n$ for $n \geq 2$.

      ii. $R = \mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 2$.