Homework Assignment 9

Due Friday, April 9

This assignment will fill in many details from lecture, and do a few hands on classifications. To begin we will confirm that the semidirect product is indeed a group. First recall the definition.

Definition 1. *Let H, K be groups, and $\varphi : K \to \mathrm{Aut}(H)$ a group homomorphism. Denote the induced action of K on H by:*

$$k \cdot h = \varphi(k)(h).$$

The semidirect product of H and K with respect to φ is the set $H \rtimes K = \{(h, k) : h \in H, k \in K\}$, where multiplication is defined by the rule:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1 k_2).$$

1. Let's make sure that $H \rtimes K$ is a group.
   (a) Show that $(1, 1) \in H \rtimes K$ is the identity. (Remember you have to check both sides).
   (b) Show that $(h, k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$. (As above, you have to check both sides).
   (c) Prove that multiplication is associative.

Studying semidirect products reduces to the study of automorphism groups, so it is useful to be able to to decompose them.

Lemma 2. *Let H and K be finite groups whose orders are coprime. Then*

$$\mathrm{Aut}(H \times K) \cong \mathrm{Aut}\, H \times \mathrm{Aut}\, K.$$

The following definition will be useful.

Definition 3. *Let $\varphi : G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H : H \to G'$ given by evaluating φ on elements of H.*

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

2. Let's prove Lemma 2.
   (a) Let G be a group and let $H \operatorname{char} G$ be a *characteristic subgroup* (recall the definition from HW8 Definition 1). Fix any automorphism $\varphi \in \mathrm{Aut}\, G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).
   (b) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\mathrm{Aut}\, G \to \mathrm{Aut}\, H$.
   (c) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.
   (d) With H, K as in (c), construct an isomorphism $\mathrm{Aut}(H \times K) \to \mathrm{Aut}\, H \times \mathrm{Aut}\, K$.

Recall that any homomorphism $\varphi : K \to \mathrm{Aut}\, H$ allows us to build a semidirect product $H \rtimes_\varphi K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used the following lemma.

Lemma 4. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \operatorname{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:*

$$\begin{tikzcd} K \arrow[dd, "\gamma"'] \arrow[dr, "\varphi"] & \\ & \operatorname{Aut} H \\ K \arrow[ur, "\psi"'] & \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 5. *Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitely, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_\alpha : \beta \mapsto \alpha\beta\alpha^{-1}$, and suppose that $\psi = \sigma_\alpha \circ \varphi$:*

$$\begin{tikzcd} & \operatorname{Aut} H \arrow[dd, "\sigma_\alpha"] \\ K \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & \operatorname{Aut} H \end{tikzcd}$$

Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$.

3. Lemmas 4 and 5 say that if we alter φ by an automorphism of K, or an inner automorphism of $\operatorname{Aut} H$, (or both), we don't change the semidirect products. Let's prove this.

   (a) Consider the setup of Lemma 4. Show that the map:

   $$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (h, \gamma(k)) \end{array}$$

   is an isomorphism, thereby proving the lemma.

   (b) Consider the setup of Lemma 5. Show that the map:

   $$\begin{array}{ccc} H \rtimes_\varphi K & \longrightarrow & H \rtimes_\psi K \\ (h,k) & \mapsto & (\alpha(h), k) \end{array}$$

   is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_α is an automorphism of $\operatorname{Aut} H$, given by conjugation by α. In unweildy notation, this says $\sigma_\alpha \in \operatorname{Aut}(\operatorname{Aut} H)$.)

   (c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

   $$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & \operatorname{Aut} H \arrow[d, "\sigma"] \\ K \arrow[r, "\psi"] & \operatorname{Aut} H. \end{tikzcd}$$

That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_\varphi K \cong H \rtimes_\psi K$. (Hint: This should follow formally from Lemmas 4 and 5, so you shouldn't have to do any lengthy computations).

4. A lot of studying semidirect products comes down to enumerating and classifying homomorphisms. Let's record a useful fact.

   (a) Show that giving a homomorphism $Z_n \to G$ is the same as selecting an element $g \in G$ with $|g|$ dividing n. That is, give a bijection between the following sets:

$$\left\{ \begin{array}{c} \text{Homomorphisms} \\ Z_n \to G \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Elements } g \in G \\ \text{where } |g| \text{ divide } n \end{array} \right\}$$

   (b) If p is prime show that giving a *nontrivial* map $Z_p \to G$ is the same as choosing an element of order p in G. (Note: the trivial map is the one that sends every element to the identity of G).

   (c) Show that giving a homomorphism $Z_{n_1} \times \cdots \times Z_{n_r} \to G$ is the same as chosing elements $g_1, \cdots, g_r \in G$ such that all the g_i commute with eachother and each $|g_i|$ divides n_i.

   (d) Suppose G is abelian and p is prime. Describe the set of homomorphisms $Z_p \times Z_p \to G$ as a subset of $G \times G$.

We finish with a couple of classification problems. You will find HW8#3 useful, as well as the following facts (you proved the third one in HW7, the other two you can freely use).

Facts (Automorphisms of abelian groups of order p and p^2)**.** *Let p a prime number. Then:*

- $\operatorname{Aut} Z_p \cong Z_{p-1}$
- $\operatorname{Aut} Z_{p^2} \cong Z_{p(p-1)}$.
- $\operatorname{Aut}(Z_p \times Z_p) \cong GL_2(\mathbb{F}_p)$.

We'll walk through the first one together and then leave the second one to you!

5. In this problem we classify all groups of order 75 up to isomorphism. (There should be 3 total).

   (a) List all the abelian groups of order 75 using the fundamental theorem of finite abelian groups.

   (b) Prove that a group of order 75 is isomorphic to $P \rtimes Q$ where P is a Sylow 5-subgroup and Q is a Sylow 3-subgroup.

   (c) Prove that if a group of order 75 has a *cyclic* Sylow 5-subgroup, then it is abelian.

   (d) Show that there is a unique nonabelian group of order 75. (*Hint:* Show that 3 is a maximal 3-divisor of $|GL_2(\mathbb{F}_5)|$. Then use Sylow's theorems and 3(c).)

6. Classify all groups of order 20 up to isomorphism. (There should be 5 total).