Homework Assignment 6

Due Friday, March 6

1. There is an absolute value on the complex numbers given by $||a+bi|| = \sqrt{a^2+b^2}$, where we use $||\cdot||$ rather than $|\cdot|$ so not confuse notation with order of a group element. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : ||z|| = 1\}$. This is called the *circle group*.
   (a) Show that $||\cdot|| : \mathbb{C}^\times \to \mathbb{R}^\times$ is a homomorphism.
   (b) Show that the circle group is a normal subgroup of the multiplicative group $\mathbb{C}^\times$.
   (c) Draw the graph of the circle group in the complex plane. Justify your answer.
   (d) Show that $\varphi : \mathbb{R} \to \mathbb{S}^1$ defined by the rule $\varphi(x) = e^{2\pi i x}$ is a surjective homomorphism (where the binary operation on $\mathbb{R}$ is addition).
   (e) Deduce that the additive quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{S}^1$

2. A root of unity ξ is a complex number such that $\xi^n = 1$ for some positive integer n. The set of roots of unity is often denoted by μ.
   (a) ± 1 are roots of unity. Give 3 more examples of roots of unity.
   (b) Show that if ξ is a root of unity, then $||\xi|| = 1$.
   (c) Show that $\mu = (\mathbb{S}^1)^{\text{tors}}$ (recall the definition from HW 4 Problem 2(b)). Deduce that μ is a subgroup of $\mathbb{S}^1$.

3. Consider the additive group quotient $\mathbb{Q}/\mathbb{Z}$.
   (a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ has exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.
   (b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order, but that there are elements of arbitrary large order.
   (c) Show that $\mathbb{Q}/\mathbb{Z} = (\mathbb{R}/\mathbb{Z})^{\text{tors}}$. Conclude that $\mathbb{Q}/\mathbb{Z} \cong \mu$.

4. Let $N \trianglelefteq G$ be a normal subgroup of a group G. Let $\pi : G \to G/N$ be the natural projection.
   (a) Let $H \leq G/N$. Show that the preimage $\pi^{-1}(H)$ is a subgroup of G containing N.
   (b) Let $H \leq G$. Show that its image $\pi(H)$ is a subgroup of G/N.
   (c) These constructions do not give a bijection between subgroups of G and subgroups of G/N. Give an example showing why.
   (d) If we restrict our attention to certain subgroups of G we do get a bijection. Indeed, show that there is a bijection:

$$\left\{ \begin{array}{c} \text{Subgroups } H \leq G \\ \text{such that } N \leq H \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Subgroups} \\ \overline{H} \leq G/N \end{array} \right\}$$

5. Let G be a group and $Z(G)$ its center.
   (a) Show that $Z(G)$ is a normal subgroup.
   (b) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   (c) Let p and q be prime numbers (not necessarily distinct), and G a group of order pq. Show that if G is not abelian, than $Z(G) = \{1\}$.

6. Let G be a group. Let $[G, G] = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$.
   (a) Show that $[G, G]$ is a normal subgroup of G.
   (b) Show that $G/[G, G]$ is abelian.

   $[G, G]$ is called the *commutator subgroup* of G, and $G/[G, G]$ is called the *abelianization* of G, denoted G^{ab}. The rest of this exercise explains why.

   (c) Let $\varphi : G \to H$ be a homomorhism with H abelian. Show $[G, G] \subseteq \ker \varphi$.
   (d) Denote the natural projection to the quotient group by $\pi : G \to G^{\text{ab}}$. Prove that φ induces a unique homomorphism $\tilde{\varphi} : G^{\text{ab}} \to H$ such that $\pi \circ \tilde{\varphi} = \varphi$.
   (e) Conclude that for H an abelian group there is a bijection:

$$\left\{ \text{ Homomorphisms } \varphi : G \to H \right\} \Longleftrightarrow \left\{ \text{ Homomorphisms } \tilde{\varphi} : G^{\text{ab}} \to H \right\}$$

7. Let's now compute D_{2n}^{ab}. We should begin computing $xyx^{-1}y^{-1}$. There are 3 cases.
   (a) Compute $x^{-1}y^{-1}xy$ in each of the following 3 cases.
      (i) x, y both reflections. So $x = sr^i$ and $y = sr^j$. Recall that reflections always have order 2.
      (ii) x a reflection and y not a reflection. So $x = sr^i$ and $y = r^j$.
      (iii) Neither x nor y are reflections. So $x = r^i$ and $y = r^j$.
   (b) Prove that $[D_{2n}, D_{2n}] = \langle r^2 \rangle$. If n is odd, there is another generator. What is it?
   (c) Now prove that D_{2n}^{ab} is either V_4 or Z_2 depending on whether n is odd or even. Note that since this is so small we should interpret this as suggesting that D_{2n} is far from abelian.

Bonus In Problem 1 we could have gone in a different direction after part (a). If you're interested, compose the complex absolute value with the log map to construct an isomorphism between $\mathbb{C}^\times/\mathbb{S}^1$ and the additive group $\mathbb{R}$. Describe in words the $\mathbb{S}^1$ cosets and how they correspond to elements of $\mathbb{R}$ (hint, it looks like a target!). I can't promise many extra points for this, but I do think it's a fun exercise.