Homework Assignment 9 Due Friday, April 9

This assignment will fill in many details from lecture, and do a few hands on classifications. To begin we will confirm that the semidirect product is indeed a group. First recall the definition.

Definition 1. Let H, K be groups, and $\varphi : K \to \operatorname{Aut}(H)$ a group homomorphism. Denote the induced action of K on H by:

$$k \cdot h = \varphi(k)(h).$$

The semidirect product of H and K with respect to φ is the set $H \rtimes K = \{(h,k) : h \in H, k \in K\}$, where multiplication is defined by the rule:

$$(h_1, k_1)(h_2, k_2) = (h_1(k_1 \cdot h_2), k_1k_2).$$

- 1. Let's make sure that $H \rtimes K$ is a group.
 - (a) Show that $(1,1) \in H \times K$ is the identity. (Remember you have to check both sides).
 - (b) Show that $(h,k)^{-1} = (k^{-1} \cdot h^{-1}, k^{-1})$. (As above, you have to check both sides).
 - (c) Prove that multiplication is associative.

Studying semidirect products reduces to the study of automorphism groups, so it is useful to be able to to decompose them.

Lemma 2. Let H and K be finite groups whose orders are coprime. Then

$$\operatorname{Aut}(H \times K) \cong \operatorname{Aut} H \times \operatorname{Aut} K$$
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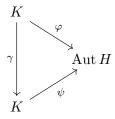
The following definition will be useful.

Definition 3. Let $\varphi: G \to G'$ be a homomorphism, and let $H \leq G$. The restriction of φ to H is the map $\varphi|_H: H \to G'$ given by evaluating φ on elements of H.

Let's consider it obvious that $\varphi|_H$ is a homomorphism (why?), and so you may use this fact without proof.

- 2. Let's prove Lemma 2.
 - (a) Let G be a group and let H char G be a characteristic subgroup (recall the definition from HW8 Definition 1). Fix any automorphism $\varphi \in \operatorname{Aut} G$. Show that $\varphi|_H$ is an automorphism of H. (Hint: you must first show its image lands in H so you can consider it as a map from H to itself).
 - (b) With H and G as in part (a), show that the rule $\varphi \mapsto \varphi|_H$ is a homomorphism $\operatorname{Aut} G \to \operatorname{Aut} H$.
 - (c) Let H, K be finite groups of coprime orders. Show that H and K are characteristic in $H \times K$.
 - (d) With H, K as in (c), construct an isomorphism $\operatorname{Aut}(H \times K) \to \operatorname{Aut} H \times \operatorname{Aut} K$.

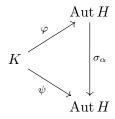
Recall that any homomorphism $\varphi: K \to \operatorname{Aut} H$ allows us to build a semidirect product $H \rtimes_{\varphi} K$. An interesting question is when different maps give us isomorphic semidirect products. In class we stated and used the following lemma. **Lemma 4.** Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they differ by an automorphism of K. That is, suppose there is some $\gamma \in \operatorname{Aut}(K)$ such that $\psi \circ \gamma = \varphi$:



Then $H \rtimes_{\varphi} K \cong H \rtimes_{\psi} K$.

One could ask if this is the only thing that could allow different φ to give different semidirect products. The answer would be no, as the following lemma shows.

Lemma 5. Let $\varphi, \psi : K \to \operatorname{Aut} H$ be two homomorphisms, and suppose they are conjugate in $\operatorname{Aut} H$. Explicitly, suppose there is some $\alpha \in \operatorname{Aut} H$, corresponding to the inner automorphism $\sigma_{\alpha} : \beta \mapsto \alpha \beta \alpha^{-1}$, and suppose that $\psi = \sigma_{\alpha} \circ \varphi$:



Then $H \rtimes_{\varphi} K \cong H \rtimes_{\psi} K$.

- 3. Lemmas 4 and 5 say that if we alter φ by an automorphism of K, or an inner automorphism of Aut H, (or both), we don't change the semidirect products. Let's prove this.
 - (a) Consider the setup of Lemma 4. Show that the map:

$$\begin{array}{ccc} H \rtimes_{\varphi} K & \longrightarrow & H \rtimes_{\psi} K \\ (h,k) & \mapsto & (h,\gamma(k)) \end{array}$$

is an isomorphism, thereby proving the lemma.

(b) Consider the setup of Lemma 5. Show that the map:

$$\begin{array}{cccc} H \rtimes_{\varphi} K & \longrightarrow & H \rtimes_{\psi} K \\ (h,k) & \mapsto & (\alpha(h),k) \end{array}$$

is an isomorphism, thereby proving the lemma. (Notice that $\alpha \in \operatorname{Aut} H$ is an automorphism of H, wheras σ_{α} is an automorphism of $\operatorname{Aut} H$, given by conjugation by α . In unweildy notation, this says $\sigma_{\alpha} \in \operatorname{Aut}(\operatorname{Aut} H)$.)

(c) Now suppose $\varphi, \psi : K \to \operatorname{Aut} H$ are two homomorphisms, and suppose there is an automorphism $\gamma \in \operatorname{Aut} K$ and an inner automorphism $\sigma \in \operatorname{Inn}(\operatorname{Aut}(H))$ such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & \operatorname{Aut} H \\ \uparrow \downarrow & & \downarrow \sigma \\ K & \xrightarrow{\psi} & \operatorname{Aut} H. \end{array}$$

That is, $\sigma \circ \varphi = \psi \circ \gamma$. Then $H \rtimes_{\varphi} K \cong H \rtimes_{\psi} K$. (Hint: This should follow formally from Lemmas 4 and 5, so you shouldn't have to do any lengthy computations).

- 4. A lot of studying semidirect products comes down to enumerating and classifying homomorphisms. Let's record a useful fact.
 - (a) Show that giving a homomorphism $Z_n \to G$ is the same as selecting an element $g \in G$ with |g| dividing n. That is, give a bijection between the following sets:

$$\left\{\begin{array}{c} \text{Homomorphisms} \\ Z_n \to G \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \text{Elements } g \in G \\ \text{where } |g| \text{ divide } n \end{array}\right\}$$

- (b) If p is prime show that giving a nontrivial map $Z_p \to G$ is the same as choosing an element of order p in G. (Note: the trivial map is the one that sends every element to the identity of G).
- (c) Show that giving a homomorphism $Z_{n_1} \times \cdots \times Z_{n_r} \to G$ is the same as chosing elements $g_1, \dots, g_r \in G$ such that all the g_i commute with eachother and each $|g_i|$ divides n_i .
- (d) Suppose G is abelian and p is prime. Describe the set of homomorphisms $Z_p \times Z_p \to G$ as a subset of $G \times G$.

We finish with a couple of classification problems. You will find HW8#3 useful, as well as the following facts (you proved the third one in HW7, the other two you can freely use).

Facts (Automorphisms of abelian groups of order p and p^2). Let p a prime number. Then:

- Aut $Z_p \cong Z_{p-1}$
- Aut $Z_{p^2} \cong Z_{p(p-1)}$.
- Aut $(Z_p \times Z_p) \cong GL_2(\mathbb{F}_p)$.

We'll walk through the first one together and then leave the second one to you!

- 5. In this problem we classify all groups of order 75 up to isomorphism. (There should be 3 total).
 - (a) List all the abelian groups of order 75 using the fundamental theorem of finite abelian groups.
 - (b) Prove that a group of order 75 is isomorphic to $P \rtimes Q$ where P is a Sylow 5-subgroup and Q is a Sylow 3-subgroup.
 - (c) Prove that if a group of order 75 has a cyclic Sylow 5-subgroup, then it is abelian.
 - (d) Show that there is a unique nonabelian group of order 75. (*Hint:* Show that 3 is a maximal 3-divisor of $|GL_2(\mathbb{F}_5)|$. Then use Sylow's theorems and 3(c).)
- 6. Classify all groups of order 20 up to isomorphism. (There should be 5 total).