Mathematical Illustration, Experimentation, and Exploration

Gabriel Dorfsman-Hopkins

Joint with Eliza Brown, Daniel Rostamloo, Shuchang Xu, and others.

Math Conference and Competition of Northern New York
March 2nd, 2024
Clarkson University

ST. LAWRENCE UNIVERSITY

Land Acknowledgement

Land Acknowledgement

St. Lawrence University occupies the traditional lands of the Haudenosaunee Nations.

Land Acknowledgement

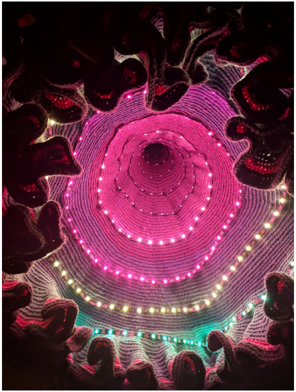
St. Lawrence University occupies the traditional lands of the Haudenosaunee Nations. We honor the heritage and existing cultures of the Haudenosaunee peoples, made up of the distinct nations that many now recognize as the Iroquois confederacy (Mohawk, Oneida, Onondaga, Cayuga, Seneca, and Tuscarora).

Land Acknowledgement

St. Lawrence University occupies the traditional lands of the Haudenosaunee Nations. We honor the heritage and existing cultures of the Haudenosaunee peoples, made up of the distinct nations that many now recognize as the Iroquois confederacy (Mohawk, Oneida, Onondaga, Cayuga, Seneca, and Tuscarora). While doing work in this space, we must acknowledge that our continued occupation of indigenous land can undermine our efforts at reparations to their communities.

I like to make art inspired by math

I like to make art inspired by math

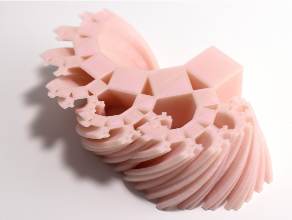


Hyperbolic Geometry

Algebraic Geometry

I like to make art inspired by math

Complex Dynamics



Fractal Deformations

Today:

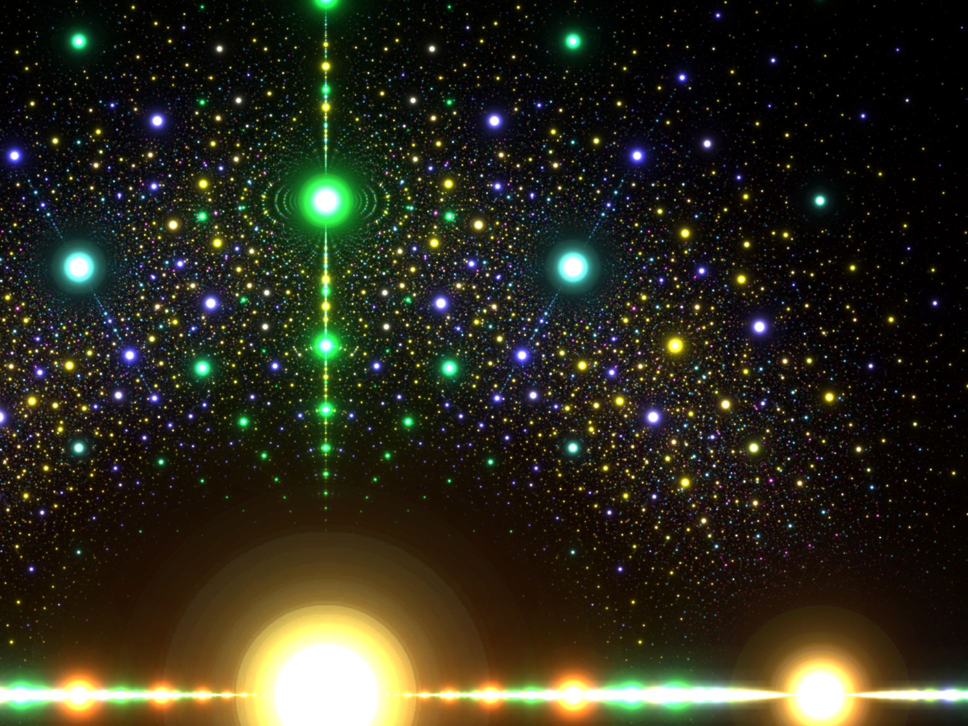
Today:

When artistic exploration can inspire **new mathematics**.

Today:

When artistic exploration can inspire **new mathematics**.

This story started with a picture…



Call this an Algebraic Starscape

Call this an Algebraic Starscape

Today I want to answer the following questions:

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?
2. What are the patterns that appear?

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?
2. What are the patterns that appear?
3. Why do they appear?

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?
2. What are the patterns that appear?
3. Why do they appear?

Short answer to the first question

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?
2. What are the patterns that appear?
3. Why do they appear?

Short answer to the first question

An algebraic starscape is a plot of algebraic integers,

Call this an Algebraic Starscape

Today I want to answer the following questions:

1. What is an algebraic starscape?
2. What are the patterns that appear?
3. Why do they appear?

Short answer to the first question

An algebraic starscape is a plot of algebraic integers, complex numbers which are roots of a monic polynomial with integer coefficients.

Algebraic Integers

Algebraic Integers

Example

Algebraic Integers

Example

- Try to solve $x^2 + 1 = 0$.

Algebraic Integers

Example

- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$.

Algebraic Integers

Example

- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!

Algebraic Integers

Example

- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!
- We call $i := \sqrt{-1}$ and we place it....

Algebraic Integers

Example

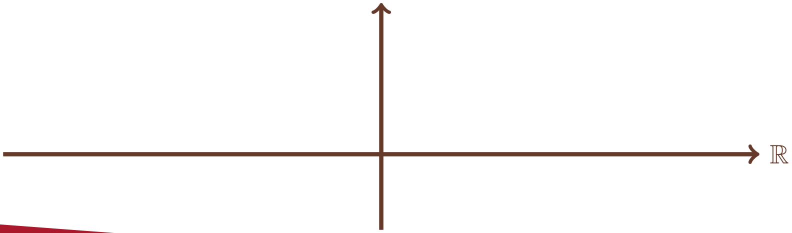
- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!
- We call $i := \sqrt{-1}$ and we place it....

$$\longrightarrow \mathbb{R}$$

Algebraic Integers

Example

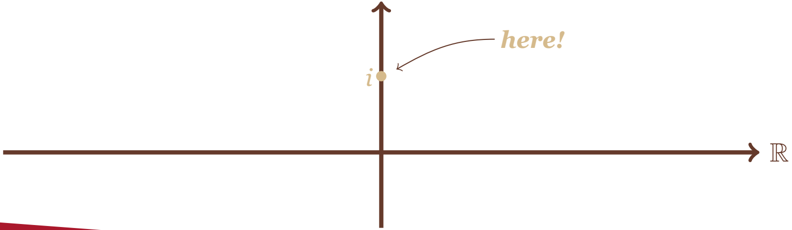
- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!
- We call $i := \sqrt{-1}$ and we place it....



Algebraic Integers

Example

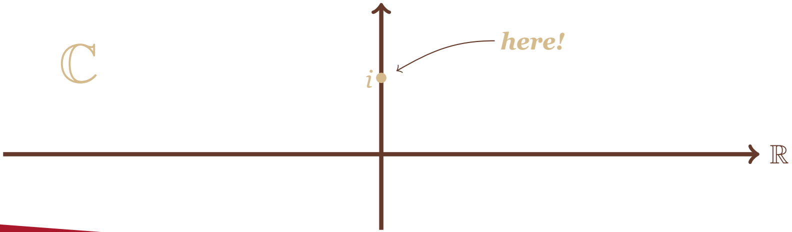
- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!
- We call $i := \sqrt{-1}$ and we place it....



Algebraic Integers

Example

- Try to solve $x^2 + 1 = 0$.
- Then $x = \pm\sqrt{-1}$. This isn't an integer, or even a real number!
- We call $i := \sqrt{-1}$ and we place it....



Exploring the New Dimension

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x =$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2}$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3}$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3}$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3}$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3} = i\sqrt{3}.$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3} = i\sqrt{3}.$$

So we can rewrite our solutions as:

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3} = i\sqrt{3}.$$

So we can rewrite our solutions as:

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3} = i\sqrt{3}.$$

So we can rewrite our solutions as:

$$x = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \approx -.5 \pm .866i.$$

Exploring the New Dimension

We can now solve things like: $x^2 + x + 1 = 0$.

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

What do we do with $\sqrt{-3}$?

$$\sqrt{-3} = \sqrt{-1 * 3} = \sqrt{-1} * \sqrt{3} = i\sqrt{3}.$$

So we can rewrite our solutions as:

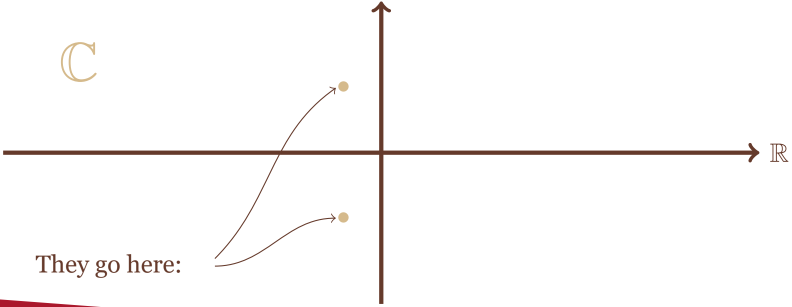
$$x = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \approx -.5 \pm .866i.$$

We give this number the coordinates $(-.5, \pm.866)$ in the complex plane.

Roots of $x^2 + x + 1$

$$x = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} \approx -.5 \pm .866i.$$

We give this number the coordinates $(-.5, \pm.866)$ in the complex plane.



This is actually an interesting number!

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$.

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1)$$

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

If we can show that $z^3 - 1 = 0$, then we know $z^3 = 1$.

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

If we can show that $z^3 - 1 = 0$, then we know $z^3 = 1$.

$$z^3 - 1$$

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

If we can show that $z^3 - 1 = 0$, then we know $z^3 = 1$.

$$z^3 - 1 = g(z)$$

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

If we can show that $z^3 - 1 = 0$, then we know $z^3 = 1$.

$$z^3 - 1 = g(z) = (z^2 + z + 1)(z - 1)$$

This is actually an interesting number!

Claim

Let $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then $z^3 = 1$.

Proof.

By definition z is a root of $x^2 + x + 1$. Consider:

$$g(x) = (x^2 + x + 1)(x - 1) = x^3 - 1.$$

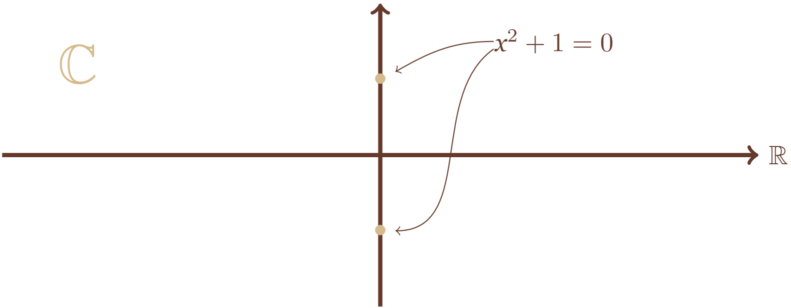
If we can show that $z^3 - 1 = 0$, then we know $z^3 = 1$.

$$z^3 - 1 = g(z) = (z^2 + z + 1)(z - 1) = 0 * (z - 1) = 0.$$

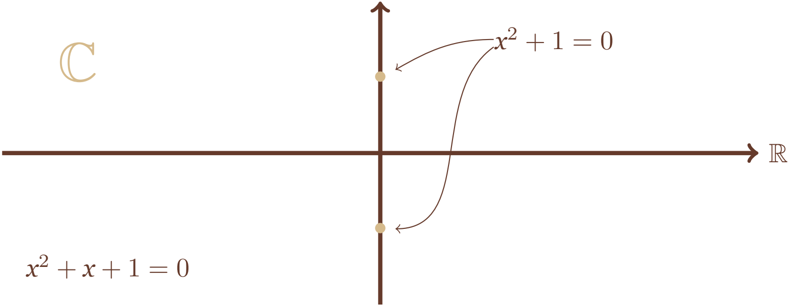
$\square$

Let’s Build our First Starscape!

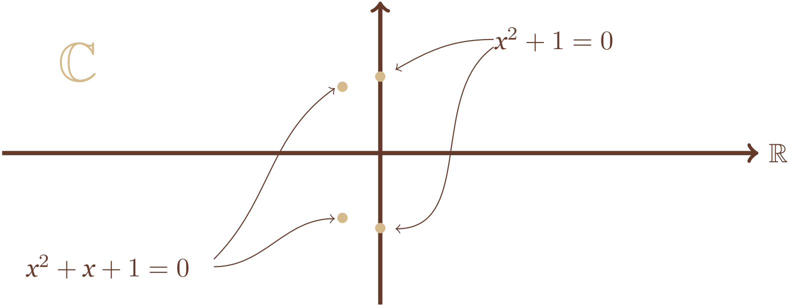
Let's Build our First Starscape!



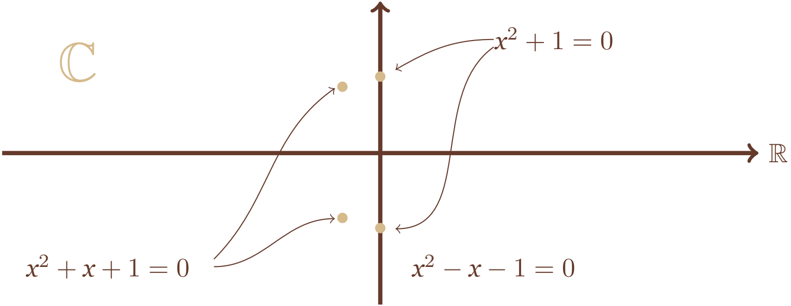
Let's Build our First Starscape!



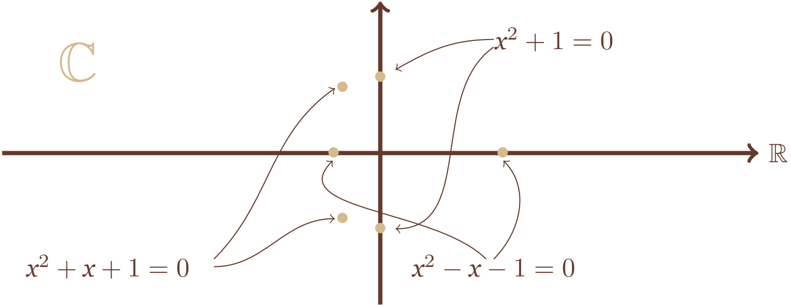
Let's Build our First Starscape!



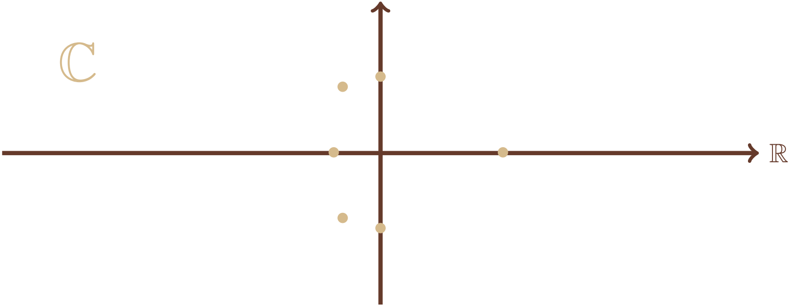
Let's Build our First Starscape!



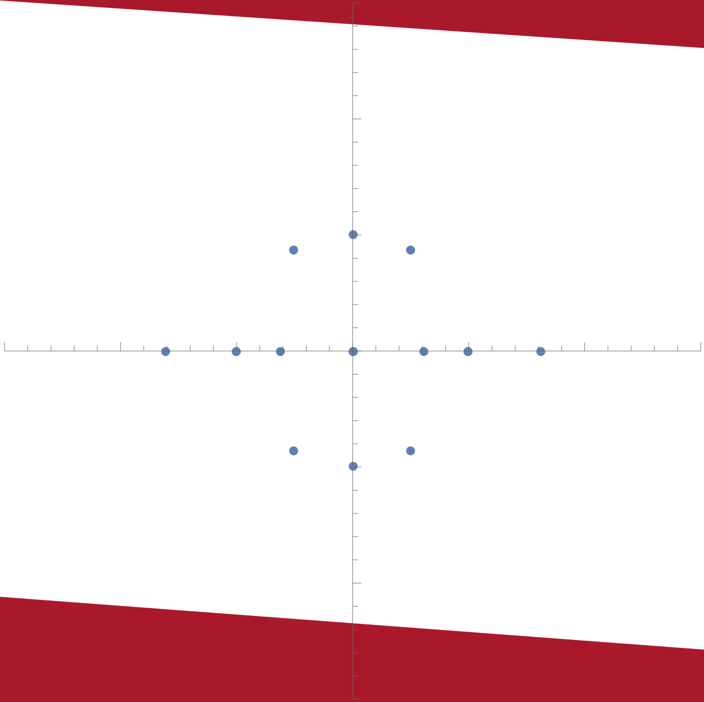
Let’s Build our First Starscape!

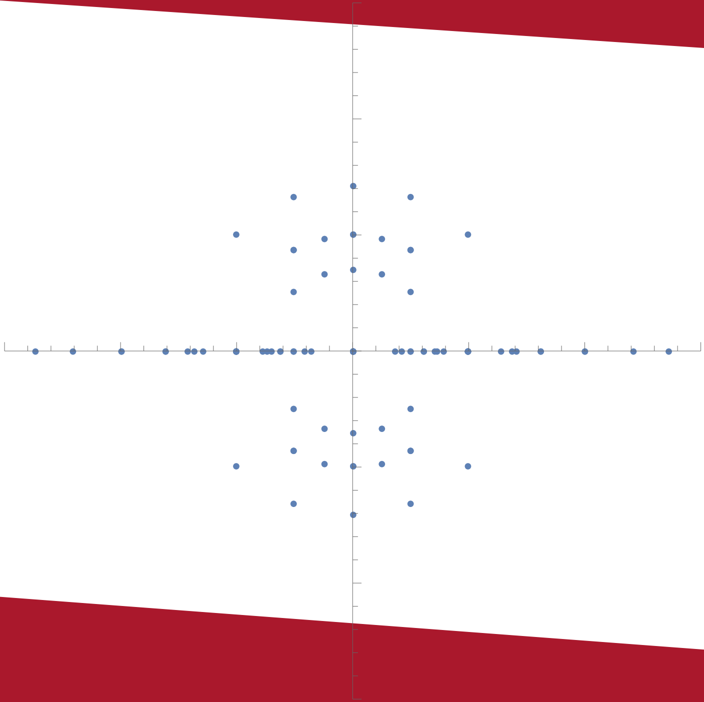


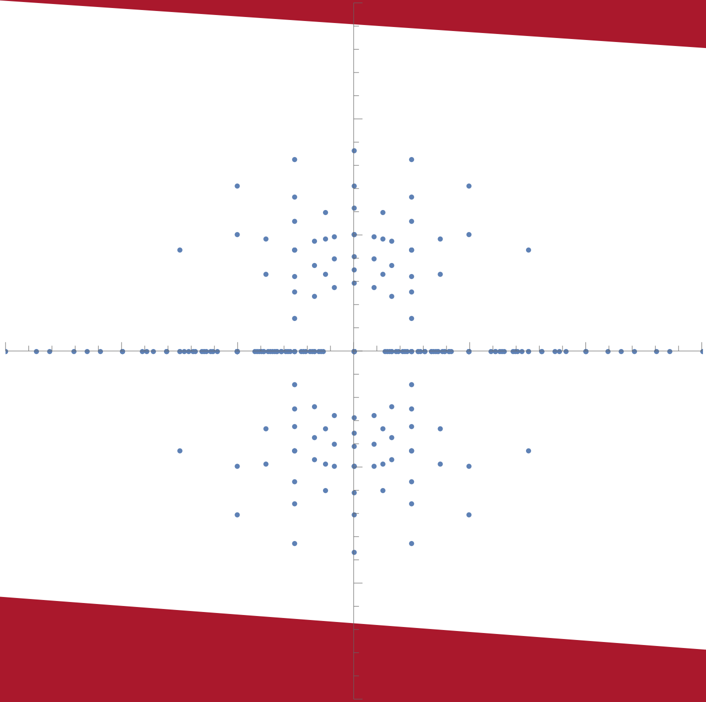
Graphing Quadratics

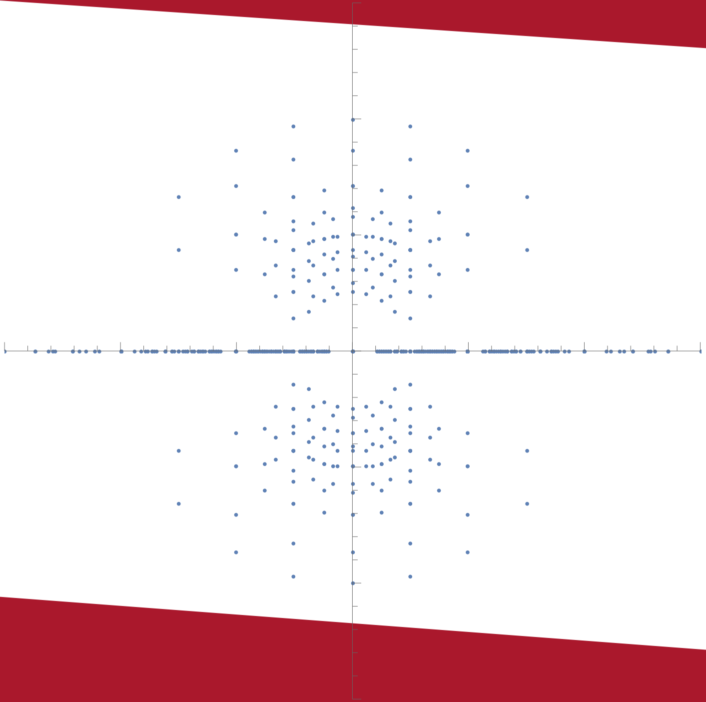


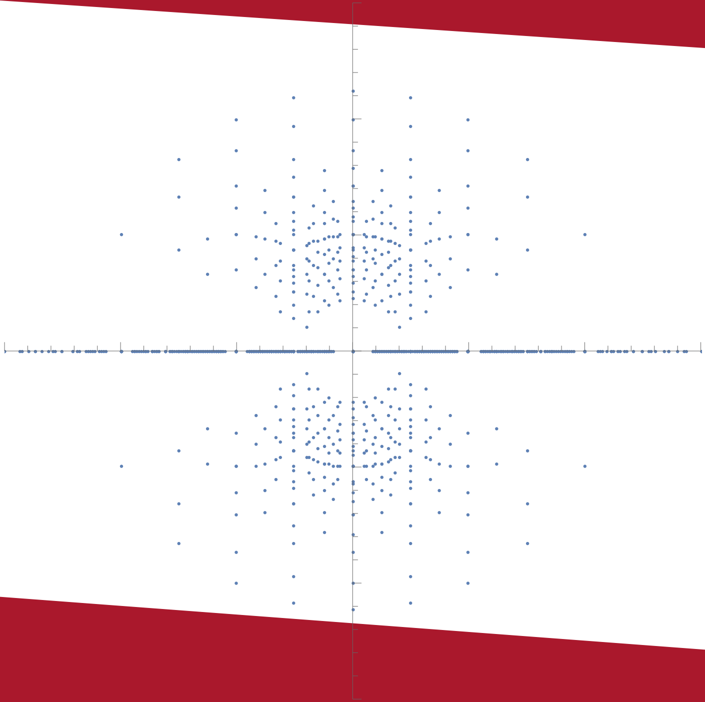
What if we do this for more and more quadratic polynomials?

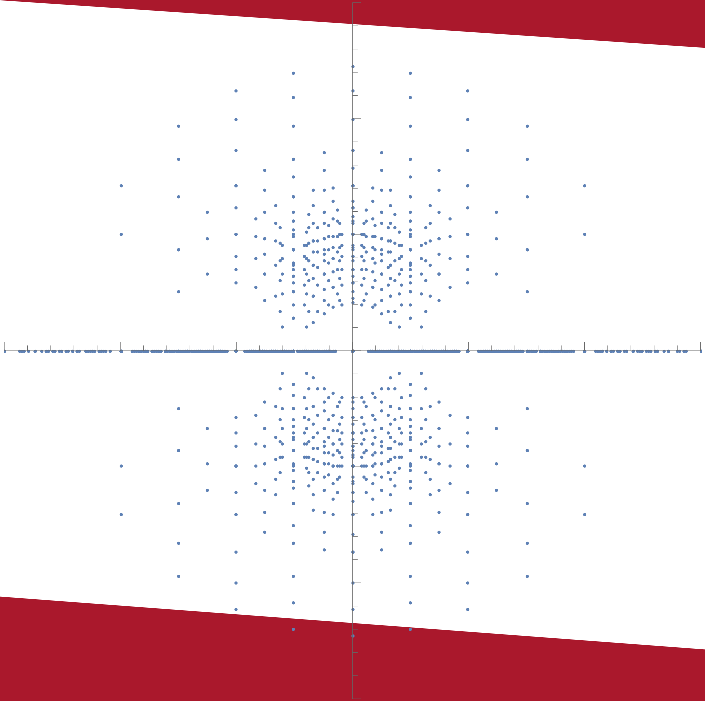


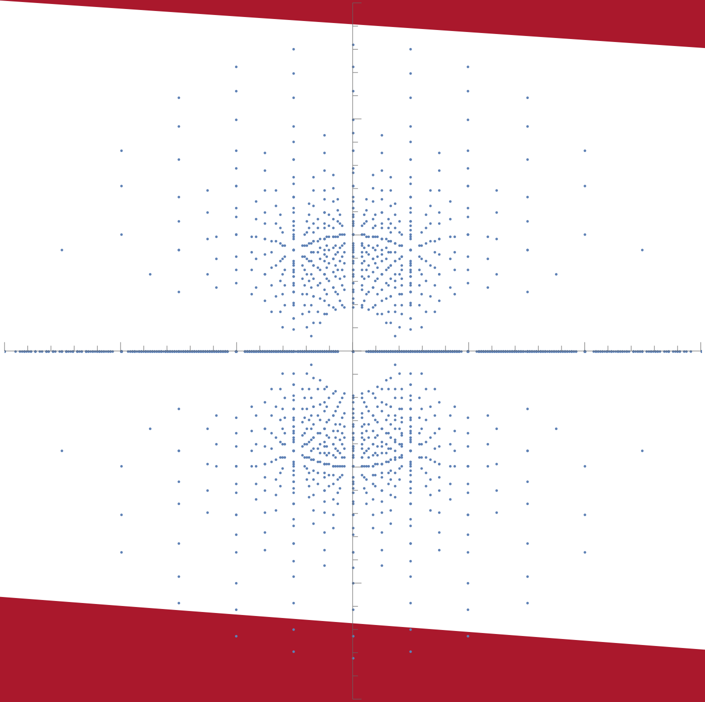


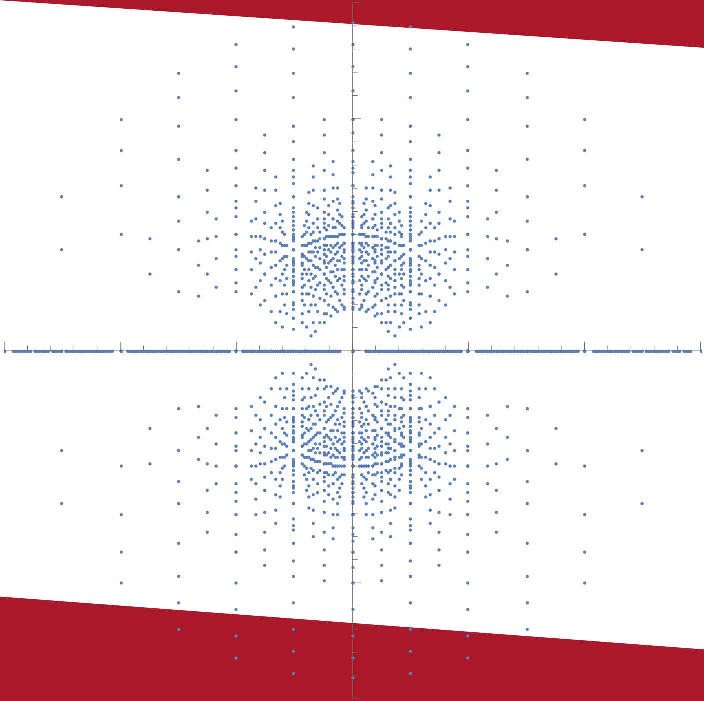


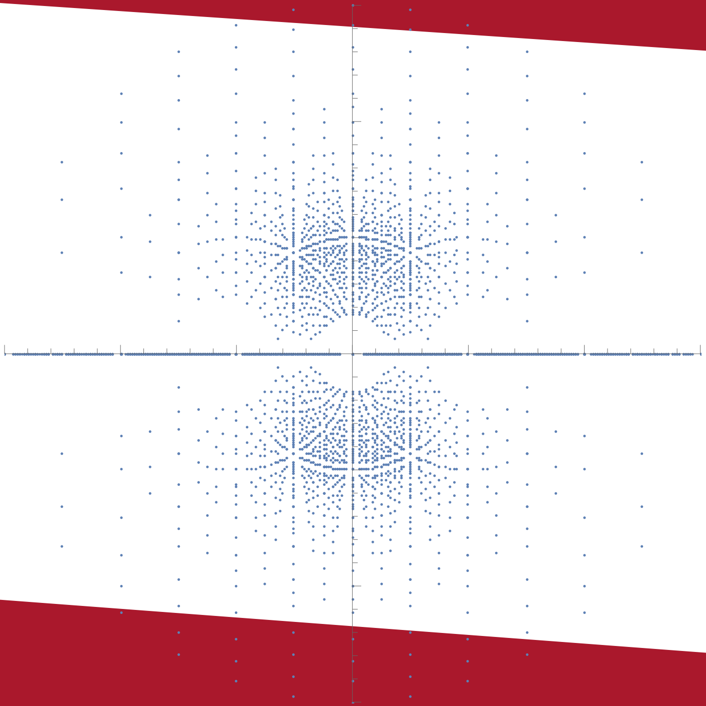


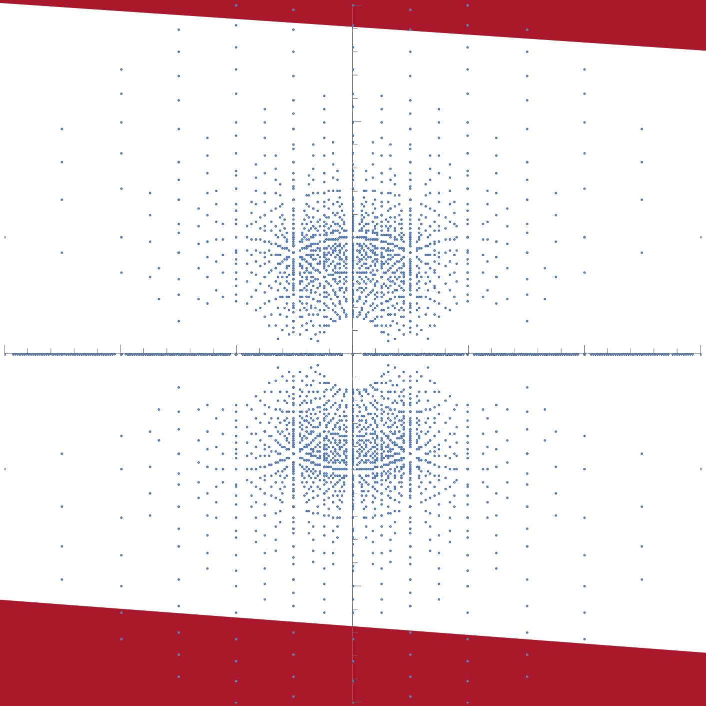


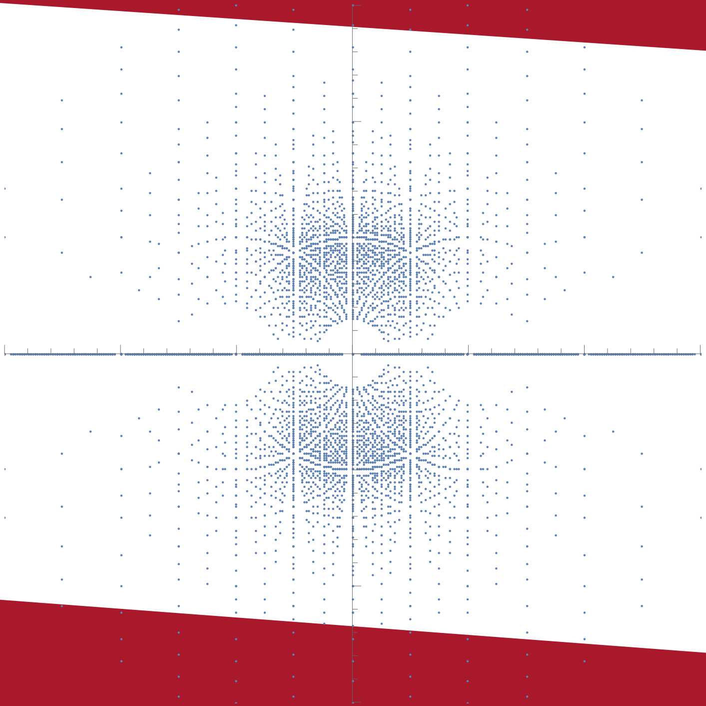


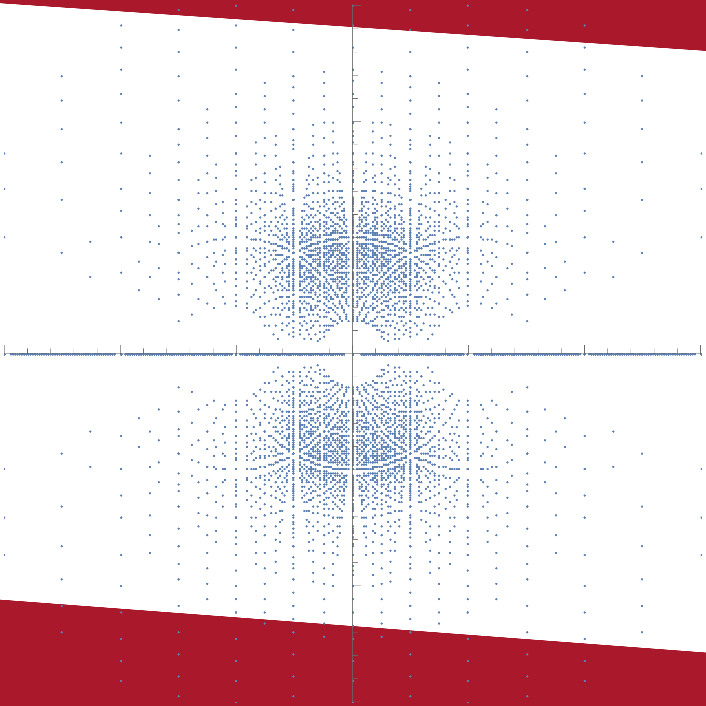


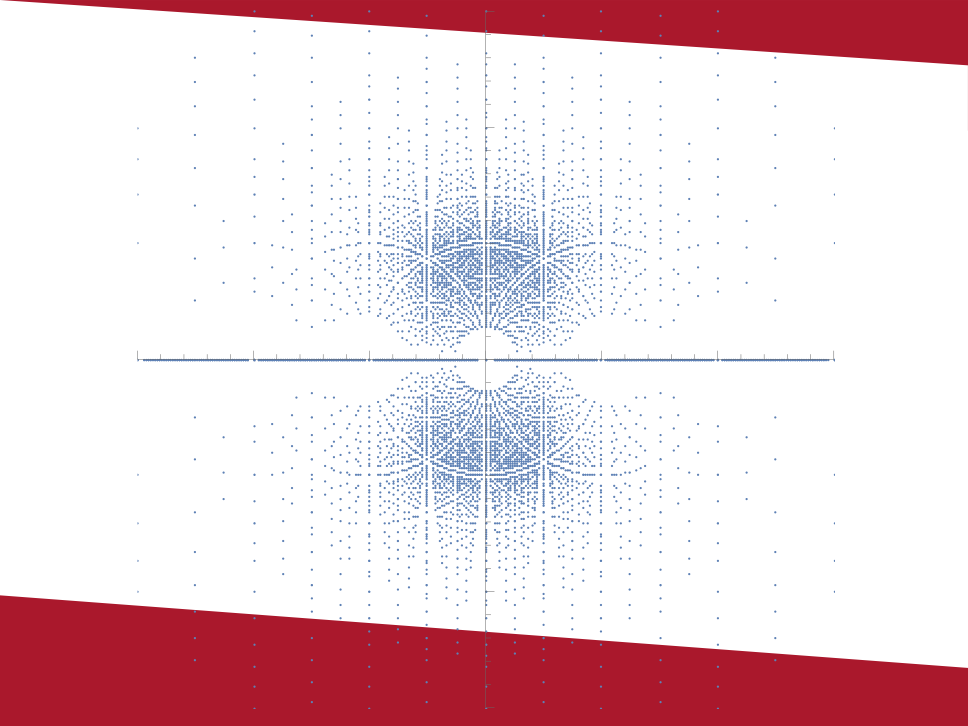


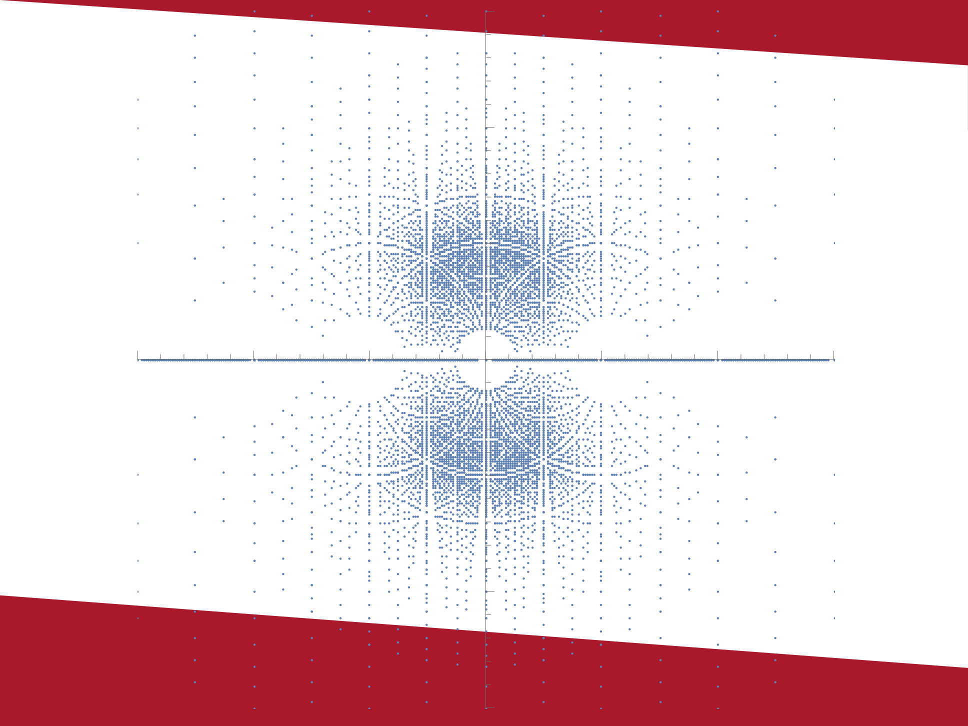


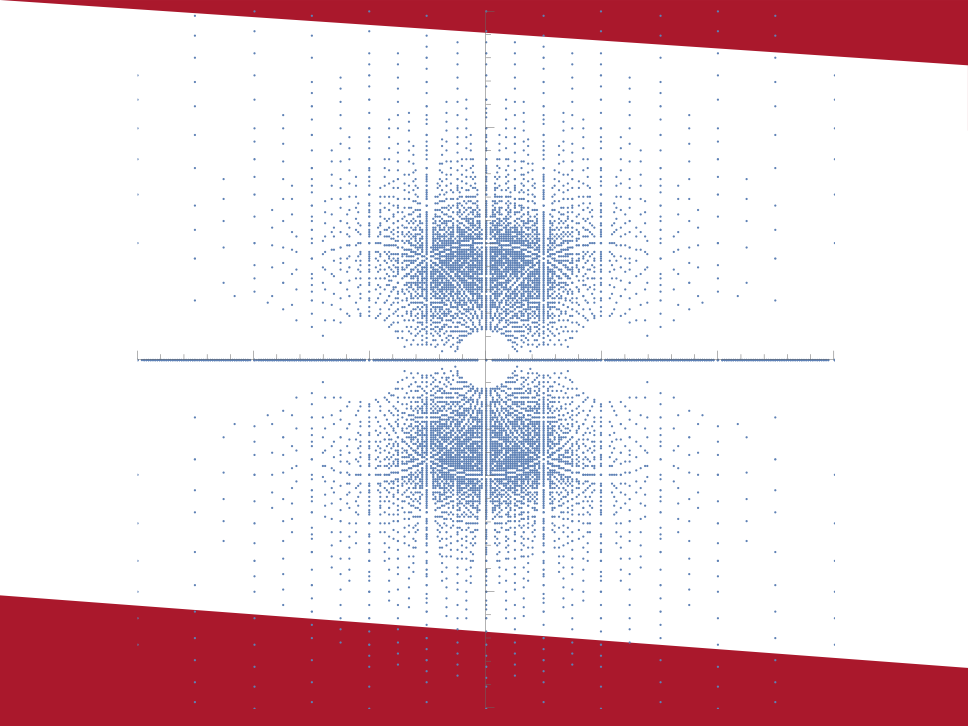


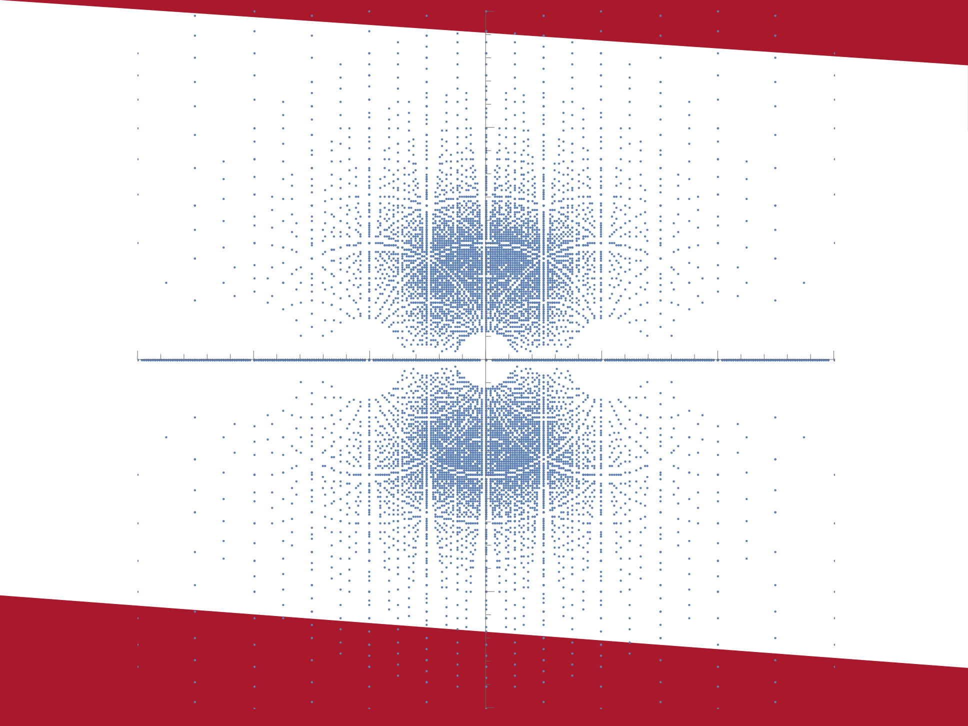


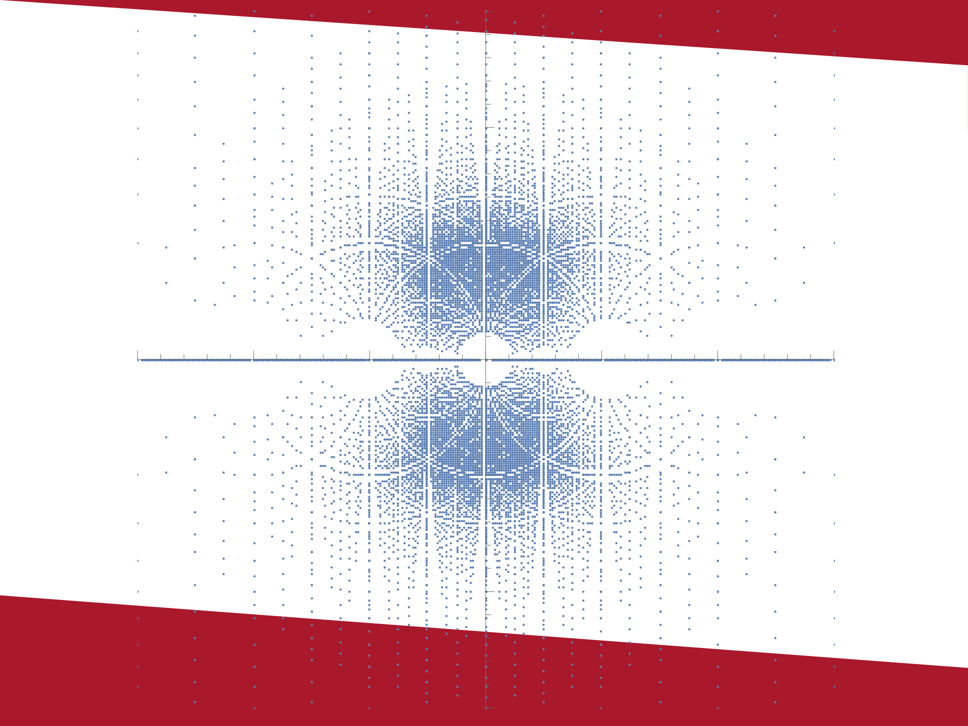


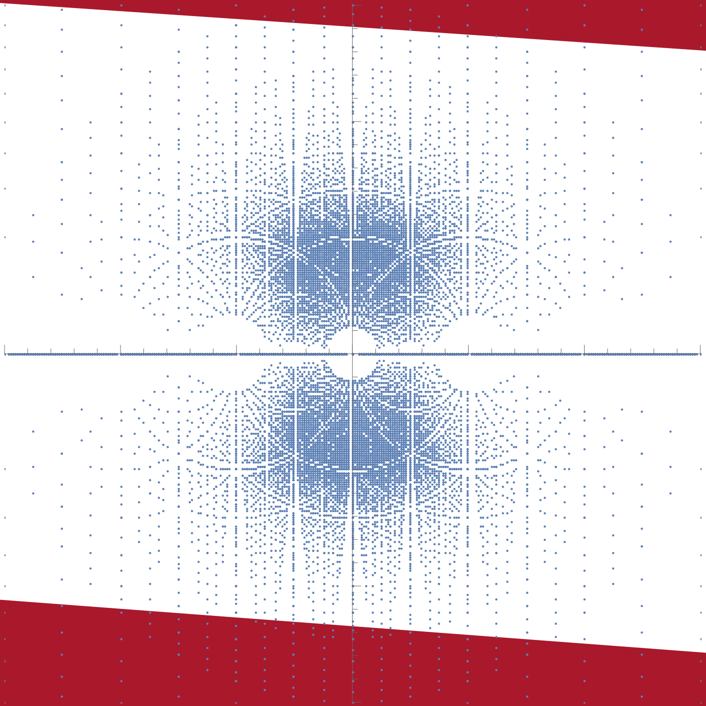


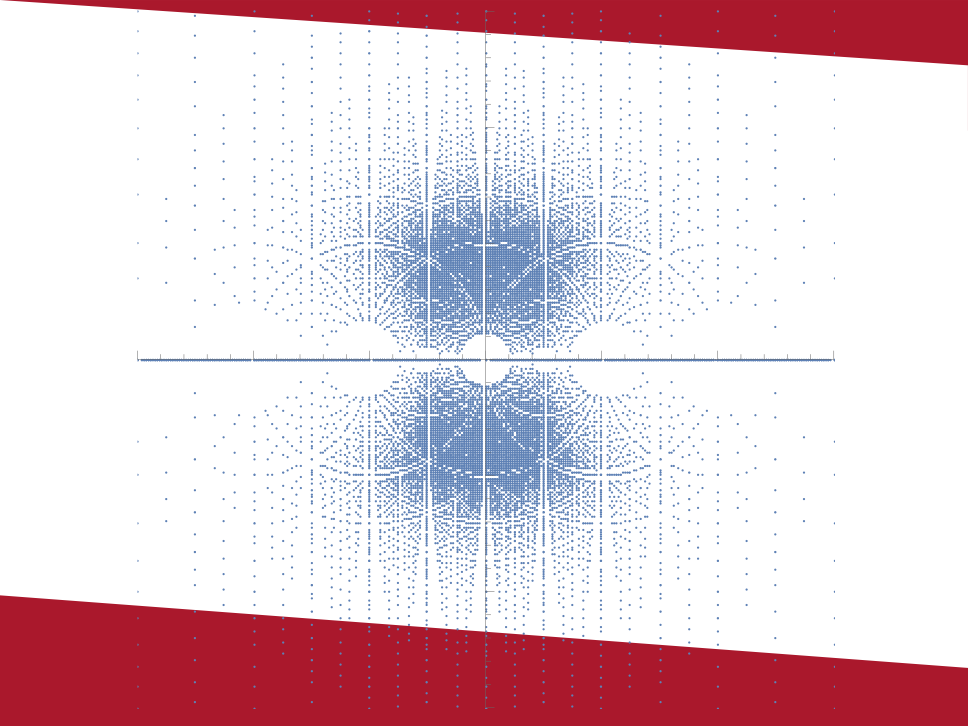


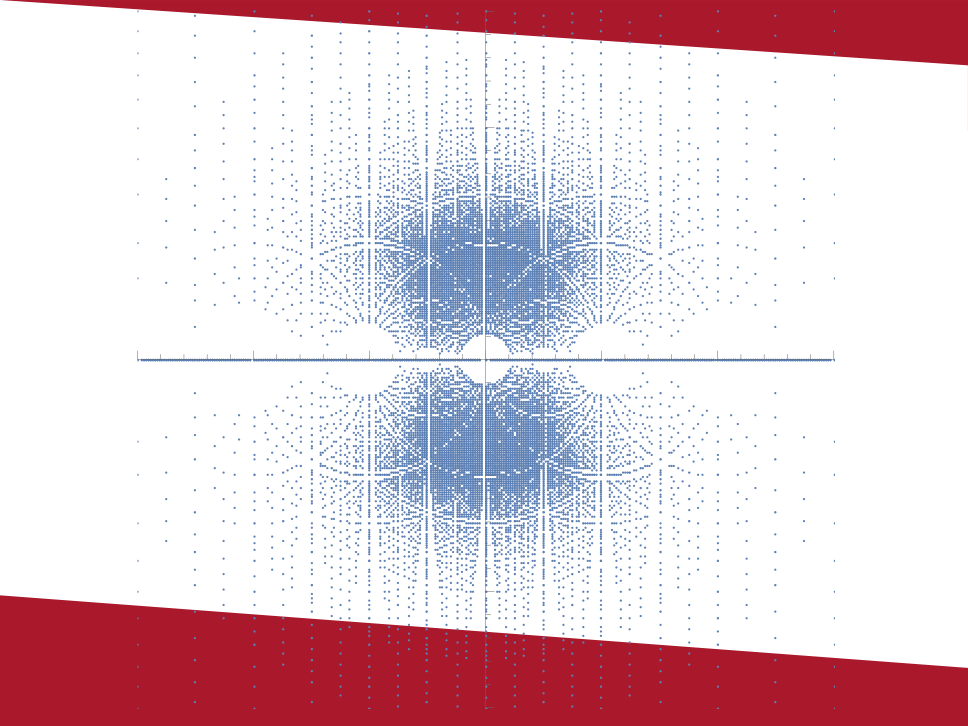












Questions

Questions

- There seems to be a hole near 0. Why is that?

Questions

- There seems to be a hole near 0. Why is that?
- As you add more roots the space seemed to fill and lose some sharpness and clarity. Can we deal with this in a meaningful way?

Questions

- There seems to be a hole near 0. Why is that?
- As you add more roots the space seemed to fill and lose some sharpness and clarity. Can we deal with this in a meaningful way?
- Can we do this with polynomials of degree >2?

Higher Degree Polynomials: An Example

Higher Degree Polynomials: An Example

- Solve $x^4 + 2x^2 + 2 = 0$.

Higher Degree Polynomials: An Example

- Solve $x^4 + 2x^2 + 2 = 0$.
- The four solutions are: $\sqrt{\pm 1 \pm i}$

Higher Degree Polynomials: An Example

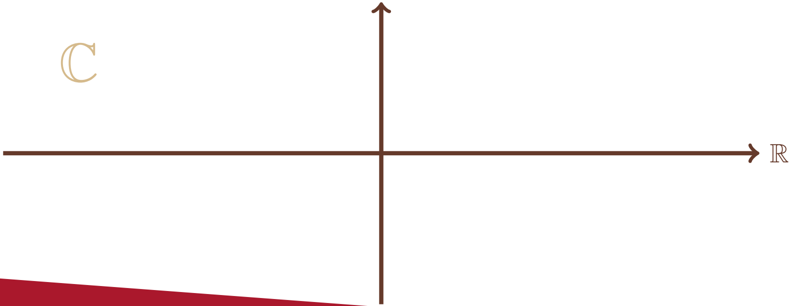
- Solve $x^4 + 2x^2 + 2 = 0$.
- The four solutions are: $\sqrt{\pm 1 \pm i} \approx \pm .46 \pm 1.10i$.

Higher Degree Polynomials: An Example

- Solve $x^4 + 2x^2 + 2 = 0$.
- The four solutions are: $\sqrt{\pm 1 \pm i} \approx \pm.46 \pm 1.10i$.
- We can plot them.

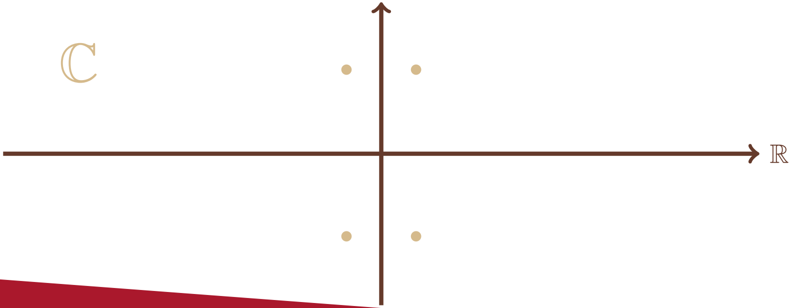
Higher Degree Polynomials: An Example

- Solve $x^4 + 2x^2 + 2 = 0$.
- The four solutions are: $\sqrt{\pm 1 \pm i} \approx \pm .46 \pm 1.10i$.
- We can plot them.

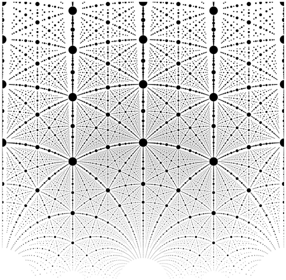


Higher Degree Polynomials: An Example

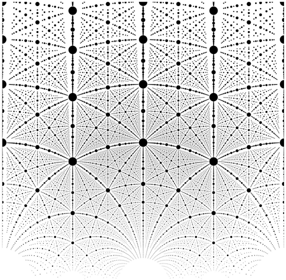
- Solve $x^4 + 2x^2 + 2 = 0$.
- The four solutions are: $\sqrt{\pm 1 \pm i} \approx \pm .46 \pm 1.10i$.
- We can plot them.



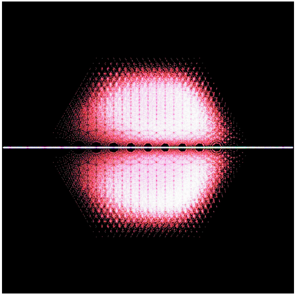
If we do this for lots and lots of polynomials we get images like…



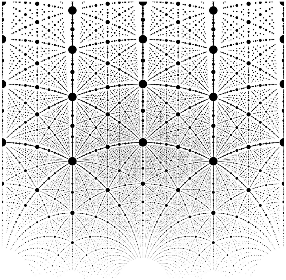
Harris, Stange, Trettel



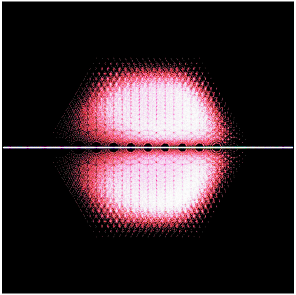
Harris, Stange, Trettel



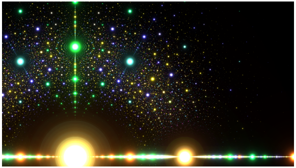
Brown, DH



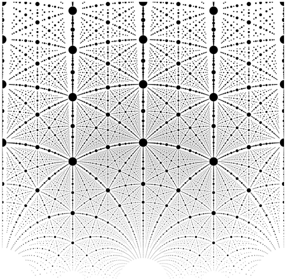
Harris, Stange, Trettel



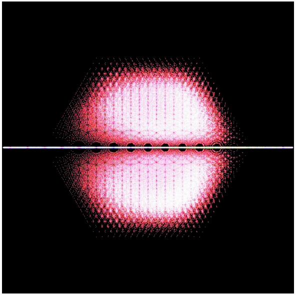
Brown, DH



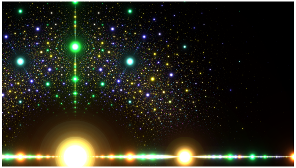
Brooks, Moore



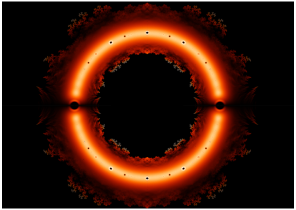
Harris, Stange, Trettel



Brown, DH



Brooks, Moore



Derbyshire

Initial takeaways from this visual exploration.

Initial takeaways from this visual exploration.

- The (complex) roots of a single polynomial form a finite set.

Initial takeaways from this visual exploration.

- The (complex) roots of a single polynomial form a finite set.
- If you plot more and more of them, they align according to some hidden geometry that a single set of roots can't reveal.

Initial takeaways from this visual exploration.

- The (complex) roots of a single polynomial form a finite set.
- If you plot more and more of them, they align according to some hidden geometry that a single set of roots can't reveal.
- We want to pin down these geometric patterns, and explain why they exist.

One detail I’ve left out…

One detail I've left out…

- Not all the points are the same size.

One detail I’ve left out…

- Not all the points are the same size. This turns out to be key.

One detail I've left out…

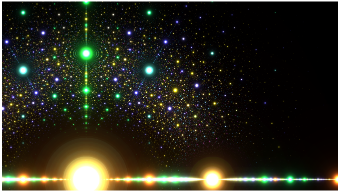
- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial.

One detail I've left out…

- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)

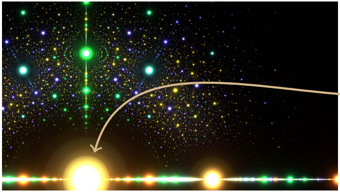
One detail I've left out...

- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)



One detail I've left out…

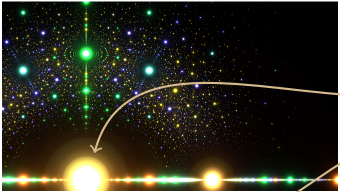
- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)



x

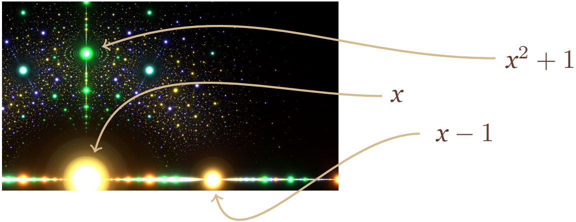
One detail I've left out...

- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)



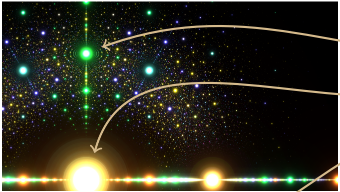
One detail I've left out…

- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)



One detail I've left out…

- Not all the points are the same size. This turns out to be key.
- Brooks and Moore size points inversely to the size of the coefficients of the generating polynomial. (More precisely, inversely to the magnitude of the vector they form.)



$x^2 + 1$

x

$x - 1$

Simpler Polynomial

=

Bigger Point

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Write $\sqrt{b^2 - 4c} = n\sqrt{\Delta}$ where n is the largest number you can pull out of the square root.

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Write $\sqrt{b^2 - 4c} = n\sqrt{\Delta}$ where n is the largest number you can pull out of the square root. I'm going to call this number Δ the discriminant.

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Write $\sqrt{b^2 - 4c} = n\sqrt{\Delta}$ where n is the largest number you can pull out of the square root. I'm going to call this number Δ the discriminant. Then

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Write $\sqrt{b^2 - 4c} = n\sqrt{\Delta}$ where n is the largest number you can pull out of the square root. I'm going to call this number Δ the discriminant. Then

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b \pm n\sqrt{\Delta}}{2}.$$

Another approach is to size points by a slightly more refined measure of algebraic complexity: the discriminant.

Example (Discriminants of Quadratics)

If we trying to solve for roots of $x^2 + bx + c$ we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Write $\sqrt{b^2 - 4c} = n\sqrt{\Delta}$ where n is the largest number you can pull out of the square root. I'm going to call this number Δ the discriminant. Then

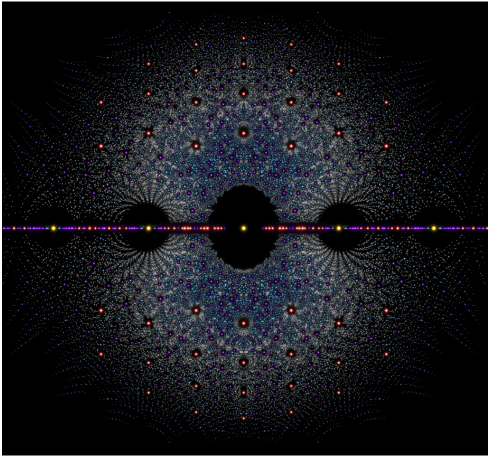
$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b \pm n\sqrt{\Delta}}{2}.$$

Δ is a measure of the irrationality of x.

Small discriminant

=

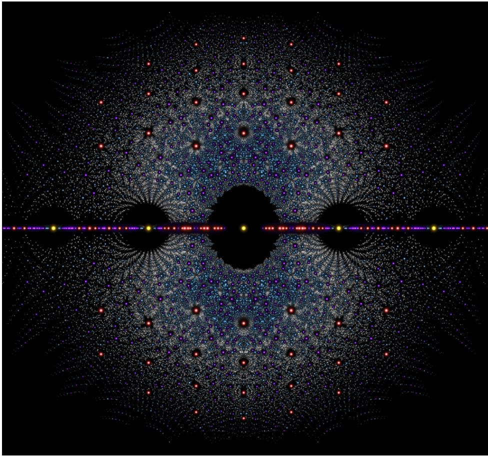
Larger point



Less irrational $\approx$ Small discriminant

$=$

Larger point



Other Approaches

Other Approaches

- One should ask what happens if you don’t vary the size of the points.

Other Approaches

- One should ask what happens if you don’t vary the size of the points.
- It turns out there are lots and lots of algebraic integers.

Other Approaches

- One should ask what happens if you don't vary the size of the points.
- It turns out there are lots and lots of algebraic integers.
- So many, that you would eventually fill the space.

Other Approaches

- One should ask what happens if you don't vary the size of the points.
- It turns out there are lots and lots of algebraic integers.
- So many, that you would eventually fill the space. That is, algebraic integers are dense in the complex numbers, so every pixel would be colored.

Other Approaches

- One should ask what happens if you don't vary the size of the points.
- It turns out there are lots and lots of algebraic integers.
- So many, that you would eventually fill the space. That is, algebraic integers are dense in the complex numbers, so every pixel would be colored.
- Instead, what if you color a pixel by the density of algebraic integers it contains?

Other Approaches

- One should ask what happens if you don't vary the size of the points.
- It turns out there are lots and lots of algebraic integers.
- So many, that you would eventually fill the space. That is, algebraic integers are dense in the complex numbers, so every pixel would be colored.
- Instead, what if you color a pixel by the density of algebraic integers it contains?

$$\begin{array}{rcl} \text{More Integers} & \longrightarrow & \text{Brighter} \\ \text{Fewer Integers} & \longrightarrow & \text{Darker} \end{array}$$

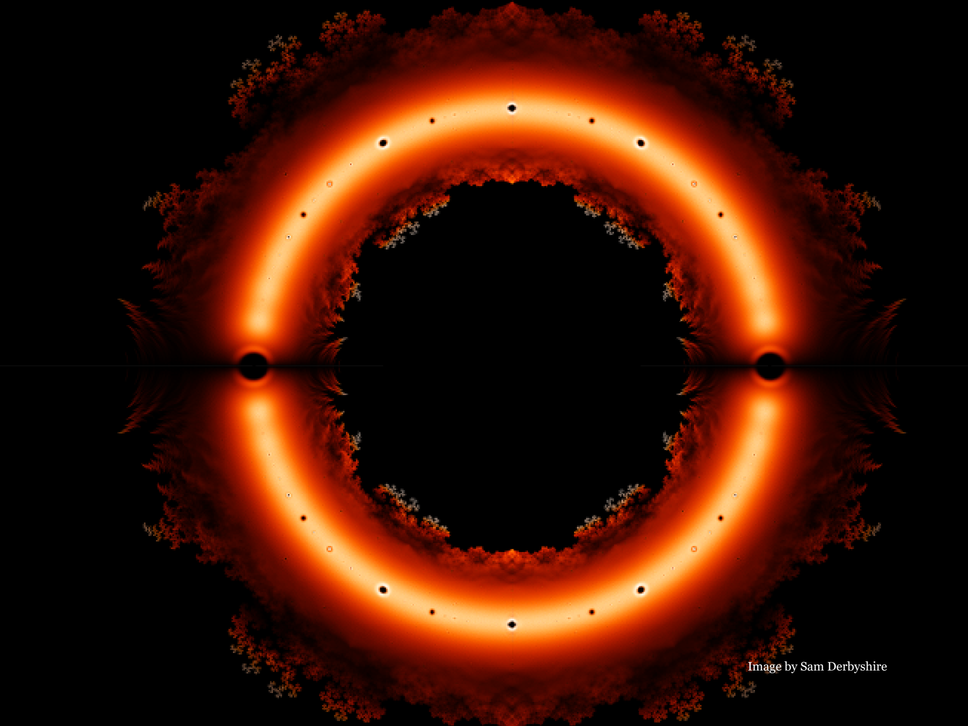


Image by Sam Derbyshire

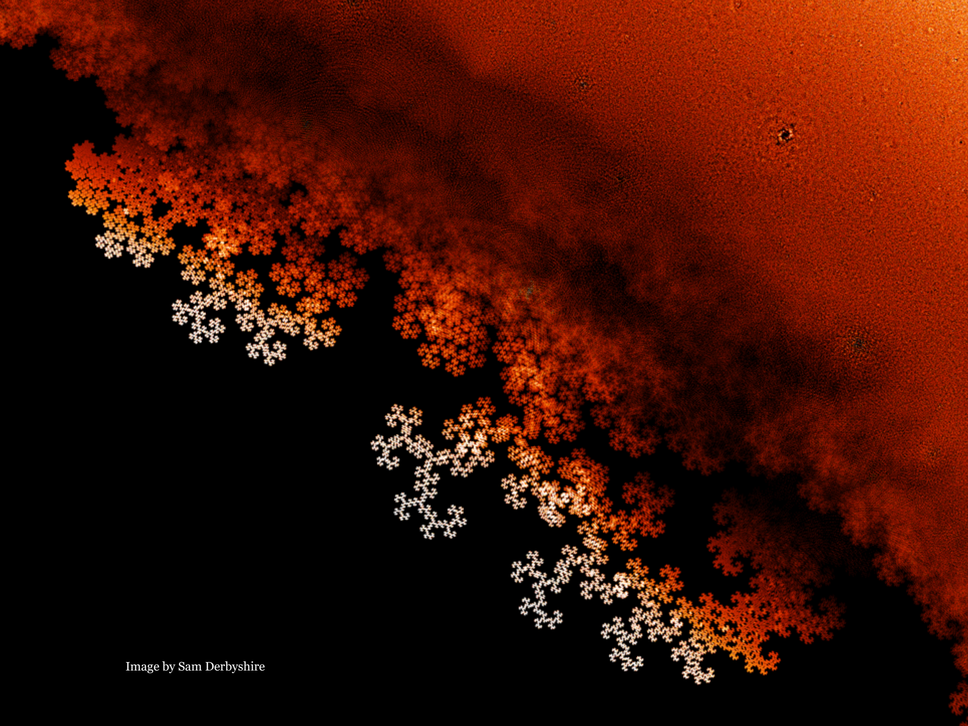
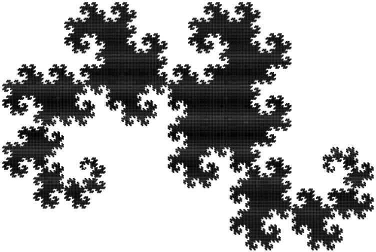
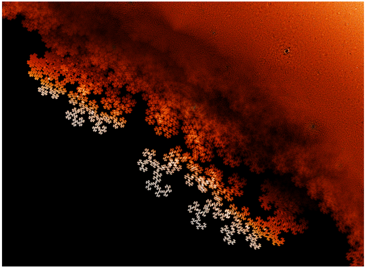


Image by Sam Derbyshire

Looks a lot like a dragon fractal!





Density Plots Reveal Analytic Properties

Density Plots Reveal Analytic Properties

- Fractal-like behavior.

Density Plots Reveal Analytic Properties

- Fractal-like behavior.
- (Greg Egan) Connections to the Julia-Mandelbrot correspondence.

Density Plots Reveal Analytic Properties

- Fractal-like behavior.
- (Greg Egan) Connections to the Julia-Mandelbrot correspondence.

On the other hand...

When you plot individual points sized by algebraic invariants (like the discriminant), you seem to get algebraic curves, like circles, parabolas and lines.

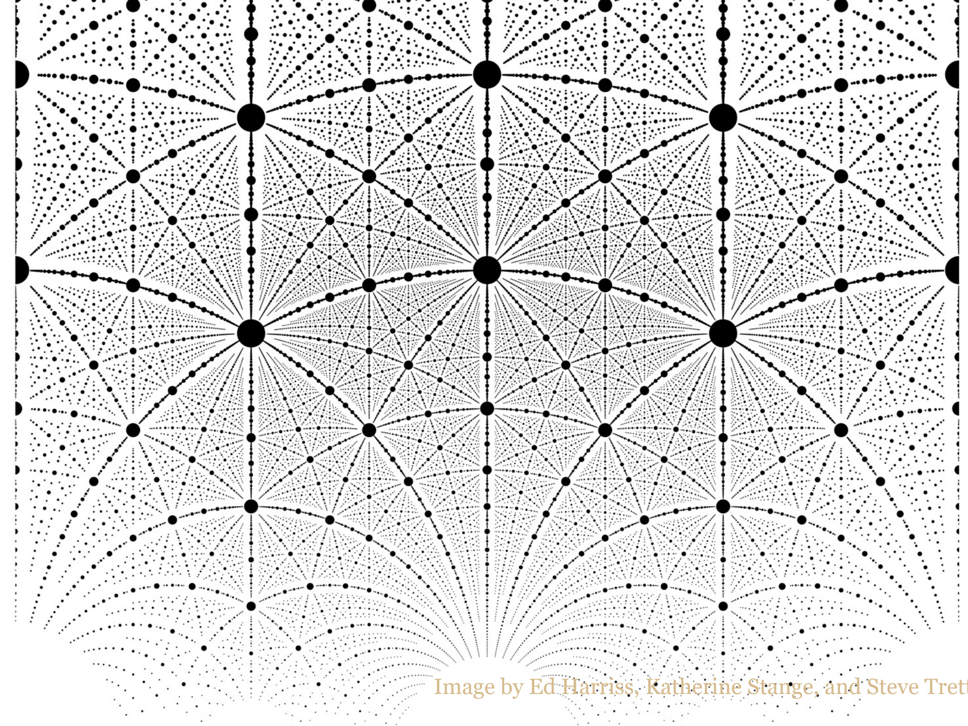


Image by Ed Harriss, Katherine Stange, and Steve Trett

Paradigm Shifts

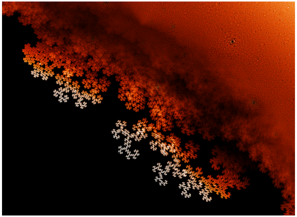
Paradigm Shifts

We saw two perspectives on drawing the same set of points.

Paradigm Shifts

We saw two perspectives on drawing the same set of points.

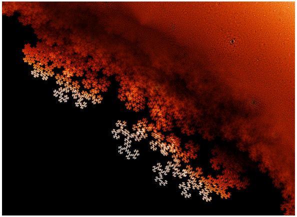
Density coloring revealed analytic properties.



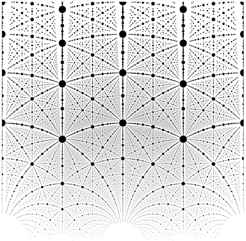
Paradigm Shifts

We saw two perspectives on drawing the same set of points.

Density coloring revealed analytic properties.



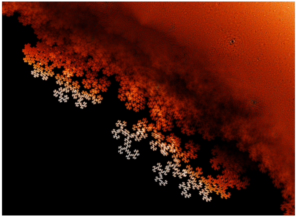
Plots sizing according to discriminant revealed algebraic properties.



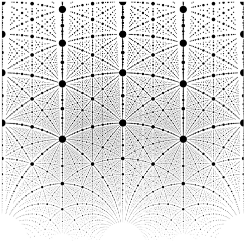
Paradigm Shifts

We saw two perspectives on drawing the same set of points.

Density coloring revealed analytic properties.



Plots sizing according to discriminant revealed algebraic properties.



Let’s explain the image on the right.

Rational Geodesics

Rational Geodesics

- What exactly are these curves?

Rational Geodesics

- What exactly are these curves?
- Why do they show up?

Rational Geodesics

- What exactly are these curves?
- Why do they show up?
- Harriss, Stange, and Trettel (a number theorist and two geometers) prove the following result inspired by their image.

Rational Geodesics

- What exactly are these curves?
- Why do they show up?
- Harriss, Stange, and Trettel (a number theorist and two geometers) prove the following result inspired by their image.

Definition

If $z = a + bi$ is a complex number, $\bar{z} = a - bi$ is its complex conjugate.

Rational Geodesics

- What exactly are these curves?
- Why do they show up?
- Harriss, Stange, and Trettel (a number theorist and two geometers) prove the following result inspired by their image.

Definition

If $z = a + bi$ is a complex number, $\bar{z} = a - bi$ is its complex conjugate.

Proposition [HST21]

The set of complex numbers $z \in \mathbb{C}$ such that $\{1, z + \bar{z}, z\bar{z}\}$ are linearly dependent over $\mathbb{Q}$

Rational Geodesics

- What exactly are these curves?
- Why do they show up?
- Harriss, Stange, and Trettel (a number theorist and two geometers) prove the following result inspired by their image.

Definition

If $z = a + bi$ is a complex number, $\bar{z} = a - bi$ is its complex conjugate.

Proposition [HST21]

The set of complex numbers $z \in \mathbb{C}$ such that $\{1, z + \bar{z}, z\bar{z}\}$ are linearly dependent over $\mathbb{Q}$ form a collection of vertical lines and circles of rational radius.

Rational Geodesics

- What exactly are these curves?
- Why do they show up?
- Harriss, Stange, and Trettel (a number theorist and two geometers) prove the following result inspired by their image.

Definition

If $z = a + bi$ is a complex number, $\bar{z} = a - bi$ is its complex conjugate.

Proposition [HST21]

The set of complex numbers $z \in \mathbb{C}$ such that $\{1, z + \bar{z}, z\bar{z}\}$ are linearly dependent over $\mathbb{Q}$ form a collection of vertical lines and circles of rational radius. Call these curves rational geodesics.

We can now explain their image.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

$$\bar{z} = \frac{-b - n\sqrt{\Delta_z}}{2}.$$

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

$$\bar{z} = \frac{-b - n\sqrt{\Delta_z}}{2}.$$

In either case, $1, z, \bar{z} \in \mathbb{Q}(\sqrt{\Delta_z}) = \{c + d\sqrt{\Delta_z} : c, d \in \mathbb{Q}\}$.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

$$\bar{z} = \frac{-b - n\sqrt{\Delta_z}}{2}.$$

In either case, $1, z, \bar{z} \in \mathbb{Q}(\sqrt{\Delta_z}) = \{c + d\sqrt{\Delta_z} : c, d \in \mathbb{Q}\}$. Therefore so are $z\bar{z}$ and $z + \bar{z}$.

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

$$\bar{z} = \frac{-b - n\sqrt{\Delta_z}}{2}.$$

In either case, $1, z, \bar{z} \in \mathbb{Q}(\sqrt{\Delta_z}) = \{c + d\sqrt{\Delta_z} : c, d \in \mathbb{Q}\}$. Therefore so are $z\bar{z}$ and $z + \bar{z}$. But $\mathbb{Q}(\sqrt{\Delta_z})$ is a 2-dimensional vector space—with basis $\{1, \sqrt{\Delta_z}\}$

We can now explain their image.

Proposition

If z is a quadratic algebraic integer, then z lies on a rational geodesic.

Proof.

Let Δ_z be the discriminant of z. So:

$$z = \frac{-b + n\sqrt{\Delta_z}}{2}.$$

If z is real, then $\bar{z} = z$. Otherwise:

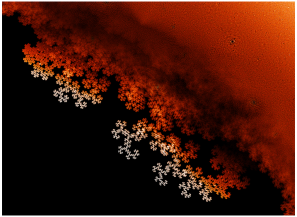
$$\bar{z} = \frac{-b - n\sqrt{\Delta_z}}{2}.$$

In either case, $1, z, \bar{z} \in \mathbb{Q}(\sqrt{\Delta_z}) = \{c + d\sqrt{\Delta_z} : c, d \in \mathbb{Q}\}$. Therefore so are $z\bar{z}$ and $z + \bar{z}$. But $\mathbb{Q}(\sqrt{\Delta_z})$ is a 2-dimensional vector space—with basis $\{1, \sqrt{\Delta_z}\}$—so any three elements must be linearly dependent. $\square$

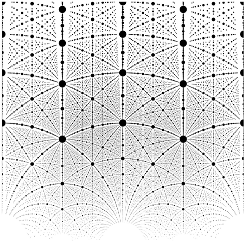
Paradigm Shifts

We saw two perspectives on drawing the same set of points.

Density coloring revealed analytic properties.



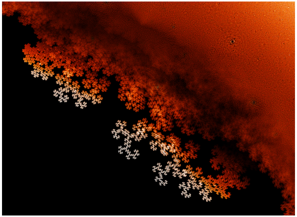
Plots sizing according to discriminant revealed algebraic properties.



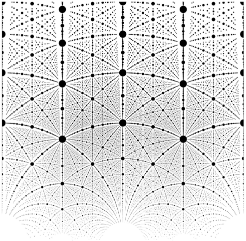
Paradigm Shifts

We saw two perspectives on drawing the same set of points.

Density coloring revealed analytic properties.



Plots sizing according to discriminant revealed algebraic properties.



In each case, the images inspired new research and new theorems!

An idea:

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special?

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns?

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups
- Ramification

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups
- Ramification
- Inertia

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups
- Ramification
- Inertia
- ...

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups
- Ramification
- Inertia
- ...

The field of algebraic number theory essentially studies how to understand and classify these invariants.

An idea:

Slogan: See interesting new patterns by increasing the size of certain special points.

What if, instead points with small discriminants, we used a different invariant to define special? This would emphasize a different set of points. Would there be new patterns? Algebraic integers have all sorts of fascinating invariants.

- Galois groups
- Ramification
- Inertia
- ...

The field of algebraic number theory essentially studies how to understand and classify these invariants. Let's focus on Galois theory.

Algebraic Symmetries and Galois Theory

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis.

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis. But actually it's even better. If $z, w \in \mathbb{C}$:

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis. But actually it's even better. If $z, w \in \mathbb{C}$:

$$\overline{z + w} = \bar{z} + \overline{w}$$

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis. But actually it's even better. If $z, w \in \mathbb{C}$:

$$\overline{z + w} = \bar{z} + \overline{w} \qquad \text{and} \qquad \overline{zw} = \bar{z} \cdot \overline{w}.$$

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis. But actually it's even better. If $z, w \in \mathbb{C}$:

$$\overline{z + w} = \bar{z} + \overline{w} \qquad \text{and} \qquad \overline{zw} = \bar{z} \cdot \overline{w}.$$

Complex conjugation is an algebraic symmetry, because it respects addition and multiplication.

Algebraic Symmetries and Galois Theory

Consider complex conjugation:

$$(z = a + bi) \mapsto (\bar{z} = a - bi).$$

This is a symmetry of the complex plane, reflecting it vertically over the real axis. But actually it's even better. If $z, w \in \mathbb{C}$:

$$\overline{z + w} = \bar{z} + \overline{w} \qquad \text{and} \qquad \overline{zw} = \bar{z} \cdot \overline{w}.$$

Complex conjugation is an algebraic symmetry, because it respects addition and multiplication. (You might also see this called an automorphism.)

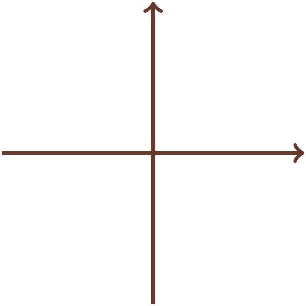
Algebraic Symmetries and Permutations

Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.

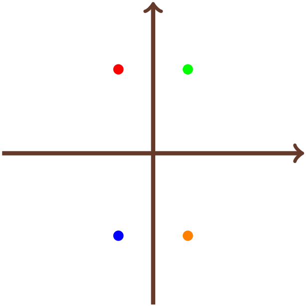
Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.



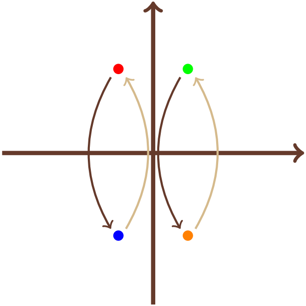
Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.



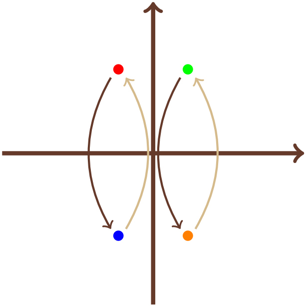
Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.

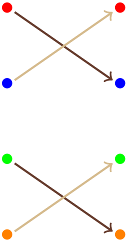


Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.

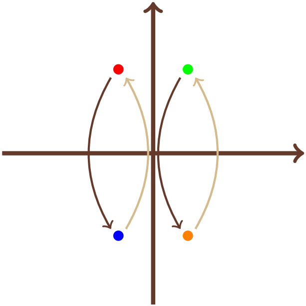


$\Rightarrow$

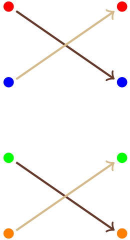


Algebraic Symmetries and Permutations

Let's consider again the polynomial $x^4 + 2x^2 + 2$, and consider complex conjugation.



$\Rightarrow$



Slogan: An algebraic symmetry gives rise to a permutation.

Does every permutation give rise to an algebraic symmetry?

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula).

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula).
Consider a permutation σ swapping ζ and ζ^2.

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula). Consider a permutation σ swapping ζ and ζ^2.

$$\sigma(\zeta^2)$$

$$\sigma(\zeta)^2$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula).
Consider a permutation σ swapping ζ and ζ^2.

$$\sigma(\zeta^2) = \zeta$$

$$\sigma(\zeta)^2$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula).
Consider a permutation σ swapping ζ and ζ^2.

$$\begin{aligned} \sigma(\zeta^2) &= \zeta \\ \sigma(\zeta)^2 &= (\zeta^2)^2 \end{aligned}$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula). Consider a permutation σ swapping ζ and ζ^2.

$$\sigma(\zeta^2) = \zeta$$

$$\sigma(\zeta)^2 = (\zeta^2)^2 = \zeta^4$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula).
Consider a permutation σ swapping ζ and ζ^2.

$$\begin{array}{ccccc} \sigma(\zeta^2) & = & \zeta & & \\ & & & \not\asymp & \\ \sigma(\zeta)^2 & = & (\zeta^2)^2 & = & \zeta^4 \end{array}$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula). Consider a permutation σ swapping ζ and ζ^2.

$$\begin{array}{ccccccc}
\sigma(\zeta^2) & = & \zeta & & & & \\
\rotatebox{90}{$\neq$} & & & \rotatebox{-30}{$\neq$} & & & \\
\sigma(\zeta)^2 & = & (\zeta^2)^2 & = & \zeta^4 & &
\end{array}$$

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula). Consider a permutation σ swapping ζ and ζ^2.

$$\begin{array}{ccccccc}
\sigma(\zeta^2) & = & \zeta & & & & \\
\nparallel & & & & \not\searrow & & \\
\sigma(\zeta)^2 & = & (\zeta^2)^2 & = & & & \zeta^4
\end{array}$$

σ does not respect multiplication, and is therefore not algebraic.

Does every permutation give rise to an algebraic symmetry?

- (Gallagher 1973) Almost always: yes!
- In general: no!

Example

The roots of $x^5 - 1$ are $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{\frac{2\pi i}{5}}$. (By Euler's formula). Consider a permutation σ swapping ζ and ζ^2.

$$\begin{array}{ccccc} \sigma(\zeta^2) & = & \zeta & & \\ \nparallel & & & \times & \\ \sigma(\zeta)^2 & = & (\zeta^2)^2 & = & \zeta^4 \end{array}$$

σ does not respect multiplication, and is therefore not algebraic. Since not every permutation of the roots of $x^5 - 1$ is algebraic, we will call the roots rigid.

Galois Theory

Galois Theory

Translating what we've said to the language of Galois theory:

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.
- Our slogan says $G \leq S_n$.

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.
- Our slogan says $G \leq S_n$.
- Gallagher's result says usually $G = S_n$.

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.
- Our slogan says $G \leq S_n$.
- Gallagher's result says usually $G = S_n$.
- Example shows sometimes $G \lneq S_n$. (This is the rigid case).

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.
- Our slogan says $G \leq S_n$.
- Gallagher's result says usually $G = S_n$.
- Example shows sometimes $G \lneq S_n$. (This is the rigid case).

Question

Do rigid algebraic integers satisfy interesting geometric relationships?

Galois Theory

Translating what we've said to the language of Galois theory:

- Group of algebraic symmetries of roots $\Leftrightarrow$ Galois Group G.
- Group of all permutations of roots $\Leftrightarrow$ Symmetric Group S_n.
- Our slogan says $G \leq S_n$.
- Gallagher's result says usually $G = S_n$.
- Example shows sometimes $G \lneq S_n$. (This is the rigid case).

Question

Do rigid algebraic integers satisfy interesting geometric relationships? To explore this we introduce an invariant which measures how far G is from S_n, called rigidity.

Rigidity

Rigidity

Rigidity measures how far G is from being all of S_n.

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Definition

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Definition

Let z be an algebraic integer of degree n, and let G be its Galois group.

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Definition

Let z be an algebraic integer of degree n, and let G be its Galois group. The rigidity of z is:

$$\mathrm{rig}(z) := \frac{n! - \#G}{n!}.$$

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Definition

Let z be an algebraic integer of degree n, and let G be its Galois group. The rigidity of z is:

$$\mathrm{rig}(z) := \frac{n! - \#G}{n!}.$$

If $\mathrm{rig}(z) > 0$ then z is rigid.

Rigidity

Rigidity measures how far G is from being all of S_n. You might see in a class on group theory that $\#S_n = n!$.

Definition

Let z be an algebraic integer of degree n, and let G be its Galois group. The rigidity of z is:

$$\mathrm{rig}(z) := \frac{n! - \#G}{n!}.$$

If $\mathrm{rig}(z) > 0$ then z is rigid.

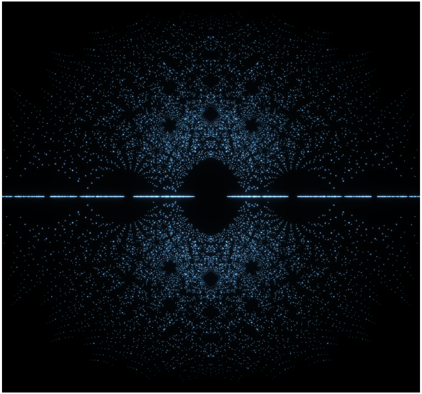
Notice that if $G = S_n$ then $\#G = n!$ so that $\mathrm{rig}(z) = 0$.

Let's plot algebraic integers again,

Let’s plot algebraic integers again, but now increasing the size of rigid points.

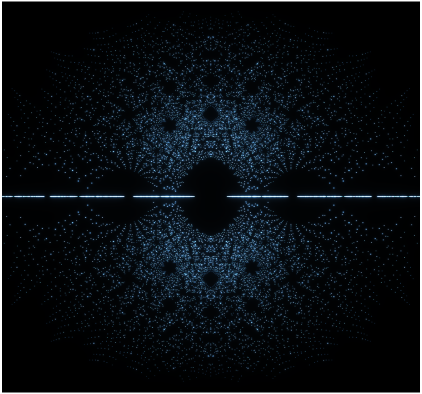
Let’s plot algebraic integers again, but now increasing the size of rigid points.

Quartics sized by discriminant.

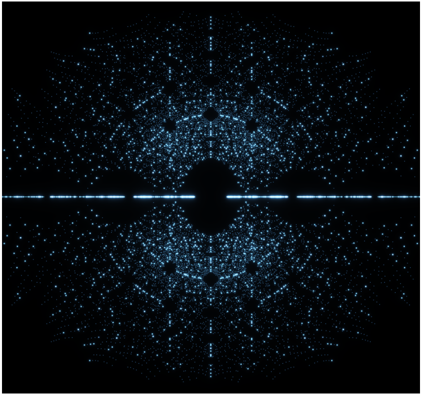


Let’s plot algebraic integers again, but now increasing the size of rigid points.

Quartics sized by discriminant.



Quartics sized by rigidity.

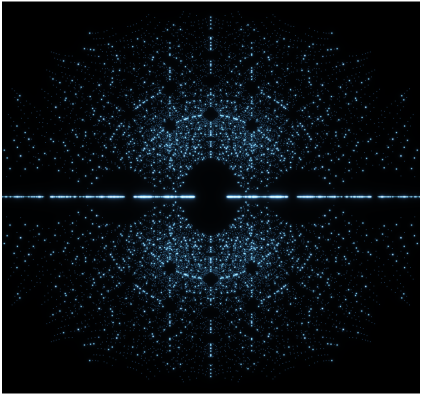


Clearly hidden geometry appears…

Clearly hidden geometry appears... but wait a minute, this looks a bit familiar.

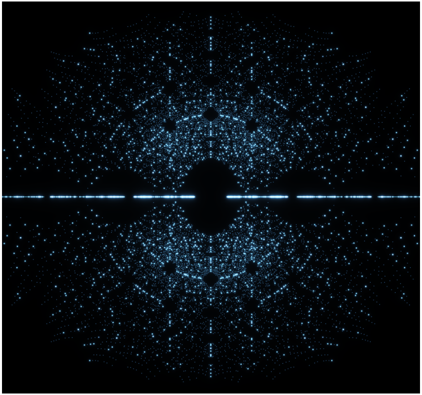
Clearly hidden geometry appears... but wait a minute, this looks a bit familiar.

Are these circles and lines...

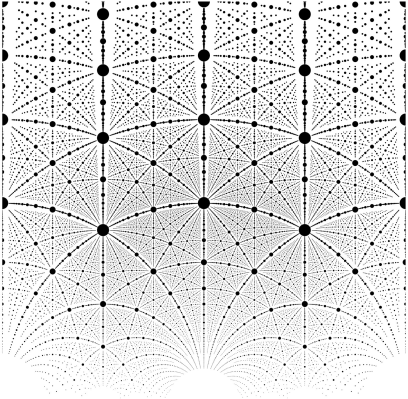


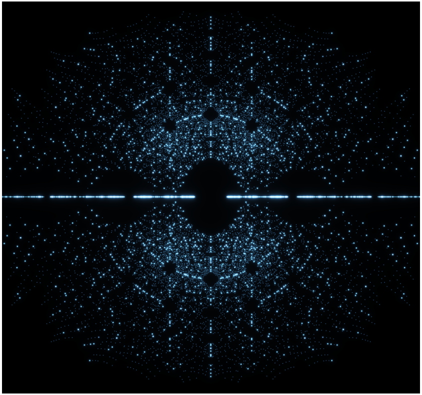
Clearly hidden geometry appears... but wait a minute, this looks a bit familiar.

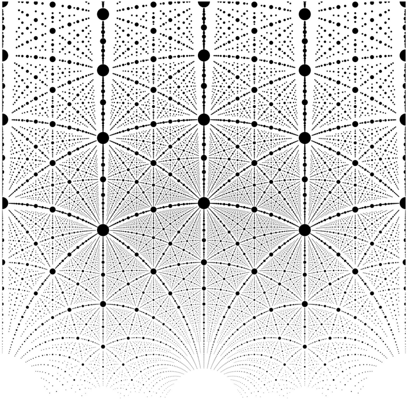
Are these circles and lines...

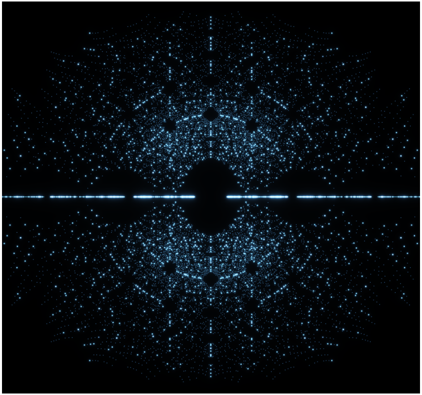


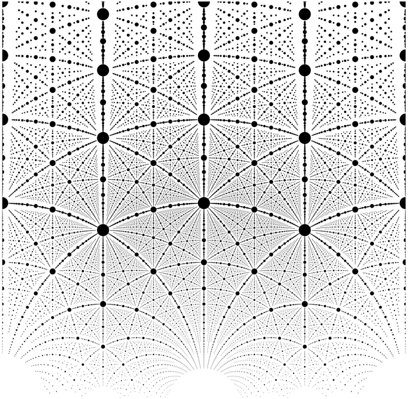
...the same as these ones?



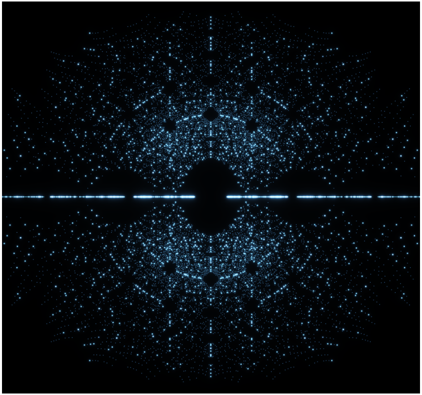




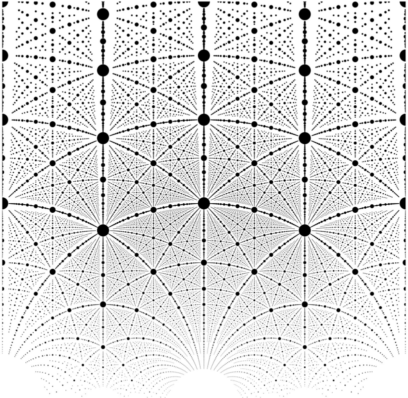




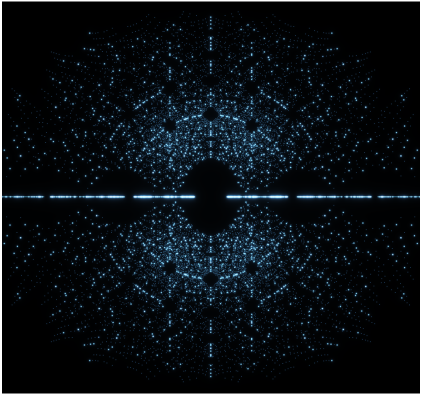
- Quartics (degree 4 polynomials).



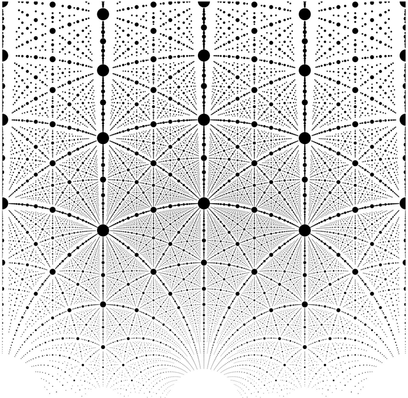
- Quartics (degree 4 polynomials).



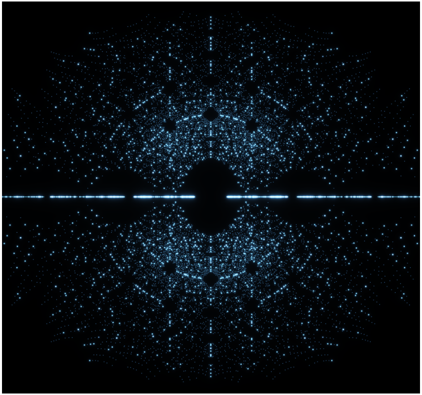
- Quadratics (degree 2 polynomials).



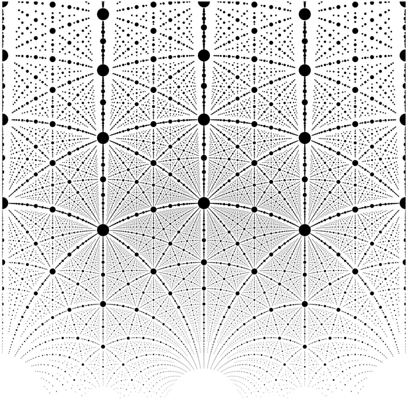
- Quartics (degree 4 polynomials).
- Uses Galois theory.



- Quadratics (degree 2 polynomials).



- Quartics (degree 4 polynomials).
- Uses Galois theory.



- Quadratics (degree 2 polynomials).
- Ignores Galois theory.

And yet…

And yet...

Theorem (DH, Xu)

And yet…

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid*

And yet...

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.

And yet...

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.
- *If a complex quartic* $z \in \mathbb{C} \setminus \mathbb{R}$

And yet…

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.
- *If a complex quartic* $z \in \mathbb{C} \setminus \mathbb{R}$ *has rigidity* $\frac{5}{6}$,

And yet...

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.
- *If a complex quartic* $z \in \mathbb{C} \setminus \mathbb{R}$ *has rigidity* $\frac{5}{6}$, *it lies on a rational geodesic.*

And yet...

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.
- *If a complex quartic* $z \in \mathbb{C} \setminus \mathbb{R}$ *has rigidity* $\frac{5}{6}$, *it lies on a rational geodesic.*

This result explains the hidden patterns in the image of the rigid quartics, and was completely inspired by that image.

And yet...

Theorem (DH, Xu)

- *If a quartic lies on a rational geodesic, it is rigid of rigidity* $\frac{2}{3}$ *or* $\frac{5}{6}$.
- *If a complex quartic* $z \in \mathbb{C} \setminus \mathbb{R}$ *has rigidity* $\frac{5}{6}$, *it lies on a rational geodesic.*

This result explains the hidden patterns in the image of the rigid quartics, and was completely inspired by that image. It also exhibits a previously hidden geometric connection between quadratics and rigid quartics.

Conclusion

Slogan

Aesthetic choices we make when we illustrate mathematics can uncover mathematical structures that could otherwise be hidden.

Conclusion

Slogan

Aesthetic choices we make when we illustrate mathematics can uncover mathematical structures that could otherwise be hidden.

- Draw starscapes with analytic measurements (e.g. density): get analytic results.

Conclusion

Slogan

Aesthetic choices we make when we illustrate mathematics can uncover mathematical structures that could otherwise be hidden.

- Draw starscapes with analytic measurements (e.g. density): get analytic results.
- Draw starscapes with arithmetic properties: get arithmetic results.

Conclusion

Question

What aesthetic choices are you making when illustrating a mathematical concept?

Conclusion

Question

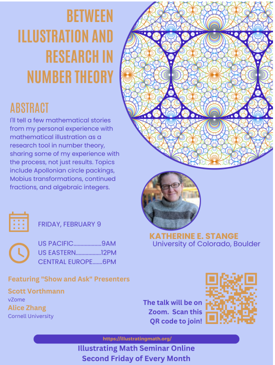
What aesthetic choices are you making when illustrating a mathematical concept?

Suggestion

Be intentional about your aesthetic choices, and try to harness them to gain new mathematical perspectives.

A plug!

Come to the Illustrating Math Seminar Online!



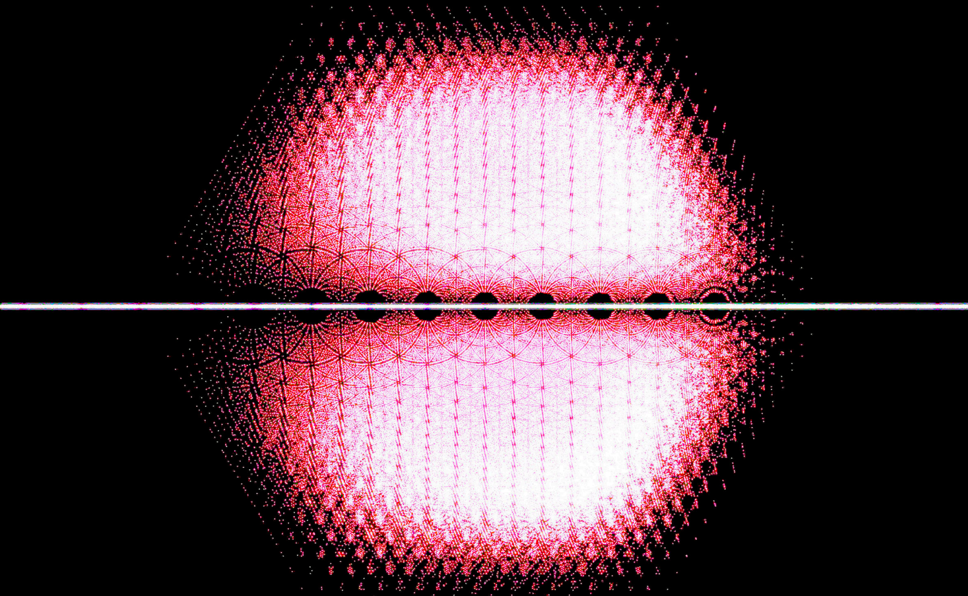
Another plug!

Join the illustrating math Discord channel.

Yet another plug!

Bohemian Eigenvalue Starscapes

One can also build algebraic starscapes from eigenvalues of integer matrices, and search for connections to linear algebra! Eliza Brown will present a poster on the subject at today's poster session. Go check it out!



Thank You!