Homework Assignment 4

Due Friday, February 18

1. In this exercise we study products of finite cyclic groups. Recall that we denote by Z_n the cyclic group of order n (written multiplicatively).

   (a) Prove that $Z_2 \times Z_2$ is not a cyclic group.

   (b) Prove that $Z_2 \times Z_3 \cong Z_6$. Conclude that $Z_2 \times Z_3$ is a cyclic group.

   Those two examples really cover all the bases. Use the intuition you gained from them to prove the following classification result.

   (c) Show that $Z_n \times Z_m$ is cyclic if and only if $\gcd(n, m) = 1$. (Hint: recall that up to isomorphism there is only one cyclic group of order N for every positive integer N).

2. Let G be a group and H a *nonempty* subset of G. Let's introduce a few tricks to speed up testing if something is a subgroup.

   (a) *(Subgroup Criterion)* Suppose that for all $x, y \in H$, $xy^{-1} \in H$. Show that H is a subgroup of G.

   (b) *(Finite Subgroup Criterion)* Show that if H is finite and closed under multiplication, then H is a subgroup of G.

3. Let G be a group. Let $H, K \leq G$ be two subgroups.

   (a) Show that the intersection $H \cap K$ is a subgroup of G.

   (b) Give an example to show that the union $H \cup K$ need not be a subgroup of G.

   (c) Show that $H \cup K$ is a subgroup of G if and only if $H \subset K$ or $K \subset H$.

   (d) Adjust your proof from part (a) to show that the intersection of an arbitrary collection of subgroups is a subgroup. That is, let $\mathcal{A}$ be a collection of subgroups of G. Show that

   $$\bigcap_{H \in \mathcal{A}} H$$

   is a subgroup of G. This completes the proof that the subgroup generated by a subset is in fact a subgroup.

   Hint. *For part (d), the proof should be very similar to part (a), with only cosmetic modifications. You won't need to use induction. In fact, since $\mathcal{A}$ is could in principle be uncountable, induction won't work without modifications (think about why this is).*

4. Given a homomorphism $\varphi : G \to H$, we obtain 2 important subgroups, one of G and one of H. They are called the *kernel of* φ and *image of* φ and are defined by the following rules:

   $$\begin{aligned} \ker \varphi &= \{g \in G : \varphi(g) = 1_H\}, \\ \operatorname{im} \varphi &= \{h \in H : h = \varphi(g) \text{ for some } g \in G\}. \end{aligned}$$

   (a) Show that $\ker \varphi$ is a subgroup of G.

   (b) Show that $\operatorname{im} \varphi$ is a subgroup of H.

   (c) *Important:* Show that φ is injective if and only if $\ker \varphi = \{1_G\}$. (This is an incredibly useful fact!)

5. The kernel has the following important generalization. For $h \in H$ define the *fiber over h* as

$$\varphi^{-1}(h) = \{g \in G : \varphi(g) = h\}.$$

This is sometimes also called the *preimage of h*. Observe that by definition, the kernel of φ is the fiber over 1.

   (a) Show that the fiber over h is a subgroup if and only if $h = 1_H$.

   (b) Show that the *nonempty* fibers of φ form a partition of G. (In particular, if φ is surjective its fibers partition G.)

   (c) Show that all nonempty fibers have the same cardinality. (Hint: if $\varphi^{-1}(h)$ is nonempty, build a bijection between it and $\ker \varphi$.) Observe that this generalizes 2(c).

6. Let G be a group and A a set, and suppose we are given homomorphism $\varphi : G \to S_A$. Show that the rule:

$$g \cdot a = \varphi(g)(a) \text{ for all } g \in G \text{ and } a \in A,$$

describes a group action of G on A, and further that the permutation representation of this action is φ itself.

7. Let G be a group acting on a set A. For an element $a \in A$, we define the *stabilizer* of a to be the collection of elements of G that act trivially on a, that is:

$$G_a := \{g \in G : g \cdot a = a\}.$$

The *kernel* of the group action is the collection of elements of G that act trivially on *all of A*, that is:

$$G_0 := \{g \in G : g \cdot a = a \text{ for all } a \in A\}.$$

   (a) Prove that G_a and G_0 are subgroups of G.

   (b) Prove that G_0 is equal to the kernel of the permutation representation associated to the action of G on A. (cf. Problem 4: This justifies the naming convention).

8. For $n \geq 2$ let $G = S_n$ be the symmetric group equipped with it's natural action on $\Omega_n = \{1, 2, \cdots, n\}$ by permutations. For $i \in \Omega_n$, let $G_i = \{\sigma \in G | \sigma(i) = i\}$ be the stabilizer of i. Describe an isomorphism between G_i and S_{n-1}.