Math 120 KAUST | Midterm 2 (80 minutes) | November 16, 2016

Name: Key

Directions:

- You have 80 minutes to complete this exam.
- No graphing calculators are allowed.
- You are allowed one hand-written sheet (two sided is ok) of notes on regular 8.5-11 paper.
- You must show ALL your work.
- Leave answers in EXACT FORM or record up to 2 DECIMAL PLACES.
- If you have any questions, raise your hand.

Question	Points	Score
1	15	
2	15	
3	10	
4	20	
5	10	
Total:	70	

1. Scientists build an enormous cooling machine. It will cool whatever object put into it. It gets more and more difficult to remove heat (kinetic energy) from a system, and so the cooling machine has diminishing returns as a function of time, and of course, no object can ever be completely cooled to absolute zero ($-273.15°$ Celsius). Therefore, temperature as a function of time can be modelled as a linear to linear rational function.

   (a) (5 points) An object is put into the machine at 100°C. After 12 hours, it has cooled all the way to -35°C. Write a function $T(x)$ which returns the temperature of the object after x hours. Specify the domain and range of the function.

2 points

① $(0, 100)$

② $(12, -35)$

Horizontal Asymptote

③ $y = -273.15$

$$T(x) = \frac{ax + b}{x + d}$$

Note:

③ $\Rightarrow a = -273.15$

① : $\frac{a \cdot 0 + b}{0 + d} = 100 \Longrightarrow b = 100d$

② : $\frac{a \cdot 12 + b}{12 + d} = -35 \Longrightarrow -3277.8 + b = -420 - 35d$

Plug ① into ②.

$135d = 2857.8$

$d \approx 21.17$

$b = 100d = 2117$

So

$$T(x) = \frac{-273.15x + 2117}{x + 21.17}$$

(b) (5 points) Write a function, $H(x)$, which will return the number of hours it takes the machine to cool the object to a temperature of x degrees Celsius. What is the domain and range of H?

Find an inverse

$$y = \frac{-273.15x + 2117}{x + 21.17}$$

Solve for x:

$$xy + 21.17y = -273.15x + 2117$$

$$x(y + 273.15) = -21.17y + 2117$$

$$H(y) = x = \frac{-21.17y + 2117}{y + 273.15}$$

Domain(H)

$= $ Range(T)

$= \{-273.15 < y \leq 100\}$

Range(H)

$=$ Domain(T)

$= \{x \geq 0\}$

(c) (5 points) How many hours will it take to cool the object to within 3 degrees Celsius of absolute zero?

$$H(-270.15) = 2612.03$$

2. Recall the following measure of exponential decay, a half life is the amount of time it takes for a quantity to reduce to half its value. There was an accident at a laboratory at the university of Washington. The entire physics building became irradiated, leaking 200 tons of radiactive material throughout the building. Suppose that the half life of the radioactive material is 36 hours.

(a) (5 points) Write a function, $R(t)$, which returns the weight of radioactive material present after t hours. State the domain and range.

2 points

① $R(0) = 200$

② $R(36) = 100$

$R(t) = A_0 b^t$

① $\Rightarrow A_0 = 200$

② $= 100 = 200 b^{36}$

$b = \left(\frac{1}{2}\right)^{1/36} \approx 0.991$

$R(t) = 200(.991)^t$

$= 200\left(\sqrt[36]{1/2}\right)^t = 200\left(\frac{1}{2}\right)^{t/36}$

(b) (5 points) Write an inverse for the function $R(t)$, state the domain and range, and briefly explain its meaning.

$y = 200\left(\frac{1}{2}\right)^{t/36}$

$\ln y = \ln 200 + \frac{t}{36}\ln\left(\frac{1}{2}\right)$

$t = 36\,\frac{\ln y - \ln 200}{\ln(1/2)}$

$= \frac{36}{\ln(1/2)} \cdot \ln\left(\frac{y}{200}\right)$

input: how long until...

output: we have y tons left.

Domain $0 < y \leq 200$

Range $t \geq 0$

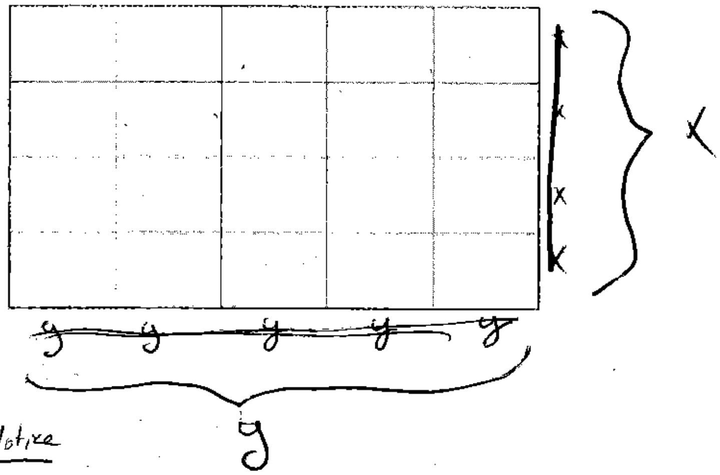
(c) (5 points) The building will not be deemed safe until there is less than 1/100 ounces of radioactive material remaining. How long with this take? (Note, 16 ounces = 1 pound, and 2000 pounds = 1 ton.)

$\frac{1}{100}$ oz $= 3.125 \times 10^{-7}$

$R^{-1}(3.125 \times 10^{-7}) = \frac{36}{\ln(1/2)} \cdot \ln\left(\frac{3.125 \times 10^{-7}}{200}\right)$

$= 1053.13$ hours

3. (10 points) A farmer is creating an enclosure for her animals. She wishes to separate the enclosure into 4 rows and 5 columns, as shown in the picture. Suppose she has 1000 ft of fencing. What is the maximum total area she can enclose?



Notice

$$6x + 5y = 1000$$

so $y = 200 - \frac{6}{5}x$

maximize $xy = x\left(200 - \frac{6}{5}x\right) = \frac{-6}{5}x^2 + 200x$

Max @ $x = \frac{-b}{2a} = \frac{-200}{-12/5} = \frac{1000}{12} = \frac{250}{3}$

Then $y = 200 - \frac{\cancel{6}^{2}}{\cancel{5}}\left(\frac{\cancel{250}^{50}}{\cancel{3}}\right) = 200 - 100 = 100$

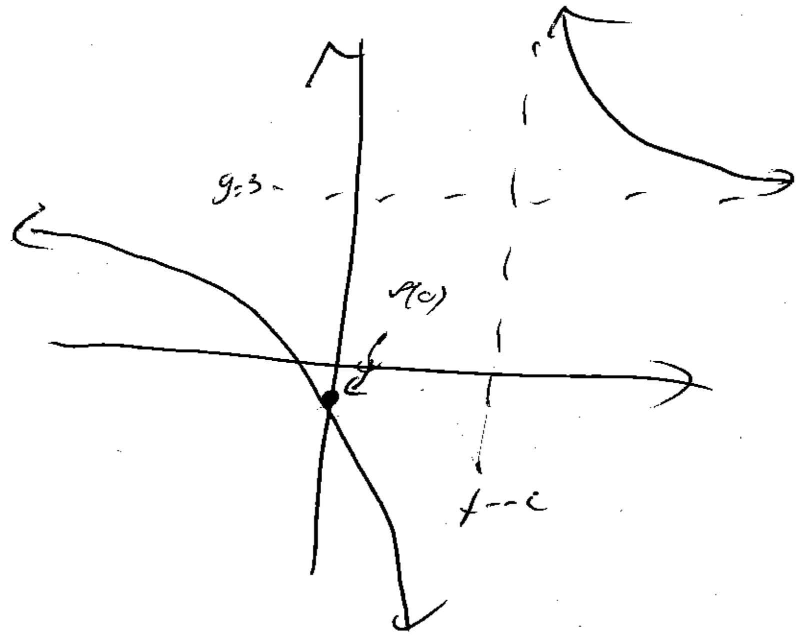
Max area $= \frac{250}{3} \cdot 100 = \frac{25000}{3}$

4. (a) (5 points) Let $f(x) = \frac{6x+5}{2x-4}$. Sketch a graph of $y = f(x)$, labeling horizontal and vertical asymptotes, as well as the point you used to determine orientation.

V.A. $x = 2$

H.A $y = 3$

point $f(0) = \frac{-5}{4}$



(b) (5 points) Invert $f(x)$. Write the algebraic rule and sketch a graph, labeling horizontal and vertical asymptotoes, as well as the point used to determine orientation.

$$y = \frac{6x+5}{2x-4}$$

$$2xy - 4y = 6x + 5$$

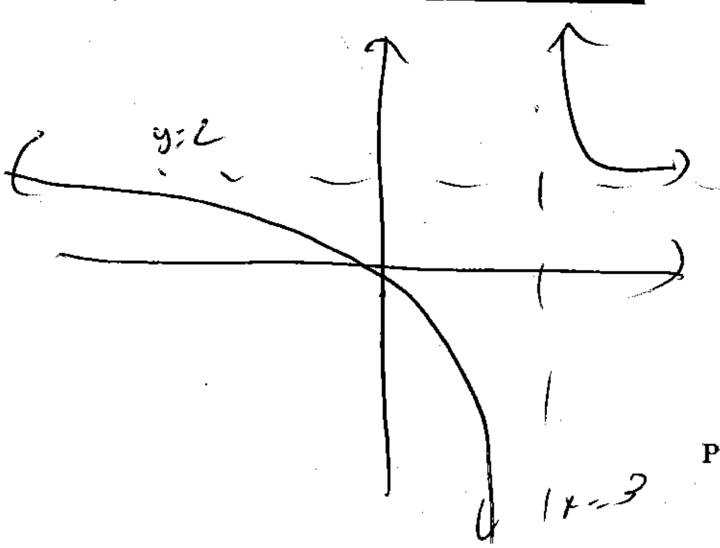
$$x(2y-6) = 4y + 5$$

$$x = \frac{4y+5}{2y-6}$$

V.A. : $x = 3$

H.A : $y = 2$

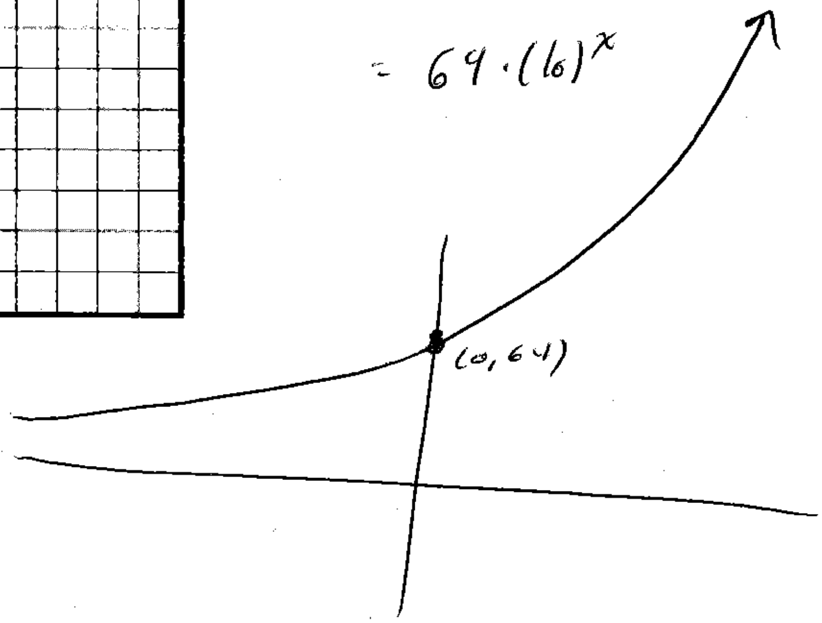
$f^{-1}(0) = \frac{-5}{6}$



(c) (5 points) Let $g(x) = 4^{2x+3}$. Convert g into standard exponential form and sketch a graph of $y = g(x)$, labeling the y intercept.

$g(x) = (4^2)^x \cdot 4^3$

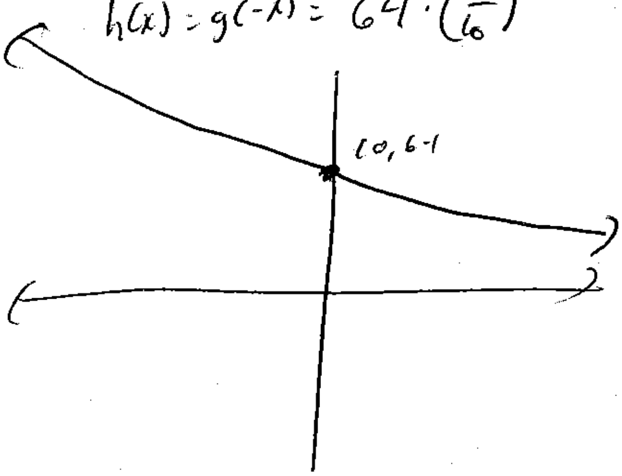
$= 64 \cdot (16)^x$



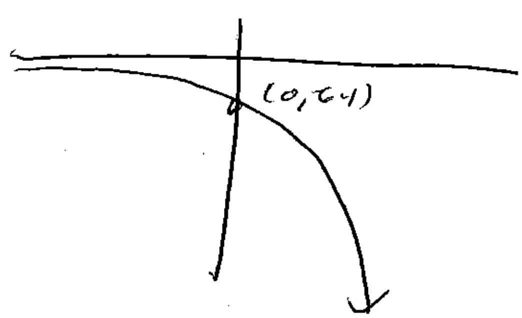
(d) (5 points) Reflect $g(x)$ over the x and y axes. Write the algebraic formulas and give sketches of both, labeling the y intercept in each case.

over x

$h(x) = g(-x) = 64 \cdot \left(\frac{1}{16}\right)^x$

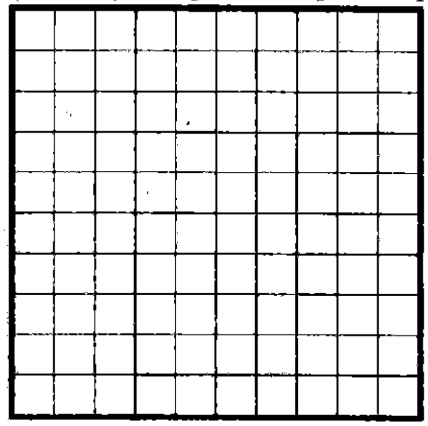


$j(x) = -g(x) = -64(16)^x$



5. Let $f(x) = 2x^2 - 12x + 23$.

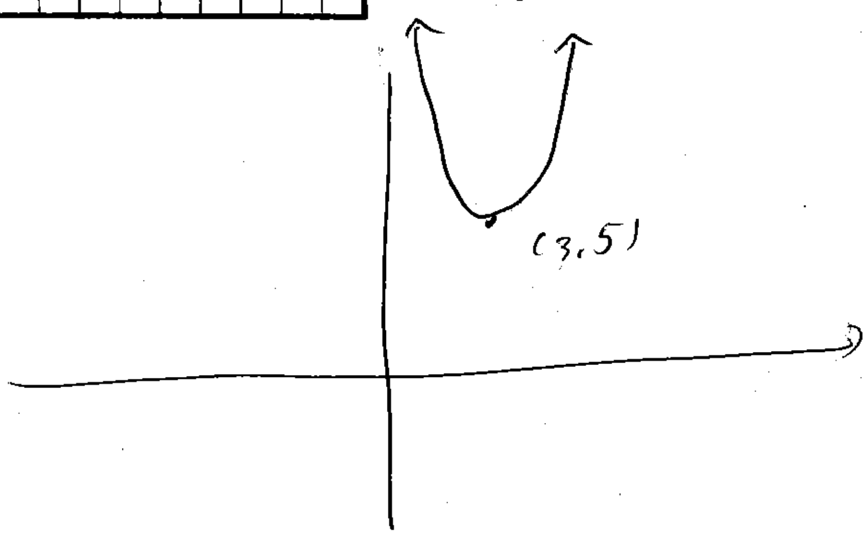
(a) (5 points) Complete the square to put f in vertex form. Then sketch f.



$$h = \frac{-b}{2a} = \frac{12}{4} = 3$$

$$k = f(3) = 18 - 36 + 23 = 5$$

$$f(x) = 2(x-3)^2 + 5$$



(b) (5 points) Restrict the domain of f so that it is one to one. Then compute an inverse. (Note, there are multiple correct answers to this question.)

Suppos $x \geq 3$

$$y = 2(x-3)^2 + 5$$

$$\frac{y-5}{2} = (x-3)^2$$

$$x - 3 = +\sqrt{\frac{y-5}{2}}$$

$$x = \sqrt{\frac{y-5}{2}} + 3$$