Homework Assignment 1

Due: Friday, January 31

1. Let S be a set with 3 elements (say {0,1,2}) and T be a set with 5 elements (say {0,1,2,3,4}).
   (a) Give an example of an injection $f : S \to T$.
   (b) Give an example of a surjection $g : T \to S$.
   (c) Can there be a bijection between S and T? Why or why not?

2. Give an example of a set S and a bijection from S to a *proper* subset of S.

3. Let S and T be two sets, and $f : S \to T$ a function between them.
   (a) Show that f is bijective *if and only if* there exists a function $g : T \to S$ so that $g \circ f = \mathrm{id}_S$ and $f \circ g = \mathrm{id}_T$.
   (b) The function g constructed above is called the *inverse* of f and is sometimes denoted f^{-1}. Show that this terminology is justified by proving that g is *unique*. That is, show that if some other h served as an inverse for f then g.

4. Show that equivalence relations are partitions are equivalent. Explicitly, let S be a set, construct a natural bijection between the partitions on S and the equivalence relations on S in the following way.
   (a) Let $\sim$ be an equivalence relation. Show that the equivalence classes of $\sim$ form a partition of S.
   (b) Conversely, let X_i be a partition of S. Show that the relation $\sim$ given by the rule

   $$x \sim y \text{ if } x, y \in X_i \text{ for the same } i$$

   is an equivalence relation for S.

5. Let d be the greatest common divisor of 792 and 275. Using Euclid's algorithm, find d and write $d = 792x + 275y$ for some x and y.

6. Fix a nonzero integer $m \in \mathbb{Z}$. Show that congruence modulo m forms an equivalence relation on $\mathbb{Z}$.

7. Let a and b be integers. Show that $a^2 + b^2$ does not have a remainder of 3 when divided by four. (Hint: First show that the squares of elements in $\mathbb{Z}/4\mathbb{Z}$ are just $\overline{0}$ and $\overline{1}$.)

8. Let p be a prime number. Show that the product of two nonzero elements in $\mathbb{Z}/p\mathbb{Z}$ is again nonzero.