Theorem 1

G fin ab. gp. $\Longrightarrow$ $G \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$ for unique $(n_1, n_2, \ldots, n_s)$ w/ $n_i \geq 2$ & $n_{i+1} | n_i$

Lemma A (Recognition)

$H, K \trianglelefteq G$. $H \cap K = 1$ $\Rightarrow HK \simeq H \times K$.

First do existence

Prop 1: A finite ab. gp. is direct prod of cyclic group.

Question 1

(Reduce to p-groups)

G fin. ab. gp.

a) Sylow subgps of G are unique.

Pf/ $P \in Syl_p G \Rightarrow P \trianglelefteq G$ (ab.)

$Q \in Syl_p G$

$\Rightarrow Q = gPg^{-1} = P$ (Syl, normal) $\varnothing$

b) $|G| = p^\alpha q^\beta$. $P \in Syl_p$. $Q \in Syl_q$

$G \simeq P \times Q$.

Pf/ $|P \cap Q|$ divide $|P| = p^\alpha$ & $|Q| = q^\beta$

$\Rightarrow P \cap Q = 1$.

$P, Q \trianglelefteq G$ b/c G abel.

$\Rightarrow PQ \simeq P \times Q$. (need $= G$.)

$|PQ| = \frac{|P|\cdot|Q|}{|P \cap Q|} = \frac{p^\alpha q^\beta}{1} = |G|$

$\Rightarrow PQ = G$. $\varnothing$

(c) $|G| = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$

$P_i \in Syl_{p_i}$

$\Rightarrow G \simeq P_1 \times \cdots \times P_t$.

Pf/ Induction

$H_i = P_1 P_2 \cdots P_i \trianglelefteq G$.

Show $H_i \simeq P_1 \times \cdots \times P_i$

Iduct on i

$i=1$ trivial.

$i=2$ is (b)

General case

$H_i = \underbrace{P_1 \cdots P_{i-1}}_{H_{i-1}} P_i$

$\Rightarrow$ induction $H_{i-1} = P_1 \times \cdots \times P_{i-1}$

If $H_{i-1} \cap P_i = 1$ *

Lemma A

$\Rightarrow H_{i-1} P_i = H_{i-1} \times P_i = P_1 \times \cdots \times P_{i-1} \times P_i$

* $|H_{i-1}| = |P_1 \times \cdots \times P_{i-1}| = |P_1| \times \cdots \times |P_{i-1}| = p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}}$

prime to $p_i^{\alpha_i}$

$\gcd(|H_{i-1}|, |P_i|) = 1$

$\Rightarrow H_{i-1} \cap P_i = 1$. $\varnothing$

$H_t = P_1 \cdots P_t = P_1 \times \cdots \times P_t$

this is G

b/c $|H_t| = |P_1| \cdots |P_t| = p_1^{\alpha_1} \cdots p_t^{\alpha_t} = |G|$. $\varnothing$

(d) Prop 1 $P_i \Rightarrow$ Prop 1 G.

Pf/ $P_i \simeq \mathbb{Z}_{i_1} \times \cdots \times \mathbb{Z}_{i_{s_i}}$

$G = P_1 \times \cdots \times P_t = (\mathbb{Z}_{1_1} \times \cdots \times \mathbb{Z}_{1_{s_1}}) \times \cdots \times (\mathbb{Z}_{t_1} \times \cdots \times \mathbb{Z}_{t_{s_t}})$

$\varnothing$

Reduced prop 1 to

Prop 2: A abelian p-group $\Rightarrow$ A product of cyclic group.

Question 2

A abelian p-group

$\phi: A \longrightarrow A$, $a \longmapsto a^p$

a) ϕ is a hom.

Pf/ A abelian.

$\phi(xy) = (xy)^p = \underbrace{xyxy \cdots xy}_{p \text{ times}} \overset{ab}{=} x^p y^p = \phi(x)\phi(y)$. $\varnothing$

(b) $A_p = \ker\phi = \{a \mid a^p = 1\}$

A_p elem ab p. gp.

Def^n A elem ab. p gp if i) $|A| = p^\alpha$ ← p gp.

ii) Abelian.

iii) $\forall x \in A$ $|x| = \{^1_p$

Pf/ A_p ab ✓

p gp ✓ L.g.

$\forall x \in A_p$ $x^p = 1 \Rightarrow |x| \leq p$ $\varnothing$

c) $A^p = \operatorname{im}\phi = \{a^p \mid a \in A\}$

$A/A^p \simeq A_p$

Pf/ A/A^p elem ab. p gp. ✓

1) Abel ✓

2) p gp ✓ Lagrange

3) $\bar{x} \in A/A^p$ $(x \in A)$

$\Rightarrow \bar{x}^p = \overline{x^p}$ but $x^p \in A^p$ $= \bar{1}$ $|\bar{x}| \leq p$

Claim $|A/A^p| = |A_p|$

Pf/ First isom thm

$A^p = \operatorname{im}\phi = A/\ker\phi = A/A_p$

$|A^p| = |A/A_p| = |A|/|A_p|$

$|A_p| = |A|/|A^p| = |A/A^p| = p^\beta$

$\Rightarrow A_p \cong \overbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}^{\beta}$

$\cong A/A^p$ $\varnothing$

(d) $|A^p| \lneq |A|$

Pf/ Goal $A/A^p \neq 1$. ($\cong A_p$)

Cauchy $\Rightarrow \exists\, x \in A$ w/ $|x| = p$

$\Rightarrow 1 \neq x \in A_p$ $\varnothing$

Question 3

Prove prop 2 on $|A| = p^\alpha$ by ind on $|A|$ or α.

(a) Base case $\alpha = 1$

Pf/ $|A| = p \Rightarrow A = \mathbb{Z}_p$ $\varnothing$

(b) Case A^p.

Pf/ $|A^p| < |A|$!

$\Rightarrow A^p = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle$ (ind)

(c) $A^p \cap A_p$. elem ab of order p^t.

Observe

$A^p \cap A_p = p$ torsion of A^p.

$= \{x \in A^p \mid x^p = 1\}$

$= (A^p)_p$.

Know (b) A^p is product of cyclic gps.

(c) understand p-torsion of product of cyclic groups.

Lemma B (p-torsion in cyclic gps)

$G = \langle x \rangle$ cyclic. $|x| = p^\ell$.

$\Rightarrow G_p = \langle x^{p^{\ell-1}} \rangle$.

Proof of lemma B

$G_p \leq G$ cyclic of order p. (subs of cyclic.)

$G_p \simeq \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (2 (b))

$\Rightarrow \mathbb{Z}_p$ or 1.

Notice $(x^{p^{\ell-1}})^p = x^{p^\ell} = 1$

$\Rightarrow x^{p^{\ell-1}} \in G_p$

$= \langle x^{p^{\ell-1}} \rangle \leq G_p \simeq \mathbb{Z}_p$

($\cong \mathbb{Z}_p$) $\varnothing$

Lemma C

$G \simeq G_1 \times \cdots \times G_n$

$G_p \simeq (G_1)_p \times \cdots \times (G_n)_p$.

Pf/ $g = (g_1, \ldots, g_n)$

$g^p = 1 \iff g_i^p = 1$ $\forall i$

$\iff g \in (G_1)_p \times \cdots \times (G_n)_p$

$\varnothing$

Proof of 3c

$A^p \cap A_p = (A^p)_p$ ← $|x| = p^{\ell_i}$

(b) $= (\langle x_1 \rangle \times \cdots \times \langle x_t \rangle)_p$

C $= \langle x_1 \rangle_p \times \cdots \times \langle x_t \rangle_p$

B $= \langle x_1^{p^{\ell_1 - 1}} \rangle \times \cdots \times \langle x_t^{p^{\ell_t - 1}} \rangle$

$= \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{t}$ $\varnothing$

d) Case 1 $A_p \leq A^p$

Eg/ $\mathbb{Z}_{p^2}$ ($\mathbb{Z}_{p^i}$ $i \geq 2$)

i) $A^p = \langle x_1 \rangle \times \cdots \times \langle x_t \rangle$

Show $x_i = y_i^p$ $y_i \in A$.

Pf/ Immediate from def of A^p. $\varnothing$

ii) $A_0 = \langle y_1, \ldots, y_t \rangle$ ($y_i^p = x_i$)

$\Rightarrow A_0 = \langle y_1 \rangle \times \cdots \times \langle y_t \rangle$.

Lemma D

G a gp. $M, N \leq G$. $MN \leq G$.

$\Rightarrow \langle M, N \rangle = MN$

Pf/ $M, N \leq MN \leq \langle M, N \rangle$

$\Rightarrow \langle M, N \rangle \leq MN$. $\varnothing$

Fix $|x_i| = p_i^{\ell_i}$

$\Rightarrow |y_i| = p_i^{\ell_i + 1}$.

Proof of d(ii) by ind.

$H_i = \langle y_1, \ldots, y_i \rangle$.

Show by ind $H_i = \langle y_1 \rangle \times \cdots \times \langle y_i \rangle$.

Base $i=1$ trivial

Ind

$H_i = \langle H_{i-1}, y_i \rangle$

$\overset{\text{Lem D}}{=} H_{i-1} \cdot y_i = H_{i-1} \times y_i$

Pf/ Apply lemma A

$H_{i-1}, \langle y_i \rangle \trianglelefteq H_i$ ✓

$H_{i-1} \cap \langle y_i \rangle = 1$

$H_{i-1} = \langle y_1 \rangle \times \cdots \langle y_{i-1} \rangle$

$a = (a_1, \ldots, a_{i-1}) \in H_{i-1}$

$b \in \langle y_i \rangle$. $a = b$

$a^p = (a_1^p, \ldots, a_{i-1}^p) \in \langle x_1 \rangle \times \cdots \times \langle x_{i-1} \rangle$

$b^p = (y_i^\gamma)^p = (y_i^p)^\gamma = x_i^\gamma$

$\Rightarrow b^p \in \langle x_i \rangle$

$a^p = b^p \in (\langle x_1 \rangle \times \cdots \times \langle x_{i-1} \rangle) \cap \langle x_i \rangle$.

$\Rightarrow a^p = b^p = 1$

$b = y_i^\gamma$

$b^p = 1 \Rightarrow b \in \langle y_i \rangle_p$

$= \langle y_i^{p_i^{\ell_i + 1 - 1}} \rangle$

$= \langle x_i^{p_i^{\ell_i - 1}} \rangle$

$a \in \langle y_1^{p_1^{\ell_1}} \rangle \times \cdots \times \langle y_{i-1}^{p_{i-1}^{\ell_{i-1}}} \rangle$

$\in \langle x_1 \rangle \times \cdots \times \langle x_{i-1} \rangle$

$\Rightarrow$

$a = b \in (\langle x_1 \rangle \times \cdots \times \langle x_{i-1} \rangle) \cap \langle x_i \rangle$

$\Rightarrow a = b = 1$ $\varnothing$

$H_i = \langle H_{i-1}, \langle y_i \rangle \rangle$

$= H_{i-1} \langle y_i \rangle$

$= H_{i-1} \times \langle y_i \rangle$

$= \langle y_1 \rangle \times \cdots \times \langle y_{i-1} \rangle \times \langle y_i \rangle$

Letting $i = t \Rightarrow A_0 = H_t$

Done $\varnothing$

Study A_0/A^p

Lemma E (Prod & quot)

$G = G_1 \times \cdots \times G_n$

$H_i \trianglelefteq G_i$ $\forall i$.

$\Rightarrow H_1 \times \cdots \times H_n \trianglelefteq G$

$G/H_1 \times \cdots \times H_n = \frac{G_1}{H_1} \times \cdots \times \frac{G_n}{H_n}$

Pf/ $\phi: G \longrightarrow \frac{G_1}{H_1} \times \cdots \times \frac{G_n}{H_n}$

$(g_1, \ldots, g_n) \longmapsto (\bar{g}_1, \ldots, \bar{g}_n)$.

Surjective ✓

$\ker\phi = H_1 \times \cdots \times H_n$

First iso thm $\varnothing$

iii) A_0/A^p is elem ab. order p^t.

Pf/ $\frac{A_0}{A^p} = \frac{\langle y_1 \rangle \times \cdots \times \langle y_t \rangle}{\langle x_1 \rangle \times \cdots \times \langle x_t \rangle}$

$\overset{E}{=} \frac{\langle y_1 \rangle}{\langle x_1 \rangle} \times \cdots \times \frac{\langle y_t \rangle}{\langle x_t \rangle} = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{t}$ $\varnothing$

$\frac{\langle y_i \rangle}{\langle y_i^p \rangle} \simeq \mathbb{Z}_p$

iv) $A_0 = A$ so done (prop 2 holds).

$A^p \leq A_0 \leq A$

$\Rightarrow \underbrace{\frac{A_0}{A^p}}_{(p^t)} \leq \frac{A}{A^p} \overset{2(c)}{\simeq} A_p \overset{\text{assume } A_p \leq A^p}{=} \underbrace{A_p \cap A^p}_{p^t \; 3(c)}$

$\Rightarrow \frac{A_0}{A^p} = \frac{A}{A^p}$ counting orders

$\Rightarrow A_0 = A$

Prop 2 holds A_0 (d ii)

$\Rightarrow$ holds for A $\varnothing$

e) Case 2

$A_p \nleq A^p$

i.e. $\exists\, x \in A_p$ $x \notin A^p$.

Ex $A = \mathbb{Z}_p$.

i) $\bar{A} = A/A^p$ $\pi: A \to \bar{A}$

$\bar{x} = \pi(x)$. $|x| = |\bar{x}| = p$.

Pf/ $x \neq 1$ but $x^p = 1 \Rightarrow |x| = p$

$\bar{x} \neq A^p = 1$ $(x \notin A^p)$

$\Rightarrow \bar{x}^p = 1$. (2 (c))

$\Rightarrow |\bar{x}| = p$ $\varnothing$

ii) $\bar{A} \simeq \langle \bar{x} \rangle \times \bar{E}$ $\bar{E} \leq \bar{A}$

Pf/ $\bar{A} \xrightarrow{\varpi} \bar{A}/\langle \bar{x} \rangle = \bar{E}$

$|\bar{A}| = p^r$ $= (\mathbb{Z}_p)^{r-1}$

$= \langle e_1 \rangle \times \cdots \times \langle e_{r-1} \rangle$

$y_i \in \varpi^{-1}(e_i)$

$\Phi: \bar{E} \times \langle \bar{x} \rangle \longrightarrow \bar{A}$

$(e_1^{\alpha_1}, \ldots, e_{r-1}^{\alpha_{r-1}}, \bar{x}^\beta) \longmapsto y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_{r-1}^{\alpha_{r-1}} \bar{x}^\beta$

Φ a hom ✓

Both sides same order ✓

Suffices to check Φ surjects

$a \in \bar{A} \to$

$\varpi(a) \in \bar{E}$

$= (e_1^{j_1}, \ldots, e_{r-1}^{j_{r-1}})$.

$\hat{a} = a y_1^{-j_1} \cdots y_{r-1}^{-j_{r-1}}$

$\varpi(\hat{a}) = 1$

$\hat{a} \in \ker\varpi = \langle \bar{x} \rangle$

$\hat{a} = \bar{x}^\gamma$

$a = y_1^{j_1} \cdots y_{r-1}^{j_{r-1}} \bar{x}^\gamma$

$= \Phi(e_1^{j_1}, \ldots, e_{r-1}^{j_{r-1}}, \bar{x}^\gamma)$

$\Rightarrow \Phi$ surjects $\varnothing$

iii) let $E = \pi^{-1}(\bar{E}) \leq A$

Show $A \simeq \langle x \rangle \times E$

Conclude A product of cyclic gps.

Pf/ 1) $\langle x \rangle E = A$ *

2) $\langle x \rangle \cap E = 1$ *

$\Rightarrow$ Lemma A $\Rightarrow A = \langle x \rangle \times E$

$a \in A$ $\pi(a) = (e, \bar{x}^k)$ ($e \in \bar{E}$)

$\Rightarrow \pi(x^{-k} a) = (e, 1)$

$x^{-k} a \in E$

$a = (x^{-k} a) x^k$ (E, $\langle x \rangle$) $\varnothing$

$E \cap \langle x \rangle \leq \langle x \rangle$ ← order p

either $\langle x \rangle$ or triv (LG).

But $x \notin E$

(b/c $\bar{x} \notin \bar{E}$) $\varnothing$

$A = E \times \langle x \rangle$

$|E| < |A|$

by ind E product of cyclic gps.

$\Rightarrow A$ is too. $\varnothing$

Any Abelian gp is the product of cyclic groups!!

Need ingredients

Lemma F (Cancellation)

M, N, K finite

$K \times M \simeq K \times N$ HW 11

$\Rightarrow M \simeq N$.

Def^n G a gp.

$Exp(G) = \min\{n \mid g^n = 1 \; \forall g\}$

Lemma G HW 10 prob 3

$G \simeq \mathbb{Z}_{n_1} \times \cdots \mathbb{Z}_{n_s}$ inv factor decomp $\Rightarrow Exp(G) = n_1$

Question 4

Uniqueness for thm 1

$G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \simeq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_t}$

$n_i, m_i \geq 2$

$n_{i+1} | n_i$ $m_{i+1} | m_i$

$\Rightarrow n_i = m_i$ $\forall i$ $(s = t)$

Pf/ Lemma G

$n_1 = Exp(G) = m_1$

Lemma F

$\underbrace{\mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}}_{G'} \simeq \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_t}$

$n_i, m_i \geq 2$

$n_{i+1} | n_i$ $m_{i+1} | m_i$

Lemma G

$n_2 = Exp(G') = m_2$

Lemma F cancel

Keep going

$\varnothing$