Takehome Assignment 3

Due Friday, May 13 **at 11:59 pm**

1. Let's start with some group theory! For this first question, let G be a finite group of order n. We'd like to understand the size of the center of G, say $|Z(G)| = z$.
   (a) Show that it is not possible for z to fall in the range $\frac{n}{4} < z < n$.
   (b) Show that these bounds are optimal. That is, give examples of a group where $z = n$, and one where $z = \frac{n}{4}$.

Now let's think about some special ideals in commutative unital rings. We remind the reader of the following definition.

Definition 1. *Let R be a commutative unital ring. An ideal $\mathfrak{p} \subseteq R$ is called a prime ideal if $\mathfrak{p} \neq R$ and for any $a, b \in R$, if $ab \in \mathfrak{p}$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*

2. Let R be a commutative ring with $1 \neq 0$. Recall that in ideal $\mathfrak{m} \subseteq R$ is maximal if and only if $R/\mathfrak{m}$ is a field. We will see there is a similar characterization of primality.
   (a) Prove that an ideal $\mathfrak{p} \subseteq R$ is prime if and only if the quotient ring $R/\mathfrak{p}$ is an integral domain.
   (b) Prove that a maximal ideal $\mathfrak{m} \subseteq R$ is prime.
   (c) What are all the prime ideals of $\mathbb{Z}$?
   (d) Prove that the ideal $(x) \subseteq \mathbb{Z}[x]$ is prime but not maximal.
3. Let $\varphi : R \to S$ be a homomorphism between commutative unital rings with $\varphi(1_R) = 1_S$.
   (a) Let $\mathfrak{q} \subseteq S$ be a prime ideal. Show that $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R.
   (b) Suppose φ is surjective, and $\mathfrak{m} \subseteq S$ is a maximal ideal. Show that $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal of R.
   (c) Give a counterexample to part (b) if φ is not surjective.
4. In this exercise we calculate the intersection of all the maximal ideals in a commutative unital ring R. Given a ring R, we define the *Jacobson radical* of R to be the ideal:
$$\mathfrak{J}(R) = \bigcap_{\mathfrak{m} \subseteq R \text{ maximal}} \mathfrak{m}.$$
   (a) Show that $\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$.
   (b) Show that an element $r \in R$ is a unit if and only if it is not contained in any maximal ideal.
   (c) Suppose $\mathfrak{m}$ is a maximal ideal and $r \in R \setminus \mathfrak{m}$. Compute the ideal $(\mathfrak{m}, r)$ generated by $\mathfrak{m}$ and r.
   (d) Prove that $r \in \mathfrak{J}(R)$ if and only if $1 - ry \in R^\times$ for every $y \in R$. (Parts (b) and (c) might help!)
5. Let's finish by exploring unit groups. For parts (a)-(c) we do not assume that R is commutative. Recall that if R is a (unital) ring, then $R^\times$ is the set of units, endowed with a group structure given by multiplication in R.

(a) Let $\varphi : R \to S$ be a (unital) homomorphism of rings. Show that if $r \in R^\times$ then $\varphi(r) \in S^\times$. Give a counterexample where φ is not unital.

(b) Show that the restriction of φ to $R^\times$ is a group homomorphism $\varphi^\times : R^\times \to S^\times$, which is injective if φ is.

(c) The analogous statement does not hold for φ surjective. Give an example of a surjective (unital) homomorphism $\varphi : R \to S$, but such that the induced map on unit groups $\varphi^\times : R^\times \to S^\times$ is not surjective.

(d) Let $\varphi : R \to S$ be a surjective (unital) homomorphism of *commutative* rings, and suppose that $\ker \varphi \subseteq \mathfrak{J}(R)$. Prove that the induced map $\varphi^\times : R^\times \to S^\times$ is surjective.

Congratulations!! We've covered a ton of material and done a ton of problems this semester. **Good work!**