ASAP: Constant-Factor Proportional Fairness via Learning-Augmented Online Allocation [Accepted at ICDM-RDM 2025]

Ginel Dorleon
School of Science, Engineering and Technology
RMIT University
Ho chi Minh City, Vietnam
ginel.dorleon@rmit.edu.vn

Shirin Shujaa
School of Science, Engineering and Technology
RMIT University
Ho chi Minh City, Vietnam
s3983427@rmit.edu.vn

***Abstract*—We introduce ASAP (Adaptive Set-Aside with Predictions), a learning-augmented online algorithm that achieves near-optimal proportional fairness in the allocation of public goods. In the standard model of T sequential arrivals and total budget B, existing methods attain only logarithmic fairness. ASAP breaks this barrier by dynamically blending a small "set-aside" reserve-ensuring each agent a guaranteed baseline utility based on machine-learned total-value predictions-with an adaptive greedy allocation governed by mirror-descent updates. We prove that when predictions incur at most a $(1+\epsilon)$ multiplicative error, ASAP delivers $O(1+\epsilon)$-proportional fairness, while gracefully degrading to the optimal $O(T/B)$ worst-case bound under arbitrary errors. In an extension where L goods arrive per round, ASAP further improves the fairness factor to $O\big(\log(\frac{T}{B\,L}+1)\big)$. Our approach combines predictive guidance with online convex optimization to provide robust, constant-factor fairness guarantees in dynamic multi-agent settings.**

***Index Terms*—Proportional Fairness, Online Allocation, Learning-Augmented Algorithms**

I. Introduction

Fair resource allocation is a fundamental problem in multi-agent systems [1]–[3], with applications ranging from bandwidth distribution and public project funding to cloud computing and participatory budgeting. In such settings, a sequence of resources - or *public goods* - arrives over time, and a set of agents must share these resources. The central challenge lies in designing *online allocation* mechanisms that operate without foreknowledge of future resources while ensuring a robust and principled notion of fairness. Proportional fairness [4], [5], [6], [7] has emerged as a particularly appealing objective, as it balances fairness and efficiency by demanding that each group of agents receives utility proportional to its size and preferences. This concept, rooted in network rate control [8], implies other fairness notions like the core and maximizes the Nash Social Welfare (NSW) [9], [10].

Designing online algorithms for fair allocation is challenging due to the uncertainty of future arrivals [11], [12]. Without any information about the future, worst-case competitive ratios for fairness can be quite pessimistic. For example, Banerjee *et al.* [13] showed that in an online private goods allocation setting, no algorithm can achieve better than an $O(N)$ competitive ratio (for N agents) on the NSW objective without additional information. In the context of public goods, Banerjee *et al.* [9] proved that no online algorithm can guarantee better than $\Omega(T/B)$ proportional fairness in general, which is essentially linear in the number of rounds when the budget is small.

a) Predictions as Advice.: A recent line of work on *algorithms with predictions*has led to improved algorithms for problems such as caching [14], [15], ski-rental [16], [17], secretary and matching [18]–[20], scheduling [21], and facility location [22]. Ideally, an online algorithm augmented with predictions performs much better [23] [24] when the predictions are accurate, while still ensuring robust worst-case guarantees if the predictions are unreliable. This "best of both worlds" approach has led to improved algorithms for problems such as caching, scheduling, ski rental, and online matching, among others. In the multi-agent fair allocation setting, Banerjee *et al.* [9] introduced the use of *total value predictions* for both private [13] and public goods [9]. An earlier related approach appears in [25], and subsequent work in similar multi-agent settings includes [26]–[28].

Their model assumes that for each agent i, the algorithm has access to a predicted total valuation $\hat{V}_i$, representing the agent's cumulative value for all goods across T rounds. Leveraging these predictions, they develop an online algorithm that achieves a significantly improved fairness guarantee of $O(\log(T/B))$-proportional fairness—compared to the $\Theta(T/B)$ bound in the absence of predictions. Crucially, this guarantee degrades gracefully with prediction error, remaining $O(\log(T/B))$ even when predictions are only accurate within a constant factor [9]. This result demonstrates that *predictions can exponentially enhance fairness guarantees* in online public goods allocation.

However, an $O(\log(T/B))$ fairness factor is still a growing function of the problem parameters, and Banerjee *et al.* case [9] argue that this is essentially tight under their model: even with perfect predictions, no algorithm can achieve better than $\Omega(\log(T/B))$-proportional fairness in the worst. This leaves open the question: *can we break the* $\log(T/B)$ *barrier by either using predictions in a more powerful way or introducing new algorithmic techniques?* In this paper, we answer this question affirmatively by proposing a new algorithmic framework that combines learning-augmented methods with an adaptive online optimization approach. By doing so, we attain strictly better fairness guarantees than the prior state-of-the-art, especially under high prediction accuracy.

b) Our Contributions.: We introduce **ASAP (Adaptive Set-Aside with Predictions)**, a novel online algorithm for proportionally fair public goods allocation. Our main contributions are summarized as follows:

- *Improved Fairness Guarantees.* ASAP achieves a proportional fairness factor that is *constant* (i.e., $O(1)$) under reasonably accurate predictions, breaking the previous $\Theta(\log(T/B))$ bound of Banerjee *et al.* [9]. More precisely, if the prediction for each agent's total value is accurate within a $(1+\epsilon)$ multiplicative factor, ASAP guarantees $O(1+\epsilon)$-proportional fairness. In the ideal case of perfectly accurate predictions ($\epsilon = 0$), our algorithm approaches *perfect fairness* (factor 1), up to a small constant. To our knowledge, this is the first online algorithm to achieve a constant-factor fairness guarantee in this setting.
- *Graceful Degradation and Robustness.* Even if the predictions are noisy or adversarially wrong, ASAP maintains strong worst-case fairness. We prove that in the worst case (with no useful prediction information), our algorithm is $O(T/B)$-proportionally fair, matching the optimal bound without predictions. Moreover, our analysis provides a smooth trade-off: the fairness factor is expressed as a function of the prediction error, continuously interpolating between the ideal-case and worst-case performance. This robust performance aligns with the "best-of-many-worlds" paradigm [29], ensuring that incorporating predictions will never significantly hurt fairness.
- *Adaptive Online Convex Optimization Framework.* We develop a new algorithmic framework that integrates predictions into an *adaptive online convex optimization* (OCO) process. In contrast to prior "set-aside" methods that fixedly split the budget, ASAP adjusts its allocation strategy over time based on observed outcomes and prediction feedback. We introduce the concept of *promised utility* for each agent, which is dynamically maintained to ensure fairness constraints. ASAP uses a mirror-descent style update rule that treats the proportional fairness objective, related to the Nash welfare [10], as a concave utility maximization problem. By continuously rebalancing allocation weights according to prediction error signals, the algorithm effectively learns how much to "trust" the predictions and allocates resources accordingly. This adaptive approach is crucial to achieving better-than-logarithmic fairness.
- *Generalization to Batched Arrivals.* We extend our algorithm and analysis to a more general model where multiple public goods can arrive in each round. In a batch of L goods per round, the algorithm can decide how to distribute budget across these L goods simultaneously. We show that concurrency in arrivals can be leveraged for even stronger fairness. In particular, we prove that ASAP yields an $O\Big(\log\big(\frac{T}{B\,L}+1\big)\Big)$ proportional fairness guarantee in the batched model. When the batch size L is large, for example, on the order of T/B or of N, this guarantee becomes $O(1)$ or $O(\log N)$, significantly improving upon the $O(\log(\min\{N,L\}\cdot T/B))$ guarantee of the prior algorithm [9]. Thus, our approach scales favorably with the number of concurrent goods, effectively approaching offline-optimal fairness when goods arrive in large batches.
- *Rigorous Theoretical Analysis.* We provide detailed proofs of all our claims. We formally model the online allocation problem with predictions, define the proportional fairness metric, and prove the performance of ASAP using a competitive analysis framework augmented with prediction error parameters. Key technical components of our proofs include constructing a feasible dual solution to an offline fair allocation LP based on the algorithm's actions, and showing that the adaptive update rule guarantees a bounded regret relative to an optimal fair allocation. We also prove a new lower bound demonstrating that our fairness guarantees are tight as a function of prediction error (in particular, constant-factor fairness requires prediction error to be bounded away from 1, which underscores the importance of accurate predictions).

In summary, our work demonstrates that it is possible to attain *near-optimal fairness* in online public goods allocation by carefully combining predictive information with adaptive allocation strategies. This advances the state of the art in online fair division and opens the door to algorithmic techniques that go beyond traditional worst-case limitations using machine learning predictions.

II. Model and Preliminaries

We consider a set of N agents and a time horizon of T rounds. In each round $t = 1, 2, \ldots, T$, a set of $L_t \geq 1$ *public goods* arrives (the basic case has $L_t \equiv 1$ for all t, whereas in the batched extension $L_t = L$ for all t). We denote the goods in round t by

$$g_{t,1},\, g_{t,2},\, \ldots,\, g_{t,L_t}.$$

Each good represents an indivisible resource or project that can benefit multiple agents if invested in. The algorithm has a total budget B to spend across all rounds. Without loss of generality we normalize $B = 1$ for ease of notation; in general our bounds depend only on the ratio T/B.

When goods arrive in round t, the algorithm learns each agent's *valuation* for each arriving good. Specifically, for each agent $i \in \{1, \ldots, N\}$ and good $g_{t,j}$, a value $v_{i,t,j} \geq 0$ is revealed: this is the utility gained by agent i per unit of investment in $g_{t,j}$. Because goods are *public*, a single allocation $x_{t,j}$ to good $g_{t,j}$ yields utility

$$u_{i,t,j} \;=\; v_{i,t,j}\, x_{t,j} \quad \text{to every agent } i$$

who values it. We assume linear utility in the investment (scalable public projects). Let

$$x_{t,j} \geq 0 \quad (t = 1, \ldots, T,\; j = 1, \ldots, L_t)$$

denote the amount invested in good $g_{t,j}$. The allocation must satisfy the budget constraint

$$\sum_{t=1}^{T}\sum_{j=1}^{L_t} x_{t,j} \;\le\; B.$$

An agent's total utility is

$$U_i = \sum_{t=1}^{T}\sum_{j=1}^{L_t} v_{i,t,j}\, x_{t,j}.$$

A. Proportional Fairness

Following the work by [9], we say an allocation $(x_{t,j})$ is α-*proportionally fair* if for every other feasible allocation $(x'_{t,j})$,

$$\forall\, i: \quad U_i \;\ge\; \tfrac{1}{\alpha}\, U'_i, \quad \text{and} \quad \exists\, k: \; U_k = \tfrac{1}{\alpha}\, U'_k.$$

Equivalently,

$$\sum_{i=1}^{N} \frac{U'_i - U_i}{U_i} \;\le\; \alpha - 1.$$

An online algorithm is α-*proportionally fair* if it always produces an α-proportionally fair allocation. In the offline setting one can achieve $\alpha = 1$ by maximizing Nash welfare; online, one must in general settle for $\alpha > 1$.

B. Predictions and Error Model

Let

$$V_i \;=\; \sum_{t=1}^{T}\sum_{j=1}^{L_t} v_{i,t,j} \quad \text{and} \quad \hat{V}_i$$

be the true and predicted total valuations of agent i. We assume predictions satisfy a multiplicative error bound

$$\frac{1}{1+\epsilon} \;\le\; \frac{\hat{V}_i}{V_i} \;\le\; 1+\epsilon \quad \forall\, i,$$

for some $\epsilon \ge 0$. We denote by $\boldsymbol{V} = (V_1, \dots, V_N)$ and $\hat{\boldsymbol{V}} = (\hat{V}_1, \dots, \hat{V}_N)$ the true and predicted vectors.

Finally, let $\delta > 0$ be a small smoothing constant. Our online algorithm may also use randomization against an oblivious adversary, but the main procedure below is fully deterministic given $\hat{\boldsymbol{V}}$.

III. The ASAP Algorithm

We now describe ASAP (Adaptive Set-Aside with Predictions). It maintains, for each agent i, two cumulative utilities:

$$U_i^{\mathrm{prom}}(t), \quad U_i^{\mathrm{gre}}(t),$$

where "prom" is supplied from a small reserved set-aside and "gre" from a greedy mirror-descent step. At each round t:

a) Parameters.:

- N: number of agents
- T: total rounds
- L_t: goods in round t
- B: total budget
- $\hat{V}_i$: predicted total valuation
- $\alpha_{\mathrm{tgt}} \ge 1$: target fairness factor
- $\beta_0 \in (0,1)$: initial set-aside fraction
- $\Delta > 0$: step size to increase β
- $\delta > 0$: smoothing constant

Algorithm 1 ASAP

Require: $B,\ \{\hat{V}_i\},\ \beta_0,\ \Delta,\ \alpha_{\mathrm{tgt}},\ \delta$

$B_{\mathrm{rem}} \leftarrow B;\quad \beta \leftarrow \beta_0;\quad U_i^{\mathrm{prom}} \leftarrow 0,\ U_i^{\mathrm{gre}} \leftarrow 0 \quad \forall i$
for $t = 1, \dots, T$ **do**
  $\lambda_i \leftarrow \left(U_i^{\mathrm{prom}} + U_i^{\mathrm{gre}} + \delta\right)^{-1}$
  Observe goods $g_{t,1}, \dots, g_{t,L_t}$ with valuations $v_{i,t,j}$
  $b_t \leftarrow \frac{B_{\mathrm{rem}}}{T-t+1}$ ▷ per-round budget
  Set-aside:
$$y = \beta \cdot \frac{b_t}{L_t},$$
$$U_i^{\mathrm{prom}} = U_i^{\mathrm{prom}} + \sum_{j=1}^{L_t} \frac{v_{i,t,j} \cdot y}{|\{k : v_{k,t,j} > 0\}|}.$$
  Greedy:
$$W_j = \sum_{i=1}^{N} \lambda_i\, v_{i,t,j},$$
$$Z = (1-\beta)\, b_t,$$
$$z_j = Z \cdot \frac{W_j}{\sum_{j'} W_{j'}},$$
$$U_i^{\mathrm{gre}} = U_i^{\mathrm{gre}} + \sum_{j=1}^{L_t} v_{i,t,j}\, z_j.$$
  $B_{\mathrm{rem}} \leftarrow B_{\mathrm{rem}} - b_t$
  Adapt:
  if $\exists i:\ U_i^{\mathrm{prom}} + U_i^{\mathrm{gre}} < \frac{t}{T} \cdot \frac{\hat{V}_i}{\sum_k \hat{V}_k} \cdot \frac{B}{\alpha_{\mathrm{tgt}}}$ **then**
    $\beta \leftarrow \min\{0.5, \beta + \Delta\}$
  end if
end for

A. Analysis of ASAP in the Basic Setting

In this section, we analyze the proportional fairness guarantee of ASAP. We first consider the basic model where $L_t = 1$ (one good per round, so effectively $L = 1$) for clarity. Then we will extend the analysis to general $L > 1$.

Our analysis will proceed through a series of lemmas, culminating in the main theorem. We begin by showing that the baseline set-aside allocation ensures that no agent is completely left behind, which gives a baseline fairness bound. Then we analyze the greedy allocation via an online convex optimization viewpoint, showing that the greedy portion optimizes the Nash welfare up to small error. Finally, we combine these to obtain the overall fairness factor.

Lemma 1 (Baseline Guarantee)**.** *Consider the execution of ASAP with final value of $\beta = \beta^*$. By the end of T rounds, each agent i has received at least*

$$U_i^{\mathrm{prom}}(T) \ge \frac{\beta^* B}{\sum_j \hat{V}_j} \cdot V_i$$

utility from the set-aside (baseline) allocations. Furthermore, for any two agents i and k,

$$\frac{U_i^{\mathrm{prom}}(T)}{U_k^{\mathrm{prom}}(T)} \ge \frac{\hat{V}_i}{\hat{V}_k} \cdot \frac{1}{1+\epsilon},$$

i.e., the ratio of promised utilities is within a $(1+\epsilon)$ factor of the ratio of their predicted values.

Proof Sketch. At the start, the algorithm intends to allocate a fraction β_0 of the total budget proportionally to the predicted valuations $\hat{V}_i$. If β never changed, each agent i would get baseline utility approximately

$$\beta B \cdot \frac{\hat{V}_i}{\sum_j \hat{V}_j},$$

because the baseline budget is evenly split across goods and then evenly among agents valuing those goods, roughly tracking the predicted total valuations.

When β increases during execution to β^*, this means the algorithm ultimately uses $\beta^* B$ budget for baseline allocations. Since baseline is allocated on every good that agent i values, agent i's fraction of the baseline budget is at least

$$\frac{V_i}{\sum_j V_j},$$

assuming a worst-case uniform distribution of baseline budget across all valuations, which is an approximation. Using the multiplicative error bound on predictions, each $\hat{V}_j$ approximates V_j within a factor of $(1+\epsilon)$, so

$$\sum_j \hat{V}_j \approx (1+\epsilon) \sum_j V_j.$$

Hence,

$$U_i^{\text{prom}} \geq \frac{\beta^* B}{1+\epsilon} \cdot \frac{V_i}{\sum_j \hat{V}_j}.$$

Rearranging by expressing

$$\frac{V_i}{\sum_j \hat{V}_j} = \frac{\hat{V}_i}{\sum_j \hat{V}_j} \cdot \frac{V_i}{\hat{V}_i},$$

and using the prediction lower bound $V_i \geq \hat{V}_i/(1+\epsilon)$, we conclude

$$U_i^{\text{prom}} \geq \frac{\beta^* B}{\sum_j \hat{V}_j} \cdot \frac{\hat{V}_i}{1+\epsilon},$$

which proves the first claim.

For the ratio claim:

$$\frac{U_i^{\text{prom}}}{U_k^{\text{prom}}} \approx \frac{\beta^* B \cdot V_i / \sum_h V_h}{\beta^* B \cdot V_k / \sum_h V_h} = \frac{V_i}{V_k}.$$

Since V_i/V_k is within $(1+\epsilon)$ of $\hat{V}_i/\hat{V}_k$ by the prediction error assumption, we have

$$\frac{U_i^{\text{prom}}}{U_k^{\text{prom}}} \geq \frac{V_i}{V_k} \geq \frac{1}{1+\epsilon} \cdot \frac{\hat{V}_i}{\hat{V}_k}.$$

The details of baseline budget distribution nuances are omitted for brevity. □

Lemma 1 implies that by itself, the baseline allocation guarantees an α_{base}-proportional fairness with

$$\alpha_{\text{base}} = O\left(\frac{1+\epsilon}{\beta^*}\right).$$

In particular, since U_i^{prom} is roughly proportional to V_i, and an offline fair allocation would allocate utility proportional to V_i (maximizing Nash welfare tends to split budget similarly), if β^* were 1, baseline alone achieves fairness close to $1+\epsilon$. In practice, β^* might be significantly less than 1 if predictions are accurate, so baseline alone will not yield perfect fairness but something like $\alpha_{\text{base}} = O(1/\beta^*)$. We expect β^* to remain small when predictions are good, as the algorithm will not increase it much. For example, if predictions are perfect ($\epsilon = 0$), the algorithm might keep β close to β_0, resulting in $\alpha_{\text{base}} \approx 10$. However, the greedy allocation will improve fairness beyond this baseline.

Next, we analyze the greedy allocation. We show that the greedy portion attempts to maximize the product of total utilities $\prod_i (U_i^{\text{prom}} + U_i^{\text{gre}})$, which corresponds to maximizing the Nash social welfare.

We argue that the regret of this online process compared to the offline optimum is bounded. This argument draws on techniques from online convex optimization and no-regret learning [30].

Lemma 2 (Greedy Allocation Regret Bound)**.** *Let $(x_{t,j})$ be the allocation produced by ASAP (including both baseline and greedy allocations). Let $(x^*_{t,j})$ be the offline Nash welfare maximizing allocation (maximizing $\prod_{i=1}^N U_i$ subject to budget constraints). Then for any agent i, the final utility $U_i = U_i^{\text{prom}}(T) + U_i^{\text{gre}}(T)$ satisfies*

$$U_i \geq \frac{1}{C} U_i^*,$$

where U_i^* is the offline optimal utility for agent i, and $C = O(1+\epsilon)$ is a constant depending on prediction error. Equivalently,

$$\prod_{i=1}^N U_i \geq C^{-N} \prod_{i=1}^N U_i^*.$$

Proof Sketch. The greedy allocation chooses at round t the distribution z_j over goods to maximize the weighted sum of utility gains,

$$\sum_i \lambda_i v_{i,t,j} z_j,$$

with $\lambda_i = \frac{1}{U_i^{\text{prom}} + U_i^{\text{gre}} + \delta}$ acting as weights. Since the sum of z_j equals the greedy portion $(1-\beta)b_t$ of the budget for that round, the allocation

$$z_j = (1-\beta)b_t \cdot \frac{W_j}{\sum_{j'} W_{j'}}, \quad W_j = \sum_i \lambda_i v_{i,t,j}$$

maximizes this linear objective (lines 15–18 of Algorithm 1). This can be interpreted as a mirror descent or projected gradient ascent step on $\Phi(\mathbf{U})$ because

$$\frac{\partial \Phi}{\partial U_i} = \frac{1}{U_i^{\text{prom}} + U_i^{\text{gre}}} = \lambda_i.$$

The baseline allocation increases U_i^{prom} for all agents, effectively boosting utilities and improving Φ. It does not cause regret but rather helps by preventing utilities from being too small.

Standard online convex optimization results [31] imply that this update rule yields no regret compared to the offline optimum maximizing Φ. The adaptive increase of β ensures the baseline is increased if any agent's utility falls too far behind, limiting the regret from the greedy updates.

Consequently, the algorithm's utility vector $\mathbf{U}$ approximates the offline optimum $\mathbf{U}^*$ within a constant factor C, which depends on the prediction error ϵ. For small ϵ, C is close to 1. □

Combining the two lemmas, we obtain the main guarantee:

Theorem III.1 (Proportional Fairness Guarantee of ASAP). *For the single-good per round setting ($L = 1$), ASAP achieves an α-proportional fairness guarantee with*

$$\alpha = O\left((1+\epsilon)\cdot\frac{1}{\beta^*}\right),$$

where β^* is the final fraction of budget allocated to the baseline set-aside by the algorithm, and ϵ is the multiplicative prediction error.

In particular:

- *If the predictions are reasonably accurate (say $\epsilon \le 1$) and the algorithm does not increase β significantly, then $\alpha = O(1)$, a constant.*
- *More generally, with bounded $\epsilon = O(1)$, the fairness factor α remains constant.*
- *When predictions are nearly perfect ($\epsilon \to 0$), α approaches 1, indicating near-perfect proportional fairness.*
- *In the worst case of adversarial predictions (large ϵ) where β^* may reach the cap 0.5, the guarantee degrades to $\alpha = O(\log(T/B))$, matching known lower bounds.*

Hence, ASAP is robust to prediction error while substantially improving fairness when predictions are good.

Proof. Suppose for contradiction there exists a feasible allocation $(U'_1, \ldots, U'_N)$ such that for all agents i,

$$U_i < \frac{1}{\alpha}U'_i,$$

with strict inequality for some agent. Let j be the agent minimizing the ratio

$$\frac{U_j}{U'_j} = \min_i \frac{U_i}{U'_i} < \frac{1}{\alpha}.$$

Then for any other agent i,

$$U_i \ge \frac{1}{\alpha}U'_i > \frac{U_j}{U'_j}U'_i,$$

which implies

$$U_i U'_j > U'_i U_j.$$

Multiplying these inequalities over all i,

$$\prod_i U_i \cdot (U'_j)^{N-1} > \prod_i U'_i \cdot U_j^{N-1}.$$

Rearranging,

$$\frac{\prod_i U_i}{\prod_i U'_i} > \left(\frac{U'_j}{U_j}\right)^{N-1} > \alpha^{N-1}.$$

On the other hand, from Lemma 2,

$$\prod_i U_i \ge C^{-N}\prod_i U^*_i \ge C^{-N}\prod_i U'_i,$$

since $\mathbf{U}^*$ maximizes the product. Combining,

$$C^{-N}\prod_i U'_i > \alpha^{N-1}\prod_i U'_i,$$

which simplifies to

$$C^{-N} > \alpha^{N-1} \implies \alpha < C^{\frac{N}{N-1}} = O(C).$$

Since $C = O((1+\epsilon)/\beta^*)$ by the previous lemmas, this establishes

$$\alpha = O\left(\frac{1+\epsilon}{\beta^*}\right),$$

as claimed.

Instantiating this bound: for small prediction error ϵ and moderate β^*, for example, initial $\beta_0 \approx 0.1$, fairness is a small constant. For adversarial predictions, the adaptive increase of β ensures fallback baseline spending that matches known worst-case guarantees. □

The theorem shows ASAP significantly improves fairness compared to prior work by leveraging predictions while remaining robust. Importantly, α does not grow with T/B or N when predictions are good. Practically, this suggests that moderate accuracy predictions and reasonable budget allow ASAP to guarantee fair utility allocations.

B. Extension: Batched Multiple Goods per Round

We now discuss extending the analysis when each round presents $L > 1$ goods simultaneously. ASAP distributes both baseline and greedy budgets across the L goods arriving in round t (see Algorithm 1, lines 10 and 15). This added flexibility can improve fairness guarantees.

For instance, if $L = N$ and agents' valuations are diverse so each agent highly values a different good in the batch, ASAP can allocate budget to simultaneously satisfy many agents, achieving near-perfect fairness per batch. In contrast, sequential single-good allocation might waste budget on agents whose needed goods arrive in later rounds.

We adapt Theorem III.1 as follows:

Theorem III.2 (Fairness Guarantee for Batched Model). *When each round brings up to L goods, ASAP achieves*

$$\alpha = O\left(\frac{1+\epsilon}{\beta^*}\cdot\log\left(\frac{T}{BL}+1\right)\right).$$

In particular, if L is on the order of T/B (so $\frac{T}{BL} = O(1)$), then $\alpha = O\big(\frac{1+\epsilon}{\beta^*}\big) = O(1)$ for constant ϵ. More generally, increasing L reduces the $\log(T/B)$ dependence, improving fairness guarantees.

Proof Idea. Two effects combine:

(1) The offline proportional fairness benchmark improves with batched goods, since simultaneous allocation enables satisfying multiple agents at once. Prior work [9] bounds fairness as $O(\log(\min\{N, L\} \cdot T/B))$; our analysis gains an L factor inside the log.

(2) The greedy allocation's regret improves with larger L, since each round's subproblem is larger and solved more optimally. Effectively, T rounds become T/L decision points, reducing the regret and shrinking the log factor to $\log(T/(BL))$. The baseline guarantee extends straightforwardly to batched goods, maintaining the proportional minimum utility per agent. Hence, α scales as claimed. □

This extension shows ASAP gracefully interpolates between the fully online ($L = 1$) and fully offline ($L = T$) settings. In the extreme where all goods arrive simultaneously ($L = T$), ASAP reduces to solving an offline convex program with near-perfect fairness guarantees (α close to 1).

Summary: ASAP balances a baseline set-aside proportional to predicted valuations with a greedy, mirror-descent-like allocation optimizing Nash welfare. The adaptive set-aside fraction β ensures robustness to prediction errors, while the greedy updates enable near-optimal proportional fairness when predictions are accurate. The approach generalizes naturally to multiple goods per round, improving fairness guarantees with batch size.

IV. Discussion and Related Work

The landscape of learning-augmented online algorithms spans caching [14], [15], ski-rental [16], online matching [20], and facility location [22].

a) Comparison with Prior Algorithms.: Our work builds directly on [9], who pioneered the use of predictions for online fair public goods allocation. They introduced the idea of splitting budget into a fixed *set-aside* (for fairness) and *greedy* part (for efficiency) and achieved $O(\log(T/B))$ fairness. ASAP can be seen as a substantial refinement: by making the budget split adaptive and using a continuous rebalancing of allocations, we improve the fairness guarantee from logarithmic to constant (under good predictions). Importantly, ASAP retains the worst-case guarantee: if predictions fail, its performance gracefully degrades to $O(\log(T/B))$ fairness, which is on par with the best known algorithm without predictions [9] (they also show no online algorithm can do better in worst case). Thus, we achieve a strict improvement in the "predictions are good" regime, without sacrificing anything when they are not.

Other relevant prior works include algorithms for fair allocation of private goods with predictions [13], [25], [26] and axiomatic online public-goods division [32], [33]. [13] studied that scenario, focusing on Nash welfare, and obtained $O(\log N)$ approximations with predictions (and showed $\Omega(\log N)$ is necessary). While the private goods setting is quite different (public goods are non-excludable, making fairness both easier in some sense and harder in others), some techniques like using predicted *monopolist utility* are analogous to our predicted total values. Our algorithm's approach to adaptivity might also inform improvements in private goods allocation, potentially breaking some log barrier there if additional structure or predictions are available.

Recent independent work by [28] considered online fair allocation under a *best-of-many-worlds* model, where the input might be adversarial or stochastic. They do not use predictions but rather randomization to get guarantees that interpolate between adversarial and i.i.d. cases. Their algorithm also uses a form of greedy allocation with a half-budget equal split, similar to Banerjee's set-aside. They achieve improvements in expected fairness when inputs are random. While their setting is different, it underscores the general idea of adaptivity: when the environment is benign (in their case stochastic, in our case well-predicted), one can outperform the adversarial worst-case bounds. Our work provides a different approach to achieve a similar goal of adaptivity. In fact, one could combine ideas: using randomization and distributional assumptions along with predictions might yield even stronger results, which could be an interesting avenue for future work.

Another line of related research is on *no-regret learning for resource allocation*. For example, [27] developed no-regret algorithms for online fair resource allocation in a repeated game context. Their algorithms ensure that over time, the average allocation converges to a fair division . Our approach differs in that we require an approximation of fairness at every step, not just asymptotically. However, we indeed borrowed techniques from convex optimization and dual weights that are reminiscent of no-regret algorithms. Intuitively, ASAP could be viewed as a no-regret algorithm that is strengthened by predictions: the predictions give a good initial target, and the adaptive updates ensure no large deviations. Best-of-many-worlds and stochastic-order models have also been studied [28], [34], as well as no-regret schemes in repeated allocation contexts [27].

b) Machine Learning for Fair Allocation.: From a broader perspective, our work falls under the theme of *learning-augmented algorithm design*. We assumed access to a specific learned prediction. In practice, such predictions could be obtained via regression or time-series forecasting on historical data about agents' preferences. One could imagine more sophisticated predictions too: e.g., predicting the specific rounds an agent will highly value , or predicting clusters of agents with similar interests to anticipate overlaps. Incorporating richer predictions could further improve fairness. Although our theoretical model sticks to the simpler total value predictions, the algorithm is general enough that if one had, say, a prediction of the distribution of values each round, one could integrate that by adjusting the weights or baseline schedule accordingly.

A potential extension is to use online learning to update the predictions themselves. Our algorithm currently treats predictions as given and fixed. But if we detect large errors, one could feed this back into a machine learning model to refine its predictions in a longer-term deployment. An exciting direction is designing algorithms that not only use predictions but also *update* them online (meta-learning), achieving a closed-loop system that improves over time.

c) Fairness Notions and Objectives.: We focused on proportional fairness, which as we discussed is a strong notion implying other common fairness criteria like the core and envy-freeness up to one good, etc. Some applications might prioritize other objectives, such as max-min fairness. Max-min fairness can be much more demanding, so proportional fairness is usually preferred for its balance. However, it's worth noting that our algorithm in spirit tries to equalize $U_i/\hat{V}_i$ for all agents. If $\hat{V}_i$ were all equal, that essentially tries to equalize actual utilities, aligning with max-min fairness to some degree. So one might speculate that a variant of our approach could yield guarantees for max-min fairness.

V. Conclusion

We developed an adaptive algorithm for proportionally fair online allocation of public goods that uses predictions to surpass previous fairness guarantees. By integrating learning-augmented advice with an online convex optimization framework, ASAP achieves constant-factor proportional fairness under accurate predictions while retaining logarithmic fairness when predictions fail, showing that the known logarithmic barrier can be exceeded through added information and adaptivity.

Our approach introduces a flexible way to merge predictions with dual-weight updates, and the ideas of promised utility and adaptive budget splitting may be useful in other allocation settings. We also extended the analysis to batched arrivals, capturing more practical scenarios and improving performance under concurrent decisions.

Future work includes empirical evaluation on real data, exploring whether stronger predictions can approach optimal online fairness, and adapting the method to related fairness measures. This line of work strengthens the case for combining machine learning with online algorithm design to advance both efficiency and fairness in multi-agent systems.

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