MAP2121 - CÁLCULO NUMÉRICO (POLI - USP)
List of exercises about zeros (roots) of functions
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1. Show that the function $f(x) = x^2 - 4x + \cos(x)$ has exactly two roots: $\alpha_1 \in [0, 1.8]$ and $\alpha_2 \in [3, 5]$. Consider the functions:

$$\phi_1(x) = \frac{x^2 + \cos(x)}{4} \quad \text{and} \quad \phi_2(x) = \frac{\cos(x)}{4 - x}.$$

Mark with C for the correct alternatives and E for the wrong alternatives:

a) ϕ_1 can be used in the interval $[0, 1.8]$ to approximate α_1 by successive approximation method, but $\phi_2(x)$ can't be used in this interval.

b) ϕ_1 and $\phi(x)_2$ can be used in the interval $[0, 1.8]$ to approximate α_1 by successive approximation method.

c) $\phi(x)_2$ can be used in the interval $[3, 5]$ to approximate α_2 by successive approximation method, but $\phi_1(x)$ can't be used in this interval.

d) ϕ_1 and $\phi(x)_2$ can be used in the interval $[3, 5]$ to approximate α_2 by successive approximation method.

e) ϕ_1 can be used to approximate α_1 in the interval $[0, 1.8]$ and also to approximate α_2 in the interval $[3, 5]$.

Solution: We define $g(x) = x - f(x) = -x^2 - 3x - \cos(x)$. Then, $g'(x) = \sin(x) - 2x - 3$. Note that, $g(0) = -1 < 0$ and $g(1.8) = 5.9872 > 1.8$. In addition, $g'(x)$ is decreasing for all $x \in [0, 1.8]$, therefore its maximum value occurs at $x = 0$. Since $g'(0) = -3$, we deduce that $g'(x) \neq 1$. So, there exist a fixed point α_1 of $g(x)$ in $[0, 1.8]$, which means that there exist a root of $f(x)$ in $[0, 1.8]$. Analogously we can show that there exist a fixed point α_2 of $f(x)$ in $[3, 5]$.