TASK 1 (Implementation for Python 3)
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LU decomposition for a tridiagonal matrix

Let $A \in \mathbb{R}^{n\times n}$ be the tridiagonal matrix given by

$$A = \begin{bmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & & \\ & a_2 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & & a_{n-1} & b_{n-1} \end{bmatrix}.$$

Let $a, b, c \in \mathbb{R}^n$ be vectors that store the diagonals of the matrix A in the following way:

$$a = (0, a_1, \ldots, a_{n-1}), \quad b = (b_0, b_1, \ldots, b_{n-1}), \quad c = (c_0, \ldots, c_{n-2}, 0).$$

Assuming that the LU decomposition of A exists, we have that

$$A = \begin{bmatrix} 1 & & & & & \\ l_0 & 1 & & & & \\ & l_1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & l_{n-3} & 1 & \\ & & & & l_{n-2} & 2 \end{bmatrix} \begin{bmatrix} u_0 & c_0 & & & & \\ & u_1 & c_1 & & & \\ & & u_2 & c_2 & & \\ & & & \ddots & \ddots & \\ & & & & u_{n-2} & c_{n-2} \\ & & & & & u_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} u_0 & c_0 & & & & \\ l_0\, u_0 & l_0\, c_0 + u_1 & c_1 & & & \\ & l_1\, u_1 & l_1\, c_1 + u_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & l_{n-3}\, u_{n-3} & l_{n-3}\, c_{n-3} + u_{n-2} & c_{n-2} \\ & & & & l_{n-2}\, u_{n-2} & l_{n-2}\, c_{n-2} + u_{n-1} \end{bmatrix}.$$

Then, comparing term by term we get

- $b_0 = u_0$.
- $a_i = l_{i-1}\, u_{i-1} \Rightarrow l_{i-1} = a_i/u_{i-1}$, for all $i = 1, \ldots, n-1$.
- $b_i = l_{i-1}\, c_{i-1} + u_i \Rightarrow u_i = b_i - l_{i-1}\, c_{i-1}$, for all $i = 1, \ldots, n-1$.

Solving the linear system $Ax = d$

Using the LU decomposition of A we can solve the linear system $Ax = d$ in the following way: