1.1.1

For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y

$$f(x, y) = x^2 + y^2 + \beta xy + x + 2y.$$

Which of these stationary points are global minima?

1.1.2

In each of the following problems fully justify your answer using optimality conditions.

(a) Show that the 2-dimensional function $f(x, y) = (x^2 - 4)^2 + y^2$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.

(b) Find all local minima of the 2-dimensional function $f(x, y) = \frac{1}{2}x^2 + x \cos y$.

(c) Find all local minima and all local maxima of the 2-dimensional function $f(x, y) = \sin x + \sin y + \sin(x + y)$ within the set $\{(x, y) \mid 0 < x < 2\pi, 0 < x < 2\pi\}$.

(d) Show that the 2-dimensional function $f(x, y) = (y - x^2)^2 - x^2$ has only one stationary point, which is neither a local maximum nor a local minimum.

(e) Consider the minimization of the function f in part (d) subject to no constraint on x and the constraint $-1 \leq y \leq 1$ on y. Show that there exists at least one global minimum and find all global minima.

1.1.3 [Hes75]

Let $f : \Re^n \mapsto \Re$ be a differentiable function. Suppose that a point x^* is a local minimum of f along every line that passes through x^*; that is, the function

$$g(\alpha) = f(x^* + \alpha d)$$

is minimized at $\alpha = 0$ for all $d \in \Re^n$.

(a) Show that $\nabla f(x^*) = 0$.

(b) Show by example that x^* need not be a local minimum of f. *Hint*: Consider the function of two variables $f(y, z) = (z - py^2)(z - qy^2)$, where $0 < p < q$; see Fig. 1.1.5. Show that $(0, 0)$ is a local minimum of f along every line that passes through $(0, 0)$. Furthermore, if $p < m < q$, then $f(y, my^2) < 0$ if $y \neq 0$ while $f(0, 0) = 0$.

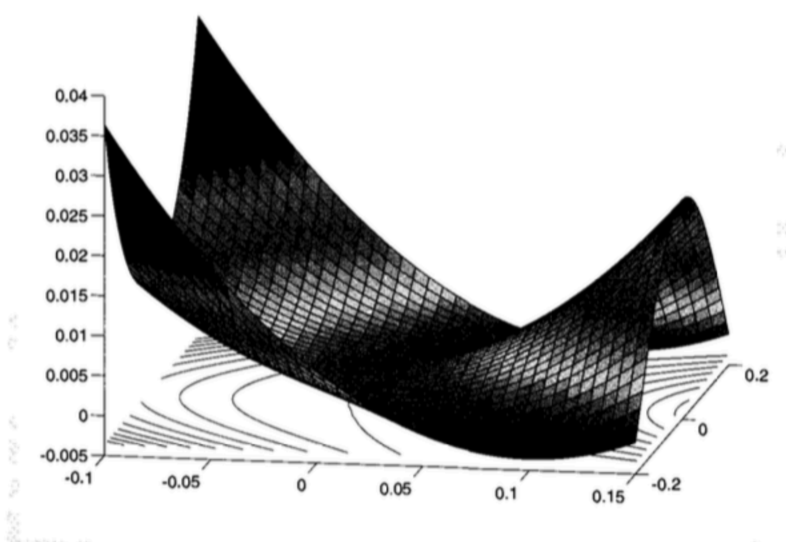

Figure 1.1.5. Three-dimensional graph of the function $f(y, z) = (z - py^2)(z - qy^2)$ for $p = 1$ and $q = 4$ (cf. Exercise 1.1.3). The origin is a local minimum with respect to every line that passes through it, but is not a local minimum of f.