Principles of Quantum Mechanics

George Lee
Girton College

October 7, 2015

1 Introduction

1.1 Blah

sergserhiugyhieru

2 Dirac Formalism

2.1 States and Operators

State of a quantum system described by a ket member $|\psi\rangle$ of a $\mathbb{C}$-Hilbert space $\mathcal{H}$. We have the corresponding dual space of bra vectors $\langle\phi| \in \mathcal{H}^\dagger$. The space is suffieciently well behaved that for each member $|\psi\rangle$ of the space there is a corresponding dual vector $\langle\psi| = (|\psi\rangle)^*$. Have sesquilinear inner product $\langle\sim | \sim\rangle : \mathcal{H}^2 \to \mathbb{C}$ etc.

Physically $|\psi\rangle \mapsto \alpha |\psi\rangle$ transforms the ket to an equivalent state. For a particle we often choose $\langle\psi|\psi\rangle = 1$, note that we often study systems whioch are idealisations, eg infinite plane waves where $\langle\psi|\psi\rangle = \infty$. Even then given $\theta \in \mathbb{R}$ the map $|\psi\rangle \mapsto e^{i\theta} |psi\rangle$ gives equivalent systems, though we find phase differences do matter.

What with this being a vector space we have linear operators on it. Given $Q : \mathcal{H} \to \mathcal{H}$ linear we can define $Q^\dagger : \mathcal{H}^\dagger \to \mathcal{H}^\dagger$ by $Q^\dagger : \langle\psi| \mapsto (Q |\psi\rangle)^\dagger$.

Eigenstates of Q with eigenvalue λ of course satisfy the equation $Q |\psi\rangle = \lambda |\psi\rangle$. Oh, also $[A, B] = AB - BA$ is the commutator. Neat identities are associated with it.

2.2 Observables and measurements

Operator Q Adjoint/Hermitian if we have $Q^\dagger = Q$ (THIS DOES NOT MAKE SENSE UNLESS WE'RE LIKE, LOL, Say two maps are the same if for all $|\psi\rangle\, |\phi\rangle$ we have $\langle\phi|Q|\psi\rangle = \langle\phi|Q'|\psi\rangle$). Can show for Q Hermitian:

- $Q\left|\psi\right\rangle = \lambda\left|\psi\right\rangle \implies \lambda \in \mathbb{R}$
- $\lambda \neq \lambda', Q\left|\psi\right\rangle = \lambda\left|\psi\right\rangle, Q\left|\phi\right\rangle = \lambda'\left|\phi\right\rangle \implies \left\langle\psi|\phi\right\rangle = 0$
- Writing $\mathbb{V}_\lambda$ for the eigenstates of value λ, we have $\bar{span}(\mathbb{V}_\lambda : \lambda \in \mathbb{R}) = \mathcal{H}$

Maybe I'll write out a bit of a rambling proof later. Or maybe just check out linear ananlysis or something?

Diagonalising Q is a matter of rewriting the states we care about in terms of Q's eigenstates. For q Q with a discrete spectrum of eigenvalues with $\forall\lambda : dim(\mathbb{V}_\lambda)$ we can write

$$\left|\psi\right\rangle = \sum_\lambda \sum_{n=1}^{dim(\mathbb{V}_\lambda)} \left|n,\lambda\right\rangle \left\langle n,\lambda|\psi\right\rangle = \sum_\lambda \sum_{n=1}^{dim(\mathbb{V}_\lambda)} \alpha_{n,\lambda}\left|n,\lambda\right\rangle$$

ie, we have that the identity operator $Id = \sum_\lambda \sum_{n=1}^{dim(\mathbb{V}_\lambda)} \left|n,\lambda\right\rangle \left\langle n,\lambda\right|$. This simplifies, for a nondegenerate spetrum, ie, when $\forall\lambda : dim(\mathbb{V}_\lambda) = 1$ to

$$\left|\psi\right\rangle = \sum_\lambda \left|n,\lambda\right\rangle \left\langle\lambda|\psi\right\rangle = \sum_\lambda \alpha_\lambda \left|\lambda\right\rangle$$

Here $\alpha_X = \left\langle X|\psi\right\rangle$.

Physically Q may correspond to a physical quantity, eg when $Q = \hat{\mathcal{H}}$, the Hamiltonian. When a measurement of the quantity is made we find that $\mathbb{P}(\lambda = \lambda') = \sum_{n=1}^{dim(\mathbb{V}'_\lambda)} \left\langle n,\lambda'|\psi\right\rangle^2$. Upon measurement of value λ the wave-function's non-λ-eigenvalue components will vanish, and the other components may be renormalised as appropriate. Thus the expected value of λ is

$$\mathbb{E}(Q) = \left\langle\psi|Q|\psi\right\rangle = \sum_\lambda \mathbb{P}(\lambda) = \sum_\lambda \sum_{n=a}^{dim(\lambda)} \lambda\alpha_{n,\lambda}$$