Probability and Measure

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1 Measures

1.1 Definitions

Let E be a set. A σ-*algebra* $\mathcal{E}$ on E is a collection of subsets of E s.t.

$$\forall n \in \mathbb{N}\ A_n \in \mathcal{E} \implies \bigcup_{\mathbb{N}} A_n \in \mathcal{E},$$
$$A \in \mathcal{E} \implies A^c \in \mathcal{E}$$

Pair $(E, \mathcal{E})$ a measurable space, $\mathcal{E}$ the collection of measurable sets.
$\mu : (E, \mathcal{E}) \to [0, \infty]$ called a measure if for all disjoint $\{A_n\}_{\mathbb{N}} \subset \mathcal{E}$ we have that

$$\mu(\bigcup_{\mathbb{N}} A_n) = \sum_{\mathbb{N}} \mu(A_n).$$

ie, countable additivity. $(E, \mathcal{E}, \mu)$ a measure space.

1.2 Discrete measure theory

Given $f : E \to [0, \infty]$ can do measure theory on measurable space $(E, 2^E)$ via $\mu(A) = \sum_A f(a)$.

1.3 Generated σ - algebras

Let $\mathcal{A} \subset 2^E$. Define

$$\sigma(\mathcal{A}) = \bigcap \{\sigma\ algebras \supseteq \mathcal{A}\}$$

Then (easy to check) $\sigma(\mathcal{A})$ a σ-algebra.

1.4 π-systems and d-systems

Let $\emptyset \in \mathcal{A} \subseteq 2^E$. Have $\mathcal{A}$ a π-system if $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$. We say that $E \in \mathcal{A} \subseteq 2^E$ is a d-system if

$$A, B \in \mathcal{A}, A \subseteq B \implies B \backslash A \in \mathcal{A},$$
$$(A_n)_{\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}, A_1 \subseteq A_2 \subseteq ... \implies \bigcup_{\mathbb{N}} A_n \in \mathcal{A}$$

Note then that $\mathcal{A}$ both of these $\implies A$ a σ-algebra.

Lemma 1.1. *Dynkin's π system lemma: Let $\mathcal{A}$ ba a π-system. Then any d-system containing $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

Proof. Let $\mathcal{D} = \bigcap\{d\text{-}systems \supseteq \mathcal{A}\}$. Then $\mathcal{D}$ a d-system. We show $\mathcal{D}$ also a π system and thus a σ-algebra as required. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

Then $\mathcal{D} \subseteq \mathcal{D}'$ ($\mathcal{A}$ a d-system). Check $\mathcal{D}'$ a d-system: have $E \in \mathcal{D}'$, and let $B, C \in \mathcal{D}',\ B \subseteq C$, then given $A \in \mathcal{A}$ we have

$$(C \backslash B) \cap A = (C \cap A) \backslash (B \cap A) \in \mathcal{D} \implies C \backslash B \in \mathcal{D}'$$

Write $B_n \uparrow B$ if $B_1 \subseteq B_2 \subseteq ...$ and $B = \bigcup_{\mathbb{N}} B_n$. Let $(B_n)_{\mathbb{N}} \in \mathcal{D}'^{\mathbb{N}}$ be increasing, then $B_n \cap A \uparrow B \cap A$. Thus $B \cap A \in \mathcal{D} \implies B \in \mathcal{D}' \implies B \in \mathcal{D}' \implies \mathcal{D} = \mathcal{D}'$. Then let

$$\mathcal{D}'' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

$A, A' \in \mathcal{A} \implies A \cup A' \in \mathcal{D}$ and thus $\mathcal{A} \subseteq \mathcal{D}''$. Check $\mathcal{D}''$ a d-system, like with $\mathcal{D}'$. Then $\mathcal{D}'' = \mathcal{D} \implies \mathcal{D}$ a π-system. $\square$

1.5 Set functions and properties

Let $\emptyset \in \mathcal{A} \subseteq 2^E$. We call any $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$ a set function. Let μ be such a function.

- $(A, B \in \mathcal{A}, A \subseteq B \implies \mu(A) \leq \mu(B)) \implies \mu$ *increasing.*
- $(A, B, A \dot{\cup} B \in \mathcal{A} \implies \mu(A \cup B) = \mu(A) + \mu(B)) \implies \mu$ *additive.*
- $(A_1, A_2, ..., \dot{\bigcup}_{\mathbb{N}} A_n \in \mathcal{A} \implies \mu(\bigcup_{\mathbb{N}} A_n) = \sum_{\mathbb{N}} \mu(A_n)) \implies \mu$ *countably additive.*

1.6 Construction of measures

Let $\mathcal{A} \subseteq 2^E$. Say $\mathcal{A}$ a *ring on E* if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies B \backslash A, A \cup B \in \mathcal{A}$

Say $\mathcal{A}$ an *algebra on* E if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies A^c, A \cup B \in \mathcal{A}$

Theorem 1.2. *Carathéodory's extention theorem: Let $\mathcal{A}$ a ring on E and μ a countably additive set function on $\mathcal{A}$. Then can extend μ to a measure on $\sigma(\mathcal{A})$.*

Proof. Will write up later □

1.7 Uniqueness of measures

Theorem 1.3. *Uniqueness of extention:srlgxdfkghdfg*

Proof. shtghfghcfgh □

1.8 Borel sets and measures

Let (E, τ) a topological space. Then the Borel σ-algebra on E is defined by $\mathcal{B}(E) = \sigma(\tau)$. For $E = \mathbb{R}$ we may write $\mathcal{B}(E) = \mathcal{B}$. A corresponding measure on such a space is called a Borel measure. If further all compact sets have a finite measure we say it is a Radon measure.

1.9 Probability measures, finite and σ-finite measures.

For $(E, \mathcal{E}, \mu)$ a measure space, we say μ is a probability measure if $\mu(E) = 1$, and tend to write instead $(\Omega, \mathcal{F}, \mathbb{P})$. $\mu(E) < \infty \implies \mu$ a finite measure, and finally if there is a countable collection of measurable sets of finite measure which cover E then we say the space is σ-finite.

1.10 Lebesgue measure

1.11 Existence of a non-Lebesgue-measurable subset of $\mathbb{R}$

1.12 Independence

1.13 Borel-Cantelli lemmas

2 Measurable functions and Random Variables

2.1 Measurable functions

Let $(E, \mathcal{E}), (F, \mathcal{F})$ be measurable spaces. We say $f : E \to F$ is measurable if $U \in \mathcal{F} \implies f^{-1}U \in \mathcal{E}$.

- $(G, \mathcal{G}) = (\mathbb{R}, \mathcal{B}) \implies f$ is a measurable function on E
- $(G, \mathcal{G}) = ([0, \infty], \mathcal{B}([0, \infty])) \implies f$ is a non-negative measurable function on E

note neither definitions are more strict than the other. Now seeing as inverse images preserve union and complementation, to show that a function is measurable it is sufficient to show that the inverse images of a collection of sets generating the σ-algebra are measurable. For example in the case where f is a measurable function on E it suffices to show that the collection $\{f^{-1}(-\infty, x] : x \in \mathbb{R}\} = \{\{f(\omega) \leq x\} : x \in \mathbb{R}\} \subseteq \mathcal{F}$. Note that if the measures are Borel and the function is continuous it immediately follows it is measurable.

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Theorem 2.1. *Monotone Class theorem: Let $(E, \mathcal{E})$ be a measurable space and let $\mathcal{A}$ ba a π-system with $\sigma(\mathcal{A}) = \mathcal{E}$. Let $\mathbb{V}$ be an $\mathbb{R}$-vector space of bounded $f : E \to \mathbb{R}$ such that the following hold:*

- $x \mapsto 1 \in \mathbb{V}$
- $(f_n)_{\mathbb{N}} \in \mathbb{V}^{\mathbb{N}}$ *with* $0 \leq f_n \uparrow f \implies f \in \mathbb{V}$

then $\{bounded\ measurable\ functions\} \subseteq \mathbb{V}$.

Proof. Let $\mathcal{D} = \{A \in \mathcal{E} : 1_A \in \mathbb{V}\}$. Then we have $\mathcal{A} \subseteq \mathcal{D}$ a d-system $\implies \mathcal{D} = \mathcal{E}$. Thus all finite linear combinations of indicator functions are contained in $\mathbb{V}$, in particular given a nonnegative bounded measurable f the series $(x \mapsto 2^{-n}\lfloor 2^n f(x) \rfloor)_{\mathbb{N}}$ is contained in $\mathcal{A}$ and converges from below to f so $f \in \mathbb{V}$. Then for an arbitrary bounded measurable function, decompose into positive and negative components both of which are in $\mathbb{V}$ to see that $f \in \mathbb{V}$. □

2.2 Image measures

Just like with topologies, you can get induced measures using inverse images!

Lemma 2.2. *Let $g : \mathbb{R} \to \mathbb{R}$ nonconstant, right continuous and non-decreasing, and extend to $\pm\infty$ via limits, and write $I = (g(-\infty), g(\infty))$. Define $f : I \to \mathbb{R}$ by $f(x) = inf\{y \in \mathbb{R} : x \leq g(y)\}$. Then f is left continuous and non-decreasing.*

Proof. Let $x \in I$ and $J_x = \{y \in \mathbb{R} : x \leq g(y)\}$. Have $\emptyset \subset J_x \subset \mathbb{R}$. g nondecreasing $\implies J_x = [x', \infty]$ for some x'. g right-continuous means that $y_n \in J_x, y_n \downarrow y \implies y \in J_x$. Thus we have $J_x = [f(x), \infty)$ and $x \leq g(y)$ exactly when $f(x) \leq y$. Given $x \leq x'$ have $J_x \supseteq J_{x'}$so that $f(x) \leq f(x')$. For $x_n \uparrow x$ have $J_x = \bigcup_{\mathbb{N}} J_{x_n}$ and so $f(x_n) \to f(x)$. Thus f is left-continuous and non-decreasing as required. □

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2.3 Random variables

For $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $(E, \mathcal{E})$ a measurable space, a measurable $X : \Omega \to E$ called a random variable in E, which is interpreted as a state of a system influenced by chance. Where E isn't mentioned we assume $E = \mathbb{R}$. blah blah blah

2.4 Rademacher functions

Blaaaaahhhh

2.5 Convergence of measurable functions and Random Variables