

# Probability and Measure

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## 1 Measures

### 1.1 Definitions

Let  $E$  be a set. A  $\sigma$ -algebra  $\mathcal{E}$  on  $E$  is a collection of subsets of  $E$  s.t.

$$\begin{aligned}\forall n \in \mathbb{N} \ A_n \in \mathcal{E} &\implies \bigcup_{\mathbb{N}} A_n \in \mathcal{E}, \\ A \in \mathcal{E} &\implies A^c \in \mathcal{E}\end{aligned}$$

Pair  $(E, \mathcal{E})$  a measurable space,  $\mathcal{E}$  the collection of measurable sets.  
 $\mu : (E, \mathcal{E}) \rightarrow [0, \infty]$  called a measure if for all disjoint  $\{A_n\}_{\mathbb{N}} \subset \mathcal{E}$  we have that

$$\mu\left(\bigcup_{\mathbb{N}} A_n\right) = \sum_{\mathbb{N}} \mu(A_n).$$

ie, countable additivity.  $(E, \mathcal{E}, \mu)$  a measure space.

### 1.2 Discrete measure theory

Given  $f : E \rightarrow [0, \infty]$  can do measure theory on measurable space  $(E, 2^E)$  via  $\mu(A) = \sum_A f(a)$ .

### 1.3 Generated $\sigma$ - algebras

Let  $\mathcal{A} \subset 2^E$ . Define

$$\sigma(\mathcal{A}) = \bigcap \{\sigma \text{ algebras } \supseteq \mathcal{A}\}$$

Then (easy to check)  $\sigma(\mathcal{A})$  a  $\sigma$ -algebra.

## 1.4 $\pi$ -systems and $d$ -systems

Let  $\emptyset \in \mathcal{A} \subseteq 2^E$ . Have  $\mathcal{A}$  a  $\pi$ -system if  $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$ . We say that  $E \in \mathcal{A} \subseteq 2^E$  is a  $d$ -system if

$$\begin{aligned} A, B \in \mathcal{A}, A \subseteq B &\implies B \setminus A \in \mathcal{A}, \\ (A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}, A_1 \subseteq A_2 \subseteq \dots &\implies \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A} \end{aligned}$$

Note then that  $\mathcal{A}$  both of these  $\implies \mathcal{A}$  a  $\sigma$ -algebra.

**Lemma 1.1.** *Dynkin's  $\pi$  system lemma: Let  $\mathcal{A}$  be a  $\pi$ -system. Then any  $d$ -system containing  $\mathcal{A}$  also contains  $\sigma(\mathcal{A})$ .*

*Proof.* Let  $\mathcal{D} = \bigcap \{d\text{-systems} \supseteq \mathcal{A}\}$ . Then  $\mathcal{D}$  a  $d$ -system. We show  $\mathcal{D}$  also a  $\pi$ -system and thus a  $\sigma$ -algebra as required. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

Then  $\mathcal{D} \subseteq \mathcal{D}'$  ( $\mathcal{A}$  a  $d$ -system). Check  $\mathcal{D}'$  a  $d$ -system: have  $E \in \mathcal{D}'$ , and let  $B, C \in \mathcal{D}'$ ,  $B \subseteq C$ , then given  $A \in \mathcal{A}$  we have

$$(C \setminus B) \cap A = (C \cap A) \setminus (B \cap A) \in \mathcal{D} \implies C \setminus B \in \mathcal{D}'$$

Write  $B_n \uparrow B$  if  $B_1 \subseteq B_2 \subseteq \dots$  and  $B = \bigcup_{n \in \mathbb{N}} B_n$ . Let  $(B_n)_{n \in \mathbb{N}} \in \mathcal{D}'^{\mathbb{N}}$  be increasing, then  $B_n \cap A \uparrow B \cap A$ . Thus  $B \cap A \in \mathcal{D} \implies B \in \mathcal{D}' \implies B \in \mathcal{D}' \implies \mathcal{D} = \mathcal{D}'$ . Then let

$$\mathcal{D}'' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

$A, A' \in \mathcal{A} \implies A \cup A' \in \mathcal{D}$  and thus  $\mathcal{A} \subseteq \mathcal{D}''$ . Check  $\mathcal{D}''$  a  $d$ -system, like with  $\mathcal{D}'$ . Then  $\mathcal{D}'' = \mathcal{D} \implies \mathcal{D}$  a  $\pi$ -system.  $\square$

## 1.5 Set functions and properties

Let  $\emptyset \in \mathcal{A} \subseteq 2^E$ . We call any  $\mu : \mathcal{A} \rightarrow [0, \infty]$  with  $\mu(\emptyset) = 0$  a set function. Let  $\mu$  be such a function.

- $(A, B \in \mathcal{A}, A \subseteq B \implies \mu(A) \leq \mu(B)) \implies \mu \text{ increasing.}$
- $(A, B, A \dot{\cup} B \in \mathcal{A} \implies \mu(A \cup B) = \mu(A) + \mu(B)) \implies \mu \text{ additive.}$
- $(A_1, A_2, \dots, \dot{\bigcup}_{n \in \mathbb{N}} A_n \in \mathcal{A} \implies \mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)) \implies \mu \text{ countably additive.}$

## 1.6 Construction of measures

Let  $\mathcal{A} \subseteq 2^E$ . Say  $\mathcal{A}$  a *ring on  $E$*  if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies B \setminus A, A \cup B \in \mathcal{A}$

Say  $\mathcal{A}$  an algebra on  $E$  if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies A^c, A \cup B \in \mathcal{A}$

**Theorem 1.2.** *Caratheodory's extention theorem: Let  $\mathcal{A}$  a ring on  $E$  and  $\mu$  a countably additive set function on  $\mathcal{A}$ . Then can extend  $\mu$  uniquely to a measure on  $\sigma(\mathcal{A})$ .*

*Proof.*

□