Linear Analysis

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1 Introduction

2 Normed vector spaces

2.1 Vector spaces

Vector things

2.2 Normed vector spaces: the definition

definition 2.1. *A normed vector space* $\mathbb{V}$ *is an* $\mathbb{R}$ *or* $\mathbb{C}$*-vector space along with a norm* $\| \sim \| : \mathbb{V} \to \mathbb{R}$ *such that*

1. $\forall v \in \mathbb{V} : 0 \leq \|v\|$
2. $\forall \lambda \in \mathbb{R}, v \in \mathbb{V} : \|\lambda v\| = |\lambda| \|v\|$
3. $\forall v, w \in \mathbb{V} : \|v + w\| \leq \|v\| + \|w\|$

3 The relation with the topology

Clear that $\| - \|$ defines a metric via taking the norm of the difference between vectors, and hence a notion of topology. Thus can use Met+Top etc a load in this course. Might use $\tau_{\mathbb{V}}$ to denote the collection of open sets.

Lemma 3.1. *Let* $(\mathbb{V}, | - |)$ *be a normed vector space. Then the operations of addition and scalar multiplication are continuous.*

Proof. For $+$: let $U \in \tau_{\mathbb{V}}$, want $+^{-1}U \in \mathbb{V}^2$. Let $(v_1, v_2) \in +^{-1}U$, ie, $v_1 + v_2 \in U$. For some $0 \leq \epsilon$ we have $v_1 + v_2 + \mathcal{B}(\epsilon) = v_1 + \mathcal{B}(\frac{\epsilon}{2}) + v_2 + \mathcal{B}(\frac{\epsilon}{2}) \subseteq U$. This is exactly the image of $(v_1 + \mathcal{B}(\frac{\epsilon}{2}), v_1 + \mathcal{B}(\frac{\epsilon}{2})) \in \tau_{\mathbb{V}^2}$ under $+$. $\square$

Corollary 3.1.1. *Let* $\mathbb{V}$ *be a normed vector space. transplations and dilations are homeomorphisms.*

3.1 More abstraction: topological vector spaces

definition 3.2. *A topological vector space is a vector space* $\mathbb{V}$ *together with a topology such that addition and scalar multiplication are continuous, and each singleton set is closed.*