Probability and Measure

George Lee
Girton College

October 7, 2015

1 Measures

1.1 Definitions

Let E be a set. A *σ-algebra* $\mathcal{E}$ on E is a collection of subsets of E s.t.

$$\forall n \in \mathbb{N}\ A_n \in \mathcal{E} \implies \bigcup_{\mathbb{N}} A_n \in \mathcal{E},$$
$$A \in \mathcal{E} \implies A^c \in \mathcal{E}$$

Pair $(E, \mathcal{E})$ a measurable space, $\mathcal{E}$ the collection of measurable sets.
$\mu : (E, \mathcal{E}) \to [0, \infty]$ called a measure if for all disjoint $\{A_n\}_{\mathbb{N}} \subset \mathcal{E}$ we have that

$$\mu(\bigcup_{\mathbb{N}} A_n) = \sum_{\mathbb{N}} \mu(A_n).$$

ie, countable additivity. $(E, \mathcal{E}, \mu)$ a measure space.

1.2 Discrete measure theory

Given $f : E \to [0, \infty]$ can do measure theory on measurable space $(E, 2^E)$ via $\mu(A) = \sum_A f(a)$.

1.3 Generated σ - algebras

Let $\mathcal{A} \subset 2^E$. Define

$$\sigma(\mathcal{A}) = \bigcap \{\sigma\ algebras \supseteq \mathcal{A}\}$$

Then (easy to check) $\sigma(\mathcal{A})$ a σ-algebra.

1.4 π-systems and d-systems

Let $\emptyset \in \mathcal{A} \subseteq 2^E$. Have $\mathcal{A}$ a π-system if $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$. We say that $E \in \mathcal{A} \subseteq 2^E$ is a d-system if

$$A, B \in \mathcal{A}, A \subseteq B \implies B \backslash A \in \mathcal{A},$$
$$(A_n)_{\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}, A_1 \subseteq A_2 \subseteq ... \implies \bigcup_{\mathbb{N}} A_n \in \mathcal{A}$$

Note then that $\mathcal{A}$ both of these $\implies A$ a σ-algebra.

Lemma 1.1. *Dynkin's π system lemma: Let $\mathcal{A}$ ba a π-system. Then any d-system containing $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

Proof. Let $\mathcal{D} = \bigcap\{d\text{-}systems \supseteq \mathcal{A}\}$. Then $\mathcal{D}$ a d-system. We show $\mathcal{D}$ also a π system and thus a σ-algebra as required. Consider

$$\mathcal{D}' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

Then $\mathcal{D} \subseteq \mathcal{D}'$ ($\mathcal{A}$ a d-system). Check $\mathcal{D}'$ a d-system: have $E \in \mathcal{D}'$, and let $B, C \in \mathcal{D}',\ B \subseteq C$, then given $A \in \mathcal{A}$ we have

$$(C \backslash B) \cap A = (C \cap A) \backslash (B \cap A) \in \mathcal{D} \implies C \backslash B \in \mathcal{D}'$$

Write $B_n \uparrow B$ if $B_1 \subseteq B_2 \subseteq ...$ and $B = \bigcup_{\mathbb{N}} B_n$. Let $(B_n)_{\mathbb{N}} \in \mathcal{D}'^{\mathbb{N}}$ be increasing, then $B_n \cap A \uparrow B \cap A$. Thus $B \cap A \in \mathcal{D} \implies B \in \mathcal{D}' \implies B \in \mathcal{D}' \implies \mathcal{D} = \mathcal{D}'$. Then let

$$\mathcal{D}'' = \{B \in \mathcal{D} : \forall A \in \mathcal{A} : B \cap A \in \mathcal{D}\} \subseteq \mathcal{D}$$

$A, A' \in \mathcal{A} \implies A \cup A' \in \mathcal{D}$ and thus $\mathcal{A} \subseteq \mathcal{D}''$. Check $\mathcal{D}''$ a d-system, like with $\mathcal{D}'$. Then $\mathcal{D}'' = \mathcal{D} \implies \mathcal{D}$ a π-system. □

1.5 Set functions and properties

Let $\emptyset \in \mathcal{A} \subseteq 2^E$. We call any $\mu : \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$ a set function. Let μ be such a function.

- $(A, B \in \mathcal{A}, A \subseteq B \implies \mu(A) \leq \mu(B)) \implies \mu$ *increasing.*
- $(A, B, A \dot{\cup} B \in \mathcal{A} \implies \mu(A \cup B) = \mu(A) + \mu(B)) \implies \mu$ *additive.*
- $(A_1, A_2, ..., \dot{\bigcup}_{\mathbb{N}} A_n \in \mathcal{A} \implies \mu(\bigcup_{\mathbb{N}} A_n) = \sum_{\mathbb{N}} \mu(A_n)) \implies \mu$ *countably additive.*

1.6 Construction of measures

Let $\mathcal{A} \subseteq 2^E$. Say $\mathcal{A}$ a *ring on E* if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies B \backslash A, A \cup B \in \mathcal{A}$

Say $\mathcal{A}$ an *algebra on E* if

- $\emptyset \in \mathcal{A}$
- $A, B \in \mathcal{A} \implies A^c, A \cup B \in \mathcal{A}$

Theorem 1.2. *Caratheodory's extention theorem: Let* $\mathcal{A}$ *a ring on* E *and* μ *a countably additive set function on* $\mathcal{A}$*. Then can extend* μ *uniquely to a measure on* $\sigma(\mathcal{A})$*.*

Proof. $\square$