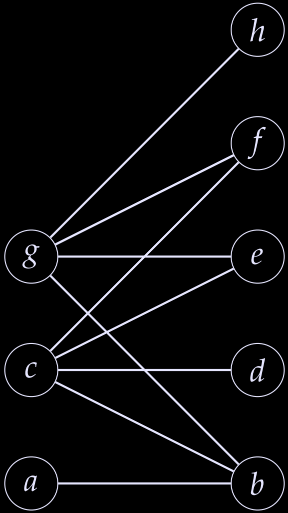
Bipartite graphs



Bipartite graphs

Lemma 1

Let $G = \langle V, E \rangle$ be a graph. Then G is bipartite if and only if it does not have any odd cycle.

Lemma 2

Let $G = \langle V, E \rangle$ be a graph and let M be its **incidence** matrix. Then G is bipartite if and only if M is totally unimodular.

Max cardinality matching

Instance

An unweighted graph $G = \langle V, E \rangle$ with bipartition A, B

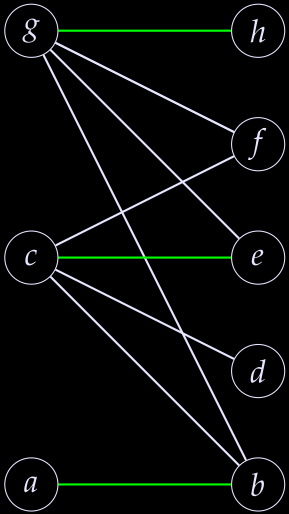
$$\max \sum_{(v,w)\in E} e_{v,w} \quad \text{subject to} \tag{1}$$

$$\sum_{w\in B} e_{v,w} \leq 1 \quad \forall v \in A \tag{2}$$

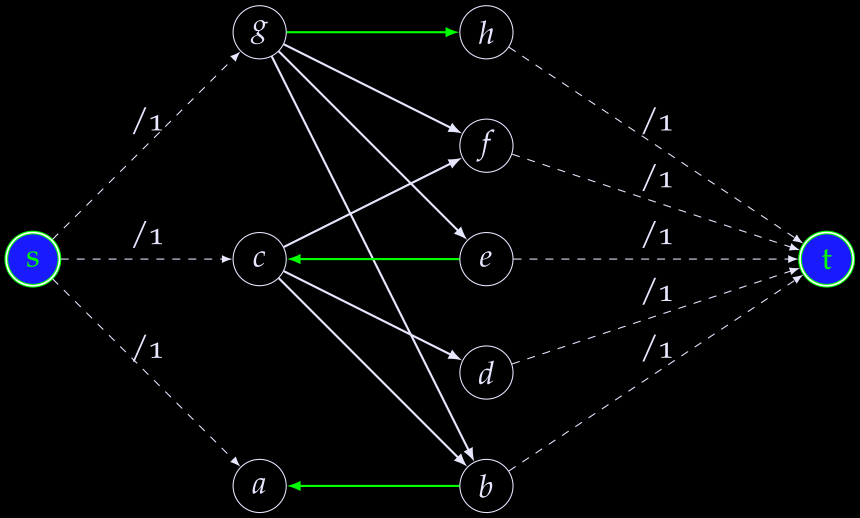
$$\sum_{v\in A} e_{v,w} \leq 1 \quad \forall w \in B \tag{3}$$

$$e_{v,w} \in \{0, 1\} \quad \forall (v, w) \in E$$

Max cardinality matching



Max cardinality matching as Max flow



Min weighted perfect matching

Instance

An edge-weighted graph $G = \langle V, E \rangle$ with bipartition S, T, with weight function c

$$
\begin{aligned}
\min \sum_{(v,w)\in E} c(v,w) e_{v,w} \quad & \text{subject to} \\
\sum_{w\in T} e_{v,w} \leq 1 \quad & \forall v \in S \\
\sum_{v\in S} e_{v,w} \leq 1 \quad & \forall w \in T \\
e_{v,w} \in \{0,1\} \quad & \forall (v,w) \in E
\end{aligned}
$$

Max weighted matching

Instance

An edge-weighted graph $G = \langle V, E \rangle$ with bipartition S, T, with weight function c

Primal

$$\min \sum_{(v,w)\in E} c(v,w) e_{v,w} \quad \text{subject to}$$

$$\sum_{w \in N(v)} e_{v,w} \leq 1 \quad \forall v \in V$$

$$e_{v,w} \geq 0 \quad \forall (v,w) \in E$$

Dual

$$\max \sum_{v \in V} l(v) \quad \text{subject to}$$

$$l(v) + l(w) \leq c_{v,w} \quad \forall (v,w) \in E$$

Invariants

- Always a dual feasible solution $l(v) + l(w) \leq c_{v,w}$
- $R_S \subseteq S$, $R_T \subseteq T$ are the **uncovered** vertices
- G_l: take only the edges $(v, w) \in E$ of G such that $l(v) + l(w) = c_{v,w}$.
  - Arcs in M are from T to S.
  - Arcs not in M are from S to T.
  - G_l has all the edges of M.
- Z: vertices reachable from R_S in G_l
- $(v, w) \in M \Rightarrow l(v) + l(w) = c_{v,w}$

Algorithm

Data: Undirected bipartite weighted graph $G = \langle V, E \rangle$, bipartition (S, T), cost c

$M \leftarrow \emptyset$, $l = \bar{0}$;

while *M is not perfect* **do**

  if $R_T \cap Z \neq \emptyset$ **then**

    Found an alternating path;

    Extend M;

  else

    $\Delta = \min_{c \in S \cap Z, t \in T \setminus Z} \{c(s,t) - c(s) - c(t)\}$;

    foreach $v \in V$ **do**

      if $v \in S \cap Z$ **then** $l(v) \leftarrow l(v) + \Delta$;

      if $v \in T \cap Z$ **then** $l(v) \leftarrow l(v) - \Delta$;

return M

Algorithm correctness

Lemma

Let M, l be a current solution such that M is not a perfect matching. Then there is an augmenting path in G, which implies:

1. there is an augmenting path in G_l or
2. there is a loose-tailed path in G

Faster and incremental algorithm

We have an optimal solution on the vertices $S_{j-1} \cup T$ and we add the vertex j.

Data: M current solution on $S_{j-1} \cup T$, vertex j to add

$Z \leftarrow \{j\}$;

$\Delta = \min_{t \in T \setminus Z} \left\{ c(j,t) - c(j) - c(t) \right\}$;

x is the argmin;

foreach $v \in V$ **do**

- **if** $v \in S \cap Z$ **then** $l(v) \leftarrow l(v) + \Delta$;
- **if** $v \in T \cap Z$ **then** $l(v) \leftarrow l(v) - \Delta$;

if x *is covered by* M **then**

- Add x and its mate to Z, then iterate;

else

- Found an augmenting path;

return M