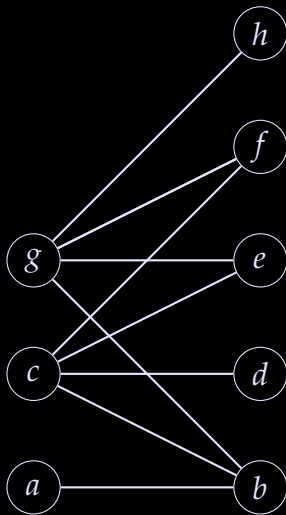


Bipartite graphs



Bipartite graphs

Lemma 1

Let $G = \langle V, E \rangle$ be a graph. Then G is bipartite if and only if it does not have any odd cycle.

Lemma 2

Let $G = \langle V, E \rangle$ be a graph and let M be its **incidence** matrix. Then G is bipartite if and only if M is totally unimodular.

Max cardinality matching

Instance

An unweighted graph $G = \langle V, E \rangle$ with bipartition A, B

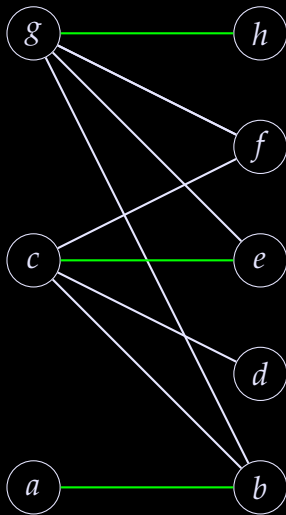
$$\max \sum_{(v,w) \in E} e_{v,w} \quad \text{subject to} \quad (1)$$

$$\sum_{w \in B} e_{v,w} \leq 1 \quad \forall v \in A \quad (2)$$

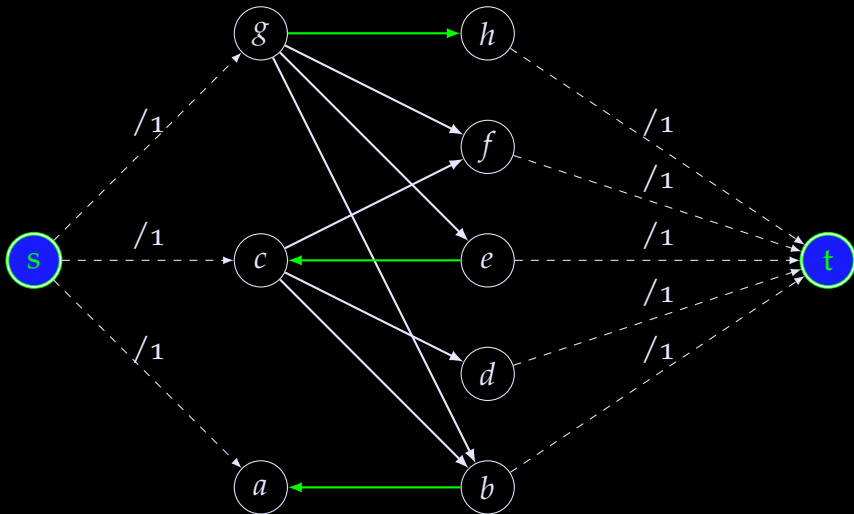
$$\sum_{v \in A} e_{v,w} \leq 1 \quad \forall w \in B \quad (3)$$

$$e_{v,w} \in \{0, 1\} \quad \forall (v, w) \in E$$

Max cardinality matching



Max cardinality matching as Max flow



Min weighted perfect matching

Instance

An edge-weighted graph $G = \langle V, E \rangle$ with bipartition S, T , with weight function c

$$\begin{aligned} \min \quad & \sum_{(v,w) \in E} c(v,w) e_{v,w} \quad \text{subject to} \\ & \sum_{w \in T} e_{v,w} \leq 1 \quad \forall v \in S \\ & \sum_{v \in S} e_{v,w} \leq 1 \quad \forall w \in T \\ & e_{v,w} \in \{0, 1\} \quad \forall (v,w) \in E \end{aligned}$$

Max weighted matching

Instance

An edge-weighted graph $G = \langle V, E \rangle$ with bipartition S, T , with weight function c

Primal

$$\begin{aligned} \min \quad & \sum_{(v,w) \in E} c(v,w) e_{v,w} \quad \text{subject to} \\ & \sum_{w \in N(v)} e_{v,w} \leq 1 \quad \forall v \in V \\ & e_{v,w} \geq 0 \quad \forall (v,w) \in E \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & \sum_{v \in V} l(v) \quad \text{subject to} \\ & l(v) + l(w) \leq c_{v,w} \quad \forall (v,w) \in E \end{aligned}$$

Invariants

- Always a dual feasible solution $l(v) + l(w) \leq c_{v,w}$
- $R_S \subseteq S, R_T \subseteq T$ are the **uncovered** vertices
- G_l : take only the edges $(v, w) \in E$ of G such that $l(v) + l(w) = c_{v,w}$.
 - Arcs in M are from T to S .
 - Arcs not in M are from S to T .
 - G_l has all the edges of M .
- Z : vertices reachable from R_S in G_l
- $(v, w) \in M \Rightarrow l(v) + l(w) = c_{v,w}$

Algorithm

Data: Undirected bipartite weighted graph $G = \langle V, E \rangle$, bipartition (S, T) , cost

```
 $c$   
 $M \leftarrow \emptyset, l = \bar{0};$   
while  $M$  is not perfect do  
  if  $R_T \cap Z \neq \emptyset$  then  
    Found an alternating path;  
    Extend  $M$ ;  
  else  
     $\Delta = \min_{c \in S \cap Z, t \in T \setminus Z} \{c(s, t) - c(s) - c(t)\};$   
    foreach  $v \in V$  do  
      if  $v \in S \cap Z$  then  $l(v) \leftarrow l(v) + \Delta;$   
      if  $v \in T \cap Z$  then  $l(v) \leftarrow l(v) - \Delta;$   
return  $M$ 
```

Algorithm correctness

Lemma

Let M, l be a current solution such that M is not a perfect matching. Then there is an augmenting path in G , which implies:

- 1 there is an augmenting path in G_l or
- 2 there is a loose-tailed path in G

Faster and incremental algorithm

We have an optimal solution on the vertices $S_{j-1} \cup T$ and we add the vertex j .

Data: M current solution on $S_{j-1} \cup T$, vertex j to add

$Z \leftarrow \{j\};$

$\Delta = \min_{t \in T \setminus Z} \{c(j, t) - c(j) - c(t)\};$

x is the argmin;

foreach $v \in V$ **do**

if $v \in S \cap Z$ **then** $l(v) \leftarrow l(v) + \Delta;$

if $v \in T \cap Z$ **then** $l(v) \leftarrow l(v) - \Delta;$

if x is covered by M **then**

 Add x and its mate to Z , then iterate;

else

 Found an augmenting path;

return M

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