Expectation Argument

Lemma

Let X be a random variable on a discrete space S such that $E(X) = \mu$. Then $Pr[X \geq \mu] > 0$ and $Pr[X \leq \mu] > 0$.

Max cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$| E \cap (A \times B) |$

Max cut

Data: graph $G = \langle V, E \rangle$
foreach *vertex* $v \in V$ **do**
    Assign v to A or to B with probability $1/2$
return (A, B)

Max cut

Data: graph $G = \langle V, E \rangle$
foreach *vertex* $v \in V$ **do**
    Assign v to A or to B with probability $1/2$
return (A, B)

Lemma 1 (to prove)

The expected value of the number of edges in the cut (A, B) is at least $m/2$

Lemma 2 (to prove)

The probability that the cut (A, B) has at least $m/2$ edges is $\geq \frac{1}{m/2+1}$

Proof of Lemma 2

Lemma 2

$$p = \Pr\left(C(A,B) \geq \frac{m}{2}\right) \geq \frac{1}{m/2+1}$$

Proof of Lemma 2

Lemma 2

$$p = \Pr\left(C(A,B) \geq \frac{m}{2}\right) \geq \frac{1}{m/2+1}$$

$$\frac{m}{2} = \mathbf{E}[C(A,B)] = \sum_{i \leq \frac{m}{2}-1} i \Pr\left(C(A,B)=i\right) + \sum_{i \geq \frac{m}{2}} i \Pr\left(C(A,B)=i\right)$$

$$\Rightarrow p \geq \frac{1}{m/2+1}$$

Proof of Lemma 2

Lemma 2

$$p = \Pr\left(C(A,B) \geq \frac{m}{2}\right) \geq \frac{1}{m/2+1}$$

$$\begin{aligned}
\frac{m}{2} = \mathbf{E}[C(A,B)] &= \sum_{i \leq \frac{m}{2}-1} i \Pr\left(C(A,B) = i\right) + \sum_{i \geq \frac{m}{2}} i \Pr\left(C(A,B) = i\right) \\
&\leq (1-p)\left(\frac{m}{2} - 1\right) + pm \\
&\Rightarrow p \geq \frac{1}{m/2+1}
\end{aligned}$$

Sample and modifying

Lemma (to prove)

Let $G = \langle V, E \rangle$ be a undirected graph. Then $G = \langle V, E \rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

Sample and modifying

Lemma (to prove)

Let $G = \langle V, E\rangle$ be a undirected graph. Then $G = \langle V, E\rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

Hint 1: $d = \frac{2m}{n}$ is the average degree.
Hint 2: the proof is a probabilistic algorithm

Sample and modifying

1. Sample $S \leftarrow$ keep each vertex with probability $\frac{1}{d}$
2. For each edge in the sample, remove one of its endpoints.

Sample and modifying

1. Sample $S \leftarrow$ keep each vertex with probability $\frac{1}{d}$
2. For each edge in the sample, remove one of its endpoints.

Proof: after step 1

$$\mathbf{E}[|S|] = \frac{n}{d}; \qquad \mathbf{E}[\text{number of edges in } S] \geq \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$$

Sample and modifying

1. Sample $S \leftarrow$ keep each vertex with probability $\frac{1}{d}$
2. For each edge in the sample, remove one of its endpoints.

Proof: after step 1

$$\mathbf{E}[|S|] = \frac{n}{d}; \qquad \mathbf{E}[\text{number of edges in } S] \geq \frac{nd}{2} \cdot \frac{1}{d^2} = \frac{n}{2d}$$

Proof: after step 2

For each edge, we remove at most a vertex. Hence

$$\mathbf{E}[\text{number of surviving vertices}] = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d} = \frac{n^2}{4m}$$

Min cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$\max |E \cap (A \times B)|$

Karger's Algorithm

Data: Undirected graph $G = \langle V, E \rangle$
while $|V| > 2$ **do**
  Pick a random edge (v, w);
  Merge v and w;
$a, b \leftarrow$ the two vertices of G;
return *(vertices merged into a, vertices merged into b)*

Karger, Stein Algorithm

Data: Undirected graph $G = \langle V, E \rangle$, parameter $t \geq 2$
while $|V| > t$ **do**
    Pick a random edge (v, w);
    Merge v and w;
return *optimal solution on G*