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- Large-Scale Graph Algorithms
- Ufficio U14-2041
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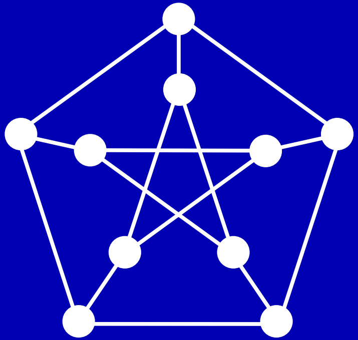
- https://github.com/gdv/large-scale-graph-algorithms
- Everything at https://elearning.unimib.it/

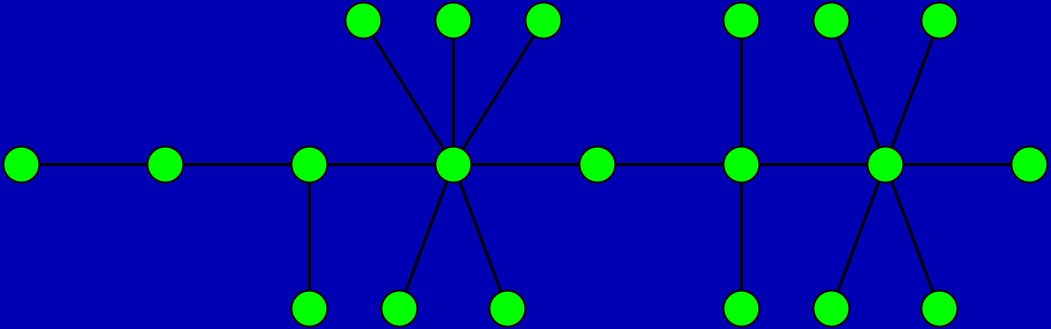
Example



Notation

- number of vertices: n
- number of edges/arcs: m

Better representation



Almost a path

- Compact representation

Breadth-first visit

Data: graph G, vertex root
Q ← a queue;
label root as explored;
Q.enqueue(root);
while Q $\neq \emptyset$ **do**
    v ← Q.dequeue();
    foreach *edge* (v, w) **do**
        if *w is not labeled as explored* **then**
            label w as explored;
            Q.enqueue(w)

Depth-first visit

Data: graph G, vertex root

```
S ← a stack;
S.push(root);
while S ≠ ∅ do
    v ← S.pop();
    if v is not labeled as explored then
        label v as explored;
        foreach edge (v, w) do
            S.push(w)
```

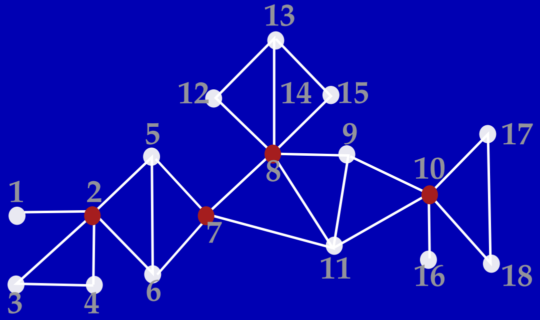
Dijkstra's algorithm

```
Data: graph G, vertex source
Q ← a queue;
foreach vertex v do
    dist[v] ← ∞;
    Q.enqueue(v)
dist[source] ← 0;
while Q ≠ ∅ do
    u ← vertex in Q minimizing dist[u];
    Q.deque(u);
    foreach neighbor v of u still in Q do
        alt ← dist[u] + Graph.Edges(u, v);
        if alt < dist[v] then
            dist[v] ← alt;
            prev[v] ← u;
return dist[], prev[];
```

Dijkstra's algorithm — priority queue

```
Data: graph G, vertex source
Q ← a priority queue;
foreach vertex v do
    dist[v] ← ∞;
    Q.add_with_priority(v, dist[v])
dist[source] ← 0;
while Q ≠ ∅ do
    u ← Q.extract_min;
    foreach neighbor v of u still in Q do
        alt ← dist[u] + Graph.Edges(u, v);
        if alt < dist[v] then
            dist[v] ← alt;
            prev[v] ← u;
            Q.decrease_priority(v, alt)
return dist[], prev[];
```

Biconnected components

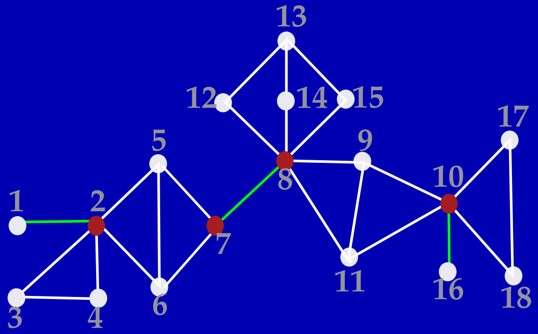


Find articulation points

Data: connected graph G, vertex root

$S \leftarrow$ a stack;
S.push((root, nil));
$d \leftarrow -1$;
while $S \neq \emptyset$ **do**
  $d \leftarrow d + 1$;
  $(v, p) \leftarrow S$.peek();
  if v *is not explored* **then**
    label v as explored; parent$(v) \leftarrow p$; depth$[v] \leftarrow d$;
    lowpoint$[v]$ =depth$[v]$;
    foreach *edge* (v, w) **do**
      S.push((w, v));
  else
    lowpoint$[v]$ =
    $\min\left\{\text{depth}[v], \min_{w \in N(v), w \neq p}\{\text{depth}[w]\}, \min_{w \in N(v), parent(w)=v}\{\text{lowpoint}[w]\}\right\}$
  $d \leftarrow d - 1$;
  $v \leftarrow S$.pop();

2-edge connected components



Find bridges

Data: connected graph G, vertex root

$S \leftarrow$ a stack, S.push(root, nil);

$x \leftarrow -1$;

while $S \neq \emptyset$ **do**

    $(v, p) \leftarrow S.\text{peek}()$;

    if v *is not explored* **then**

        $x \leftarrow x + 1$;

        label v as explored; $P(v) \leftarrow p$; $num[v] \leftarrow x$;

        foreach *edge* (v, w) **do**

            $S.\text{push}((w, v))$;

    else

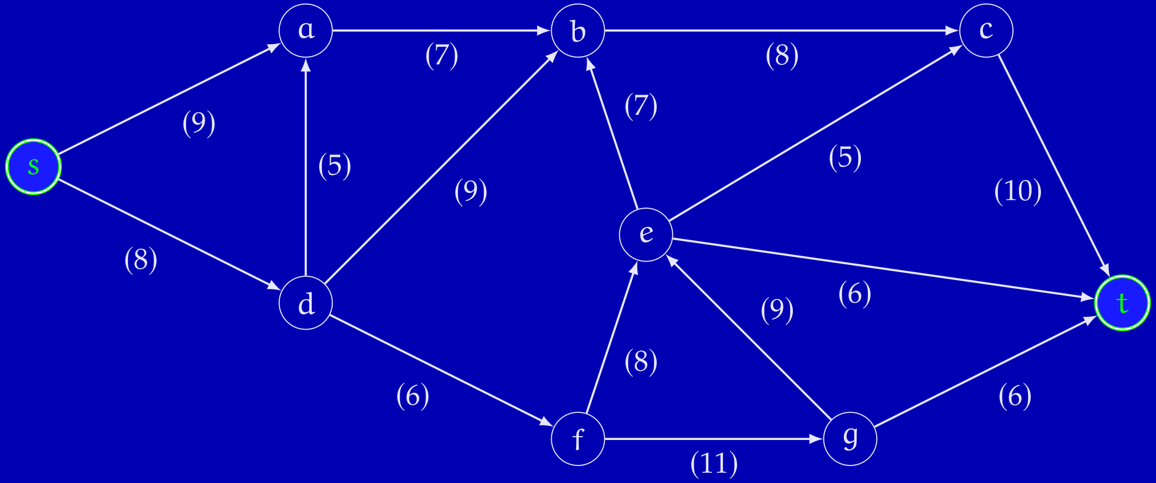
        $nd[v] = 1 + \sum_{w \in N(v), P(w)=v} nd[w]$;

        $l[v] =$
        $\min\left\{num[v], \min_{w \in N(v), w \neq p, P(w) \neq v}\{num[w]\}, \min_{w \in children(v)}\{l[w]\}\right\}$;

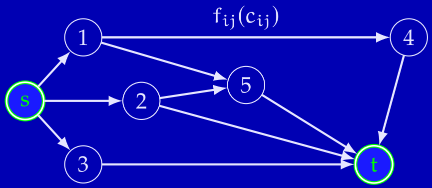
        $h[v] =$
        $\max\left\{num[v], \max_{w \in N(v), w \neq p, P(w) \neq v}\{num[w]\}, \max_{w \in children(v)}\{h[w]\}\right\}$;

    $v \leftarrow S.\text{pop}()$;

Max flow



Max flow



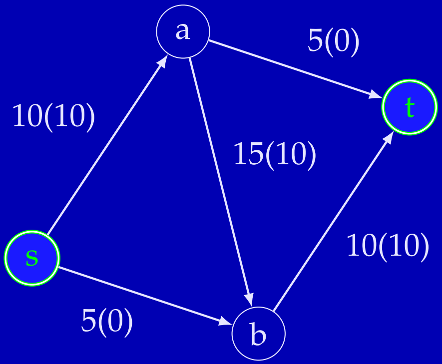
- c_{ij} : capacity of the arc (i, j)
- f_{ij} : flow through the arc (i, j)
- v_i : flow imbalance of the vertex i (< 0 incoming, > 0 outgoing)

$$\max \quad F$$

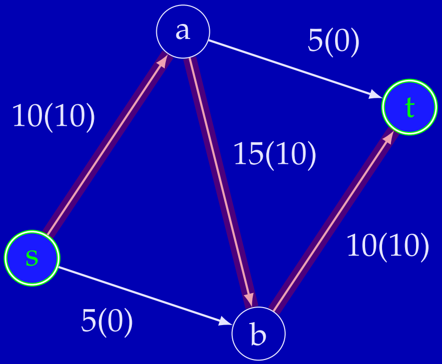
$$\text{s.t} \sum_{j:(i,j)\in E} f_{ij} - \sum_{k:(i,k)\in E} f_{ki} = \begin{cases} F & \text{if } i = s \\ -F & \text{if} i = t \\ 0 & i \text{ otherwise} \end{cases} \qquad \forall i \in V$$

$$0 \leqslant f_{ij} \leqslant c_{ij} \qquad \forall (i, j) \in E$$

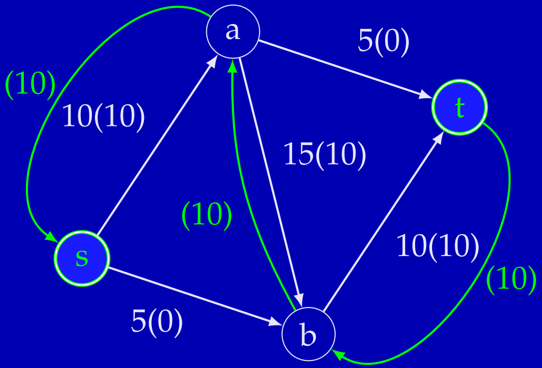
Residual network



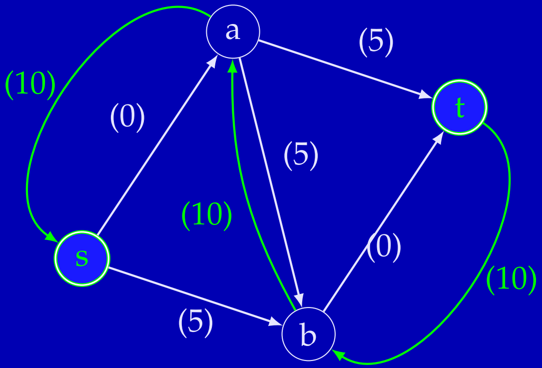
Residual network



Residual network



Residual network



Cut

Let $G = \langle V, E \rangle$ be a directed graph and let $S \subseteq V$. Then:

- $(S, \bar{S}) = E \cap (S \times \bar{S})$ is a forward cut,
- $(\bar{S}, S) = E \cap (\bar{S} \times S)$ is a backward cut,
- $E \cap \big((S \times \bar{S}) \cup (\bar{S} \times S)\big)$ is a cut.

Flow — Cut

Lemma 1

Let $G = \langle V, E \rangle$ be a directed graph and let $(S, \bar{S})$ be a bipartition of V, with $s \in S$, $t \notin S$. Let f be an (s, t)-flow with total flow F. Then

$$F = \sum_{e \in (S, \bar{S})} f(e) - \sum_{e \in (\bar{S}, S)} f(e)$$

Lemma 2

Let $G = \langle V, E \rangle$ be a directed graph and let $(S, \bar{S})$ be a bipartition of V, with $s \in S$, $t \notin S$. Let f be an (s, t)-flow with total flow F. Then

$$F \leqslant \sum_{e \in (S, \bar{S})} c(e)$$

Max flow – Min cut theorem

Let f be a flow of a graph $G = (V, E)$. Then the following three conditions are equivalent:

1. f is a maximum flow
2. the residual graph has no augmenting path
3. there is a cut (S, T) of G such that $c(S, T) = |f|$

Ford–Fulkerson

1. Find an augmenting path
2. Use it!

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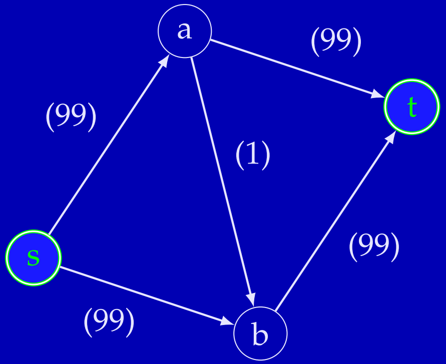
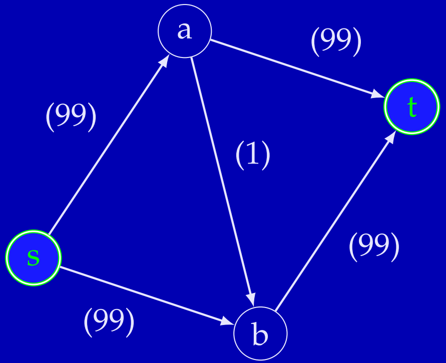


Ford–Fulkerson

1. Find an augmenting path
2. Use it!

Edmonds–Karp

BFS to find the augmenting path



Residual network

Figures

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