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- Large-Scale Graph Algorithms
- Ufficio U14-2041
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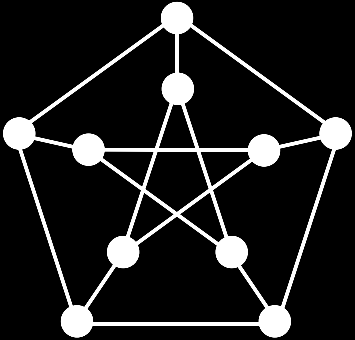
- https://github.com/gdv/large-scale-graph-algorithms
- Everything at https://elearning.unimib.it/

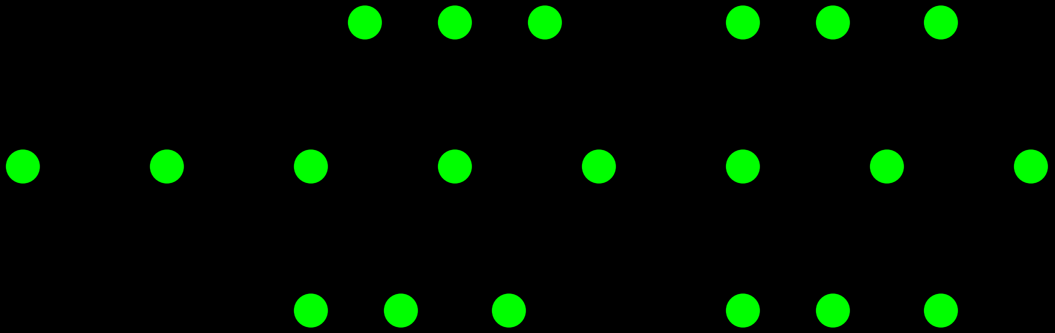
Example



Notation

- number of vertices: n
- number of edges/arcs: m

Better representation



Almost a path

- Compact representation

Breadth-first visit

```
Data: graph $G$, vertex $root$
$Q \leftarrow$ a queue;
label root as explored;
$Q$.enqueue(root);
while $Q \neq \emptyset$ do
    $v \leftarrow Q$.dequeue();
    foreach edge $(v, w)$ do
        if $w$ is not labeled as explored then
            label $w$ as explored;
            $Q$.enqueue($w$)
```

Depth-first visit

```
Data: graph $G$, vertex $root$
$S \leftarrow$ a stack;
$S$.push(root);
while $S \neq \emptyset$ do
    $v \leftarrow S$.pop();
    if $v$ is not labeled as explored then
        label $v$ as explored;
        foreach edge $(v, w)$ do
            $S$.push($w$)
```

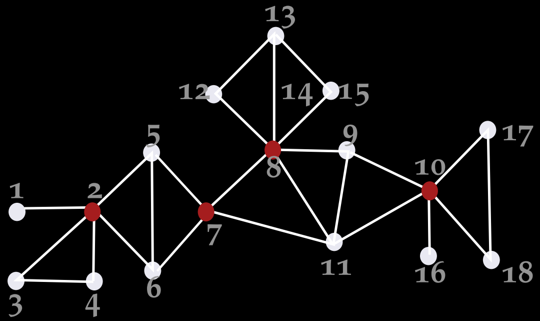
Dijkstra's algorithm

```
Data: graph $G$, vertex $source$
$Q \leftarrow$ a queue;
foreach vertex $v$ do
    dist[$v$] $\leftarrow \infty$;
    $Q$.enqueue(v)
dist[source] $\leftarrow 0$;
while $Q \neq \emptyset$ do
    $u \leftarrow$ vertex in $Q$ minimizing dist[$u$];
    $Q$.deque(u);
    foreach neighbor $v$ of $u$ still in $Q$ do
        $alt \leftarrow dist[u] + Graph.Edges(u, v)$;
        if $alt < dist[v]$ then
            dist[v] $\leftarrow$ alt;
            prev[v] $\leftarrow u$;
return dist[], prev[];
```

Dijkstra's algorithm — priority queue

```
Data: graph $G$, vertex $source$
$Q \leftarrow$ a priority queue;
foreach vertex $v$ do
    dist[$v$] $\leftarrow \infty$;
    $Q$.add_with_priority(v, dist[v])
dist[source] $\leftarrow 0$;
while $Q \neq \emptyset$ do
    $u \leftarrow Q$.extract_min;
    foreach neighbor $v$ of $u$ still in $Q$ do
        $alt \leftarrow dist[u] + Graph.Edges(u, v)$;
        if $alt < dist[v]$ then
            dist[v] $\leftarrow$ alt;
            prev[v] $\leftarrow u$;
            $Q$.decrease_priority(v, alt)
return dist[], prev[];
```

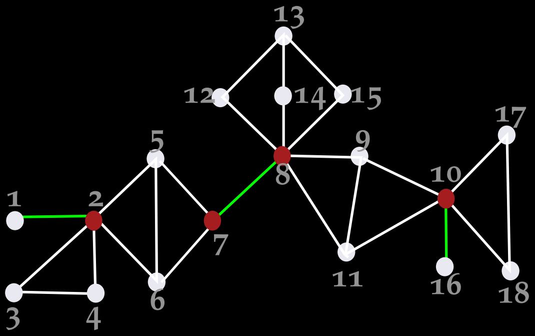
Biconnected components



Find articulation points

```
Data: connected graph $G$, vertex $root$
$S \leftarrow$ a stack;
$S$.push((root, nil));
$d \leftarrow -1$;
while $S \neq \emptyset$ do
    $d \leftarrow d + 1$;
    $(v, p) \leftarrow S$.peek();
    if $v$ is not explored then
        label $v$ as explored; parent($v$) $\leftarrow p$; $depth[v] \leftarrow d$;
        lowpoint[$v$] =depth[$v$];
        foreach edge $(v, w)$ do
            $S$.push($(w, v)$);
    else
        lowpoint[$v$] =
         $\min \left\{depth[v], \min_{w \in N(v), w \neq p}\{depth[w]\}, \min_{w \in N(v), parent(w)=v}\{lowpoint[w]\}\right\}$
    $d \leftarrow d - 1$;
    $v \leftarrow S$.pop();
```

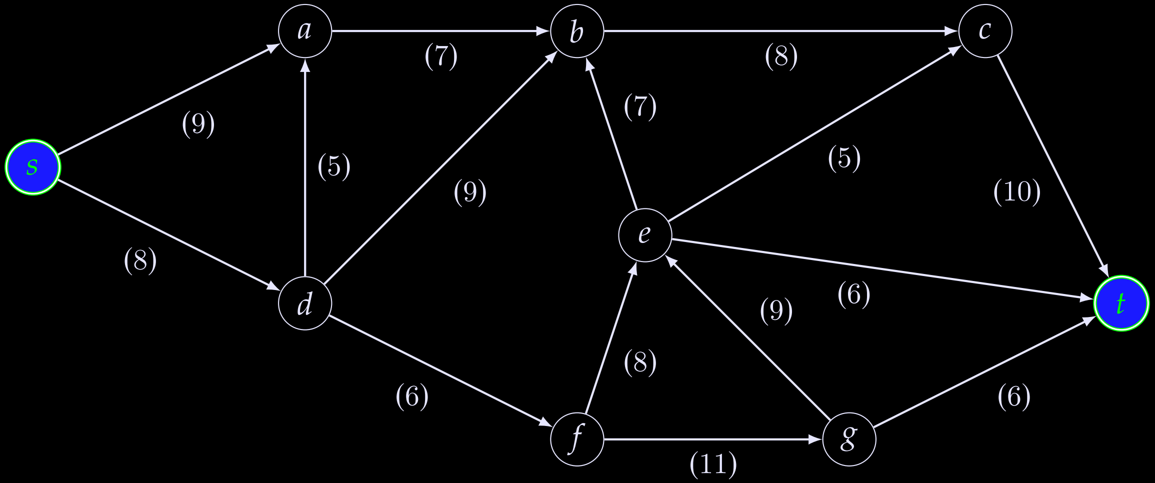
2-edge connected components



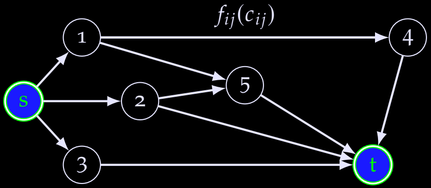
Find bridges

```
Data: connected graph $G$, vertex $root$
$S \leftarrow$ a stack, $S$.push(root, nil);
$x \leftarrow -1$;
while $S \neq \emptyset$ do
    $(v, p) \leftarrow S$.peek();
    if $v$ is not explored then
        $x \leftarrow x + 1$;
        label $v$ as explored; P($v$) $\leftarrow p$; $num[v] \leftarrow x$;
        foreach edge $(v, w)$ do
            $S$.push($(w, v)$);
    else
        $nd[v] = 1 + \sum_{w \in N(v), P(w)=v} nd[w]$;
        $l[v] =$
         $\min\left\{num[v], \min_{w \in N(v), w \neq p, P(w) \neq v}\{num[w]\}, \min_{w \in children(v)}\{l[w]\}\right\}$;
        $h[v] =$
         $\max\left\{num[v], \max_{w \in N(v), w \neq p, P(w) \neq v}\{num[w]\}, \max_{w \in children(v)}\{h[w]\}\right\}$;
    $v \leftarrow S$.pop();
```

Max flow



Max flow



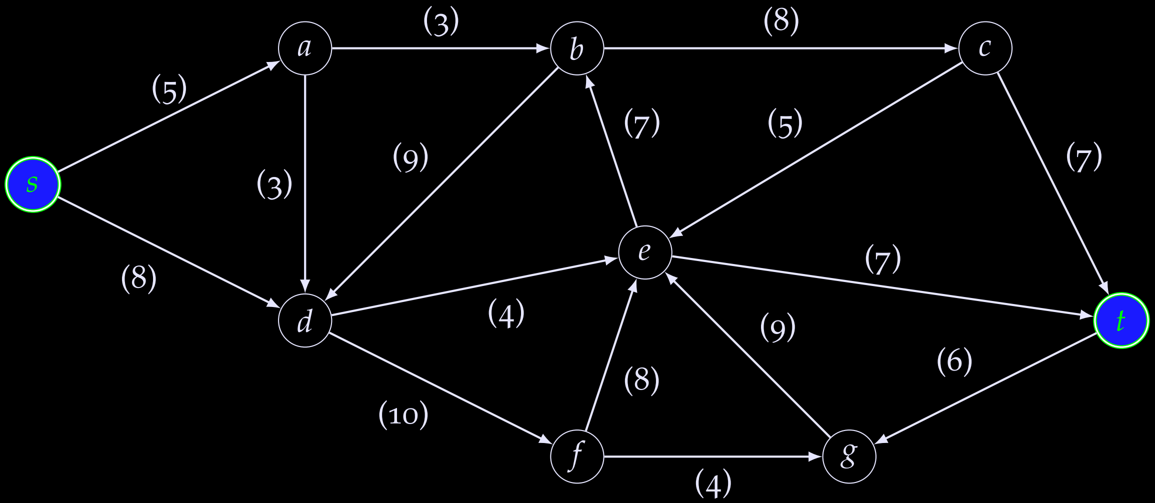
- c_{ij} : capacity of the arc (i, j)
- f_{ij} : flow through the arc (i, j)
- v_i : flow imbalance of the vertex i (< 0 incoming, > 0 outgoing)

$$\max \quad F$$

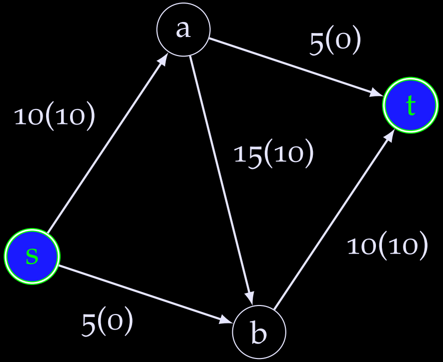
$$\text{s.t} \sum_{j:(i,j)\in E} f_{ij} - \sum_{k:(i,k)\in E} f_{ki} = \begin{cases} F & \text{if } i = s \\ -F & \text{if } i = t \\ 0 & i \text{ otherwise} \end{cases} \qquad \forall i \in V$$

$$0 \leq f_{ij} \leq c_{ij} \qquad \forall (i,j) \in E$$

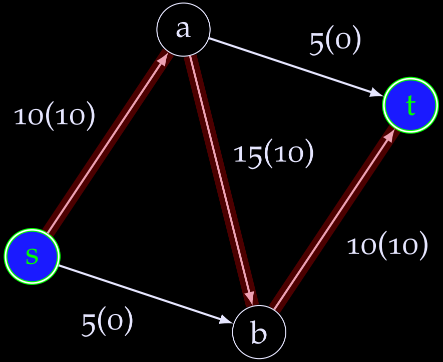
Example



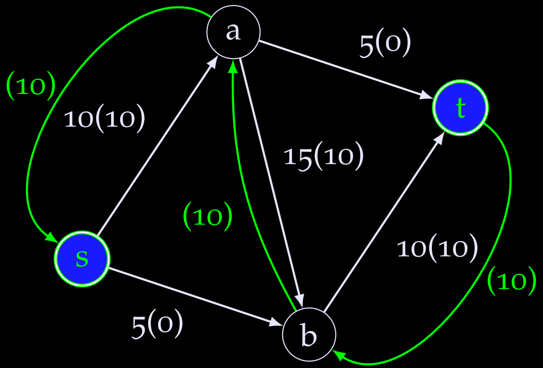
Residual network



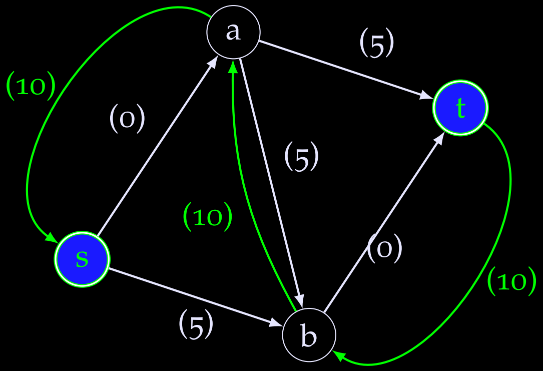
Residual network



Residual network



Residual network



Cut

Let $G = \langle V, E \rangle$ be a directed graph and let $S \subseteq V$. Then:

- $(S, \bar{S}) = E \cap (S \times \bar{S})$ is a forward cut,
- $(\bar{S}, S) = E \cap (\bar{S} \times S)$ is a backward cut,
- $E \cap \big((S \times \bar{S}) \cup (\bar{S} \times S)\big)$ is a cut.

Flow — Cut

Lemma 1

Let $G = \langle V, E\rangle$ be a directed graph and let $(S, \bar{S})$ be a bipartition of V, with $s \in S$, $t \notin S$. Let f be an (s, t)-flow with total flow F. Then

$$F = \sum_{e \in (S,\bar{S})} f(e) - \sum_{e \in (\bar{S},S)} f(e)$$

Lemma 2

Let $G = \langle V, E\rangle$ be a directed graph and let $(S, \bar{S})$ be a bipartition of V, with $s \in S$, $t \notin S$. Let f be an (s, t)-flow with total flow F. Then

$$F \leq \sum_{e \in (S,\bar{S})} c(e)$$

Max flow – Min cut theorem

Let f be a flow of a graph $G = (V, E)$. Then the following three conditions are equivalent:

1. f is a maximum flow
2. the residual graph has no augmenting path
3. there is a cut (S, T) of G such that $c(S, T) = |f|$

Ford–Fulkerson

1. Find an augmenting path
2. Use it!

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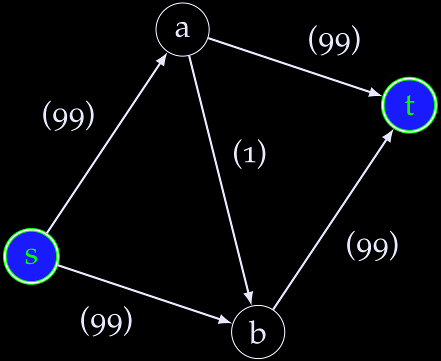
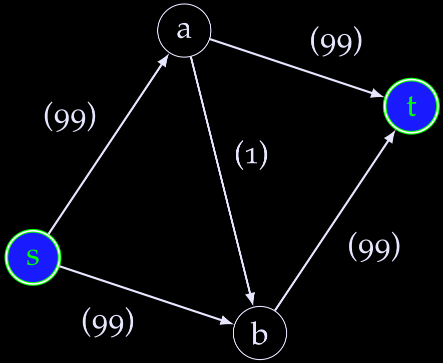


Ford–Fulkerson

1. Find an augmenting path
2. Use it!

Edmonds–Karp

BFS to find the augmenting path



Figures

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