

Expectation Argument

Lemma

Let X be a random variable on a discrete space S such that $E(X) = \mu$. Then $Pr[X \geq \mu] > 0$ and $Pr[X \leq \mu] > 0$.

Max cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$$| E \cap (A \times B) |$$

Max cut

```
Data: graph  $G = \langle V, E \rangle$   
foreach vertex  $v \in V$  do  
    | Assign  $v$  to  $A$  or to  $B$  with probability  $1/2$   
return  $(A, B)$ 
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Lemma 1 (to prove)

The expected value of the number of edges in the cut (A, B) is at least $m/2$

Lemma 2 (to prove)

The probability that the cut (A, B) has at least $m/2$ edges is $\geq \frac{1}{m/2+1}$

Proof of Lemma 2

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$$\frac{m}{2} = \mathbf{E}[C(A, B)] = \sum_{i \leq \frac{m}{2} - 1} i \Pr(C(A, B) = i) + \sum_{i \geq \frac{m}{2}} i \Pr(C(A, B) = i)$$

$$\Rightarrow p \geq \frac{1}{m/2 + 1}$$

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$$\begin{aligned} \frac{m}{2} &= \mathbf{E}[C(A, B)] = \sum_{i \leq \frac{m}{2} - 1} i \Pr(C(A, B) = i) + \sum_{i \geq \frac{m}{2}} i \Pr(C(A, B) = i) \\ &\leq (1 - p) \left(\frac{m}{2} - 1 \right) + pm \\ &\Rightarrow p \geq \frac{1}{m/2 + 1} \end{aligned}$$

Sample and modifying

Lemma (to prove)

Let $G = \langle V, E \rangle$ be a undirected graph. Then $G = \langle V, E \rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

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Let $G = \langle V, E \rangle$ be a undirected graph. Then $G = \langle V, E \rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

Hint 1: $d = \frac{2m}{n}$ is the average degree.

Hint 2: the proof is a probabilistic algorithm

Sample and modifying

- 1 Sample $S \leftarrow$ keep each vertex with probability $\frac{1}{d}$
- 2 For each edge in the sample, remove one of its endpoints.

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Proof: after step 1

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Proof: after step 2

For each edge, we remove at most a vertex. Hence

$$\mathbb{E}[\text{number of surviving vertices}] = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d} = \frac{n^2}{4m}$$

Min cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$\max | E \cap (A \times B) |$

Karger's Algorithm

Data: Undirected graph $G = \langle V, E \rangle$

while $|V| > 2$ **do**

 | Pick a random edge (v, w) ;

 | Merge v and w ;

$a, b \leftarrow$ the two vertices of G ;

return (*vertices merged into a , vertices merged into b*)

Karger, Stein Algorithm

Data: Undirected graph $G = \langle V, E \rangle$, parameter $t \geq 2$

while $|V| > t$ **do**

 Pick a random edge (v, w) ;

 Merge v and w ;

return *optimal solution on G*

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