Expectation Argument

Lemma

Let X be a random variable on a discrete space S such that $E(X) = \mu$. Then $\Pr[X \geqslant \mu] > 0$ and $\Pr[X \leqslant \mu] > 0$.

Max cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$| E \cap (A \times B) |$

Max cut

Data: graph $G = \langle V, E \rangle$
foreach *vertex* $v \in V$ **do**
    Assign v to A or to B with probability $1/2$
return (A, B)

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Lemma 1 (to prove)

The expected value of the number of edges in the cut (A, B) is at least $m/2$

Lemma 2 (to prove)

The probability that the cut (A, B) has at least $m/2$ edges is $\geqslant \frac{1}{m/2+1}$

Proof of Lemma 2

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$$\frac{m}{2} = \mathbf{E}[C(A,B)] = \sum_{i \leqslant \frac{m}{2}-1} i \Pr(C(A,B) = i) + \sum_{i \geqslant \frac{m}{2}} i \Pr(C(A,B) = i)$$

$$\Rightarrow p \geqslant \frac{1}{m/2+1}$$

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$$\begin{aligned}
\frac{m}{2} = \mathbf{E}[C(A,B)] &= \sum_{i \leqslant \frac{m}{2}-1} i \Pr\left(C(A,B) = i\right) + \sum_{i \geqslant \frac{m}{2}} i \Pr\left(C(A,B) = i\right) \\
&\leqslant (1-p)\left(\frac{m}{2} - 1\right) + pm \\
&\Rightarrow p \geqslant \frac{1}{m/2+1}
\end{aligned}$$

Sample and modifying

Lemma (to prove)

Let $G = \langle V, E \rangle$ be a undirected graph. Then $G = \langle V, E \rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

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Let $G = \langle V, E \rangle$ be a undirected graph. Then $G = \langle V, E \rangle$ has an independent set with at least $\frac{n^2}{4m}$ vertices.

Hint 1: $d = \frac{2m}{n}$ is the average degree.
Hint 2: the proof is a probabilistic algorithm

Sample and modifying

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2. For each edge in the sample, remove one of its endpoints.

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Proof: after step 1

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Proof: after step 2

For each edge, we remove at most a vertex. Hence

$$\mathbf{E}[\text{number of surviving vertices}] = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d} = \frac{n^2}{4m}$$

Min cut

Instance

Undirected graph $G = \langle V, E \rangle$

Feasible solutions

Bipartition (A, B) of V

Objective function

$\max |\, E \cap (A \times B) \,|$

Karger's Algorithm

Data: Undirected graph $G = \langle V, E \rangle$
while $|V| > 2$ **do**
  Pick a random edge (v, w);
  Merge v and w;
$a, b \leftarrow$ the two vertices of G;
return *(vertices merged into* a*, vertices merged into* b*)*

Karger, Stein Algorithm

Data: Undirected graph $G = \langle V, E \rangle$, parameter $t \geqslant 2$
while $|V| > t$ **do**
    Pick a random edge (v, w);
    Merge v and w;
return *optimal solution on* G

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