Mathematics IA

Calculating the total displacement with a velocity-time graph using the Taylor Series and comparing its accuracy to the Trapezoidal rule and Monte Carlo integration.

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Introduction

Integration is “the process of finding the expression of a function (antiderivative) from an expression of the derivative” (Paul, n.d.). This technique is especially essential in physics and other Mathematical application courses, which include the subjects of kinematics, engineering, and stochastic calculus. In this case, kinematics will be applied in the exploration, and the displacement will be calculated from a velocity-time graph. Displacement, the change in the position of an object, is a fundamental element in physics, as it is considered in the work done equation ($W = Force \times displacement \times \cos\theta$) and the kinematic equation ($Displacement\ = Initial\ velocity \times Time + \frac{1}{2} \times Acceleration \times (Time)^2$). Therefore, this exploration is especially important in the field of engineering and physics.

Additionally, in the universe of mathematics, there is an infinite number of functions. However, some of the equations cannot be integrated, such as $f(x) = \frac{x}{\sqrt{1-x^3}}$. Whereas, numerous functions cannot be integrated using regular integrational methods, such as the substitution and by-part methods. Though, some of these functions may be integrated with the help of nonelementary integrational methods, such as the Taylor Series and Trapezoidal methods, where the resultant function cannot be constructed as a finite composition of known functions like algebraic, exponential, trigonometric, and logarithmic functions (Nonelementary integral, n.d.). It is also possible for it to be calculated via programming language, through Monte Carlo Simulation, where the program averages the areas of rectangles created by imputing randomly selected coordinates (Victor, 2020).

Rationale

Since it is impossible to integrate some functions using normal substitution and by-parts, a nonelementary method or complex integration method must be used. Thus, choosing the methods with the highest accuracy may be strictly required for engineering inventions that require incredibly high accuracy. This topic aroused my interest since it requires practical mathematic skills (nonelementary integration by using Taylor series, trapezoidal rules, and Monte Carlo integration) which is applied in multiple real-life scenarios and across multiple mathematical applications courses. The topic is also extremely important for future development and technological inventions, where the methods' accuracy could be dependent on the safety and uncertainty of future inventions. Thus, choosing the method with the highest accuracy can reduce failure chances and ensure safety. In addition, as a mathematical, especially calculus lover, I am intrigued to learn further about the nonelementary integration method, since it is more complex and uncertain, making it more exciting. Therefore, I am interested in working out and examining which nonelementary integration method has a higher accuracy. Furthermore, I believe that it would be interesting to develop mathematical skills and research on this topic further.

Aims

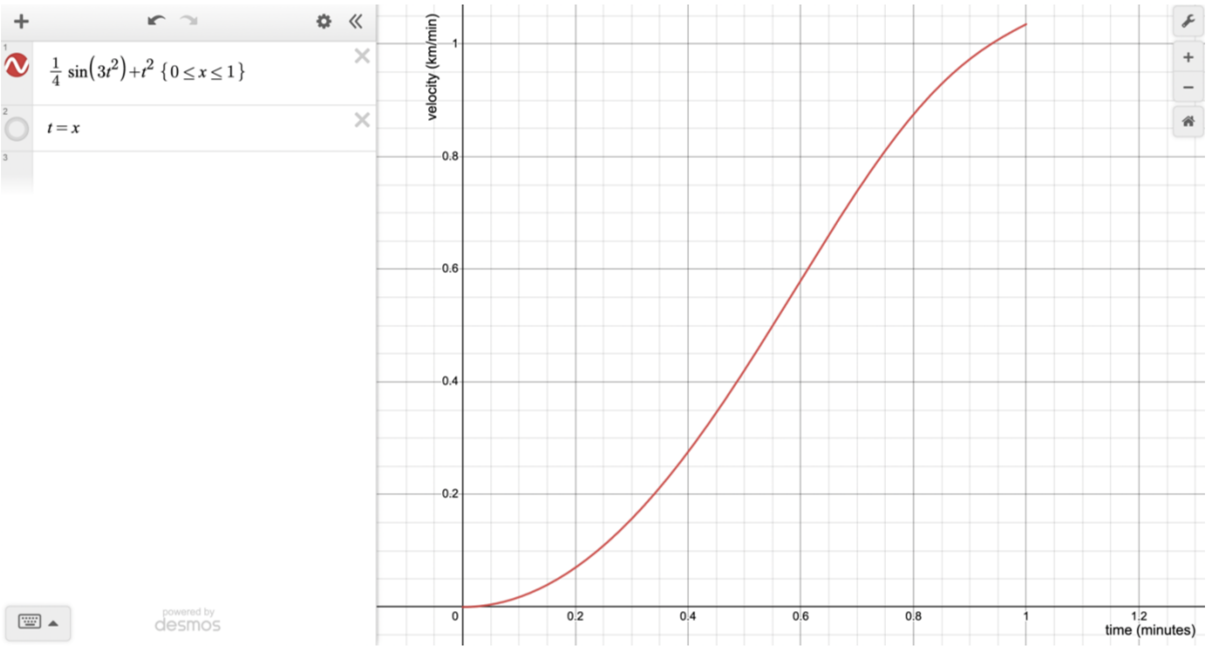

Figure 1 Velocity-time graph for a vehicle in the first minute (Desmos, n.d.)

In this essay, assume the velocity-time graph for a vehicle in the very first minute can be expressed as $v = \frac{1}{4}\sin(3t^2) + t^2$ (shown in Figure 1), where 'v' is the velocity (kilometre per minute) on the y-axis and 't' is time (minute) on the x-axis. The total displacement will be calculated for the first minute.

This function cannot be integrated via elementary integration methods, such as substitutions and by-parts. Therefore, the Taylor series will be used in the calculation while comparing its accuracy to the results from the Monte Carlo Integration and Trapezoidal rules methods. Lastly, it will conclude which one of the three will produce the most accurate results and evaluate the effectiveness of the investigation.

Taylor series

In this section of the investigation, the displacement is calculated through the use of definite integration, while using Taylor Series to expand into a nonelementary function into exponential form so that it would be possible to integrate.

Firstly, it is essential to use the Taylor Formula to expand the function of $\sin(x)$ into exponential form. To do so, the original Taylor Formula will be needed:

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $n \in \mathbb{Z}^+$

The formula could also be written in the form of:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

To derive the function focused on, we must derive Taylor series formula in terms of $\sin(x)$ first. Therefore, it is necessary to compute the results of its derivatives first.

$$The\ first\ derivative: sin'\,x = cosx$$

$$The\ second\ derivative: sin''x = -sinx$$

$$The\ third\ derivative: sin^{(3)}x = -cosx$$

$$The\ fourth\ derivative: sin^{(4)}x = sinx$$

After the first four derivatives of $\sin(x)$, it is clear that the pattern will repeat and form a loop since the fourth derivative is equal to the original function.

Next, "a" could be set to an infinite range of numbers, since it is the number where the series is centred, which means that the function converges on some interval centred at $x = a$. However, it is common to set $a = 0$ (Bill, 2015), which could be called a 'Maclaurin Series.' To substitute the into the original formula, the derivatives and the original function will be calculated when $a = 0$ due to simplicity:

$$The\ original\ function: \sin(0) = 0$$

$$The\ first\ derivative: sin'(0) = \cos(0) = 1$$

$$The\ second\ derivative: sin''(0) = -\sin(0) = 0$$

$$The\ third\ derivative: sin^{(3)}(0) = -\cos(0) = -1$$

$$The\ fourth\ derivative: sin^{(4)}(0) = \sin(0) = 0$$

Once again, a loop is formed where the fourth derivate equals to the original function when $a = 0$. Consequently, sufficient information is gathered to substitute into the original Taylor Series formula. Thus, creating the Taylors Series expansion for $\sin(x)$:

$$\sin(x) = 0 + 1(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4 + \cdots \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}, where\ (2n+1) is\ odd\ number$$

From the expansion, it is clear that the powers series converges for any given values of x, so the radius of convergence (R) is infinity. This is because as the denominator gets larger, the terms will get extremely small when n tends to infinity.

Calculating total displacement:

Now that the Taylors Series expression for $\sin(x)$ is derived, we can utilise that in the investigation of the expression for total displacement that the car has travelled in the first minute:

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt$$

Since the car started to accelerate from the 0^{th} second, the lower bound of the definite integration is 0, and since the displacement is calculated till the end of the first minute, the upper bound is 1.

To simplify and for better visualisation, the calculation will be separated into several parts:

$$\frac{1}{4}\int_0^1 (\sin(3t^2))dt + \int_0^1 t^2\, dt$$

However, the first step is to derive the expression for $\sin(3t^2)$. Since in the velocity-time function, it is $\sin(3t^2)$ instead of $\sin(x)$ as expressed earlier using the Taylors Series expansion. Therefore, by substituting $3t^2 = x$ into the derived expression, a new expression of $\sin(3t^2)$ will be created, which will look like this:

$$\sin(3t^2) = \sum_{n=0}^{\infty}(-1)^n \times \frac{(3t^2)^{2n+1}}{(2n+1)!} = 3t^2 - \frac{(3t^2)^3}{3!} + \frac{(3t^2)^5}{5!} - \frac{(3t^2)^7}{7!} + \cdots + \frac{(3t^2)^{2n+1}}{(2n+1)!}$$

Or it can be written as:

$$= 3t^2 - \frac{3^3t^6}{3!} + \frac{3^5t^{10}}{5!} - \frac{3^7t^{14}}{7!} + \frac{3^9t^{18}}{9!} - \frac{3^{11}t^{22}}{11!} + \cdots$$

After substituting and simplifying, the Taylor Series expression for $\sin(3t^2)$ is formed. The next step is using this expression in the first half of the definite integration:

$$\frac{1}{4}\int_0^1 (\sin(3t^2))dt = \frac{1}{4}\int_0^1 \left(3t^2 - \frac{3^3t^6}{3!} + \frac{3^5t^{10}}{5!} - \frac{3^7t^{14}}{7!} + \frac{3^9t^{18}}{9!} - \frac{3^{11}t^{22}}{11!} + \cdots\right)dt$$

By using exponential integration rules:

$$= \frac{1}{4}\left[\frac{3t^3}{3} - \frac{3^3t^7}{7\times 3!} + \frac{3^5t^{11}}{11\times 5!} - \frac{3^7t^{15}}{15\times 7!} + \frac{3^9t^{19}}{19\times 9!} - \frac{3^{11}t^{23}}{23\times 11!} + \cdots\right]_0^1$$

Next, by subtracting the series that is substituted with $t = 0$ from the series that is substituted with $t = 1$, the results of the definite integration will be expressed:

$$= \frac{1}{4}\left(\left(\frac{3\times 1^3}{3} - \frac{3^3\times 1^7}{7\times 3!} + \frac{3^5\times 1^{11}}{11\times 5!} - \frac{3^7\times 1^{15}}{15\times 7!} + \frac{3^9\times 1^{19}}{19\times 9!} - \frac{3^{11}\times 1^{23}}{23\times 11!} + \cdots\right)\right.$$

$$\left. - \left(\frac{3\times 0^3}{3} - \frac{3^3\times 0^7}{7\times 3!} + \frac{3^5\times 0^{11}}{11\times 5!} - \frac{3^7\times 0^{15}}{15\times 7!} + \frac{3^9\times 0^{19}}{19\times 9!} - \frac{3^{11}\times 0^{23}}{23\times 11!} + \cdots\right)\right)$$

Since the lower bound, where $t = 0$ is substituted, all have $3^n t^m,\ n \in \mathbb{Z}^+\ and\ m \in \mathbb{Z}^+$, on the numerator, the sequence sums up to 0 as all terms equal to 0. Therefore:

$$= \frac{1}{4}\left(\left(\frac{3 \times 1^3}{3} - \frac{3^3 \times 1^7}{7 \times 3!} + \frac{3^5 \times 1^{11}}{11 \times 5!} - \frac{3^7 \times 1^{15}}{15 \times 7!} + \frac{3^9 \times 1^{19}}{19 \times 9!} - \frac{3^{11} \times 1^{23}}{23 \times 11!} + \cdots\right) - 0\right)$$

By expanding the bracket, it can be written as:

$$= \frac{3 \times 1^3}{4 \times 3} - \frac{3^3 \times 1^7}{4 \times 7 \times 3!} + \frac{3^5 \times 1^{11}}{4 \times 11 \times 5!} - \frac{3^7 \times 1^{15}}{4 \times 15 \times 7!} + \frac{3^9 \times 1^{19}}{4 \times 19 \times 9!} - \frac{3^{11} \times 1^{23}}{4 \times 23 \times 11!} + \cdots$$

Which simplifies to:

$$= \frac{1}{4} - \frac{3^3}{28 \times 3!} + \frac{3^5}{44 \times 5!} - \frac{3^7}{60 \times 7!} + \frac{3^9}{76 \times 9!} - \frac{3^{11}}{92 \times 11!} + \cdots$$

With the help of a calculator:

$$= \frac{1}{4} - \frac{9}{56} + \frac{81}{1760} - \frac{81}{11200} + \frac{243}{340480} - \frac{2187}{45337600} + \cdots$$

From this half of the definite integration, it is once again proven that this is a convergence series. In simpler words, this means that as the number of terms tends to infinity, the values of the terms in the series are getting smaller and smaller. Therefore, if more terms are being considered, the more accurate the result will be for the definite integration, and so will the results for the displacement travelled. To prove so, comparisons can be made between the results calculated with different numbers of terms that are considered. However, before that, there is still another half of the integration that needs to be done:

$$\int_0^1 t^2\, dt$$

By using the power integration method, this equals to:

$$= \left[\frac{t^3}{3}\right]_0^1$$

Then, by substituting both the upper bound and lower bound:

$$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Then, by adding it to the series, the displacement will be calculated:

$$\left(\frac{1}{4}-\frac{9}{56}+\frac{81}{1760}-\frac{81}{11200}+\frac{243}{340480}-\frac{2187}{45337600}+\cdots\right)+\frac{1}{3}$$

To investigate how the accuracy changes, the results by considering the first two, four terms and six terms in the series (by using addition and subtraction on a normal calculator) will be compared to the results obtained from a graphical display calculator (GDC). However, a question can be raised on the method which the GDC uses and its result accuracy (For better accuracy, all digits will be recorded from the display of the calculator (10 decimals)):

$$with\ the\ first\ two\ terms: \frac{1}{4}-\frac{9}{56}+\frac{1}{3}=0.4226190476\ km$$

$$with\ the\ first\ four\ terms: \frac{1}{4}-\frac{9}{56}+\frac{81}{1760}-\frac{81}{11200}+\frac{1}{3}=0.4614096320\ km$$

$$with\ the\ first\ six\ terms: \frac{1}{4}-\frac{9}{56}+\frac{81}{1760}-\frac{81}{11200}+\frac{243}{340480}-\frac{2187}{45337600}+\frac{1}{3}$$

$$=0.4620750944\ km$$

$$GDC\ value: \int_0^1\left(\frac{1}{4}\sin(3t^2)+t^2\right)dt=0.4620773770\ km$$

For better visualisation of the comparison, the percentage of accuracy (the yield) can be calculated by dividing the value from the terms by the actual value from the GDC.

Percentage of accuracy with the first two terms:

$$\frac{0.4226190476}{0.4620773770}=0.914607=91.4607\%$$

Percentage of accuracy with the first four terms:

$$\frac{0.4614096320}{0.4620773770}=0.998554=99.8554\%$$

Percentage of accuracy with the first six terms:

$$\frac{0.4620750944}{0.4620773770}=0.999995=99.9995\%$$

From the percentage calculation, the estimation which included six terms from the series appears to be more accurate with a percentage of 99.9995%, which is extremely close to the actual value of the displacement travelled by the vehicle. Therefore, in the Taylor Series, as the number of terms to be calculated increases, the estimation will be more accurate.

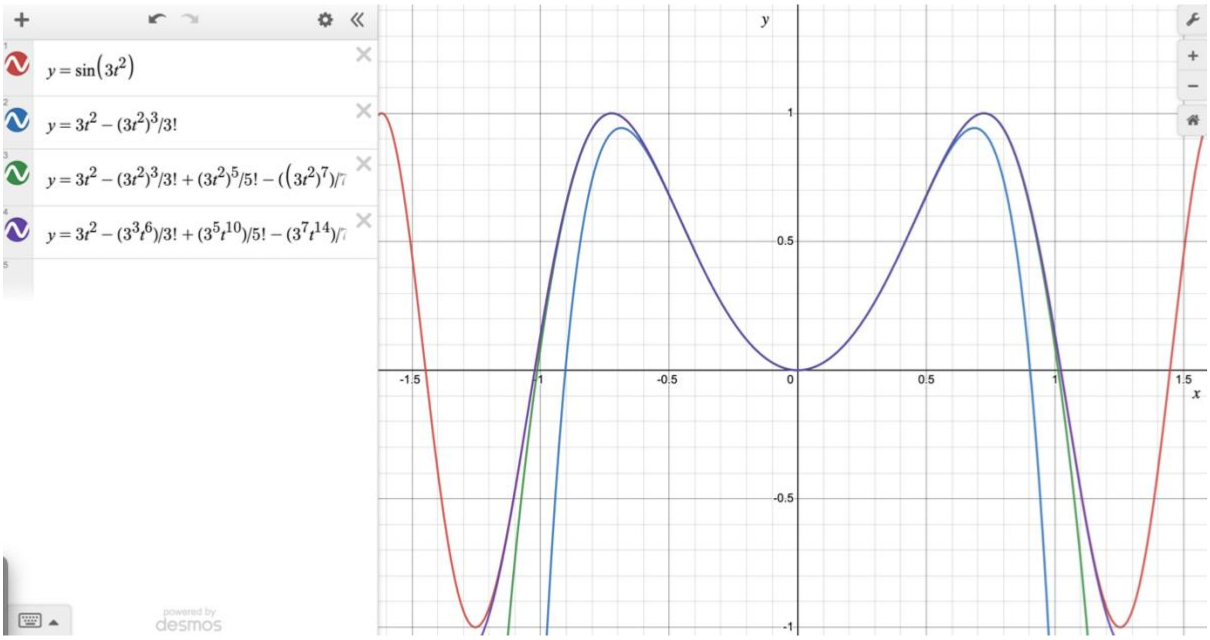

Figure 2 The Accuracy of Taylor Series (with different no. terms) to the Original Function (y=sin(3x2)) (Desmos, n.d.)

This can also be shown through the shapes of the graphs shown in Figure 2: as the number of exponential terms added to the series increases, the larger the interval of convergence is. Therefore, resulting in a more overlapping and accurate modified exponential function to the original function (only a part of the original function, $y = \sin(3x^2)$, is evaluated in Figure 2). Consequently, this is important in studies such as Physics and data analysis. If enough terms are taken, the risk of uncertainty and inaccuracy will decrease. Nevertheless, there will always be some inaccuracy if it is calculated manually, as it would be impossible to calculate an infinite number of terms without using technologies.

Overall, this section of the exploration enhanced my analytical abilities, as calculating definite integration using the Taylor series requires being able to work with mathematical expressions and understand the relationships between a function and its antiderivative.

Trapezoidal Rules

In the second part of the investigation, the displacement will be calculated using the Trapezoidal Rule. Instead of modifying and expanding the function into an elementary function like the Taylor Series, the Trapezoidal Rule directly provides an estimation of the value for a definite integral by dividing the area under the curve (within the domain) into small trapezoids and calculating their sum. Therefore, this method can only be used in estimating the definite integration unfortunately, while the Taylor Series may be used elsewhere as it transforms nonelementary integrals into elementary functions.

To work out the definite integration, the formula for the trapezoidal equation can be derived from the sum of all the trapeziums, which can be expressed as (Trapezoidal Rule, n.d.):

$$\boldsymbol{Area = \int_a^b y\,dx \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n]}$$

$$\boldsymbol{where\ the\ function\ is\ continuous\ on\ [a, b]\ and\ n\ is\ the\ number\ of\ subintervals,}$$

$$\boldsymbol{so\ that\ h\ (the\ height\ of\ trapezium) = \Delta x = \frac{b-a}{n}}$$

Applying to the scenario, where the displacement for the car of the first minute is given by $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt$. To show how the accuracy of the investigation changes by the number of subintervals, three values for n (5, 10, 15) will be used to calculate the estimated total displacement.

However, before calculating the displacement, it is beneficial to list out the required variables:

$$from\ Displacement = \int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt\,, b = 1\ and\ a = 0$$

The Estimated displacement when $n = 5$:

The height of the small trapeziums can be calculated by substituting the values in:

$$h = \frac{1-0}{5} = \frac{1}{5}$$

Now that the values for the v values need to be calculated for the estimation. To do so, the sampling t values need to be calculated. There would be 6 values available for t as the interval (the height of each trapezium) is $\frac{1}{5}$. Therefore, the t values are:

Instead of using the left/right-end rule or the mid-points rule, the trapezoidal rule will utilise all six values listed as $t = 0, 0.2, 0.4, 0.6, 0.8, 1$. Then, the values will be used for the calculation of the corresponding v values by substituting individual t values into the function $v = \frac{1}{4}\sin(3t^2) + t^2$:

$$when\ t = 0, v_0 = \frac{1}{4}\sin(3 \times 0^2) + 0^2 = 0$$

$$when\ t = 0.2, v_1 = \frac{1}{4}\sin(3 \times 0.2^2) + 0.2^2 = \frac{1}{4}\sin(0.12) + 0.04,$$

$$when\ t = 0.4, v_2 = \frac{1}{4}\sin(3 \times 0.4^2) + 0.4^2 = \frac{1}{4}\sin(0.48) + 0.16,$$

$$when\ t = 0.6, v_3 = \frac{1}{4}\sin(3 \times 0.6^2) + 0.6^2 = \frac{1}{4}\sin(1.08) + 0.36,$$

$$when\ t = 0.8, v_4 = \frac{1}{4}\sin(3 \times 0.8^2) + 0.8^2 = \frac{1}{4}\sin(1.92) + 0.64,$$

$$when\ t = 1, v_5 = \frac{1}{4}\sin(3 \times 1^2) + 1^2 = \frac{1}{4}\sin(3) + 1,$$

For accuracy, the expression for ‘v’ was listed instead of the decimal values.

The next step to estimate the displacement is to substitute the values into the formula:

$$\int_a^b y\,dx \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n]$$

$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx \frac{1}{2} \times \frac{1}{5} \times \left[0 + 2\left(\left(\frac{1}{4}\sin(0.12) + 0.04\right) + \left(\frac{1}{4}\sin(0.48) + 0.16\right) + \right.\right.$

$\left.\left.\left(\frac{1}{4}\sin(1.08) + 0.36\right) + \left(\frac{1}{4}\sin(1.92) + 0.64\right)\right) + \left(\frac{1}{4}\sin(3) + 1\right)\right]$

Which simplifies to:

$\approx \frac{1}{10} \times \left[\frac{1}{2}\sin(0.12) + 0.08 + \frac{1}{2}\sin(0.48) + 0.32 + \frac{1}{2}\sin(1.08) + 0.72 + \frac{1}{2}\sin(1.92) + \right.$

$\left. 1.28 + \frac{1}{4}\sin(3) + 1\right]$

After summing up all the rational numbers (highlighted in grey), this can be written as:

$$\approx \frac{1}{10} \times \left[\frac{1}{2}\sin(0.12) + \frac{1}{2}\sin(0.48) + \frac{1}{2}\sin(1.08) + \frac{1}{2}\sin(1.92) + \frac{1}{4}\sin(3) + 3.4\right]$$

So that:

$$\approx \frac{1}{20}\sin(0.12) + \frac{1}{20}\sin(0.48) + \frac{1}{20}\sin(1.08) + \frac{1}{20}\sin(1.92) + \frac{1}{40}\sin(3) + 0.34$$

After calculating through a calculator:

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4636827334km$$

To achieve the best precision, all the decimal places that appear on the calculator are kept.

The estimated displacement when $n = 10$ and 15:

The estimation for when $n = 10$ *and* 15 will follow the same algorithm as when $n = 5$ (see in Appendix), which gives the estimation of:

$$when\ n = 10\text{:} \int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4624996366km$$

$$when\ n = 15\text{:} \int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4622667579km$$

Comparison:

After estimating the displacement by using three different numbers of intervals, the accuracy can be compared and evaluated. The results are listed below:

The estimation when $n = 5$: $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4636827334\, km$

The estimation when $n = 10$: $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4624996366 km$

The estimation when $n = 15$: $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4622667579\, km$

The value according to GDC: $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt = 0.4620773770\ km$

For better visualisation of the comparison, the percentage of accuracy can be calculated:

$$percentage\ of\ accuracy\ when\ n = 5{:}\ \frac{0.4636827334}{0.4620773770} = 1.003474 = 100.3474\%$$

$$percentage\ of\ accuracy\ when\ n = 10{:}\ \frac{0.4624996366}{0.4620773770} = 1.000914 = 100.0914\%$$

$$percentage\ of\ accuracy\ when\ n = 15{:}\frac{0.4622667579}{0.4620773770} = 1.000410 = 100.0410\%$$

From the percentage of accuracy, we can see that when $n = 15$, the estimation of the displacement of the car is the most accurate, with 100.0410% accuracy. On contrast, when $n = 5$, the estimation accuracy is the lowest of the three, with 100.3474% accuracy. This is due to the area being neglected and excessed from the trapezium, depending on the concavity of the function between the subintervals. This inaccuracy would be reduced if more subintervals and more trapeziums were divided. Therefore, it is fair to conclude that if more trapeziums are divided, the estimation of the definite integration will be more accurate.

To reflect, applying the trapezoidal rule has deepened my understanding of numerical integration methods, as it approximates the area under a curve by dividing it into trapezoids.

Monte Carlo Integration

In the third section of the investigation, the displacement will be calculated, or rather approximated using Monte Carlo integration methods through the programming language, Python. This differs from the previous two methods as this method takes statistics and probability, specifically the idea of the law of large numbers into account, which suggests that as the sample size increases, the average will be closer to the true population average. (The Investopedia Team, 2022).

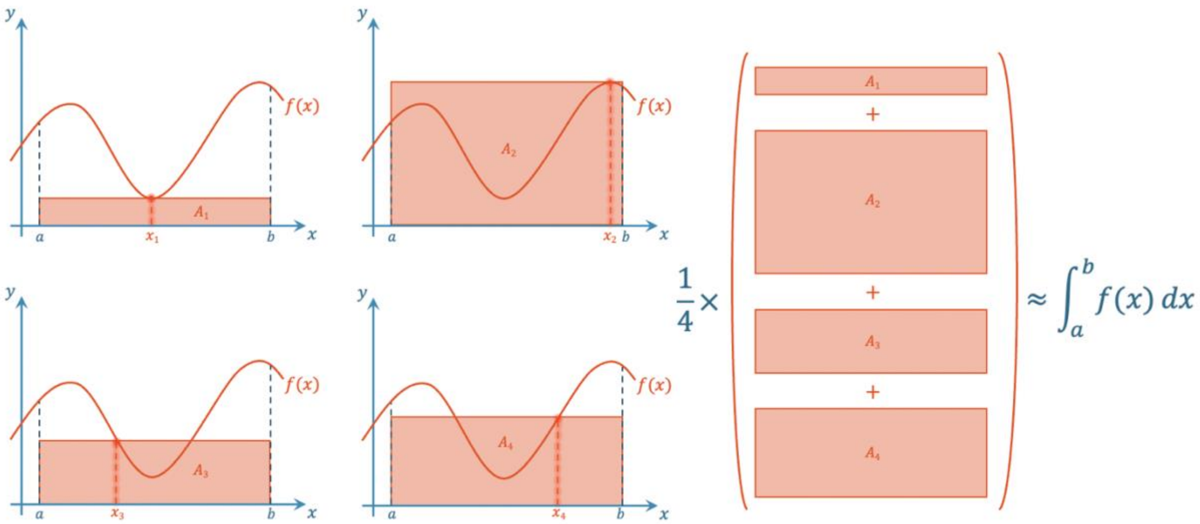

Figure 3 Monte Carlo Integration visualisation

In short, Monte Carlo integration involves (an example is shown in Figure 3, the example graph trace is not the function that will be integrated) (Monte Carlo Methods in Practice, n.d.):

1. Randomly sample multiple coordinates (depending on the sample size) on the function (when $x = x_1, x_2, x_3, x_4$ in Figure 3).
2. Calculate the areas of the rectangles formed with the height being the y-coordinate of the point and the width being the distance between the two limits (expressed as A_1, A_2, A_3, and A_4 in Figure 3).
3. Sum up all the areas for each of the sampling points ($A_1 + A_2 + A_3 + A_4$in Figure 3)

4. Divide the sum by the sample size to obtain the average area, which is also the approximation/estimation of the value of the definite integration. ($\frac{1}{4} \times (A_1 + A_2 + A_3 + A_4) \approx \int_a^b f(x)\, dx$ Figure 3)

Therefore, the equation of Monte Carlo Integration can be expressed as (Monte Carlo Methods in Practice, n.d.):

$$\int_a^b f(x)\, dx \approx \frac{1}{N}(b-a)\sum_{i=0}^{N-1} f(x_i),$$

$$where\ N\ is\ the\ number\ of\ samples\ used\ (generated\ randomly)$$

In this part of the investigation, Python will be used to approximate a section of the definite integral that expresses the velocity-time graph of a vehicle in the first minute (highlighted in yellow). During the process, three different N values will be used to examine how it may affect the accuracy of the integration ($when\ N = 100, 1000, 10000$, 100000 was going to be used, but it took too long to run the program).

$$\frac{1}{4}\int_0^1 (\sin(3t^2))dt + \int_0^1 t^2\, dt$$

Then, the elementary integration method will be used to calculate the rest of the definite integration. Thus, estimating the total displacement of the car travelled in the first minute by adding both sections together.

In Figure 4, the codes for Monte Carlo integration are written in Python, by using the Jupiter notebook, an extension to anaconda (anaconda, n.d.)

```
%matplotlib inline

from numpy import random
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1 #limits of definite integration
N = 1000
xrand = np.zeros(N)

for i in range(len(xrand)) :
    xrand[i] = random.uniform(a,b)

def func(x) :
    return np.sin(3*(x**2))

integral = 0.0

for i in range(N) :
    integral += func(xrand[i])

answer = (b-a)/float(N)*integral
print("The integral of sin(3x^2) from 0 to 1 when n=1000: ", answer)
```

Figure 4 Monte Carlo Python value approximation

- First, all the necessary libraries required for this programming are imported. This includes access to a random number generator by accessing the ‘NumPy’ library, which is imported as ‘np’ and “matplotlib.pyplot” is imported as “plt”.
- Then, the limits of the function are set, where $a = 1$ is the upper limits and $n = 0$ is the lower limits. The number of samples is also set as ‘N’, where three values ($N = 100, 1000, and\ 10000$) will be examined to evaluate the accuracy.
- After, an array ‘xrand’ is created to generate random numbers, where the numbers were first set to zeros and then changed into random numbers between the two limits as ‘random.uniform(a,b)’
- Next, the function that will be examined, ‘sin(3*(x**2))’ is defined as ‘func(x)’
- The following step is to substitute the numbers into the formula of Monte Carlo estimations, by summing up all the multiples of the interval and the randomly generated coordinate and dividing everything by ‘N.’ This is done by first setting the variable of ‘integral’ to 0.0 and adding the evaluated function at the random coordinates. After, it is multiplied by ‘b-a,’ the intervals of two limits and divided by ‘float(N)’ (to avoid integer division) to finalise the estimation of the area beneath the curve.
- Finally, the result of the estimation is printed out so that it is visible.

After running the code for different N values (100, 1000, and 10000), the accuracy can be compared. However, the actual displacement needs to be calculated by substituting the value into the equation of:

$$\frac{1}{4}\int_0^1 (\sin(3t^2))dt + \int_0^1 t^2\, dt$$

As the answer for $\int_0^1 t^2\,dt = \frac{1}{3}$ is already worked out in the Taylor Series, the displacement of the vehicle in the first minute can be calculated by multiplying the output number from the code (written at the bottom in Figure 5, 6, 7) by $\frac{1}{4}$ and add $\frac{1}{3}$ to it. Therefore:

For when $N = 100$, the estimated displacement of the vehicle in the first minute is:

```
%matplotlib inline

from numpy import random
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1 #limits of definite integration
N = 100
xrand = np.zeros(N)

for i in range(len(xrand)) :
    xrand[i] = random.uniform(a,b)

def func(x) :
    return np.sin(3*(x**2))

integral = 0.0

for i in range(N) :
    integral += func(xrand[i])

answer = (b-a)/float(N)*integral
print("The integral of sin(3x^2) from 0 to 1 when n=100: ", answer)
The integral of sin(3x^2) from 0 to 1 when n=100:  0.5397030038186578
```

Figure 5 Monte Carlo estimation when N=100

$$\frac{1}{4} \times 0.5397030038186578 + \frac{1}{3}$$

$$= 0.4682590843$$

For when $N = 1000$, the estimation is:

```
%matplotlib inline

from numpy import random
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1 #limits of definite integration
N = 1000
xrand = np.zeros(N)

for i in range(len(xrand)) :
    xrand[i] = random.uniform(a,b)

def func(x) :
    return np.sin(3*(x**2))

integral = 0.0

for i in range(N) :
    integral += func(xrand[i])

answer = (b-a)/float(N)*integral
print("The integral of sin(3x^2) from 0 to 1 when n=1000: ", answer)
The integral of sin(3x^2) from 0 to 1 when n=1000:  0.5197767572876835
```

Figure 6 Monte Carlo estimation when N=1000

$$\frac{1}{4} \times 0.5197767572876835 + \frac{1}{3}$$

$$= 0.4632775227$$

For when $N = 10000$, the estimation is:

```
%matplotlib inline

from numpy import random
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1 #limits of definite integration
N = 10000
xrand = np.zeros(N)

for i in range(len(xrand)) :
    xrand[i] = random.uniform(a,b)

def func(x) :
    return np.sin(3*(x**2))

integral = 0.0

for i in range(N) :
    integral += func(xrand[i])

answer = (b-a)/float(N)*integral
print("The integral of sin(3x^2) from 0 to 1 when n=10000: ", answer)
The integral of sin(3x^2) from 0 to 1 when n=10000:  0.5161215745015587
```

Figure 7 Monte Carlo estimation when N=10000

$$\frac{1}{4} \times 0.5161215745015587 + \frac{1}{3}$$

$$= 0.462363727$$

(All decimal places are kept from the output from the program and calculator)

Compared to the value from the GDC, which is:

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt = 0.4620773770\ km$$

The percentage of accuracy can be calculated by dividing the estimations by the GDC value:

Percentage of accuracy when $N = 100$:

$$Percentage\ of\ accuracy\ when\ N = 100\text{: } \frac{0.4682590843}{0.4620773770} = 1.013378 = 101.3378\%$$

$$Percentage\ of\ accuracy\ when\ N = 1000\text{: } \frac{0.4632775227}{0.4620773770} = 1.002597 = 100.2597\%$$

$$Percentage\ of\ accuracy\ when\ N = 10000\text{: } \frac{0.4623637270}{0.4620773770} = 1.000620 = 100.0620\%$$

According to the percentage of accuracy, as the sample size increases, the accuracy of the estimation improves as the accuracy when $N = 10000$ is the closest to 100%. This can be explained through the law of large numbers, which matches the results that as the sample size increases, it is closer to the GDC value (given by the GDC).

However, this may not always be the case, as the coordinates are randomly generated. So, if more coordinates tend toward a smaller area, the estimation would be smaller compared to the actual values. Vice versa, more coordinates tend toward a greater sized area would make the estimation larger compared to the actual values. This is possible even when the probability of selecting one coordinate is the same as the rest. Therefore, there is a possibility that the value may be more accurate when a smaller sample size is examined, as the randomly generated coordinates may tend closer to the averaged areas. However, using a larger sampling size would improve the accuracy and eliminate these types of errors according to the law of large numbers.

Overall, from the Monte Carlo section of the exploration, I improved my grasp of understanding of how random sampling can be used to approximate definite integrations, especially in the case of non-elementary integration, when analytical methods or other numerical techniques are impractical.

Comparison and Conclusion

All three methods give an approximation of the total displacement travelled by the vehicle in the first minute with almost 100% accuracy. However, by comparing the accuracy between the three methods, a general pattern appears. As the terms, the intervals, or the sampling numbers increase, the estimation would be more accurate. However, comparing the individual calculations, the estimation displacement of the car with the first six terms in the Taylor Series was the most accurate, with 99.9995% of accuracy, making the Taylor Series the most accurate. In addition, the trapezoidal method took more effort in writing and calculation as more intervals will have to be calculated to achieve reasonably accurate data. Nevertheless, the accuracy for all methods would increase further if the terms, the intervals, or the sampling numbers increased further.

Evaluation

For the Taylor Series integration, hundreds and thousands of terms would need to be used when examining a longer range to be extremely accurate, since the method expands the function into infinite terms of an exponential function. Therefore, more terms need to be considered to achieve a relatively accurate answer if the investigation is examining a longer time (as only one unit is changed in this investigation). For the Trapezoidal rule, numerous intervals are needed to acquire a decent accuracy of estimation, thus making it harder to calculate manually. Although the Monte Carlo Integration method is fast to set up a program, it would take a long time, even up to hours to run the codes, if better accuracy is desired (such as when $N = 100000$ in this scenario). In addition, this investigation only examines the function: $\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt$, which means that this might not be as accurate as other functions.

Lastly, it is also important to consider the definite integration value obtained from the GDC. Unfortunately, I am unable to acquire accurate information on the algorithm the calculator uses. However, similar methods (to Taylor Series, Trapezoidal Rule and Monte Carlos integration) may have been used to estimate the definite integration while just considering more terms or sample size, such as the Riemann Method, which is similar to the trapezoidal method as it divides the area under the curve into small rectangles and calculates their sums.

Reflection and Further Applications

From this investigation, I was able to grasp the concept of calculus, especially non-elementary definite integration more comprehensively. More specifically, I now am able to calculate non-elementary integration through three different approaches: Taylor Series, Trapezoidal Rule, and Monte Carlo Integration.

One notable aspect during this exploration was realising how these methods are extensively applied in the scientific and data analysis fields. These methods are used to approximate functions, make predictions, and illustrate useful mathematical concepts in a variety of fields, including economics and physics. For example, the law of big numbers is taken into consideration when the Monte Carlo theory is applied in data analysis, finance, and actuarial science. While the other methods may be utilised in calculators' algorithms to calculate definite integration and may be applied in the field of kinematics.

To take this exploration further, more terms, intervals, or sample sizes could be considered and calculated to achieve a potentially more accurate estimation. Additionally, multiple functions may be examined, as they may provide a differing accuracy to this specific exploration, which would allow me to form a more precise conclusion on the accuracy of these methods.

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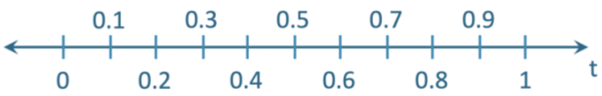
Appendix

The estimated displacement when $n = 10$ by using the Trapezoidal rule:

First, the height of the small trapeziums can be calculated:

$$h = \frac{1-0}{10} = \frac{1}{10}$$

Next, the t values when $h = \frac{1}{10}$ are:



Therefore, the corresponding t ($t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$) values will be used by substituting into the function $v = \frac{1}{4}\sin(3t^2) + t^2$:

$$when\ t = 0, v_0 = \frac{1}{4}\sin(3 \times 0^2) + 0^2 = 0$$

$$when\ t = 0.1, v_1 = \frac{1}{4}\sin(3 \times 0.1^2) + 0.1^2 = \frac{1}{4}\sin(0.03) + 0.01,$$

$$when\ t = 0.2, v_2 = \frac{1}{4}\sin(3 \times 0.2^2) + 0.2^2 = \frac{1}{4}\sin(0.12) + 0.04,$$

$$when\ t = 0.3, v_3 = \frac{1}{4}\sin(3 \times 0.3^2) + 0.3^2 = \frac{1}{4}\sin(0.27) + 0.09,$$

$$when\ t = 0.4, v_4 = \frac{1}{4}\sin(3 \times 0.4^2) + 0.4^2 = \frac{1}{4}\sin(0.48) + 0.16,$$

$$when\ t = 0.5, v_5 = \frac{1}{4}\sin(3 \times 0.5^2) + 0.5^2 = \frac{1}{4}\sin(0.75) + 0.25,$$

$$when\ t = 0.6, v_6 = \frac{1}{4}\sin(3 \times 0.6^2) + 0.6^2 = \frac{1}{4}\sin(1.08) + 0.36,$$

$$when\ t = 0.7, v_7 = \frac{1}{4}\sin(3 \times 0.7^2) + 0.7^2 = \frac{1}{4}\sin(1.47) + 0.49,$$

$$when\ t = 0.8, v_8 = \frac{1}{4}\sin(3 \times 0.8^2) + 0.8^2 = \frac{1}{4}\sin(1.92) + 0.64,$$

$$when\ t = 0.9, v_9 = \frac{1}{4}\sin(3 \times 0.9^2) + 0.9^2 = \frac{1}{4}\sin(2.43) + 0.81,$$

$$when\ t = 1, v_{10} = \frac{1}{4}\sin(3 \times 1^2) + 1^2 = \frac{1}{4}\sin(3) + 1,$$

For better accuracy, the expressions for v were listed instead of the decimal values.

The next step to estimate the displacement is to substitute the values into the formula:

$$\int_a^b y\,dx \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n]$$

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt$$

$$\approx \frac{1}{2} \times \frac{1}{10}$$

$$\times \Bigg[0$$

$$+ 2\left(\left(\frac{1}{4}\sin(0.03) + 0.01\right) + \left(\frac{1}{4}\sin(0.12) + 0.04\right) + \left(\frac{1}{4}\sin(0.27) + 0.09\right)\right.$$

$$+ \left(\frac{1}{4}\sin(0.48) + 0.16\right) + \left(\frac{1}{4}\sin(0.75) + 0.25\right) + \left(\frac{1}{4}\sin(1.08) + 0.36\right)$$

$$\left. + \left(\frac{1}{4}\sin(1.47) + 0.49\right) + \left(\frac{1}{4}\sin(1.92) + 0.64\right) + \left(\frac{1}{4}\sin(2.43) + 0.81\right)\right)$$

$$+ \left(\frac{1}{4}\sin(3) + 1\right)\Bigg]$$

Which simplifies to:

$$\approx \frac{1}{20} \times \left[\frac{1}{2}\sin(0.03) + 0.02 + \frac{1}{2}\sin(0.12) + 0.08 + \frac{1}{2}\sin(0.27) + 0.18 + \frac{1}{2}\sin(0.48)\right.$$

$$+ 0.32 + \frac{1}{2}\sin(0.75) + 0.5 + \frac{1}{2}\sin(1.08) + 0.72 + \frac{1}{2}\sin(1.47) + 0.98$$

$$\left. + \frac{1}{2}\sin(1.92) + 1.28 + \frac{1}{2}\sin(2.43) + 1.62 + \frac{1}{4}\sin(3) + 1\right]$$

Adding up all the rational numbers (highlighted in grey):

$$\approx \frac{1}{20} \times \left[\frac{1}{2}\sin(0.03) + 0.02 + \frac{1}{2}\sin(0.12) + 0.08 + \frac{1}{2}\sin(0.27) + 0.18 + \frac{1}{2}\sin(0.48)\right.$$

$$+ 0.32 + \frac{1}{2}\sin(0.75) + 0.5 + \frac{1}{2}\sin(1.08) + 0.72 + \frac{1}{2}\sin(1.47) + 0.98$$

$$\left. + \frac{1}{2}\sin(1.92) + 1.28 + \frac{1}{2}\sin(2.43) + 1.62 + \frac{1}{4}\sin(3) + 1\right]$$

Which could be written as:

$$\approx \frac{1}{20} \times \left[\frac{1}{2}\sin(0.03) + \frac{1}{2}\sin(0.12) + \frac{1}{2}\sin(0.27) + \frac{1}{2}\sin(0.48) + \frac{1}{2}\sin(0.75)\right.$$

$$\left. + \frac{1}{2}\sin(1.08) + \frac{1}{2}\sin(1.47) + \frac{1}{2}\sin(1.92) + \frac{1}{2}\sin(2.43) + \frac{1}{4}\sin(3) + 6.7\right]$$

Expanding the bracket:

$$\frac{1}{40}\sin(0.03) + \frac{1}{40}\sin(0.12) + \frac{1}{40}\sin(0.27) + \frac{1}{40}\sin(0.48) + \frac{1}{40}\sin(0.75) + \frac{1}{40}\sin(1.08)$$

$$+ \frac{1}{40}\sin(1.47) + \frac{1}{40}\sin(1.92) + \frac{1}{40}\sin(2.43) + \frac{1}{80}\sin(3) + \frac{6.7}{20}$$

By using a calculator:

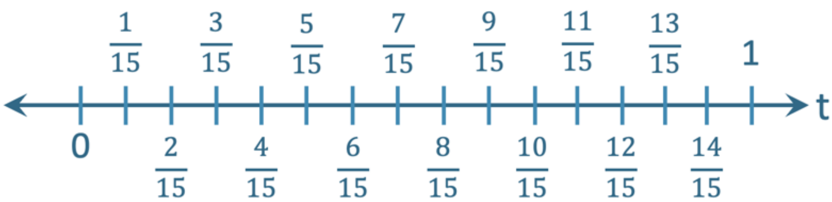
$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4624996366km$$

The estimated displacement when $n = 15$ by using the Trapezoidal rule:

First, the height of the small trapeziums can be calculated:

$$h = \frac{1-0}{15} = \frac{1}{15}$$

The next step is to list out the t values that are needed in order to calculate the corresponding v values. Since the interval (h) equals to $\frac{1}{15}$, the t values needed are:



Then, these t values ($t = 0, \frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}, \frac{6}{15}, \frac{7}{15}, \frac{8}{15}, \frac{9}{15}, \frac{10}{15}, \frac{11}{15}, \frac{12}{15}, \frac{13}{15}, \frac{14}{15}, 1$) will be used to find the corresponding v values by substituting into the function $v = \frac{1}{4}\sin(3t^2) + t^2$:

$$when\ t = 0, v_0 = \frac{1}{4}\sin(3 \times 0^2) + 0^2 = 0$$

$$when\ t = \frac{1}{15}, v_1 = \frac{1}{4}\sin\left(3 \times \left(\frac{1}{15}\right)^2\right) + \left(\frac{1}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{1}{75}\right) + \frac{1}{225},$$

$$when\ t = \frac{2}{15}, v_2 = \frac{1}{4}\sin\left(3 \times \left(\frac{2}{15}\right)^2\right) + \left(\frac{2}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{4}{75}\right) + \frac{4}{225},$$

$$when\ t = \frac{3}{15}, v_3 = \frac{1}{4}\sin\left(3 \times \left(\frac{3}{15}\right)^2\right) + \left(\frac{3}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{3}{25}\right) + \frac{1}{25},$$

$$when\ t = \frac{4}{15}, v_4 = \frac{1}{4}\sin\left(3 \times \left(\frac{4}{15}\right)^2\right) + \left(\frac{4}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{16}{75}\right) + \frac{16}{225},$$

$$when\ t = \frac{5}{15}, v_5 = \frac{1}{4}\sin\left(3 \times \left(\frac{5}{15}\right)^2\right) + \left(\frac{5}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{1}{3}\right) + \frac{1}{9},$$

$$when\ t = \frac{6}{15}, v_6 = \frac{1}{4}\sin\left(3 \times \left(\frac{6}{15}\right)^2\right) + \left(\frac{6}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{12}{25}\right) + \frac{4}{25},$$

$$when\ t = \frac{7}{15}, v_7 = \frac{1}{4}\sin\left(3 \times \left(\frac{7}{15}\right)^2\right) + \left(\frac{7}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{49}{75}\right) + \frac{49}{225},$$

$$when\ t = \frac{8}{15}, v_8 = \frac{1}{4}\sin\left(3 \times \left(\frac{8}{15}\right)^2\right) + \left(\frac{8}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{64}{75}\right) + \frac{64}{225},$$

$$when\ t = \frac{9}{15}, v_9 = \frac{1}{4}\sin\left(3 \times \left(\frac{9}{15}\right)^2\right) + \left(\frac{9}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{27}{25}\right) + \frac{9}{25},$$

$$when\ t = \frac{10}{15}, v_{10} = \frac{1}{4}\sin\left(3 \times \left(\frac{10}{15}\right)^2\right) + \left(\frac{10}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{4}{3}\right) + \frac{4}{9},$$

$$when\ t = \frac{11}{15}, v_{11} = \frac{1}{4}\sin\left(3 \times \left(\frac{11}{15}\right)^2\right) + \left(\frac{11}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{121}{75}\right) + \frac{121}{225},$$

$$when\ t = \frac{12}{15}, v_{12} = \frac{1}{4}\sin\left(3 \times \left(\frac{12}{15}\right)^2\right) + \left(\frac{12}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{48}{25}\right) + \frac{16}{25},$$

$$when\ t = \frac{13}{15}, v_{13} = \frac{1}{4}\sin\left(3 \times \left(\frac{13}{15}\right)^2\right) + \left(\frac{13}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{169}{75}\right) + \frac{169}{225},$$

$$when\ t = \frac{14}{15}, v_{14} = \frac{1}{4}\sin\left(3 \times \left(\frac{14}{15}\right)^2\right) + \left(\frac{14}{15}\right)^2 = \frac{1}{4}\sin\left(\frac{196}{75}\right) + \frac{196}{225},$$

$$when\ t = 1, v_{15} = \frac{1}{4}\sin(3 \times 1^2) + 1^2 = \frac{1}{4}\sin(3) + 1,$$

For better accuracy, the expressions for v were listed instead of the decimal values.

To estimate the displacement, their expression will be substituted into the trapezoidal formula:

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt$$

$$\approx \frac{1}{2} \times \frac{1}{15}$$

$$\times \Bigg[0$$

$$+ 2\left(\left(\frac{1}{4}\sin\left(\frac{1}{75}\right) + \frac{1}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{4}{75}\right) + \frac{4}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{3}{25}\right) + \frac{1}{25}\right)\right.$$

$$+ \left(\frac{1}{4}\sin\left(\frac{16}{75}\right) + \frac{16}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{1}{3}\right) + \frac{1}{9}\right) + \left(\frac{1}{4}\sin\left(\frac{12}{25}\right) + \frac{4}{25}\right)$$

$$+ \left(\frac{1}{4}\sin\left(\frac{49}{75}\right) + \frac{49}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{64}{75}\right) + \frac{64}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{27}{25}\right) + \frac{9}{25}\right)$$

$$+ \left(\frac{1}{4}\sin\left(\frac{4}{3}\right) + \frac{4}{9}\right) + \left(\frac{1}{4}\sin\left(\frac{121}{75}\right) + \frac{121}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{48}{25}\right) + \frac{16}{25}\right)$$

$$\left. + \left(\frac{1}{4}\sin\left(\frac{169}{75}\right) + \frac{169}{225}\right) + \left(\frac{1}{4}\sin\left(\frac{196}{75}\right) + \frac{196}{225}\right)\right) + \left(\frac{1}{4}\sin(3) + 1\right)\Bigg]$$

Which simplifies to:

$$\approx \frac{1}{30} \times \left[\frac{1}{2}\sin\left(\frac{1}{75}\right) + \frac{2}{225} + \frac{1}{2}\sin\left(\frac{4}{75}\right) + \frac{8}{225} + \frac{1}{2}\sin\left(\frac{3}{25}\right) + \frac{2}{25} + \frac{1}{2}\sin\left(\frac{16}{75}\right) + \frac{32}{225}\right.$$

$$+ \frac{1}{2}\sin\left(\frac{1}{3}\right) + \frac{2}{9} + \frac{1}{2}\sin\left(\frac{12}{25}\right) + \frac{8}{25} + \frac{1}{2}\sin\left(\frac{49}{75}\right) + \frac{98}{225} + \frac{1}{2}\sin\left(\frac{64}{75}\right) + \frac{128}{225}$$

$$+ \frac{1}{2}\sin\left(\frac{27}{25}\right) + \frac{18}{25} + \frac{1}{2}\sin\left(\frac{4}{3}\right) + \frac{8}{9} + \frac{1}{2}\sin\left(\frac{121}{75}\right) + \frac{242}{225} + \frac{1}{2}\sin\left(\frac{48}{25}\right) + \frac{32}{25}$$

$$\left. + \frac{1}{2}\sin\left(\frac{169}{75}\right) + \frac{338}{225} + \frac{1}{2}\sin\left(\frac{196}{75}\right) + \frac{392}{225} + \frac{1}{4}\sin(3) + 1\right]$$

Summing up all the rational numbers (highlighted in grey):

$$\approx \frac{1}{30} \times \left[\frac{1}{2}\sin\left(\frac{1}{75}\right) + \frac{1}{2}\sin\left(\frac{4}{75}\right) + \frac{1}{2}\sin\left(\frac{3}{25}\right) + \frac{1}{2}\sin\left(\frac{16}{75}\right) + \frac{1}{2}\sin\left(\frac{1}{3}\right) + \frac{1}{2}\sin\left(\frac{12}{25}\right)\right.$$

$$+ \frac{1}{2}\sin\left(\frac{49}{75}\right) + \frac{1}{2}\sin\left(\frac{64}{75}\right) + \frac{1}{2}\sin\left(\frac{27}{25}\right) + \frac{1}{2}\sin\left(\frac{4}{3}\right) + \frac{1}{2}\sin\left(\frac{121}{75}\right)$$

$$\left. + \frac{1}{2}\sin\left(\frac{48}{25}\right) + \frac{1}{2}\sin\left(\frac{169}{75}\right) + \frac{1}{2}\sin\left(\frac{196}{75}\right) + \frac{1}{4}\sin(3) + \frac{451}{45}\right]$$

Expand the bracket:

$$\approx \frac{1}{60}\sin\left(\frac{1}{75}\right) + \frac{1}{60}\sin\left(\frac{4}{75}\right) + \frac{1}{60}\sin\left(\frac{3}{25}\right) + \frac{1}{60}\sin\left(\frac{16}{75}\right) + \frac{1}{60}\sin\left(\frac{1}{3}\right) + \frac{1}{60}\sin\left(\frac{12}{25}\right)$$

$$+ \frac{1}{60}\sin\left(\frac{49}{75}\right) + \frac{1}{60}\sin\left(\frac{64}{75}\right) + \frac{1}{60}\sin\left(\frac{27}{25}\right) + \frac{1}{60}\sin\left(\frac{4}{3}\right) + \frac{1}{60}\sin\left(\frac{121}{75}\right)$$

$$+ \frac{1}{60}\sin\left(\frac{48}{25}\right) + \frac{1}{60}\sin\left(\frac{169}{75}\right) + \frac{1}{60}\sin\left(\frac{196}{75}\right) + \frac{1}{120}\sin(3) + \frac{451}{1350}$$

By using a calculator, the estimation for the displacement of the car in the first minute is:

$$\int_0^1 \left(\frac{1}{4}\sin(3t^2) + t^2\right) dt \approx 0.4622667579$$