SWE on sphere formulation in SWEET

Martin Schreiber

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This documents describes the particular SWE on sphere implementation in SWEET.

1 Vorticity-divergence formulation

We use the vorticity-divergence formulation (See Williamson et al. paper on SWE test cases) with the divergence given by

$$\delta = \nabla \cdot \vec{v}$$

and the relative vorticity by

$$\xi = \vec{k} \cdot (\nabla \times \vec{v})$$

with $\vec{v} = (u, v)$ the velocity vector. The vorticity-divergence formulation is then given by

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi &= -\nabla \cdot (\Phi \vec{v}) \\
\frac{\partial}{\partial t}\xi &= -\nabla \cdot ((\xi + f)\, \vec{v}) \\
\frac{\partial}{\partial t}\delta &= \vec{k} \cdot \nabla \times ((\xi + f)\, \vec{v}) - \nabla^2 \left(\Phi + \frac{\vec{v} \cdot \vec{v}}{2} \right)
\end{aligned}$$

2 Spherical harmonics

We use state variables in physical space without a tilde, e.g. x, and variables in spectral space denoted with a tilde such as $\tilde{x}$. Let the function converting data from spectral to physical space be given by

$$x = s2p(\tilde{x})$$

and from physical to spectral space by

$$\tilde{x} = p2s(x).$$

The Laplace operator is then provided by

$$\nabla^2(u) = lap(u).$$

We also exploit the feature of the SHTNS library to compute the divergence and vorticity in spectral space for given velocities in physical space and denote this function by

$$(\tilde{\xi}, \tilde{\delta}) = vel2vd(u, v)$$

and vice versa

$$(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta}).$$

3 Linear shallow-water equations

3.1 l_erk

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Evaluate temporary vector $(u^*, v^*) = f\vec{v}$ in physical space
   (c) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (d) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (e) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}' - lap(\tilde{\Phi})$
2. Geopotential:
   (a) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\overline{\Phi}\tilde{\delta}$

3.2 lc_erk_lg_erk

3.2.1 lc_erk

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Evaluate temporary vector $(u^*, v^*) = f\vec{v}$ in physical space
   (c) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (d) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (e) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}'$
2. Geopotential:
   (a) Zero geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = 0$

3.2.2 lg_erk

1. Vorticity/divergence:
   (a) Zero vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = 0$
   (b) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = -lap(\tilde{\Phi})$
2. Geopotential:
   (a) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\overline{\Phi}\tilde{\delta}$

4 Nonlinear shallow-water equations

4.1 ln_erk

Next, we discuss the full non-linear time tendency computation. The different steps for the time integration are given as follows:

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Convert ξ from spectral to physical space: $\xi = s2p(\tilde{\xi})$
   (c) Evaluate temporary vector $(u^*, v^*) = (\xi + f)\,\vec{v}$ in physical space
   (d) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (e) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (f) Convert geopotential $\Phi = s2p(\tilde{\Phi})$
   (g) Compute $e = \Phi + \frac{\vec{v}\cdot\vec{v}}{2}$
   (h) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}' - lap(p2s(e))$
2. Geopotential:
   (a) Prepare divergence $(u^t, v^t) = (u\Phi, v\Phi)$
   (b) Compute divergence $(\tilde{\xi}^*, \tilde{\delta}^*) = vel2vd(u^t, v^t)$
   (c) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\tilde{\delta}^*$

5 Semi-Lagrangian shallow-water equations

For the Lagrangian formulation we use the total derivative

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + \vec{v} \cdot \nabla().$$

We write the geopotential as

$$\frac{\partial}{\partial t}\Phi = -\nabla \cdot (\Phi\vec{v})\,.$$

As usual, this is split into the linearized form $\Phi = \overline{\Phi} + \Phi'$, yielding

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi' &= -\nabla \cdot \left(\left(\overline{\Phi} + \Phi'\right)\vec{v}\right) \\
&= -\overline{\Phi}\nabla \cdot \vec{v} - \nabla \cdot (\Phi'\vec{v}) \\
&= -\overline{\Phi}\nabla \cdot \vec{v} - \vec{v} \cdot (\nabla\Phi') - (\nabla \cdot \vec{v})\,\Phi' \\
\frac{d}{dt}\Phi' &= \underbrace{-\overline{\Phi}\delta}_{\text{linear}} - \underbrace{(\nabla \cdot \vec{v})\,\Phi'}_{\text{non-linear advection}} \\
&= -\delta\Phi'
\end{aligned}$$

1. Advective terms
   (a) Regarding the advective terms, since $\frac{d}{dt}u$ and $\frac{d}{dt}v$ would be entirely linear on the right-hand side, so are the vorticity and divergence. Therefore, the remaining terms are the ones which can be treated linearly (see l_erk above).
2. Geopotential
   (a) Convert states to spectral space: $\delta = s2p(\tilde{\delta})$ and $\Phi' = s2p(\tilde{\Phi}')$
   (b) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -p2s\,(\delta\Phi')$