SWE on sphere formulation in SWEET

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This documents describes the particular SWE on sphere implementation in SWEET. (See also Williamson et al. benchmark test cases for some more details).

1 Changelog

- 2020-05-11: Various updates witin the last 2 months.
- 2021-07-23: P.Peixoto updates on how Barotropic Vorticity Equation is calculated in SWEET

2 Velocity-based formulation (Advective form)

We start with the velocity-based formulation.

2.1 Eulerian formulation

The Eulerian formulation is given by

$$\begin{aligned}\frac{\partial h}{\partial t} &= -h\nabla\cdot\vec{V}-\vec{V}\cdot\nabla h=-\nabla\cdot\left(h\vec{V}\right)\\ \frac{\partial\vec{V}}{\partial t} &= -f\vec{k}\times\vec{V}-g\nabla h-\vec{V}\cdot\nabla\vec{V}\end{aligned}$$

or in terms of the geopotential $\Phi=hg$

$$\begin{aligned}\frac{\partial\Phi}{\partial t} &= -\Phi\nabla\cdot\vec{V}-\vec{V}\cdot\nabla\Phi=-\nabla\cdot\left(\Phi\vec{V}\right)\\ \frac{\partial\vec{V}}{\partial t} &= -f\vec{k}\times\vec{V}-\nabla\Phi-\vec{V}\cdot\nabla\vec{V}.\end{aligned}$$

We briefly describe the notation $\vec{V}\cdot\nabla\vec{V}$ which applies a gradient element-wise on $\vec{V}$, followed by a dot product:

$$\begin{aligned}\vec{V}\cdot\nabla\vec{V} &= \vec{V}\cdot\nabla\left(u,v\right)\\ &= \vec{V}\cdot\left(\nabla u,\nabla v\right)\\ &= \left(\vec{V}\cdot\nabla u,\vec{V}\cdot\nabla v\right).\end{aligned}$$

2.2 Lagrangian formulation

Writing this in terms of a total material derivative

$$\frac{d\left(.\right)}{dt}=\frac{\partial\left(.\right)}{\partial t}+\vec{V}\cdot\nabla\left(.\right)$$

we get

$$\begin{aligned}\frac{dh}{dt} &= -h\nabla\cdot\vec{V}\\ \frac{d\vec{V}}{dt} &= -f\vec{k}\times\vec{V}-g\nabla h\end{aligned}$$

and in geopotential form

$$\frac{d\Phi}{dt} = -\Phi\nabla \cdot \vec{V}$$
$$\frac{d\vec{V}}{dt} = -f\vec{k} \times \vec{V} - g\nabla\Phi.$$

2.3 Building blocks

We split each term of the equation into corresponding linear and non-linear parts with respect to using the velocity as the variable being advected by the non-linear advection term. This will make a significant difference for the vorticity-divergence formulation as we will see. Using a perturbed formulation $\Phi = \overline{\Phi} + \Phi'$ of the geopotential leads to

$$\frac{\partial\Phi}{\partial t} = -\overline{\Phi}\nabla \cdot \vec{V} - \Phi'\nabla \cdot \vec{V} - \vec{V} \cdot \nabla\Phi$$

and we can write

$$\frac{\partial}{\partial t}\begin{pmatrix} \Phi \\ \vec{V} \end{pmatrix} = \underbrace{\begin{pmatrix} -\overline{\Phi}\nabla \cdot \vec{V} \\ -\nabla\Phi \end{pmatrix}}_{\text{linear gravity}} + \underbrace{\begin{pmatrix} \\ -f\vec{k} \times \vec{V} \end{pmatrix}}_{\text{linear Coriolis}} + \underbrace{\begin{pmatrix} -\vec{V} \cdot \nabla\Phi \\ -\vec{V} \cdot \nabla\vec{V} \end{pmatrix}}_{\text{nonlinear advection}} + \underbrace{\begin{pmatrix} -\Phi'\nabla \cdot \vec{V} \\ 0 \end{pmatrix}}_{\text{nonlinear rest}}.$$

3 Vorticity-divergence (VD) formulation

We use the vorticity-divergence formulation by applying the divergence and curl operator

$$div\,(\vec{.}) \quad = \quad \nabla \cdot (\vec{.})$$

$$curl\,(\vec{.}) \quad = \quad (\nabla \times (\vec{.}))$$

to the velocity equations, yielding the vorticity and divergence

$$\delta \quad = \quad div(\vec{V}) = \nabla \cdot \vec{V}$$

$$\xi \quad = \vec{k} \cdot \quad curl(\vec{V}) = \vec{k} \cdot \left(\nabla \times \vec{V}\right)$$

with $\vec{V} = (u, v)$ the velocity vector.

3.1 Warmup

In the following, we will use the identity

$$\vec{V} \cdot \nabla\vec{V} = \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) - \vec{V} \times \left(\nabla \times \vec{V}\right)$$

to reformulate the velocity equation

$$\begin{aligned} \frac{\partial\vec{V}}{\partial t} &= -f\vec{k} \times \vec{V} - \nabla\Phi - \vec{V} \cdot \nabla\vec{V} \\ &= -f\vec{k} \times \vec{V} - \nabla\Phi - \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) + \vec{V} \times \left(\nabla \times \vec{V}\right) \\ &= -\nabla\Phi - \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) - f\vec{k} \times \vec{V} + \vec{V} \times \left(\nabla \times \vec{V}\right). \end{aligned}$$

Next, we rewrite the term

$$\begin{aligned}-f\vec{k}\times\vec{V}-\vec{V}\times\left(\nabla\times\vec{V}\right) &= -f\vec{k}\times\vec{V}-\left(\nabla\times\vec{V}\right)\times\vec{V}\\ &= \left(-\left(\nabla\times\vec{V}\right)-f\vec{k}\right)\times\vec{V}.\end{aligned}$$

We can utilize that $\nabla\times\vec{V}$ is always parallel to $\vec{k}$, hence we can write

$$\begin{aligned}\left(\nabla\times\vec{V}\right)\times V &= \left(\left(\left(\nabla\times\vec{V}\right)\cdot\vec{k}\right)\vec{k}\right)\times\vec{V}\\ &= \xi\left(\vec{k}\times\vec{V}\right)\end{aligned}$$

to finally get

$$\frac{\partial\vec{V}}{\partial t} = -\nabla\left(\Phi+\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\right)-(\xi+f)\,\vec{k}\times\vec{V}$$

or written with the total vorticity

$$\eta = \xi + f$$

we get

$$\frac{\partial\vec{V}}{\partial t} = -\nabla\left(\Phi+\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\right)-\eta\vec{k}\times\vec{V}.$$

3.2 Divergence equation

Using the divergence operator on the velocity equations we then get

$$\begin{aligned}\nabla\cdot\left(\frac{\partial\vec{V}}{\partial t}\right) &= -\nabla^2\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)+\nabla\cdot\left(\eta\vec{k}\times\vec{V}\right)\\ &= -\nabla^2\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)+\nabla\cdot\left(\vec{k}\times\eta\vec{V}\right)\end{aligned}$$

and we use

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A).$$

to get

$$\nabla\cdot\left(\vec{k}\times\eta\vec{V}\right) = \vec{k}\cdot\left(\nabla\times\eta\vec{V}\right).$$

Finally, we have

$$\frac{\partial\delta}{\partial t} = -\nabla^2\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)+\vec{k}\cdot\nabla\times\left(\eta\vec{V}\right). \tag{1}$$

3.3 Vorticity equation

Using the divergence operator on the velocity equations we then get

$$\begin{aligned}\vec{k}\cdot\left(\nabla\times\frac{\partial\vec{V}}{\partial t}\right) &= \vec{k}\cdot\nabla\times\left[-\nabla\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)-\left(\eta\vec{k}\times\vec{V}\right)\right]\\ \frac{\partial\xi}{\partial t} &= -\vec{k}\cdot\nabla\times\nabla\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)-\vec{k}\cdot\nabla\times\left(\eta\vec{k}\times\vec{V}\right)\end{aligned}$$

and furthermore

$$\begin{aligned}\frac{\partial\xi}{\partial t} &= -\vec{k}\cdot\left[\nabla\times\left(\eta\left(\vec{k}\times\vec{V}\right)\right)\right]\\ &= -\vec{k}\cdot\left[\nabla\times\left(\vec{k}\times\eta\vec{V}\right)\right].\end{aligned}$$

Using the identity

$$\vec{k}\cdot(\nabla\times B) = B\cdot\left(\vec{k}\times\nabla\right)$$

we get

$$\begin{aligned}\frac{\partial\xi}{\partial t} &= -\nabla\cdot\left[\vec{k}\times\left(\vec{k}\times\eta\vec{V}\right)\right]\\ &= -\nabla\cdot\left(\eta\vec{V}\right).\end{aligned}$$

Finally, we obtain

$$\frac{\partial\xi}{\partial t} = -\nabla\cdot\left(\eta\vec{V}\right) = -\nabla\cdot\left(\left(\xi+f\right)\vec{V}\right). \tag{2}$$

3.4 Eulerian formulation

The vorticity-divergence formulation in Eulerian formulation is then given by

$$\begin{aligned}\frac{\partial}{\partial t}\Phi &= -\nabla\cdot(\Phi\vec{V})\\ \frac{\partial}{\partial t}\xi &= -\nabla\cdot\left(\left(\xi+f\right)\vec{V}\right)\\ \frac{\partial}{\partial t}\delta &= \vec{k}\cdot\nabla\times\left(\left(\xi+f\right)\vec{V}\right) - \nabla^2\left(\Phi+\frac{\vec{V}\cdot\vec{V}}{2}\right)\end{aligned}$$

with the total vorticity $\eta = \xi + f$.

3.5 Lagrangian formulation

Now we rearrange the formulation into a Lagrangian form with (Φ, ξ, δ) the advected quantities. The geopotential remains unchanged, hence

$$\frac{d\Phi}{dt} = -\Phi\nabla\cdot\vec{V} = -\Phi\delta.$$

For the vorticity we get

$$\begin{aligned}\frac{\partial}{\partial t}\xi &= -\nabla\cdot\left(\left(\xi+f\right)\vec{V}\right)\\ &= -\nabla\cdot\left(\xi\vec{V}\right) - \nabla\cdot\left(f\vec{V}\right)\\ &= -\vec{V}\cdot\nabla\xi - \xi\nabla\cdot\vec{V} - \nabla\cdot\left(f\vec{V}\right)\\ \frac{d}{dt}\xi &= -\xi\nabla\cdot\vec{V} - \nabla\cdot\left(f\vec{V}\right)\\ &= -\xi\delta - \nabla\cdot\left(f\vec{V}\right).\end{aligned}$$

For the divergence term we first write the nonlinear advection in a different way

$$\begin{aligned}\vec{V}\cdot\nabla\delta &= \vec{V}\cdot\nabla\delta + \left(\nabla\cdot\vec{V}\right)\delta - \left(\nabla\cdot\vec{V}\right)\delta\\ &= \nabla\cdot\left(\delta\vec{V}\right) - \delta^2.\end{aligned}$$

We can then write the divergence formulation as

$$
\begin{aligned}
\frac{\partial\delta}{\partial t} &= -\nabla^2\left(\Phi + \frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{k}\cdot\nabla\times\left((f+\xi)\,\vec{V}\right) \\
&= -\nabla^2\left(\Phi + \frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{k}\cdot\nabla\times\left((f+\xi)\,\vec{V}\right) - \vec{V}\cdot\nabla\delta + \vec{V}\cdot\nabla\delta \\
&= -\nabla^2\left(\Phi + \frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{k}\cdot\nabla\times\left((f+\xi)\,\vec{V}\right) - \vec{V}\cdot\nabla\delta + \nabla\cdot\left(\delta\vec{V}\right) - \delta^2 \\
&= -\nabla^2\Phi + \vec{k}\cdot\nabla\times\left(f\vec{V}\right) - \vec{V}\cdot\nabla\delta - \nabla^2\frac{\vec{V}\cdot\vec{V}}{2} + \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right) + \nabla\cdot\left(\delta\vec{V}\right) - \delta^2 \\
\frac{d}{dt}\delta &= -\nabla^2\Phi + \vec{k}\cdot\nabla\times\left(f\vec{V}\right) + \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right) + \nabla\cdot\left(\delta\vec{V}\right) - \nabla^2\frac{\vec{V}\cdot\vec{V}}{2} - \delta^2
\end{aligned}
$$

Note, that **in contrast to the velocity**-based formulation, **there are now nonlinear terms on the RHS** of the vorticity and divergence quations.

3.6 Building blocks

Similar to the velocity-based formulation, we can split up the vorticity divergence equation into different parts.

Since we are overall interested in a Lagrangian treatment and with the vorticity and divergence now used as prognostic variables, hence the nonlinear advection terms being given by

$$
\vec{V}\cdot\nabla\xi \text{ and } \vec{V}\cdot\nabla\delta,
$$

also a splitting into basic blocks will lead to different results. Using the previous Lagrangian reformulatino for the vorticity-divergence formulation, we can identify the building blocks as

$$
\frac{\partial}{\partial t}\begin{pmatrix}\Phi\\ \xi\\ \delta\end{pmatrix} = \underbrace{\begin{pmatrix}-\overline{\Phi}\delta\\ 0\\ -\nabla^2\Phi\end{pmatrix}}_{\text{linear gravity}} + \underbrace{\begin{pmatrix} \\ -\nabla\cdot\left(f\vec{V}\right)\\ \vec{k}\cdot\nabla\times\left(f\vec{V}\right)\end{pmatrix}}_{\text{linear Coriolis}} + \underbrace{\begin{pmatrix}-\vec{V}\cdot\nabla\Phi\\ -\vec{V}\cdot\nabla\xi\\ -\vec{V}\cdot\nabla\delta\end{pmatrix}}_{\text{nonlinear advection}} + \underbrace{\begin{pmatrix}-\Phi'\delta\\ -\xi\delta\\ \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right) + \nabla\cdot\left(\delta\vec{V}\right) + \nabla^2\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) - \delta^2\end{pmatrix}}_{\text{nonlinear rest}}.
$$

Note, that e.g. $\vec{V}\cdot\nabla\psi$ can be evaluated by

$$
\begin{aligned}
\vec{V}\cdot\nabla\psi &= \vec{V}\cdot\nabla\psi + \left(\nabla\cdot\vec{V}\right)\psi - \left(\nabla\cdot\vec{V}\right)\psi \\
&= \nabla\cdot\left(\vec{V}\psi\right) - \delta\psi
\end{aligned}
$$

to avoid computing the gradient of a scalar in physical space.

3.7 Barotropic Vorticity Equation from SWE (added by P.Peixoto)

Using the vorticity-divergence formulation it is simple to derive the Barotropic Vorticity Equations, and a obtain a natural implementation for sweet.

Details here: https://www.gfdl.noaa.gov/wp-content/uploads/files/user_files/pjp/barotropic.pdf

The barotropic vorticity equations have as hipothesis a non-divergent horizontal wind acting on a constant density imprempressible fluid on the surface of the sphere. Therefore, following the have:

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi &= 0 \\
\frac{\partial}{\partial t}\xi &= -\nabla\cdot\left((\xi+f)\,\vec{V}\right) \\
\frac{\partial}{\partial t}\delta &= 0
\end{aligned}
$$

Therefore the barotropic vorticity equations can be solved as a special case of the SWE in this form, simply assuming constant Φ and zero δ.

4 Velocity-based formulation with vorticity-divergence as prognostic

For sake of completeness and implementation, we also provide the building blocks using vorticity-divergence fields as prognistic variables as if we would do the splitting with the velocity-based formulation

$$\frac{\partial}{\partial t}\begin{pmatrix}\Phi\\ \vec{V}\end{pmatrix} = \underbrace{\begin{pmatrix}-\overline{\Phi}\nabla\cdot\vec{V}\\ -\nabla\Phi\end{pmatrix}}_{\text{linear gravity}} + \underbrace{\begin{pmatrix}\\ -f\vec{k}\times\vec{V}\end{pmatrix}}_{\text{linear Coriolis}} + \underbrace{\begin{pmatrix}-\vec{V}\cdot\nabla\Phi\\ -\vec{V}\cdot\nabla\vec{V}\end{pmatrix}}_{\text{nonlinear advection}} + \underbrace{\begin{pmatrix}-\Phi'\delta\\ 0\end{pmatrix}}_{\text{nonlinear rest}}.$$

For the linear gravity waves we get

$$\begin{aligned}\frac{\partial}{\partial t}\delta &= div\left(-\nabla\Phi\right)\\ &= -\nabla^2\Phi\\ \frac{\partial}{\partial t}\xi &= 0\end{aligned}$$

hence no change.

The linear Coriolis term is given by

$$\begin{aligned}\frac{\partial}{\partial t}\delta &= \nabla\cdot\left(-f\vec{k}\times\vec{V}\right)\\ &= -\nabla\cdot\left(\vec{k}\times\left(f\vec{V}\right)\right)\end{aligned}$$

Using

$$\nabla\cdot(A\times B) = (\nabla\times A)\cdot B - (\nabla\times B)\cdot A$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}\delta &= \underbrace{-\left(\nabla\times\vec{k}\right)\cdot\left(f\vec{V}\right)}_{0} - \left(\nabla\times\left(f\vec{V}\right)\right)\vec{k}\\ &= -\left(\nabla\times\left(f\vec{V}\right)\right)\cdot\vec{k}\end{aligned}$$

which matches the one for the resular vort/div formulation. Similarly, we get the one for the vorticity.

Regarding the nonlinear remainder terms in the velocity formulation we know that there are no terms left in the "nonlinear rest" and given Eq. 2 we get

$$\left.\frac{\partial\xi}{\partial t}\right|_{\text{nonlinear advection}} = -\nabla\cdot\left(\xi\vec{V}\right)$$

and based on Eq. 1 we get

$$\left.\frac{\partial\delta}{\partial t}\right|_{\text{nonlinear advection}} = -\nabla^2\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right).$$

Finally we get

$$\frac{\partial}{\partial t}\begin{pmatrix}\Phi\\ \xi\\ \delta\end{pmatrix} = \underbrace{\begin{pmatrix}-\overline{\Phi}\delta\\ 0\\ -\nabla^2\Phi\end{pmatrix}}_{\text{linear gravity}} + \underbrace{\begin{pmatrix}0\\ -\nabla\cdot\left(f\vec{V}\right)\\ \vec{k}\cdot\nabla\times\left(f\vec{V}\right)\end{pmatrix}}_{\text{linear Coriolis}} + \underbrace{\begin{pmatrix}-\vec{V}\cdot\nabla\Phi\\ -\nabla\cdot\left(\xi\vec{V}\right)\\ -\nabla^2\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) + \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right)\end{pmatrix}}_{\text{nonlinear advection}} + \underbrace{\begin{pmatrix}-\Phi'\delta\\ 0\\ 0\end{pmatrix}}_{\text{nonlinear rest}}.$$

5 Spherical harmonics

We use state variables in physical space without a tilde, e.g. x, and variables in spectral space denoted with a tilde such as $\tilde{x}$. Let the function converting data from spectral to physical space be given by

$$x = s2p(\tilde{x})$$

and from physical to spectral space by

$$\tilde{x} = p2s(x).$$

The Laplace operator is then provided by

$$\nabla^2(u) = lap(u).$$

We also exploit the feature of the SHTNS library to compute the divergence and vorticity in spectral space for given velocities in physical space and denote this function by

$$(\tilde{\xi}, \tilde{\delta}) = vel2vd(u, v)$$

and vice versa

$$(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta}).$$

6 Implementation: Linear shallow-water equations

6.1 l_erk

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Evaluate temporary vector $(u^*, v^*) = f\vec{v}$ in physical space
   (c) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (d) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (e) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}' - lap(\tilde{\Phi})$
2. Geopotential:
   (a) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\overline{\Phi}\tilde{\delta}$

6.2 lc_erk_lg_erk

6.2.1 lc_erk

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Evaluate temporary vector $(u^*, v^*) = f\vec{v}$ in physical space
   (c) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (d) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (e) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}'$
2. Geopotential:
   (a) Zero geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = 0$

6.2.2 lg_erk

1. Vorticity/divergence:
   (a) Zero vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = 0$
   (b) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = -lap(\tilde{\Phi})$
2. Geopotential:
   (a) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\overline{\Phi}\tilde{\delta}$

7 Implementation: Nonlinear shallow-water equations

7.1 ln_erk

Next, we discuss the full non-linear time tendency computation. The different steps for the time integration are given as follows:

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Convert ξ from spectral to physical space: $\xi = s2p(\tilde{\xi})$
   (c) Evaluate temporary vector $(u^*, v^*) = (\xi + f)\,\vec{v}$ in physical space
   (d) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (e) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (f) Convert geopotential $\Phi = s2p(\tilde{\Phi})$
   (g) Compute $e = \Phi + \frac{\vec{v}\cdot\vec{v}}{2}$
   (h) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}' - lap(p2s(e))$
2. Geopotential:
   (a) Prepare divergence $(u^t, v^t) = (u\Phi, v\Phi)$
   (b) Compute divergence $(\tilde{\xi}^*, \tilde{\delta}^*) = vel2vd(u^t, v^t)$
   (c) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\tilde{\delta}^*$

7.2 l_na_erk (without nonlinear divergence)

Next, we discuss the full non-linear time tendency computation. The different steps for the time integration are given as follows:

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u, v) = vd2vel(\tilde{\xi}, \tilde{\delta})$
   (b) Convert ξ from spectral to physical space: $\xi = s2p(\tilde{\xi})$
   (c) Evaluate temporary vector $(u^*, v^*) = (\xi + f)\,\vec{v}$ in physical space
   (d) Compute vorticity and divergence updates $(\tilde{\xi}', \tilde{\delta}') = vel2vd(u^t, v^t)$
   (e) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi} = -\tilde{\delta}'$
   (f) Convert geopotential $\Phi = s2p(\tilde{\Phi})$
   (g) Compute $e = \Phi + \frac{\vec{V}\cdot\vec{V}}{2}$
   (h) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta} = \tilde{\xi}' - lap(p2s(e))$
2. Geopotential:
   (a) Prepare full nonlinear term $(u^n, v^n) = (u\Phi, v\Phi)$
   (b) Compute divergence $(\tilde{\xi}^*, \tilde{\delta}^*) = vel2vd(u^n, v^n)$
   (c) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -\tilde{\delta}^*$

7.3 Nonlinear advection terms in VD formulation

Considering solely the nonlinear advection terms for the VD updates, we get

$$\begin{aligned}\frac{\partial}{\partial t}\xi &= -\nabla\cdot\left(\xi\vec{V}\right)\\ \frac{\partial}{\partial t}\delta &= \vec{k}\cdot\nabla\times\left(\xi\vec{V}\right)-\nabla^2\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\end{aligned}$$

1. Vorticity/divergence:
   (a) Compute the velocity based on the vorticity and divergence: $(u,v)=vd2vel(\tilde{\xi},\tilde{\delta})$
   (b) Convert ξ from spectral to physical space: $\xi = s2p(\tilde{\xi})$
   (c) Evaluate temporary vector $(u^*,v^*)=\xi\vec{V}$ in physical space **(Coriolis f removed here)**
   (d) Compute vorticity and divergence updates $(\tilde{\xi}',\tilde{\delta}')=vel2vd(u^t,v^t)$
   (e) Assign vorticity update: $\frac{\partial}{\partial t}\tilde{\xi}=-\tilde{\delta}'$
   (f) Compute $e=\frac{\vec{V}\cdot\vec{V}}{2}$ **(No geopotential)**
   (g) Assign divergence tendency: $\frac{\partial}{\partial t}\tilde{\delta}=\tilde{\xi}'-lap(p2s(e))$

For the geopotential, we get

$$\begin{aligned}\frac{\partial\Phi}{\partial t}&=-\nabla\cdot(\Phi\vec{V})\\ &=-\nabla\cdot\left(\left(\overline{\Phi}+\Phi'\right)\vec{V}\right)\\ &=-\overline{\Phi}\nabla\cdot\vec{V}-\nabla\cdot(\Phi'\vec{V})\\ &=\underbrace{-\overline{\Phi}\nabla\cdot\vec{V}}_{\text{linear}}\quad\underbrace{-\Phi'\nabla\cdot\vec{V}}_{\text{nonlinear divergence}}\quad\underbrace{-\nabla\Phi'\cdot\vec{V}}_{\text{nonlinear advection}}\end{aligned}$$

evaluating the nonlinear advection term $\nabla\Phi'\cdot\vec{V}$ (a gradient) might result into artificial modes due to spectral issues.

Therefore, we search for an alternative given as follows: We can compute the VD formulation simply by evaluating the full nonlinear term and then by subtracting the nonlinear divergence

$$\begin{aligned}\frac{\partial\Phi}{\partial t}&=-\vec{V}\cdot\nabla\Phi'\\ &=\underbrace{-\nabla\cdot\left(\Phi'\vec{V}\right)}_{\text{full nonlinear term}}+\underbrace{\Phi'\nabla\cdot\vec{V}}_{\text{nonlinear divergence}}.\end{aligned}$$

2 Geopotential:
   (a) Convert perturbed geopotential $\Phi'=s2p(\tilde{\Phi}')$ **(Perturbed geopotential)**
   (b) Prepare perturbed geopotential nonlinear term $(u^n,v^n)=(u\Phi',v\Phi')$ **(Use perturbation here)**
   (c) Compute divergence $(\tilde{\xi}^*,\tilde{\delta}^*)=vel2vd(u^n,v^n)$
   (d) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi}'=-\tilde{\delta}^*$
   (e) **(Start 2nd term)** Convert divergence to physical space $\delta=s2p(\tilde{\delta})$
   (f) Evaluate $\Phi'\delta$ and back to spectral space: $\tilde{e}=s2p\left(\Phi'\delta\right)$
   (g) Assemble final geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi}=-\tilde{\delta}^*+\tilde{e}$

8 Implementation: Semi-Lagrangian shallow-water equations

For the Lagrangian formulation we use the total derivative

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + \vec{V} \cdot \nabla().$$

This leads to a formulation splitting the equation into linear parts, Lagrangian-treated parts and the remaining (typically) non-linear terms

$$\frac{\partial}{\partial t}X = LX + \vec{V} \cdot \nabla X + N^*(X).$$

8.1 Non-linear geopotential update

For the geopotential, we can write it as

$$\frac{\partial}{\partial t}\Phi = -\nabla \cdot \left(\Phi\vec{V}\right).$$

As usual, this is split into the linearized form $\Phi = \overline{\Phi} + \Phi'$, yielding

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi' &= -\nabla \cdot \left(\left(\overline{\Phi} + \Phi'\right)\vec{V}\right) \\
&= -\overline{\Phi}\nabla \cdot \vec{V} - \nabla \cdot \left(\Phi'\vec{V}\right) \\
&= -\overline{\Phi}\nabla \cdot \vec{V} - \vec{V} \cdot (\nabla\Phi') - \left(\nabla \cdot \vec{V}\right)\Phi' \\
\frac{d}{dt}\Phi' &= \underbrace{-\overline{\Phi}\delta}_{\text{linear}} - \underbrace{\left(\nabla \cdot \vec{V}\right)\Phi'}_{\text{non-linear divergence}} \\
&= -\delta\Phi'
\end{aligned}$$

1. (Step SLPHI) geopotential
   (a) Convert states to spectral space: $\delta = s2p(\tilde{\delta})$ and $\Phi' = s2p(\tilde{\Phi}')$
   (b) Assign geopotential tendency: $\frac{\partial}{\partial t}\tilde{\Phi} = -p2s\left(\delta\Phi'\right)$

8.2 Coriolis effect in non-linear treatment

We can also treat the Coriolis effect as part of the nonlinearities. Hence, we add the lc_erk contributions to $N^*(U)$.

8.3 Coriolis in trajectory

An alternative approach is to treat the Coriolis effect as part of the trajectory.