

# Seminar: Advanced Topics in Quantum Computing On efficient encodings for QAOA solutions to vehicle routing problems

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October 20, 2023

## Abstract

This report details recent advances in the optimisation of computing resources for quantum approaches to solving the vehicle routing problem (VRP) and its variants. This set of problems is of significant importance with regard to logistical applications in industry. In accordance with the input constraints of the quantum hardware, the problem is formulated as a quadratic unconstrained binary optimisation (QUBO). A simple approach known as the full encoding results in each solution represented by a unique basis state, thereby requiring one qubit per classical variable. Due to this inefficiency in resource allocation when considering the at worst factorial search space, a more optimised minimal encoding is suggested that offers a logarithmic reduction in necessary computing power. In spite of certain drawbacks incurred by employing this minimal encoding, experiments have shown that the solution quality is not heavily impacted.

## I. Introduction

The Vehicle Routing Problem (VRP) is concerned with finding optimal routes for vehicles to deliver goods to a set of customers with some geographical distance between them. It is obvious that this type of problem has far reaching practical applications in various contexts, in fact the need to find a reasonable solution is almost ubiquitous when dealing with logistical planning, for example to determine efficient road, rail, shipping, and air routes for commercial or public interest.

VRP, being itself a more general version of the Travelling Salesman Problem (TSP), is similarly an NP hard combinatorial optimisation problem with a solution space scaling factorially in the number of customers, and thus finding the definitive optimal solution already becomes intractable at two digit customer counts. The most optimal classical algorithms known to date and used in practice employ heuristics and greedy methods to construct routes within single digit percentage tolerances.

More recently, the noisy intermediate scale quantum (NISQ) era has paved the way for the development of variational quantum algorithms alongside the potent quantum annealing method as a contender for computing reliable and near optimal solutions to combinatorial optimisation problems, exploiting the unique inherent property of

quantum algorithms to operate on the entire solution space at once, albeit incrementally and probabilistically.

Initial tests have shown some promise in applying these quantum algorithms to the VRP against classical solvers, achieving comparable accuracy on small problem instances. As quantum hardware continues to improve, it makes sense to also look at utilising it as efficiently as possible. Solving VRP naively using quantum methods maps one classical variable to one whole qubit, quickly exhausting the already limited computing resources when introducing additional constraints such as vehicle capacity or time windows on larger problem scales more commonly encountered in practical scenarios.

This report discusses recent research into an idea to mitigate this uneconomical use of computing power through a more clever and refined encoding of the input problem, in order to achieve a logarithmic correlation between problem size and resource consumption. Section II outlines a formal description of the VRP and some of its variants. Section III explains the QUBO and its application to solving the VRP on quantum hardware, along with a brief mention of the closely related Ising model and in particular its Hamiltonian function. Section IV details the encoding approaches onto the quantum system and lists benefits and drawbacks for each. Published experimental observations de-

scribing the impact of the encodings are discussed in Section V, and the report concludes in Section VI.

## II. Vehicle Routing Problem

The vehicle routing problem (VRP) is a generalisation of the perhaps more well known NP hard Travelling Salesman Problem (TSP). VRP seeks the optimal route or routes for a number of vehicles to traverse in order to deliver certain goods to customers in various locations.

The problem is modelled intuitively by a graph  $G = (V, E)$  where each node or vertex  $v_i \in V$  represents the location of a customer and each edge  $(i, j) \in E$  connecting two vertices corresponds to a path traversible by a delivery vehicle.  $\|V\| = N$  is then the number of customers considered. Traversing an edge  $(i, j)$  typically incurs a cost represented by  $c_{ij} \in \mathbb{R}_{\geq 0}$ , of which the total value summed up across the entire journey is to be optimised, ie. made as small as possible. This cost may be set based on travel time, distance, or other concerns with economic consequences.

Vehicles may only travel across edges. For simplicity, one can assume  $G$  is a complete graph. Those edges directly connecting two nodes  $(i, j) \in E$  that are absent in reality can be labelled with an edge cost  $c_{ij} = c_{in_1} + \dots + c_{n_mj}$  corresponding to the most efficient path available between them. This preprocessing step does not necessarily complicate matters, and classical algorithms exist for all pairs shortest path with known polynomial runtime performance, such as the Dijkstra algorithm. Furthermore, each graph contains a designated node  $v_0 \in V$  known as the depot. Valid routes must always begin and end at the depot. A valid route is then a tuple  $r = (v_n, v_{n+1}, \dots, v_m)$  s.t.  $v_n = v_m = v_0 \wedge \forall n. (v_n, v_{n+1}) \in E$ .

For a problem to be classified as VRP, it should fulfil at least the above minimal constraints. However, additional constraints may be imposed as needed to better reflect a practical use case at the expense of slightly complicating the model. For example, the capacitated VRP or CVRP stipulates a fixed upper bound on the carrying capacity of each vehicle leaving the depot, where in most cases this value is consistent across all vehicles. Customers  $v_i$  are then assigned a score  $d_i \in \mathbb{R}_{\geq 0}$  reflecting their demand quantity. This introduces the complication of optimising for capacity and demand as well as edge cost, and the ideal solution for a given graph will most likely differ from the simple VRP.

On the other hand, the VRP with time windows (VRPTW) introduces a secondary time parameter. Customers  $v_i$  are assigned a certain time window  $[t_i^0, t_i^f] \subseteq \mathbb{R}_{\geq 0}$  in which they expect a delivery. For

the depot  $v_0$  this interval is  $[0, \infty)$  for simplicity. Hence the edge costs  $c_{ij}$  represent travel time between two nodes in this formulation, and the objective shifts to finding the optimal route that serves all customers whilst respecting these time windows, or failing this attempting to maximise the number of customers or total goods supplied.

## III. Problem Formulation

The quadratic unconstrained binary optimisation (QUBO) is concerned with finding a binary vector  $|x^*\rangle \in \{0, 1\}^n$ ,  $n \in \mathbb{N}$  that fulfils the following optimal condition:

$$|x^*\rangle = \underset{|x\rangle \in \{0,1\}^n}{\operatorname{argmin}} \langle x|Q|x\rangle \quad (1)$$

where the linear operator  $Q \in \mathbb{R}^{n \times n}$  is a symmetric matrix. This can be interpreted as an objective function:

$$f_Q(x) = \langle x|Q|x\rangle = \sum_{i=1}^n \sum_{j=i}^n Q_{ij}x_i x_j \quad (2)$$

In general, QUBO is also NP hard due to the exponential scaling of the solution space in  $\|\{0, 1\}^n\| \in \mathcal{O}(2^n)$  with respect to the number of dimensions  $n$ . Many combinatorial optimisation problems have conversions into QUBO [1] [2], not least the VRP and its variants. These conversions are useful to establish a uniform problem description for solvers to work with, however in the context of quantum solvers, they are further motivated by the equivalence between QUBO and the Hamiltonian of the Ising model for ferromagnetism in particle physics.

The Ising model describes a lattice structure  $\Lambda$  in which each lattice site houses a particle. The spin of each particle is represented by a discrete variable  $\sigma_i \in \{-1, 1\}$ ,  $i \in \Lambda$ . This spin value governs the local magnetic moment of the particle according to the shell model (appendix). Neighbouring lattice sites  $\langle i, j \rangle$ ,  $i, j \in \Lambda$  influence each other, termed nearest neighbour interactions, whose interaction strength is represented by  $J_{ij} \in \mathbb{R}$ . Furthermore, one may consider the influence of an external field  $h_i$  at site  $i \in \Lambda$ , such that the spin wants to align with this field.

Thus the Hamiltonian reads:

$$H = - \sum_{\langle i, j \rangle} J_{ij} \sigma_i \sigma_j - \mu \sum_i h_i \sigma_i \quad (3)$$

where  $\mu$  is the magnetic moment.

Replacing  $\sigma$  with the Pauli operators yields the quantum mechanical description:

$$H = - \sum_{\langle i, j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \mu \sum_i h_i^z \sigma_i^z - \mu \sum_i h_i^x \sigma_i^x \quad (4)$$

where the second term with  $\sigma^z$  describes the external longitudinal field, and the final term with  $\sigma^x$  describes the transverse field per lattice site.

Notably, the Ising model is typically simplified to exclude the transverse field, resulting in a classical Hamiltonian where the constituent terms commute i.e. a diagonal operator in the  $Z$  basis.

$$H = \sum_{z \in \{0,1\}^n} E_z |z\rangle\langle z| \quad (5)$$

This means the ground state  $E_0$  is described by a corresponding eigenvector  $|\phi\rangle$  with  $\phi \in \{0,1\}^n$ , allowing for simple measurement to obtain an eigenvalue mappable to a specific bit string solution.

Through the reversible transformation  $\sigma \mapsto 2x - 1$ , where  $x \in \{0,1\}$  s.t.  $-1 \mapsto 0$  and  $1 \mapsto 1$ , one obtains the equivalent QUBO formulation for free, and hence optimising for the ground state of the Ising Hamiltonian is equivalent to optimising the QUBO objective function.

With this knowledge, one can formulate the VRP as follows. We define the set of constraints that a valid solution must adhere to:

**once** Each customer  $v_i$  in a route  $r_j$  is visited exactly once.

**step** For each time step, each vehicle is either at a customer node  $v_i$  or at the depot  $v_0$ .

Additional constraints may be specified depending on the problem variant. For CVRP the following is useful:

**cap** All routes  $\{r_i\}$  respect the maximum capacity  $m \in \mathbb{R}$  of a vehicle.

Note that for simplicity, we do not distinguish between vehicles and routes, under the premise that each route will be navigated by exactly one vehicle. Since in CVRP the routes are optimised for the vehicle capacity  $m$ , assuming homogeneous vehicles this simplification is unlikely to affect the final result. Furthermore, some CVRP variants consider multiple depot nodes. This variant will also not be further investigated here.

Let  $x_{t,i}^r \in \{0, 1\}$  denote the assignment of the customer node  $v_i$  to be visited along route  $r \in \mathbb{N}_0$  at time step  $t \in \mathbb{N}_0$ . The first two constraints can then be described mathematically as follows:

$$H_{\text{once}} = \sum_i (1 - \sum_{r,t} x_{t,i}^r)^2 \quad (6)$$

$$H_{\text{step}} = \sum_{r,t} (1 - \sum_i x_{t,i}^r)^2 \quad (7)$$

The idea is to coerce each squared term to evaluate to zero, implying each inner summation should

evaluate to one. To illustrate, consider for example  $H_{\text{step}}$  which encodes the constraint **step**. We sum over all node indices  $i$ , such that for a given step  $t$  along the route  $r$ , precisely one customer node or the depot is assigned to the schedule. In general, once a constraint is violated it follows that the calculation yields a positive result, after which penalty values may be applied accordingly.

Encoding the constraint **cap** is a little more involved:

$$H_{\text{cap}} = \sum_r (m - (\Delta + \sum_{t,i} d_i x_{t,i}^r))^2 \quad (8)$$

where  $d_i$  represents the demand score of node  $v_i$  and

$$\Delta = \sum_{b=0}^{\lceil \log_2 1+m \rceil - 1} 2^b a_b^r \quad (9)$$

While the principle remains identical to the first two constraints, The additional term  $\Delta$  in  $H_{\text{cap}}$  is a result of **cap** being somewhat more relaxed, in that any value  $\sum_{t,i} d_i x_{t,i}^r \leq m$  for the loading of the vehicle is acceptable, instead of a hard requirement. This is conveyed through the auxiliary variables  $a_b^r \in \{0, 1\}$ , which may be freely set to perfectly zero out unused vehicle capacity for the route  $r$ . Intuitively,  $\Delta$  represents the tolerable deviation of the amount loaded onto the vehicle on route  $r$  compared to maximum capacity. Naturally not fully loading the vehicle and forcing an early return to depot may be suboptimal in many cases, however the edge cost metric represented by the objective function  $H_c$  is sufficient to deter this.

$$H_c = \sum_{i,j,t,r} c_{ij} x_{i,t}^r x_{j,t+1}^r \quad (10)$$

Coupled with the objective function given by  $H_c$ , the complete CVRP Hamiltonian is the linear combination of these constraints:

$$H_p = H_c + p_1 H_{\text{once}} + p_2 H_{\text{step}} + p_3 H_{\text{cap}} \quad (11)$$

Here  $p_k \in \mathbb{R}_{\geq 0}$  with  $k \in \{1, 2, 3\}$  corresponds to a large penalty value that serves to mark a potential solution as undesirable, in the case that one or more of the corresponding constraints are violated. The exact values of  $p_k$  are dependent on the problem instance and must be heuristically determined and optimised.

Crucially, this description of the CVRP is only one of several possible formulations. Extensive research has been conducted into determining the most optimal description that benefits solution quality as well as speed of convergence. In particular, the formulation in (11) turns out not to be suitable with regard to current limitations of NISQ

era processors, due to the large amount of free binary variables and comparative lack of computing resources available to account for them all. The CVRP must as a result be split into two distinct phases, a clustering phase that groups customer nodes together similar to the knapsack problem, and a routing phase to determine the ideal route in each of these clusters as in the TSP.

The clustering phase aims to discern routes by grouping nodes together in such a way that **once** and **cap** hold true for each cluster. Furthermore, intuition suggests neighbouring nodes should not be split apart but rather be assigned to the same cluster to minimise edge costs.

**cluster** The cumulative edge cost  $c_{ij}$  between all pairs of nodes  $(i, j)$  within a cluster should be minimal.

This results in the following problem Hamiltonian:

$$H_{\text{kn}} = q_1 H_{\text{once}} + q_2 H_{\text{cap}} + q_3 H_{\text{cluster}} \quad (12)$$

with penalty values  $q_i$ , where  $H_{\text{cluster}}$  is defined as:

$$H_{\text{cluster}} = \sum_r \sum_{i, j} c_{ij} x_i^r x_j^r \quad (13)$$

and  $x_i^r = \sum_t x_{i,t}^k$ .

It is worth noting that the penalty value  $q_3$  should be set more modestly in comparison to the others, given that **cluster** is not a hard constraint but rather a metric measuring the optimality of clusters. It should always be more appealing to relax **cluster** over violating **once** or **cap**.

Unfortunately, quantum solvers have been shown to perform relatively poorly when applied to the multiple knapsack problem (refs), hence at present the clustering phase should be deferred to a classical algorithm as suggested in (refs).

The routing phase takes the partitioned clusters and determines the optimal route for each cluster locally. The problem is thus reduced to the TSP, which conveniently has a QUBO formulation given in [1] as follows

$$H_{\text{tsp}} = q_1 H_{\text{once}} + q_2 H_{\text{step}} + q_3 H_c \quad (14)$$

with the exception that, as we have broken up the problem and there is locally only one route to optimise, in all terms we do not sum over  $r$ .

Overall, the TSP requires  $\mathcal{O}(\|V\|^2)$  binary variables to account for all possible nodes and all possible time steps. Here [1] suggests a resource optimisation saving to  $(\|V\| - 1)^2$  variables by fixing the start node to be the depot  $v_0$ .

## IV. Quantum Algorithms

Traditionally, quantum solvers for combinatorial optimisation problems have been implemented using the quantum annealing approach. This method relies on the theory of adiabatic quantum computation to evolve a prepared system state of the Hamiltonian  $H_0$  gradually towards the problem Hamiltonian  $H_p$ . With sufficiently slow evolution, the system is guaranteed to arrive in the ground state by the adiabatic theorem.

More precisely, the system evolves according to

$$H(t) = \left(1 - \frac{t}{T}\right) \cdot H_0 + \frac{t}{T} \cdot H_p \quad (15)$$

with  $0 \leq t \leq T$ , where the total algorithm runtime  $T$  is bound by  $T \in \mathcal{O}(\frac{1}{g^2})$ . Here  $g$  denotes the spectral gap of  $H$ , the minimal energy difference between the ground state and first excited state throughout the evolution. Consequently, Hamiltonians with narrower spectral gaps have to be evolved more slowly for the sake of precision.

On the other hand, the more common circuit model processor operates based on a series of quantum gates connected in a circuit in order to manipulate the quantum state to a desired outcome. This principle of operation is founded upon the discretised solution to the Schrödinger equation for negligible small time intervals:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \iff U(t) = e^{-\frac{i}{\hbar} H t} \quad (16)$$

where  $U$  denotes the unitary time evolution operator. This unitary can be decomposed into several building blocks termed quantum gates, which are implemented in the hardware and operate on a small subset of qubits. The quantum circuit thus realises the Hamiltonian.

The advances in circuit model quantum processors that have brought about the NISQ era have offered an alternative approach to tackle optimisation problems, namely that of supplementing a quantum algorithm with classically tuned parameters to iteratively arrive at a good approximation of the ground state, known as variational algorithms. NISQ era processors are characterised by a primary limitation of restricted circuit depth due to their sensitivity to noise as well as the inability to correct the faults introduced. By offloading some of the computation to a classical algorithm and proceeding iteratively, this limitation can be worked around to a degree that allows one to obtain reasonable estimations of the ideal solutions to optimisation problems.

## V. Encoding Approaches

We assume the use of a conventional NISQ processor to tackle a VRP involving  $n_c$  classical binary variables. A simple mapping from the VRP to QUBO uses what is known as the complete encoding, where each classical binary variable is represented by a single qubit. The quantum state takes the form:

$$|\psi(\theta)\rangle = U(\theta)|\psi_0\rangle \quad (17)$$

$$= \sum_{x \in \{0,1\}^{n_c}} \alpha_x |x\rangle \quad (18)$$

In this case, sampling the quantum state yields any given bitstring solution  $x \in \{0,1\}^{n_c}$  with probability  $\|\alpha_x\|^2$ , where  $\alpha$  is parametrised by a set of angles  $\theta$ . In variational algorithms,  $\theta$  are heuristically determined values that serve to tune arrays of rotation unitaries, which are adjusted after each pass to attain the optimised state  $|\psi^*\rangle = U(\theta^*)|\psi_0\rangle$ .

The QUBO objective function in the complete encoding is described by the expectation of the equivalent Ising Hamiltonian:

$$C_{cpl}(\theta) = \langle H \rangle = \langle \psi(\theta) | H | \psi(\theta) \rangle \quad (19)$$

The advantage of the complete encoding is its ability to capture all possible correlations between classical variables. This however sacrifices scalability, requiring computational resources that scale proportionally to the problem size as each binary classical variable is mapped to a single qubit, hence  $n_q = n_c$ . As a result, feasible applications of the complete encoding are limited to toy examples. Even as quantum hardware becomes more powerful, it is prudent to examine encoding schemes that can more efficiently take advantage of the available technology.

We consider the minimal encoding as presented in [3]. This scheme allows for  $n_c$  classical variables to be mapped to  $n_q = 1 + \log_2 n_c$  qubits, using a  $n_r = \log_2 n_c$  wide register and  $n_a = 1$  auxiliary qubit.

Let  $\{|\phi_i\rangle\}$  and  $\{|0\rangle, |1\rangle\}$  denote a basis for the register and auxiliary qubits respectively. The system state  $|\psi\rangle$  is then defined as:

$$|\psi(\theta)\rangle = \sum_{i=1}^{n_c} \beta_i (a_i |0\rangle + b_i |1\rangle) |\phi_i\rangle \quad (20)$$

where the coefficients  $\beta_i, a_i, b_i$  are parametrised by  $\theta$ , as in the complete encoding.

This encodes in the register the probability  $\beta_i$  of measuring each register state  $|\phi_i\rangle$  that corresponds to an individual classical variable  $x_i$ , and

in the auxiliary the probability of each of these binary variables taking on the value 0 or 1 according to the Born rule:  $\Pr(x_i = 0) = \|a_i\|^2$  and  $\Pr(x_i = 1) = \|b_i\|^2$ . Taken together, the probability of measuring a certain bitstring solution  $x$  for a given  $\theta$  is expressed by:

$$\Pr x = \prod_{i=1}^{n_c} \Pr x_i = \prod_{i=1}^{n_c} \|b_i\|^2 \quad (21)$$

As an example, consider  $n_c = 4, n_r = 2$ . The normalised system state representing a particular bitstring  $x^* = (1, 1, 1, 0)^\dagger$  would be  $|\psi^*\rangle = \frac{1}{2}(|1\rangle|00\rangle + |1\rangle|01\rangle + |1\rangle|10\rangle + |0\rangle|11\rangle)$ .

The variational algorithm is run for several trials to obtain an estimate of the coefficients  $a_i$  and  $b_i$ , from which the probability distribution for bitstring solutions as given in (21) may be constructed.

As stated previously, with the minimal encoding a logarithmic scaling of qubits to classical variables is achieved. However, this comes at the expense of a more limited expressiveness in the quantum state. As seen in (21), the probability distribution is sufficient to encode only statistically independent classical variables, ie. those s.t.  $\forall i, j. \Pr(x_i \wedge x_j) = \Pr x_i \Pr x_j$ , which tend not to be the case for highly complex and intertwined problems as in the VRP.

Furthermore, the number of trials required to obtain a solution of sufficient quality is necessarily increased. Depending on the exact problem configuration, particularly small coefficient values in the quantum state may result in certain bitstring outcomes being sampled too infrequently if at all while evaluating the objective function, leading to inaccuracies.

Substituting (21) into (2) and incorporating the probabilistic component, we obtain the following objective function for the minimal encoding:

$$C_{min}(\theta) = \sum_{i \neq j}^{n_c} Q_{ij} \|b_i\|^2 \|b_j\|^2 + \sum_{i=1}^{n_c} Q_{ii} \|b_i\|^2 \quad (22)$$

$$= \sum_{i \neq j}^{n_c} Q_{ij} \frac{\langle P_i^1 \rangle}{\langle P_i \rangle} \frac{\langle P_j^1 \rangle}{\langle P_j \rangle} + \sum_{i=1}^{n_c} Q_{ii} \frac{\langle P_i^1 \rangle}{P_i} \quad (23)$$

expressed as projectors  $P_i = |\phi_i\rangle\langle\phi_i|$  and  $P_i^1 = |1\rangle\langle 1| \otimes P_i$ .

## VI. Discussion

## VII. Conclusion

## References

- [1] A. Lucas, "Ising formulations of many NP problems," *Frontiers in Physics*, 2014.

- [2] F. Glover, G. Kochenberger, R. Hennig, and Y. Du, “A tutorial on formulating and using QUBO models,” *Ann Oper Res*, 2022.
- [3] B. Tan, M.-A. Lemonde, S. Thanasilp, J. Tangpanitanon, and D. G. Angelakis, “Qubit-efficient encoding schemes for binary optimisation problems,” *Quantum*, 2021.