Technische Universität München
Department of Mathematics

Experimental Evaluation of Robust Revenue-Maximizing Auctions

Yichen Lou

Examiner: Prof. Dr. Andreas S. Schulz

Advisor: Dr. Yiannis Giannakopoulos

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I hereby confirm that this is my own work, and that I used only the cited sources and materials.

München, February 19, 2021

Yichen Lou

Abstract

Here we give a short summary of the project or thesis of length at most a quarter of a page. This could be e. g. as follows:

This document is an introduction to the use of the LaTeX-package `tumthesis.cls`, with which theses can be written in the TUM style. The basic structure of the example files is explained and some optional components are mentioned briefly. There are also some tips for LaTeXbeginners (and also for more advanced users who want to learn some more) as well as suggested reading for individual study.

Zusammenfassung

Hier schreibt man eine kurze Zusammenfassung der Arbeit im Umfang von maximal einer Viertelseite. Das kann z. B. so aussehen:

Die Arbeit führt in die Verwendung des LaTeX-Pakets `tumthesis.cls` ein, mit dem Abschlussarbeiten im TUM-Stil gesetzt werden können. Die grundlegende Gliederung der Beispieldateien wird erklärt und auf optionale Bestandteile wird kurz eingangen. Außerdem enthält der Text ein paar Tipps für LaTeX-Anfänger (und auch für Fortgeschrittene, die noch etwas dazulernen wollen) sowie Literaturhinweise zum Selbststudium.

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1 Introduction

Introduction about sbsi

then single multiple items

introduce the auction model

truthful mechanisms

notation of revenue, optimal and welfare

myerson optimal auction introduction

It is impossible to collect the information about the true prior of bidders valuation. In addition, it may impact on bidders' incentive and performance of the auction mechanism. Therefore, it is interesting to design a mechanism with less assumption on the prior distribution/with limited information of this prior. Normally we start to consider the worst case analysis.

2 Single-bidder and Single-item Deterministic Auctions

In this chapter, we evaluate DAPX under single-bidder and single-item deterministic auction experiment with some well-known probability distribution. We use Myerson optimal acution as our auction mechanism, that seller make a reserve price for the item and if the bidder's bid higher than the reserve price, she gets the item and charged by the reserve price, otherwise the item remains un-sale. This is also called *take-it-or-leave-it* auction. We start with some Definitions and euqtions from Robust paper.

Definition 2.1 (Function ρ)
For any $r \geqslant 0$, let $\rho(r) = \rho_D$ be the unique positive solution of equation

$$\frac{(\rho - 1)^3}{(2\rho - 1)^2} = r^2$$

where $r = \frac{\sigma}{u}$ is called *coefficient of variation (CV)*. If we look at the left-hand side expression $\frac{(\rho_D - 1)^3}{(2\rho_D - 1)^2}$ is increasing and goes from 0 at $\rho_D = 1$, and to ∞ at $\rho_D \to \infty$, so that for any non-negative r there is a unique solution $\rho_D \in [1, \infty)$ to the above equation.
The paper proposed a reserve price in terms of ρ_D for the *take-it-or-leave-it* auction

$$p_D = \mu \cdot \frac{\rho_D}{2\rho_D - 1} \tag{2.1}$$

where ρ_D is given in Definition 2.1.
We know from Myerson [50] that for single-item settings the optimum revenue can be achieved by a deterministic mechanism by setting a reserve price p and we can write the optimal revenue

$$\mathrm{OPT}(F) = \sup_{p \geqslant 0} \mathrm{REV}(p; F) = \sup_{p \geqslant 0} p \times (1 - F(p-)) \tag{2.2}$$

where $F(p-) = Pr[x < p]$.
We denote $\mathrm{OPT}(F)$ as Myerson optimal operator. We can use $\mathrm{OPT}(F)$ to determine the optimal reserve price. Given a valuation probability distribution F, we are able to find a reserve price $p = \arg\max_{p \geqslant 0} p \times (1 - F(p-))$, and we denote this price as

optimal reserve price p_{opt}. Our experiment simulates over 100000 *take-it-or-leave-it* auctions with reserve price p_D and p_{opt} separately, and we compute the expected revenues. We denote the expected revenue with reserve price p_D as $\mathrm{REV}(F)$, and the expected revenue with reserve price p_{opt} as $\mathrm{OPT}(F)$. Thus our experimental DAPX for distribution F is: $\mathrm{DAPX} = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(F)}$.
Procedure 1 shows our design for the auction experiment. Detailed Python code can be viewed on Github*.

Algorithm 1 **Auction Experiment**

```
procedure AUCTION EXPERIMENT(p, n, F)          ▷ n = number of experiments
    bid_data ← n random numbers generated from distribution F
    REV(F) = 0
    for each bid in bid_data do                ▷ stop until the last bid in bid_data
        if bid ⩾ p then
            REV(F) = REV(F) + p                ▷ bidder wins and pay price p
    return REV(F)/n
```

2.1 Uniform distribution

For an uniform distribution $U[a, b], 0 \leqslant a \leqslant b$, we know:

- mean $\mu = \frac{b+a}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$
- CDF $F(x) = \frac{x-a}{b-a}$
- PDE $f(x) = \frac{1}{b-a}$

Using Myerson optimal operator, we can write:

$$\mathrm{OPT}(F) = \sup_{p \geqslant 0} \mathrm{REV}(p; F) = \sup_{p \geqslant 0} p \times (1 - \frac{p-a}{b-a})$$

$p \times (1 - \frac{p-a}{b-a})$ is a concave function, therefore, there exists a maximum point. Then we take its first derivative

$$(p \times (1 - \frac{p-a}{b-a}))' = (\frac{pb-p^2}{b-a})' = \frac{b-2p}{b-a}$$

Because $p \in [a, b]$, we need to divide it into two cases.

- Case 1: if $\frac{b}{2} < a$, $\frac{b-2p}{b-a}$ is negative then it means this function is monotone decreasing on $[a, b]$. Thus when $p = a$, we get $\mathrm{OPT}(F)$.

- Case 2: if $\frac{b}{2} \geqslant a$, then the maximum expected revenue can be achieved when $p = \frac{b}{2}$

Combine two cases above, when $p_{opt} = \max\{a, \frac{b}{2}\}$, we have $\mathrm{OPT}(F) = p_{opt} \times (1 - \frac{p_{opt}-a}{b-a})$. For a given uniform distribution, i.e. a and b are known, therefore r^2 is also known: $r^2 = (\frac{\sigma}{u})^2 = \frac{(b-a)^2}{3(b+a)^2}$, then we can determine ρ_D by solving equation in Definition 2.1 and compute reserve price p_D using Equation 2.1, Then our expected revenue

$$\mathrm{REV}(F) = \frac{\rho_D}{2\rho_D - 1} \cdot \mu \cdot (1 - \frac{\frac{\rho_D}{2\rho_D-1} \cdot \mu - a}{b-a})$$

However, we do not determine $\mathrm{OPT}(F)$ and $\mathrm{REV}(F)$ using above expressions during the experiments. We only use their reserve price p_{opt} and p_D during the experiments.

2.1.1 The Bound of r and DAPX

Based on our experiments, we notice that no matter how we change a and b, the *coefficience of variation* r of uniform distribution is way smaller than 1. Which is expected, as uniform distribution is well defined distribution and actually indeed we can write r in terms of a and b explicitly:

$$r = \frac{\mu}{\sigma} = \frac{b-a}{\sqrt{3}(a+b)} = \frac{a+b-2a}{\sqrt{3}(a+b)} = \frac{1}{\sqrt{3}}(1 - \frac{2}{1+\frac{b}{a}})$$

when a = 0. then $r = \frac{1}{\sqrt{3}}$, otherwise, when $b \to \infty$ and $a \leqslant b$, we can have:

$$\sup r = \lim_{\frac{b}{a}\to\infty} (\frac{1}{\sqrt{3}}(1 - \frac{2}{1+\frac{b}{a}})) = \frac{1}{\sqrt{3}}$$

as we can see the CV of uniform distribution is at most $\frac{1}{\sqrt{3}}$.

From equation 2.1, ρ_D is monotonically increasing with r, then ρ_D is also upper bounded. we can compute its upper bound using numerical solver by setting $r = \frac{1}{\sqrt{3}}$ in equation 2.1. We know the equation of computing the reserve price, $p_D = (\frac{a+b}{2}) \cdot \frac{\rho_D}{2\rho_D - 1} = \frac{1}{2 - \frac{1}{\rho_D}} \cdot (\frac{a+b}{2})$.

Now let's write DAPX explicitly, when $\frac{b}{2} \geqslant a$ so $p_{opt} = \frac{b}{2}$:

$$\mathrm{DAPX} = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(F)} = \frac{\frac{b^2}{4(b-a)}}{\frac{\rho_D}{2\rho_D-1} \cdot \mu \cdot (1 - \frac{\frac{\rho_D}{2\rho_D-1} \cdot \mu - a}{b-a})}$$

simplified we get

$$\mathrm{DAPX} = \frac{b^2}{\frac{2\rho_D}{2\rho_D-1} \cdot 2\mu \cdot (b - \frac{\rho_D}{2\rho_D-1} \cdot \mu)}$$

subsititute $b = \sqrt{3}\sigma + \mu$

$$\begin{aligned}
\text{DAPX} &= \frac{(\sqrt{3}\sigma+\mu)^2}{\frac{2\rho_D}{2\rho_D-1}\cdot 2\mu \cdot (\sqrt{3}\sigma+\mu-\frac{\rho_D}{2\rho_D-1}\cdot\mu)} \\
&= \frac{\mu^2\cdot(\sqrt{3}r+1)^2}{\frac{2\rho_D}{2\rho_D-1}\cdot 2\mu^2\cdot(\sqrt{3}r+1-\frac{\rho_D}{2\rho_D-1})} \\
&= \frac{(\sqrt{3}r+1)^2}{4\cdot\frac{\rho_D}{2\rho_D-1}\cdot(\sqrt{3}r+1-\frac{\rho_D}{2\rho_D-1})}
\end{aligned}$$

when $\frac{b}{2} \leqslant a$ so $p_{opt} = a$

$$\text{DAPX} = \frac{\text{OPT}}{\text{REV}} = \frac{a}{\frac{\rho_D}{2\rho_D-1}\cdot\mu\cdot(1-\frac{\frac{\rho_D}{2\rho_D-1}\cdot\mu-a}{b-a})}$$

simplified we get

$$\text{DAPX} = \frac{a(b-a)}{\frac{\rho_D}{2\rho_D-1}\cdot\mu\cdot(b-\frac{\rho_D}{2\rho_D-1}\cdot\mu)}$$

subsititute $b = \mu + \sqrt{3}\sigma$ and $a = \mu - \sqrt{3}\sigma$

$$\begin{aligned}
\text{DAPX} &= \frac{2\sqrt{3}\sigma(\mu-\sqrt{3}\sigma)}{\frac{\rho_D}{2\rho_D-1}\cdot\mu\cdot(\sqrt{3}\sigma+\mu-\frac{\rho_D}{2\rho_D-1}\cdot\mu)} \\
&= \frac{\mu^2\cdot r(1-\sqrt{3}r)}{\mu^2\cdot\frac{\rho_D}{2\rho_D-1}\cdot(\sqrt{3}r+1-\frac{\rho_D}{2\rho_D-1})} \\
&= \frac{r(1-\sqrt{3}r)}{\frac{\rho_D}{2\rho_D-1}\cdot(\sqrt{3}r+1-\frac{\rho_D}{2\rho_D-1})}
\end{aligned}$$

Below we represent the result in Fig. 2.1, which compares DAPX of uniform distribution against theoretical value ρ_D from the paper with different r values.(show example, set a =1, change r value, since for each r there is a corresponding value of b, which is a valid uniform distribution, then we can get following plot
$\frac{\rho_D}{2\rho_D-1} = \frac{1}{2-\frac{1}{\rho_D}}$ as $\rho_D \geqslant 1$ because optimal revenue is always greater than the expected revenue, then $\frac{\rho_D}{2\rho_D-1} \leqslant 1$. ρ_d is also a function of r, it seems that DAPX is a quadratic equation in terms of r????? not too sure, since I cannot derive the pd using r **note**: another interesting things when a uniform ditribution has $\frac{b}{a} = 2.44224957$, then DAPX = 1, by setting $p = p_{opt}$ which is equavlent to $\frac{\rho_D}{2\rho_D-1}\cdot(\frac{a+b}{2}) = \frac{b}{2}$ then solving equation 2.1.

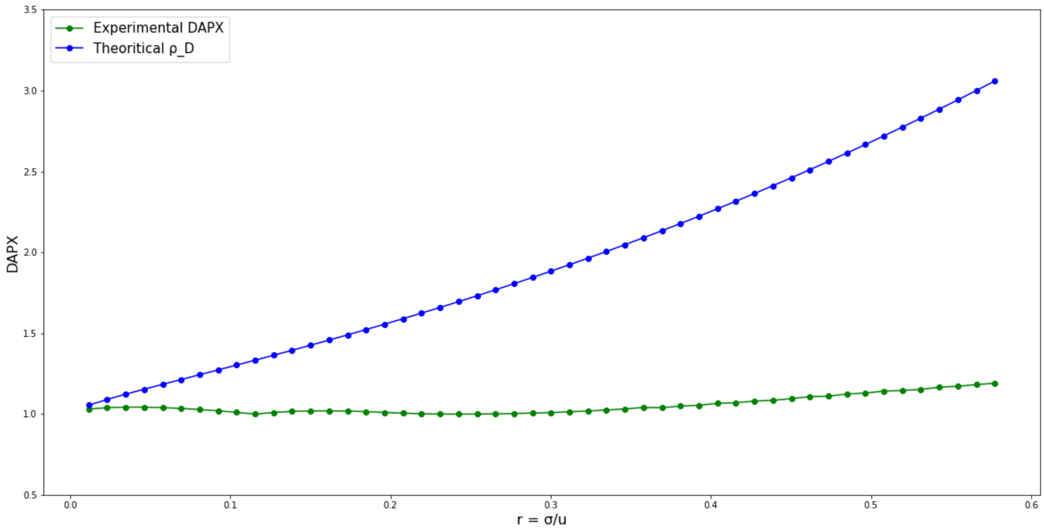

Figure 2.1: DAPX of uniform distribution versus ρ_D

2.2 Exponential and Poisson distribution

These two distributions have very special property that the mean and the standard deviation are the same, which results in constant CV, for exponential distribution $r = \frac{\sigma}{\mu} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}} = 1$ and for poisson distribution $r = \frac{\sigma}{\mu} = \frac{\lambda}{\lambda} = 1$, therefore ρ_D is also constant by solving equation 2.1. Thus from these two distributions we cannot find a useful relation between DAPX and r.

We can still perform some insight of this type distribution, and let us look at exponential distribution for example. For any exponential distribution, we denote as $exp(\lambda)$, its mean and standard deviation are $\frac{1}{\lambda}$, then using Meyerson optimal operator, we can determine the optimal reserve price is $\frac{1}{\lambda}$ and the optimal revenue is $\frac{1}{\lambda e}$. Let's denote $\hat{\rho_D}$ as the the value of ρ_D when $r = 1$, and our expected revenue is $\frac{\hat{\rho_D}}{\lambda(2\hat{\rho_D}-1)}e^{-\frac{\hat{\rho_D}}{(2\hat{\rho_D}-1)}}$, then:

$$
\begin{aligned}
\mathrm{DAPX} &= \frac{\mathrm{OPT}(F)}{\mathrm{REV}(F)} \\
&= \frac{\frac{1}{\lambda \cdot e}}{\frac{\hat{\rho_D}}{\lambda(2\hat{\rho_D}-1)} e^{-\frac{\hat{\rho_D}}{(2\hat{\rho_D}-1)}}} \\
&= \frac{1}{\frac{\hat{\rho_D}}{(2\hat{\rho_D}-1)} e^{1-\frac{\hat{\rho_D}}{(2\hat{\rho_D}-1)}}}
\end{aligned}
$$

From above expression we can see DAPX is indepedent of λ, and it is a constant as well. we will not explore any futher on these two distributions.

2.3 Truncated Normal distribution

Another interesting probability distribution to explore is the truncated normal distribution. The definition of truncated normal distribution is: suppose X is from a normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$ and lies within the interval $[a, b]$, then X conditional on $a \leqslant X \leqslant b$ has a truncated normal distribution $\mathrm{TN}(\hat{\mu}, \hat{\sigma^2}, a, b)$. Here we assume $\hat{\mu} \geqslant 0$, because if $\hat{\mu} < 0$, that means the density function of truncated normal distribution is monotone decreasing, which means we have high probability of low valuation bidders and low probability of high valuation bidders. This kind of characteristics can be captured by another probability distribution which we introduce in the next section called Pareto distribution. To notice here $\hat{\mu}$ and $\hat{\sigma}$ are the mean and standard deviation of the normal distribution before the truncation, and we denote μ and σ are the mean and standard deviation of the truncated normal distribution. We also assume the truncated range is $[0, \infty)$, thus set $a = 0$ and $b = \infty$. The corresponding PDF and CDF of $\mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)$: (using the notation from S. Kotz, N. L. Johnson, and N. Balakrishnan [**kotz1994continuous**]):

$$
f_t(x) = \begin{cases} \frac{1}{\hat{\sigma}} \frac{\phi(\frac{x-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})} & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
F_t(x) = \begin{cases} \frac{\Phi(\frac{x-\hat{\mu}}{\hat{\sigma}})-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})} & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}
$$

where $\Phi(\cdot)$ the cumulative distribution function and $\phi(\cdot)$ the probability density function of the standard normal distribution, and the mean and variance of $\mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)$

$$E(X|X \geqslant 0) = \mu = \hat{\mu} + \frac{\hat{\sigma}\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1 - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}$$
$$Var(X|X \geqslant 0) = \sigma^2 = \hat{\sigma}^2 \left(1 + \frac{\frac{-\hat{\mu}}{\hat{\sigma}}\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1 - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})} - \left(\frac{\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1 - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}\right)^2\right)$$

then our r for a given $\mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)$

$$r = \frac{\sigma}{\mu} = \frac{\hat{\sigma}\sqrt{1 + \frac{\frac{-\hat{\mu}}{\hat{\sigma}}\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})} - (\frac{\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})})^2}}{\hat{\mu} + \frac{\hat{\sigma}\phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}}$$

To simplify the expression, we denote $\hat{h}(x) = \frac{\phi(x)}{1-\Phi(x)}$ Then we get:

$$r = \frac{\hat{\sigma}\sqrt{1 - \frac{\hat{\mu}}{\hat{\sigma}}\hat{h}(-\frac{\hat{\mu}}{\hat{\sigma}}) - \hat{h}(-\frac{\hat{\mu}}{\hat{\sigma}})^2}}{\hat{\mu} + \hat{\sigma}\hat{h}(-\frac{\hat{\mu}}{\hat{\sigma}})}$$
$$= \frac{\sqrt{1 - \frac{\hat{\mu}}{\hat{\sigma}}\hat{h}(-\frac{\hat{\mu}}{\hat{\sigma}}) - \hat{h}(-\frac{\hat{\mu}}{\sigma})^2}}{\frac{\hat{\mu}}{\hat{\sigma}} + \hat{h}(-\frac{\hat{\mu}}{\hat{\sigma}})}$$

Let's set $y = -\frac{\hat{\mu}}{\hat{\sigma}} \in (-\infty, 0]$, then

$$r = \frac{\sqrt{1 + y\hat{h}(y) - \hat{h}(y)^2}}{-y + \hat{h}(y)}$$

If y increases then $\phi(y)$ increases, and $1 - \Phi(y)$ decreases, thus $\hat{h}(y)$ is monotone increasing on $(-\infty, 0]$. We can investigate the upper bound and lower bound for $\hat{h}(y)$ on $(-\infty, 0]$: when $y = 0$, $\hat{h}(0) = \frac{\phi(0)}{1-\Phi(0)} = 2\phi(0)$ $(2\phi(0) \approx)$; when $y \to -\infty$, $\hat{h}(-\infty) = \frac{0}{1-0} = 0$. We find out that $0 \leqslant \hat{h}(y) \leqslant 2\phi(0)$. Now we consider two extreme cases

- Case 1: when $y \to -\infty$ then:

$$
\begin{aligned}
r &= \lim_{y\to-\infty} \frac{\sqrt{1+y\hat{h}(y)-\hat{h}(y)^2}}{-y+\hat{h}(y)} \\
&= \lim_{y\to-\infty} \sqrt{\frac{1+y\hat{h}(y)-\hat{h}(y)^2}{\left(-y+\hat{h}(y)\right)^2}} \\
&= \lim_{y\to-\infty} \sqrt{\frac{1-\hat{h}(y)(\hat{h}(y)-y)}{\left(\hat{h}(y)-y\right)^2}} \\
&= \lim_{y\to-\infty} \sqrt{\frac{1-\hat{h}(y)}{\hat{h}(y)-y}} \\
&= \sqrt{\frac{1-\hat{h}(-\infty)}{\hat{h}(-\infty)-(-\infty)}} \\
&= \sqrt{\frac{1-0}{0+\infty}} \\
&= 0
\end{aligned}
$$

- Case 2: when $y \to 0$ then:

$$
\begin{aligned}
r &= \lim_{y\to 0} \frac{\sqrt{1+y\hat{h}(y)-\hat{h}(y)^2}}{-y+\hat{h}(y)} \\
&= \frac{\sqrt{1+0\cdot\hat{h}(0)-\hat{h}(0)^2}}{-0+\hat{h}(0)} \\
&= \frac{\sqrt{1-4\phi(0)^2}}{2\phi(0)}
\end{aligned}
$$

If we plot r in terms of y, as we can see from Fig. 2.2, r is monotone increasing. Then r is upper bounded when $y = 0$. Since we know $\phi(0) \approx 0.3989$, substitute it into Case 2 equation, the upper bound of $r \approx 0.7555$.

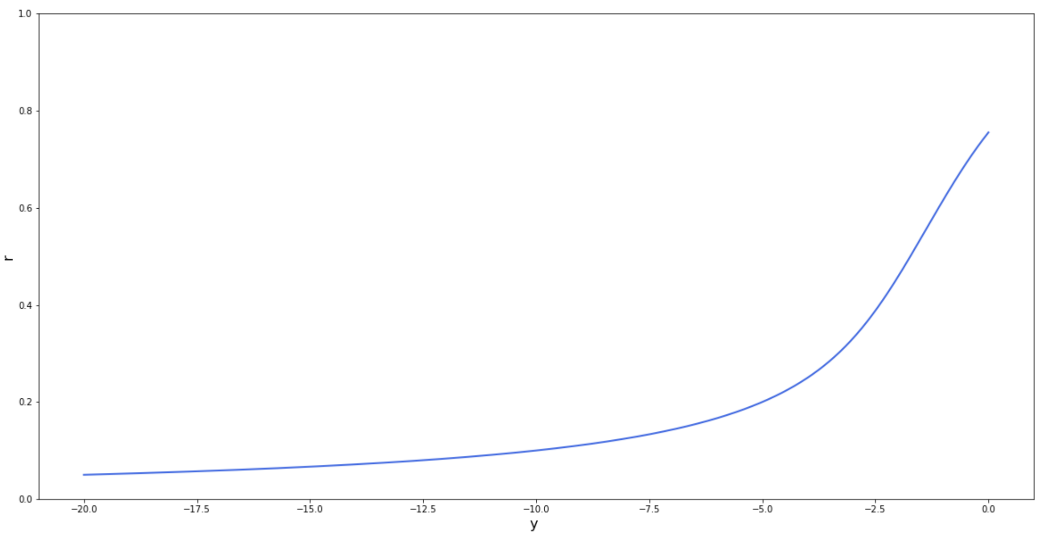

Figure 2.2: The relation between r and y

To find optimal revenue for truncated normal distribution, we consider an alternative way by using *virtual valuation*. Previously we compute the optimal reserve price by maximize the expected revenue formula, because of the simplicity of uniform distribution, the equation of expected revenue is concave and simple to compute the derivative. However for truncated normal distribution, it is not easy to programme this way, therefore based on Myerson optimal auction the optimal reserve price is the price which makes the *virtual valuation* equal 0. The *virtual valuation* is defined as

Definition 2.2 The *virtual valuation* of bidder i with valuation v_i is

$$\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$$

In our case, we only have one bidder, so just use v instead of v_i, then we can write $\psi(v) = v - \frac{1-F_t(v)}{f_t(v)}$, and the optimal reserve price $p_{opt} = \psi^{-1}(0)$. During the experiments, we use numerical solver to solve this equation $\psi(v) = 0$ to get p_{opt}.

2.3.1 Truncated Normal distribution is MHR

One assumption of Myerson optimal auction is that the *virtual valuation* needs to be regular, which means the *virtual valuation* is non-decreasing in values. Thus we need to prove the regularity of the *virtual valuation* for truncated normal distribution and to prove the regularity we can also prove that the truncated normal distribution is a MHR distribution, because MHR implies regularity. Below we show our proof for $\mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)$ is MHR distribution,

Proof. Hazard rate for this truncated normal distribution is $h(v) = \frac{f_t(v)}{1-F_t(v)}$, to prove it is non-decreasing, we can take its first derivative and see if the first derivative is nonnagative or not (here we assume $v \geqslant 0$):

$$\begin{aligned} h^{'}(v) &= \frac{f_t^{'}(v)}{1-F_t(v)} + \frac{f_t(v)\cdot F_t^{'}(v)}{(1-F_t(v))^2} \\ &= \frac{f_t^{'}(v)}{1-F_t(v)} + \frac{f_t^2(v)}{(1-F_t(v))^2} \\ &= \frac{f_t^{'}(v)(1-F_t(v)) + f_t^2(v)}{(1-F_t(v))^2} \end{aligned}$$

Clearly the denominator is nonnegative, so we only need to check the nominator, also $f_t(v) = \frac{1}{\hat{\sigma}}\frac{\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}$ and $F_t(v) = \frac{\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}$,then:

$$\begin{aligned} f_t^{'}(v)(1-F_t(v)) + f_t^2(v) &= \frac{1}{\hat{\sigma}}\frac{-\frac{v-\hat{\mu}}{\hat{\sigma}^2}\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}\cdot\frac{1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})}{1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}})} + \frac{\phi^2(\frac{v-\hat{\mu}}{\hat{\sigma}})}{\hat{\sigma}^2(1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}}))^2} \\ &= \frac{-\frac{v-\hat{\mu}}{\hat{\sigma}}\cdot\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})\left(1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})\right)+\phi^2(\frac{v-\hat{\mu}}{\hat{\sigma}})}{\hat{\sigma}^2(1-\Phi(\frac{-\hat{\mu}}{\hat{\sigma}}))^2} \end{aligned}$$

Now again we only need to prove the nominator is nonnegative or not. We consider it into two cases:

$$-\frac{v-\hat{\mu}}{\hat{\sigma}}\cdot\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})\left(1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})\right)+\phi^2(\frac{v-\hat{\mu}}{\hat{\sigma}})$$

- Case 1: if $v \leqslant \hat{\mu}$, then $-\frac{v-\hat{\mu}}{\hat{\sigma}} \geqslant 0$,
  we know $\phi(\frac{v-\hat{\mu}}{\hat{\sigma}}), 1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})$ is nonnegative, then $-\frac{v-\hat{\mu}}{\hat{\sigma}}\cdot\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})(1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}})) + \phi^2(\frac{-\hat{\mu}}{\hat{\sigma}})$ is nonnegative
- Case 2: if $v > \hat{\mu}$, then $\frac{v-\hat{\mu}}{\hat{\sigma}} > 0$ and set $x = \frac{v-\hat{\mu}}{\hat{\sigma}}$,
  we know that Q-funtion is defined as $Q(x) = 1-\Phi(x)$, and it is bounded when $x > 0$,

  $$\left(\frac{x}{1+x^2}\right)\phi(x) < Q(x) < \frac{\phi(x)}{x}$$

  thus
  $-\frac{v-\hat{\mu}}{\sigma}\cdot\phi(\frac{v-\hat{\mu}}{\hat{\sigma}})(1-\Phi(\frac{v-\hat{\mu}}{\hat{\sigma}}))+\phi^2(\frac{v-\hat{\mu}}{\hat{\sigma}})$
  $= -x\phi(x)(1-\Phi(x)) + \phi^2(x)$
  $= -x\phi(x)Q(x) + \phi^2(x) > -x\phi(x)\cdot\frac{\phi(x)}{x} + \phi^2(x) = -\phi^2(x)+\phi^2(x) = 0$
  the expression is nonnegative.

In both case, we prove that the nominator is nonnegative, thus the first derivative of truncated normal hazard rate is nonnegative, which means its hazard rate is monotone non-decreasing, and truncated normal distribution is a MHR distribution. Regularity assumption satisfied. □

2.3.2 Result of experiment: TBC

We experiment different truncated normal distributions with different r values, and evaluate the corresponding DAPXs. In the implementation, we need to assign values to $\hat{\mu}, \hat{\sigma}$, in order to have different distributions, first we can set either of them to a fix value, in our case, we set $\hat{\mu}$ fixed and gradually increase $\hat{\sigma}$. As we derive in previous section, r is a function of $\frac{\hat{\sigma}}{\hat{\mu}}$, only related to the ratio of $\hat{\mu}$ and $\hat{\sigma}$, so without loss we can set $\hat{\mu} = 1$. We plot the DAPX of truncated normal distributions and the theoretical ρ_D in two plots: the left plot with small values of r and the right with lager values of r. Fig. 2.3 shows how these two values increase with r increases. ρ_D increases really fast while the experimental DAPX remains small. Since we gradually increase $\hat{\sigma}$, we can see the increment in Fig. 2.3 is not uniform. Thus we also plot the results with parameter $\hat{\sigma}$. The left figure in Fig. 2.4 shows the results with small $\hat{\sigma}$, while the right one is with larger $\hat{\sigma}$. Previously we find out r is upper bounded, therefore ρ_D is also upper bounded, and we can observe a clear convergence in Fig. 2.3 and Fig. 2.4. We can also notice that the DAPX of truncated normal is also bounded. Our experiment shows the upper bound of DAPX of truncated normal is approximate 1.1522 for $r = 0.7555$.

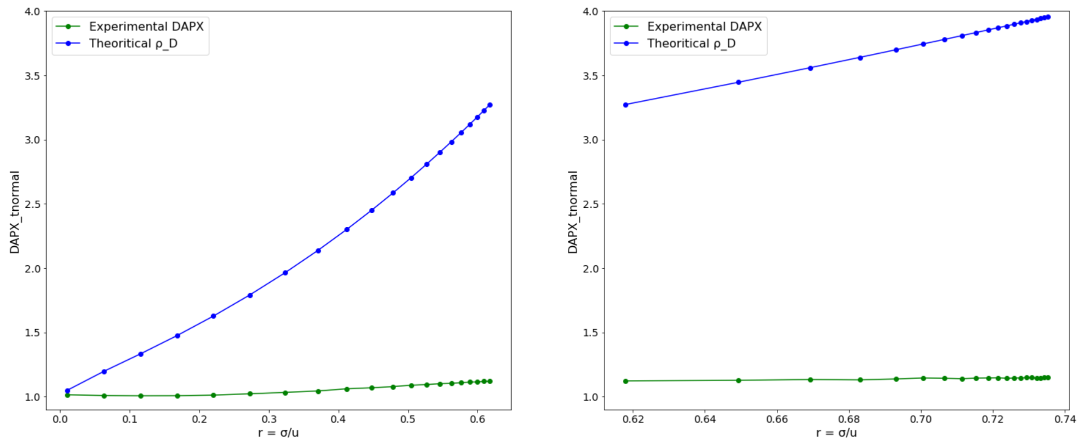

Figure 2.3: DAPX of truncated normal distribution versus ρ_D with different r values

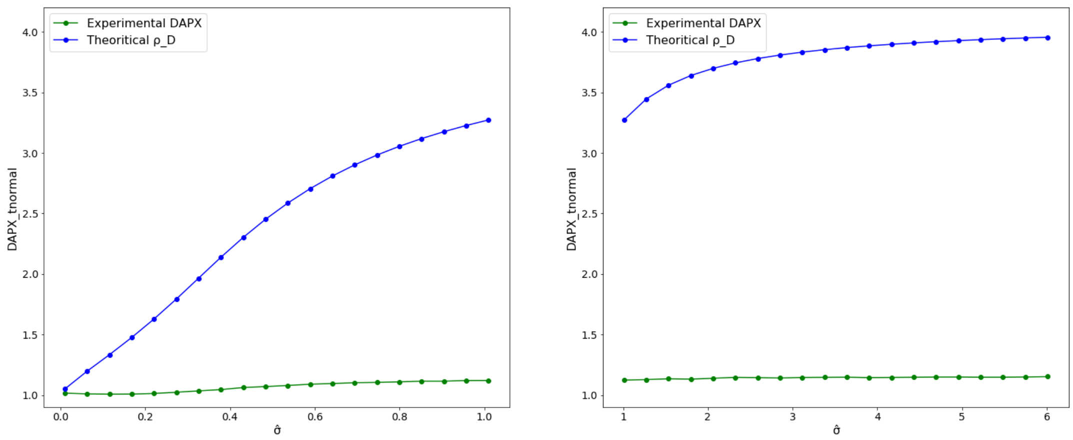

Figure 2.4: DAPX of truncated normal distribution versus ρ_D with different $\hat{\sigma}$ values

2.4 Pareto distribution

Here we also want to consider Pareto distribution. This distribution can possible be used in description of auction data when most bids are consetrated at lower values while high bid is hardly to occur.
Pareto distribution is defined by two parameters $x_m > 0, c > 0$ and $x \in [x_m, \infty]$. To simplify, we choose the scale parameter $x_m = 1$ and denote this Pareto distribution as Pareto(1) due to its support range. Next we can write corresponding PDF and CDF as following

$$f(x) = \frac{c}{x^{c+1}}$$

and

$$F(x) = 1 - \frac{1}{x^c}$$

Another special property of Pareto distribution is that when parameter $c \leqslant 1$, it does not have a mean (the mean is infinite), in addition, when $c \leqslant 2$ its variance is infinite too. Thus in order to have a valid r, we assume the parameter $c > 2$. Then the mean and standard deviation of this Pareto distribution are

$$\mu = \tfrac{c}{c-1} \qquad\qquad \sigma = \sqrt{\tfrac{c}{(c-1)^2(c-2))}}$$

We first determine the optimal reserve price for this Pareto distribution. We use Myersons optimal operator again

$$\mathrm{OPT}\,(F) = \sup_{p\geqslant 1} \mathrm{REV}(p;F) = \sup_{p\geqslant 1} p \cdot \left(1 - \left(1 - \frac{1}{p^c}\right)\right) = \sup_{p\geqslant 1} \frac{1}{p^{c-1}}$$

since $c > 2$, then $\frac{1}{p^{c-1}}$ is monotone decreasing. When reserve price $p = 1$, we can achieve the maximum expected revenue, thus $\mathrm{OPT}(F) = 1$. Now let's check whether r is bounded or not

$$r = \frac{\sigma}{\mu} = \frac{\sqrt{\frac{c}{(c-1)^2(c-2))}}}{\frac{c}{c-1}} = \frac{1}{\sqrt{c(c-2)}}$$

As we can see from the expression, when $c \to 2$, $r \to \infty$, thus r is unbounded and our theoretical bound ρ_D is also unbounded. However our experiment shows the DAPX of the Pareto distribution is not unbounded, as we can see figure....., so we write down the expression for DAPX explicitly:

$$\begin{aligned}
\mathrm{DAPX} = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(F)} &= \frac{1}{p_D(1 - F(p_D))} = \frac{1}{p_D\left(1 - \left(1 - \frac{1}{p_D^c}\right)\right)} \\
&= p_D^{c-1} = (\mu\frac{\rho_D}{2\rho_D - 1})^{c-1} = \left(\frac{c}{c-1} \cdot \frac{1}{2 - \frac{1}{\rho_D}}\right)^{c-1} \\
&= \left((1 + \frac{1}{c-1}) \cdot \frac{1}{2 - \frac{1}{\rho_D}}\right)^{c-1}
\end{aligned}$$

From the expression, when $c \to 2$, we know $r \to \infty$ and $\rho_D \to \infty$ also, then $\mathrm{DAPX} \to 1$; when $c \to \infty$, and $r \to 0$ and $\rho_D \to 1$, so $\mathrm{DAPX} \to 1$ also. This matches the results in both Fig. 2.6 and Fig. 2.7. Fig. 2.5 shows the experimental DAPX comparing to ρ_D, as we can see, the DAPX of Pareto(1) remains small while ρ_D increases exponentially. Fig. 2.6 shows the experimental DAPX by itself, as we can see, the DAPX remains around 1 and we observe a clear upper bound. To see more details how DAPX values under small r values, we run the experiment with 50 steps at range (0, 2) of r. Results are represented in Fig. 2.7, and from the experiment the upper bound of Pareto(1) DAPX ≈ 1.1416 to 4 decimals for $r = 0.2700$.

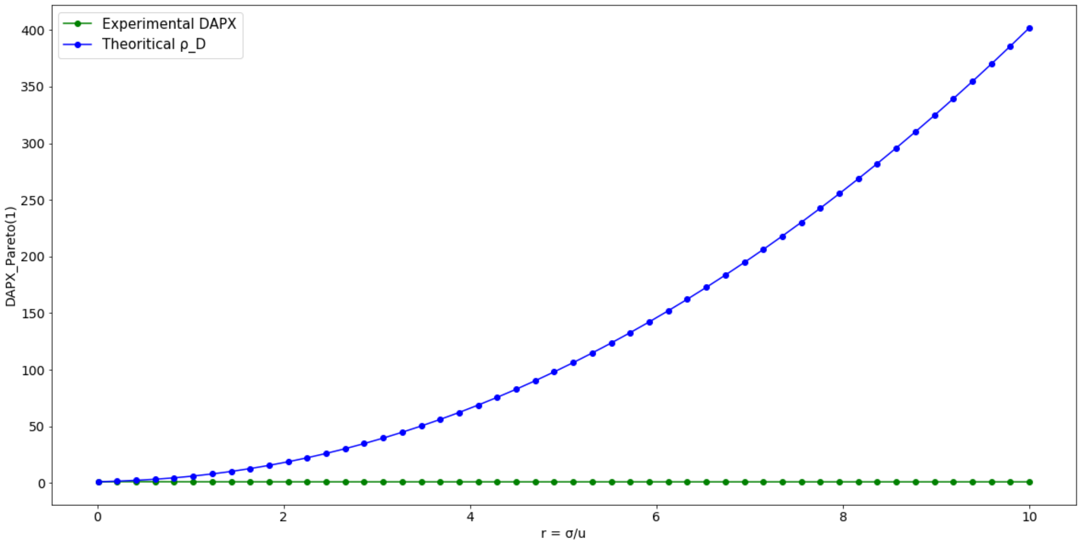

Figure 2.5: DAPX of Pareto(1) distribution versus ρ_D

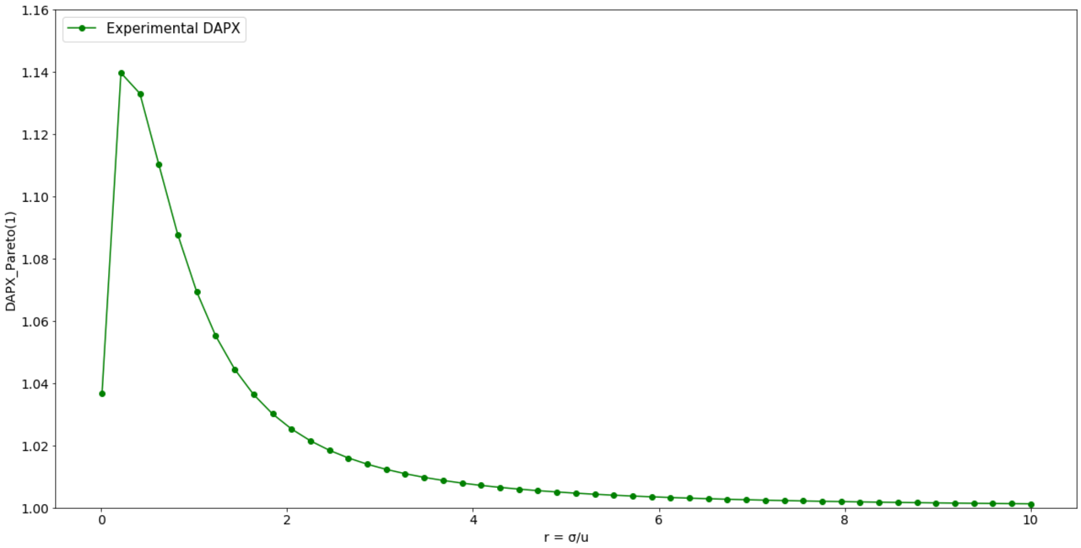

Figure 2.6: DAPX of Pareto(1) distribution with r

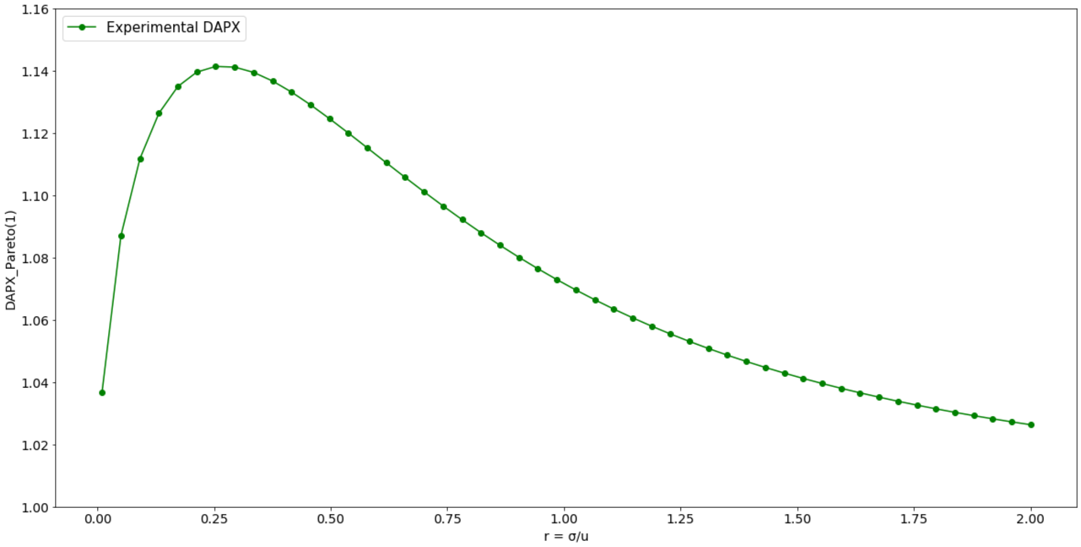

Figure 2.7: DAPX of Pareto(1) distribution when r remains small

2.4.1 Pareto Extension

We usually consider all the valuation is non-negative, therefore we would like to modify the previous Pareto distribution to have support $[0, \infty)$, which means shifting the distribution to the left by 1. Let's denote this distribution as Pareto(0) and its PDF and CDF are

$$f(x) = \frac{c}{(x+1)^{c+1}}$$

and

$$F(x) = 1 - \frac{1}{(x+1)^c}$$

Below we make sure it is a valid distribution:

$$\int_0^\infty f(x)dx = 1 - \frac{1}{(x+1)^c}\Bigg|_0^\infty = 1 - 0 - (1 - \frac{1}{1}) = 1$$

$$F(0) = 1 - \frac{1}{(0+1)^c} = 0$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} 1 - \frac{1}{(x+1)^c} = 1$$

We can derive mean and standard deviation of Pareto(0) from Pareto(1). We denote $E[X], Var[X]$ are mean and variance of Pareto(1) , then $E[X-1], Var[X-1]$ is the mean and variance for Pareto(0). We get $E[X-1] = E[X]-1$ and $Var[X-1] = Var[X]$, thus from this relation, we can write down the formula for μ, σ

$$\mu = \tfrac{1}{c-1} \qquad \text{and} \qquad \sigma = \sqrt{\tfrac{c}{(c-1)^2(c-2)}}$$

This time $p_{opt} = \arg\max_{p\geqslant 0} p(1-(1-\frac{1}{(p+1)^c})) = \arg\max_{p\geqslant 0} \frac{p}{(p+1)^c}$, to find p_{opt}, we take the first derivative of $\frac{p}{(p+1)^c}$, and check if the maximum exists or not.

$$\frac{p}{(p+1)^c}' = \frac{1}{(p+1)^c} - \frac{pc}{(p+1)^{c+1}} = \frac{1-p(c-1)}{(p+1)^{c+1}}$$

Since $p \geqslant 0, c > 2$, when $p < \frac{1}{c-1}$, then $\frac{p}{(p+1)^c}' > 0$, which means $\frac{p}{(p+1)^c}$ monotonously increases at $[0,\frac{1}{c-1})$, while $p > \frac{1}{c-1}$, then $\frac{p}{(p+1)^c}' < 0$ means $\frac{p}{(p+1)^c}$ monotonously decreases at $(\frac{1}{c-1},\infty)$. Therefore the maximum value can be achieved when $p = \frac{1}{c-1}$. Then $p_{opt} = \frac{1}{c-1}$, we can also write down the optimal revenue:

$$\mathrm{OPT}(F) = \frac{1}{c-1} \cdot \frac{1}{(\frac{1}{c-1}+1)^c} = \frac{(c-1)^{c-1}}{c^c}$$

and r

$$r = \frac{\sigma}{\mu} = \frac{\sqrt{\frac{c}{(c-1)^2(c-2)}}}{\frac{1}{c-1}} = \sqrt{\frac{c}{c-2}} = \sqrt{\frac{1}{1-\frac{2}{c}}}$$

We can get similar conclusion as before: if $c \to 2$, $r \to \infty$; if $c \to \infty$, $r \to 1$. Therefore r is also unbounded in this case. Then we write down DAPX for Pareto(0) explicitly

$$\mathrm{DAPX} = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(F)} = \frac{\frac{(c-1)^{c-1}}{c^c}}{p_D(1-F(p_D))}$$

$$= \frac{(c-1)^{c-1}}{c^c} \cdot \frac{(p_D+1)^c}{p_D}$$

where $p_D = \frac{1}{c-1} \cdot \frac{\rho_D}{2\rho_D - 1}$, from above expression, it is hard to see if an upper bound for DAPX of Pareto(0) exists or not. If we plot above expression, from Fig. 2.8, we can see DAPX is a function of r which is monotone decreasing, so it is bounded when $r \to 1$. We can find this upper bound using the experiment by setting parameter c to a very large number, so $r \to 1$, and our result shows the Pareto(0) DAPX is 1.1638 to 4 decimal places for $c = 10^{10}$.

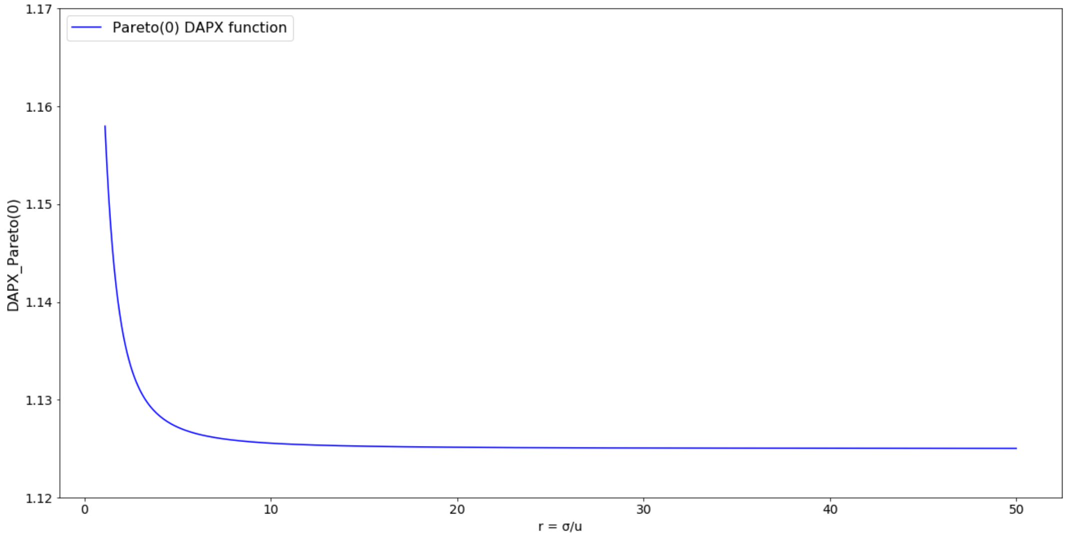

Figure 2.8: Pareto(0) DAPX function

2.4.2 Result

First we show the results of the experimental DAPX and ρ_D with $r \in (0, 10)$. From Fig. 2.9, as we can see ρ_D increase exponentially with r, while the experimental DAPX remain small(around 1). In the second Fig. 2.10, we can see more details of these two values when $r \to 1$.

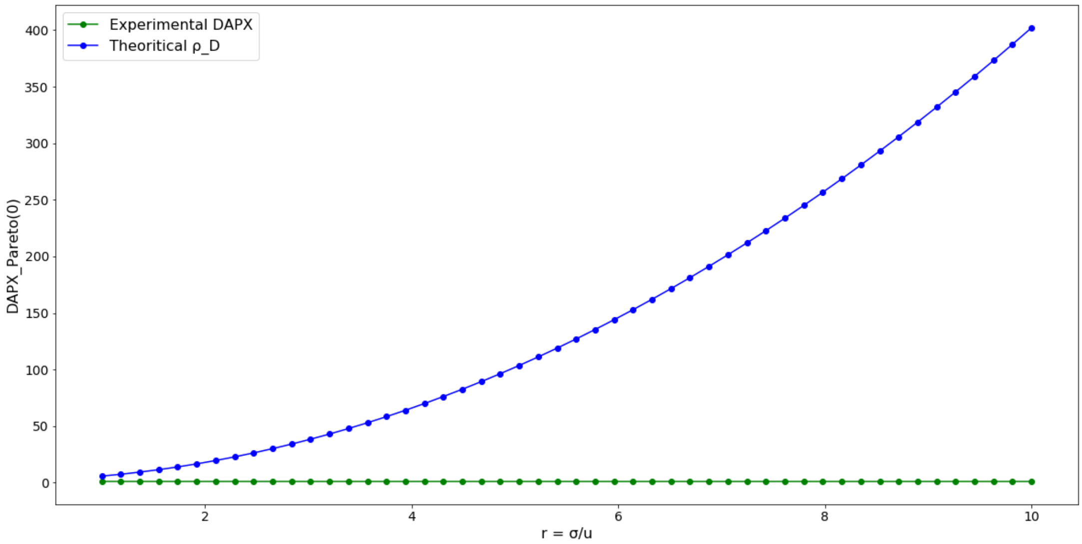

Figure 2.9: Pareto(0) DAPX when $r \in (0, 10)$

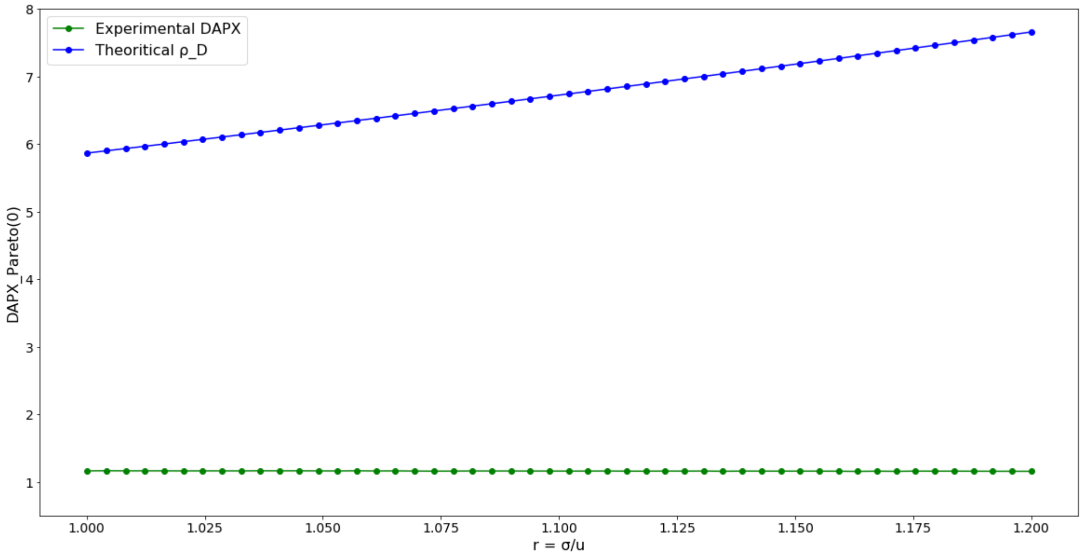

Figure 2.10: Pareto(0) DAPX when $r \in (1, 1.2)$

2.5 Summary of Deterministic SBSI Auction

From above evaluation on different distributions, we notice that for most of these distributions, r is upper bounded, which also means, our theoretical ρ_D and our experimental DAPX have also an upper bound. Table 2.1 represent our foundlings from our evaluation(keep 4 decimals):

Distribution	**r upper bound**	**Theoritical ρ_D upper bound**	**Experimental DAPX upper bound**
Uniform distribution	$\frac{1}{\sqrt{3}}$	3.0593	1.1919
Truncated normal distribution	$\frac{\sqrt{1-4\phi(0)^2}}{2\phi(0)}$	4.0836	1.1511
Pareto distribution ($x \in [1, \infty)$)	∞	∞	1.1416
Pareto distribution ($x \in [0, \infty)$)	∞	∞	1.1638

Table 2.1: Upper bound table

We discuss the Uniform distribution DAPX in . There are two cases, since the upper bound of r is $\frac{1}{\sqrt{3}}$ when $\frac{b}{a} \to \infty$, therefore in this case, $\frac{b}{2} \geqslant a$, then DAPX $= \frac{(\sqrt{3}r+1)^2}{4\cdot\frac{\rho_D}{2\rho_D-1}\cdot(\sqrt{3}r+1-\frac{\rho_D}{2\rho_D-1})}$. We subsititute $r = \frac{1}{\sqrt{3}}$ and $\rho_D = 3.5936$ into this equation, The upper bound for DAPX of uniform distribution is 1.1931. Our experimental number in Table 2.1 matches this number. We can also explicitly compute the DAPX upper bound for Truncated normal distribution using following equation:

$$\text{DAPX} = \frac{\text{OPT}(F)}{\text{REV}(F)} = \frac{p_{opt}(1 - F_t(p_{opt}))}{p_D(1 - F_t(p_D))}$$

where p_{opt} and p_D can be determined explicitly with $r = 0.7555$. Then $\sup \text{DAPX} = 1.1515$, our experimental value matches this number.

3 Single-bidder and Single-item Randomization Auction

In this chapter, we evaluate APX with different probability distribution under single-bidder and single-item randomization auction experiment. We still use *take-it-or-leave-it* auction mechanism. The robust paper defines a specific *randomized* selling mechanism, which essentially corresponds to the lottery proposed by Carrasco et al. [paper reference]

Definition 3.1 (Log-Lottery)
Fix any $\mu > 0$ and $\sigma \geqslant 0$. A log-lottery is a randomized mechanism that sells at a price $P^{log}_{\mu,\sigma}$, which is distributed over the nonnegative interval support $[\pi_1, \pi_2]$ according to the cdf

$$F^{log}_{\mu,\sigma}(x) = \frac{\pi_2 \ln \frac{x}{\pi_1} - (x - \pi_1)}{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)}$$

where parameters π_1, π_2 are the (unique) solutions of the system

$$\begin{cases} \pi_1(1 + \ln \frac{\pi_2}{\pi_1}) = \mu \\ \pi_1(2\pi_2 - \pi_1) = \mu^2 + \sigma^2 \end{cases} \tag{3.1}$$

We will sometimes slightly abuse notation and use $P^{log}_{\mu,\sigma}$ to refer both the log-lottery mechanism and the corresponding variable of the prices.

This time, we sample N $P^{log}_{\mu,\sigma}$s from log-lottery distribution (i.e. $N = 10000$), and for each $P^{log}_{\mu,\sigma}$ we perform our Auction Experiment Procedure 1 from Chapter 2. We just need replace p_D with $P^{log}_{\mu,\sigma}$ in the procedure.

3.1 Log-Lottery Randomization

The log-lottery distribution defined in Definition 3.1 is not a regular distribution, therefore there is no sampling function in Python we can use directly. For this distribution we can use rejection a sampling technique. First we need to check its pdf

$$f^{log}_{\mu,\sigma}(x) = \frac{\pi_2 \frac{1}{x} - 1}{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)}$$

$f^{log}_{\mu,\sigma}(x)$ is monotone decreasing on $[\pi_1, \pi_2]$, so $\max(f^{log}_{\mu,\sigma}(x)) = f^{log}_{\mu,\sigma}(\pi_1)$. Since this distribution is bounded on bounded support, we can indeed use a rejection sampling technique by proposing an uniform density

$$q(x) = \begin{cases} \frac{1}{\pi_2 - \pi_1} & \text{if } x \in [\pi_1, \pi_2] \\ 0 & \text{otherwise} \end{cases}$$

A constant $A = f^{log}_{\mu,\sigma}(\pi_1) \cdot (\pi_2 - \pi_1)$, then we know $A \cdot q(x) \geqslant f^{log}_{\mu,\sigma}(x), \forall x \in [\pi_1, \pi_2]$. Our rejection sampling steps can be described in Algorithm 2 below.

Algorithm 2 Rejection Sampling Algorithm

```
procedure REJECTION SAMPLING(N)
    n ← 0
    x_list is empty
    while n ⩽ N do                                        ▷ N is sample size
        draw x ~ q(x)
        compute acception probability a := f^log_{μ,σ}(x) / (A q(x))
        draw a random sample u ~ U[0, 1]
        if u ⩽ a then
            accept x, add it to x_list
            n ← n + 1
    return x_list
```

For example, if we have a $\mu = 1$ and $\sigma = 1$ log-lottery distribution, we can solve π_1, π_2 using system of equation Section 3.1. We get $\pi_1 = 0.2778$ and $\pi_2 = 3.7383$ from the numerical solver. Histogram in Fig. 3.1 shows the density of samples from our rejection sampling algorithm and the green line represents the corresponding pdf of the log-lottery distribution.

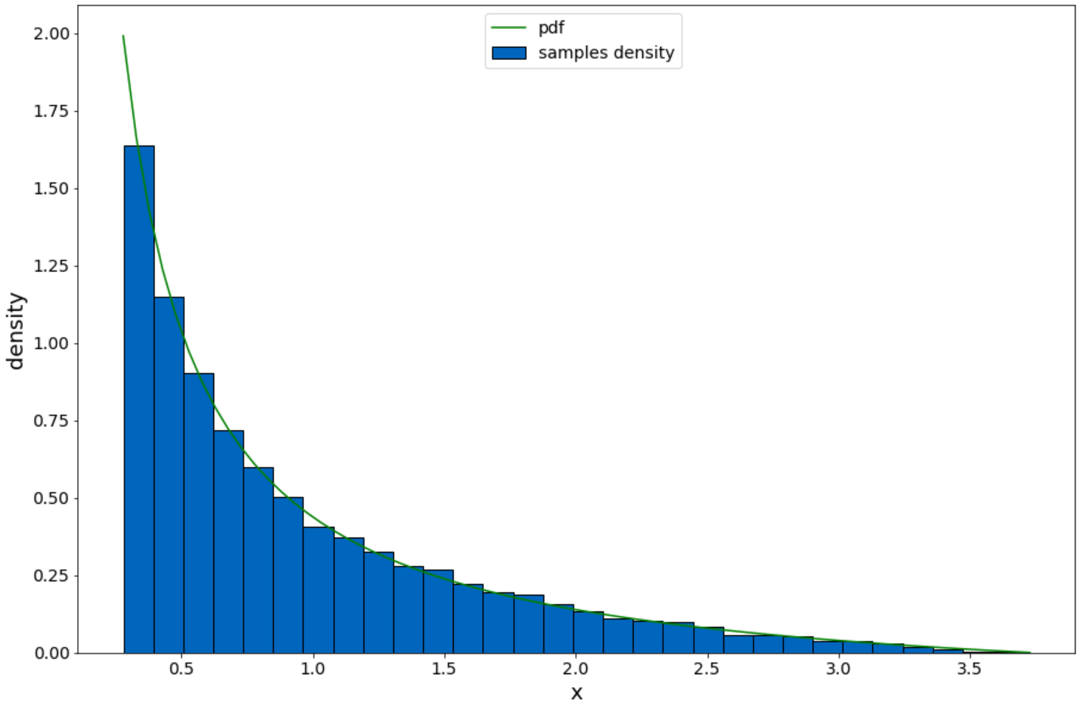

Figure 3.1: The actual log-lottery pdf and the results from rejection sampling algorithm

In our new randomised auction experiment, we keep exact same steps from Auction Experiment 1 to simulate the $\mathrm{OPT}(F)$. The only difference is simulating the expected revenue $\mathrm{REV}(F)$, since now our p_D is not a fixed value but a random number from a log-lottery distribution. Thus in order to simulate the expected revenue, we first draw x_{list} (contains N random samples) using above Algorithm 2, and for each reserve price in x_{list} we simulate n auction experiments. Then we will have N revenue outputs from the auction experiments, and we take the average of these N values and this average is our final expected revenue $\mathrm{REV}(F)$.

3.2 Different Valuation Distributions

In this section, we want to see how the expected revenue and APX will be under different valuation distributions.

3.2.1 Uniform distribution

If our bidder's valuation is from a uniform distribution, this time our reserve price is randomly draw from $P^{log}_{\mu,\sigma}$ while other parameters remain the same, given $U[a,b]$, under the log-lottery randomization selling mechanism, we can write down our expected revenue with reserve price $P^{log}_{\mu,\sigma}$

$$
\begin{aligned}
\mathrm{REV}(P^{log}_{\mu,\sigma}; U[a,b]) &= \mathop{\mathbb{E}}_{p\sim F^{log}_{\mu,\sigma}}[p(1-F_{uniform}(p))] \\
&= \int_{\pi_1}^{\pi_2} p(1-\frac{p-a}{b-a}) f^{log}_{\mu,\sigma}(p)dp \\
&= \int_{\pi_1}^{\pi_2} p(1-\frac{p-a}{b-a}) \cdot \frac{\pi_2\frac{1}{p}-1}{\pi_2 \ln\frac{\pi_2}{\pi_1}-(\pi_2-\pi_1)}dp \\
&= \int_{\pi_1}^{\pi_2} \frac{p(b-p)}{b-a} \cdot \frac{\pi_2-p}{p(\pi_2 \ln\frac{\pi_2}{\pi_1}-(\pi_2-\pi_1))}dp \\
&= \frac{1}{(b-a)(\pi_2 \ln\frac{\pi_2}{\pi_1}-(\pi_2-\pi_1))}\int_{\pi_1}^{\pi_2}(b-p)(\pi_2-p)dp \\
&= \frac{1}{(b-a)\left(\pi_2 \ln\frac{\pi_2}{\pi_1}-(\pi_2-\pi_1)\right)} \cdot \left(\frac{p^3}{3}-\frac{bp^2}{2}-\frac{\pi_2 p^2}{2}+b\pi_2 p\Big|_{\pi_2}^{\pi_1}\right)
\end{aligned}
$$

From Chapter 2, we also know $\mathrm{OPT}(U[a,b]) = \frac{b^2}{4(b-a)}$, then

$$
\mathrm{APX} = \frac{\mathrm{OPT}(U[a,b])}{\mathrm{REV}(P^{log}_{\mu,\sigma}; U[a,b])} = \frac{b^2\left(\pi_2 \ln\frac{\pi_2}{\pi_1}-(\pi_2-\pi_1)\right)}{4\left(\frac{p^3}{3}-\frac{bp^2}{2}-\frac{\pi_2 p^2}{2}+b\pi_2 p\Big|_{\pi_2}^{\pi_1}\right)}
$$

As we can see, the expected revenue expression under randomization gets more complex comparing to the deterministic one. We will use our experiment simulation to determine APX experimentally, and results are presented in next section.

3.2.2 Truncated Normal Distribution

Similarly the expected revenue of a truncated normal distribution $\mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)$

$$
\begin{aligned}
\mathrm{REV}(P^{log}_{\mu,\sigma}; \mathrm{TN}(\hat{\mu}, \hat{\sigma}^2, 0, \infty)) &= \mathop{\mathbb{E}}_{p\sim F^{log}_{\mu,\sigma}} [p(1 - F_{\mathrm{TN}}(p))] \\
&= \int_{\pi_1}^{\pi_2} p(1 - F_{\mathrm{TN}}(p)) f^{log}_{\mu,\sigma}(p) dp \\
&= \int_{\pi_1}^{\pi_2} p(1 - F_{\mathrm{TN}}(p)) \cdot \frac{\pi_2 - p}{p(\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1))} dp \\
&= \frac{1}{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} (1 - F_{\mathrm{TN}}(p))(\pi_2 - p) dp
\end{aligned}
$$

where $F_{\mathrm{TN}} = \frac{\Phi(\frac{x-\hat{\mu}}{\hat{\sigma}}) - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}{1 - \Phi(\frac{-\hat{\mu}}{\hat{\sigma}})}$.
For truncated normal distribution, there is a nonlinear integration term in its expected revenue expression, which is even more complex. Thus we will not explore further on the expression of APX in this case.

3.2.3 Pareto Distribution

We first consider Pareto(1) distribution, whose support is $[1, \infty)$.

$$
\begin{aligned}
\mathrm{REV}(P^{log}_{\mu,\sigma}; \mathrm{Pareto}(1)) &= \mathop{\mathbb{E}}_{p\sim F^{log}_{\mu,\sigma}} [p\,(1 - F(p))] \\
&= \int_{\pi_1}^{\pi_2} p \left(1 - (1 - \frac{1}{p^c})\right) f^{log}_{\mu,\sigma}(p) dp \\
&= \int_{\pi_1}^{\pi_2} p^{1-c} \cdot \frac{\pi_2 - p}{p(\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1))} dp \\
&= \frac{1}{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} p^{-c} (\pi_2 - p) dp
\end{aligned}
$$

We know the $\mathrm{OPT}(\mathrm{Pareto}(1)) = 1$, then

$$
\mathrm{APX} = \frac{\mathrm{OPT}\,(\mathrm{Pareto}(1))}{\mathrm{REV}(P^{log}_{\mu,\sigma}; \mathrm{Pareto}(1))} = \frac{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)}{\int_{\pi_1}^{\pi_2} p^{-c} (\pi_2 - p) dp}
$$

We can also write down the expected revenue and APX for Pareto(0) distribution.

$$
\begin{aligned}
\mathrm{REV}(P^{log}_{\mu,\sigma}; \mathrm{Pareto}(0)) &= \mathop{\mathbb{E}}_{p\sim F^{log}_{\mu,\sigma}} [p\,(1 - F(p))] \\
&= \int_{\pi_1}^{\pi_2} p \left(1 - (1 - \frac{1}{(p+1)^c})\right) f^{log}_{\mu,\sigma}(p) dp \\
&= \int_{\pi_1}^{\pi_2} \frac{p}{(p+1)^c} \cdot \frac{\pi_2 - p}{p(\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1))} dp \\
&= \frac{1}{\pi_2 \ln \frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)} \int_{\pi_1}^{\pi_2} \frac{\pi_2 - p}{(p+1)^c} dp
\end{aligned}
$$

We know the OPT(Pareto(0)) $= \frac{(c-1)^{c-1}}{c^c}$, then

$$\mathrm{APX} = \frac{\mathrm{OPT}(\mathrm{Pareto}(0))}{\mathrm{REV}(P^{log}_{\mu,\sigma};\mathrm{Pareto}(0))} = \frac{\frac{(c-1)^{c-1}}{c^c} \cdot \Big(\pi_2 \ln\frac{\pi_2}{\pi_1} - (\pi_2 - \pi_1)\Big)}{\int_{\pi_1}^{\pi_2} \frac{\pi_2 - p}{(p+1)^c} dp}$$

It is clear that the expression of expected revenue and APX are getting more complex under randomisation situation. Thus using our experiment simulation becomes more convenient and delicate here to compute the experimental APX.

3.3 Observation and Results

We present four figures which are the results from different valuation distributions. In each figure, we plot the r with the corresponding APX against the corresponding DAPX. As we can see, for these four distributions: uniform, truncated normal, Pareto(1) and Pareto(2), the experimental DAPX is better/smaller than APX. The robust paper proposed both deterministic selling mechanism and log-lottery randomised selling mechanism, and our experiment shows that for single-bidder and single-item auction the proposed deterministic mechanism performs better than the proposed randomization mechanism. The possible reason to this result will be: for single-bidder and single-item auction, a good deterministic mechanism can perform better than a specific randomised mechanism, in our case, that means log-lottery randomization is probably not a good randomization choice comparing to the proposed deterministic mechanism.

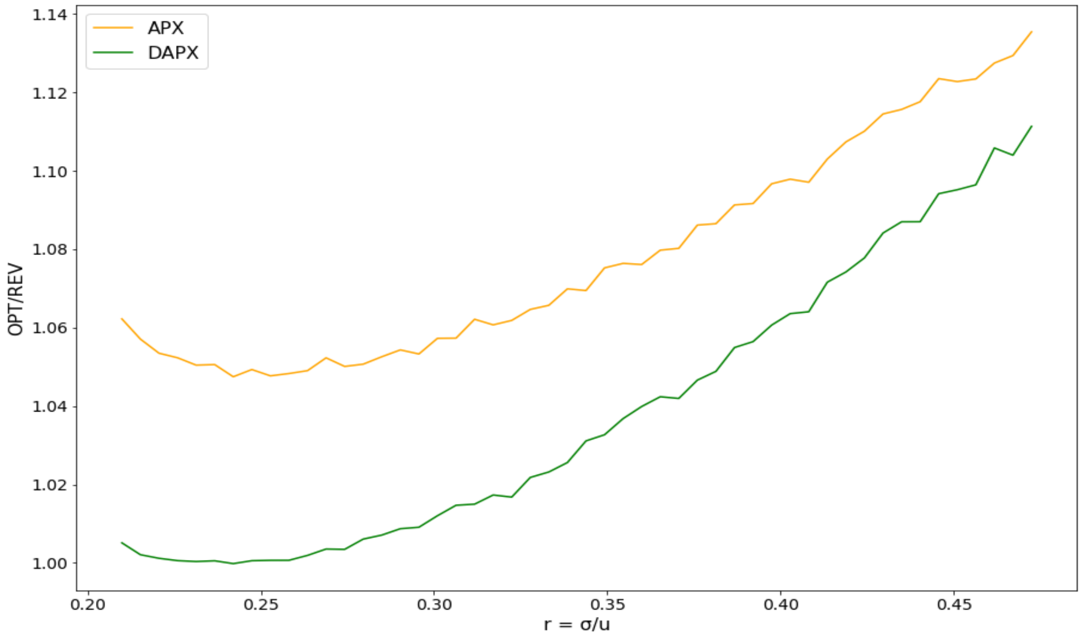

Figure 3.2: Uniform distribution: experimental APX versus DAPX

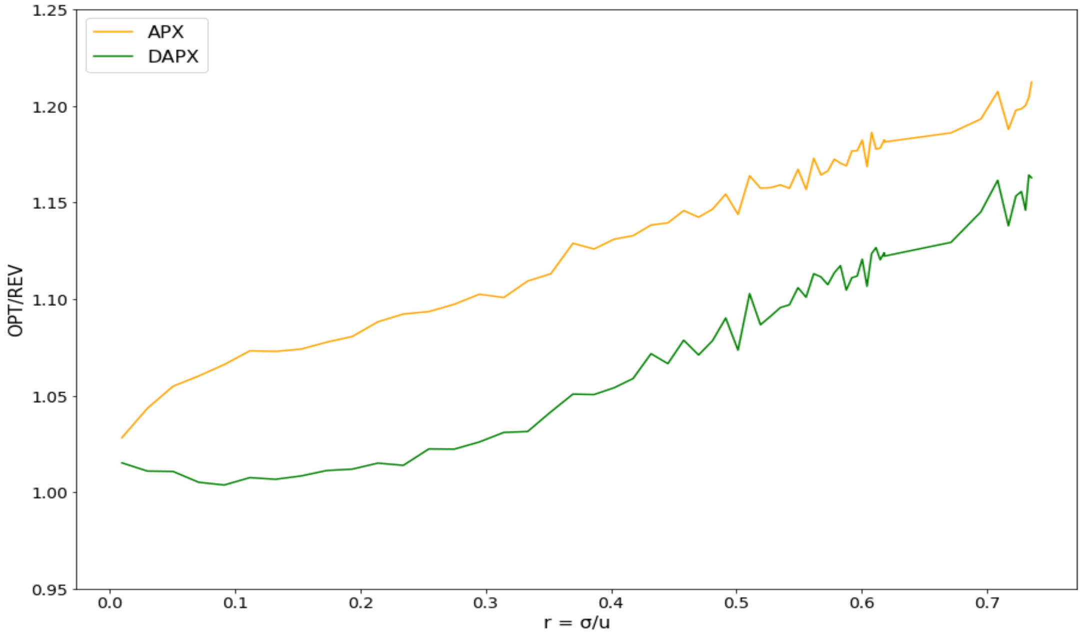

Figure 3.3: Truncated normal distribution: experimental APX versus DAPX

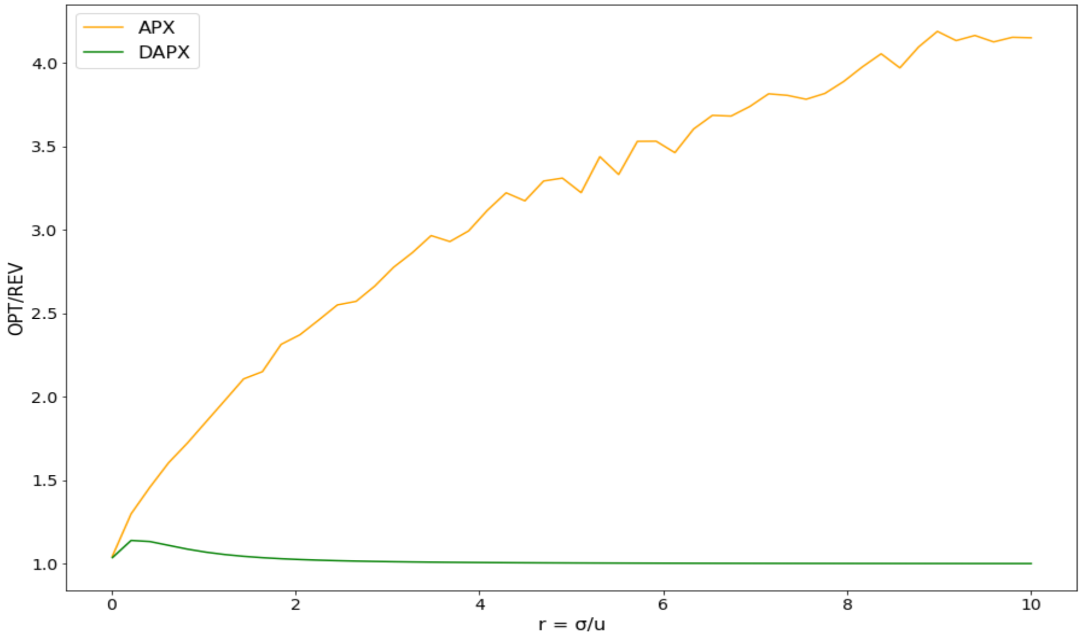

Figure 3.4: Pareto(1) distribution: experimental APX versus DAPX

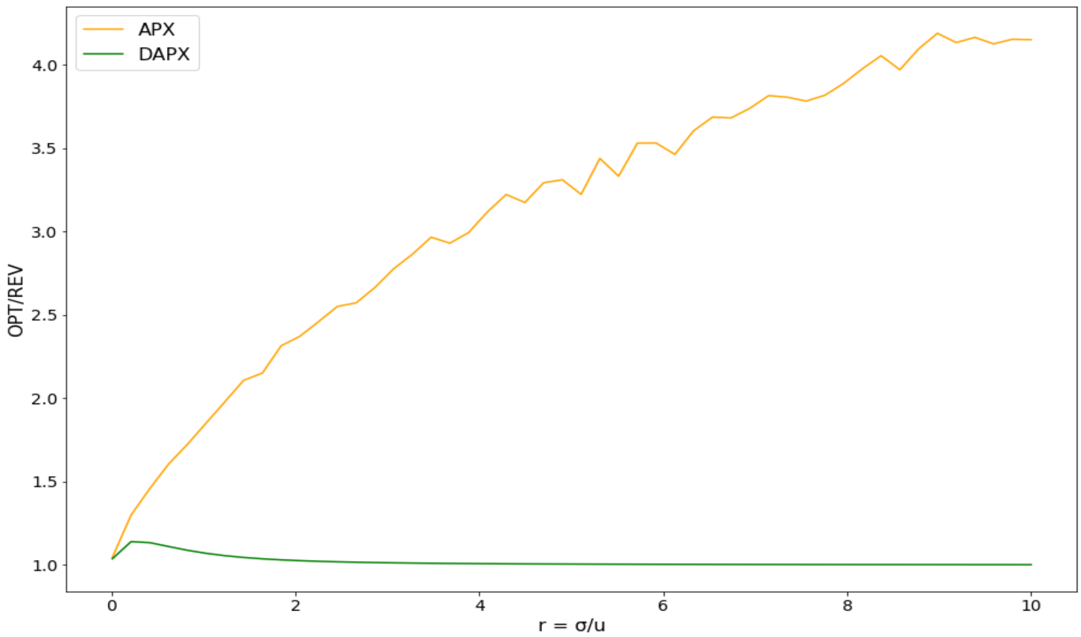

Figure 3.5: Pareto(0) distribution: experimental APX versus DAPX

However here also comes our questions: in general, randomization is better than deterministic mechanism. Here we compared one deterministic mechanism with one randomization mechanism given a certain valuation distribution. In the paper, APX is the ratio under worst case (worst probability distribution) with log lottery randomization.

1.Randomization is better than deterministic, but given a valuation distribution, using one specific randomization may not be better. Thus whether log lottery is the best randomization distribution. How about using uniform or truncated normal distribution for price randomization? Especially is log lottery is the best for uniform and truncated normal distribution? 2.During the experiment, we use rejection sampling to generate random log lottery price, is the sample size sufficient enough? 3.A better deterministic mechanism can result a better approximate robust ratio than a specific randomization mechanism, i.e. better than log lottery randomization. However in worse case, APX is always better than DAPX.

4 Multiple Items

In this chapter, we consider a single additive buyer with valuations for m items. We start with a simple setting by assuming these m items are independently identical distributed items. First challenge for us is to determine the optimal revenue, the robust paper simply use the welfare bound in the proof for APX upper bound. First one, we are interested in $\mathrm{OPT}(F)$, we propose two feasible deterministic mechanisms to evaluate the lower bound of $\mathrm{OPT}(F)$ in section Section 4.1

4.1 Optimal Revenue

In paper...., the optimal revenue is proved up to 5 uniform i.i.d items. Thus even when the items are i.i.d distributed, it is difficult to determine the optimal revenue. We know the upper bound of the optimal revenue is the optimum welfare, $\mathrm{OPT}(F) \leqslant \mathrm{VAL}(F)$. Now we propose two possible "optimal" auction mechanisms for a single additive bidder with m i.i.d items, and we run our experiment according to these two auction mechanisms with different probability distribution. Then we will get two lower bounds for the optimal revenue.
Let us denote $A1$ is the mechanism by selling m items separately, i.e. perform m Myerson optimal auctions. Since the bidder's valuation for the items are i.i.d distributed, we have $\mathrm{REV}(A1, F) = \sum_{j=1}^{m} \mathrm{REV}(p_{opt}, F) = m\mathrm{REV}(p_{opt}, F)$, thus we only need to run our auction experiment algorithm once with the optimal reserve price for the given probability distribution, which we have already done in Section 2. Multiplying the experimental expected revenue with m we get the expected revenue for $A1$ mechanism. Following table shows different distributions we want to test and the optimal reserve price we will use in the experiment and the corresponding expression of $\mathrm{REV}(F)$. The next table shows the tested specific distributions and the corresponding p_{opt} and the experimental values for $\mathrm{OPT}(F)$

Distribution	**optimal reserve price** p_{opt}	**REV**(F) **for m items**
Uniform distribution	$\max\{a, \frac{b}{2}\}$	$m \cdot p_{opt}(1 - \frac{p_{opt}-a}{b-a})$
Truncated normal distribution	$\psi_{TN}^{-1}(0)$	$m \cdot p_{opt}(1 - F_{TN}(F))$
Pareto distribution $(x \in [1, \infty))$	1	m
Pareto distribution $(x \in [0, \infty))$	$\frac{1}{c-1}$	$m \cdot \frac{(c-1)^{c-1}}{c^c}$

Table 4.1: A1: Optimal revenue expression

Distribution	**optimal reserve price** p_{opt}	**Experimental REV**(F) **for 1 item**
Uniform distribution	$\max\{a, \frac{b}{2}\}$	$m \cdot p_{opt}(1 - \frac{p_{opt}-a}{b-a})$
Truncated normal distribution	$\psi_{TN}^{-1}(0)$	$m \cdot p_{opt}(1 - F_{TN}(F))$
Pareto distribution $(x \in [1, \infty))$	1	m
Pareto distribution $(x \in [0, \infty))$	$\frac{1}{c-1}$	$m \cdot \frac{(c-1)^{c-1}}{c^c}$

Table 4.2: A1: Experimental Optimal revenue results

Let us denote $A2$ is the mechanism by selling m items all together, i.e. perform one Myerson optimal auction. To compute the expected revenue for $A2$ we can run our auction experiment algorithm and within the algorithm we generate m random numbers parallel from the probability distribution F and treat the sum of these m random numbers as one bid.

4.2 DAPX and APX

therefore we can try to evaluate the upper bound of APX. Let's denote our auction mechanism is A, and valuation distribution F (i.e. a unifrom distribution) with mean μ and standard deviation σ, then the robust approximation ratio of selling m independently (μ, σ) - distributed items is:

$$\mathrm{APX}(\vec{\mu}, \vec{\sigma}) = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(A; F)} \leqslant \frac{m\mu}{\mathrm{REV}(A; F)}$$

From the Robust paper, it proposes two feasible randomization auction mechanisms, which are:

- $A1$ is the mechanism that sells independently each i.i.d item using log-lottery

$$\mathrm{APX}(\vec{\mu},\vec{\sigma}) = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(A1;F)} \leqslant \frac{m\mu}{m\mathrm{REV}(P^{log}_{\mu,\sigma};F)} = \frac{\mu}{\mathrm{REV}(P^{log}_{\mu,\sigma};F)}$$

- $A2$ is the mechanism that sells all m items together, i.e. price their sum of valuations Y and using log-lottery

$$\mathrm{APX}(\vec{\mu},\vec{\sigma}) = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(A2;F_Y)} \leqslant \frac{m\mu}{\mathrm{REV}(P^{log}_{\bar{\mu},\bar{\sigma}};F_Y)}$$

Robust paper does not mention it explicitly, but we can also consider another two feasible deterministic auction mechanisms

- $A3$ is the mechanism that sells independently each i.i.d item using a fixed reserve price p_D, where $p_D = \frac{\rho_D(r)}{2\rho_D(r)-1}\cdot\mu$, and $r = \frac{\sigma}{\mu}$

$$\mathrm{DAPX}(\vec{\mu},\vec{\sigma}) = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(A3;F)} \leqslant \frac{m\mu}{m\mathrm{REV}(p_D;F)} = \frac{\mu}{\mathrm{REV}(p_D;F)}$$

- $A4$ is the mechanism that sells all m items together using a fixed reserve price $\bar{p}_D$, where $\bar{p}_D = \frac{\rho_D(\bar{r})}{2\rho_D(\bar{r})-1}\cdot\bar{\mu}$, and $\bar{r} = \frac{\bar{\sigma}}{\bar{\mu}}$, i.e. price their sum of valuations Y and

$$\mathrm{DAPX}(\vec{\mu},\vec{\sigma}) = \frac{\mathrm{OPT}(F)}{\mathrm{REV}(A4;F_Y)} \leqslant \frac{m\mu}{\mathrm{REV}(\bar{p}_D;F_Y)}$$

5 Additional notes

5.1 truncated normal distribution part

$$f_t(x) = \begin{cases} kf(x) & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a normalizing constant. We can determine k by using the knowledge of the sum of the probablity equals to 1. Then we have:

$$1 = \int_R f_t(x)dx = \int_0^\infty kf(x)dx = k \cdot \int_0^\infty f(x)dx$$

$$\implies k = \frac{1}{\int_0^\infty f(x)dx} = \frac{1}{1-F(0)}$$

Here $x \geqslant 0$, then the coresponding cdf is:

$$F_t(x) = \int_0^x kf(t)dt = k \cdot \int_0^x f(t)dt = k(F(x) - F(0))$$

question to answer:
1.two random variables, X,Y, let Y = c*X where c>0 constant. let F_X, f_X be the cdf, pdf of X what is the relation of F_Y, f_y to F_X, f_X
$F_Y(y) = P(Y \geqslant y) = P(cX \leqslant y) = P(X \leqslant \frac{y}{c}) = F_X(\frac{y}{c})$
we know $f_Y(y) = f_X(g^{-1}(y))|\frac{dg^{-1}(y)}{dy}|$ thus:
$f_Y(y) = f_X(\frac{y}{c}) \cdot |\frac{1}{c}| = \frac{1}{c}f_X(\frac{y}{c})$
2. what is the relation of myerson(Y) and myerson(X)
Myerson(X) optimal revenue can be achieved by a deterministic mechanism, let denot v_x of optimal reverse price for X, then OPT(X) = $v_x(1 - F_X(v_x))$, what if optimal reserve price for Y? we can compute optimla reserve price by setting virtual valuation for Y equal to 0:

$$v - \frac{1 - F_X(\frac{v}{c})}{\frac{1}{c}f_X(\frac{v}{c})} = 0$$

$$\frac{v}{c} - \frac{1 - F_X(\frac{v}{c})}{f_X(\frac{v}{c})} = 0$$

set $v^{'} = \frac{v}{c}$, above equation has solution v_x. Then denote optimal reserve price for Y: $v_y = cv_x$ then OPT(Y) = $v_y(1 - F_Y(v_y)) = cv_x(1 - F_X(\frac{cv_x}{c})) = cv_x(1 - F_X(v_x))$.

Therefore OPT(Y) = cOPT(X) 3. what is the relation of REV(Y) and REV(x), given v as reserve price
REV(Y) $= v(1 - F_Y(v)) = v(1 - F_X(\frac{v}{c}))$
REV(X) $= v(1 - F_X(v))$

$$v(1 - F_X(\frac{v}{c})) \overset{0<c<1}{<} v(1 - F_X(v))$$
$$\overset{c=1}{=}$$
$$\overset{c>1}{>}$$

Experiment Evaluation of Robust Revenue-Maximizing Auctions

6 Conclusion

A Remarks on Implementation

In the appendix you can include e. g. computer codes or further remarks which would disturb the flow of the main text. If you do not need an appendix, you can simply leave out this file (in which case you should also delete the `\include` command in `thesis.tex`).

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All figures in this document were created by the author using TikZ, the excellent TEX-package by Till Tantau, see [**Tantau2007**].

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