Mixed Models

Accounting for clustered/correlated data

ERHS 732

Assumptions for linear models

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  - **Independence**: The Y_i's are independent random variables

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- This correlation has to be accounted for in the model by assuming a certain covariance structure

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- In a traditional linear regression model we assume that any two observations are independent and the covariance is zero. However with correlated data $\sigma_{12} \neq 0$

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- Mixed models separate the types of effects into 'fixed' and 'random', but also quantify the contributions of these types of variation

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- For each participant the intercept is $\beta_0 + b_i$, or in other words the intercept varies randomly around β_0 by a factor of b_i

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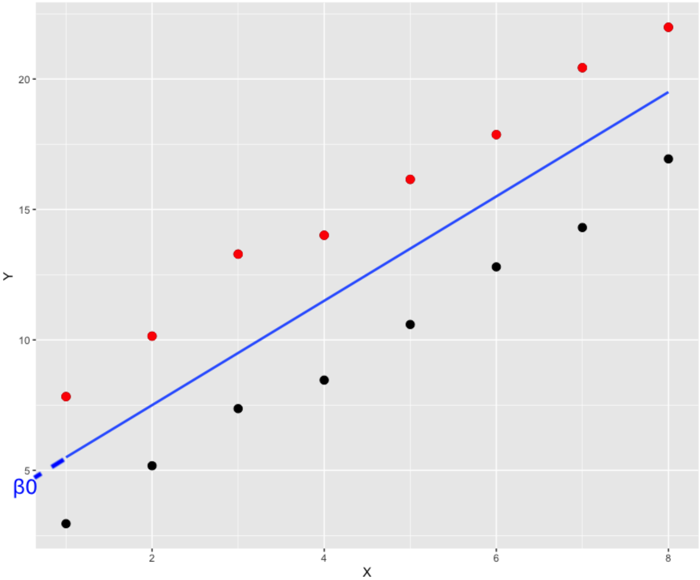

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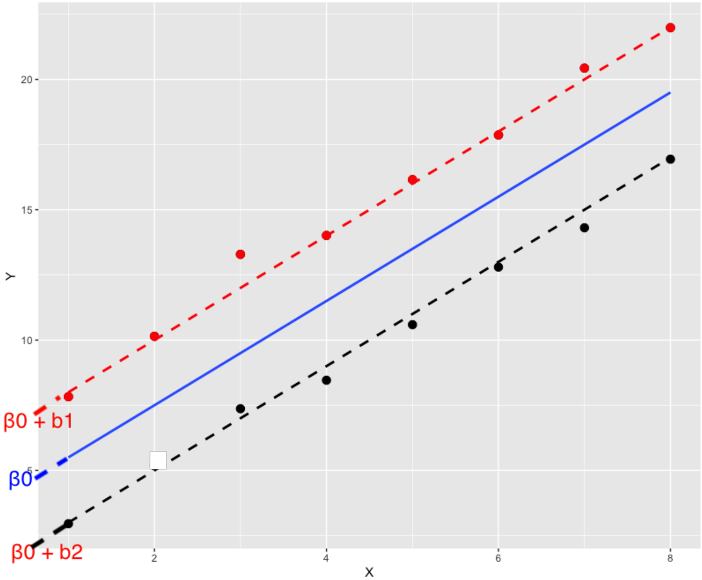

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- The total variance is $\sigma_b^2 + \sigma^2$ and σ_b^2 represents between-participant variability while σ^2 is within-participant variability
- σ_b^2 also represents the covariance in this structure and the correlation between two observations from the same participant is $\rho = \frac{\sigma_b^2}{\sigma_b^2 + \sigma^2}$

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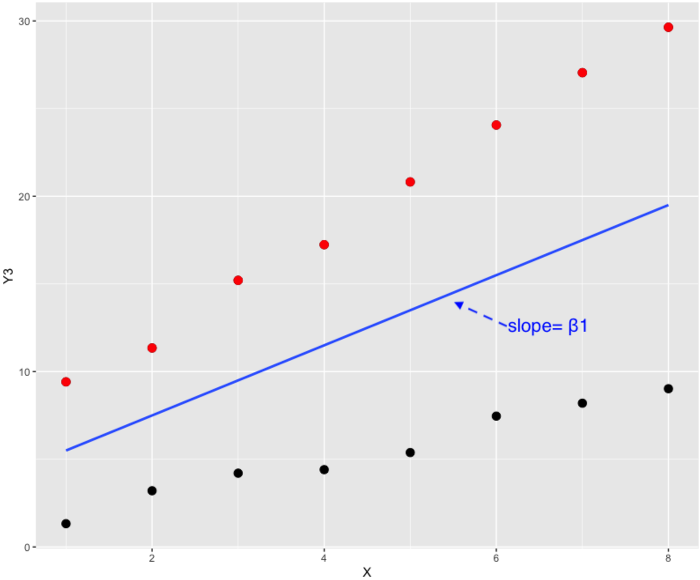

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