University of Texas at Austin, Cockrell School of Engineering
Data Mining – EE 380L

Problem Set # 2 (2b)
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Discussed Homework with Following Students:
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Q1] Part a]
FE (Fuel Efficiency) is a good candidate for dependent variable.
EngDispl, NumCyl, Transmission, NumGears, IntakeValvePerCyl, ExhaustValvesPerCyl are continuous variables.
AirAspirationMethod, TransLockup, TransCreeperGear, DriveDesc, CarlineClassDesc, VarValveTiming, VarValveLift are categorical variables.

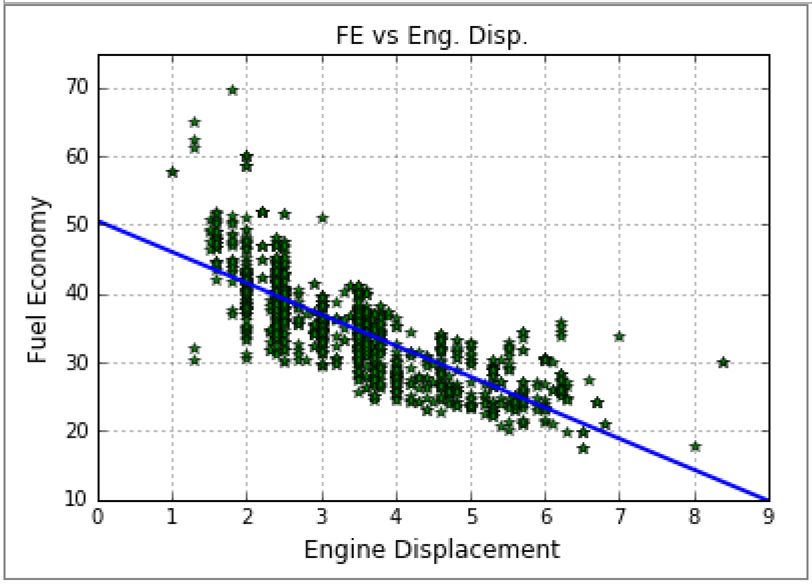
Part b]

In [5]:
```
#Part b
# http://docs.scipy.org/doc/numpy-1.10.0/reference/generated/numpy.linalg.lstsq.html
# B1 = m; B0 = c
# y = B1x + B0
# y = Ap, where A = [[x 1]] and p = [[B1], [B0]]
x = df["EngDispl"]
A = np.vstack([x, np.ones(len(x))]).T
#print A
y = np.array(df["FE"])
#m, c = np.linalg.lstsq(A, y)[0]
#B1, B0 = np.linalg.lstsq(A, y)[0]
model, resid = np.linalg.lstsq(A, y)[:2]
B1, B0 = model
#np.linalg.lstsq(A, y)
print("m(B1) is: {}.".format(round(B1,8)))
print("c(B0) is: {}.".format(round(B0,8)))
r2 = 1 - resid / (y.size * y.var())
print("R2 is: {}.".format(round(r2[0],8)))
```

```
m(B1) is: -4.52092928.
c(B0) is: 50.56322991.
R2 is: 0.61998904.
```

In [6]:
```
#Part b
# Add bestfit line to ScatterPlot using B1 and B0 values
# y = B1x + B0

part_b, = plt.plot(df["EngDispl"],df["FE"],'g*')
plt.grid(True)
plt.ylabel("Fuel Economy",fontsize=12)
plt.xlabel("Engine Displacement",fontsize=12)
plt.title("FE vs Eng. Disp.")
plt.xlim(0, 9)
plt.ylim(10, 75)
xVals = np.arange(0, 325)
yVals = (B1*xVals) + B0
plt.plot(xVals,yVals,'b-',linewidth=2.0)
plt.show()
```



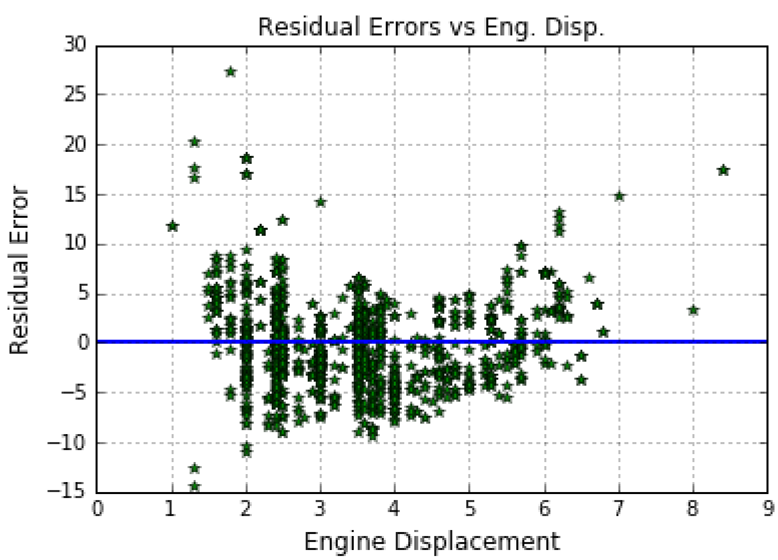
Part c]

```
In [7]: # Part C
        # Plot the residual errors. Now sum the residual errors. What do you find?
        # compute the predicted value for EngDispl
        def predictedVals(coeff,intercept,engDisp):
            y_vals=list()
            for x in np.nditer(engDisp):
                y_vals.append((coeff * x )+ intercept)
            return y_vals
```

```
In [8]: yPredict=predictedVals(B1,B0,df.as_matrix(["EngDispl"]))
```

```
In [9]: # compute residual errors
        residualErrors=[]
        Y_FE = df.as_matrix(["FE"])
        for i in np.arange(len(Y_FE)):
            residualErrors.append(Y_FE[i]-yPredict[i])
```

```
In [10]: part_c, = plt.plot(df["EngDispl"],residualErrors,'g*')
         plt.grid(True)
         plt.ylabel("Residual Error",fontsize=12)
         plt.xlabel("Engine Displacement",fontsize=12)
         plt.title("Residual Errors vs Eng. Disp.")
         plt.xlim(0, 9)
         xVals1 = np.arange(0, 325)
         yVals1 = (0*xVals)
         plt.plot(xVals1,yVals1,'b-',linewidth=2.0)
         plt.show()
```




```
In [11]: #Sum of residual errors
         print "Sum of Residual Errors: ",np.sum(residualErrors)

         Sum of Residual Errors:  -2.2055246518e-11
```

Part d]

In [12]:
```
# Part D
# Now do the same where you fit a linear regression for FE on both EngDispl and NumCyl.
# Report the R2 value. Sum the residual errors. What do you find?
X1 = df.as_matrix(["EngDispl","NumCyl"])
FE_Y = df.as_matrix(["FE"])
model_lr = linear_model.LinearRegression()
model_lr.fit(X1,FE_Y)
print "Coeff, B0:", model_lr.coef_, model_lr.intercept_
print "r2 = ", model_lr.score(X1,FE_Y)
```

```
Coeff, B0: [[-3.74535214 -0.58802919]] [ 51.35414193]
r2 =  0.623958939799
```

In [13]:
```
def predictedVals1(coeff,intercept,X):
    y_vals1=list()
    for x in np.arange(len(X)):
        y_vals1.append(np.dot(coeff,X[x]) + intercept )
    return y_vals1
```

In [14]:
```
#yPredict=predictedVals(B1,B0,df.as_matrix(["EngDispl"]))
yPredict_D=predictedVals1(model_lr.coef_,model_lr.intercept_,X1)
```

In [15]:
```
# compute residual errors
residualErrors_D=[]
for i in np.arange(len(FE_Y)):
    residualErrors_D.append(FE_Y[i]-yPredict_D[i])
```

In [16]:
```
#Sum of residual errors
print "Sum of Residual Errors: ",np.sum(residualErrors_D)
```

```
Sum of Residual Errors:  0.0
```

Part e]

In [17]:
```
#PART E
#Now solve the OLS regression problem for FE against all the variables. Report the R2
#value. Sum the residual errors. What do you find?
```

In [18]:
```
X1 = df.as_matrix(["EngDispl","NumCyl","Transmission","NumGears","IntakeValvePerCyl",
                   "ExhaustValvesPerCyl","AirAspirationMethod","TransLockup",
                   "TransCreeperGear","DriveDesc","CarlineClassDesc","VarValveTiming",
                   "VarValveLift"])
FE_Y = df.as_matrix(["FE"])
model_lr = linear_model.LinearRegression()
model_lr.fit(X1,FE_Y)
print "Coeff, B0:", model_lr.coef_, model_lr.intercept_
print "r2 = ", model_lr.score(X1,FE_Y)
```

```
Coeff, B0: [[-3.73881937 -0.59854429  0.09853067 -0.26103154 -0.46677908 -1.41290715
  -0.32029424 -0.74947468 -0.75500417  0.83699599 -0.27189826  1.63506899
   0.8440168 ]] [ 54.26927826]
r2 =  0.707406484867
```

In [19]:
```
yPredict_E=predictedVals1(model_lr.coef_,model_lr.intercept_,X1)
```

In [20]:
```
# compute residual errors
residualErrors_E=[]
for i in np.arange(len(FE_Y)):
    residualErrors_E.append(FE_Y[i]-yPredict_E[i])
```

In [21]:
```
#Sum of residual errors
print "Sum of Residual Errors: ",np.sum(residualErrors_D)
```

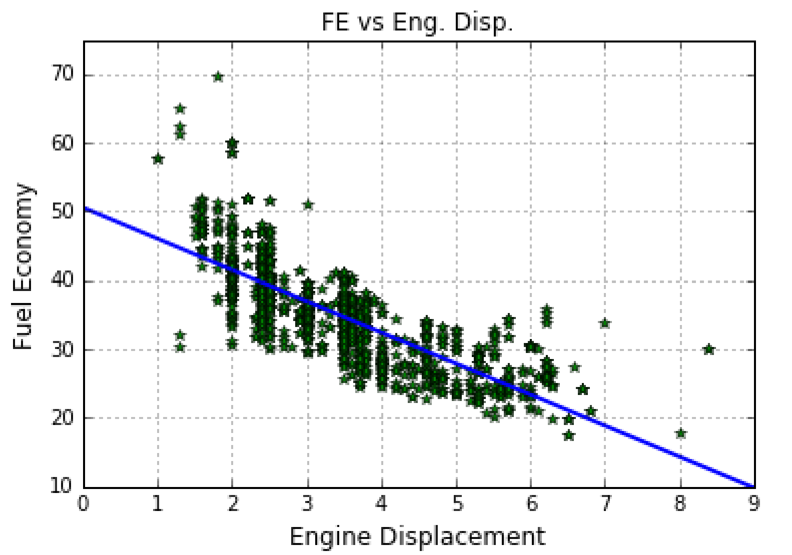
```
Sum of Residual Errors:  0.0
```

Q2. Part A]

```
In [4]: df=pd.read_stata("cars2010.dta")
```

```
In [5]: X = df["EngDispl"]
        A = np.vstack([X, np.ones(len(X))]).T
        #print A
        y = np.array(df["FE"])
        #m, c = np.linalg.lstsq(A, y)[0]
        #B1, B0 = np.linalg.lstsq(A, y)[0]
        model, resid = np.linalg.lstsq(A, y)[:2]
        B1, B0 = model
        #np.linalg.lstsq(A, y)
        print("m(B1) is: {}.".format(round(B1,8)))
        print("c(B0) is: {}.".format(round(B0,8)))
        r2 = 1 - resid / (y.size * y.var())
        print("R2 is: {}.".format(round(r2[0],8)))
```

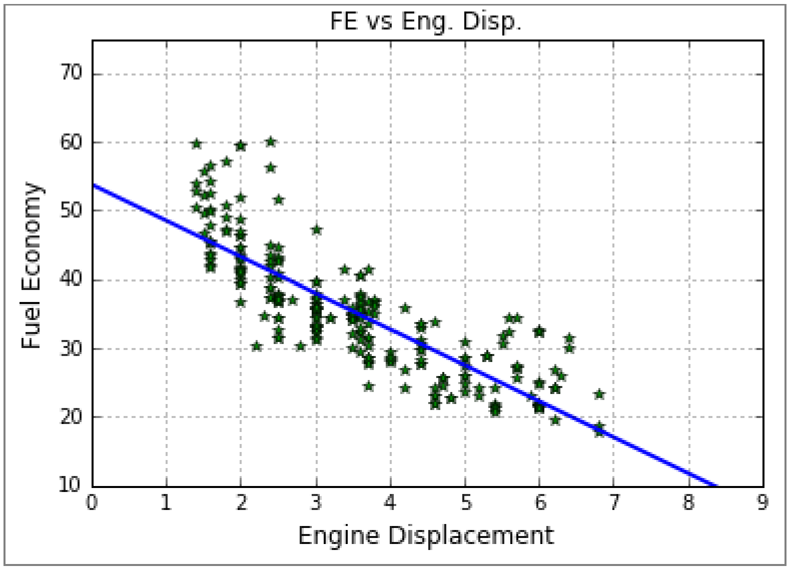
```
m(B1) is: -4.52092928.
c(B0) is: 50.56322991.
R2 is: 0.61998904.
```




```
In [7]: df2=pd.read_stata("cars2011.dta")
```

```
In [9]: X = df2["EngDispl"]
        A = np.vstack([X, np.ones(len(X))]).T
        #print A
        y = np.array(df2["FE"])
        #m, c = np.linalg.lstsq(A, y)[0]
        #B1, B0 = np.linalg.lstsq(A, y)[0]
        model, resid = np.linalg.lstsq(A, y)[:2]
        B1, B0 = model
        #np.linalg.lstsq(A, y)
        print("m(B1) is: {}.".format(round(B1,8)))
        print("c(B0) is: {}.".format(round(B0,8)))
        r2 = 1 - resid / (y.size * y.var())
        print("R2 is: {}.".format(round(r2[0],8)))
```

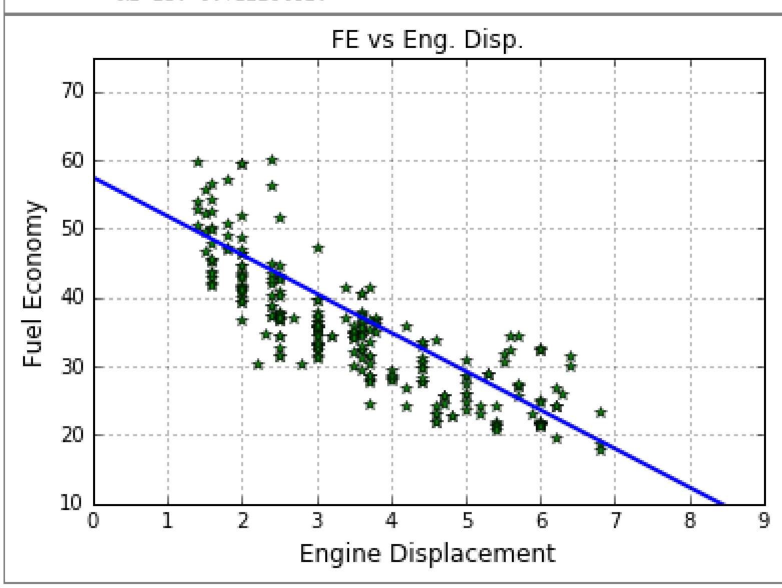
```
m(B1) is: -5.24210076.
c(B0) is: 53.78837489.
R2 is: 0.70186418.
```




```
In [11]: df3=pd.read_stata("cars2012.dta")
```

```
In [12]: X = df3["EngDispl"]
         A = np.vstack([X, np.ones(len(X))]).T
         #print A
         y = np.array(df3["FE"])
         #m, c = np.linalg.lstsq(A, y)[0]
         #B1, B0 = np.linalg.lstsq(A, y)[0]
         model, resid = np.linalg.lstsq(A, y)[:2]
         B1, B0 = model
         #np.linalg.lstsq(A, y)
         print("m(B1) is: {}.".format(round(B1,8)))
         print("c(B0) is: {}.".format(round(B0,8)))
         r2 = 1 - resid / (y.size * y.var())
         print("R2 is: {}.".format(round(r2[0],8)))
```

```
m(B1) is: -5.63071065.
c(B0) is: 57.47228974.
R2 is: 0.71225482.
```



Q2 Part B]

Given:

$\mathrm{E}[w_i] = 0$

$\mathrm{E}[w_i w_k] = \mathrm{E}[w_i]\mathrm{E}[w_k] = 0$

$\mathrm{E}\left[w_i^2\right] = \sigma^2$

$$
\begin{aligned}
Cov(w) = {} & \mathrm{E}[ww^T] \\
& = \mathrm{E}\left[\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \begin{pmatrix} w_1 & \cdots & w_n \end{pmatrix}\right] \\
& = \mathrm{E}\begin{bmatrix} w_1^2 & w_1 w_i & w_i w_n \\ w_j w_1 & \ddots & w_j w_n \\ w_n w_1 & w_n w_i & w_n^2 \end{bmatrix} \\
& = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \\
& = \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \sigma^2 I
\end{aligned}
$$

Q2 Part C]

```
In [13]: #Part C
df=pd.read_stata("cars2010.dta")
df1=pd.read_stata("cars2011.dta")
df2=pd.read_stata("cars2012.dta")
X_2010 = df.as_matrix(["EngDispl","NumCyl"])
X_2011 = df1.as_matrix(["EngDispl","NumCyl"])
X_2012 = df2.as_matrix(["EngDispl","NumCyl"])
FE_Y_2010 = df.as_matrix(["FE"])
FE_Y_2011 = df1.as_matrix(["FE"])
FE_Y_2012 = df2.as_matrix(["FE"])
```

```
In [14]: X0 = np.column_stack((X_2010,np.ones(FE_Y_2010.shape[0])))
X1 = np.column_stack((X_2011,np.ones(FE_Y_2011.shape[0])))
X2 = np.column_stack((X_2012,np.ones(FE_Y_2012.shape[0])))
```

```
In [15]: # Estimate Sigma
def sigma_estimate(B,X,Y):
    b0 = B[2]
    b1 = B[1]
    b2 = B[0]
    sum = 0.0
    for i in np.arange(len(X)):
        sum += np.square(Y[i]-(b0 + np.dot(X[i][0],b1) + np.dot(X[i][1],b2)))
    return (sum / (len(X) - 2))
```

```
In [16]: # Compute Beta_hat 2010
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X0.transpose(),X0)),X0.transpose()),FE_Y_2010)
         print B_hat

         [[ -3.74535214]
          [ -0.58802919]
          [ 51.35414193]]
```

```
In [17]: print "Sigma Estimate Cars 2010: ", sigma_estimate(B_hat,X0,FE_Y_2010)

         Sigma Estimate Cars 2010:  [ 89.94853481]
```

```
In [18]: # Compute Beta_hat 2011
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X1.transpose(),X1)),X1.transpose()),FE_Y_2011)
         print B_hat

         [[ -3.96769213]
          [ -1.13215237]
          [ 56.10527854]]
```

```
In [19]: print "Sigma Estimate Cars 2011: ", sigma_estimate(B_hat,X1,FE_Y_2011)

         Sigma Estimate Cars 2011:  [ 80.00672933]
```

```
In [20]: # Compute Beta_hat 2012
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X2.transpose(),X2)),X2.transpose()),FE_Y_2012)
         print B_hat

         [[ -5.29556177]
          [ -0.27929841]
          [ 58.00480858]]
```

```
In [21]: print "Sigma Estimate Cars 2012: ", sigma_estimate(B_hat,X2,FE_Y_2012)

         Sigma Estimate Cars 2012:  [ 213.23969024]
```

Q2 Part D]

```
In [22]: #PART D
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X0.transpose(),X0)),X0.transpose()),FE_Y_2010)
         sigma_2010 = sigma_estimate(B_hat,X0,FE_Y_2010)
         cov_B = (np.square(sigma_2010)) * (np.linalg.inv(np.dot(X0.transpose(),X0)))
         print cov_B

         [[ 24.00506369 -14.94797628   5.06021902]
          [-14.94797628  11.33329747 -15.24353096]
          [  5.06021902 -15.24353096  80.58099929]]
```

Q2 Part E]

I think there is some relationship between the standard deviation for each hear and the computed beta hat. The 2012 dataset had the highest standard deviation of the three datasets.

Q4]

PROBABILITY SECTION

4. For X, Y, and Z random variables, the *Covariance Matrix* of the random vector (X, Y, Z) is defined as:

$$\mathbb{E}\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\begin{pmatrix} X & Y & Z \end{pmatrix}\right] = \mathbb{E}\begin{bmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{bmatrix}.$$

Suppose X, Y and Z represent the result of rolling three independent dice. What is the covariance matrix of (X, Y, Z)?

PMF of rolling a dice

$$f_X(y) = \begin{cases} \frac{1}{6} & if\ y \in \{1,2,3,4,5,6\} \\ 0 & otherwise \end{cases}$$

$$\mu_X = E(X) = \sum_{y=1}^{6} y \cdot f_Z(y) = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right)$$

$$\mu_X = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

$$\sigma_X{}^2 = var(X) = \textstyle\sum_{y=1}^{6}(y - \mu_X)^2 \cdot f_Z(y)$$

$$\sigma_X{}^2 = \left(1-\frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(2-\frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(3-\frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(4-\frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(5-\frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(6-\frac{7}{2}\right)^2\left(\frac{1}{6}\right)$$

$$\sigma_X{}^2 = \left(-\frac{5}{2}\right)^2\left(\frac{1}{6}\right) + \left(-\frac{3}{2}\right)^2\left(\frac{1}{6}\right) + \left(-\frac{1}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{3}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{2}\right)^2\left(\frac{1}{6}\right)$$

$$\sigma_X{}^2 = \left(\frac{25}{24}\right) + \left(\frac{9}{24}\right) + \left(\frac{1}{24}\right) + \left(\frac{1}{24}\right) + \left(\frac{9}{24}\right) + \left(\frac{25}{24}\right) = \left(\frac{70}{24}\right) = \frac{35}{12} = 2\frac{11}{12}$$

Rules:

$E(XY) = E(X)E(Y)$

$E(X \pm a) = E(X) \pm a$

Covariance Matrix is Σ

$$\Sigma = \begin{pmatrix} E(X-3.5)^2 & E(X-3.5)(Y-3.5) & E(X-3.5)(Z-3.5) \\ E(X-3.5)(Y-3.5) & E(Y-3.5)^2 & E(Y-3.5)(Z-3.5) \\ E(X-3.5)(Z-3.5) & E(Y-3.5)(Z-3.5) & E(Z-3.5)^2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2\frac{11}{12} & E(X-3.5)E(Y-3.5) & E(X-3.5)E(Z-3.5) \\ E(X-3.5)E(Y-3.5) & 2\frac{11}{12} & E(Y-3.5)E(Z-3.5) \\ E(X-3.5)E(Z-3.5) & E(Y-3.5)E(Z-3.5) & 2\frac{11}{12} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2\frac{11}{12} & 0 & 0 \\ 0 & 2\frac{11}{12} & 0 \\ 0 & 0 & 2\frac{11}{12} \end{pmatrix}$$

Q7]

7. Consider flipping a fair coin. Let Z denote the random variable that is the number of Heads that come up in a row. Thus, if the first flip comes up tails, $Z = 0$. If the flip sequence is

$$HHTHHHHT....$$

then $Z = 2$, and so on.

- Write the probability mass function of Z.
- Compute the mean and variance of Z.

For a Fair Coin.

P(H) = $p = \frac{1}{2}$

P(T) = $(1-p) = \frac{1}{2}$

Random Variable

Compound Event	Elementary Event	Probability
(Z=0)	(T..) =	$\frac{1}{2}$
(Z=1)	(HT..)	$\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right) = \frac{1}{4}$
(Z=2)	(HHT..)	$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{8}$
(Z=3)	(HHHT..)	$\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{16}$
(Z=4)	(HHHHT..)	$\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$

PMF:

y	$f_Z(y)$	
0	$\frac{1}{2}$	$p^0(1-p)$
1	$\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right) = \frac{1}{4}$	$p^1(1-p)$
2	$\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{8}$	$p^2(1-p)$
3	$\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{16}$	$p^3(1-p)$
4	$\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$	$p^4(1-p)$

So **PMF** for Z is $f_Z(y) = p^y(1-p)$ or for fair coin where $p = (1-p)$

$$f_Z(y) = p^{y+1} = \left(\tfrac{1}{2}\right)^{y+1} \qquad \text{where } y = 0,1,2,\ldots,n$$

Compute Mean & Variance:

$$\mu = E(Z) = \sum_{y=0}^{n} y \cdot f_Z(y) = \sum_{y=0}^{n} y \cdot \left(\frac{1}{2}\right)^{y+1} = \sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y+1} = \frac{1}{4}\sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y-1}$$

From http://www.trans4mind.com/personal_development/mathematics/series/airthmeticGeometricSeries.htm

$$\boxed{1 + 2\,r + 3\,r^2 \ldots n\,r^{(n-1)} = \frac{\left(1 - (n+1)\,r^n + n\,r^{(n+1)}\right)}{(1-r)^2}}$$

$r = \frac{1}{2}$

$$\mu = E(Z) = \frac{1}{4}\sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y-1} = \frac{1}{4}\frac{\left(1 - (n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}\right)}{\left(\frac{1}{2}\right)^2}$$

$$\mu = \frac{1}{4}\frac{\left(1 - (n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}\right)}{\frac{1}{4}} = \left(1 + n\left(\frac{1}{2}\right)^{n+1} - (n+1)\left(\frac{1}{2}\right)^n\right) =$$

$$\mu = \left(1 + n\left(\frac{1}{2}\right)^{n+1} - (n+1)\left(\frac{1}{2}\right)^n\right) = \left(1 + \left(\frac{1}{2}\right)n\left(\frac{1}{2}\right)^n - (n+1)\left(\frac{1}{2}\right)^n\right)$$

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, then $\mu = 1$

Variance:
Use:

Properties of Variance

- $\mathrm{var}(X) = E\{(X - E[X])^2\} = E[X^2] - (E[X])^2$

Take advantage of:
$E[Z^2] = E[Z(Z-1)] + E[Z]$

Compute:
$E[Z(Z-1)] = \sum_{y=1}^{n} y(y-1) \cdot \left(\frac{1}{2}\right)^{y+1}$

$\sigma_Z{}^2 = E[(Z-\mu)^2] = E[Z^2] - (E[Z])^{\,2} = E[Z^2] - \mu^{\,2}$

$$\sum_{y=0}^{n}(y-\mu)^2 f_Z(y) = \sum_{y=0}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1} = \sum_{y=0}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1}$$

$$\sigma_Z{}^2 = (0-1)^2\left(\frac{1}{2}\right)^{0+1} + \sum_{y=1}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1} = \frac{1}{2} + \sum_{y=1}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1}$$

Use from http://math2.org/math/expansion/power.htm:

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \quad \rightarrow \quad \sum_{j=1}^{n}(j-1)^2 = (1-1)^2 + \sum_{j=2}^{n}(j-1)^2 = \sum_{k=1}^{n} k^2$

Use from http://math2.org/math/expansion/geom.htm:

$$\sum_{k=1}^{n} k^2$$

Revisit with:
http://arnoldkling.com/apstats/geometric.html