# University of Texas at Austin, Cockrell School of Engineering
# Data Mining – EE 380L

## Problem Set # 2 (2b)

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Discussed Homework with Following Students:
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**Q1] Part a]**

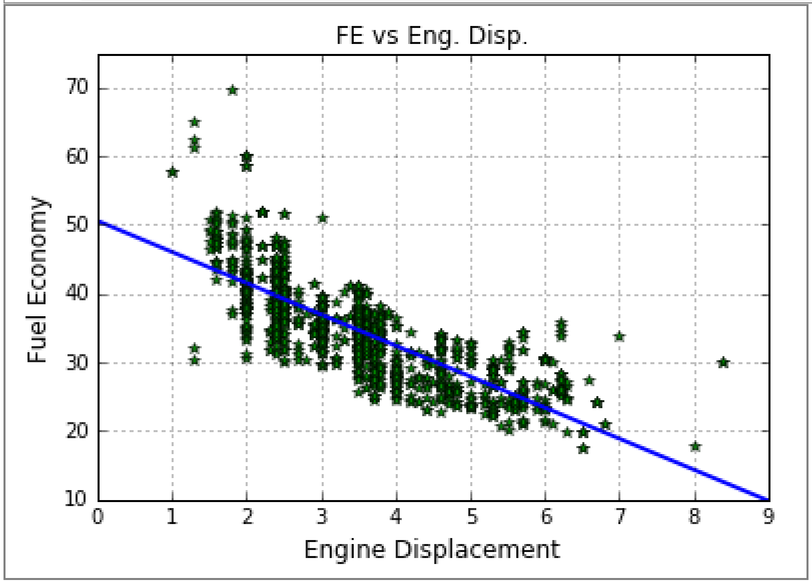
FE (Fuel Efficiency) is a good candidate for dependent variable.
EngDispl, NumCyl, Transmission, NumGears, IntakeValvePerCyl, ExhaustValvesPerCyl are continuous variables.
AirAspirationMethod, TransLockup, TransCreeperGear, DriveDesc, CarlineClassDesc, VarValveTiming, VarValveLift are categorical variables.

**Part b]**

In [5]:
```
#Part b
# http://docs.scipy.org/doc/numpy-1.10.0/reference/generated/numpy.linalg.lstsq.html
# B1 = m; B0 = c
# y = B1x + B0
# y = Ap, where A = [[x 1]] and p = [[B1], [B0]]
x = df["EngDispl"]
A = np.vstack([x, np.ones(len(x))]).T
#print A
y = np.array(df["FE"])
#m, c = np.linalg.lstsq(A, y)[0]
#B1, B0 = np.linalg.lstsq(A, y)[0]
model, resid = np.linalg.lstsq(A, y)[:2]
B1, B0 = model
#np.linalg.lstsq(A, y)
print("m(B1) is: {}.".format(round(B1,8)))
print("c(B0) is: {}.".format(round(B0,8)))
r2 = 1 - resid / (y.size * y.var())
print("R2 is: {}.".format(round(r2[0],8)))
```

```
m(B1) is: -4.52092928.
c(B0) is: 50.56322991.
R2 is: 0.61998904.
```

In [6]:
```
#Part b
# Add bestfit line to ScatterPlot using B1 and B0 values
# y = B1x + B0

part_b, = plt.plot(df["EngDispl"],df["FE"],'g*')
plt.grid(True)
plt.ylabel("Fuel Economy",fontsize=12)
plt.xlabel("Engine Displacement",fontsize=12)
plt.title("FE vs Eng. Disp.")
plt.xlim(0, 9)
plt.ylim(10, 75)
xVals = np.arange(0, 325)
yVals = (B1*xVals) + B0
plt.plot(xVals,yVals,'b-',linewidth=2.0)
plt.show()
```



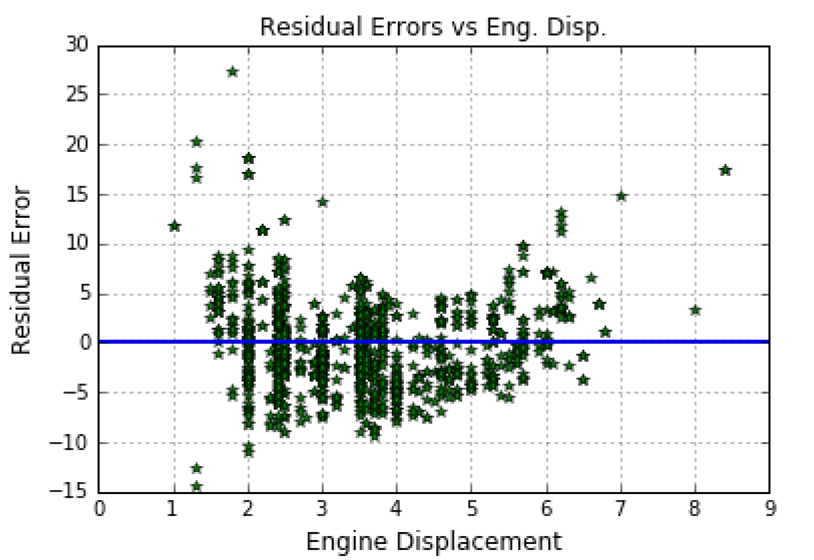
**Part c]**

In [7]:
```
# Part C
# Plot the residual errors. Now sum the residual errors. What do you find?
# compute the predicted value for EngDispl
def predictedVals(coeff,intercept,engDisp):
    y_vals=list()
    for x in np.nditer(engDisp):
        y_vals.append((coeff * x )+ intercept)
    return y_vals
```

In [8]:
```
yPredict=predictedVals(B1,B0,df.as_matrix(["EngDispl"]))
```

In [9]:
```
# compute residual errors
residualErrors=[]
Y_FE = df.as_matrix(["FE"])
for i in np.arange(len(Y_FE)):
    residualErrors.append(Y_FE[i]-yPredict[i])
```

In [10]:
```
part_c, = plt.plot(df["EngDispl"],residualErrors,'g*')
plt.grid(True)
plt.ylabel("Residual Error",fontsize=12)
plt.xlabel("Engine Displacement",fontsize=12)
plt.title("Residual Errors vs Eng. Disp.")
plt.xlim(0, 9)
xVals1 = np.arange(0, 325)
yVals1 = (0*xVals)
plt.plot(xVals1,yVals1,'b-',linewidth=2.0)
plt.show()
```




In [11]:
```
#Sum of residual errors
print "Sum of Residual Errors: ",np.sum(residualErrors)
```

```
Sum of Residual Errors:  -2.2055246518e-11
```

## Part d]

In [12]:
```
# Part D
# Now do the same where you fit a linear regression for FE on both EngDispl and NumCyl.
# Report the R2 value. Sum the residual errors. What do you find?
X1 = df.as_matrix(["EngDispl","NumCyl"])
FE_Y = df.as_matrix(["FE"])
model_lr = linear_model.LinearRegression()
model_lr.fit(X1,FE_Y)
print "Coeff, B0:", model_lr.coef_, model_lr.intercept_
print "r2 = ", model_lr.score(X1,FE_Y)
```

```
Coeff, B0: [[-3.74535214 -0.58802919]] [ 51.35414193]
r2 =  0.623958939799
```

In [13]:
```
def predictedVals1(coeff,intercept,X):
    y_vals1=list()
    for x in np.arange(len(X)):
        y_vals1.append(np.dot(coeff,X[x]) + intercept )
    return y_vals1
```

In [14]:
```
#yPredict=predictedVals(B1,B0,df.as_matrix(["EngDispl"]))
yPredict_D=predictedVals1(model_lr.coef_,model_lr.intercept_,X1)
```

In [15]:
```
# compute residual errors
residualErrors_D=[]
for i in np.arange(len(FE_Y)):
    residualErrors_D.append(FE_Y[i]-yPredict_D[i])
```

In [16]:
```
#Sum of residual errors
print "Sum of Residual Errors: ",np.sum(residualErrors_D)
```

```
Sum of Residual Errors:  0.0
```

## Part e]

In [17]:
```
#PART E
#Now solve the OLS regression problem for FE against all the variables. Report the R2
#value. Sum the residual errors. What do you find?
```

In [18]:
```
X1 = df.as_matrix(["EngDispl","NumCyl","Transmission","NumGears","IntakeValvePerCyl",
                   "ExhaustValvesPerCyl","AirAspirationMethod","TransLockup",
                   "TransCreeperGear","DriveDesc","CarlineClassDesc","VarValveTiming",
                   "VarValveLift"])
FE_Y = df.as_matrix(["FE"])
model_lr = linear_model.LinearRegression()
model_lr.fit(X1,FE_Y)
print "Coeff, B0:", model_lr.coef_, model_lr.intercept_
print "r2 = ", model_lr.score(X1,FE_Y)
```

```
Coeff, B0: [[-3.73881937 -0.59854429  0.09853067 -0.26103154 -0.46677908 -1.41290715
  -0.32029424 -0.74947468 -0.75500417  0.83699599 -0.27189826  1.63506899
   0.8440168 ]] [ 54.26927826]
r2 =  0.707406484867
```

In [19]:
```
yPredict_E=predictedVals1(model_lr.coef_,model_lr.intercept_,X1)
```

In [20]:
```
# compute residual errors
residualErrors_E=[]
for i in np.arange(len(FE_Y)):
    residualErrors_E.append(FE_Y[i]-yPredict_E[i])
```

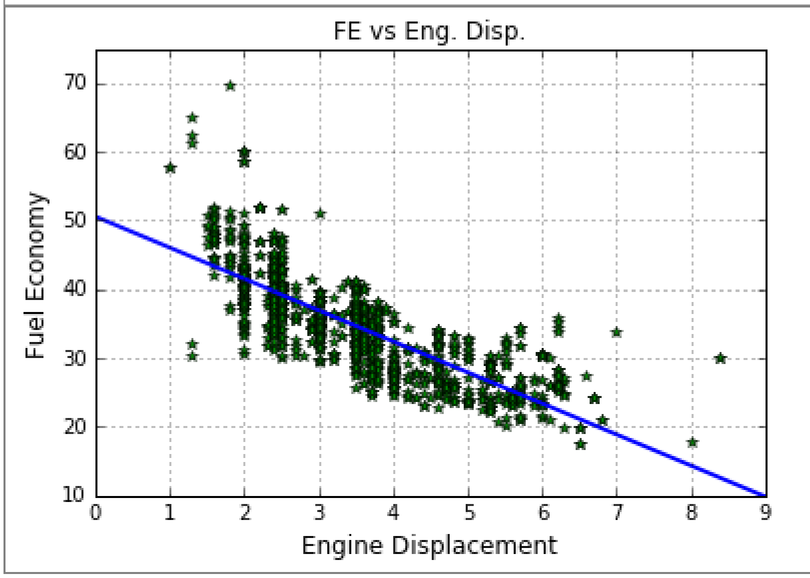
In [21]:
```
#Sum of residual errors
print "Sum of Residual Errors: ",np.sum(residualErrors_D)
```

```
Sum of Residual Errors:  0.0
```

**Q2. Part A]**

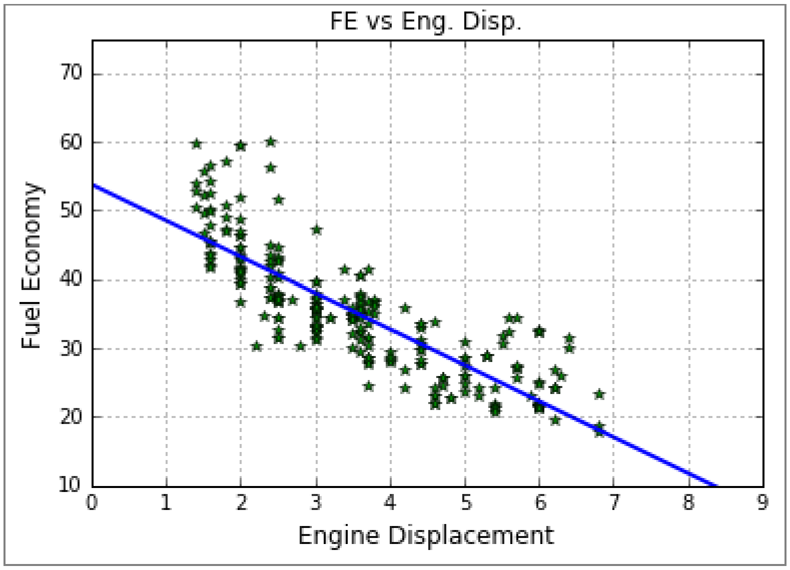
```
In [4]: df=pd.read_stata("cars2010.dta")
```

```
In [5]: X = df["EngDispl"]
        A = np.vstack([X, np.ones(len(X))]).T
        #print A
        y = np.array(df["FE"])
        #m, c = np.linalg.lstsq(A, y)[0]
        #B1, B0 = np.linalg.lstsq(A, y)[0]
        model, resid = np.linalg.lstsq(A, y)[:2]
        B1, B0 = model
        #np.linalg.lstsq(A, y)
        print("m(B1) is: {}.".format(round(B1,8)))
        print("c(B0) is: {}.".format(round(B0,8)))
        r2 = 1 - resid / (y.size * y.var())
        print("R2 is: {}.".format(round(r2[0],8)))

        m(B1) is: -4.52092928.
        c(B0) is: 50.56322991.
        R2 is: 0.61998904.
```




```
In [7]: df2=pd.read_stata("cars2011.dta")
```

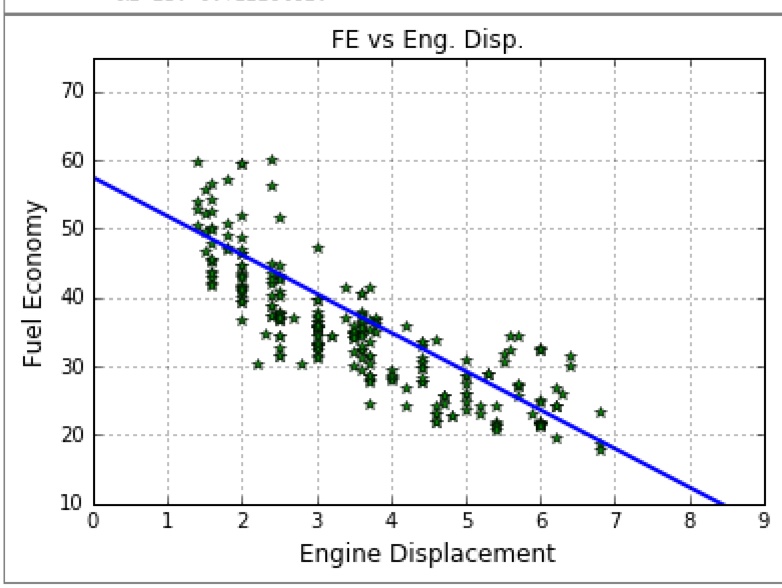
```
In [9]: X = df2["EngDispl"]
        A = np.vstack([X, np.ones(len(X))]).T
        #print A
        y = np.array(df2["FE"])
        #m, c = np.linalg.lstsq(A, y)[0]
        #B1, B0 = np.linalg.lstsq(A, y)[0]
        model, resid = np.linalg.lstsq(A, y)[:2]
        B1, B0 = model
        #np.linalg.lstsq(A, y)
        print("m(B1) is: {}.".format(round(B1,8)))
        print("c(B0) is: {}.".format(round(B0,8)))
        r2 = 1 - resid / (y.size * y.var())
        print("R2 is: {}.".format(round(r2[0],8)))

        m(B1) is: -5.24210076.
        c(B0) is: 53.78837489.
        R2 is: 0.70186418.
```




In [11]:
```
df3=pd.read_stata("cars2012.dta")
```

In [12]:
```
X = df3["EngDispl"]
A = np.vstack([X, np.ones(len(X))]).T
#print A
y = np.array(df3["FE"])
#m, c = np.linalg.lstsq(A, y)[0]
#B1, B0 = np.linalg.lstsq(A, y)[0]
model, resid = np.linalg.lstsq(A, y)[:2]
B1, B0 = model
#np.linalg.lstsq(A, y)
print("m(B1) is: {}.".format(round(B1,8)))
print("c(B0) is: {}.".format(round(B0,8)))
r2 = 1 - resid / (y.size * y.var())
print("R2 is: {}.".format(round(r2[0],8)))
```

```
m(B1) is: -5.63071065.
c(B0) is: 57.47228974.
R2 is: 0.71225482.
```



**Q2 Part B]**

**Given:**

$E[w_i] = 0$

$E[w_i w_k] = E[w_i]E[w_k] = 0$

$E[w_i^2] = \sigma^2$

$$Cov(w) = E[ww^T]$$

$$= E\left[\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} (w_1 \quad \cdots \quad w_n)\right]$$

$$= E\begin{bmatrix} w_1^2 & w_1 w_i & w_i w_n \\ w_j w_1 & \ddots & w_j w_n \\ w_n w_1 & w_n w_i & w_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \sigma^2 I$$

**Q2 Part C]**

In [13]:
```
#Part C
df=pd.read_stata("cars2010.dta")
df1=pd.read_stata("cars2011.dta")
df2=pd.read_stata("cars2012.dta")
X_2010 = df.as_matrix(["EngDispl","NumCyl"])
X_2011 = df1.as_matrix(["EngDispl","NumCyl"])
X_2012 = df2.as_matrix(["EngDispl","NumCyl"])
FE_Y_2010 = df.as_matrix(["FE"])
FE_Y_2011 = df1.as_matrix(["FE"])
FE_Y_2012 = df2.as_matrix(["FE"])
```

In [14]:
```
X0 = np.column_stack((X_2010,np.ones(FE_Y_2010.shape[0])))
X1 = np.column_stack((X_2011,np.ones(FE_Y_2011.shape[0])))
X2 = np.column_stack((X_2012,np.ones(FE_Y_2012.shape[0])))
```

In [15]:
```
# Estimate Sigma
def sigma_estimate(B,X,Y):
    b0 = B[2]
    b1 = B[1]
    b2 = B[0]
    sum = 0.0
    for i in np.arange(len(X)):
        sum += np.square(Y[i]-(b0 + np.dot(X[i][0],b1) + np.dot(X[i][1],b2)))
    return (sum / (len(X) - 2))
```

```
In [16]: # Compute Beta_hat 2010
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X0.transpose(),X0)),X0.transpose()),FE_Y_2010)
         print B_hat

         [[ -3.74535214]
          [ -0.58802919]
          [ 51.35414193]]

In [17]: print "Sigma Estimate Cars 2010: ", sigma_estimate(B_hat,X0,FE_Y_2010)

         Sigma Estimate Cars 2010:  [ 89.94853481]
```

```
In [18]: # Compute Beta_hat 2011
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X1.transpose(),X1)),X1.transpose()),FE_Y_2011)
         print B_hat

         [[ -3.96769213]
          [ -1.13215237]
          [ 56.10527854]]

In [19]: print "Sigma Estimate Cars 2011: ", sigma_estimate(B_hat,X1,FE_Y_2011)

         Sigma Estimate Cars 2011:  [ 80.00672933]
```

```
In [20]: # Compute Beta_hat 2012
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X2.transpose(),X2)),X2.transpose()),FE_Y_2012)
         print B_hat

         [[ -5.29556177]
          [ -0.27929841]
          [ 58.00480858]]

In [21]: print "Sigma Estimate Cars 2012: ", sigma_estimate(B_hat,X2,FE_Y_2012)

         Sigma Estimate Cars 2012:  [ 213.23969024]
```

**Q2 Part D]**

```
In [22]: #PART D
         B_hat =np.dot(np.dot(np.linalg.inv(np.dot(X0.transpose(),X0)),X0.transpose()),FE_Y_2010)
         sigma_2010 = sigma_estimate(B_hat,X0,FE_Y_2010)
         cov_B = (np.square(sigma_2010)) * (np.linalg.inv(np.dot(X0.transpose(),X0)))
         print cov_B

         [[ 24.00506369 -14.94797628   5.06021902]
          [-14.94797628  11.33329747 -15.24353096]
          [  5.06021902 -15.24353096  80.58099929]]
```

**Q2 Part E]**

I think there is some relationship between the standard deviation for each hear and the computed beta hat.  The 2012 dataset had the highest standard deviation of the three datasets.

## Q3 Part A]

In [4]:
```
# PART A - 2010 data
```

In [5]:
```
import numpy as np
import matplotlib.pyplot as plt
from sklearn import datasets, linear_model
import pandas as pd
from pandas import DataFrame, Series
from __future__ import division
```

In [6]:
```
df=pd.read_stata("cars2010.dta")
```

In [7]:
```
# get XMean
X0 = df.as_matrix(["EngDispl","NumCyl"])
FE_Y_2010 = df.as_matrix(["FE"])
Xmean=[]
Xmean.append(np.mean(X0[:,0]))
Xmean.append(np.mean(X0[:,1]))
print Xmean
```

```
[3.5074074074074071, 5.9710930442637764]
```

In [8]:
```
sigma=[]
sigma.append(np.std(X0[:,0]))
sigma.append(np.std(X0[:,1]))
print sigma
Ymean = np.mean(FE_Y_2010)
print Ymean
```

```
[1.3053151226197799, 1.8997158959884082]
34.7064890696
```

In [9]:
```
std_X0=X0
for i in np.arange(len(X0)):
    for j in np.arange(len(X0[0])):
        std_X0[i][j]=(X0[i][j] - Xmean[j])/sigma[j]
print std_X0
```

```
[[ 0.91364344  1.06800546]
 [ 0.91364344  1.06800546]
 [ 0.53059417  1.06800546]
 ...,
 [-0.23550436  0.01521646]
 [-0.23550436  0.01521646]
 [ 0.68381388  1.06800546]]
```

In [10]:
```
Y_Center = FE_Y_2010
for i in np.arange(len(FE_Y_2010)):
    Y_Center[i]=FE_Y_2010[i]-Ymean
print Y_Center
```

```
[[-6.68668907]
 [-9.09708907]
 [-7.90648907]
 ...,
 [-4.21388907]
 [-4.96338907]
 [-8.50648907]]
```

## Q3 Part B]

```
In [9]:  #Part B
         lamda = 0.01
         model_rr = linear_model.Ridge(alpha =lamda)
         model_rr.fit(std_X0, Y_Center)
         coeff=model_rr.coef_
         print 'When Lambda=', lamda
         print 'B1 =', coeff[0][0]
         print 'B2 =', coeff[0][1]
         print 'B0 ', model_rr.intercept_
         print 'r2 = ', model_rr.score(std_X0, Y_Center)

         When Lambda= 0.01
         B1 = -4.88866883941
         B2 = -1.11725589258
         B0  [ -8.38049530e-16]
         r2 =  0.623958939675
```

```
In [10]: lamda = 5
         model_rr = linear_model.Ridge(alpha =lamda)
         model_rr.fit(std_X0, Y_Center)
         coeff=model_rr.coef_
         print 'When Lambda=', lamda
         print 'B1 =', coeff[0][0]
         print 'B2 =', coeff[0][1]
         print 'B0 ', model_rr.intercept_
         print 'r2 = ', model_rr.score(std_X0, Y_Center)

         When Lambda= 5
         B1 = -4.79507488508
         B2 = -1.19668137988
         B0  [ -8.77082684e-16]
         r2 =  0.62393043522
```

```
In [11]: lamda = 10000
         model_rr = linear_model.Ridge(alpha =lamda)
         model_rr.fit(std_X0, Y_Center)
         coeff=model_rr.coef_
         print 'When Lambda=', lamda
         print 'B1 =', coeff[0][0]
         print 'B2 =', coeff[0][1]
         print 'B0 ', model_rr.intercept_
         print 'r2 = ', model_rr.score(std_X0, Y_Center)

         When Lambda= 10000
         B1 = -0.542643100054
         B2 = -0.503905302656
         B0  [ -1.64280226e-15]
         r2 =  0.194972254986
```

## Q3 Part C]

```
In [12]: # Part C
         df1=pd.read_stata("cars2011.dta")
         X1 = df1.as_matrix(["EngDispl","NumCyl"])
         FE_Y_2011 = df1.as_matrix(["FE"])
         X1mean=[]
         X1mean.append(np.mean(X1[:,0]))
         X1mean.append(np.mean(X1[:,1]))
```

```
In [13]: sigma1=[]
         sigma1.append(np.std(X1[:,0]))
         sigma1.append(np.std(X1[:,1]))
         Ymean1 = np.mean(FE_Y_2011)
```

```
In [14]: std_X1=X1
         for i in np.arange(len(X1)):
             for j in np.arange(len(X1[0])):
                 std_X1[i][j]=(X1[i][j] - X1mean[j])/sigma1[j]
```

```
In [15]: Y_Center1 = FE_Y_2011
         for i in np.arange(len(FE_Y_2011)):
             Y_Center1[i]=FE_Y_2011[i]-Ymean1
         #print Y_Center1
```

```
In [23]: lambdaList = [0,0.001,0.01,0.1,1,10,100]
         for L in lambdaList:
             model_rr = linear_model.Ridge(alpha = L)
             model_rr.fit(std_X0, Y_Center)
             print "Coefficients [L=",L,"] ",model_rr.coef_
             print "Intercept [L=",L,"] ",model_rr.intercept_
             print "r2 =",model_rr.score(X0,FE_Y_2010)

         Coefficients [L= 0 ]  [[-4.88886479 -1.1170884 ]]
         Intercept [L= 0 ]  [ -8.37967528e-16]
         r2 = 0.623958939799
         Coefficients [L= 0.001 ]  [[-4.88884519 -1.11710515]]
         Intercept [L= 0.001 ]  [ -8.37975729e-16]
         r2 = 0.623958939798
         Coefficients [L= 0.01 ]  [[-4.88866884 -1.11725589]]
         Intercept [L= 0.01 ]  [ -8.38049530e-16]
         r2 = 0.623958939675
         Coefficients [L= 0.1 ]  [[-4.88690687 -1.11876173]]
         Intercept [L= 0.1 ]  [ -8.38786838e-16]
         r2 = 0.623958927422
         Coefficients [L= 1 ]  [[-4.86944216 -1.13366626]]
         Intercept [L= 1 ]  [ -8.46090060e-16]
         r2 = 0.623957721019
         Coefficients [L= 10 ]  [[-4.70893834 -1.26868796]]
         Intercept [L= 10 ]  [ -9.12750278e-16]
         r2 = 0.623853611485
         Coefficients [L= 100 ]  [[-3.82749986 -1.90671901]]
         Intercept [L= 100 ]  [ -1.25458771e-15]
         r2 = 0.619847045333
```

```
In [24]: lambdaList = [0,0.001,0.01,0.1,1,10,100]
         for L in lambdaList:
             model_rr = linear_model.Ridge(alpha = L)
             model_rr.fit(std_X1, Y_Center1)
             print "Coefficients [L=",L,"] ",model_rr.coef_
             print "Intercept [L=",L,"] ",model_rr.intercept_
             print "r2 =",model_rr.score(X1,FE_Y_2011)

         Coefficients [L= 0 ]  [[-5.8258967  -2.04233847]]
         Intercept [L= 0 ]  [ -6.67111699e-16]
         r2 = 0.709795966141
         Coefficients [L= 0.001 ]  [[-5.82579615 -2.04242227]]
         Intercept [L= 0.001 ]  [ -6.67089827e-16]
         r2 = 0.70979596612
         Coefficients [L= 0.01 ]  [[-5.82489157 -2.04317601]]
         Intercept [L= 0.01 ]  [ -6.66893075e-16]
         r2 = 0.709795964137
         Coefficients [L= 0.1 ]  [[-5.81588574 -2.05067383]]
         Intercept [L= 0.1 ]  [ -6.64934802e-16]
         r2 = 0.709795767266
         Coefficients [L= 1 ]  [[-5.72963895 -2.12187227]]
         Intercept [L= 1 ]  [ -6.46233628e-16]
         r2 = 0.709777458332
         Coefficients [L= 10 ]  [[-5.12405807 -2.58007675]]
         Intercept [L= 10 ]  [ -5.18552931e-16]
         r2 = 0.70872791014
         Coefficients [L= 100 ]  [[-3.56542497 -2.92115038]]
         Intercept [L= 100 ]  [ -2.62862347e-16]
         r2 = 0.683238364945
```

**Q3 Part D]**

**Q3 Part E]**

We need three separated datasets for the following reasons:

- Training – Develop optimal parameters for the model we intend to use
- Cross Validation – Test the quality of the model against data not sampled to test predicted result against known results
- Testing – Exercise model against test samples to predict results using the model we developed

**Q4]**

**PROBABILITY SECTION**

4. For $X$, $Y$, and $Z$ random variables, the *Covariance Matrix* of the random vector $(X, Y, Z)$ is defined as:

$$\mathbb{E}\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\begin{pmatrix} X & Y & Z \end{pmatrix}\right] = \mathbb{E}\begin{bmatrix} X^2 & XY & XZ \\ XY & Y^2 & YZ \\ XZ & YZ & Z^2 \end{bmatrix}.$$

Suppose $X$, $Y$ and $Z$ represent the result of rolling three independent dice. What is the covariance matrix of $(X, Y, Z)$?

PMF of rolling a dice

$$f_X(y) = \begin{cases} \frac{1}{6} \; if \; y \in \{1,2,3,4,5,6\} \\ 0 \qquad otherwise \end{cases}$$

$$\mu_X = E(X) = \sum_{y=1}^{6} y \cdot f_Z(y) = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right)$$

$$\mu_X = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5$$

$$\sigma_X{}^2 = var(X) = \textstyle\sum_{y=1}^{6}(y - \mu_X)^2 \cdot f_Z(y)$$

$$\sigma_X{}^2 = \left(1 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(2 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(3 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(4 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(5 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right) + \left(6 - \frac{7}{2}\right)^2\left(\frac{1}{6}\right)$$

$$\sigma_X{}^2 = \left(-\frac{5}{2}\right)^2\left(\frac{1}{6}\right) + \left(-\frac{3}{2}\right)^2\left(\frac{1}{6}\right) + \left(-\frac{1}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{3}{2}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{2}\right)^2\left(\frac{1}{6}\right)$$

$$\sigma_X{}^2 = \left(\frac{25}{24}\right) + \left(\frac{9}{24}\right) + \left(\frac{1}{24}\right) + \left(\frac{1}{24}\right) + \left(\frac{9}{24}\right) + \left(\frac{25}{24}\right) = \left(\frac{70}{24}\right) = \frac{35}{12} = 2\frac{11}{12}$$

Rules:

$E(XY) = E(X)E(Y)$

$E(X \pm a) = E(X) \pm a$

Covariance Matrix is $\Sigma$

$$\Sigma = \begin{pmatrix} E(X-3.5)^2 & E(X-3.5)(Y-3.5) & E(X-3.5)(Z-3.5) \\ E(X-3.5)(Y-3.5) & E(Y-3.5)^2 & E(Y-3.5)(Z-3.5) \\ E(X-3.5)(Z-3.5) & E(Y-3.5)(Z-3.5) & E(Z-3.5)^2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2\frac{11}{12} & E(X-3.5)E(Y-3.5) & E(X-3.5)E(Z-3.5) \\ E(X-3.5)E(Y-3.5) & 2\frac{11}{12} & E(Y-3.5)E(Z-3.5) \\ E(X-3.5)E(Z-3.5) & E(Y-3.5)E(Z-3.5) & 2\frac{11}{12} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2\frac{11}{12} & 0 & 0 \\ 0 & 2\frac{11}{12} & 0 \\ 0 & 0 & 2\frac{11}{12} \end{pmatrix}$$

**Q7]**

7. Consider flipping a fair coin. Let $Z$ denote the random variable that is the number of Heads that come up in a row. Thus, if the first flip comes up tails, $Z = 0$. If the flip sequence is

$$HHTHHHHT....$$

then $Z = 2$, and so on.

- Write the probability mass function of $Z$.
- Compute the mean and variance of $Z$.

For a Fair Coin.

P(H) = $p = \frac{1}{2}$

P(T) = $(1-p) = \frac{1}{2}$

Random Variable

| Compound Event | Elementary Event | Probability |
|---|---|---|
| (Z=0) | (T..) = | $\frac{1}{2}$ |
| (Z=1) | (HT..) | $\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right) = \frac{1}{4}$ |
| (Z=2) | (HHT..) | $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{8}$ |
| (Z=3) | (HHHT..) | $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{16}$ |
| (Z=4) | (HHHHT..) | $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$ |

PMF:

| $y$ | $f_Z(y)$ | |
|---|---|---|
| 0 | $\frac{1}{2}$ | $p^0(1-p)$ |
| 1 | $\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right) = \frac{1}{4}$ | $p^1(1-p)$ |
| 2 | $\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{8}$ | $p^2(1-p)$ |
| 3 | $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{16}$ | $p^3(1-p)$ |
| 4 | $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$ | $p^4(1-p)$ |

So **PMF** for Z is $f_Z(y) = p^y(1-p)$ or for fair coin where $p = (1-p)$

$$f_Z(y) = p^{y+1} = \left(\tfrac{1}{2}\right)^{y+1} \qquad \text{where } y = 0,1,2,\dots,n$$

Compute Mean & Variance:

$$\mu = E(Z) = \sum_{y=0}^{n} y \cdot f_Z(y) = \sum_{y=0}^{n} y \cdot \left(\frac{1}{2}\right)^{y+1} = \sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y+1} = \frac{1}{4}\sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y-1}$$

From http://www.trans4mind.com/personal_development/mathematics/series/airthmeticGeometricSeries.htm

$$\boxed{1 + 2\,r + 3\,r^2 \ldots n\,r^{(n-1)} = \frac{\left(1 - (n+1)\,r^n + n\,r^{(n+1)}\right)}{(1-r)^2}}$$

$r = \frac{1}{2}$

$$\mu = E(Z) = \frac{1}{4}\sum_{y=1}^{n} y \cdot \left(\frac{1}{2}\right)^{y-1} = \frac{1}{4}\frac{\left(1 - (n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}\right)}{\left(\frac{1}{2}\right)^2}$$

$$\mu = \frac{1}{4}\frac{\left(1 - (n+1)\left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^{n+1}\right)}{\frac{1}{4}} = \left(1 + n\left(\frac{1}{2}\right)^{n+1} - (n+1)\left(\frac{1}{2}\right)^n\right) =$$

$$\mu = \left(1 + n\left(\frac{1}{2}\right)^{n+1} - (n+1)\left(\frac{1}{2}\right)^n\right) = \left(1 + \left(\frac{1}{2}\right)n\left(\frac{1}{2}\right)^n - (n+1)\left(\frac{1}{2}\right)^n\right)$$

$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, then $\mu = 1$

Variance:
Use:

**Properties of Variance**

- $\operatorname{var}(X) = E\{(X - E[X])^2\} = E[X^2] - (E[X])^2$

Take advantage of:
$E[Z^2] = E[Z(Z-1)] + E[Z]$

Compute:
$E[Z(Z-1)] = \sum_{y=1}^{n} y(y-1) \cdot \left(\frac{1}{2}\right)^{y+1}$

$\sigma_Z{}^2 = E[(Z-\mu)^2] = E[Z^2] - (E[Z])^{\,2} = E[Z^2] - \mu^{\,2}$

$$\sum_{y=0}^{n}(y-\mu)^2 f_Z(y) = \sum_{y=0}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1} = \sum_{y=0}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1}$$

$$\sigma_Z{}^2 = (0-1)^2\left(\frac{1}{2}\right)^{0+1} + \sum_{y=1}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1} = \frac{1}{2} + \sum_{y=1}^{n}(y-1)^2\left(\frac{1}{2}\right)^{y+1}$$

Use from http://math2.org/math/expansion/power.htm:

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \quad \rightarrow \quad \sum_{j=1}^{n}(j-1)^2 = (1-1)^2 + \sum_{j=2}^{n}(j-1)^2 = \sum_{k=1}^{n} k^2$

Use from http://math2.org/math/expansion/geom.htm:

$$\sum_{k=1}^{n} k^2$$

Revisit with:
http://arnoldkling.com/apstats/geometric.html