Advanced Artificial Intelligence: Spring 2016 Hoang Nguyen - 15M54097
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Professors: Nitta, Inoue, and Shinoda Quiz 3 - Lecture 14 (Prof. Shinoda)

Quiz 3 - Lecture 14 (Prof. Shinoda)

1. Prove that $p(\mathbf{x}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$
2. Discuss the future prospect of deep learning and its related techniques.

Collaborators: None.

Exercise 3-1. Prove that $p(\mathbf{x}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$

Solution: By definition, $\mathbf{z}$ is one-hot encoding representation, we have:

$$p(\mathbf{z}) = \prod_{k=1}^{K} \pi_k^{z_k}$$
$$p(\mathbf{x}|\mathbf{z}) = \prod_{k=1}^{K} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)^{z_k}$$

By the product rule, we have the join probability of $\mathbf{x}$ and $\mathbf{z}$ as follow:

$$p(\mathbf{x}, \mathbf{z}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z})$$

Using the sum product to compute the marginal $p(\mathbf{x})$:

$$\begin{aligned} p(\mathbf{x}) &= \sum_{\mathbf{z}} p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) \\ &= \sum_{\mathbf{z}} \prod_{k=1}^{K} \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)^{z_k} \pi_k^{z_k} \quad = \sum_{j=1}^{K} \prod_{k=1}^{K} (\pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)^{\delta_{jk}}, \end{aligned}$$

where δ_{jk} is the Kronecker delta. Simply rewrite the product keeping not 1 values, we have the desired result:

$$p(\mathbf{x}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

Exercise 3-2. Prove $a \perp\!\!\!\perp b \mid c$

Solution: We have the conditional probabilities:

$$P(a, b \mid c) = \frac{P(a, b, c)}{P(c)}$$
$$P(a \mid c) = \frac{\sum_b P(a, b, c)}{P(c)}$$
$$P(b \mid c) = \frac{\sum_a P(a, b, c)}{P(c)}$$

We also have:

$$P(c = 1) = \sum_{a,b} P(a, b, c = 1) = 0.216 + 0.144 + 0.064 + 0.096 = 0.52$$
$$P(c = 0) = \sum_{b,b} P(a, b, c = 1) = 1 - 0.52 = 0.48$$

We have table of join probability conditioned on c:

a	b	c	$P(a, b \mid c)$	$P(a \mid c)P(b \mid c)$
0	0	0	0.4	0.4
0	0	1	0.277	0.277
0	1	0	0.1	0.1
0	1	1	0.415	0.415
1	0	0	0.4	0.4
1	0	1	0.123	0.123
1	1	0	0.1	0.1
1	1	1	0.185	0.185

Table 1: Join probability conditioned on c

From Table 2, we have:

$$P(a, b \mid c) = P(a \mid c)P(b \mid c)$$

Therefore, $a \perp\!\!\!\perp c \mid c$.

Exercise 3-3. **Show** $P(a, b, c) = P(a)P(c \mid a)P(b \mid c)$

Solution: Using sum rule similar to the previous questions, we have these following probability tables:

Multiply the probabilities in Table 3 gives us the result:

$$P(a, b, c) = P(a)P(c \mid a)P(b \mid c)$$

Exercise 3-4. Illustrate a DAD corresponds to question 3

Solution:

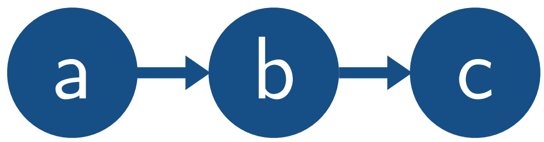

Figure 1: DAG corresponding to question 3.

a	$P(a)$
0	0.6
1	0.4

(a) $P(a)$

c	a	$P(c \mid a)$
0	0	0.4
0	1	0.6
1	0	0.6
1	1	0.4

(b) $P(a)$

b	c	$P(b \mid c)$
0	0	0.8
0	1	0.4
1	0	0.2
1	1	0.6

(c) $P(a)$

Table 2: Conditional probability tables