Quiz 2 - Lecture 13 (Prof. Shinoda)

a	b	c	P(a,b,c)
0	0	0	0.192
0	0	1	0.144
0	1	0	0.048
0	1	1	0.216
1	0	0	0.192
1	0	1	0.064
1	1	0	0.048
1	1	1	0.096

Table 1: Join probability

Consider the probability table of random variables a, b, and c given in Table 1. Answer these following questions:

1. Prove that $a \not\perp\!\!\!\perp b$.
2. Prove that $a \perp\!\!\!\perp b \mid c$.
3. Show $P(a, b, c) = P(a)P(c \mid a)P(b \mid c)$.
4. Illustrate a DAG correspondes to question 3.

Collaborators: None.

Exercise 2-1. Prove $a \not\perp\!\!\!\perp b$

Solution: Assume: $a \perp\!\!\!\perp b \Leftrightarrow P(a, b) = P(a)P(b)$. We have a counter example:

$$\begin{aligned}
P(b=0) &= \sum_{a,c} P(a, b=0, c) = 0.192 + 0.144 + 0.192 + 0.064 = 0.592 \\
P(a=0) &= \sum_{b,c} P(a=0, b, c) = 0.192 + 0.064 + 0.048 + 0.096 = 0.4 \\
P(a=0, b=0) &= \sum_{c} P(a=0, b=0, c) = 0.192 + 0.144 = 0.336 \\
&\neq P(a=0)P(b=0) = 0.4 \times 0.592 = 0.2368
\end{aligned}$$

Therefore, $a \not\perp\!\!\!\perp b$.

Exercise 2-2. Prove $a \perp\!\!\!\perp b \mid c$

Solution: We have the conditional probabilities:

$$\begin{aligned}
P(a, b \mid c) &= \frac{P(a, b, c)}{P(c)} \\
P(a \mid c) &= \frac{\sum_b P(a, b, c)}{P(c)} \\
P(b \mid c) &= \frac{\sum_a P(a, b, c)}{P(c)}
\end{aligned}$$

We also have:

$$P(c=1) = \sum_{a,b} P(a,b,c=1) = 0.216 + 0.144 + 0.064 + 0.096 = 0.52$$
$$P(c=0) = \sum_{b,b} P(a,b,c=1) = 1 - 0.52 = 0.48$$

We have table of join probability conditioned on c:

a	b	c	$P(a,b \mid c)$	$P(a \mid c)P(b \mid c)$
0	0	0	0.4	0.4
0	0	1	0.277	0.277
0	1	0	0.1	0.1
0	1	1	0.415	0.415
1	0	0	0.4	0.4
1	0	1	0.123	0.123
1	1	0	0.1	0.1
1	1	1	0.185	0.185

Table 2: Join probability conditioned on c

From Table 2, we have:

$$P(a,b \mid c) = P(a \mid c)P(b \mid c)$$

Therefore, $a \perp\!\!\!\perp c \mid c$.

Exercise 2-3. Show $P(a,b,c) = P(a)P(c \mid a)P(b \mid c)$

Solution: Using sum rule similar to the previous questions, we have these following probability tables:

a	$P(a)$
0	0.6
1	0.4

(a) $P(a)$

c	a	$P(c \mid a)$
0	0	0.4
0	1	0.6
1	0	0.6
1	1	0.4

(b) $P(a)$

b	c	$P(b \mid c)$
0	0	0.8
0	1	0.4
1	0	0.2
1	1	0.6

(c) $P(a)$

Table 3: Conditional probability tables

Multiply the probabilities in Table 3 gives us the result:

$$P(a,b,c) = P(a)P(c \mid a)P(b \mid c)$$

Exercise 2-4. Illustrate a DAD corresponds to question 3

Solution:

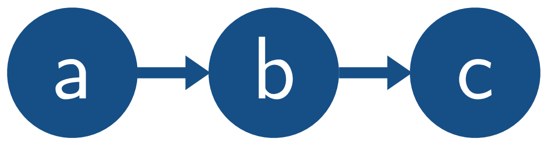

Figure 1: DAG corresponding to question 3.