

COMPLEX NETWORK - ASSIGNMENT 2

NGUYEN T. Hoang - SID: 15M54097

Fall 2015, W833 Mon. Period 5-6

Due date: 2016/02/01

Problem

Given a network as in Figure 1. Complete these following tasks:

1. Compute eigenvector centrality and betweenness centrality of each vertex in the given network.
2. Find eigenvectors of matrices, construct the Laplacian and the modularity matrix for this small network:
 - (a) Find the eigenvector of the Laplacian corresponding to the second smallest eigenvalue and hence perform a spectral bisection of the network into two equally sized parts.
 - (b) Find the eigenvector of the modularity matrix corresponding to the largest eigenvalue and hence divide the network into two communities.
3. Explain quantitatively why “*your friends have more friends than you do*” in the configuration model.
4. Write examples of parameters β and γ of SIR model when:
 - (a) There is an epidemic.
 - (b) There is no epidemic.

Answer

In this assignment, I will use SNAP.PY, a network analysis developed by Stanford University, as a computational tool. All illustrations (figures, graphs ...) are drawn using D3 Javascript Framework.

Question 1: Compute eigenvector centrality and betweenness centrality of each vertex.

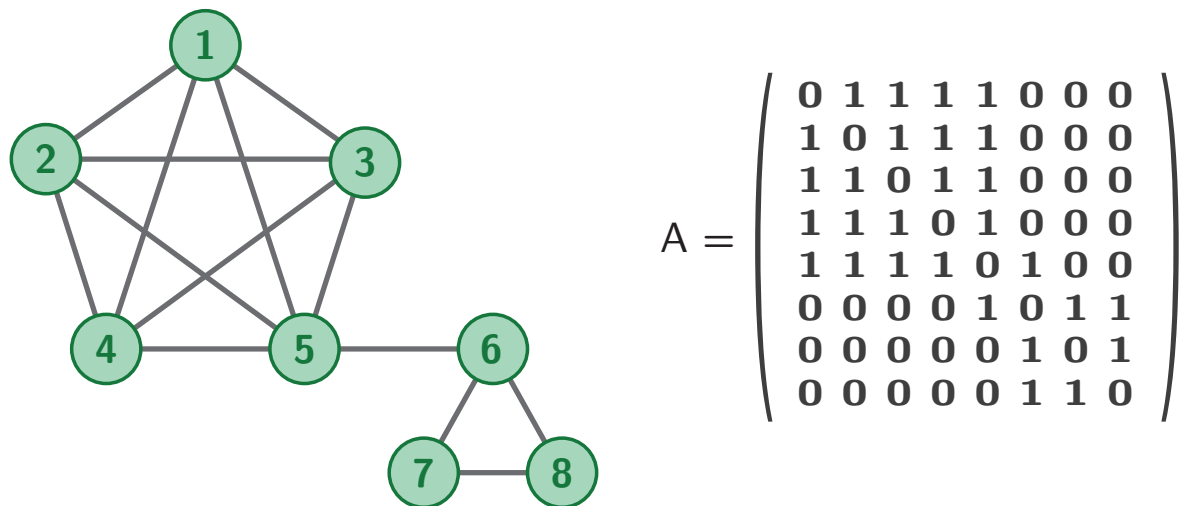


Figure 1: Simple network and its corresponding adjacency matrix.

Eigenvector centrality is given by the formula:

$$x_i^{(t)} = \sum_{j \neq i} A_{ij} \times x_j^{(t)}$$

Rewrite in final matrix form:

$$A\mathbf{x} = k_1\mathbf{x}$$

where \mathbf{x} is a vector storing score of all nodes, and k_1 is the largest eigenvalue of the adjacency matrix A .

Listing 1: Eigenvector centrality computation with SNAP.PY

```
1 # Extracted from UnweightedUndirectedGraph class - File: cn_a2_p1.py
2 ...
3 import snap as sn
4 self._graph = sn.LoadEdgeList(sn.PUNGraph, edge_list, 0, 1, 'u')
5 ...
6 # Compute eigenvector centrality and store to a hash table.
7 def EigenvectorCentrality(self):
8     # Create a hash map: Int -> Float
9     NIdEigenH = sn.TIntFltH()
10    sn.GetEigenvector(self._graph, NIdEigenH)
11    return NIdEigenH
```

The vector result of Listing 1 is shown as follow:

$$\begin{pmatrix} 0.437 & 0.437 & 0.437 & 0.437 & 0.464 & 0.136 & 0.044 & 0.044 \end{pmatrix}$$

Figure 2 shows the result in the graph. Eigenvector centrality of each vertex is shown by a blue decimal number next to it. As we can see, vertex number 5 has the highest eigenvector centrality means that vertex number 5 is the most *central* vertex. By looking at the graph, we can intuitively understand this fact.

Betweenness centrality is another metric to measure how important a vertex is within the network. Different from other metric, betweenness centrality measure how important a vertex is in the information flow between other vertices. In [1], the author defines betweenness centrality x_i of vertex i as follow:

$$x_i = \sum_{st} n_{st}^i, \text{ or } x_i = \sum_{st} \frac{n_{st}^i}{g_{st}}$$

where n_{st}^i is the number of geodesic paths between vertex s and vertex t that go through i , and g_{st} is the total number of geodesic paths between s and t . Besides the normal betweenness centrality, in [1] the author also mentioned 2 other types of betweenness: *flow betweenness* and *random walk betweenness*. However, in this assignment, I will only compute the standard betweenness for the given network.

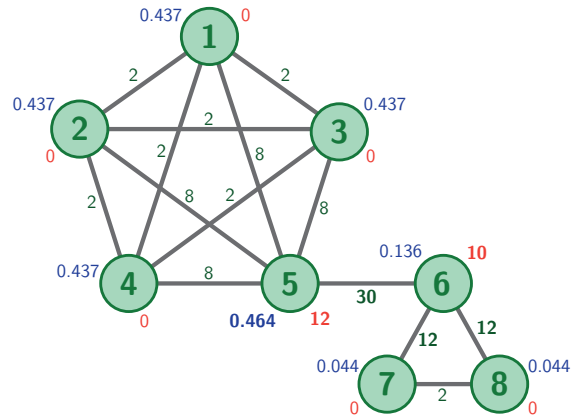


Figure 2: The given network with eigenvector centrality (blue), and betweenness centrality of vertices (red) and edges (green)

Listing 2: Betweenness centrality computation with SNAP.PY

```
1 # Extracted from UnweightedUndirectedGraph class - File: cn_a2_p1.py
2 ...
3 import snap as sn
4 self._graph = sn.LoadEdgeList(sn.PUNGraph, edge_list, 0, 1, 'u')
5 ...
6 # Compute betweenness centrality of vertices and edges and store to a hash table.
7 def BetweennessCentrality(self, isNode = true):
8     # Create 2 hash maps: Int -> Float
9     NodeScore = sn.TIntFltH()
10    EdgeScore = sn.TIntFltH()
11    sn.GetBetweennessCentr(self._graph, NodeScore, EdgeScore, 1.0)
12    if (isNode):
13        return NodeScore
14    else:
15        return EdgeScore
```

The vector result of Listing 2 is shown as follow:

$$\text{NodeScore} = \begin{pmatrix} 0 & 0 & 0 & 0 & 12 & 10 & 0 & 0 \end{pmatrix}$$

$$\text{EdgeScore} = \begin{pmatrix} 2 & 2 & 2 & 8 & 2 & 2 & 8 & 2 & 8 & 8 & 30 & 12 & 12 & 2 \end{pmatrix}$$

Figure 2 shows the result in the graph. Betweenness centrality of each vertex is shown by a red integer next to it. As we can see, vertex number 5 has the highest betweenness centrality since it is the connection between two cliques.

Question 2: Construct Laplacian matrix and Modularity matrix for the given network. Laplacian matrix and Modularity matrix of the graph is computed with *numpy* as follow:

Listing 3: Laplacian matrix and Modularity matrix computation with Numpy

```

1 # Extracted from UnweightedUndirectedGraph class - File: cn_a2_p1.py
2 ...
3 import numpy as np
4 ...
5 self._adj_matrix = edge_list_to_np(edge_list)
6 ...
7 # Compute and return Laplacian matrix
8 def LaplacianMatrix(self):
9     D = np.diag(np.sum(self._adj_matrix, 0))
10    return D - self._adj_matrix
11 ...
12 # Compute and return Modularity matrix
13 def ModularityMatrix(self):
14     B = np.zeros(self._adj_matrix.shape)
15     d = np.sum(self._adj_matrix, 0)
16     m = sum(d)
17     for i in range(B.shape[0]):
18         for j in range(B.shape[1]):
19             B[i,j] = A[i,j] - (d[i] * d[j]) / float(m)
20     return B

```

The result of Listing 3 is:

$$\mathcal{L} = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -4 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -4 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} -0.571 & 0.428 & 0.428 & 0.428 & 0.281 & -0.443 & -0.429 & -0.289 \\ 0.428 & -0.571 & 0.428 & 0.428 & 0.285 & -0.423 & -0.229 & -0.229 \\ 0.428 & 0.428 & -0.571 & 0.428 & 0.285 & -0.423 & -0.229 & -0.229 \\ 0.428 & 0.428 & 0.428 & -0.571 & 0.285 & -0.423 & -0.229 & -0.229 \\ 0.285 & 0.285 & 0.285 & 0.285 & -0.892 & 0.464 & -0.356 & -0.356 \\ 0.428 & -0.428 & -0.428 & -0.428 & 0.464 & -0.327 & 0.729 & 0.729 \\ 0.285 & -0.285 & -0.285 & -0.285 & -0.357 & 0.785 & -0.144 & 0.856 \\ 0.285 & -0.285 & -0.285 & -0.285 & -0.357 & 0.785 & 0.856 & -0.144 \end{pmatrix}$$

(a) Compute the eigenvalue of the Laplacian Matrix, deduce the second smallest eigenvalue, from there perform a spectral bisection on the graph into 2 equal parts.

Listing 4: Perform Spectral Bisection on the graph

```

1 # Extracted from UnweightedUndirectedGraph class - File: cn_a2_p1.py
2 ...
3 import numpy as np
4 ...
5 self._adj_matrix = edge_list_to_np(edge_list)
6 ...
7 # Compute spectral bisection
8 def SpectralBisection(self):
9     L = self.LaplacianMatrix()
10    # Get eigenvectors of \L
11    u , v = np.linalg.eig(L)
12    # Get the second smallest eigenvalue and its eigenvector
13    i = np.argsort(u)[1]
14    eigv = v[:,i]
15    partition = np.ones(eigv.shape[0])
16    index_sorted_eigv = np.argsort(eigv)
17    for i in range(partition.shape[0] / 2, partition.shape[0]):
18        partition[index_sorted_eigv[i]] = -1
19    return partition

```

The vector result for this bisection is demonstrated as follow:

$$\text{partition} = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

(b) Compute the eigenvalue of the Modularity Matrix, deduce the largest eigenvector, from there divide the network into 2 communities. In spectral modularity, we decide groups by positiveness of each element in the eigenvector. Maximum modularity is achieved when there is a densense connection within a community and sparse between communities. The computation is performed as follow:

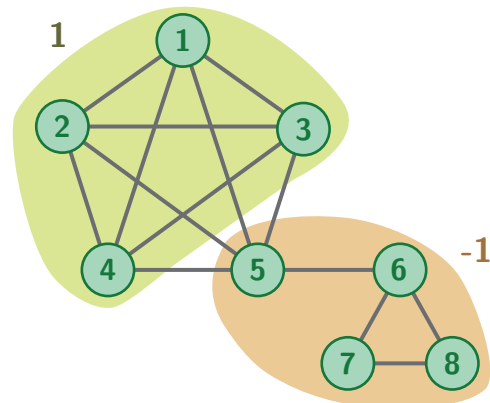


Figure 3: Illustration of spectral bisection.

Listing 5: Perform Spectral Modularity on the graph

```

1 # Extracted from UnweightedUndirectedGraph class - File: cn_a2_p1.py
2 ...
3 import numpy as np
4 ...
5 self._adj_matrix = edge_list_to_np(edge_list)
6 ...
7 # Compute spectral modularity
8 def SpectralBisection(self):
9     B = self.LaplacianMatrix()
10    # Get eigenvectors of \L
11    u , v = np.linalg.eig(L)
12    # Get the second smallest eigenvalue and its eigenvector
13    i = np.argsort(u)[1]
14    eigv = v[:,i]
15    partition = np.ones(eigv.shape[0])
16    index_sorted_eigv = np.argsort(eigv)
17    for i in range(partition.shape[0] / 2, partition.shape[0]):
18        partition[index_sorted_eigv[i]] = -1
19    return partition

```

The vector result for this bisection is demonstrated as follow:

$$\text{partition} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Comparing result between Figure 3 and Figure 4, we can see that the spectral modularity gives us a more *appropriate* result since it divided the given network into two *clique*. Both technique I used here minimizes the number of connection between communities. However, by forcing spectral bisection to divide network into 2 *equal* parts, we obtained an unoptimized result. In conclusion, it is clear that modularity method gives us a better clustering in both computational value and common sense.

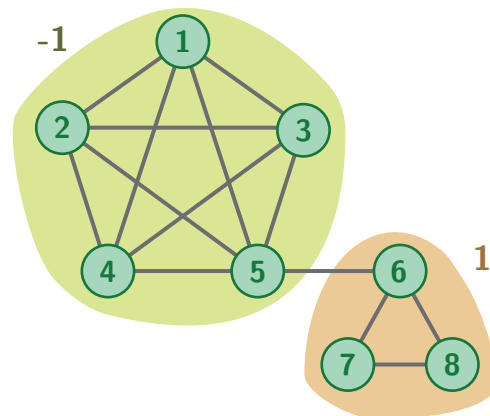


Figure 4: Illustration of spectral modularity.

Question 3: Explain quantitatively why “*your friends have more friends than you do*” in the configuration model. *Proof:* Suppose we are given a vector $k = \{k_0, k_1, \dots, k_{n-1}\}$ contains the degree of all vertices in the configuration model with n vertices and m edges. Without changing the generality of the statement, we assume that $k_i > 0 \forall k_i \in k$ and $n > 0$. In this context, the word “friend” of a vertex i means the adjacency vertices of vertex i . Also, k_i is the number of friend that vertex i has. Denote F_i as the number of friends that some vertex i has, and FF_i as the number of friends that friends of vertex i has. Quantitatively, the average number of friend that an arbitrary vertex i has is:

$$\mathbb{E}(F_i) = \frac{1}{n} \sum_{i=0}^{n-1} k_i = \frac{2m}{n}$$

Therefore, in this configuration model, the average friend that “you” have is $2m/n$. On the other hand, the probability that there is a connection between vertex i and vertex j (i and j are friends) is:

$$p(i \leftrightarrow j) = \frac{k_i k_j}{2m - 1} \approx \frac{k_i k_j}{2m}$$

The average number of friends that a friend of vertex i has is:

$$\begin{aligned} \mathbb{E}(FF_i) &= \frac{1}{k_i} \sum_{j=0}^{n-1} p(i \leftrightarrow j) \times k_j \\ &= \frac{1}{k_i} \times k_i \sum_{j=0}^{n-1} \frac{k_j}{2m} \times k_j \\ &= \frac{1}{2m} \sum_{j=0}^{n-1} k_j^2 \end{aligned}$$

Consider the different \mathcal{D} between $\mathbb{E}(F_i)$ and $\mathbb{E}(FF_i)$. Note that $2m = \sum_j k_j$:

$$\begin{aligned} \mathcal{D} &= \mathbb{E}(FF_i) - \mathbb{E}(F_i) \\ &= \frac{\sum_{j=0}^{n-1} k_j^2}{2m} - \frac{k_i k_j}{2m} \\ &= \frac{n \sum_{j=0}^{n-1} k_j^2 - 2 \left(\sum_{j=0}^{n-1} k_j \right)^2}{2mn} \end{aligned}$$

Expand the square term $2(\dots)^2$, we have the following result:

$$\mathcal{D} = \frac{(n-1) \sum_{j=0}^{n-1} k_j^2 - 2 \sum_{i < j} k_i k_j}{2mn}$$

Using the Cauchy-Schwarz inequality, we have:

$$k_i^2 + k_j^2 \geq 2k_i k_j$$

Therefore, using the general form of Cauchy-Schwarz we have:

$$(n-1) \sum_{j=0}^{n-1} k_j^2 \geq 2 \sum_{i < j} k_i k_j$$

Hence,

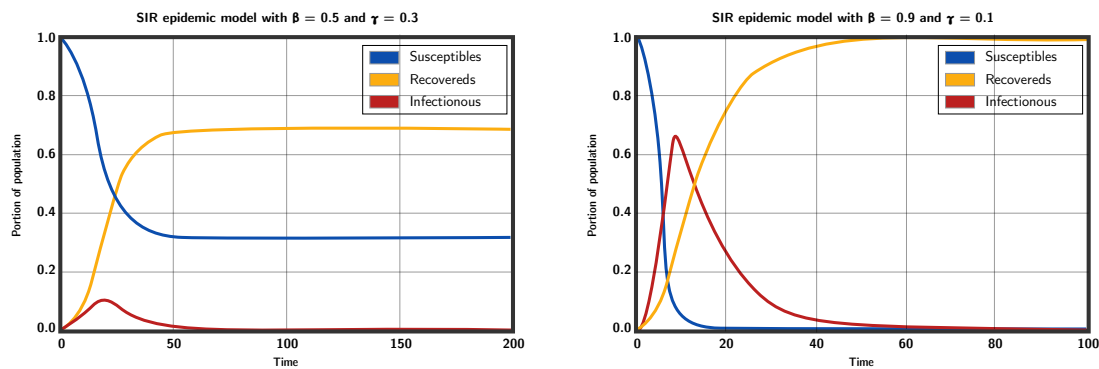
$$\mathcal{D} = \frac{(n-1) \sum_{j=0}^{n-1} k_j^2 - 2 \sum_{i < j} k_i k_j}{2mn} \geq 0$$

In conclusion, the different between average number of a friend’s friend and your friend is $\mathcal{D} \geq 0$, therefore the statement “(on average) your friends have more friends than you do” holds true. QED.

Question 4: Example of parameter β and γ of SIR model. In the SIR epidemic model, β is the infection rate and γ is recover (or death) rate. The model is represented as a system of differential equation:

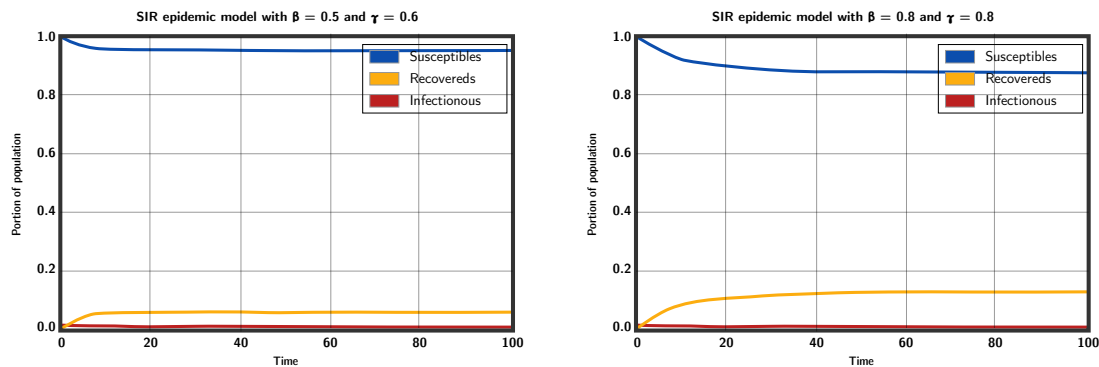
$$\begin{aligned}\frac{ds}{dt} &= -\beta sx, \\ \frac{dx}{dt} &= \beta sx - \gamma x, \\ \frac{dr}{dt} &= \gamma x\end{aligned}$$

where s, x, r are fraction of population that are *subceptible*, *recovered*, and *infectionous* respectively. Since the solution of these diffirential equation is not analytical, numerical method is applied to analyze this SIR model. In real-life application, the parameter β and γ is determined empirically by trying to fit known data with some trial value of β and γ . The result for each value (s, x, r) in this exercise is stored in an array, which is indexed my time step. According to [1], the epidemic happens when $\beta > \gamma$, which means the disease spreads faster than human recovery. I plotted the time graph for 4 cases:



(a) SIR model, case $\beta = 0.5$ and $\gamma = 0.3$ (b) SIR model, case $\beta = 0.9$ and $\gamma = 0.1$

Figure 5: SIR model in case of epidemic happens.



(a) SIR model, case $\beta = 0.5$ and $\gamma = 0.6$ (b) SIR model, case $\beta = 0.8$ and $\gamma = 0.8$

Figure 6: SIR model in case of no epidemic happens.

References

- [1] NEWMAN, M. E. J. *Networks: An Introduction*. Oxford University Press, 2010.