CRLB下限

前提：$p(x;\theta)$ 满足正则条件

$$E\left[\frac{\partial \ln(p(x;\theta))}{\partial\theta}\right]=0$$

结论 $$var(\hat{\theta}) \geq -\frac{1}{E\left[\frac{\partial^2 \ln(p(x;\theta))}{\partial\theta^2}\right]}$$

当且仅当 $\frac{\partial \ln p(x;\theta)}{\partial\theta} = I(\theta)(g(x)-\theta)$ 时达到下限

此时 $\hat{\theta}=g(x)$， $var(\hat{\theta})=\frac{1}{I(\theta)}$

高斯噪声的CRLB

$$x[n]=s[n;\theta]+w[n],\quad n=0,1,2\cdots N-1$$

$$var(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial s[n;\theta]}{\partial\theta}\right)^2}$$

参数变换

$$\alpha=g(\theta)$$

$$var(\hat{\alpha})=\frac{\left(\frac{\partial g}{\partial\theta}\right)^2}{-E\left[\frac{\partial^2 \ln(p(x;\theta))}{\partial\theta^2}\right]}$$

扩展至矢量参数

前提：$p(x;\theta)$ 满足正则条件 $\theta=[\theta_1\ \theta_2\ \theta_3\ \cdots\ \theta_p]^T$

$$E\left[\frac{\partial \ln(p(x;\theta))}{\partial\theta}\right]=0$$

结论 $$C_{\hat{\theta}}-I^{-1}(\theta)\geq 0$$

$$[I(\theta)]_{ij}=-E\left[\frac{\partial^2 \ln p(x;\theta)}{\partial\theta_i\partial\theta_j}\right]$$

当且仅当 $\frac{\partial \ln p(x;\theta)}{\partial\theta} = I(\theta)(g(x)-\theta)$ 时达到下限

此时 $\hat{\theta}=g(x)$， $C_{\hat{\theta}}=I^{-1}(\theta)$

矢量参数变换

$$\alpha = g(\theta)$$

$$C_\alpha - \frac{\partial g(\theta)}{\partial \theta} I^{-1}(\theta) \frac{\partial g(\theta)^T}{\partial \theta} \geq 0$$

矢量高斯噪声的 CRLB

$$x \sim N(\mu(\theta), C(\theta))$$

$$I(\theta) = \left[\frac{\partial \mu(\theta)}{\partial \theta}\right]^T C^{-1}(\theta) \left[\frac{\partial \mu(\theta)}{\partial \theta}\right] + \frac{1}{2} tr\left[\left(C^{-1}(\theta) \frac{\partial C(\theta)}{\partial \theta}\right)^2\right]$$

若是 WGN 则

$$[I(\theta)]_{ij} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \frac{\partial s[n;\theta]}{\partial \theta_i} \frac{\partial s[n;\theta]}{\partial \theta_j}$$

线性模型的 MVU

$$x = H\theta + w, \quad w \sim N(0, \sigma^2 I)$$

则 $\hat{\theta} = (H^T H)^{-1} H^T x$

$$C_{\hat{\theta}} = \sigma^2 (H^T H)^{-1}$$

$$\hat{\theta} \sim N(\theta, \sigma^2 (H^T H)^{-1})$$

一般线性模型的 MVU

$$x = H\theta + s + w, \quad w \sim N(0, C)$$

则 $\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} (x - s)$

$$C_{\hat{\theta}} = (H^T C^{-1} H)^{-1}$$

Neyman-Fisher 因子分解

若能把 $p(x;\theta)$ 分解为

$$p(x;\theta) = g(T(x), \theta)\, h(x)$$

则 $T(x)$ 是 θ 的充分统计量

RBLS 流程

已知 $T(x)$ 是一个充分统计量，且 $\check{\theta}$ 是一个无偏估计.

则 MVU 为 $\hat{\theta} = E[\check{\theta} \mid T(x)]$ 与 θ 无关

且 1) $E[\hat{\theta}] = \theta$

2) $var(\hat{\theta}) \leq var(\check{\theta})$

方法：求得 $T(x)$ 后，寻找 $g(T(x))$ 使其无偏

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考虑 $\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n] = a^T x$

且 $E[x] = s\theta$

限定 $E[\hat{\theta}] = \theta \Leftrightarrow a^T s = 1$

则 $a_{opt} = \dfrac{C^{-1}s}{s^T C^{-1} s}$

$$var(\hat{\theta}) = \frac{1}{s^T C^{-1} s}$$

$$\hat{\theta} = \frac{s^T C^{-1} x}{s^T C^{-1} s}$$

高斯马尔可夫定理

$$x = H\theta + w, \quad w \text{ 协方差为 } C, \text{ 任意pdf}$$

$$\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} x$$

$$Var(\hat{\theta}_i) = [(H^T C^{-1} H)^{-1}]_{ii}$$

$$C_{\hat{\theta}} = (H^T C^{-1} H)^{-1}$$

MLE

对于固定的 x，找使 $p(x;\theta)$ 最大的 $\hat{\theta}$

MLE 渐近特性

对于足够多的数据，$\hat{\theta} \overset{a}{\sim} N(\theta, I^{-1}(\theta))$

MLE 的不变性

$$\alpha = g(\theta), \quad \hat{\alpha} = g(\hat{\theta})$$

线性模型的 MLE

$$x = H\theta + w, \quad w \sim N(0, C)$$

$$\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1} x \text{ 为 MLE}$$

贝叶斯 MSE

$$mse(\hat{A}) = \int (\hat{A} - A)^2 p(x;A)\, dx$$

$$Bmse(\hat{A}) = \iint (\hat{A} - A)^2 p(x;A)\, dx\, dA$$

$$\hat{A} = E[A|x] = \int A p(A|x)\, dA$$

$$p(A|x) = \frac{p(x|A)p(A)}{\int p(x|A) p(A)\, dA}$$

2维高斯条件pdf

$$\mu = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix} \quad C = \begin{bmatrix} \mathrm{var}(x) & \mathrm{cov}(y,x) \\ \mathrm{cov}(y,x) & \mathrm{var}(y) \end{bmatrix}$$

$$E(y|x) = E(y) + \frac{\mathrm{cov}(y,x)}{\mathrm{var}(x)}(x - E(x))$$

$$\mathrm{var}(y|x) = \mathrm{var}(y) - \frac{\mathrm{cov}^2(x,y)}{\mathrm{var}(x)}$$

多维高斯条件pdf

x, y 联合高斯.

x 是 $k\times1$
y 是 $l\times1$ 均值矢量为 $\begin{bmatrix} E(x) \\ E(y) \end{bmatrix}$, 协方差矩阵 $\begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix}$

$$E(y|x) = E(y) + C_{yx} C_{xx}^{-1}(x - E(x))$$

$$C_{y|x} = C_{yy} - C_{yx} C_{xx}^{-1} C_{xy}$$

贝叶斯 MSE 高斯先验 pdf

$$x \sim N(0, \sigma^2) \quad A \sim N(\mu_A, \sigma_A^2)$$

$$p(x|A)\cdot p(A) = \frac{1}{(2\pi\sigma^2)^{N/2}(2\pi\sigma_A^2)} \exp\left(-\frac{1}{2\sigma^2}\sum x^2[n]\right) \exp\left(-\frac{1}{2}Q(A)\right)$$

$$Q(A) = \frac{1}{\sigma_{A|x}^2}(A - \mu_{A|x})^2 - \frac{\mu_{A|x}^2}{\sigma_{A|x}^2} + \frac{\mu_A^2}{\sigma_A^2}$$

贝叶斯线性模型

$$x = H\theta + w, \quad \theta \sim N(\mu_\theta, C_\theta)$$
$$w \sim N(0, C_w)$$

$$E(\theta|x) = \mu_\theta + C_\theta H^T (H C_\theta H^T + C_w)^{-1} (x - H\mu_\theta)$$
$$= \mu_\theta + (C_\theta^{-1} + H^T C_w^{-1} H)^{-1} H^T C_w^{-1} (x - H\mu_\theta)$$

$$C_{\theta|x} = C_\theta - C_\theta H^T (H C_\theta H^T + C_w)^{-1} H C_\theta$$
$$= (C_\theta^{-1} + H^T C_w^{-1} H)^{-1}$$

最大后验估计量

$$\hat{\theta} = \arg\max_\theta p(\theta|x)$$
$$= \arg\max_\theta p(x|\theta)\, p(\theta)$$

贝叶斯高斯马尔可夫定理

$x = H\theta + w$, 已知 $C_{\theta\theta}$, C_w. $p(w, \theta)$ 任意, LMMSE估计为

$$\hat{\theta} = \mu_\theta + (C_\theta^{-1} + H^T C_w^{-1} H)^{-1} H^T C_w^{-1} (x - H\mu_\theta)$$
$$C_\varepsilon = (C_\theta^{-1} + H^T C_w H)^{-1}$$

序贯LMMSE估计

$$\hat{A}[N] = \hat{A}[N-1] + K[N](x[N] - \hat{A}[N-1])$$
$$K[N] = \frac{B_{mse}(\hat{A}[N-1])}{B_{mse}(\hat{A}[N-1]) + \sigma^2}$$

$$B_{mse}(\hat{A}[N]) = (1 - K[N])\, B_{mse}(\hat{A}[N-1])$$

$$(X|Y) \sim N\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$$

$$E(y|x) = E(y) + \frac{\mathrm{cov}(y,x)}{\mathrm{var}(x)}(x - E(x))$$

$$\mathrm{var}(y|x) = \mathrm{var}(y) - \frac{\mathrm{cov}^2(x,y)}{\mathrm{var}(x)}$$

$$E(y|x) = E(y) + C_{yx} C_{xx}^{-1}(x - E(x))$$

$$C_{y|x} = C_{yy} - C_{yx} C_{xx}^{-1} C_{xy}$$

$$\hat{\theta} = \mu_\theta + (C_\theta^{-1} + H^T C_w^{-1} H)^{-1} H^T C_w^{-1}(x - H\mu_\theta)$$

$$C_\varepsilon = (C_\theta^{-1} + H^T C_w^{-1} H)^{-1}$$