09 级一期 A 卷参考解答

一.(每小题 6 分,共 12 分)求下列极限:

1. $\lim\limits_{x\to\infty} x\left(e^{\frac{2}{x}}-1\right)$;

解 $\lim\limits_{x\to\infty} x\left(e^{\frac{2}{x}}-1\right)=\lim\limits_{x\to\infty}\dfrac{e^{2/x}-1}{1/x}=\lim\limits_{x\to\infty}\dfrac{e^{2/x}\cdot\left(-2/x^2\right)}{-1/x^2}=2\lim\limits_{x\to\infty}e^{2/x}=2e^0=2$

2. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

解 令 $y=\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$,则

$\lim\limits_{x\to 0}\ln y=\lim\limits_{x\to 0}\dfrac{1}{x^2}\ln\left(\dfrac{\sin x}{x}\right)=\lim\limits_{x\to 0}\dfrac{\ln\sin x-\ln x}{x^2}=\lim\limits_{x\to 0}\dfrac{\dfrac{\cos x}{\sin x}-\dfrac{1}{x}}{2x}$

$=\lim\limits_{x\to 0}\dfrac{x\cos x-\sin x}{2x^2\sin x}=\lim\limits_{x\to 0}\dfrac{\cos x-x\sin x-\cos x}{4x\sin x+2x^2\cos x}=-\lim\limits_{x\to 0}\dfrac{\sin x}{4\sin x+2x\cos x}$

$=-\lim\limits_{x\to 0}\dfrac{\cos x}{4\cos x+2\cos x-2x\sin x}=-\dfrac{1}{6}$, 于是 $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}=\lim\limits_{x\to 0}e^y=e^{-1/6}$.

二.(每小题 6 分,共 24 分)完成如下各题

1. $\displaystyle\int\dfrac{2x^2+1}{x^2(1+x^2)}\,dx$;

解 原式$=\displaystyle\int\left(\dfrac{x^2+x^2+1}{x^2(x^2+1)}\right)dx=\int\dfrac{1}{x^2}\,dx+\int\dfrac{1}{x^2+1}\,dx=-\dfrac{1}{x}+\arctan x+C$

2. $\displaystyle\int\dfrac{dx}{1+\sqrt[3]{x+2}}$;

解 令 $x+2=t^3, dx=3t^2 dt$,则

原式 $= \int \frac{3t^2}{1+t} dt = 3\int \frac{t^2-1+1}{1+t} dt = 3\int (t-1)dt + 3\int \frac{1}{1+t} dt$

$= \frac{3t^2}{2} - 3t + 3\ln|t+1| + C \quad = \frac{3}{2}(x+2)^{\frac{2}{3}} - 3(x+2)^{\frac{1}{3}} + 3\ln\left|(x+2)^{\frac{1}{3}}+1\right| + C.$

3. $\int_0^4 e^{\sqrt{x}} dx$;

解　令 $t = \sqrt{x}$, 则

$$\int_0^4 e^{\sqrt{x}} dx = 2\int_0^2 te^t dt = 2\left[te^t \Big|_0^2 - \int_0^2 e^t dt \right] = 2\left(2e^2 - e^t \Big|_0^2 \right) = 2(e^2+1).$$

4.求证: $\int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx,$ 并求此积分.

证明　令 $t = \frac{\pi}{2} - x$, 则

左边 $= \int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \int_{\frac{\pi}{2}}^0 \frac{\sin^{2010}\left(\frac{\pi}{2}-t\right)}{\sin^{2010}\left(\frac{\pi}{2}-t\right) + \cos^{2010}\left(\frac{\pi}{2}-t\right)} d\left(\frac{\pi}{2}-t\right)$

$= -\int_{\frac{\pi}{2}}^0 \frac{\cos^{2010} t}{\sin^{2010} t + \cos^{2010} t} dt = \int_0^{\frac{\pi}{2}} \frac{\cos^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = $右边.

而,左边+右边$= \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2},$　故　$\int_0^{\frac{\pi}{2}} \frac{\sin^{2010} x}{\sin^{2010} x + \cos^{2010} x} dx = \frac{\pi}{4}.$

三.(每小题 7 分,共 21 分)完成如下各题:

1.设 $u(x,y) = \ln\sqrt{1+x^2+y^2}$,求 $du\big|_{(1,2)}$.

解　$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1+x^2+y^2}} \cdot \frac{2x}{2\sqrt{1+x^2+y^2}} = \frac{x}{1+x^2+y^2},\quad \frac{\partial u}{\partial y} = \frac{y}{1+x^2+y^2},$　于是

$\frac{\partial u}{\partial x}\bigg|_{(1,2)} = \frac{x}{1+x^2+y^2}\bigg|_{(1,2)} = \frac{1}{1+1+4} = \frac{1}{6}, \frac{\partial u}{\partial y}\bigg|_{(1,2)} = \frac{y}{1+x^2+y^2}\bigg|_{(1,2)} = \frac{2}{6} = \frac{1}{3},$

$$du\big|_{(1,2)} = \frac{\partial u}{\partial x}\bigg|_{(1,2)} dx + \frac{\partial u}{\partial y}\bigg|_{(1,2)} dy = \frac{dx + 2dy}{6}.$$

2.已知 $f(x,y,z) = 2xy - z^2$ 及点 $A(2,-1,1), B(3,1,-1)$,求函数 $f(x,y,z)$ 在点 A 处沿

由 A 到 B 方向的方向导数,并求此函数在点 A 处方向导数的最大值.

解　　$\boldsymbol{l} = \overrightarrow{AB} = (1,2,-2),\quad (\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right),$

$$\frac{\partial f}{\partial x} = 2y, \frac{\partial f}{\partial y} = 2x, \frac{\partial f}{\partial z} = -2z.\quad \frac{\partial f}{\partial x}\bigg|_{(2,-1,1)} = -2,\ \frac{\partial f}{\partial y}\bigg|_{(2,-1,1)} = 4,\ \frac{\partial f}{\partial z}\bigg|_{(2,-1,1)} = -2.$$

因此, $\dfrac{\partial f}{\partial \boldsymbol{l}} = \dfrac{\partial f}{\partial x}\cos\alpha + \dfrac{\partial f}{\partial y}\cos\beta + \dfrac{\partial f}{\partial z}\cos\gamma = (-2)\cdot\dfrac{1}{3} + 4\cdot\dfrac{2}{3} + (-2)\cdot\left(-\dfrac{2}{3}\right) = \dfrac{10}{3}.$

而在点 A 处方向导数的最大值为　　$|\boldsymbol{g}| = \sqrt{(-2)^2 + 4^2 + (-2)^2} = 2\sqrt{6}.$

3.设函数 $z = z(x,y)$ 由方程 $z^3 - 3xyz = 1$ 给出,求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ 及 $\dfrac{\partial^2 z}{\partial x^2}$.

解　　令 $F(x,y) = z^3 - 3xyz - 1,\quad \dfrac{\partial F}{\partial x} = -3yz, \dfrac{\partial F}{\partial y} = -3xz, \dfrac{\partial F}{\partial z} = 3z^2 - 3xy,$

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial z} = -\frac{-3yz}{3z^2 - 3xy} = \frac{yz}{z^2 - xy},$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\bigg/\frac{\partial F}{\partial z} = -\frac{-3xz}{3z^2 - 3xy} = \frac{xz}{z^2 - xy},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{yz}{z^2 - xy}\right) = \frac{y\dfrac{\partial z}{\partial x}\cdot(z^2 - xy) - yz\left(2z\dfrac{\partial z}{\partial x} - y\right)}{(z^2 - xy)^2},$$

$$= \frac{y^2 z - y(xy + z^2)\dfrac{\partial z}{\partial x}}{(z^2 - xy)^2} = \frac{y^2 z - y(xy + z^2)\cdot\dfrac{yz}{z^2 - xy}}{(z^2 - xy)^2}$$

$$= -\frac{2xy^3 z}{(z^2 - xy)^3}.$$

四.(第一小题 4 分,第二小题 6 分,共 10 分)

1.已知点 $A(2,2,2), B(4,4,2), C(4,2,4)$, 求向量 $\overrightarrow{AB}, \overrightarrow{AC}$ 的夹角.

解　$\overrightarrow{AB} = (2,2,0), \overrightarrow{AC} = (2,0,2)$, 设所求夹角为 α, 则

$$\cos\alpha = \frac{\overrightarrow{AB} \bullet \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{2\times2+2\times0+0\times2}{\sqrt{2^2+2^2+0^2}\cdot\sqrt{2^2+0^2+2^2}} = \frac{1}{2}, \quad \alpha = \frac{\pi}{3}.$$

2.求经过直线 $L_1 : \begin{cases} x+y=0, \\ x-y-z-2=0, \end{cases}$ 且平行于直线 $L_2 : x=y=z$ 的平面方程.

解　L_1 的参数方程为 $x=t, y=-t, z=2(t-1)$, 化为标准方程为

$$\frac{x}{1} = \frac{y}{-1} = \frac{z+2}{2},$$

其方向向量为 $\boldsymbol{l}_1 = (1,-1,2)$, 而直线 L_2 的方向向量为 $\boldsymbol{l}_2 = (1,1,1)$, 故所求平面法向量为

$$\boldsymbol{n} = \boldsymbol{l}_1 \times \boldsymbol{l}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k} = (-3,1,2).$$

所求平面过点 $(0,0,-2)$, 故所求平面方程为 $-3x+y+2(z+2)=0$, 即 $3x-y-2z=4$.

五.(7 分)求函数 $f(x) = \int_0^x (t-1)(t-2)^2 dt$ 的极值.

解　$f'(x) = (x-1)(x-2)^2$, 从而驻点为 $x_1=1, x_2=2$. 列表如下

x	$(-\infty,1)$	1	（1，2）	2	（2，$+\infty$）
$f'(x)$	—	0	+	0	+
$f(x)$	↘	极小值	↗	非极值	↗

所求函数最小值为

$$f(1) = \int_0^1 (t-1)(t-2)^2 dt = \int_0^1 (t^3 d - 5t^2 + 8t - 4)dt$$

$$= \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}.$$

六.(12 分)设函数 $f(x) = \dfrac{x^3}{2(1+x)^2}$,求(1)函数的单调区间与极值点;(2)函数的凹凸区间与拐点;(3)函数的渐近线.

解 函数的定义域为 $(-\infty,\ -1) \cup (-1, +\ \infty)$,且

$$f'(x) = \frac{3x^2(1+x)^2 - x^3 \cdot 2(1+x)}{2(1+x)^4} = \frac{x^2(x+3)}{2(1+x)^3}.$$

$$f''(x) = \frac{(3x^2+6x)(1+x)^3 - (x^3+3x^2) \cdot 3(1+x)^2}{2(1+x)^6} = \frac{3x}{(1+x)^4}.$$

从而函数的驻点为 0, −3.又二阶导数为零的点为 0,列表如下:

x	$(-\infty,-3)$	-3	$(-3,-1)$	$(-1,0)$	0	$(0,+\infty)$
$f'(x)$	+	0	−	+	0	+
$f''(x)$	−	−	−	−	0	+
$f(x)$	凸↗	极大	凸↘	凸↗	拐点	凹↗

函数的单调增加区间为 $(-\infty,-3)$ 和 $(0,+\infty)$,单调减少区间为 $(-3,\ -1)$.极小值点为−3.凹区间为 $(0,+\infty)$,凸区间为 $(-\infty,-1)$ 和 $(-1,0)$,拐点为 $(0,0)$.下面再求渐近线.

显然,直线 $x=-1$ 是垂直渐近线.而

$$\lim_{x \to \infty} \frac{x^3}{2(1+x)^2} = \infty$$

因而曲线无水平渐近线,但

$$\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^3}{2x(1+x)^2} = \frac{1}{2},$$

$$\lim_{x \to \infty} \left[f(x) - \frac{x}{2} \right] = \lim_{x \to \infty} \left[\frac{x^3}{2(1+x)^2} - \frac{x}{2} \right] = -1,$$

因而曲线有斜渐近线 $y = \dfrac{1}{2}x - 1.$

七.(每小题 7 分,共 14 分)

1.求证:$1+x\ln\left(x+\sqrt{1+x^2}\right)\geq\sqrt{1+x^2}, x\in R.$

证　令 $f(x)=1+x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}$,则当 $x>0$ 时,

$$f'(x)=\ln\left(x+\sqrt{1+x^2}\right)+\frac{x\left(1+\dfrac{x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}}-\frac{x}{\sqrt{1+x^2}}=\ln\left(x+\sqrt{1+x^2}\right)>0,$$

故此函数单调增加.而容易验证 $f(0)=0$,故当 $x>0$ 时,$f(x)\geq0$,此即

$$1+x\ln\left(x+\sqrt{1+x^2}\right)\geq\sqrt{1+x^2}, x>0.$$

又, $f(-x)=1-x\ln\left(-x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}=1-x\ln\left(\frac{1}{x+\sqrt{1+x^2}}\right)-\sqrt{1+x^2}=f(x),$

从而　　　　$1+x\ln\left(x+\sqrt{1+x^2}\right)\geq\sqrt{1+x^2}, x\in R.$

2.设函数 $f(x)$ 在闭区间[0,1]上连续,在开区间(0,1)内可导,且 $f(0)=0, f(1)=1,$ 求证:

(1)存在 $\alpha\in(0,1)$,使得$f(\alpha)=1-\alpha$;

(2)存在两个不同的点 $\xi\in(0,1),\eta\in(0,1)$,满足 $f'(\xi)f'(\eta)=1.$

证　(1)令 $g(x)-f(x)+x-1,$　则 $g(0)=f(0)-1<0, g(1)=f(1)+1-1=1>0.$

于是由介值定理,存在 $\alpha\in(0,1)$,使得$g(\alpha)=0,$ 即 $f(\alpha)=1-\alpha$;

(2)由 Largranger 定理,在区间$(0,\alpha)$内存在ξ,使得

$$f'(\xi)=\frac{f(\alpha)-f(0)}{\alpha-0}=\frac{1-\alpha}{\alpha}.$$

在区间$(\alpha,1)$内,存在η,使得

$$f'(\eta)=\frac{f(1)-f(\alpha)}{1-\alpha}=\frac{\alpha}{1-\alpha}.$$

于是存在两个不同的点 $\xi\in(0,1),\eta\in(0,1)$,满足 $f'(\xi)f'(\eta)=1.$