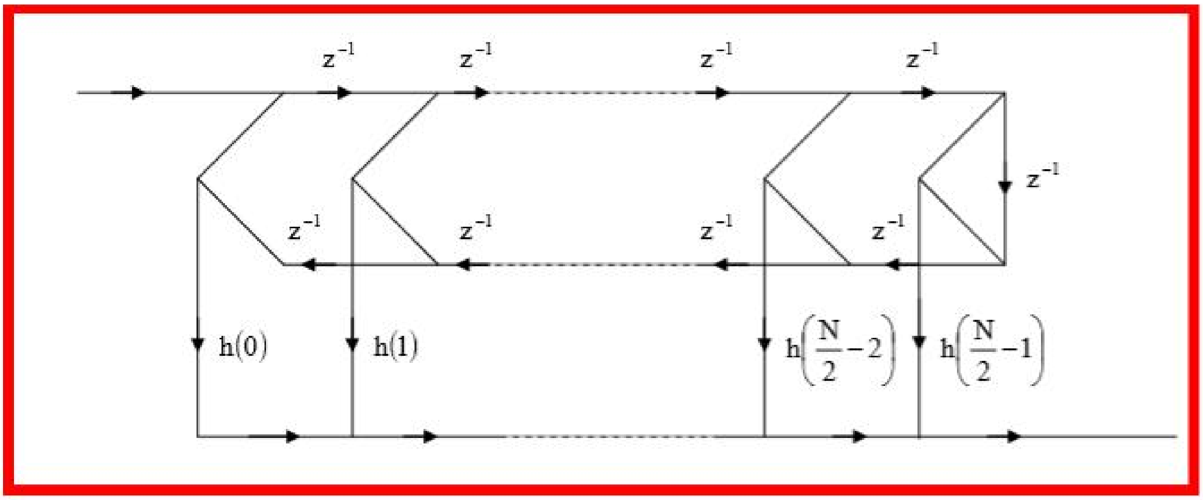
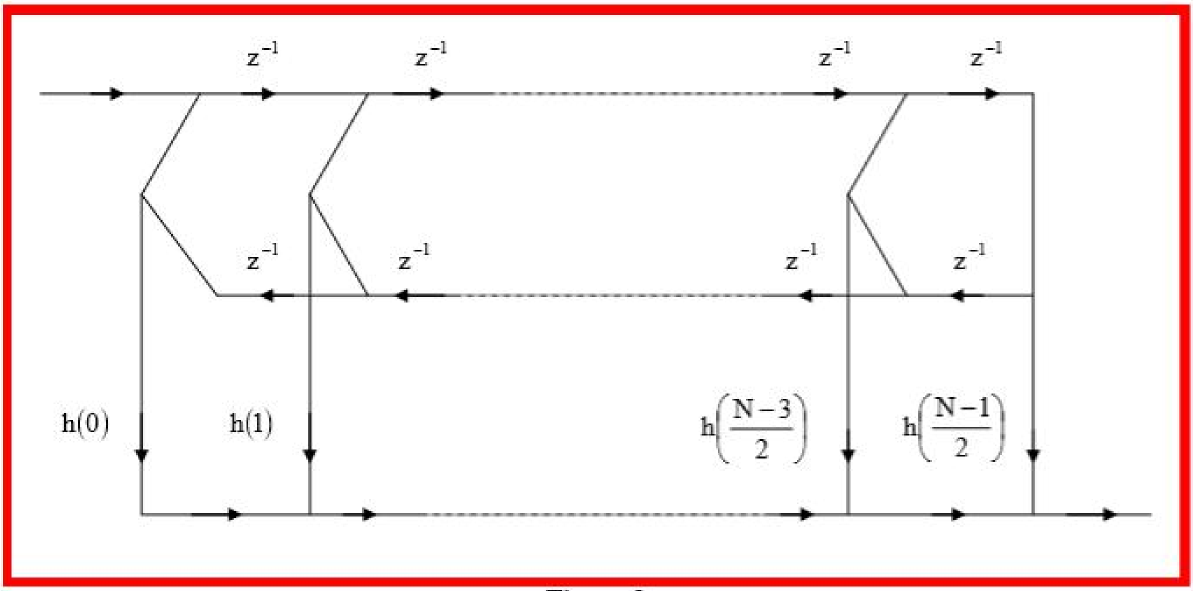
一、Let $h(n)$ be the impulse response of a FIR filter with $h(n)=h(N-1-n)$, please draw the direct form of its network structure when N is odd/even.

Direct Forms The network structure of a linear phase FIR filter can be realized directly from its system function:

(1) for N even, $H(z)=\sum_{n=0}^{\frac{N}{2}-1}h(n)z^{-n}+\sum_{n=0}^{\frac{N}{2}-1}h(n)z^{-(N-1-n)}$



(2) for N odd, $H(z)=\sum_{n=0}^{\frac{N-1}{2}-1}h(n)z^{-n}+\sum_{n=0}^{\frac{N-1}{2}-1}h(n)z^{-(N-1-n)}+h\left(\frac{N-1}{2}\right)z^{-\frac{N-1}{2}}$



二、Suppose the function system of a filter is

$$H(z)=\sum_{n=0}^{2}h(n)z^{-n}\ ,\quad H(z)=\frac{1}{\sum_{n=0}^{2}h(n)z^{-n}}$$

where $h(0)=1$, $h(2)\neq -1$. realize $H(z)$ using an all-zero grid network structure.

Example Suppose the function system of a IIR filter is

$$H(z)=\frac{1}{\sum_{n=0}^{2}h(n)z^{-n}},\ \text{where } h(0)=1,\ h(2)\neq -1$$

realize $H(z)$ using an all-pole grid network structure.

Solution:

Since the transfer function of a two-order all-pole grid network structure is of the form

$$G(z)=\frac{1}{[1,0]\begin{bmatrix}1 & \alpha_2 z^{-1}\\ \alpha_2 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & \alpha_1 z^{-1}\\ \alpha_1 & z^{-1}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}}=\frac{1}{1+\alpha_1(1+\alpha_2)z^{-1}+\alpha_2 z^{-2}}$$

the parameters of $G(z)$ is then seen to be

$$\alpha_2=h(2),\ \alpha_1=\frac{h(1)}{1+h(2)}\ \#$$

Example Suppose the function system of a FIR filter is

$$H(z)=\sum_{n=0}^{2}h(n)z^{-n},\ \text{where } h(0)=1,\ h(2)\neq -1$$

realize $H(z)$ using an all-zero grid network structure.

Solution:

Since the transfer function of a two-order all-zero grid network structure is of the form

$$G(z)=[1,0]\begin{bmatrix}1 & \alpha_2 z^{-1}\\ \alpha_2 & z^{-1}\end{bmatrix}\begin{bmatrix}1 & \alpha_1 z^{-1}\\ \alpha_1 & z^{-1}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=1+\alpha_1(1+\alpha_2)z^{-1}+\alpha_2 z^{-2}$$

the parameters of $G(z)$ is then seen to be

$$\alpha_2=h(2),\ \alpha_1=\frac{h(1)}{1+h(2)}\ \#$$

三、Analog Butterworth Filters：cutoff frequency：Ω_c，order：$N=6$，write out the system function of analog Butterworth filter：$H_a(s)=?$

$$H_a(s)H_a(-s)=\frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

(1) The $H_a(s)H_a(-s)$ has $2N$ poles, which are equally spaced in angle on a circle of radius Ω_c in the s-plane:

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N}=0 \Rightarrow s^{2N}+(j\Omega_c)^{2N}=0$$

$$\Rightarrow s^{2N}=-(j\Omega_c)^{2N}=e^{j(2k+1)\pi}(j\Omega_c)^{2N}=\left(j\Omega_c e^{j\frac{2k+1}{2N}\pi}\right)^{2N}$$

$$\Rightarrow s_k=j\Omega_c e^{j\frac{2k+1}{2N}\pi},\ k=0,1,\cdots,2N-1$$

The phrase difference of two adjacent poles is

$$\frac{2(k+1)+1}{2N}\pi-\frac{2k+1}{2N}\pi=\frac{\pi}{2}$$

(2) If s_k is a pole of $H_a(s)H_a(-s)$, then $-s_k$, s_k^* and $-s_k^*$ must be also the poles of $H_a(s)H_a(-s)$. This means that the distribution of poles between 0 and $\frac{\pi}{2}$ can determine the distribution of poles over 0 and 2π.

$$H_a(s)H_a(-s)=\frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}=\frac{(j\Omega_c)^{2N}}{s^{2N}+(j\Omega_c)^{2N}}=\frac{(-1)^N\,\Omega_c^{2N}}{\left(\prod_{k=1}^{N}(s-\alpha_k)\right)\left(\prod_{k=1}^{N}(s+\alpha_k)\right)}$$

$$=\left(\prod_{k=1}^{N}\frac{\Omega_c}{s-\alpha_k}\right)\left(\prod_{k=1}^{N}\frac{-\Omega_c}{s+\alpha_k}\right)=\left(\prod_{k=1}^{N}\frac{\Omega_c}{s-\alpha_k}\right)\left(\prod_{k=1}^{N}\frac{\Omega_c}{-s-\alpha_k}\right)$$

where $H_a(s)=\prod_{k=1}^{N}\frac{\Omega_c}{s-\alpha_k}$. It is clear that, if we arrange the poles $\alpha_1,\alpha_2,\cdots,\alpha_N$ to be all located in the left half of the s-plane, then the Butterworth filter as described will be stable and causal.

四、Let $H_{proto}(z)$ be the system function of a digital lowpass filter with cutoff frequency θ_c, please find out a function $f(z)$ such that $H(z)=H_{proto}(f(z))$ is a lowpass/highpass filter with cutoff frequency ω_c

(1) Let $H_{proto}(z)$ be the system function of the prototype digital lowpass filter with θ_c as its cutoff frequency, by bilinear transformation, we obtain the system function $H_1(s)$ of an analog lowpass filter

$$H_1(s)=H_{proto}(z)\Big|_{z=\frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}}=H_{proto}\left(\frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}\right)$$

Note that the cutoff frequency of $H_1(s)$ corresponding to that of $H_{proto}(z)$ is

$\Omega_c=\frac{2}{T}tg\left(\frac{\theta_c}{2}\right)$.

(2) Let $p=\frac{s}{\Omega_c}$, we obtain the normalized system function $H_2(p)$:

$$H_2(p)=H_1(s)\big|_{s=\Omega_c p}=H_{proto}\left(\frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}\right)\Bigg|_{s=\Omega_c p}=H_{proto}\left(\frac{1+\frac{T\Omega_c}{2}p}{1-\frac{T\Omega_c}{2}p}\right)$$

Note that the cutoff frequency of $H_2(p)$ corresponding to that of $H_1(s)$ becomes 1.

(3) Let $q=W_c p$, where $W_c=\frac{2}{T}tg\left(\frac{\omega_c}{2}\right)$, we obtain the denormalized system function $H_3(q)$:

$$H_3(q)=H_2(p)\big|_{p=\frac{q}{W_c}}=H_{proto}\left(\frac{1+\frac{T\Omega_c}{2}p}{1-\frac{T\Omega_c}{2}p}\right)\Bigg|_{p=\frac{q}{W_c}}=H_{proto}\left(\frac{1+\frac{T}{2}\frac{\Omega_c}{W_c}q}{1-\frac{T}{2}\frac{\Omega_c}{W_c}q}\right)$$

Note that the cutoff frequency of $H_3(q)$ corresponding to that of $H_2(p)$ becomes W_c.

(4) By bilinear transformation again, we obtain the system function $H_d(z)$ of the desired digital lowpass filter:

$$H_d(z)=H_3(q)\big|_{q=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}=H_{proto}\left(\frac{1+\frac{T}{2}\frac{\Omega_c}{W_c}q}{1-\frac{T}{2}\frac{\Omega_c}{W_c}q}\right)\Bigg|_{q=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}=H_{proto}\left(\frac{1+\frac{\Omega_c}{W_c}\frac{1-z^{-1}}{1+z^{-1}}}{1-\frac{\Omega_c}{W_c}\frac{1-z^{-1}}{1+z^{-1}}}\right)$$

$$= H_{proto}\left(\frac{z^{-1} - \dfrac{\Omega_c + W_c}{\Omega_c - W_c}}{1 - \dfrac{\Omega_c + W_c}{\Omega_c - W_c} z^{-1}} \right) = H_{proto}(f(z))$$

where

$$f(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \text{ and } \alpha = \frac{\Omega_c + W_c}{\Omega_c - W_c} = \frac{tg\dfrac{\theta_c}{2} + tg\dfrac{\omega_c}{2}}{tg\dfrac{\theta_c}{2} - tg\dfrac{\omega_c}{2}} = \frac{sin\dfrac{\theta_c + \omega_c}{2}}{sin\dfrac{\theta_c - \omega_c}{2}}$$

Note that the cutoff frequency of $H_d(z)$ is ω_c.

五、Please prove that $H(e^{j\omega}) = e^{-j\frac{N-1}{2}\omega} \sum_{m=0}^{\frac{N-1}{2}} \alpha_m cos(m\omega)$ is the frequency response of a linear phase FIR filter when even symmetric and N odd.

For the case of even symmetry and N being odd, we have

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\frac{N-1}{2}\omega} + \sum_{n=\frac{N-1}{2}+1}^{N-1} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\frac{N-1}{2}\omega} + \sum_{n=\frac{N-1}{2}+1}^{N-1} h(N-1-n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-1}{2}-1} h(n) e^{-j(N-1-n)\omega} + h\left(\frac{N-1}{2}\right) e^{-j\frac{N-1}{2}\omega}$$

$$= e^{-j\frac{N-1}{2}\omega} \left[\sum_{n=0}^{\frac{N-1}{2}-1} h(n) \left[e^{j\left(\frac{N-1}{2}-n\right)\omega} + e^{-j\left(\frac{N-1}{2}-n\right)\omega} \right] + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\frac{N-1}{2}\omega} \left[\sum_{n=0}^{\frac{N-1}{2}-1} 2h(n) cos\left[\omega\left(\frac{N-1}{2} - n\right) \right] + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\frac{N-1}{2}\omega} A_N(\omega)$$

where

$$A_N(\omega) = \sum_{n=0}^{\frac{N-1}{2}-1} 2h(n)\cos\left[\omega\left(\frac{N-1}{2}-n\right)\right] + h\left(\frac{N-1}{2}\right)$$

$A_N(\omega)$ can be further simplified

$$A_N(\omega) = \sum_{n=0}^{\frac{N-1}{2}-1} 2h(n)\cos\left[\omega\left(\frac{N-1}{2}-n\right)\right] + h\left(\frac{N-1}{2}\right)$$

$$\underset{m=\frac{N-1}{2}-n}{=} \sum_{m=\frac{N-1}{2}}^{1} 2h\left(\frac{N-1}{2}-m\right)\cos(m\omega) + h\left(\frac{N-1}{2}\right) = \sum_{m=0}^{\frac{N-1}{2}} \alpha_m \cos(m\omega)$$

where the parameters $\alpha_0, \alpha_1, \cdots, \alpha_{\frac{N-1}{2}}$ are given by

$$\alpha_m = \begin{cases} h\left(\frac{N-1}{2}\right) & m = 0 \\ 2h\left(\frac{N-1}{2}-m\right) & m \neq 0 \end{cases}$$