一阶高斯马尔可夫过程

$$s[n] = a s[n-1] + u[n], \quad n \geq 0$$

$$s[-1] \sim N(\mu_s, \sigma_s^2) \qquad u[n] \sim N(0, \sigma_u^2)$$

$$\Rightarrow \quad s[n] = a^{n+1} s[-1] + \sum_{k=0}^{n} a^k u[n-k]$$

可证得 $E[s[n]] = a^{n+1} E[s[-1]] = a^{n+1}\mu_s$

$$C_s[m,n] = E\left[\left(s[m] - E(s[m])\right)\left(s[n] - E[s[n]]\right)\right]$$

$$= E\left[\left(a^{m+1}(s[-1]-\mu_s) + \sum_{k=0}^{m} a^k u[m-k]\right) \cdot \left(a^{n+1}(s[-1]-\mu_s) + \sum_{l=0}^{n} a^l u[n-l]\right)\right]$$

$$= a^{m+n+2}\sigma_s^2 + \sum_{k=0}^{m}\sum_{l=0}^{n} a^{k+l} E[u[m-k]u[n-l]]$$

$$= a^{m+n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=m-n}^{m} a^{2k+n-m}$$

$$= a^{m+n+2}\sigma_s^2 + \sigma_u^2 a^{m-n} \sum_{k=0}^{n} a^{2k}$$

$$\operatorname{var}(s[n]) = C_s[n,n] = a^{2n+2}\sigma_s^2 + \sigma_u^2 \sum_{k=0}^{n} a^{2k}$$

1 与m和n的取值相关，不是广义平稳过程

2 $n \to \infty$ 时 $E[s[n]] \to 0$

$$C_s[m,n] \to \frac{\sigma_u^2 a^{m-n}}{1-a^2}$$ 趋近于平稳

3 递推公式

$$E[s[n]] = aE[s[n-1]] + u[n])$$

$$var(s[n]) = E[(s[n] - E(s[n]))^2]$$

$$= E[(as[n-1] + u[n] - aE(s[n-1]))^2]$$

$$= a^2 var(s[n-1]) + \sigma_u^2$$

标量卡尔曼滤波

问题

有 标量状态方程 $s[n] = as[n-1] + u[n]$

标量观测方程 $x[n] = s[n] + w[n]$

其中有
$$\left.\begin{array}{l} s[-1] \sim N(\mu_s, \sigma_s^2) \\ u[n] \sim N(0, \sigma_u^2) \\ w[n] \sim N(0, \sigma_n^2) \end{array}\right\} \text{三者相互独立}$$

求 $s[n]$ 给定 $x[0], x[1] \cdots x[n]$ 的 LMMSE 估计

分析

1

$\hat{s}[n|n]$ 是线性表示递推的形式.

$$\hat{s}[n|n] = \sum_{i=0}^{n} a_i x[i] + b$$

$s[n]$ 是高斯马尔可夫过程. LMMSE = MMSE

满足以下准则

正交准则 $E[(s[n] - \hat{s}[n|n])x[i]] = 0$

均方误差 $M[n|n] = E[(s[n] - \hat{s}[n|n])^2]$

$$= E[(s[n] - \hat{s}[n|n])s[n]]$$

2 最终结

$$\hat{s}[n|n] = a(1-K[n])\,\hat{s}[n-1|n-1] + K[n]x[n]$$

$$K[n] = \frac{M[n|n]}{\sigma_n^2}$$

推导　利用正交订则

设 $a_n = a(1-K[n])$

则 $\hat{s}[n|n] = a_n \hat{s}[n-1|n-1] + K[n]x[n]$

$$E[(s[n] - a_n\hat{s}_{n-1} - K[n]x[n])\,x[i]] = 0$$

为了去掉 $\hat{s}_{n-1}$ 项，加入 $a_n s[n-1]$，拆分 $x[n]$

$$E[(s[n] - a_n s[n-1] + \underbrace{a_n(s[n-1] - \hat{s}_{n-1})} - K[n]s[n] - \underbrace{K[n]w[n]})\,x[i]] = 0$$

又有 $E[w[n]x[i]] = 0$

$E[(s[n-1] - \hat{s}_{n-1})x[i]] = 0$ 所以划线部分为0

整理得 $$E[((1-K[n])s[n] - a_n s[n-1])\,x[i]] = 0$$

拆分 $x[i]$，去掉 $w[i]$，即 $$E[((1-K[n])s[n] - a_n s[n-1])\,s[i]] = 0$$

因此 $$(1-K[n])\,r_s[n,i] - a_n r_s[n-1,i] = 0$$

$$\frac{a_n}{1-K[n]} = \frac{r_s[n,i]}{r_s[n-1,i]}$$

$$\frac{r_s[n,i]}{r_s[n-1,i]} = \frac{C_s[n,i] + a^{n+i+2}\mu_s^2}{C_s[n-1,i] + a^{n+i+1}\mu_s^2} = 0$$

再用正交原则 $E\left[\left(s[n]-\alpha_n\hat{s}_{n-1}-K[n]x[n]\right)\left(s[n]+w[n]\right)\right]=0$

$$r_s[n,n]-a_n E[\hat{s}_{n-1}s[n]]-K[n]\left(r_s[n,n]+\sigma_n^2\right)=0 \quad ①$$

误差矩阵 $M[n|n]=E[(s[n]-\hat{s}[n|n])s[n]]$

$$=E[(s[n]-\alpha_n\hat{s}_{n-1}-K[n]x[n])s[n]]$$

$$=r_s[n,n]-\alpha_n E[\hat{s}_{n-1}s[n]]-k[n]r_s[n,n] \quad ②$$

联立①②得

$$M[n|n]=K[n]\sigma_n^2$$

结论

预测 $\hat{s}[n|n-1]=a\hat{s}[n-1|n-1]$

最小预测MSE $M[n|n-1]=a^2M[n-1|n-1]+\sigma_u^2$

卡尔曼增益 $K[n]=\dfrac{M[n|n-1]}{\sigma_n^2+M[n|n-1]}$

修正 $\hat{s}[n|n]=\hat{s}[n|n-1]+k[n]\left(x[n]-\hat{s}[n|n-1]\right)$

最小MSE $M[n|n]=(1-k[n])M[n|n-1]$