线性模型的MVU估计量

$$x = H\theta + w$$

其中 $w \sim N(0, \sigma^2 I)$

若要求 x 的似然函数，先写出 w 的分布

$$p(w) = \frac{\exp(-\frac{1}{2}[w^T \sigma^{-2} I w])}{(2\pi)^{N/2} \sigma^N}$$ 即

$$p(w) = \frac{1}{(2\pi)^{N/2}\sigma^N} \exp(-\frac{1}{2\sigma^2} w^T w)$$

$$w \sim N(0, \sigma^2 I)$$

$$X \sim N(H\theta, \sigma^2 I)$$

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}(x - H\theta)^T(x - H\theta)\right)$$

$$\ln p(x;\theta) = C - \frac{1}{2\sigma^2}(x - H\theta)^T(x - H\theta)$$

$$\frac{\partial \ln p}{\partial \theta} = -\frac{1}{2\sigma^2}\left(-2H^T(x - H\theta)\right)$$

$$= \frac{1}{\sigma^2}(H^T x - (H^T H)\theta)$$

$$= \frac{1}{\sigma^2}(H^T H)(H^T H)^{-1}(H^T x - (H^T H)\theta)$$

$$= \frac{1}{\sigma^2}(H^T H)[(H^T H^{-1})H^T x - \theta)$$

其中 $I(\theta) = \frac{1}{\sigma^2} H^T H$，$g(x) = (H^T H)^{-1} H x$

协方差矩阵为 $I^{-1}(\theta) = \sigma^2 (H^T H)^{-1}$

证明 rank(H) = rank(H^TH)

先证 rank(H) = rank(H^TH)

要证 rank(H) = rank(H^TH)

即证 $Hx=0$ 与 $H^THx=0$ 同解

$Hx=0$ 左右乘 H^T $\Rightarrow$ $H^THx=0$

$H^THx=0$ 左右乘 x^T $\Rightarrow$ $x^TH^THx=0$

$y^Ty=0$ 其中 $y=Hx$

要使 $y^Ty=0$

$\therefore$ 须有 $y=Hx=0$

所以 $Hx=0 \Leftrightarrow H^THx=0$

即 rank(H) = rank(H^TH) 得证

傅丽叶分析

傅丽叶分析

$$x[n] = \sum_{k=1}^{M} a_k\cos\left(\frac{2\pi kn}{N}\right) + \sum_{k=1}^{M} b_k\sin\left(\frac{2\pi kn}{N}\right) + w[n]$$

证明傅丽叶级数正好是线性模型的估计

设 $\theta = [a_1\ a_2\ a_3\ \cdots a_m\ b_1\ b_2\ b_3 \cdots\ b_m]^T$

$$H = \begin{bmatrix} 1 & 1 & \cdots & 0 & 0 & \cdots \\ \cos(\ ) & \cdots & & \sin & & \\ \vdots & & & & \ddots & \end{bmatrix} = [h_1\ h_2\ h_3 \cdots h_{2m}]$$

由于 $\sum_{n=0}^{N-1} \cos\left(\frac{2\pi i n}{N}\right)\cos\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$

$$\sum_{n=0}^{N-1} \sin\left(\frac{2\pi i n}{N}\right)\sin\left(\frac{2\pi j n}{N}\right) = \frac{N}{2}\delta_{ij}$$

$$\sum_{n=0}^{N-1} \cos\left(\frac{2\pi i n}{N}\right)\sin\left(\frac{2\pi j n}{N}\right) = 0$$

所以 $h_i^T h_j = 0 \quad (i \neq j)$

所以 $H^T H = \frac{N}{2} I$

$$\hat{\theta} = (H^T H)^{-1} H^T x = \frac{2}{N} \begin{bmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_{2M}^T \end{bmatrix} x = \frac{2}{N} \begin{bmatrix} h_1^T x \\ h_2^T x \\ \vdots \\ h_{2M}^T x \end{bmatrix}$$

所以 $\hat{a}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi k n}{N}\right)$

$$\hat{b}_k = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \sin\left(\frac{2\pi k n}{N}\right)$$

扩展至非白噪声

$$w \sim N(0, C)$$

1) 证明 C 正定

$$C = E[(w - E[w])(w - E[w])^T]$$
$$= E[w w^T]$$

$$x^T C x = x^T E[w w^T] x$$
$$= E[x^T w w^T x]$$

设 $y = w^T x$

$= E[y^T y] > 0$

$\Rightarrow$ 所以 C 正定，n个特征值均大于0。

2) 对 C 进行变化

C^{-1} 也正定

存在 $C^{-1} = P^{-1} \Lambda P = P^T \Lambda P$

相似对角化　合同对角化

$\Rightarrow P^T = P^{-1}$，P为正交矩阵

$$C^{-1} = P^T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} P$$

$$= P^T \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} P$$

$$= D^T D$$

3) 对数据白化

$$x = H\theta + w$$

$$Dx = DH\theta + Dw$$

$E[Dw] = DE[w] = 0$ → 均值

$$\mathrm{var}(Dw) = E[(Dw)(Dw)^T]$$
$$= D E[ww^T] D^T$$
$$= DCD^T$$

又有 $C^{-1} = D^T D$ 即 $C = (D^T D)^{-1}$

$$= D \cdot (D^T D)^{-1} D^T$$
$$= I$$

说明 Dw 为高斯白噪声

令 $x' = Dx$ 可得 $\hat{\theta} = (H^T D^T D H)^{-1} H^T D^T D x$

$H' = DH$ $\quad I(\theta) = (H^T C^{-1} H)^{-1}$

$w' = Dw$