BLUE的扩展

BLUE回顾

若 $E[x[n]] = s[n]\theta$

$x[n] = E(x[n]) + [x[n] - E(x[n])] = \theta s[n] + w[n]$

估计 $\hat{\theta} = a^T x$

则 $\mathrm{var}(\hat{\theta}) = \dfrac{1}{s^T C^{-1} s}$

$$a_{opt} = \frac{C^{-1}s}{s^T C^{-1} s}$$

扩展BLUE

若 $E[x[n]] = s[n]\theta + \beta$

估计 $\hat{\theta} = a^T x + b$

$$E[\hat{\theta}] = E[\sum_n a_n x[n] + b]$$

$$= \sum a_n (\theta s[n] + \beta) + b$$

$$= \theta$$

$\Rightarrow \quad \sum_n a_n s[n] = 1, \quad \beta \sum_n a_n = -b$

则 $E[\hat{\theta}] = \sum a_n (\theta s[n] + \beta) + b$

$$\hat{\theta} = \sum a_n x[n] - \beta \sum a_n$$

$$= \sum a_n (x[n] - \beta)$$

令 $x'[n] = x[n] - \beta$ 得

$$\hat{\theta} = \frac{s^T C^{-1}(x-\beta)}{s^T C^{-1} s} \qquad var(\hat{\theta}) = \frac{1}{s^T C^{-1} s}$$

证明BLUE线性变化不变性，B为p×p矩阵 $E(ww^T)=C$

$x = H\theta + w$. $\hat{\theta}$为θ的BLUE

现有 $\alpha = B\theta + b$ 证 $\hat{\alpha} = B\hat{\theta} + b$

证. $\theta = B^{-1}(\alpha - b)$

$$x = HB^{-1}(\alpha - b) + w$$

$$x = HB^{-1}\alpha - HB^{-1}b + w$$

$$\underbrace{x + HB^{-1}b}_{x'} = \underbrace{HB^{-1}}_{H'}\alpha + w$$

$$\hat{\alpha} = (H'^T C^{-1} H')^{-1} H'^T C^{-1} x'$$

$$= (B^{-1T} H^T C^{-1} H B^{-1})^{-1} B^{-1T} H^T C^{-1} (x + HB^{-1}b)$$

$$= B(H^T C^{-1} H)^{-1} H^T C^{-1} (x + HB^{-1}b)$$

$$= B\hat{\theta} + BB^{-1}b = B\hat{\theta} + b$$

一般线性模型

$x = H\theta + s + w$, $w \sim N(0, C)$

求θ的BLUE

$$x - s = H\theta + w$$

$$\hat{\theta} = (H^T C^{-1} H)^{-1} H^T C^{-1}(x - s)$$

$\alpha = A\theta$, A为 $r \times p$. 估计 $\hat{\alpha}$

$$x = H\theta + w$$

$$A(H^T H)^{-1} H^T x = A\theta + A(H^T H)^{-1} H^T w$$

$$x' = A\theta + A(H^T H)^{-1} H^T w$$

$$= I\alpha + w'$$

$$C = E[w' w'^T] = E[A(H^T H)^{-1} H^T w w^T H (H^T H)^{-1} A^T]$$

$$= \sigma^2 A (H^T H)^{-1} A^T$$

A满秩. 所以C正定. C^{-1}存在

$$\hat{\alpha} = (H'^T C'^{-1} H')^{-1} H'^T C'^{-1} x'$$

$$= C C^{-1} x'$$

$$= A(H^T H)^{-1} H^T x$$

$$= A\hat{\theta}$$

非线性模型的MVU $x = H\theta + w$

$w \sim N(0, \sigma^2 I)$

先证明 $(x - H\theta)^T (x - H\theta) = (x - H\hat{\theta})^T (x - H\hat{\theta}) + (\theta - \hat{\theta})^T H^T H (\theta - \hat{\theta})$

$$(x - H\hat{\theta})^T (x - H\hat{\theta}) + (\theta - \hat{\theta})^T H^T H (\theta - \hat{\theta})$$

$$= (x - H(H^T H)^{-1} H^T x)^T (x - H(H^T H)^{-1} H^T x) +$$

$$(\theta-(H^TH)^{-1}H^Tx)^T H^TH(\theta-(H^TH)^{-1}H^Tx)$$

$$= x^T(I-H(H^TH)^{-1}H^T)(I-H(H^TH)^{-1}H^T)x +$$

$$\theta^TH^TH\theta - \theta^TH^TH(H^TH^{-1})H^Tx -$$

$$x^TH(H^TH)^{-1}H^TH\theta + x^TH^T(H^TH)^{-1}H^TH(H^TH)^{-1}H^Tx$$

$$= x^Tx - x^TH(H^TH)^{-1}H^Tx + \theta^TH^TH\theta - 2\theta^TH^Tx + x^TH(H^TH)^{-1}H^Tx$$

$$= (x-H\theta)^T(x-H\theta)$$

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(x-H\theta)^T(x-H\theta)}$$

$$= \underbrace{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}(\theta-\hat{\theta})^TH^TH(\theta-\hat{\theta})}}_{g(T(x),\theta)} \underbrace{e^{-\frac{1}{2\sigma^2}(x-H\hat{\theta})^T(x-H\hat{\theta})}}_{h(x)}$$

由于 $T(x)=\hat{\theta}$ 无偏，所以 $\hat{\theta}$ 是充分的

线性模型，$x=H\theta+w$，当 H 为随机变量时

证明 $\hat{\theta}=(H^TH)^{-1}H^Tx$ 讨论 H 与 w 独立/非独立情况

$E(\hat{\theta})=\theta$

$C_{\hat{\theta}} = \sigma^2 E_H[(H^TH)^{-1}]$

当之者独立则

$$E[\hat{\theta}] = E[(H^TH^{-1})H^Tx]$$

$$= E[(H^TH)^{-1}H^T(H\theta+w)]$$

$$= E[((H^TH)^{-1}H^TH)\theta + (H^TH)^{-1}H^Tw]$$

$$= E[\theta] + E[w] \cdot ((H^T H)^{-1} H^T)$$

$$= \theta + 0 = \theta$$

$$C_{\hat{\theta}} = E[(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T]$$

$$= E\left[\left\{(H^T H)^{-1} H^T x - \theta\right\}\left\{(H^T H)^{-1} H^T x - \theta\right\}^T\right]$$

$$= E\left[\left\{(H^T H)^{-1} H^T (x - H\theta)\right\}\left\{(H^T H)^{-1} H^T (x - H\theta)\right\}^T\right]$$

$$= E[(H^T H)^{-1} H^T w w^T H (H^T H)^{-1}]$$

$$= E_{H|w} E_w[(H^T H)^{-1} H^T w w^T H (H^T H)^{-1}]$$

$$= E_{H|w}[(H^T H)^{-1} H^T \sigma^2 I H (H^T H)^{-1}]$$

$$= E_{H|w}[\sigma^2 (H^T H)^{-1}] = \sigma^2 E_H[(H^T H)^{-1}]$$ 得证

若不独立，则

$$E[\hat{\theta}] = E[(H^T H^{-1}) H^T x]$$

$$= E[(H^T H)^{-1} H^T (H\theta + w)]$$

$$= E[((H^T H)^{-1} H^T H)\theta + (H^T H)^{-1} H^T w]$$

$$= \theta + E[(H^T H)^{-1} H^T w]$$

估计量不一定无偏