1. Let

$$x(n)=s(n)+v(n);\quad y(n)=\sum_{k=0}^{N-1} h_n(k)\,x(n-k)=\vec{h}_n^T\vec{x}_n$$

where

$$\vec{h}_n=\begin{bmatrix} h_n(0) & \cdots & h_n(N-1)\end{bmatrix}^T,\quad \vec{x}_n=\begin{bmatrix} x(n) & \cdots & x(n-N+1)\end{bmatrix}^T$$

prove that

$$f\left(\vec{h}_n\right)=E\left[\left(y(n)-s(n)\right)^2\right]=\vec{h}_n^T R_X \vec{h}_n-2\vec{p}^T\vec{h}_n+E\left[s^2(n)\right]$$

where

$$R_x=\begin{bmatrix} R_x(0) & \cdots & R_x(N-1) \\ \vdots & \ddots & \vdots \\ R_x(N-1) & \cdots & R_x(0)\end{bmatrix},\quad \vec{p}=\begin{bmatrix} R_{xs}(0) \\ \vdots \\ R_{xs}(N-1)\end{bmatrix}$$

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2. Suppose the random discrete signal $x(n)$ is modeled by ARMA, i.e.,

$$x(n)=-\sum_{i=1}^{N}\alpha_i x(n-i)+\sum_{j=0}^{M}\beta_j u(n-j)$$

prove that

$$R_x(m)=E\left[x(n)x(n+m)\right]=-\sum_{i=1}^{N}\alpha_i R_x(m-i)+\sigma^2\sum_{j=0}^{M}\beta_j h(j-m)$$

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3. Let

$$E=\sum_{i=1}^{N_{out}}\left(y_i-d_i\right)^2$$

where

$$y_i = f_{neu}\left(\hat{y}_i\right), \quad \hat{y}_i = \sum_{j=1}^{N_K} \tilde{w}_{ij}^{out} h_j^K + \tilde{b}_i^{out}$$

$$h_j^K = f_{neu}\left(\hat{h}_j^K\right), \quad \hat{h}_j^K = \sum_{t=1}^{N_{K-1}} w_{jt}^K h_t^{K-1} + b_j^K$$

prove that

$$\frac{\partial E}{\partial w_{pq}^K} = 2\sum_{i=1}^{N_{out}} \left(y_i - d_i\right) f'_{neu}\left(\hat{y}_i\right) \tilde{w}_{ip}^{out} f'_{neu}\left(\hat{h}_p^K\right) h_q^{K-1}$$

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4. Let

$$N\left(x \middle| \mu_1, \Sigma_1\right) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}} \left|\Sigma_1\right|^{\frac{1}{2}}} e^{-\frac{\left(x-\mu_1\right)^T \Sigma_1^{-1} \left(x-\mu_1\right)}{2}}, \quad N\left(x \middle| \mu_2, \Sigma_2\right) = \frac{1}{\left(2\pi\right)^{\frac{n}{2}} \left|\Sigma_2\right|^{\frac{1}{2}}} e^{-\frac{\left(x-\mu_2\right)^T \Sigma_2^{-1} \left(x-\mu_2\right)}{2}}$$

Prove that

$$N\left(x \middle| \mu_1, \Sigma_1\right) N\left(x \middle| \mu_2, \Sigma_2\right) = C \frac{1}{\left(2\pi\right)^{\frac{n}{2}} \left|\Sigma_0\right|^{\frac{1}{2}}} e^{-\frac{\left(x-\mu_0\right)^T \Sigma_0^{-1} \left(x-\mu_0\right)}{2}}$$

where

$$\Sigma_0 = \left(\Sigma_1^{-1} + \Sigma_2^{-1}\right)^{-1}; \quad \mu_0 = \Sigma_0 \left(\Sigma_1^{-1} \mu_1 + \Sigma_2^{-1} \mu_2\right)$$

or further

$$\Sigma_0 = \Sigma_1 - \Sigma_1 \left(\Sigma_1 + \Sigma_2\right)^{-1} \Sigma_1; \quad \mu_0 = \mu_1 + \Sigma_1 \left(\Sigma_1 + \Sigma_2\right)^{-1} \left(\mu_2 - \mu_1\right)$$