(https://www.cut-the-knot.org/manifesto/index.shtml)

GCD of Fibonacci Numbers

Elsewhere (https://www.cut-the-knot.org/arithmetic/algebra/FibonacciMatrix.shtml#proposition) we proved that, for $m,n\ge 1$, f_m divides f_{mn}, where f_m is the Fibonacci number defined recursively with

$$f_0=0, f_1=1, f_{n+1}=f_n+f_{n-1}, n\ge 1.$$

We shall employ this fact to establish a stronger

Proposition

For For $m,n\ge 1$,

$$\gcd(f_m,f_n)=f_{\gcd(m,n)}.$$

The proof will be based on four lemmas; before stating and proving those I'd like to give an example. Doing that with wolframalpha is a snap.

Example

$f_{34}=5702887=1597\times 3571$, $f_{51}=20365011074=2\times 1597\times 637021$, while $f_{\gcd(34,51)}=f_{17}=1597=\gcd(f_{34},f_{51})$.

Lemma 1

For integers n,m, $\gcd(m,n)=\gcd(m,n\pm m)$.

This because any common factor of n,m is also a factor of $n\pm m$.

Lemma 2

For $n>0$, $\gcd(f_n,f_{n-1})=1$,

In other words, any two consecutive Fibonacci numbers are *mutually prime (https://www.cut-the-knot.org/arithmetic/Divisibility.shtml)*.

The easiest proof is by *induction (https://www.cut-the-knot.org/induction.shtml)*. There is no question about the validity of the claim at the beginning of the Fibonacci sequence: $1,1,2,3,5,\ldots$ Let for some $k>1$, $\gcd(f_k,f_{k-1})=1$. Then, by Lemma 1,

$$\gcd(f_k,f_{k+1})=\gcd(f_k,f_k+f_{k-1})=\gcd(f_k,f_{k-1})=1$$

Lemma 3

Assume, $\gcd(m,n)=1$, *then* $\gcd(m,nk)=\gcd(m,k)$.

This is because any mutual factor of m and nk divides k, because it can't divide n.

Lemma 4

$$f_{m+n}=f_mf_{n+1}+f_{m-1}f_n.$$

This has been proved *elsewhere (https://www.cut-the-knot.org/arithmetic/algebra/FibonacciMatrix.shtml#nice)*.

From Lemma 1, (or *Euclid's algorithm (https://www.cut-the-knot.org/blue/Euclid.shtml#alg)*), $\gcd(m,mk+r)=\gcd(m,r)$.

Proof of Proposition

Let then $n = mk + r, 0 \le r < m$. We have

$$\begin{aligned} \gcd(f_m, f_n) &= \gcd(f_m, f_{mk+1}f_r + f_{mk}f_{r-1}) && \textit{due to } \text{Lemma 4} \\ &= \gcd(f_m, f_{mk+1}f_r) && \textit{because } f_m|f_{mk} \\ &= \gcd(f_m, f_r) && \textit{due to } \text{Lemma 3.} \end{aligned}$$

If we lose the functional symbol f, the subscripts form a step in *Euclid's algorithm (https://www.cut-the-knot.org/blue/Euclid.shtml)*. As there, the process can continue until the remainder r becomes zero. At this step we claim that the previous (the last non-zero) remainder is exactly the greatest common divisor of the two original numbers. What we showed so far is that applying Euclid's algorithm to f_m and f_n goes exactly in parallel with applying it to the subscripts. So when eventually we arrive at, say, $\gcd(m, n) = \gcd(s, 0)$, and declare that $\gcd(m, n) = s$, we can at the same time to declare that $\gcd(f_m, f_n) = \gcd(f_s, 0)$and, hence, that

$$\gcd(f_m, f_n) = \gcd(f_s, 0) = f_s = f_{\gcd(m,n)}$$

Corollary

If $f_m|f_n$ then $m|n$.

Indeed, if $f_m|f_n$ then $f_m = \gcd(f_m, f_n)$. By the proposition just proved, it follows that $m = \gcd(m, n)$. But this is exactly what was claimed.

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