FORMAL POWER SERIES

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1. Introduction. Our purpose is to develop a systematic theory of formal power series. Such a theory is known, or at least presumed, by many writers on mathematics, who use it to avoid questions of convergence in infinite series. What is done here is to formulate the theory on a proper logical basis and thus to reveal the absence of the convergence question. Thus "hard" analysis can be replaced by "soft" analysis in many applications.

John Riordan [4] has discussed these matters in a chapter on generating functions, but his interest is in the applications to combinatorial problems. A more abstract discussion is given by de Branges and Rovnyak [1]. Many examples of the use of formal power series could be cited from the literature; we mention only two, one by John Riordan [5] the other by David Zeitlin [6].

The scheme of the paper is as follows. The theory of formal power series is developed in Sections 3, 4, 5, 6, 7, 11, and 12. Applications to number theory and combinatorial analysis are discussed in Sections 2, 8, 9, 10, and in the last part of 11.

The paper is self-contained insofar as it pertains to the theory of formal power series. However, in the applications of this theory, especially in the application to partitions in Section 9, we do not repeat here the fundamental results needed from number theory. Thus Sections 9 and 10 may be difficult for a reader who is not too familiar with the basic theory of partitions and the sum of divisors function. This difficulty can be removed by use of the specific references given in these sections; only a few pages of fairly straightforward material are needed as background. In Section 11 on the other hand, the background material is set forth in detail because the source is not too readily available.

2. An example from algebra. To motivate the theory we begin with an illustration from algebra, to be found in Jacobson [2, p. 19]. Let q_n denote the number of ways of associating an n-product $a_1a_2a_3 \cdots a_n$ in a nonassociative system. For example $q_3=2$ because $a_1(a_2a_3)$ and $(a_1a_2)a_3$ are the only possibilities. Similarly $q_4=5$ because of the cases $a_1(a_2(a_3a_4))$, $a_1((a_2a_3)a_4)$, $(a_1a_2)(a_3a_4)$, $(a_1(a_2a_3))a_4$, $((a_1a_2)a_3)a_4$. For $n\geqq 2$ it is easy to establish the recursive formula

$$q_n = \sum_{j=1}^{n-1} q_jq_{n-j}, \tag{1}$$

by the following argument. In imposing a system of parentheses on $a_1a_2a_3 \cdots a_n$

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to make it a well-defined n-product, we can begin by writing

$$(a_1a_2 \cdots a_j)(a_{j+1}a_{j+2} \cdots a_n). \tag{2}$$

Now the number of ways of associating the product $a_1a_2 \cdots a_j$ is q_j by definition, and likewise the second factor in (2) can be associated in q_{n-j} ways. Hence (2) can be associated in q_jq_{n-j} ways, and formula (1) follows by considering the possible values for j. Now define the power series

$$f(x) = \sum_{j=1}^{\infty} q_jx^j. \tag{3}$$

Taking for granted (for the moment) the multiplication of power series, we see that for $n \geqq 2$ the coefficient of x^n in $\{f(x)\}^2$ is

$$q_1q_{n-1} + q_2q_{n-2} + q_3q_{n-3} + \cdots + q_{n-1}q_1.$$

But this is q_n by (1), and so we see that $\{f(x)\}^2 = f(x) - x$ or $f^2 - f + x = 0$.

Solving this quadratic equation for f we get

$$f(x) = f = \tfrac{1}{2}\{1 \pm (1 - 4x)^{1/2}\}. \tag{4}$$

The binomial theorem gives

$$(1 - 4x)^{1/2} = 1 + \frac{1}{2}(-4x) + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2!}(-4x)^2 + \cdots$$
$$+ \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2) \cdots (\frac{1}{2} - n + 1)}{n!}(-4x)^n + \cdots.$$

The coefficient of x^n here can be simplified by multiplying numerator and denominator by 2^n to give

$$\begin{aligned}\frac{(1)(-1)(-3)(-5) \cdots (-2n + 3)}{2^n \cdot n!}(-4)^n &= -\frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{n!} \cdot 2^n \\ &= -\frac{(2n - 2)!}{(n!)2^{n-1}(n - 1)!} 2^n \\ &= -2\frac{(2n - 2)!}{n!(n - 1)!}.\end{aligned}$$

In view of the minus sign here we see that (4) holds with the minus sign and not the plus sign. Comparing coefficients of x^n in (4) we get the simple formula for q_n,

$$q_n = \frac{(2n - 2)!}{n!(n - 1)!}. \tag{5}$$

This analysis, however, leaves a number of questions unanswered. Why can we solve the quadratic to derive (4)? Why can we equate coefficients on the two sides of (4) to obtain (5)? To avoid hard analysis in answering such questions, we now develop a theory of formal power series that involves no questions of convergence or divergence. At the end of Section 5 we shall return to the question of the validity of the procedure leading to formula (5).

3. Formal power series. Define α to be an infinite sequence of complex numbers

$$\alpha = [a_0, a_1, a_2, a_3, \cdots]. \tag{6}$$

By P we denote the class of all such infinite sequences α, and these are the formal power series. There are three subsets of P that play a significant role:

P_r: those sequences α all of whose components a_j are real numbers;

P_1: those sequences α with $a_0=1$;

P_0: those sequences α with $a_0=0$.

Although we have specified that the components a_j in the elements of P are complex numbers, the theory could be developed with the a_j in any integral domain.

If $\beta\in P$, say $\beta=[b_0, b_1, b_2, b_3, \cdots]$, define addition by

$$\alpha+\beta = [a_0+b_0, a_1+b_1, a_2+b_2, \cdots].$$

Define multiplication by

$$\alpha\beta = \left[a_0b_0,\ a_1b_0+a_0b_1,\ a_2b_0+a_1b_1+a_0b_2,\ \cdots,\ \sum_{j=0}^{n} a_jb_{n-j},\ \cdots\right].$$

The definition of equality is that $\alpha=\beta$ if and only if $a_j=b_j$ for all j, i.e., $j=0, 1, 2, 3, \cdots$.

It is not difficult to establish that the set P is a commutative ring with a unit. The zero element and the unit element are

$$z = [0, 0, 0, 0, \cdots] \quad \text{and} \quad u = [1, 0, 0, 0, \cdots].$$

Given any $\alpha=[a_0, a_1, a_2, a_3, \cdots]$ the additive inverse of α is $-\alpha=[-a_0, -a_1, -a_2, -a_3, \cdots]$. The verification of the associative property of multiplication is not difficult, and it is the only property of any depth in establishing that P is a commutative ring.

Moreover, $\alpha\beta=z$ if and only if $\alpha=z$ or $\beta=z$. If $\alpha=z$ or $\beta=z$ it is obvious that $\alpha\beta=z$. To establish the converse, suppose that $\alpha\beta=z$ but $\alpha\neq z$ and $\beta\neq z$. Let j be the least nonnegative integer such that $a_j\neq 0$, and similarly let k be the least nonnegative integer such that $b_k\neq 0$. Then the component in the $(j+k+1)$-th position in $\alpha\beta$ is

$$\sum_{r=0}^{j+k} a_rb_{j+k-r} = a_jb_k \neq 0,$$

which contradicts $\alpha\beta=z$.

It follows that if $\alpha\beta=\alpha\gamma$ and $\alpha\neq z$ then $\beta=\gamma$, and P is an integral domain.

Given any α in P, there corresponds a multiplicative inverse α^{-1} if there is an element α^{-1} in P such that

$$\alpha\cdot\alpha^{-1} = \alpha^{-1}\cdot\alpha = u = [1, 0, 0, 0, \cdots].$$

THEOREM 1. *If* $\alpha=[a_0, a_1, a_2, \cdots]$, α^{-1} *exists if and only if* $a_0\neq 0$.

Proof. Denote α^{-1} by $[c_0, c_1, c_2, \cdots]$. We see that $\alpha\,\alpha^{-1}=u$ amounts to an infinite system of equations

$$a_0c_0 = 1,\ a_1c_0 + a_0c_1 = 0, \cdots, \sum_{j=0}^{n} a_jc_{n-j} = 0.$$

These equations can be solved successively for $c_0, c_1, c_2, \cdots$ if and only if $a_0\neq 0$.

LEMMA 2. *Let* $\beta\in P_1$, *so that* β *is of the form* $[1, b_1, b_2, b_3, \cdots]$. *Then for any positive integer* n *we see that* $\beta^n\in P_1$, *say* $\beta^n=[1, c_1, c_2, c_3, \cdots]$. *Also* $c_1=n\,b_1$ *and for each* $k\geqq 2$ *we have* $c_k=n\,b_k+f_{n,k}\,(b_1, b_2, \cdots, b_{k-1})$ *where* $f_{n,k}$ *is an appropriate polynomial in* $b_1, b_2, \cdots, b_{k-1}$.

Proof. This result can be readily established by induction on n.

THEOREM 3. *Let* $\alpha\in P_1$, *say* $\alpha=[1, a_1, a_2, a_3, \cdots]$, *and let* n *be any positive integer. Then there is a unique* $\beta\in P_1$, *say* $\beta=[1, b_1, b_2, b_3, \cdots]$, *such that* $\beta^n=\alpha$. *Define* $\alpha^{1/n}=\beta$.

Proof. Using Lemma 2 we can solve the equations

$$nb_1 = a_1,\ nb_2 + f_{2,n}(b_1) = a_2, \cdots, nb_k + f_{k,n}(b_1, b_2, \cdots, b_{k-1}) = a_k, \cdots,$$

successively for $b_1, b_2, b_3, \cdots$.

THEOREM 4. *For any positive integer* n *and* $\alpha\in P_1$, *we have* $(\alpha^{-1})^n=(\alpha^n)^{-1}$. *Define* $\alpha^{-n}=(\alpha^n)^{-1}$ *and* $\alpha^0=u$.

Proof. We see that $\alpha^n(\alpha^{-1})^n=\alpha\cdot\alpha\cdots\alpha\cdot\alpha^{-1}\cdot\alpha^{-1}\cdots\cdot\alpha^{-1}=u$. (Another way of establishing Theorem 4 is to observe that P_1 is a multiplicative group.)

THEOREM 5. *Let* m *and* n *be any integers,* $n>0$. *To any* $\alpha\in P_1$ *there corresponds a unique* $\beta\in P_1$ *such that* $\alpha^m=\beta^n$, *i.e.,* $\beta=\alpha^{m/n}$.

Proof. This is a corollary of Theorem 3 with α in that theorem replaced by α^m.

4. A power series notation. Let λ denote the particular element $[0, 1, 0, 0, 0, \cdots]$ of P so that

$$\lambda^2 = [0, 0, 1, 0, 0, \cdots],\quad \lambda^3 = [0, 0, 0, 1, 0, 0, \cdots],$$

and in general λ^{n-1} is the sequence with zeros in all positions except the nth, where 1 occurs. We now introduce the notation

$$\sum_{j=0}^{\infty} a_j\lambda^j = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \cdots \tag{7}$$

for $\alpha=[a_0, a_1, a_2, \cdots]$. What this amounts to is an agreement that a_j in (7) stands for $[a_j, 0, 0, 0, \cdots]$ and that $\lambda^0=[1, 0, 0, 0, \cdots]$. Thus we are *not* extending the integral domain P to a vector space by introducing scalar multiplication; this could be done, but all we intend by (7) is an alternative, convenient notation for the elements of P. Thus z and u can now be written simply as 0 and 1. The definitions of addition, multiplication, and equality of elements of P can be rewritten as follows. With α as in (7) and

$$\beta = [b_0, b_1, b_2, \cdots] = \sum_{j=0}^{\infty} b_j\lambda^j,$$

then

$$\alpha + \beta = \sum_{j=0}^{\infty} (a_j + b_j)\lambda^j, \quad \alpha\beta = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k b_{j-k}\right)\lambda^j,$$

and $\alpha=\beta$ if and only if $a_j=b_j$ for all $j=0, 1, 2, 3, \cdots$.

For example, in the earlier notation we could write

$$[1, -1, 0, 0, 0, \cdots]\cdot[1, 1, 1, 1, 1, \cdots] = [1, 0, 0, 0, 0, \cdots].$$

This can now be written as $(1-\lambda)(1+\lambda+\lambda^2+\lambda^3+\cdots)=1$, or $(1-\lambda)^{-1}=1+\lambda+\lambda^2+\lambda^3+\cdots$. A general binomial theorem is established later, in Theorems 11 and 17.

THEOREM 6. *Let n be any positive integer, let $\alpha\in P_r$ and $\beta\in P_r$, so that α and β are real sequences. If n is odd, $\alpha^n=\beta^n$ implies $\alpha=\beta$. If n is even, $\alpha^n=\beta^n$ implies $\alpha=\beta$ or $\alpha=-\beta$.*

Proof. We may presume $\alpha\neq 0$ and $\beta\neq 0$. For if $\alpha=0$, for example, then $\alpha^n=0$, $\beta^n=0$ and so $\beta=0$, $\alpha=\beta$. Let ω denote the nth root of unity

$$\omega = e^{2\pi i/n} = \cos(2\pi/n) + i\sin(2\pi/n).$$

Then $\alpha^n-\beta^n=0$ can be factored $\alpha^n-\beta^n=\prod_{j=1}^{n}(\alpha-\omega^j\beta)=0$. If ω^j is not real then $\alpha-\omega^j\beta\neq 0$ because α and β are real sequences with $\alpha\neq 0$ and $\beta\neq 0$. If n is odd, ω^j is real only in the case $j=n$ and hence

$$\alpha - \omega^n\beta = 0, \quad \alpha - \beta = 0, \quad \alpha = \beta.$$

If n is even, ω^j is real in the two cases $j=n$ and $j=n/2$, leading to the conclusion that $\alpha=\beta$ or $\alpha=-\beta$.

Consider an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ of elements of P, say

$$\alpha_k = \sum_{j=0}^{\infty} a_{jk}\lambda^j, \qquad k = 1, 2, 3, \cdots. \tag{8}$$

DEFINITION. *A sequence $\alpha_1, \alpha_2, \alpha_3, \cdots$ as in* (8) *is said to be a sequence admitting addition if corresponding to any integer $r\geq 0$ there is an integer $N=N(r)$ such that for all $n\geq N$, $a_{0n}=a_{1n}=a_{2n}=\cdots=a_{rn}=0$.*

If this condition is satisfied we also say that $\sum \alpha_j$ is an *admissible sum*, and we can write

$$\sum_{j=1}^{\infty} \alpha_j = \sum s_r \lambda^r,$$

where for each integer $r \geqq 0$ the coefficient s_r is the coefficient of λ^r in the finite sum $\alpha_1+\alpha_2+\cdots+\alpha_N$, i.e.,

$$s_r = a_{r1} + a_{r2} + \cdots + a_{rN}.$$

We note that s_r is the coefficient of λ^r in every finite sum $\alpha_1+\alpha_2+\cdots+\alpha_n$ with $n \geqq N$.

LEMMA 7. *Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of elements of P admitting addition. Let $\beta_1, \beta_2, \beta_3, \cdots$ be a rearrangement of the α's in the sense that given any j there exists a unique k such that $\alpha_j = \beta_k$. Then $\beta_1, \beta_2, \beta_3, \cdots$ is also a sequence admitting addition, and*

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots = \beta_1 + \beta_2 + \beta_3 + \cdots.$$

Proof. Let r be any given nonnegative integer. For n sufficiently large the coefficient of λ^r in $\alpha_1+\alpha_2+\alpha_3+\cdots$ equals the coefficient of λ^r in the finite sum $\alpha_1+\alpha_2+\cdots+\alpha_n$. Similarly for n sufficiently large the coefficient of λ^r in $\beta_1+\beta_2+\beta_3+\cdots$ equals the coefficient of λ^r in $\beta_1+\beta_2+\cdots+\beta_n$. And clearly $\alpha_1+\alpha_2+\cdots+\alpha_n$ and $\beta_1+\beta_2+\cdots+\beta_n$ have identical terms in λ^r.

Next we get a result analogous to Lemma 7 for multiplication. Consider an infinite sequence $\gamma_1, \gamma_2, \gamma_3, \cdots$ of elements of P of the form

$$\gamma_k = \sum_{j=1}^{\infty} c_{jk} \lambda^j, \qquad k = 1, 2, 3, \cdots. \tag{9}$$

Note that the sums begin with $j=1$. If this is a sequence admitting addition, then we say that the related sequence

$$1 + \gamma_1, \ 1 + \gamma_2, \ 1 + \gamma_3, \cdots \tag{10}$$

is a sequence admitting multiplication. Furthermore, we write

$$\prod_{k=1}^{\infty} (1 + \gamma_k) = 1 + \sum_{j=1}^{\infty} q_j \lambda^j,$$

where q_r is the coefficient of λ^r in any finite product $\prod_{k=1}^{n} (1+\gamma_k)$ with n sufficiently large that $c_{jk}=0$ for $1 \leqq j \leqq r$ if $k > n$. Then it is clear that we can state a result analogous to Lemma 7 as follows:

LEMMA 7a. *If* (10) *is a sequence admitting multiplication, so is any rearrangement* $1+\delta_1, 1+\delta_2, 1+\delta_3, \cdots$ *of* (10), *and*

$$\prod_{k=1}^{\infty}(1+\gamma_k)=\prod_{k=1}^{\infty}(1+\delta_k).$$

5. Formal derivatives. Given any α in P, say $\alpha=\sum_{j=0}^{\infty} a_j\lambda^j$, define the derivative $D(\alpha)$ and the scalar $S(\alpha)$ by

$$D(\alpha)=\sum_{j=1}^{\infty} ja_j\lambda^{j-1},\qquad S(\alpha)=a_0. \tag{11}$$

Define $D^2(\alpha)=D(D(\alpha))$, and in general for any positive integer n, the nth derivative is $D^n(\alpha)$. Taking $D^0(\alpha)=\alpha$ for convenience, we can now write a McLaurin series expansion.

THEOREM 8. $\alpha=\sum_{n=0}^{\infty}(1/n!)\ S(D^n(\alpha))\cdot\lambda^n$.

The proof of this is quite easy.

THEOREM 9. *If $\alpha\in P$, $\beta\in P$ then $D(\alpha+\beta)=D(\alpha)+D(\beta)$ and $D(\alpha\beta)=\alpha D(\beta)+\beta D(\alpha)$, and $D(\alpha^n)=n\alpha^{n-1}D(\alpha)$ for any positive integer n. Also if α^{-1} exists then $D(\alpha^{-1})=-\alpha^{-2}D(\alpha)$ and $D(\alpha^{-n})=-n\alpha^{-n-1}D(\alpha)$.*

Proof. The formula for $D(\alpha\beta)$ can be established easily by comparing coefficients of λ^n. By using induction on n we get the formula for $D(\alpha^n)$. Next if we differentiate $\alpha\alpha^{-1}=1$ we get the formula for $D(\alpha^{-1})$. Finally, $\alpha^{-n}=(\alpha^{-1})^n$ can be used to write

$$D(\alpha^{-n})=D((\alpha^{-1})^n)=n(\alpha^{-1})^{n-1}D(\alpha^{-1})=-n\alpha^{-n-1}D(\alpha).$$

THEOREM 10. *Let $\alpha\in P_1$ so that $S(\alpha)=1$. For any rational number r, $D(\alpha^r)=r\alpha^{r-1}D(\alpha)$.*

Proof. By Theorem 5 there is a unique meaning for α^r. If $r=m/n$ where m and n are integers we can write

$$D((\alpha^r)^n)=n(\alpha^r)^{n-1}D(\alpha^r),\quad D((\alpha^r)^n)=D(\alpha^m)=m\alpha^{m-1}D(\alpha),$$

by Theorem 9. The result follows at once.

A simple version of the binomial theorem can be easily obtained from Theorems 8 and 10, as follows:

THEOREM 11. *For any rational number r and any complex number k,*

$$(1+k\lambda)^r=1+r(k\lambda)+\frac{r(r-1)}{2!}(k\lambda)^2+\cdots$$
$$+\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}(k\lambda)^n+\cdots.$$

Proof. First note that $D(1+k\lambda)^r=r(1+k\lambda)^{r-1}D(1+k\lambda)=rk(1+k\lambda)^{r-1}$, and so by induction on n,

$$D^n(1+k\lambda)^r = r(r-1)(r-2)\cdots(r-n+1)k^n(1+k\lambda)^{r-n}.$$

Now $(1+k\lambda)^{r-n}$ is a unique element of P_1 by Theorems 3 and 5, and so $S(1+k\lambda)^{r-n}=1$. It follows that

$$S(D^n(1+k\lambda)^r) = r(r-1)(r-2)\cdots(r-n+1)k^n.$$

Now use Theorem 8 with α replaced by $(1+k\lambda)^r$, and the result follows.

The form of the binomial theorem just established is sufficient in most applications, for example, to justify the argument given in Section 2. To see this, we replace equation (3) with this definition of α,

$$\alpha = \sum_{j=1}^{\infty} q_j\lambda^j,$$

where the q_j have the same meaning as in Section 2. Then the analysis following equation (3) leads to $\alpha^2=\alpha-\lambda$. From this we can write $4\alpha^2-4\alpha+1=1-4\lambda$, or

$$(1-2\alpha)^2 = ((1-4\lambda)^{1/2})^2.$$

By Theorem 6 it follows that $1-2\alpha=(1-4\lambda)^{1/2}$, and so by Theorem 11 we conclude that

$$\begin{aligned} 1 - 2q_1\lambda - 2q_2\lambda^2 - 2q_3\lambda^3 - \cdots \\ = 1 + \frac{1}{2}(-4\lambda) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-4\lambda)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-4\lambda)^3 + \cdots. \end{aligned}$$

From the definition of equality in Section 3 we can now equate the coefficients of λ^n to get equation (5).

We want to get a more general form of the binomial theorem, namely the expansion of $(1+\alpha)^r$ where $\alpha\in P_0$, so that $S(\alpha)=0$. To do this we define a formal logarithm. But first we establish one more result about derivatives.

THEOREM 12. *If $\alpha_1+\alpha_2+\alpha_3+\cdots$ is an admissible sum of elements of P in the sense of Section 4, then*

$$D(\alpha_1+\alpha_2+\alpha_3+\cdots) = D(\alpha_1)+D(\alpha_2)+D(\alpha_3)+\cdots.$$

Proof. For any nonnegative integer r the coefficient of λ^r in the infinite sum $\alpha_1+\alpha_2+\alpha_3+\cdots$ equals the coefficient of λ^r in the finite sum $\alpha_1+\alpha_2+\cdots+\alpha_n$ provided $n\geqq N=N(r)$. Hence the coefficients of λ^{r-1} are equal in the equation in Theorem 12. But this holds for all r, so the result follows.

6. Logarithms and the binomial theorem. A formal logarithm is not defined for any element of P, but only for $\alpha\in P_1$, so that $S(\alpha)=1$. For any $\alpha\in P_1$, say $\alpha=1+\beta$ with $\beta\in P_0$, define

$$L(\alpha) = L(1+\beta) = \beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - \tfrac{1}{4}\beta^4 + \cdots = \sum_{j=1}^{\infty}(-1)^{j+1}\beta^j/j,$$

noting that this is an admissible sum as in Section 4. Thus L is a formal logarithmic function from P_1 to P_0.

THEOREM 13. $D(L(\alpha))=\alpha^{-1}D(\alpha)$.

Proof. With $\alpha=1+\beta$ we use Theorem 12 to write

$$\begin{aligned} D(L(\alpha)) = D(L(1+\beta)) &= D[\beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - \tfrac{1}{4}\beta^4 + \cdots \\ &= D(\beta) + D(-\tfrac{1}{2}\beta^2) + D(\tfrac{1}{3}\beta^3) + D(-\tfrac{1}{4}\beta^4) + \cdots \\ &= D(\beta) - \beta D(\beta) + \beta^2 D(\beta) - \beta^3 D(\beta) + \cdots \\ &= D(\beta)[1 - \beta + \beta^2 - \beta^3 + \cdots] \\ &= D(\beta)\cdot(1+\beta)^{-1} = D(\alpha)\cdot\alpha^{-1}, \end{aligned}$$

because $D(\alpha)=D(\beta)$ by definition.

THEOREM 14. *If* $\alpha \in P_1$ *and* $\gamma \in P_1$ *then* $L(\alpha\gamma)=L(\alpha)+L(\gamma)$.

Proof. We use Theorems 13 and 9 to observe that

$$\begin{aligned} D(L(\alpha\gamma)) = (\alpha\gamma)^{-1}D(\alpha\gamma) &= (\alpha\gamma)^{-1}\{\alpha D(\gamma) + \gamma D(\alpha)\} \\ &= \alpha^{-1}D(\alpha) + \gamma^{-1}D(\gamma) \\ &= D(L(\alpha)) + D(L(\gamma)) \\ &= D(L(\alpha) + L(\gamma)). \end{aligned}$$

Now $L(\alpha\gamma)$ and $L(\alpha)+L(\gamma)$ are elements in P_0, and it is clear from the definition of a derivative that if $\theta_1 \in P_0$ and $\theta_2 \in P_0$ and $D(\theta_1)=D(\theta_2)$, then $\theta_1=\theta_2$.

THEOREM 15. *For any rational number* r, $L(\alpha^r)=rL(\alpha)$.

Proof. By definition $L(1)=0$. Then $\alpha\cdot\alpha^{-1}=1$ implies $L(\alpha)+L(\alpha^{-1})=L(\alpha\cdot\alpha^{-1})$ $=L(1)=0$ and so $L(\alpha^{-1})=-L(\alpha)$. For any integer n we have $L(\alpha^n)=nL(\alpha)$ by induction. If $r=m/n$ where m and n are integers we see that

$$mL(\alpha) = L(\alpha^m) = L((\alpha^r)^n) = nL(\alpha^r).$$

THEOREM 16. $L(\alpha)=0$ *if and only if* $\alpha=1$. *Also if* $L(\alpha)=L(\beta)$ *then* $\alpha=\beta$.

Proof. If $L(\alpha)=0$ then $D(L(\alpha))=D(0)=0$ and so $\alpha^{-1}D(\alpha)=0$. But $\alpha^{-1}\neq 0$ and hence $D(\alpha)=0$ and $\alpha=1$.

THEOREM 17. *If* r *is rational, if* β *is an element of* P_0 *so that* $S(\beta)=0$, *then*

$$(12) \qquad (1+\beta)^r = 1 + r\beta + \frac{r(r-1)}{2!}\beta^2 + \cdots + \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\beta^n + \cdots.$$

Proof. For convenience we write

$$\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

Let γ denote the right side of equation (12) so that

$$\begin{aligned}
D(\gamma) &= D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1},\\
(1+\beta)D(\gamma) &= D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1} + D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j}\\
&= D(\beta)\sum_{j=1}^{\infty} j\binom{r}{j}\beta^{j-1} + D(\beta)\sum_{j=2}^{\infty} (j-1)\binom{r}{j-1}\beta^{j-1}\\
&= D(\beta)\cdot r + D(\beta)\sum_{j=2}^{\infty}\left\{j\binom{r}{j} + (j-1)\binom{r}{j-1}\right\}\beta^{j-1}\\
&= D(\beta)\cdot r + D(\beta)\sum_{j=2}^{\infty} r\binom{r}{j-1}\beta^{j-1}\\
&= rD(\beta)\left[1 + \sum_{j=1}^{\infty}\binom{r}{j}\beta^{j}\right] = r\gamma D(\beta).
\end{aligned}$$

Multiplying by $\gamma^{-1}(1+\beta)^{-1}$ we get $\gamma^{-1}D(\gamma) = r(1+\beta)^{-1}D(\beta) = r(1+\beta)^{-1}D(1+\beta)$. But $D(L(\gamma)) = \gamma^{-1}D(\gamma)$ and $D(L((1+\beta)^r)) = D(rL(1+\beta)) = r(1+\beta)^{-1}D(1+\beta)$, and so $D(L(\gamma)) = D(L((1+\beta)^r))$. Since $L(\gamma)$ and $L(1+\beta)^r$ are in P_0 it follows that $L(\gamma) = L((1+\beta)^r)$, and so $\gamma = (1+\beta)^r$ by Theorem 16.

7. The exponential function. Let β be an element of P_0, so that $S(\beta) = 0$. Then we define

$$E(\beta) = 1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\beta^n}{n!},$$

so that E is a function from P_0 to P_1. Since $E(\beta)$, as defined, is an admissible sum, we can apply Theorem 12 to get

$$D(E(\beta)) = D(\beta)\left\{1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots\right\} = D(\beta)\cdot E(\beta).$$

THEOREM 18. *If $E(\beta) = E(\gamma)$ then $\beta = \gamma$.*

Proof. We observe that $D(E(\beta)) = D(E(\gamma))$, so that $D(\beta)\cdot E(\beta) = D(\gamma)\cdot E(\gamma)$. But $E(\beta) \neq 0$ so that $E(\beta)$ and $E(\gamma)$ can be cancelled giving $D(\beta) = D(\gamma)$, and hence $\beta = \gamma$.

THEOREM 19. *If $\beta \in P_0$ then $L(E(\beta)) = \beta$. If $\alpha \in P_1$ then $E(L(\alpha)) = \alpha$. Thus L and E are inverse functions, L being one-to-one from P_1 onto P_0, and E one-to-one from P_0 onto P_1.*

Proof. By Theorem 13 we see that

$$D(L(E(\beta))) = \{E(\beta)\}^{-1}\cdot D(E(\beta)) = \{E(\beta)\}^{-1}\cdot E(\beta)\cdot D(\beta) = D(\beta).$$

It follows that $L(E(\beta))=\beta$. Next, given any α in P_1 suppose that $E(L(\alpha))=\alpha_1$. Then $L(E(L(\alpha)))=L(\alpha_1)$ and so $L(\alpha)=L(\alpha_1)$. Hence $\alpha=\alpha_1$ by Theorem 16.

THEOREM 20. *Given $\beta\in P_0$, $\gamma\in P_0$, then $E(\beta+\gamma)=E(\beta)\cdot E(\gamma)$.*

Proof. By Theorems 14 and 19 we see that

$$L(E(\beta)\cdot E(\gamma)) = L(E(\beta)) + L(E(\gamma)) = \beta + \gamma.$$

Taking the exponential function of each side, and using Theorem 19 again, we get the result.

By Theorems 15 and 19 we see that $\alpha^r=E(rL(\alpha))$ for any $\alpha\in P_1$ and any rational r. This equation we take as the definition of α^r for any complex number r, so that such properties of exponents as $\alpha^r\cdot\alpha^s=\alpha^{r+s}$ follow at once for complex numbers r and s. Also by use of this definition we note that Theorem 10 can be extended to any complex number r; thus

$$D(\alpha^r) = D(E(rL(\alpha))) = E(rL(\alpha))\cdot D(rL(\alpha)) = \alpha^r\cdot r\alpha^{-1}D(\alpha) = r\alpha^{r-1}D(\alpha).$$

Also Theorem 15 extends to any complex r by use of Theorem 19. Finally, Theorem 17 holds for complex r; in fact the proof of this result needs no alteration for this generalization in view of the extended versions of Theorems 10 and 15 just mentioned.

8. An application to recurrence functions. For any given a, b, x_0, x_1 define a sequence $x_0, x_1, x_2, x_3, \cdots$ by the recurrence relation $x_{n+1}=ax_n+bx_{n-1}$ for $n=1, 2, 3, \cdots$. The Fibonacci sequence is the special case with $a=b=x_0=x_1=1$. The problem is to determine x_n explicitly in terms of a, b, x_0, x_1. If we define $\alpha=x_0+x_1\lambda+x_2\lambda^2+x_3\lambda^3+\cdots$ we see that

$$\alpha - a\lambda\alpha - b\lambda^2\alpha = x_0 + (x_1 - ax_0)\lambda. \tag{13}$$

If k_1 and k_2 are the roots of $k^2-ak-b=0$ we see that (13) can be written as

$$\alpha(1 - k_1\lambda)(1 - k_2\lambda) = x_0 + (x_1 - ax_0)\lambda. \tag{14}$$

CASE 1. Suppose that $k_1=k_2$. Then we see that

$$\alpha = \{x_0 + (x_1 - ax_0)\lambda\}\cdot(1 - k_1\lambda)^{-2}. \tag{15}$$

Now by Theorem 11 or Theorem 17 we have

$$(1 - k_1\lambda)^{-2} = 1 + 2k_1\lambda + 3k_1^2\lambda^2 + 4k_1^3\lambda^3 + 5k_1^4\lambda^4 + \cdots,$$

and so equating coefficients of λ^n in (15) we get

$$\begin{aligned} x_n &= x_0(n+1)k_1^n + n(x_1 - ax_0)k_1^{n-1} \quad \text{or} \\ x_n &= nx_1k_1^{n-1} - (n-1)x_0k_1^n. \end{aligned} \tag{16}$$

CASE 2. Suppose that $k_1 \neq k_2$. Multiplying the identity

$$k_1 - k_2 = k_1(1 - k_2\lambda) - k_2(1 - k_1\lambda)$$

by $(1-k_1\lambda)^{-1}(1-k_2\lambda)^{-1}$ we get

$$(k_1 - k_2)(1 - k_1\lambda)^{-1}(1 - k_2\lambda)^{-1} = k_1(1 - k_1\lambda)^{-1} - k_2(1 - k_2\lambda)^{-1}.$$

Multiplying this into (14) we have

$$(17) \qquad (k_1 - k_2)\alpha = \{x_0 + (x_1 - ax_0)\lambda\}\{k_1(1 - k_1\lambda)^{-1} - k_2(1 - k_2\lambda)^{-1}\}.$$

Also we use $k_1(1-k_1\lambda)^{-1} = k_1 + k_1^2\lambda + k_1^3\lambda^2 + k_1^4\lambda^3 + \cdots + k_1^{n+1}\lambda^n + \cdots$. Equating coefficients of λ^n in (17) we have

$$(k_1 - k_2)x_n = x_0(k_1^{n+1} - k_2^{n+1}) + (x_1 - ax_0)(k_1^n - k_2^n),$$

or

$$(18) \qquad x_n = \{x_0(k_1^{n+1} - k_2^{n+1}) + (x_1 - ax_0)(k_1^n - k_2^n)\}/(k_1 - k_2).$$

The results (16) and (18) are well known; an alternative derivation is given in [**3**, page 100]. An entirely different way of treating equation (13) is as follows. We can write

$$(19) \qquad \alpha = (1 - a\lambda - b\lambda^2)^{-1}\{x_0 + (x_1 - ax_0)\lambda\}.$$

Now by Theorem 17 we have

$$(1 - a\lambda - b\lambda^2)^{-1} = 1 + (a\lambda + b\lambda^2) + (a\lambda + b\lambda^2)^2 + (a\lambda + b\lambda^2)^3 + \cdots.$$

The coefficient of λ^n here is

$$a^n + \binom{n-1}{1}a^{n-2}b + \binom{n-2}{2}a^{n-4}b^2 + \binom{n-3}{3}a^{n-6}b^3 + \cdots$$
$$= \sum_{j=0}^{[n/2]} \binom{n-j}{j} a^{n-2j}b^j.$$

Equating coefficients of λ^n in (19) gives therefore

$$x_n = x_0 \sum_{j=0}^{[n/2]} \binom{n-j}{j} a^{n-2j}b^j + (x_1 - ax_0) \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} a^{n-1-2j}b^j.$$

Finally, let us return to the method used for deriving (16) and (18). This method can be used with recurrence relations of higher order. Consider for example any given real (or complex) numbers x_0, x_1, x_2, a, b, c and a recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n + cx_{n-1}, \qquad n = 1, 2, 3, \cdots.$$

If we define $\alpha = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + \cdots$ we note that

$$(20) \qquad \alpha(1 - a\lambda - b\lambda^2 - c\lambda^3) = x_0 + (x_1 - ax_0)\lambda + (x_2 - ax_1 - bx_0)\lambda^2.$$

If the equation $k^3-ak^2-bk-c=0$ has roots k_1, k_2, k_3 say, then (13) can be rewritten as

$$(21)\quad \alpha(1-k_1\lambda)(1-k_2\lambda)(1-k_3\lambda) = x_0+(x_1-ax_0)\lambda+(x_2-ax_1-bx_0)\lambda^2.$$

There are now three cases depending on the nature of the roots k_1, k_2, k_3: three equal roots, two equal roots, or distinct roots. The case of equal roots follows the pattern of equation (15),

$$\alpha = [x_0+(x_1-ax_0)\lambda+(x_2-ax_1-bx_0)\lambda^2]\cdot(1-k_1\lambda)^{-3}.$$

In the other two cases it is a matter of partial fraction expansions, in the sense that constants q_1, q_2, q_3, q_4, q_5, q_6 can be found so that

$$(1-k_1\lambda)^{-2}(1-k_2\lambda)^{-1} = q_1(1-k_1\lambda)^{-1}+q_2(1-k_1\lambda)^{-2}+q_3(1-k_2\lambda)^{-1},$$
$$(1-k_1\lambda)^{-1}(1-k_2\lambda)^{-1}(1-k_3\lambda)^{-1}=q_4(1-k_1\lambda)^{-1}+q_5(1-k_2\lambda)^{-1}+q_6(1-k_3\lambda)^{-1},$$

in the case of two equal roots or the case of distinct roots, respectively.

For example if $a=6$, $b=-11$, $c=6$ then we find that $k_1=1$, $k_2=2$, $k_3=3$, $q_4=\frac{1}{2}$, $q_5=-4$, $q_6=9/2$. Then (21) implies that

$$\begin{aligned}\alpha &= [x_0+(x_1-6x_0)\lambda+(x_2-6x_1+11x_0)\lambda^2]\\ &\qquad\cdot[\tfrac{1}{2}(1-\lambda)^{-1}-4(1-2\lambda)^{-1}+\tfrac{9}{2}(1-3\lambda)^{-1}],\\ x_n &= x_0(\tfrac{1}{2}-4\cdot 2^n+\tfrac{9}{2}\cdot 3^n)+(x_1-6x_0)(\tfrac{1}{2}-4\cdot 2^{n-1}+\tfrac{9}{2}\cdot 3^{n-1})\\ &\qquad+(x_2-6x_1+11x_0)(\tfrac{1}{2}-4\cdot 2^{n-2}+\tfrac{9}{2}\cdot 3^{n-2}).\end{aligned}$$

9. An application to partitions. The notation $p(n)$ represents the number of ways that a positive integer n can be written as a sum of positive integers. Two partitions are not different if they differ only in the order of their summands. As usual, we define $p(0)=1$.

Let α_j denote $1+\lambda^j+\lambda^{2j}+\lambda^{3j}+\cdots$ for every positive integer j. Then $\alpha_1, \alpha_2, \alpha_3, \cdots$ is a sequence admitting multiplication in the sense of (10) in Section 4. By the standard argument, for example in [3, pp. 226, 227], we have

$$(22)\qquad \alpha_1\cdot\alpha_2\cdot\alpha_3\cdot\cdots = \prod_{j=1}^{\infty}\alpha_j = \sum_{k=0}^{\infty}p(k)\lambda^k.$$

But also we see that $\alpha_j(1-\lambda^j)=1$ so that $\alpha_j=(1-\lambda^j)^{-1}$, and

$$(23)\qquad \prod_{j=1}^{\infty}\alpha_j = \prod_{j=1}^{\infty}(1-\lambda^j)^{-1}.$$

Next let $q^e(n)$ denote the number of partitions of any positive integer n into an even number of distinct summands, and similarly let $q^0(n)$ be the number of partitions of n into an odd number of distinct summands. It is customary to take $q^e(0)=1$ and $q^0(0)=0$. Then the coefficient of λ^n in the expansion of the admissible product

$$(1-\lambda)(1-\lambda^2)(1-\lambda^3)\cdots = \prod_{j=1}^{\infty}(1-\lambda^j)$$

is seen to be $q^e(n)-q^0(n)$ by a simple combinatorial argument. It follows that

$$\prod_{j=1}^{\infty}(1-\lambda^j) = \sum_{n=0}^{\infty}\{q^e(n)-q^0(n)\}\lambda^n. \tag{24}$$

By use of graphs of partitions it can be proved, cf. [**3**, pp. 224–226], that $q^e(n)-q^0(n)=(-1)^j$ if n is of the form $(3j^2+j)/2$ or $(3j^2-j)/2$ for some nonnegative integer j, and $q^e(n)-q^0(n)=0$ otherwise. It is easy to prove that the sets of positive integers

$$\{(3j^2+j)/2; j=1,2,3,\cdots\},\ \{(3j^2-j)/2; j=1,2,3,\cdots\}$$

are distinct, and hence (24) can be written as

$$\begin{aligned}\prod_{j=1}^{\infty}(1-\lambda^j) &= 1+\sum_{j=1}^{\infty}(-1)^j(\lambda^{(3j^2+j)/2}+\lambda^{(3j^2-j)/2}) \\ &= 1-\lambda-\lambda^2+\lambda^5+\lambda^7-\lambda^{12}-\lambda^{15}+\cdots.\end{aligned} \tag{25}$$

This with (22) and (23) implies that

$$\left\{1+\sum_{j=1}^{\infty}(-1)^j(\lambda^{(3j^2+j)/2}+\lambda^{(3j^2-j)/2})\right\}\sum p(k)\lambda^k = 1,$$

$$(1-\lambda-\lambda^2+\lambda^5+\lambda^7-\lambda^{12}-\lambda^{15}+\cdots)\sum p(k)\lambda^k = 1.$$

For any positive integer n, the coefficient of λ^n on the left side of this equation is $p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-p(n-12)-p(n-15)+\cdots$. Thus we have proved the following well-known result of Euler [**3**, p. 235].

THEOREM 21. *For any positive integers n,*

$$\begin{aligned}p(n) &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+\cdots \\ &= \sum_{j=1}^{\infty}(-1)^{j+1}\{p(n-(3j^2+j)/2)+p(n-(3j^2-j)/2)\}\end{aligned}$$

with $p(t)=0$ if $t<0$, so that the sum is finite.

It should be emphasized that the proof given here of Theorem 21 is not new. The proof above is simply the usual one formulated in terms of the "soft" analysis of formal power series.

10. An application to the sum of divisors function. For any positive integer n let $\sigma(n)$ denote the sum of the positive divisors of n; for example $\sigma(6)=1+2+3+6$. We establish a known recurrence relation [**3**, p. 236] for $\sigma(n)$, and again the positive integers of the form $(3k^2-k)/2$ and $(3k^2+k)/2$ play a role, namely, the positive integers $1, 2, 5, 7, 12, 15, 22, 26, \cdots$.

THEOREM 22. *For any positive integer k,*

$$\sigma(k) - \sigma(k-1) - \sigma(k-2) + \sigma(k-5) + \sigma(k-7) - \cdots = \begin{cases} (-1)^{j+1}k \text{ if } k = (3j^2+j)/2 \quad \text{or } k = (3j^2-j)/2, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Define $\beta = \prod_{j=1}^{k} (1-\lambda^j)$ so that $L(\beta) = \sum_{j=1}^{k} L(1-\lambda^j)$,

$$\begin{aligned} -D(L(\beta)) = -\beta^{-1}D(\beta) &= \sum_{j=1}^{k} j(1-\lambda^j)^{-1}\lambda^{j-1} \\ &= \sum_{j=1}^{k} \{j\lambda^{j-1} + j\lambda^{2j-1} + j\lambda^{3j-1} + j\lambda^{4j-1} + \cdots\} \\ &= \sum_{n=1}^{\infty} f(n)\lambda^{n-1}, \end{aligned}$$

where $f(n)$ is seen to be the sum of all positive divisors of n that do not exceed k. Thus we have $f(n) = \sigma(n)$ if $n \leqq k$, and so we can write

$$-\beta^{-1}D(\beta) = \sum_{n=1}^{k} \sigma(n)\lambda^{n-1} + \sum_{n=k+1}^{\infty} f(n)\lambda^{n-1}. \tag{26}$$

Now equation (24) can be written with a finite product

$$\beta = \prod_{j=1}^{k} (1-\lambda^j) = \sum_{n=0}^{\infty} \{q_k^e(n) - q_k^0(n)\}\lambda^n, \tag{27}$$

where $q_k^e(n)$ denotes the number of partitions of n into an even number of distinct summands $\leqq k$, and $q_k^0(n)$ denotes the number of partitions of n into an odd number of distinct summands $\leqq k$. Define $q_k^e(0) = 1$ and $q_k^0(0) = 0$. If $n \leqq k$ we note that $q_k^e(n) = q^e(n)$ and $q_k^0(n) = q^0(n)$, so (27) can be written as

$$\beta = \sum_{n=0}^{k} \{q^e(n) - q^0(n)\}\lambda^n + \sum_{n=k+1}^{\infty} \{q_k^e(n) - q_k^0(n)\}\lambda^n. \tag{28}$$

We now equate the coefficients of λ^{k-1} in $-D(\beta)$ and in the product $\beta(-\beta^{-1}D(\beta))$. From (28) it is clear that the coefficient of λ^{k-1} in $-D(\beta)$ is

$$-k\{q^e(k) - q^0(k)\} = \begin{cases} -(-1)^j k \text{ if } k = (3j^2 \pm j)/2, \\ 0 \text{ otherwise.} \end{cases}$$

From (28) and (26) the coefficient of λ^{k-1} in $\beta(-\beta^{-1}D(\beta))$ is

$$\begin{aligned} &\sigma(k)\{q^e(0) - q^0(0)\} + \sigma(k-1)\{q^e(1) - q^0(1)\} + \sigma(k-2)\{q^e(2) - q^0(2)\} + \cdots \\ &\qquad = \sigma(k) - \sigma(k-1) - \sigma(k-2) + \sigma(k-5) + \sigma(k-7) - \cdots, \end{aligned}$$

and so the theorem is proved.

11. Trigonometric functions and differential equations. We now return to the general theory of formal power series and make the definitions

$$\sin\alpha = \{E(i\alpha) - E(-i\alpha)\}/2i = \sum_{k=0}^{\infty}\{(-1)^k\alpha^{2k+1}\}/(2k+1)!$$

$$\cos\alpha = \{E(i\alpha) + E(-i\alpha)\}/2 = \sum_{k=0}^{\infty}\{(-1)^k\alpha^{2k}\}/(2k)!,$$

where α is any element in P_0. Thus $\sin\alpha$ is in P_0, but $\cos\alpha$ is in P_1, so we can define $\sec\alpha = (\cos\alpha)^{-1}$ and $\tan\alpha = (\sin\alpha)(\cos\alpha)^{-1}$. However, we cannot now define $\operatorname{cosec}\alpha$ and $\cot\alpha$, but in the next section we extend the theory, to encompass these two functions. All the rules of differentiation now apply, such as $D(\sin\alpha) = (\cos\alpha)D(\alpha)$.

The standard theory of homogeneous linear differential equations with constant coefficients is valid. For example, in the second order case, let a and b be any complex numbers, and let r_1 and r_2 be the roots of $x^2+ax+b=0$. Then a solution for ρ in P of the equation $D^2(\rho)+aD(\rho)+b\rho=0$ is

$$\rho = c_1E(r_1\lambda) + c_2E(r_2\lambda)$$

with arbitrary constants c_1 and c_2. It is easy to prove that this is the general solution if $r_1 \neq r_2$. If $r_1 = r_2$ the general solution is of course $\rho = c_1E(r_1\lambda) + c_2\lambda E(r_1\lambda)$.

We now give a brief sketch of the use of a differential equation to solve a combinatorial problem, as in André [7, p. 172]. Our approach differs from that of André in that we treat the differential equation in a purely formal sense, which he did not. For $n \geqq 2$ let b_n be the number of permutations $a_1, a_2, \cdots, a_n$ of $1, 2, \cdots, n$ such that $a_j > a_{j-1}$ if j is even, and $a_j < a_{j-1}$ if j is odd. Call such a permutation an E-permutation. Similarly, say that $a_1, a_2, \cdots, a_n$ is an O-permutation of $1, 2, \cdots, n$ if $a_j > a_{j-1}$ if j is odd, and $a_j < a_{j-1}$ if j is even. Note that if $a_1, a_2, \cdots, a_n$ is an O-permutation then $n+1-a_1, n+1-a_2, \cdots, n+1-a_n$ is an E-permutation, and conversely. Thus there is a one-to-one correspondence between E-permutations and O-permutations; there are b_n of each type. Define $b_0 = 1$ and $b_1 = 1$.

Next, consider the number of O-permutations with $a_1 = n$. It is not difficult to see that there are b_{n-1} of these. Also, there are no E-permutations with $a_1 = n$. Turning to permutations with $a_2 = n$, there are no O-permutations of this type. However, the number of E-permutations with $a_2 = n$ is $(n-1)b_{n-2}$, or what is the same thing $(n-1)b_1b_{n-2}$; the reason for this is that a_1 can be any element among $1, 2, \cdots, n-1$ and the rest can be set up as $a_3, a_4, \cdots, a_n$ in b_{n-2} ways. A similar argument shows that there are no E-permutations with $a_3 = n$, whereas the number of O-permutations with $a_3 = n$ is $\binom{n}{2}b_2b_{n-3}$. Thus by considering all E-permutations and all O-permutations with successively $a_1 = n$, then $a_2 = n$, then $a_3 = n, \cdots$, and finally $a_n = n$, we are led to the recurrence relation

$$2b_n = \sum_{j=0}^{n-1}\binom{n-1}{j}b_jb_{n-j-1} \quad \text{or } 2nc_n = \sum_{j=0}^{n-1}c_jc_{n-j-1},$$

where c_n is defined as $b_n/n!$ for all nonnegative integers n. Taking α to be the formal power series

$$\alpha = \sum_{n=0}^{\infty} c_n \lambda^n$$

we can readily verify that the differential equation $2D(\alpha) = \alpha^2 + 1$ holds. Now it is easy to verify from the definitions of the formal trigonometric functions that $\sin^2 \lambda + \cos^2 \lambda = 1$, $\sec^2 \lambda = 1 + \tan^2 \lambda$, $D(\tan \lambda) = \sec^2 \lambda$, $D(\sec \lambda) = \sec \lambda \tan \lambda$. Thus the unique formal solution of the differential equation is $\alpha = \tan\lambda + \sec\lambda$. (André gives the solution of the differential equation as $\alpha = \tan(\lambda/2 + \pi/4)$ which has no meaning in our formal definition of the trigonometric functions. The usual formula for $\tan(\alpha+\beta)$ in terms of $\tan \alpha$ and $\tan \beta$ is valid, but $\tan \pi/4 = 1$ cannot be established in the formal theory. In fact $\tan \pi/4$ is not even defined because $\pi/4$ is not an element of P_0, although it is an element of P.) Thus we have

$$\alpha = \sum b_n \lambda^n / n! = \tan \lambda + \sec \lambda.$$

Now the power series for $\tan \lambda$ has odd powers of λ only, with coefficients closely connected with the Bernoulli numbers [8, p. 268]. Similarly the power series for $\sec \lambda$ has even powers of λ only, with coefficients related to the Euler numbers [8, p. 269]. Thus André was able to relate the combinatorial numbers b_n to the Bernoulli numbers for odd n, and to the Euler numbers for even n. (A different approach to this problem has been given recently by R. C. Entringer [9].)

From our point of view in this paper, the important aspect of this is that André's conclusions can be drawn with only a formal use of calculus and differential equations and without any convergence questions in the use of α^2, the square of a power series, in the differential equation. The series expansions for $\tan \lambda$ and $\sec \lambda$ come from those for $\sin \lambda$ and $\cos \lambda$, and these are defined in terms of the exponential functions $E(i\lambda)$ and $E(-i\lambda)$. The formal structure carries the entire argument, with no need for the classical infinitesimal calculus. Of course, such relations as $\sin^2 \lambda + \cos^2 \lambda = 1$ have meaning only in terms of formal power series in this context and not in terms of the geometry of right-angled triangles.

12. Extension to a field. Since the set of formal power series P is a commutative integral domain, it can be imbedded in a field P^* in the classical manner by use of pairs of elements, cf. [2, pp. 87–92]. This construction is very well known in the extension of the integers to the rational numbers. Thus P^* is the field of all pairs (α, β) with $\alpha \in P$, $\beta \in P$ and $\beta \neq 0$. Addition and multiplication are defined by

$$(\alpha_1, \beta_1) + (\alpha_2, \beta_2) = (\alpha_1\beta_2 + \alpha_2\beta_1, \beta_1\beta_2),$$

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2).$$

Two elements (α_1, β_1) and (α_2, β_2) are said to be equal if and only if $\alpha_1\beta_2=\alpha_2\beta_1$.

If $\beta=1$ we agree to write α for $(\alpha, \beta)=(\alpha, 1)$, so that P is a subset of P^*. Similarly we agree to write

$$\left(\sum_{j=0}^{\infty} a_j\lambda^j, \lambda^r\right) \quad \text{as} \quad \sum_{j=0}^{\infty} a_j\lambda^{j-r}, \tag{29}$$

where r is a positive integer. We prove in Theorem 23 that every element of P can be written in this way, so that P^* can be thought of as the class of Laurent power series expansions, with a finite number of negative exponents allowed.

To do this we first define the degree of α for any α in P, $\alpha\neq 0$. If $\alpha=\Sigma a_j\lambda^j$ then the degree of α, written $\deg(\alpha)$, is the subscript of the first nonzero coefficient in the sequence of coefficients $a_0, a_1, a_2, \cdots$. If α_1 and α_2 are nonzero elements of P it follows that $\deg(\alpha_1\alpha_2)=\deg(\alpha_1)+\deg(\alpha_2)$. This definition is extended to P^* as follows: if $(\alpha, \beta)\in P^*$ with $\alpha\neq 0$ then $\deg(\alpha, \beta)=\deg(\alpha)-\deg(\beta)$. Degree is well-defined, because if $(\alpha_1, \beta_1)=(\alpha_2, \beta_2)$ then $\alpha_1\beta_2=\alpha_2\beta_1$ and so we have

$$\deg(\alpha_1) + \deg(\beta_2) = \deg(\alpha_2) + \deg(\beta_1),$$
$$\deg(\alpha_1) - \deg(\beta_1) = \deg(\alpha_2) - \deg(\beta_2).$$

Next for any (α, β) in P^* with $\alpha\neq 0$, let $\deg(\alpha)=m$, $\deg(\beta)=n$ so that $\deg(\alpha, \beta)=m-n$. Then we see that $\beta=\lambda^n\beta_1$ where β_1 has degree 0, so that β_1 has an inverse. It follows that $(\alpha, \beta)=(\alpha, \lambda^n\beta_1)=(\alpha\beta_1^{-1}, \lambda^n)$. Now $\alpha\beta_1^{-1}$ has degree m, so it can be written in the form

$$\alpha\beta_1^{-1} = \sum_{j=m}^{\infty} a_j\lambda^j, \qquad a_m \neq 0.$$

Thus we have

$$(\alpha, \beta) = \sum_{j=m}^{\infty} a_j\lambda^{j-n}, \qquad a_m \neq 0, \tag{30}$$

by virtue of (29).

THEOREM 23. *The representation* (30) *of any nonzero element* (α, β) *of* P^* *is unique.*

Proof. Suppose that (α, β) can also be written as

$$(\alpha, \beta) = \sum_{j=h}^{\infty} c_j\lambda^{j-n}, \qquad c_h \neq 0.$$

By the invariance of degree under different representations we see that $m-n=h-n$ and $m=h$. Also we have

$$(\alpha, \beta) = \left(\sum_{j=m} a_j\lambda^j, \lambda^n\right) = \left(\sum_{j=m}^{\infty} c_j\lambda^j, \lambda^n\right),$$

and so by the definition of equality in P^*,

$$\sum_{j=m}^{\infty} a_j\lambda^{j+n} = \sum_{j=m}^{\infty} c_j\lambda^{j+n}.$$

The theorem follows by the definition of equality in P.

In the preceding section we saw that the trigonometric functions sin α, cos α, tan α, and sec α could be defined for any element α of P, but not cosec α and cot α. If $\alpha \neq 0$ we can define the latter two functions from P to P^*; thus

$$\operatorname{cosec} \alpha = (1, \sin \alpha), \quad \cot \alpha = (\cos \alpha, \sin \alpha).$$

A simple calculation shows that

$$\operatorname{cosec} \lambda = \lambda^{-1} + (\lambda/6) + (7\lambda^3/360) + \cdots.$$

Finally, we note that the theory of formal power series, developed here in analogy to power series in a single variable, can be extended in a similar way to the multiple variable case.

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