POLITECNICO DI TORINO

Mathematical Engineering

Analysis of Dynamical System Stability using Gershgorin Circles, Power Method, and Lanczos Method

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Contents

1 Introduction

In this report, we analyze how *Gershgorin circles* can be applied to the study of the stability of a dynamical system, whether continuous or discrete. We discuss their effectiveness in estimating the spectral properties of a system and their limitations in providing definitive conclusions about stability.

If *Gershgorin circles* alone are not sufficient, we explore the use of the *Power Method* and the *Lanczos Method* to gain further insight into the dominant eigenvalues of the system. We implement and compare these numerical algorithms, evaluating their efficiency in terms of convergence speed, computational cost, and accuracy. Finally, we assess which method provides the best balance between precision and performance in different scenarios.

2 Dynamical Systems and Stability of Equilibria

A *dynamical system* is a mathematical model that describes the evolution of a state $\mathbf{x}(t)$ over time. There are two main classes of dynamical systems:

- **Continuous dynamical systems:** These are described by ordinary differential equations (ODE) of the form:
$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1}$$
where $\mathbf{x} \in \mathbb{R}^n$ represents the state of the system and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and differentiable function.

- **Discrete dynamical systems:** These are described by difference equations of the form:
$$\mathbf{x}_{k+1} = g(\mathbf{x}_k), \tag{2}$$
where $\mathbf{x}_k$ represents the state of the system at the discrete time step k and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a function that describes the update of the state.

2.1 Stability of Equilibria and Eigenvalue Analysis

An *equilibrium point* of a dynamical system is a state $\mathbf{x}^*$ such that:

$$f(\mathbf{x}^*) = 0 \quad \text{(continuous system)}, \qquad g(\mathbf{x}^*) = \mathbf{x}^* \quad \text{(discrete system)}. \tag{3}$$

To study the stability of an equilibrium, the system is linearized around $\mathbf{x}^*$, and the eigenvalues of the Jacobian matrix are analyzed:

- **For continuous systems:** The Jacobian matrix is $J = \frac{\partial f}{\partial \mathbf{x}}\big|_{\mathbf{x}^*}$, and its eigenvalues λ_i are examined. The equilibrium is:
  - **Asymptotically stable** if $\Re(\lambda_i) < 0$ for all i;
  - **Unstable** if there exists at least one λ_i with $\Re(\lambda_i) > 0$;
  - **Neutral** if $\Re(\lambda_i) \leq 0$ with at least one eigenvalue having a zero real part.

- **For discrete systems:** The Jacobian matrix is $J = \frac{\partial g}{\partial \mathbf{x}}\big|_{\mathbf{x}^*}$, and its eigenvalues μ_i are examined. The equilibrium is:
  - **Asymptotically stable** if $|\mu_i| < 1$ for all i;
  - **Unstable** if there exists at least one μ_i with $|\mu_i| > 1$;
  - **Neutral** if there exists at least one μ_i with $|\mu_i| = 1$.

This eigenvalue-based analysis provides a fundamental criterion for the stability of dynamical systems and forms the basis for many numerical techniques used to study the properties of a system.

3 Application of Gershgorin Circles to the Stability of Dynamical Systems

A useful method for obtaining spectral information about the matrices associated with dynamical systems is the *Gershgorin Circle Theorem.* This theory provides an estimate of the set of eigenvalues of a square matrix $A \in \mathbb{R}^{n \times n}$, which is particularly useful for analyzing the stability of an equilibrium point without explicitly calculating all the eigenvalues.

3.1 Definition and Geometric Interpretation

The *Gershgorin Theorem* states that every eigenvalue λ of the matrix A lies within at least one of the disks in the complex plane, known as the *Gershgorin circles*, defined as:

$$C_i = \{ z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i \}, \quad i = 1, \ldots, n, \tag{4}$$

where a_{ii} is the diagonal element and R_i is the radius of the circle, given by the sum of the absolute values of the non-diagonal elements of the corresponding row:

$$R_i = \sum_{j \neq i} |a_{ij}|. \tag{5}$$

Furthermore, there's an extension of the Gershgorin circle theorem, which states that if we have K_i circles for the matrix $\mathbf{A}$, and there exists an index k such that:

$$\left(\bigcup_{i=1}^{k} K_i \right) \cap \left(\bigcup_{i=k+1}^{n} K_i \right) = \emptyset,$$

then exactly k eigenvalues of $\mathbf{A}$ lie within $\bigcup_{i=1}^{k} K_i$, and exactly $n - k$ eigenvalues lie within $\bigcup_{i=k+1}^{n} K_i$. This construction provides an immediate visualization of the possible distribution of the eigenvalues in the complex plane.

3.2 Use for Stability Analysis

From the theory of stability (see the previous section), we know that:

- For a continuous system, stability depends on the signs of the real parts of the eigenvalues of the Jacobian matrix J.
- For a discrete system, it is crucial that all the eigenvalues of the Jacobian matrix J have modulus less than 1.

Gershgorin circles provide a qualitative estimate of stability without directly calculating the eigenvalues. In particular, for continuous systems:

- If all the Gershgorin circles lie entirely in the left half of the complex plane (i.e., with $\Re(z) < 0$), the system is guaranteed to be stable.
- If at least one circle lies entirely in the right half of the complex plane(i.e., with $\Re(z) > 0$), the system is guaranteed to be unstable.

- If some circles intersect the right half of the complex plane, the system may be unstable, but further verification is needed.

For discrete systems:

- If all the Gershgorin circles are contained within the unit disk $|z| < 1$, the discrete system is guaranteed to be stable.
- If at least one circle lies entirely outside the unit disk $|z| < 1$, the discrete system is guaranteed to be unstable.
- If some circles extend beyond the unit disk, the system may be unstable, but further verification is needed.

3.3 Limitations and Complementary Methods

Although the Gershgorin theory is useful for quickly obtaining information about eigenvalues, it has some limitations:

- The circles provide only an approximate estimate and, in some cases, may overestimate the region where the eigenvalues are located.
- They do not provide information about the multiplicity of the eigenvalues or the convergence of the system's states.

For these reasons, when the Gershgorin circles do not give a clear answer regarding stability, it is useful to resort to more precise numerical methods such as the *Power Method* or the *Lanczos Method*, which will be discussed in later chapters.

3.4 Implementation in Python

To implement the Gershgorin circle theorem, we have developed four different functions:

- *Gerschgorin_circles*: This function takes a square matrix as input and computes the Gershgorin circles. It returns a vector containing the coordinates of the centers and the radius of the circles.
- *Circles_overlap*: This function takes as input the circles as lists of the form [real part fo the center, imaginary part of the center, radius], and gives in output the distance between the centers.
- *Gerschgorin_stability*: This function takes as input the circles obtained from the first function and checks if any circles overlap using the second one. The overlapping circles are saved in the unions, and then using the theorem mentioned before the stability is computed evaluating the disjoint unions. The function returns *stable = True* if the system is stable and *inconclusive = True* if stability cannot be determined solely from the Gershgorin theorem.
- *Plot_gerschgorin_circles*: This function takes a square matrix as input, applies the two previously described functions, and plots the corresponding Gershgorin circles in the complex plane. Additionally, if stability can be determined, the function outputs that the system is stable. If the analysis is inconclusive, it utilizes the *numpy.linalg.eigval* function to compute the eigenvalues and provide a definitive stability assessment(if possible).

3.5 Practical Results on Interesting Examples

Now we present some practical examples of applying the Gershgorin circle theory to evaluate the stability of dynamical systems, with particular attention to cases where the estimate provides certain results and those where further analysis is needed.

3.5.1 Linear Systems

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. A linear system is defined by:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

in the continuous case, and by:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

in the discrete case.

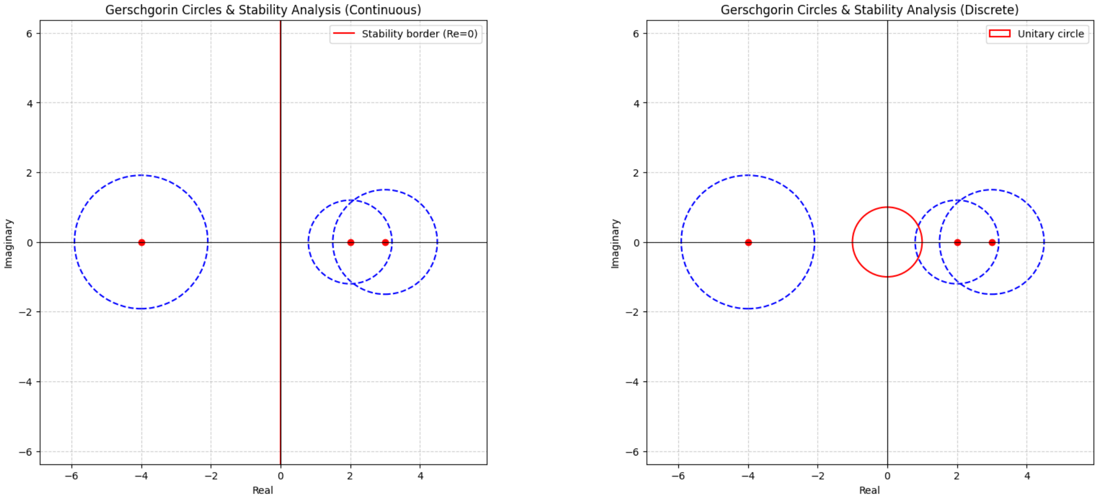
If the matrix $\mathbf{A}$ is singular, there exists a unique equilibrium point, which is the vector $\mathbf{0_n}$. As previously discussed, we need to analyze the Jacobian matrix of the system at the equilibrium point. In this case, it is straightforward to observe that the Jacobian is simply the matrix $\mathbf{A}$.

Next, using the functions described earlier, we examine some dynamical systems defined by different matrices $\mathbf{A}$ and explore how the Gershgorin circle theorem can aid in analyzing their stability.

Let us consider the following system described by:

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 0.5 \\ 0.5 & 3 & 1 \\ 1 & 0.2 & 2 \end{bmatrix}$$

The Gershgorin circles for $\mathbf{A}$ are shown in the following figures:



(a) Continuous system

(b) Discrete system

Figure 1: Gershgorin circles

For the continuous system, we observe that two of the Gershgorin circles are entirely located in the positive real part of the complex plane. Based on this observation, we can immediately conclude that the system is not stable.

Similarly, in this case, since the leftmost Gershgorin circle is entirely outside the unit circle centered at the origin and does not intersect with the other two circles, we can deduce that exactly one eigenvalue has a modulus greater than 1. Consequently, we conclude that the system is unstable.

3.5.2 Hopf System

The Hopf bifurcation describes a local bifurcation in which a system's equilibrium loses stability, leading to the emergence of a periodic solution (limit cycle). This bifurcation occurs in nonlinear dynamical systems and is fundamental in modeling oscillatory behaviors, such as in biological, chemical, and mechanical systems.

A standard normal form for a system undergoing a Hopf bifurcation is given by the two-dimensional system:

$$\frac{dx}{dt} = \mu x - \omega y + f(x, y), \tag{6}$$

$$\frac{dy}{dt} = \omega x + \mu y + g(x, y), \tag{7}$$

where μ is a bifurcation parameter, ω represents the intrinsic frequency of oscillations, and $f(x, y)$ and $g(x, y)$ capture nonlinear terms that determine the stability of the limit cycle.

Linearizing the system near the origin (neglecting nonlinear terms) results in the Jacobian matrix:

$$J = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}. \tag{8}$$

The eigenvalues of this matrix are given by:

$$\lambda = \mu \pm i\omega. \tag{9}$$

For $\mu < 0$, the real part of the eigenvalues is negative, leading to a stable equilibrium. When μ crosses zero and becomes positive, the equilibrium loses stability, and a limit cycle emerges, indicating the presence of a Hopf bifurcation.

Typical values for the parameters μ and ω depend on the physical system being modeled:

$\mu < 0$: The system is stable at the equilibrium point.

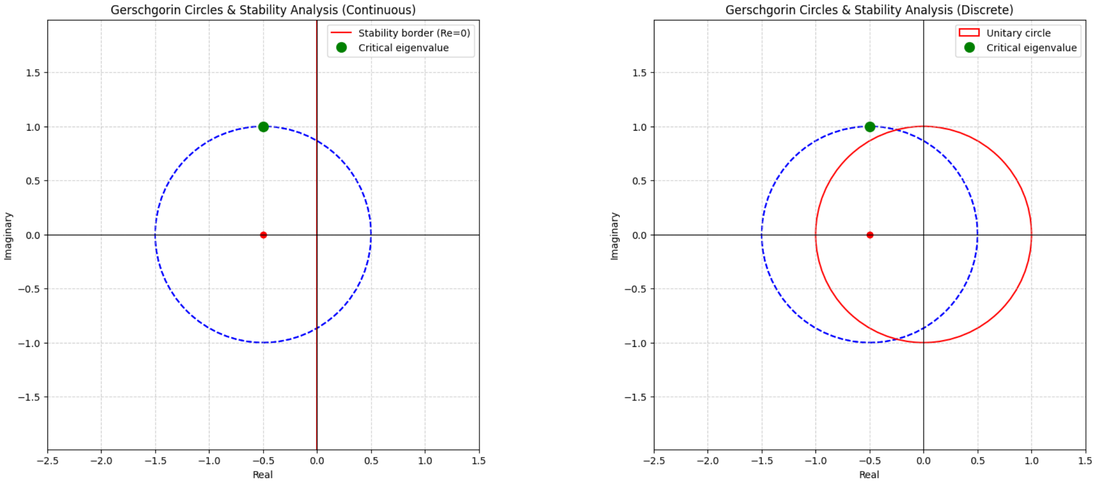
$\mu > 0$: The system undergoes a Hopf bifurcation, leading to oscillatory behavior.

$\omega > 0$: Represents the natural oscillation frequency of the system.

Common values used in simulations are:

$\mu = -0.5$ (stable state), $\mu = 0.1$ (near bifurcation)

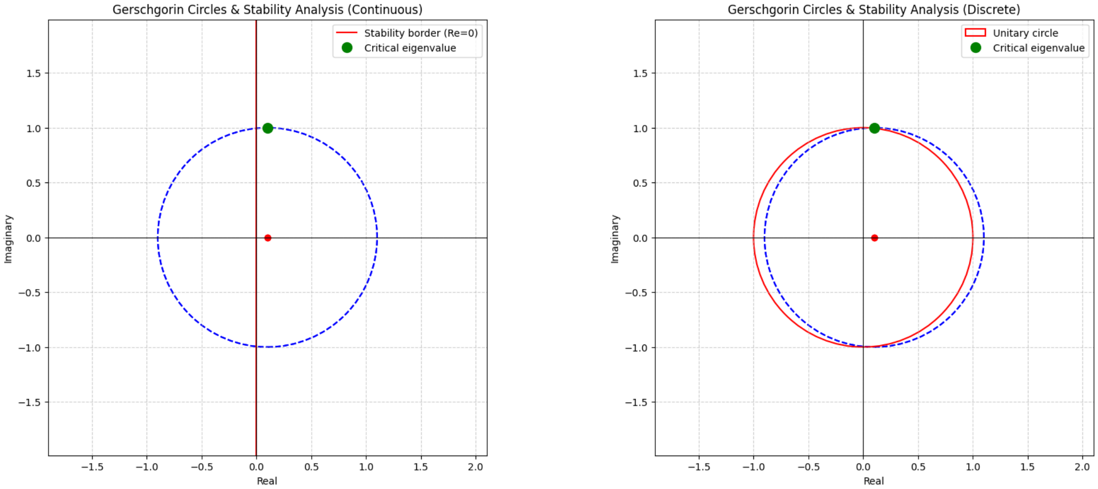
$\omega = 1$ (normalized oscillation frequency), though values can range from small perturbations ($\omega \approx 0.1$) to larger frequencies ($\omega \approx 10$) depending on the system.



(a) Continuous system with $\mu = -0.5$ and $\omega = 1$

(b) Discrete system with $\mu = -0.5$ and $\omega = 1$

Figure 2: Gershgorin circles for hopf bircuation



(a) Continuous system with $\mu = 0.1$ and $\omega = 1$

(b) Discrete system with $\mu = 0.1$ and $\omega = 1$

Figure 3: Gershgorin circles for hopf bifurcation

3.5.3 SIR System

The SIR model is a fundamental epidemiological model that describes the spread of infectious diseases in a population. It consists of three compartments:

- S (Susceptible): Individuals who can be infected.
- I (Infected): Individuals who are currently infected and can spread the disease.
- R (Recovered): Individuals who have recovered and are immune.

The model is governed by the following system of differential equations:

$$\frac{dS}{dt} = \mu N - \beta SI - \mu S, \tag{10}$$

$$\frac{dI}{dt} = \beta SI - (\gamma + \mu)I, \tag{11}$$

$$\frac{dR}{dt} = \gamma I - \mu R, \tag{12}$$

where:

- β is the transmission rate,
- γ is the recovery rate,
- μ is the birth/death rate (natural turnover in the population),
- $N = S + I + R$ is the total population size.

The Disease-Free Equilibrium (DFE) occurs when there are no infections in the population, i.e., $I = 0$. Assuming $S_0 = 1$ (normalized population), the Jacobian matrix of the system is given by:

$$J = \begin{bmatrix} -\mu & -\beta S_0 & 0 \\ 0 & \beta S_0 - (\gamma + \mu) & 0 \\ 0 & \gamma & -\mu \end{bmatrix}. \tag{13}$$

The eigenvalues of this matrix determine the stability of the equilibrium. The key parameter that influences stability is the basic reproduction number, R_0:

$$R_0 = \frac{\beta N}{\gamma + \mu}. \tag{14}$$

If $R_0 < 1$, the disease-free equilibrium is stable, meaning infections will die out over time. If $R_0 > 1$, the equilibrium becomes unstable, leading to an outbreak.

Typical values for the parameters depend on the disease being modeled:

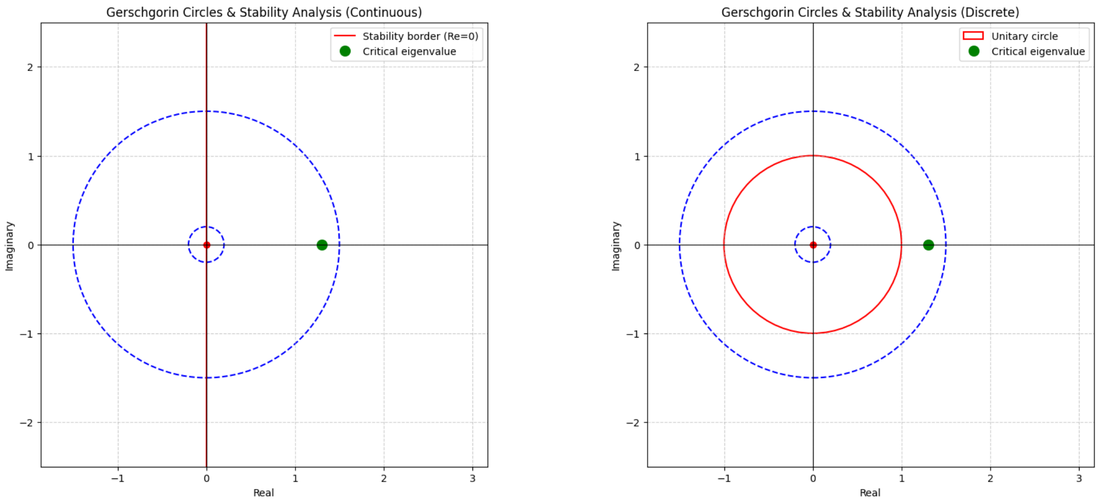
- Transmission Rate (β): Generally ranges from 0.5 to 5 per day, depending on the disease.
- Recovery Rate (γ): Given by $\gamma = 1/D$, where D is the average infectious period (e.g., $\gamma \approx 0.1$ for a 10-day infection duration).

- Birth/Death Rate (μ): For human populations, typically $\mu \approx 1/(70 \times 365) \approx 0.00004$ per day.

For example, common parameter values for different diseases are:

Disease	β	γ	μ
flu	1.5	0.2	0.00004
COVID-19	2.0	0.1	0.00004
Measles	12.0	0.1	0.00004

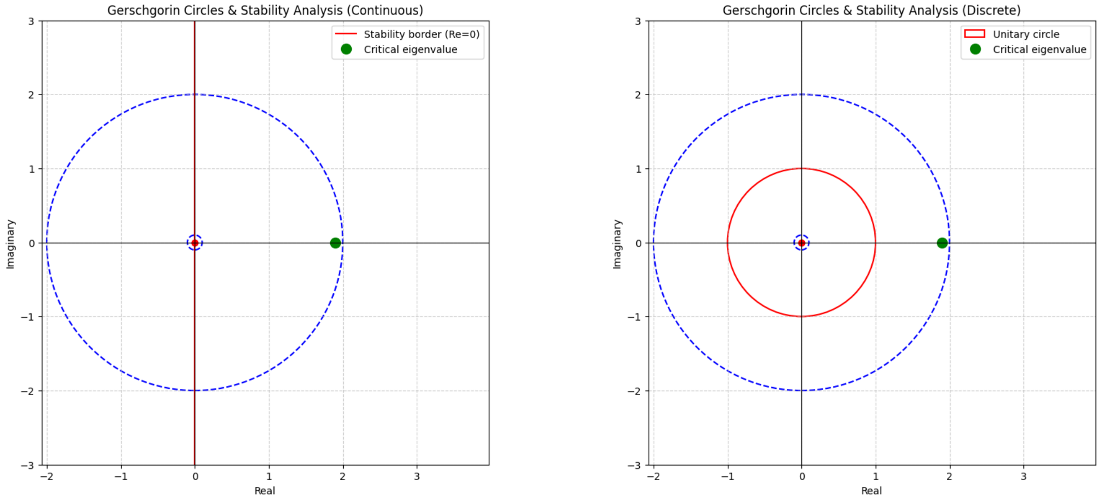
Table 1: Typical parameter values for different diseases.



(a) Continuous system with $\beta = 1.5$ and $\gamma = 0.2$

(b) Discrete system with $\beta = 1.5$ and $\gamma = 0.2$

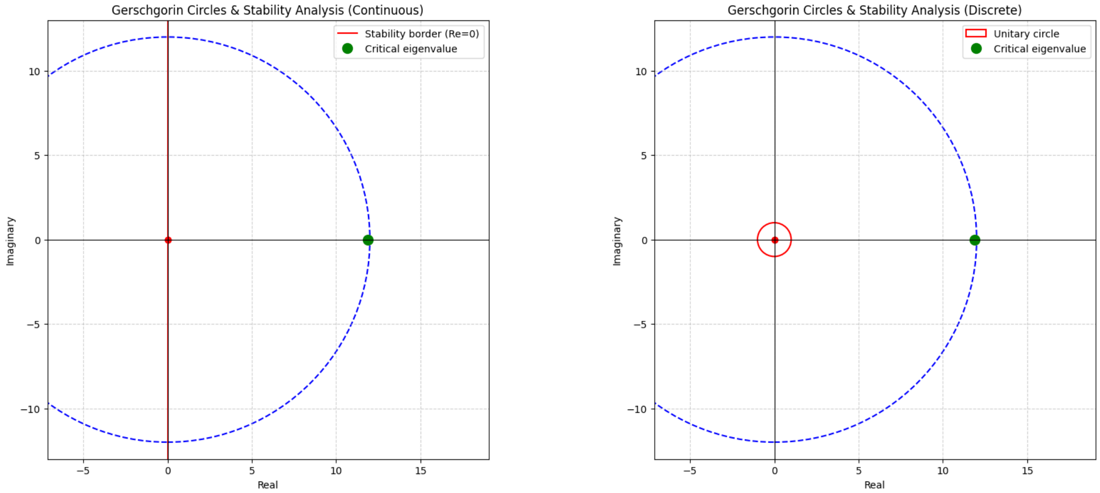
Figure 4: Gershgorin circles for flu SIR model



(a) Continuous system with $\beta = 2$ and $\gamma = 0.1$

(b) Discrete system with $\beta = 2$ and $\gamma = 0.1$

Figure 5: Gershgorin circles for COVID SIR model



(a) Continuous system with $\beta = 12$ and $\gamma = 0.1$

(b) Discrete system with $\beta = 12$ and $\gamma = 0.1$

Figure 6: Gershgorin circles for measles SIR model

3.5.4 Lotka-Volterra System

The Lotka-Volterra equations, also known as the predator-prey model, describe the dynamics of biological systems where two species interact: one as a predator and the other as a prey. The model is governed by the following system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \beta xy \\ \frac{dy}{dt} &= \delta xy - \gamma y\end{aligned} \tag{15}$$

where:
- $x(t)$ represents the prey population;
- $y(t)$ represents the predator population;
- α is the growth rate of the prey in the absence of predators;
- β is the rate at which predators capture prey;
- γ is the death rate of predators in the absence of prey;
- δ is the rate at which predators increase by consuming prey.

At the equilibrium point, we assume that the prey and predator populations no longer change. The equilibrium populations are given by:

$$x^* = \frac{\gamma}{\delta}, \quad y^* = \frac{\alpha}{\beta}.$$

The Jacobian matrix of the system describes how small perturbations around the equilibrium point evolve over time. It is calculated by taking the partial derivatives of the right-hand side of the system with respect to the population variables x and y. For the continuous system, the Jacobian matrix at the equilibrium point (x^*, y^*) is given by:

$$J = \begin{pmatrix} 0 & -\alpha\frac{\gamma}{\delta} \\ \beta\frac{\alpha}{\beta} & 0 \end{pmatrix}.$$

In this matrix: - The first row describes the rate of change of prey x with respect to small changes in both x and y, - The second row describes the rate of change of predators y with respect to small changes in both x and y.

Below are several possible values for the parameters α, β, γ, and δ that represent different ecological scenarios:

- **Scenario 1: High prey growth and low predator efficiency**

  $$\alpha = 2.0, \quad \beta = 0.5, \quad \delta = 0.3, \quad \gamma = 0.1.$$

- **Scenario 2: Balanced dynamics**

  $$\alpha = 1.5, \quad \beta = 1.0, \quad \delta = 0.5, \quad \gamma = 0.3.$$

- **Scenario 3: High predation pressure**

  $$\alpha = 1.0, \quad \beta = 2.0, \quad \delta = 0.5, \quad \gamma = 1.0.$$

- **Scenario 4: Prey thrives with few predators**

  $$\alpha = 3.0, \quad \beta = 0.2, \quad \delta = 0.1, \quad \gamma = 0.2.$$

- **Scenario 5: High predator population**

$$\alpha = 1.0, \quad \beta = 3.0, \quad \delta = 0.8, \quad \gamma = 0.5.$$

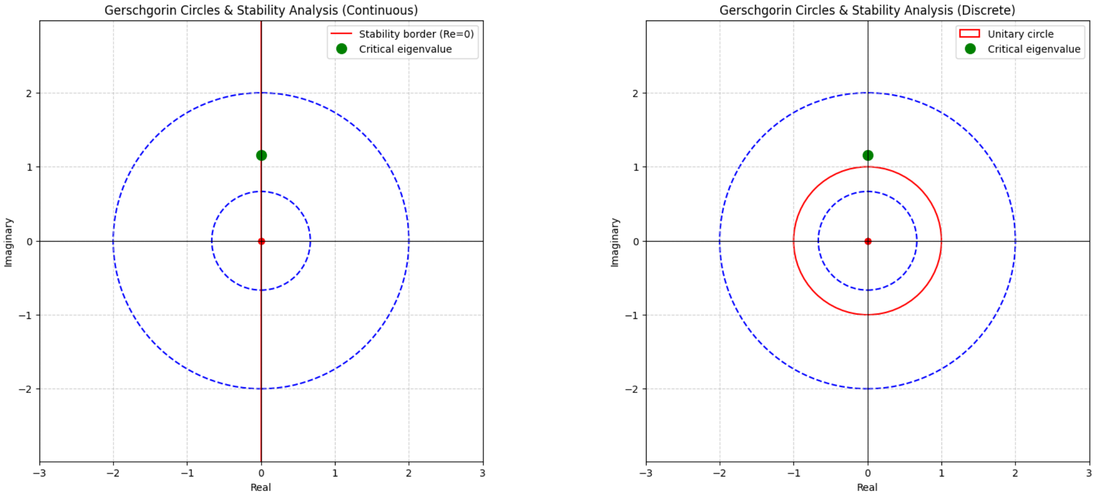
- **Scenario 6: Low prey population with efficient predators**

$$\alpha = 0.5, \quad \beta = 1.5, \quad \delta = 0.7, \quad \gamma = 0.2.$$

- **Scenario 7: Very low predator population**

$$\alpha = 2.0, \quad \beta = 0.1, \quad \delta = 0.2, \quad \gamma = 0.1.$$

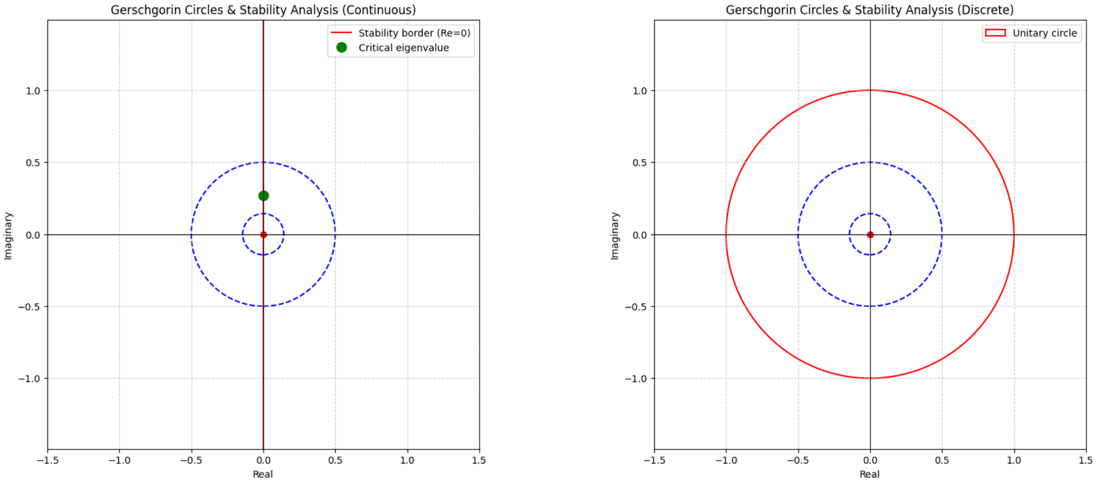
These different sets of parameter values represent various ecological scenarios, ranging from balanced predator-prey dynamics to extreme cases where one population outpaces the other. Each scenario can be studied to explore the behavior and stability of the system under different conditions.



(a) Continuous system for scenario 1

(b) Discrete system for scenario 1

Figure 7: Gershgorin circles for Lotka-Volterra scenario 1 model



(a) Continuous system for scenario 6

(b) Discrete system for scenario 6

Figure 8: Gershgorin circles for Lotka-Volterra scenario 6 model

We have demonstrated that Gershgorin's circle theorem is a powerful method to estimate the stability of a system by providing insights into the location of the eigenvalues of the Jacobian matrix. While it is a valuable tool for stability analysis, it may not always provide a complete picture, particularly when the eigenvalues are complex or when finer details are needed.

In such cases, computing the maximum (or critical) eigenvalue is crucial. For this purpose, iterative methods such as the Power Method or the Lanczos Method can be implemented. These methods allow us to efficiently approximate the largest eigenvalue, providing more accurate information on the stability of the system, especially for large matrices or more complex systems.

4 Power Method and Lanczos Method

4.1 Power Method

The *Power Method* is an iterative algorithm used to compute the dominant eigenvalue of a square matrix, which is the eigenvalue with the largest absolute value. The method relies on the idea that by repeatedly multiplying the matrix by an initial vector, the resulting vector will converge to the eigenvector corresponding to the dominant eigenvalue. The method performs well when the dominant eigenvalue is well-separated from the others in magnitude.

The algorithm for the Power Method is as follows:

Algorithm 1 Power Method for Dominant Eigenvalue Computation

H

```
Input: A square matrix $A \in \mathbb{R}^{n \times n}$, an initial vector $v_0$ (random vector), the number of maximum iterations $maxIter$ and a tolerance $tol$
Output: The dominant eigenvalue $\lambda_{max}$ and its corresponding eigenvector $v_{max}$

function PowerMethod(A, $v_0$, maxIter, tol)
    $v \leftarrow \frac{v_0}{\|v_0\|}$
    $\lambda_0 \leftarrow \infty$
    for $k \leftarrow 0$ to $maxIter$ do
        $\tilde{v}_{(k+1)} \leftarrow Av_k$
        $\tilde{\lambda}_{(k+1)} \leftarrow v_k \cdot \tilde{v}_{(k+1)}$        ▷ Estimate the eigenvalue
        $v_{(k+1)} \leftarrow \frac{\tilde{v}_{(k+1)}}{\|\tilde{v}_{(k+1)}\|}$        ▷ Normalize the new vector
        if $\frac{\|\lambda_k - \lambda_{(k-1)}\|}{\|\lambda_k\|} < tol$ then
            break        ▷ Convergence reached
        end if
    end for
    return $\lambda_k, v_k$        ▷ Return the dominant eigenvalue and eigenvector
end function
```

Pros:

- Simple and easy to implement.
- Efficient for computing the dominant eigenvalue when it is well-separated from the others.
- Works well with large matrices, as it only requires matrix-vector multiplications.

Cons:

- Convergence can be slow if the dominant eigenvalue is close to the others.
- Requires a good initial vector; otherwise, convergence may be slow or fail.
- Does not provide information about the other eigenvalues of the matrix.

4.2 Lanczos Method

The *Lanczos Method* is a more advanced iterative technique used to find multiple eigenvalues and eigenvectors of a symmetric matrix A. It constructs an orthonormal basis for the Krylov subspace and reduces A to a tridiagonal matrix.

Algorithm 2 Lanczos Method for Eigenvalue Computation

```
Input: A symmetric square matrix $A \in \mathbb{R}^{n\times n}$, an initial vector $q_0$ (random vector), the number of maximum iterations $maxIter$ and a tolerance $tol$
Output: Approximate eigenvalues and eigenvectors

function LANCZOS(A, $q_0$, $maxIter$, $tol$)
    $q_1 \leftarrow \frac{q_0}{\|q_0\|}$                              ▷ Normalize the initial vector
    $\alpha_1 \leftarrow q_1^T A q_1$                         ▷ First alpha value (Rayleigh quotient)
    $r_1 \leftarrow Aq_1 - \alpha_1 q_1$                                        ▷ Residual vector
    $\beta_1 \leftarrow \|r_1\|$                                              ▷ First beta value
    for $j \leftarrow 1$ to $k$ do
        $q_{(k+1)} \leftarrow \frac{r_k}{\beta_k}$
        $\alpha_{k+1} \leftarrow q_{(k+1)}^T A q_{(k+1)}$                                ▷ Next alpha value
        $r_{(k+1)} \leftarrow Aq_{(k+1)} - \alpha_{(k+1)} \cdot q_{(k+1)} - \beta_k \cdot q_k$          ▷ Compute residual
        $\beta_{k+1} \leftarrow \|r_{k+1}\|$                                        ▷ Next beta value
        if $\beta_{k+1} < tol$ then
            break                                               ▷ Convergence reached
        end if
    end for
    return Tridiagonal matrix with $\alpha$ and $\beta$ values      ▷ Return the tridiagonal matrix T
end function
```

The eigenvalues of the tridiagonal matrix T_m approximate those of A, making the method efficient for computing multiple eigenvalues. Compared to the Power Method, the Lanczos Method is more powerful as it can extract several eigenvalues at once, but it is applicable only to symmetric matrices.

5 Comparison Between Methods

In this study, we compare three methods for computing the largest eigenvalue of a sparse matrix, a crucial step in analyzing the stability of dynamical systems. The methods under evaluation are the Power Method, the Lanczos Method, and direct eigenvalue computation using `np.linalg.eigvals`.

To generate a sparse matrix with a density of 0.1, we use the following approach:

$$\text{density} = 0.1 \tag{16}$$

$$\text{sparse\_matrix} = \text{sp.random}(n, n, \text{density} = 0.1, \text{format} ='\ csr', \text{dtype} = \text{np.float64}) \tag{17}$$

Since the Lanczos method requires a symmetric matrix, we symmetrize it as follows:

$$\text{sparse\_matrix} = \frac{\text{sparse\_matrix} + \text{sparse\_matrix}^T}{2} \tag{18}$$

This ensures that the matrix remains sparse while being suitable for all methods under consideration.

For each method, we use the following parameters:

- **Power Method:** `tol` $= 1 \times 10^{-12}$, `max_iter` $= 10000$,
- **Lanczos Method:** `k` $= 100$, `tol` $= 1 \times 10^{-12}$,
- **Direct Eigenvalue Computation (`np.linalg.eigvals`):** No specific parameters for tolerance or iterations.

These methods will be compared to evaluate their efficiency and accuracy in computing the largest eigenvalue, which is essential for assessing the stability of dynamical systems.

Matrix Dimension	**Power Method(s)**	**Lanczos Method(s)**	**Numpy Eigenvals(s)**
500	0.0059	0.0233	0.3253
1000	0.0080	0.0401	1.0793
5000	0.0444	0.4134	46.7914
10000	0.1484	1.6530	343.3226

Table 2: Comparison of execution times (in seconds) for Power Method, Lanczos Method, and Numpy Eigenvalue Computation.

6 Conclusion

The **Gershgorin Circle Theorem** is a computationally inexpensive method that provides a quick and rough estimate of the eigenvalue locations, making it an excellent first tool for analyzing system stability. By calculating simple row-based matrix operations, it offers a theoretical understanding of eigenvalue distribution, which can help narrow down the search space. While it does not directly compute the eigenvalues, it is a useful preliminary analysis step, especially when paired with other methods to refine the results.

For more precise eigenvalue computation, the **Power Method** is a highly efficient choice, particularly when only the dominant eigenvalue is of interest. It requires only matrix-vector multiplications and is computationally inexpensive, making it well-suited for large, sparse matrices. Based on the computational times observed, the Power Method outperforms the Lanczos Method in terms of efficiency, as it converges faster and is less resource-intensive. However, it is limited to finding just the dominant eigenvalue and may not perform well if the eigenvalues are not well-separated.

On the other hand, the **Lanczos Method** is more versatile and capable of computing multiple eigenvalues, which can be particularly useful in problems where more than one eigenvalue is required. Although it is more computationally expensive and more prone to numerical instability compared to the Power Method, it remains a valuable tool for large, sparse and symmetric matrices, especially when multiple eigenvalues need to be computed. However, due to the higher complexity and additional orthogonalization steps, it is slower than the Power Method.

In summary:

- The **Gershgorin Circle Theorem** provides an efficient first analysis of system stability, offering rough estimates of eigenvalues and aiding in the initialization of more sophisticated methods.

- The **Power Method** is computationally efficient and effective for finding the dominant eigenvalue, making it a good choice for large, sparse matrices, but is limited to that single eigenvalue.

- The **Lanczos Method** is more versatile, capable of computing multiple eigenvalues, but is computationally more expensive and may require careful numerical handling.

Method	**Advantages**	**Disadvantages**	**Use Cases**
Power Method	Simple, fast for large matrices, works well for dominant eigenvalue.	Only finds the dominant eigenvalue, slow convergence for closely spaced eigenvalues.	Finding the largest eigenvalue and corresponding eigenvector.
Lanczos Method	Can find multiple eigenvalues, efficient for large, sparse symmetric matrices.	May lose orthogonality, can be numerically unstable for non-symmetric matrices.	Finding multiple eigenvalues of large, sparse matrices, especially symmetric.
Gershgorin Circles	Very fast, provides bounds on eigenvalues, computationally cheap.	Does not compute exact eigenvalues, may give loose bounds.	Spectral analysis, initializing other methods, theoretical insights.

Table 3: Comparison of Power Method, Lanczos Method, and Gershgorin Circle Theorem

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