BIDA407 - Machine Learning Fundamentals

Final Project

Modelling the Performance of a Heave-Constrained Point Absorber

prepared by

Anthony Truelove MASc, P.Eng.

PRIMED - Research Engineer

prepared for

Dr. Hossen Teimoorinia

AI Specialist - Herzberg Astronomy and Astrophysics Research Centre

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1 Motivation and Intent

Wave energy converter (WEC) technology is an emerging field within the domain of renewable energy, especially for regions with a high wave energy potential such as Vancouver Island. However, compared to other renewable energy technologies such as solar and wind, WEC dynamics are highly complex and expensive to model and simulate, and so methods of reducing this computational expense (without sacrificing an excessive amount of detail) are desirable. This is precisely the intent of this work

To that end, the approach taken in this work is inspired by the historical successes of the so-called perturbation methods. Essentially, this is the technique of obtaining a solution (albeit approximate) to a complex dynamics problem by first obtaining an exact solution to a related, reduced problem, and then seeking an appropriate correction (or *perturbation*) of this exact solution to better approximate the true, complex dynamics.[1] The canonical example of the power of these methods are their contribution to the discovery of the planet Neptune [1]:

> *Perturbation theory has been investigated by prominent mathematicians (e.g., Laplace, Poisson, Gauss), and as such the computations can be performed with very high accuracy. For example, the existence of the planet Neptune was postulated in 1848 by mathematicians J.C. Adams and U. le Verrier, with this postulation being based on the deviations (i.e. perturbations) in the observed motion of the planet Uranus. The existence of Neptune was then confirmed by way of observation by the astronomer J.G. Galle (who received coordinates from le Verrier). This represented a triumph of perturbation theory.*

The approach taken in this work can itemized as follows

1. Define an appropriate "related, reduced problem" for WEC technology.
2. Seek an exact solution to the reduced problem.
3. Identify an appropriate form for the perturbation.
4. Generate data suitable for use in mining and artificial intelligence / machine learning (AI/ML).
5. Mine the data for some initial insight into the nature of the perturbation.
6. Train an AI/ML model to serve as the perturbation.

[1] This could also be a sequence of corrections, rather than simply one.

2 Reduced Problem

2.1 Definition

While there are several classes of WEC technology currently under development, for example (see [2])

1. point absorbers (like the AquaBuoy WEC)
2. attenuators (like the Pelamis WEC)
3. terminators (like the Oyster WEC)
4. oscillating water columns (like the Mutriku WEC)
5. overtopping devices (like the Wave Dragon WEC)

the point absorber class appears to be emerging as a particularly promising candidate. As such, this work will focus exclusively on this class of tech.

That said, consider a point absorber WEC which is a cylindrical float riding a bottom-fixed piling as illustrated in Figure 2.1.[1]

[1]Think a WEC design which is intended to be installed at the waterline of existing offshore infrastructure, like wind turbines or oil rigs.

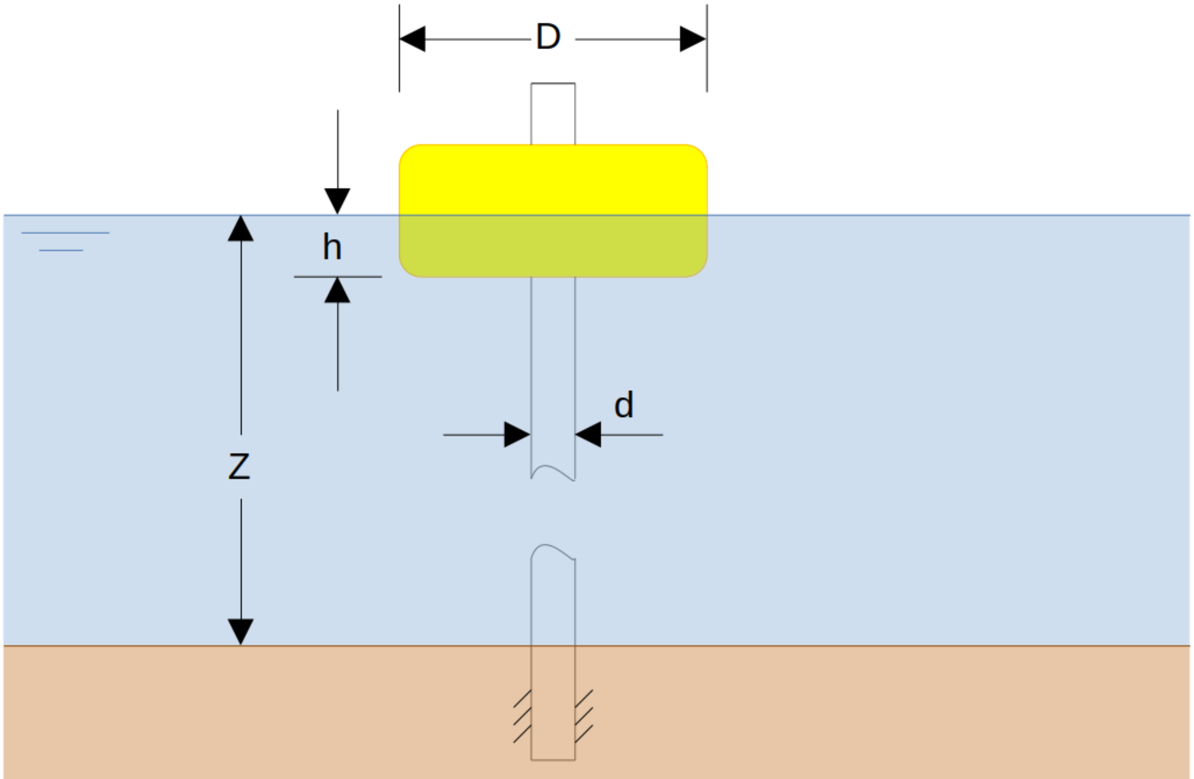

Figure 2.1: A point absorber as a cylindrical float riding a bottom-fixed piling.

That is, a float of diameter $D > d$ and with resting draft $h > 0$ is riding a bottom-fixed piling of diameter $d > 0$ in a sea of depth $Z > 0$. The basic operating principle here is that wave action causes the float to oscillate, and the relative motion between the float and piling can then drive some power takeoff device.

2.2 Simplifying Assumptions

To begin assembling the reduced problem, a number of simplifying assumptions are made; namely

1. Inviscid fluid (i.e., no drag).
2. Forces due to wave incidence, wave diffraction, and wave radiation are all negligible compared to buoyancy forces.
3. Added mass effects can be ignored.
4. Fluid memory effects can be ignored.
5. Power takeoff (PTO) dynamics are linear.

6. The WEC is heave constrained (i.e., it is a single degree-of-freedom system).

As such, the only forces acting on the WEC in this reduced case are weight, buoyancy, and the reaction from the PTO. Furthermore, the WEC can only move in heave (i.e. up and down).

2.3 Constructing the Differential Equation of Motion

Consider the free-body diagram illustrated in Figure 2.2.

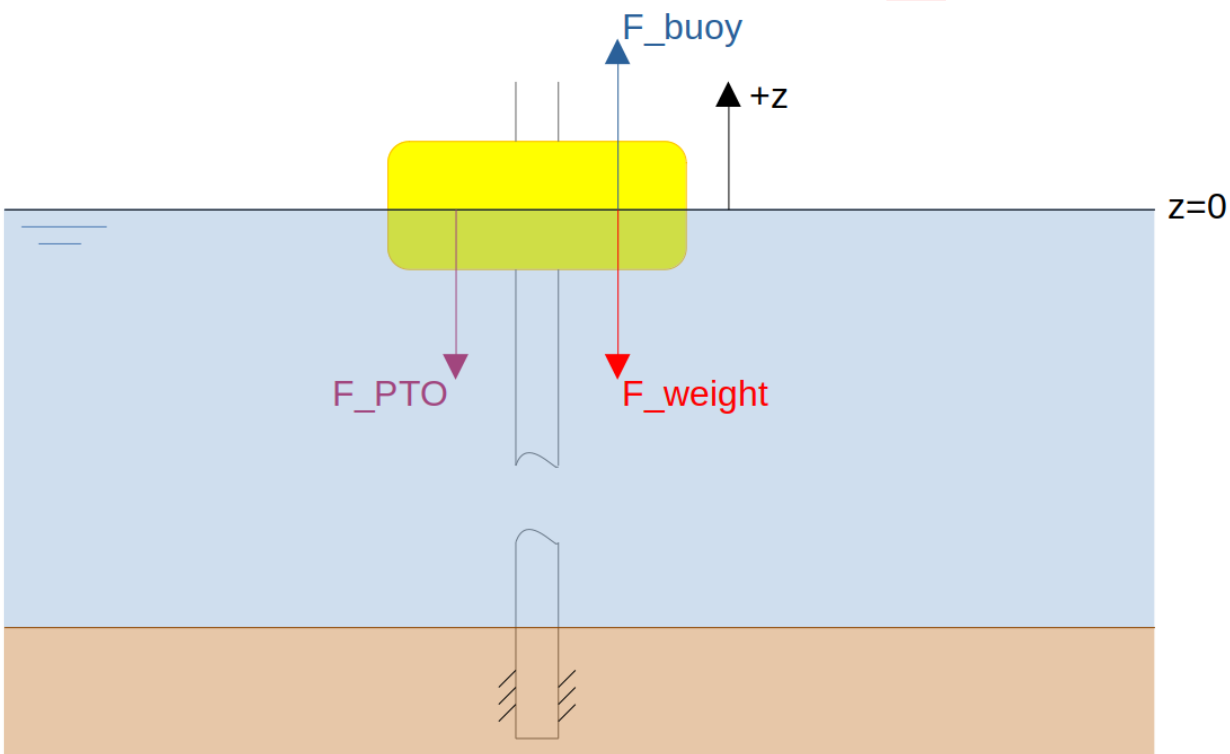

Figure 2.2: Free-body diagram for the reduced WEC dynamics.

That is, define positive float motion ($+z$) as upward, and define the origin ($z = 0$) as the mean sea level. Therefore, for positive float motion, the driving buoyancy force has positive orientation and the PTO reaction has negative orientation. Of course, weight always has negative orientation in this case.

As per Newton's Second Law (Newton II), the sum of external forces acting on a body is equal to the time rate-of-change of the body's momentum. So, in this case, application of Newton II yields

$$F_{\text{buoy}} + F_{\text{weight}} + F_{\text{PTO}} = \frac{d}{dt}\{m\dot{z}\} \tag{2.1}$$

Assuming a constant float mass $m > 0$, it then follows that

$$F_{\text{buoy}} + F_{\text{weight}} + F_{\text{PTO}} = m\ddot{z} \tag{2.2}$$

Now, the force due to weight is given simply by the product of float mass m and acceleration due to gravity g; namely

$$F_{\text{weight}} = -mg \tag{2.3}$$

As for the reaction from the PTO, if one assumes constant stiffness and damping values $k \geq 0$ and $b \geq 0$ respectively, then it follows that

$$F_{\text{PTO}} = -kz - b\dot{z} \tag{2.4}$$

Finally, the driving buoyancy force can be obtained by way of Archimedes' Principle, which states that a floating body experiences an upward buoyancy force equal to the weight of the fluid displaced. Therefore, for a float with a resting draft of h in a fluid of density $\rho > 0$, it follows in this case that

$$F_{\text{buoy}} = \frac{\pi\rho g}{4}(D^2 - d^2)(h + \overline{\eta} - z) \quad \text{for} \quad h + \overline{\eta} - z \geq 0 \tag{2.5}$$

where $\overline{\eta}$ is the deviation in sea surface elevation, from $z = 0$, averaged over the waterplane area of the float. But then, the fact that the float has a resting draft of h implies (again, by Archimedes) that

$$m = \frac{\pi\rho h}{4}(D^2 - d^2) \tag{2.6}$$

as this ensures that (2.3) and (2.5) are equal and opposite when the system is at rest (i.e., $z = \dot{z} = \ddot{z} = 0$ and $\overline{\eta} = 0$), as one might logically expect. From this, it follows that

$$F_{\text{weight}} = -\frac{\pi\rho g h}{4}(D^2 - d^2) \tag{2.7}$$

Finally, substituting (2.4) - (2.7) into (2.2) and re-arranging yields (after simplifying) the following differential equation of motion

$$m\ddot{z} + b\dot{z} + (k + k_D)\, z = k_D\overline{\eta} \quad \text{for} \quad h + \overline{\eta} - z \geq 0 \tag{2.8}$$

with

$$k_D = \frac{\pi\rho g}{4}(D^2 - d^2) \tag{2.9}$$

Note that (2.8) is a linear, second-order, ordinary differential equation with constant coefficients (and as such, can be solved exactly using classical methods).

2.4 Solving the Differential Equation of Motion

2.4.1 General Solution to the Homogeneous Equation

First, a general solution to the homogeneous equation is sought. That is, determine $z(t)$ such that

$$m\ddot{z} + b\dot{z} + (k + k_D)\, z = 0 \tag{2.10}$$

The classical approach here is by way of the trial solution

$$z(t) = C \exp[rt] \tag{2.11}$$

Substitution of (2.11) into (2.10) then yields the second-order characteristic equation (in r), which in turn then leads to the general solution

$$z(t) = C_1 \exp\left[\frac{1}{2m}\left(-b + \sqrt{b^2 - 4(k + k_D)m}\right) t\right] + C_2 \exp\left[\frac{1}{2m}\left(-b - \sqrt{b^2 - 4(k + k_D)m}\right) t\right] \tag{2.12}$$

where C_1 and C_2 are arbitrary constants (see `Maple/pdf/ODE_general_solution.pdf`). That said, assuming zero initial conditions (i.e., $z(0) = \dot{z}(0) = 0$) leads to $C_1 = C_2 = 0$. This assumption is made in this work, and so the general solution is here omitted.

2.4.2 Expressing Average Sea Surface Elevation

Since the differential equation of motion has constant coefficients, it follows that a particular solution to the nonhomogeneous equation can be obtained by way of the method of undetermined coefficients. However, in order to do so, an appropriate expression for $\overline{\eta}$ must first be obtained.

As per [3], the deviation in the sea surface elevation (from mean sea level) over an area of sea can be expressed as a cosine series (in polar form) as follows

$$\eta(r, \theta, t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T} - k_n r \cos(\psi_n - \theta) - \phi_n\right) \tag{2.13}$$

where a_n is component amplitude, $T > 0$ is some fundamental period, $k_n > 0$ is component wave number, $\psi_n \in [-\pi, \pi]$ is component direction, and $\phi_n \in [-\pi, \pi]$ is component phase (random, with uniform distribution). Applying the trig identity

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

then expands (2.13) as follows

$$\eta(r, \theta, t) = \sum_{n=1}^{\infty} a_n \left[\cos\left(\frac{2\pi n t}{T} - \phi_n\right)\cos(k_n r \cos(\psi_n - \theta)) + \sin\left(\frac{2\pi n t}{T} - \phi_n\right)\sin(k_n r \cos(\psi_n - \theta))\right] \tag{2.14}$$

Now, $\overline{\eta}$ was introduced as the deviation in the sea surface elevation (from mean sea level) averaged over the waterplane area of the float. That is

$$\overline{\eta}(t) = \frac{1}{A_{\text{water}}} \iint_{A_{\text{water}}} \eta(r, \theta, t) dA \tag{2.15}$$

where A_{water} is the waterplane area of the float. For the case of the cylindrical float considered in this work, (2.15) can be expressed as

$$\overline{\eta}(t) = \frac{4}{\pi(D^2 - d^2)} \int_0^{2\pi} \int_{\frac{d}{2}}^{\frac{D}{2}} \eta(r, \theta, t) r dr d\theta \tag{2.16}$$

Next, approximation using a *finite* cosine series allows one to express (2.16), by way of substitution of (2.14), as

$$\begin{aligned}
&\overline{\eta}(t) \cong \\
&\quad \frac{4}{\pi(D^2 - d^2)} \sum_{n=1}^{N} \left[a_n \cos\left(\frac{2\pi n t}{T} - \phi_n\right) \int_0^{2\pi} \int_{\frac{d}{2}}^{\frac{D}{2}} r \cos(k_n r \cos(\psi_n - \theta)) dr d\theta \right] + \\
&\quad \frac{4}{\pi(D^2 - d^2)} \sum_{n=1}^{N} \left[a_n \sin\left(\frac{2\pi n t}{T} - \phi_n\right) \int_0^{2\pi} \int_{\frac{d}{2}}^{\frac{D}{2}} r \sin(k_n r \cos(\psi_n - \theta)) dr d\theta \right]
\end{aligned} \tag{2.17}$$

Unfortunately, the integrals in (2.17) defy exact solution,[2] and so the best that can be done here is some partial and asymptotic analysis to get a sense of the nature of (2.17). That said, such results (luckily) are forthcoming.

To begin, attempting to solve the integrals in (2.17) exactly yields the following partial results (see `Maple/pdf/eta_overline_integrals.pdf`).

$$\int_0^{2\pi} \int_{\frac{d}{2}}^{\frac{D}{2}} r \cos(k_n r \cos(\psi_n - \theta)) dr d\theta = \frac{1}{2k_n^2} \int_0^{2\pi} [\cdots] d\theta \tag{2.18a}$$

$$\int_0^{2\pi} \int_{\frac{d}{2}}^{\frac{D}{2}} r \sin(k_n r \cos(\psi_n - \theta)) dr d\theta = -\frac{1}{2k_n^2} \int_0^{2\pi} [\cdots] d\theta \tag{2.18b}$$

So then, presumably the dimensionless term

$$\Pi_{k_n} = k_n^2 (D^2 - d^2) \tag{2.19}$$

is of significance in an exact solution to the integrals in (2.17).

[2] To the best of the author's ability.

Next, for the limiting case of $d = 0$, it can be shown (see `Maple/pdf/eta_overline_integrals.pdf`) that

$$\lim_{D \to 0^+} \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{\frac{D}{2}} r \cos(k_n r \cos(\psi_n - \theta)) dr d\theta = 1 \tag{2.20a}$$

$$\lim_{D \to 0^+} \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{\frac{D}{2}} r \sin(k_n r \cos(\psi_n - \theta)) dr d\theta = 0 \tag{2.20b}$$

And so, as $D \to 0^+$ in this case,

$$\lim_{D \to 0^+} \overline{\eta}(t) \cong \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n t}{T} - \phi_n\right) \tag{2.21}$$

as one might logically expect (as this is the expression for deviation in sea surface elevation at a point, as per [3]; simply let $r = 0$ in (2.13)).

Next, for the limiting case of $d = 0$, it can be shown (see `Maple/pdf/eta_overline_integrals.pdf`) that

$$\lim_{D \to \infty} \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{\frac{D}{2}} r \cos(k_n r \cos(\psi_n - \theta)) dr d\theta \sim \lim_{D \to \infty} \pm \frac{L_1}{\pi k_n^2 D} = 0 \tag{2.22a}$$

$$\lim_{D \to \infty} \frac{4}{\pi D^2} \int_0^{2\pi} \int_0^{\frac{D}{2}} r \sin(k_n r \cos(\psi_n - \theta)) dr d\theta \sim \lim_{D \to \infty} \mp \frac{L_2}{\pi k_n^2 D} = 0 \tag{2.22b}$$

for some constants (bounded) $L_1 \geq 0$ and $L_2 \geq 0$. And so, as $D \to \infty$ in this case,

$$\lim_{D \to \infty} \overline{\eta}(t) \cong 0 \tag{2.23}$$

as one might logically expect (over a large enough area of sea, the average deviation from mean sea level approaches zero as per [3]).

Finally, given the results laid out in (2.19) - (2.23), one *potential* form for $\overline{\eta}$ that satisfies (for small d) is

$$\overline{\eta}(t) \cong \sum_{n=1}^{N} a_n \exp\left[-\gamma k_n^2 (D^2 - d^2)\right] \cos\left(\frac{2\pi n t}{T} - \phi_n\right) \quad \text{for} \quad D \geq d \tag{2.24}$$

where $\gamma > 0$ is an undetermined parameter. It is this potential form that will be considered in this work, but note that the exp[ ] part of (2.24) could just as well be any other dimensionless potential function $\Phi(\ )$ that preserves (2.21) and (2.23), and this reveals an appropriate form for the perturbation!

2.4.3 Particular Solution to the Nonhomogeneous Equation

If one applies the same trig identity as was used previously, it follows that (2.24) can be expanded as follows

$$\overline{\eta}(t) \cong \sum_{n=1}^{N} a_n \exp[-\gamma k_n^2 (D^2 - d^2)] \left[\cos(\phi_n)\cos\left(\frac{2\pi n t}{T}\right) + \sin(\phi_n)\sin\left(\frac{2\pi n t}{T}\right)\right] \quad \text{for} \quad D \geq d \tag{2.25}$$

or, more compactly,

$$\overline{\eta}(t) \cong \sum_{n=1}^{N} \left[\alpha_n \cos\left(\frac{2\pi n t}{T}\right) + \beta_n \sin\left(\frac{2\pi n t}{T}\right)\right] \tag{2.26a}$$

$$\alpha_n = a_n \exp[-\gamma k_n^2 (D^2 - d^2)] \cos(\phi_n) \tag{2.26b}$$

$$\beta_n = a_n \exp[-\gamma k_n^2 (D^2 - d^2)] \sin(\phi_n) \tag{2.26c}$$

From this, it follows that the nonhomogeneous equation can be expressed as

$$m\ddot{z} + b\dot{z} + (k + k_D)\, z \cong k_D \sum_{n=1}^{N} \left[\alpha_n \cos\left(\frac{2\pi n t}{T}\right) + \beta_n \sin\left(\frac{2\pi n t}{T}\right)\right] \tag{2.27}$$

Therefore, given the form of the right-hand side of (2.27), the method of undetermined coefficients can be invoked by assuming a particular solution of the form

$$z(t) \cong \sum_{n=1}^{N} \left[A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right)\right] \tag{2.28}$$

where the A_n and B_n are the undetermined coefficients. This then implies that

$$\dot{z}(t) \cong \sum_{n=1}^{N} \left[-A_n \left(\frac{2\pi n}{T}\right) \sin\left(\frac{2\pi n t}{T}\right) + B_n \left(\frac{2\pi n}{T}\right) \cos\left(\frac{2\pi n t}{T}\right)\right] \tag{2.29a}$$

$$\ddot{z}(t) \cong \sum_{n=1}^{N} \left[-A_n \left(\frac{2\pi n}{T}\right)^2 \cos\left(\frac{2\pi n t}{T}\right) - B_n \left(\frac{2\pi n}{T}\right)^2 \sin\left(\frac{2\pi n t}{T}\right)\right] \tag{2.29b}$$

Substitution of (2.28) and (2.29a-b) into (2.27) then yields

$$
\begin{aligned}
m \sum_{n=1}^{N} \left[-A_n \left(\frac{2\pi n}{T} \right)^2 \cos\left(\frac{2\pi n t}{T} \right) - B_n \left(\frac{2\pi n}{T} \right)^2 \sin\left(\frac{2\pi n t}{T} \right) \right] + \\
b \sum_{n=1}^{N} \left[-A_n \left(\frac{2\pi n}{T} \right) \sin\left(\frac{2\pi n t}{T} \right) + B_n \left(\frac{2\pi n}{T} \right) \cos\left(\frac{2\pi n t}{T} \right) \right] + \\
(k + k_D) \sum_{n=1}^{N} \left[A_n \cos\left(\frac{2\pi n t}{T} \right) + B_n \sin\left(\frac{2\pi n t}{T} \right) \right] \cong \\
k_D \sum_{n=1}^{N} \left[\alpha_n \cos\left(\frac{2\pi n t}{T} \right) + \beta_n \sin\left(\frac{2\pi n t}{T} \right) \right]
\end{aligned}
\tag{2.30}
$$

Equating the sine and cosine terms in (2.30), by index n, then reveals the following system of linear equations

$$
\begin{bmatrix} k + k_D - m \left(\frac{2\pi n}{T} \right)^2 & b \left(\frac{2\pi n}{T} \right) \\ -b \left(\frac{2\pi n}{T} \right) & k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} \cong \begin{bmatrix} k_D \alpha_n \\ k_D \beta_n \end{bmatrix} \tag{2.31}
$$

Now, provided that the determinant of the 2×2 in (2.31) is non-zero, namely

$$
\left(k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \right)^2 + b^2 \left(\frac{2\pi n}{T} \right)^2 \neq 0 \tag{2.32}
$$

(which is guaranteed for $b > 0$ since $n \geq 1$) it follows that the unique solution to (2.31) is, by Cramer's rule,

$$
A_n \cong \frac{k_D \alpha_n \left(k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \right) - k_D \beta_n b \left(\frac{2\pi n}{T} \right)}{\left(k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \right)^2 + b^2 \left(\frac{2\pi n}{T} \right)^2} \tag{2.33a}
$$

$$
B_n \cong \frac{k_D \beta_n \left(k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \right) + k_D \alpha_n b \left(\frac{2\pi n}{T} \right)}{\left(k + k_D - m \left(\frac{2\pi n}{T} \right)^2 \right)^2 + b^2 \left(\frac{2\pi n}{T} \right)^2} \tag{2.33b}
$$

(2.33a-b) into (2.28) thus defines a particular solution to (2.27) (see `Maple/pdf/ODE_particular_solution.pdf`).

2.4.4 Computing Expected Power

Observe that the power captured (or dissipated) by the WEC PTO is given by

$$P = b\dot{z}^2 \tag{2.34}$$

And so, substitution of (2.29a) yields

$$\begin{aligned} P \cong b\left(\sum_{n=1}^{N}\left[-A_n\left(\frac{2\pi n}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)+B_n\left(\frac{2\pi n}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\right]\right)^2 = \\ b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nA_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi jt}{T}\right)- \\ 2b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)+ \\ b\sum_{n=1}^{N}\sum_{j=1}^{N}B_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi jt}{T}\right) \end{aligned} \tag{2.35}$$

Now, the expected value (over the fundamental period T) of (2.35) can be expressed as

$$\begin{aligned} \mathrm{E}\left\{P\right\} \cong b\mathrm{E}\left\{\left(\sum_{n=1}^{N}\left[-A_n\left(\frac{2\pi n}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)+B_n\left(\frac{2\pi n}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\right]\right)^2\right\} = \\ b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nA_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\mathrm{E}\left\{\sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi jt}{T}\right)\right\}- \\ 2b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\mathrm{E}\left\{\sin\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\right\}+ \\ b\sum_{n=1}^{N}\sum_{j=1}^{N}B_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\mathrm{E}\left\{\cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi jt}{T}\right)\right\} \end{aligned} \tag{2.36}$$

or, in integral form,

$$
\begin{aligned}
\mathrm{E}\left\{P\right\} \cong b\mathrm{E}\left\{\left(\sum_{n=1}^{N}\left[-A_n\left(\frac{2\pi n}{T}\right)\sin\left(\frac{2\pi nt}{T}\right)+B_n\left(\frac{2\pi n}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)\right]\right)^2\right\} = \\
b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nA_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\left(\frac{1}{T}\int_0^T\sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi jt}{T}\right)dt\right) - \\
2b\sum_{n=1}^{N}\sum_{j=1}^{N}A_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\left(\frac{1}{T}\int_0^T\sin\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)dt\right) + \\
b\sum_{n=1}^{N}\sum_{j=1}^{N}B_nB_j\left(\frac{2\pi n}{T}\right)\left(\frac{2\pi j}{T}\right)\left(\frac{1}{T}\int_0^T\cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi jt}{T}\right)dt\right)
\end{aligned}
\tag{2.37}
$$

But then, by orthogonality (see `Maple/pdf/orthogonality.pdf`), namely

$$\frac{1}{T}\int_0^T\sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi jt}{T}\right)dt = \frac{\delta_{nj}}{2} \tag{2.38a}$$

$$\frac{1}{T}\int_0^T\sin\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)dt = 0 \tag{2.38b}$$

$$\frac{1}{T}\int_0^T\cos\left(\frac{2\pi nt}{T}\right)\cos\left(\frac{2\pi jt}{T}\right)dt = \frac{\delta_{nj}}{2} \tag{2.38c}$$

$$\delta_{nj} = \begin{cases} 1 & \text{if} \quad n = j \\ 0 & \text{otherwise} \end{cases} \tag{2.38d}$$

it follows that (2.37) reduces significantly to

$$\mathrm{E}\left\{P\right\} \cong \frac{b}{2}\sum_{n=1}^{N}\left(A_n^2+B_n^2\right)\left(\frac{2\pi n}{T}\right)^2 \tag{2.39}$$

2.5 Summary of Results

Given environmental and WEC design parameters, together with

$$m = \frac{\pi \rho h}{4}(D^2 - d^2)$$

$$k_D = \frac{\pi \rho g}{4}(D^2 - d^2)$$

$$\overline{\eta}(t) \cong \sum_{n=1}^{N} \left[\alpha_n \cos\left(\frac{2\pi n t}{T}\right) + \beta_n \sin\left(\frac{2\pi n t}{T}\right) \right]$$

$$\alpha_n = a_n \exp[-\gamma k_n^2 (D^2 - d^2)] \cos(\phi_n)$$

$$\beta_n = a_n \exp[-\gamma k_n^2 (D^2 - d^2)] \sin(\phi_n)$$

it can be shown (see above) that

$$z(t) \cong \sum_{n=1}^{N} \left[A_n \cos\left(\frac{2\pi n t}{T}\right) + B_n \sin\left(\frac{2\pi n t}{T}\right) \right]$$

$$A_n \cong \frac{k_D \alpha_n \left(k + k_D - m \left(\frac{2\pi n}{T}\right)^2\right) - k_D \beta_n b \left(\frac{2\pi n}{T}\right)}{\left(k + k_D - m \left(\frac{2\pi n}{T}\right)^2\right)^2 + b^2 \left(\frac{2\pi n}{T}\right)^2}$$

$$B_n \cong \frac{k_D \beta_n \left(k + k_D - m \left(\frac{2\pi n}{T}\right)^2\right) + k_D \alpha_n b \left(\frac{2\pi n}{T}\right)}{\left(k + k_D - m \left(\frac{2\pi n}{T}\right)^2\right)^2 + b^2 \left(\frac{2\pi n}{T}\right)^2}$$

$$\mathrm{E}\left\{P\right\} \cong \frac{b}{2} \sum_{n=1}^{N} \left(A_n^2 + B_n^2\right) \left(\frac{2\pi n}{T}\right)^2$$

3 Data Generation

[...]

4 Data Mining

[...]

4.1 Dimensionless Analysis

[...]

$$\overline{k} = \frac{\sum_{n=1}^{N} k_n S_n}{\sum_{n=1}^{N} S_n} \tag{4.1}$$

$$k_n \cong \frac{4\pi^2 f_n^2}{g} \quad \text{for deep water} \tag{4.2}$$

$$\overline{k}_{\text{deep}} \cong \frac{4\pi^2}{g} \left(\frac{\int_0^\infty f^2 S(f) df}{\int_0^\infty S(f) df} \right) = \frac{4\pi^2}{g} \left(\frac{m_2}{m_0} \right) \tag{4.3}$$

$$\Pi_1 = \overline{k}^2 \left(D^2 - d^2 \right) \tag{4.4}$$

$$\Pi_2 = \frac{b}{\rho g H_s^2 T_p} \tag{4.5}$$

$$\Pi_3 = \frac{b}{2\sqrt{(k + k_D) m}} \tag{4.6}$$

$$\Pi_4 = T_p \sqrt{\frac{k + k_D}{m}} \tag{4.7}$$

5 Training and Testing

[...]

6 Results and Discussion

[...]

7 Conclusions and Future Work

[...]

References

[1] N. Bogolyubov, “Perturbation Theory,” 2024. [Online]. Available: https://encyclopediaofmath.org/index.php?title=Perturbation_theory

[2] Y. Zhang, Y. Zhao, W. Sun, and J. Li, “Ocean wave energy converters: Technical principle, device realization, and performance evaluation,” *Renewable and Sustainable Energy Reviews*, vol. 141, 2021.

[3] L. Holthuijsen, *Waves in Oceanic and Coastal Waters*. Cambridge University Press, 2010, iSBN-13: 978-1-13-946252-5.