Math 4610 Task Sheet 2

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1 First Order Centered Difference

Here we analyse the error of the approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

Consider the following consequences of the Mean Value theorem

$$f(x+h) = f(x) + f'(x)(h) + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(\zeta_1)h^3 \tag{1}$$

$$f(x-h) = f(x) + f'(x)(-h) + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(\zeta_2)h^3 \tag{2}$$

with $x \leq \xi_1 \leq x+h$ and $x-h \leq \xi_2 \leq x$. Now let our error term ψ be

$$\psi = f'(x) - \frac{f(x+h) - f(x-h)}{2h}.$$

Substituting in equations 1, 2 we get

$$\psi = f'(x) - \frac{1}{2h}\left(2f'(x)h + \frac{1}{6}f'''(\zeta_1)h^3 + \frac{1}{6}f'''(\zeta_2)h^3\right).$$

Which upon further simplification we see that our error term is second order for some constant c

$$\psi = h^2\left(\frac{1}{6}f'''(\zeta_1) + \frac{1}{6}f'''(\zeta_2)\right) < ch^2$$

2 Second Order Centered Difference

We want do analyze the error approximation

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

$$f(x+h) = f(x) + f'(x)(h) + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(\xi_1)h^4 \quad (3)$$
$$f(x-h) = f(x) - f'(x)(h) + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f'''(x)h^3 + \frac{1}{24}f''''(\xi_2)h^4 \quad (4)$$

with $x \leq \xi_1 \leq x+h$ and $x-h \leq \xi_2 \leq x$. Now let our error term ε be

$$\varepsilon = f''(x) - f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h}.$$

Substituting in equations 3, 4 we get

$$\varepsilon = f''(x) - \frac{1}{h^2}\left(\frac{1}{2}f''(x)h^2 + \frac{1}{24}f''''(\xi_1)h^4 + \frac{1}{24}f''''(\xi_2)h^4\right)$$

This shows our error term is

$$\varepsilon = h^2\left(\frac{1}{24}f''''(\xi_1) + \frac{1}{24}f''''(\xi_2)\right) < ch^2.$$

Which we see is second order for some constant c.