Bar Elements

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EG3111 – Finite Element Analysis and Design

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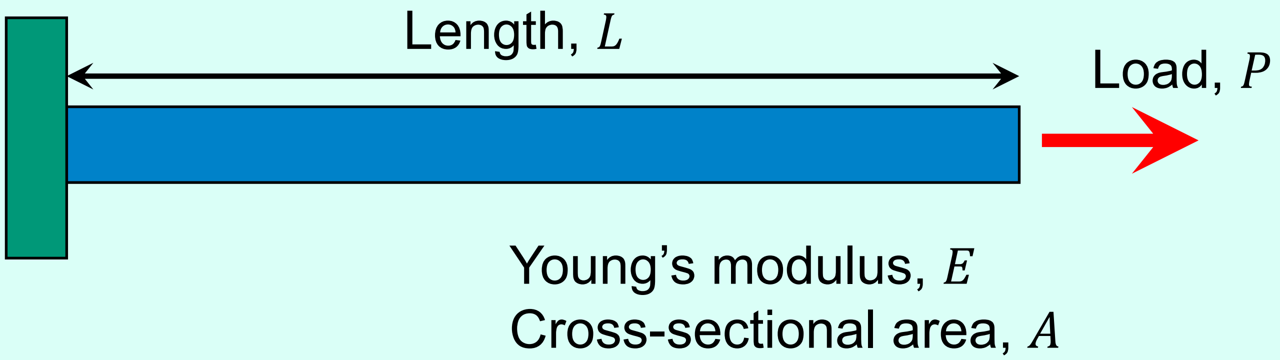
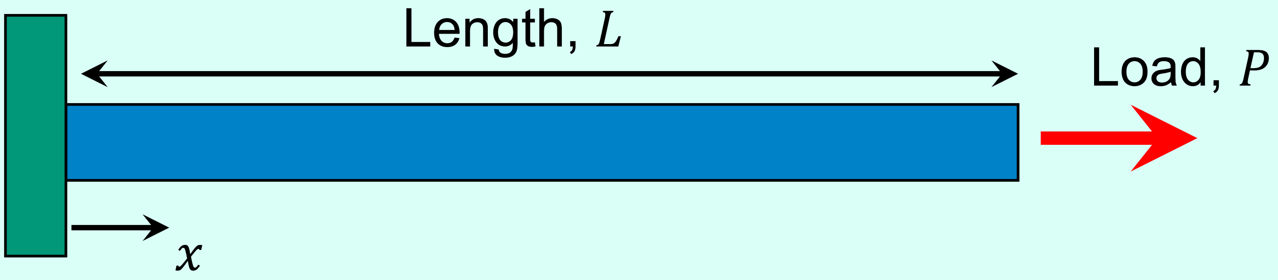
3. Bar Elements

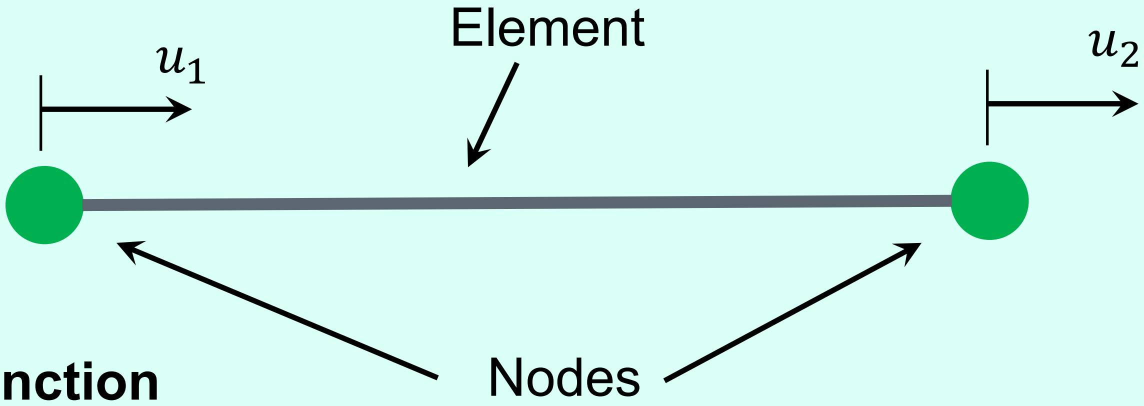


- We have already considered the problem of a bar under extension/compression in section 2.
- However, we do not wish to be formulating the total energy each time (especially for systems with many DOF) and hence wish to construct the FEM for general bar elements within a formal mathematical framework.
- This framework will also help us later on when we wish to develop other types of element, e.g beam, solid, shell.

3a. General Bar Element



Bar element

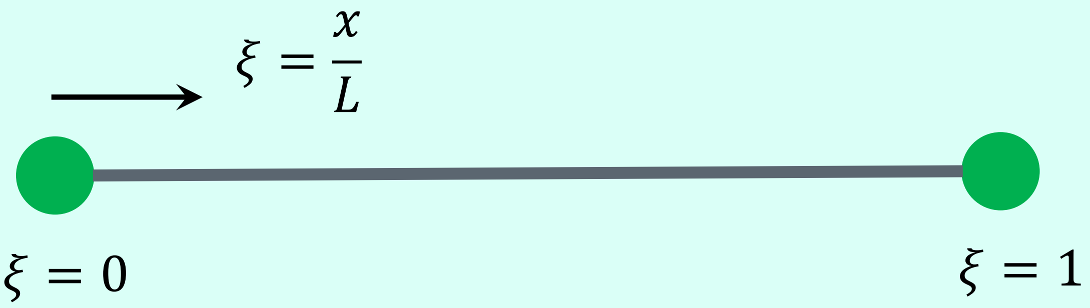


Linear shape function

$$u(x) = u_1\left(1 - \frac{x}{L}\right) + u_2\left(\frac{x}{L}\right)$$

3a. General Bar Element

Local variable, ξ



$$u(\xi) = u_1(1 - \xi) + u_2\xi$$

$$\Longrightarrow \boxed{u(\xi) = \underline{n}^{e^T} . \underline{d}^e}$$

$$\underline{n}^e(\xi) = \begin{bmatrix} 1 - \xi \\ \xi \end{bmatrix} \qquad \underline{d}^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Elemental shape function matrix

Elemental DOF matrix

3a. General Bar Element

$$\xi = \frac{x}{L} \qquad d\xi = \frac{dx}{L} \qquad \Longrightarrow \qquad dx = L d\xi$$

$$u(\xi) = u_1(1-\xi) + u_2\xi$$

$$u(\xi) = \underline{n^{e}}^{T} . \underline{d^{e}}$$

Where

$$\underline{n^{e}}(\xi) = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \qquad \underline{d^{e}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Check

$$u(\xi) = \underline{n^{e}}^{T} . \underline{d^{e}} \qquad = [1-\xi, \xi]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Longrightarrow \qquad u(\xi) = [(1-\xi)u_1 + \xi u_2]$$

3a. General Bar Element

Strain-displacement matrix

$$\epsilon_x = \frac{du}{dx} = \frac{1}{L}\frac{du}{d\xi} \qquad as\ dx = Ld\xi$$

$$\boxed{\epsilon_x = \frac{1}{L}\frac{du}{d\xi} = \frac{1}{L}\frac{d\left(\underline{n^{e}}^{T}.\underline{d^{e}}\right)}{\mathrm{d}\xi} = \frac{1}{L}\frac{d\underline{n^{e}}^{T}}{\mathrm{d}\xi}.\underline{d^{e}} = \underline{b^{e}}^{T}.\underline{d^{e}}}$$

Where

$$\boxed{\underline{b^{e}}^{T} = \frac{1}{L}\frac{d\underline{n^{e}}^{T}}{\mathrm{d}\xi} = \frac{1}{L^{e}}\frac{d}{\mathrm{d}\xi}\begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} = \frac{1}{L^{e}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\underline{n^{e}} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}$$

Elemental strain-displacement matrix
($L^{e} = L$ is the length of element e)

3a. General Bar Element

Check

$$\epsilon_x = \underline{b^{eT}}.\underline{d^e}$$

$$\epsilon_x = \frac{1}{\mathrm{L^e}}[-1,1]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\epsilon_x = \frac{1}{\mathrm{L^e}}(-u_1 + u_2)$$

$$\epsilon_x = \frac{(u_2 - u_1)}{L^e}$$

3a. General Bar Element

Elastic strain energy of an element

$$U = \frac{1}{2} EA \int_0^L \epsilon_x^2 \, dx$$

Where

$$\boxed{U = \frac{1}{2} EAL \int_0^1 \epsilon_x^2 \, d\xi} \quad \boxed{Eq\ 1}$$

$$\boxed{\begin{gathered} \underline{\epsilon} = [\epsilon_x] = \underline{b^{eT}}.\underline{d^e} \\ \underline{\epsilon^T} = [\epsilon_x^T] = \underline{d^{eT}}.\underline{b^e} \\ \epsilon_x^2 = \epsilon_x^T.\epsilon_x = \underline{d^{eT}}.\underline{b^e}.\underline{b^{eT}}.\underline{d^e} \end{gathered}}$$

Substitute ϵ_x^2 in $Eq\ 1$

$$U = \frac{1}{2} AL \int_0^1 \underline{d^{eT}}.\underline{b^e}.E.\underline{b^{eT}}.\underline{d^e} d\xi$$

3a. General Bar Element

Elastic strain energy of an element

$$\mathrm{U} = \frac{1}{2} AL \int_0^1 \underline{d^{e^T}}.\underline{b^e}.E.\underline{b^{e^T}}.\underline{d^e} d\xi$$

$$\mathrm{U} = \frac{1}{2}.\underline{d^{e^T}} \int_0^1 \{\underline{b^e}.EAL.\underline{b^{e^T}} d\xi\}.\underline{d^e}$$

$$\mathrm{U} = \frac{1}{2}.\underline{d^{e^T}}[k^e].\underline{d^e}$$

3a. General Bar Element

Elastic strain energy of an element

$$\boxed{\begin{gathered}\underline{\sigma} = [\sigma_x] = E.\underline{\epsilon} \\ \underline{\sigma^T} = \underline{d^{e^T}}.\underline{b^e}.E \\ \underline{\sigma^T}.\underline{\epsilon} = \underline{d^{e^T}}.\underline{b^e}.E.\underline{b^{e^T}}.\underline{d^e}\end{gathered}}$$

$$\boxed{U^e = \frac{1}{2}\int_{V^e} \underline{\sigma^T}.\underline{\epsilon}\ dV = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}}$$

Where

$$\boxed{[k^e] = \int_{V^e} \underline{b^e}.E.\underline{b^{e^T}}\ dV = EAL\int_0^1 \underline{b^e}.\underline{b^{e^T}}\,d\xi} \quad \boxed{Eq\ 2}$$

Elemental stiffness matrix

3a. General Bar Element

Check

$$\underline{b^e} = \frac{1}{L}\begin{bmatrix}-1\\ 1\end{bmatrix} \qquad \underline{b^{eT}} = \frac{1}{L}[-1, 1]$$

So

$$\underline{b^e}.\underline{b^{eT}} = \frac{1}{L}\begin{bmatrix}-1\\ 1\end{bmatrix}.\frac{1}{L}[-1, 1]$$

$$\underline{b^e}.\underline{b^{eT}} = \frac{1}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Substitute $\underline{b^e}.\underline{b^{eT}}$ in $Eq\ 2$

$$\boxed{\begin{aligned}[k^e] &= \int_{V^e} \underline{b^e}.E.\underline{b^{eT}}\ dV = EAL\int_0^1 \underline{b^e}.\underline{b^{eT}}\, d\xi \\ &= EAL\,\frac{1}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\cancel{\int_0^1} d\xi = \frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\end{aligned}}$$

3a. General Bar Element

Total strain energy

Elemental strain energy

$$\boxed{U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}}$$

$$U = \frac{1}{2}[u_1 - u_2].\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$U = \frac{1}{2}\frac{EA}{L}.(u_1^2 - u_2u_1 - u_1u_2 + u_2^2)$$

$$U = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

3a. General Bar Element

Total strain energy

Elemental strain energy

$$U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

Total strain energy

$$U = \sum_{all\ elements\ e} U^e$$
$$= \frac{1}{2}\underline{d^T}.[K].\underline{d}$$

The global stiffness matrix $[K]$ is the assembly of the elemental stiffness matrices $[k^e]$.

The global DOF matrix $\underline{d}$ contains all the DOF across all elemental DOF $\underline{d^e}$.

3a. General Bar Element

Finite Element Method

Potential energy of the applied loads

$$\boxed{\Omega = -\underline{d}^T . \underline{f}}$$

The global force matrix $\underline{f}$ contains all the forces acting of the respective DOF

Total energy

$$\boxed{\Pi = \frac{1}{2} \underline{d}^T . [K] . \underline{d} - \underline{d}^T . \underline{f}}$$

A quadratic function of $\underline{d}$

Minimise the total energy with respect to the DOF to find the unknowns $\underline{d}$.

Solution

$$\boxed{\frac{\partial \Pi}{\partial \underline{d}} = [K] . \underline{d} - \underline{f} = \underline{0}} \Longrightarrow \boxed{[K] . \underline{d} = \underline{f}}$$

3a. FEM Summary

Finite Element Method

To find the N degrees of freedom $\underline{d}$ solve the N simultaneous linear equations defined by

$$\boxed{[K].\underline{d} = \underline{f}}$$

All FEM problems can be written in this form.

Where

$[K]$ is the assembly of the $[k^e]$

$\underline{f}$ are the forces acting on each DOF

Elemental Matrices

Defined by assumed displacement field.

In 1D

$$u(x) = \underline{n^{e}}^{T}.\underline{d^e}$$

$$\underline{b^{e}}^{T} = \frac{1}{L}\frac{d\underline{n^{e}}^{T}}{\mathrm{d}\xi}$$

$$[k^e] = \int_{V^e} \underline{b^e}.E.\underline{b^{e}}^{T}\, dV$$

$$\Longrightarrow$$

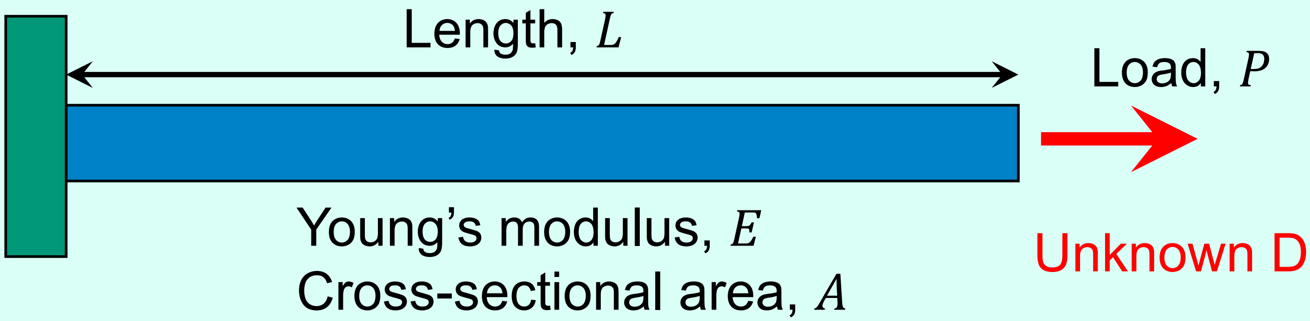
For a linear bar element

$$\underline{n^e} = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}$$

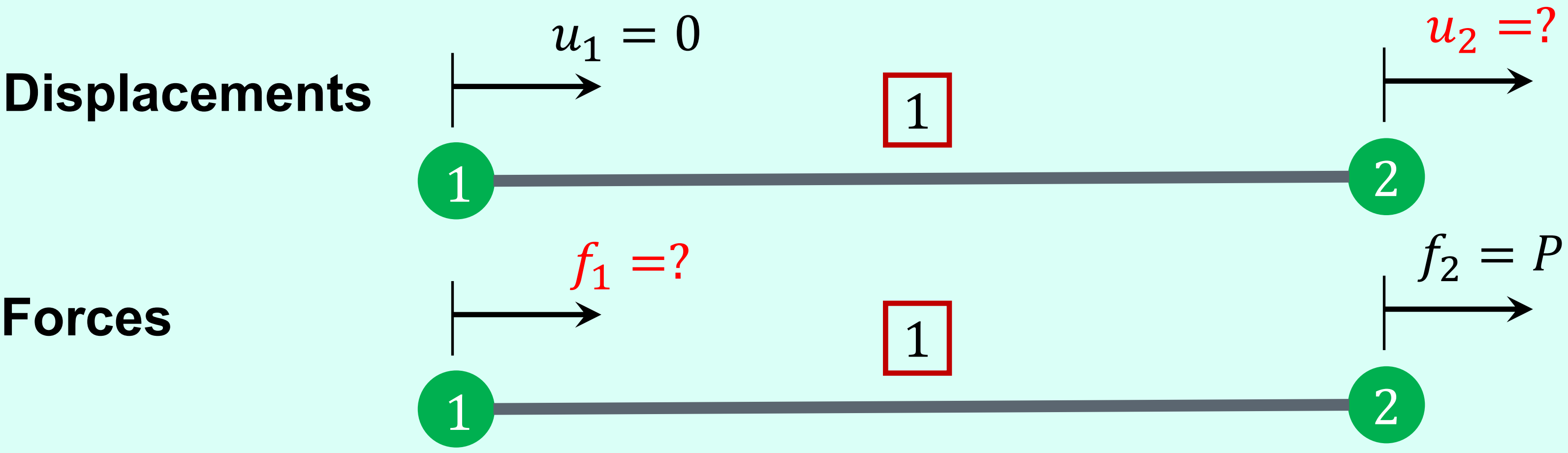
$$\underline{b^{e}}^{T} = \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[k^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

3b. Bar Element Example 1



Unknown DOF in red.



In general, either the displacement OR the force at a node is known, but never NEITHER or BOTH.

$$\underline{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

3b. Bar Element Example 1

All that is needed is the elemental stiffness matrix.

For element (1)

$$[k^{(1)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \underline{d}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

As only one element the elemental matrix is the same as the global matrix so no “assembly” required.

$$[K] = [k^{(1)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{[K].\underline{d} = \underline{f}} \qquad \Longrightarrow \qquad \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

3b. Bar Element Example 1

Two equations for 2 unknowns (u_2 and f_1)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ P \end{bmatrix}$$

$$\frac{EA}{L}(1 \times 0 - 1 \times u_2) = f_1 \qquad \boxed{Eq\ 1}$$

$$\frac{EA}{L}(-1 \times 0 + 1 \times u_2) = P \qquad \boxed{Eq\ 2}$$

3b. Bar Element Example 1

Second equation gives

$$\frac{EA}{L}(-1 \times 0 + 1 \times u_2) = P \quad \overset{\text{Partition}}{\Longrightarrow} \quad \frac{EA}{L} u_2 = P$$

$$u_2 = \frac{PL}{EA} \longleftarrow$$

Same as previous solution as same linear shape function

First equation gives

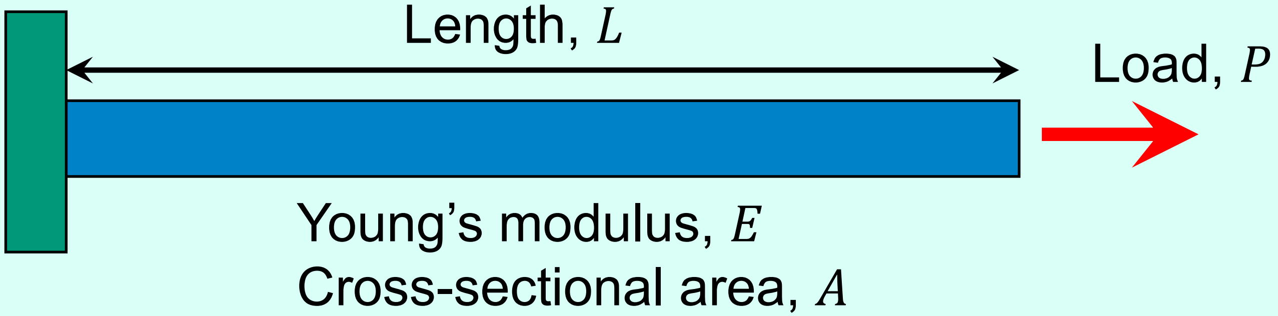
$$\frac{EA}{L}(1 \times 0 - 1 \times u_2) = f_1 \quad \Longrightarrow \quad -\frac{EA}{L} u_2 = f_1$$

Substitute u_2 in $Eq\ 1$

$$f_1 = -P \longleftarrow$$

This is the reaction force provided by the wall to keep the displacement $u_1 = 0$.

3b. Bar Element Example 1

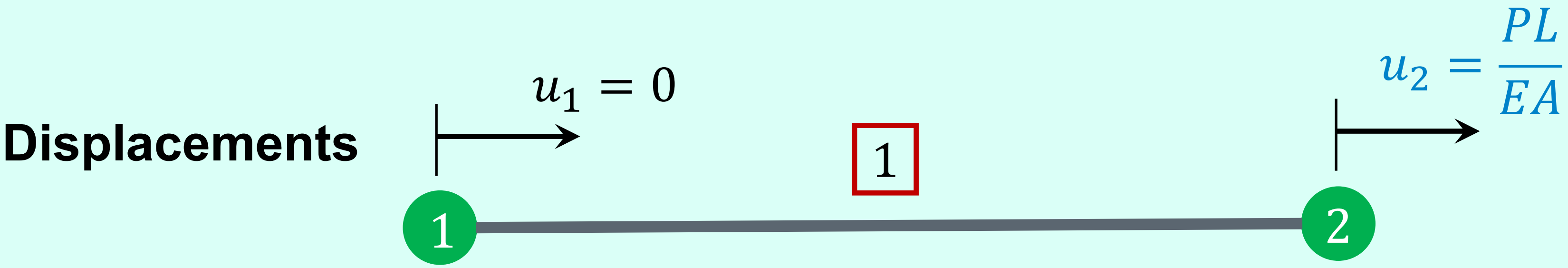


Length, L

Load, P

Young's modulus, E

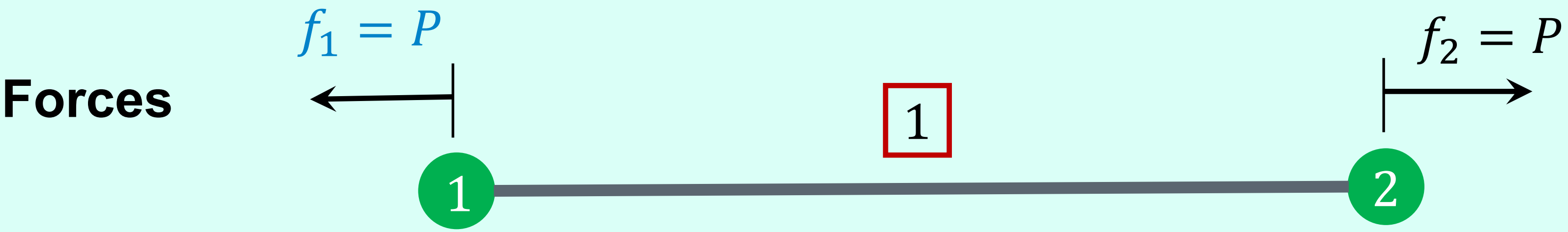
Cross-sectional area, A



Displacements

$u_1 = 0$

$u_2 = \frac{PL}{EA}$

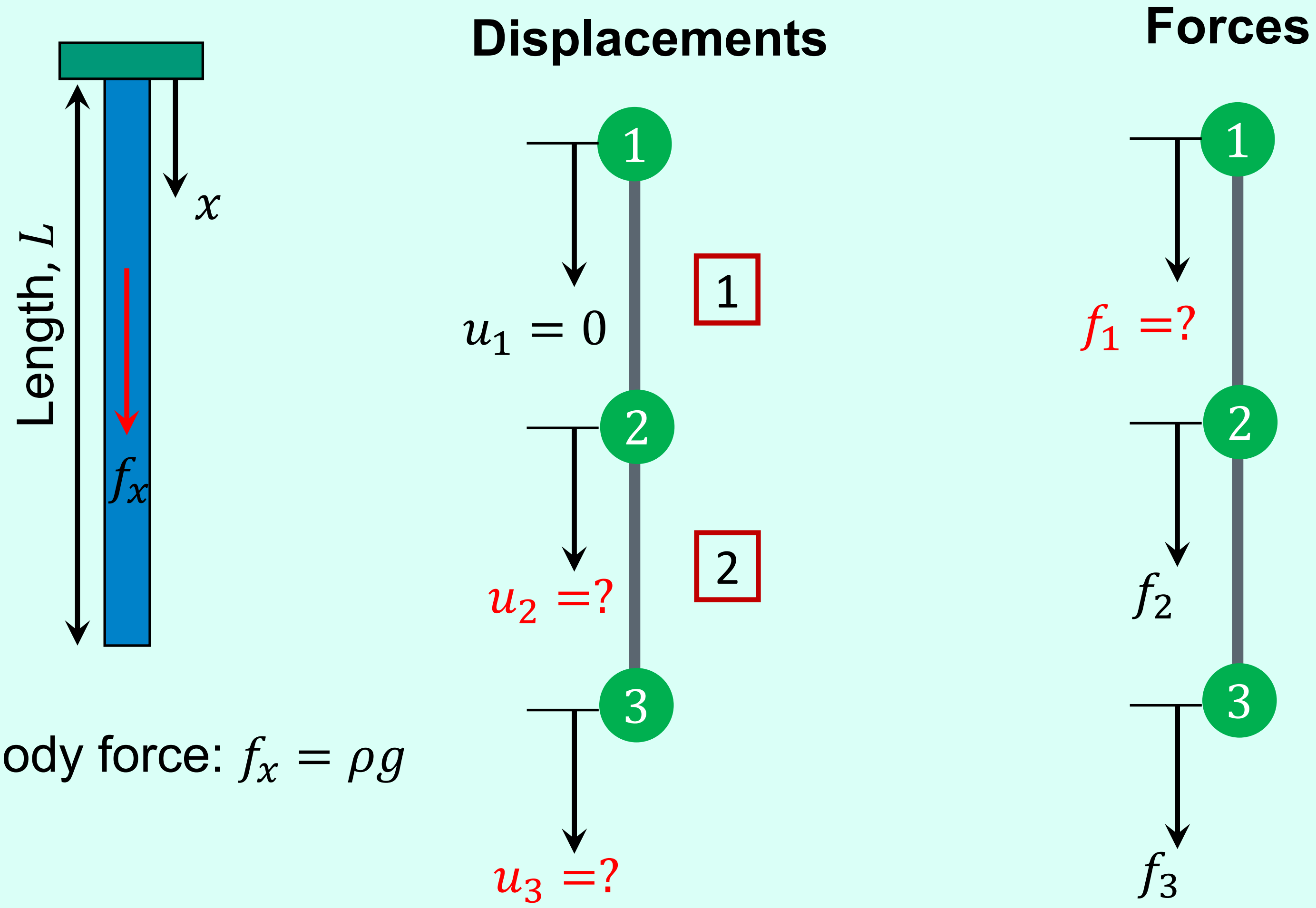


Forces

$f_1 = P$

$f_2 = P$

3b. Bar Element Example 2 : Self weight



Body force: $f_x = \rho g$

From section 2b, we know exact solution is a quadratic. Here we use two linear bar elements to get an approximate solution.

3b. Bar Element Example 2 : Self weight

Forces due to distributed load

$$\Omega^e = -\int_{V^e} f_x u(x)\, dx = -AL^e \int_0^1 \rho g . \underline{n^{e^T}}(\xi) d\xi . \underline{d^e} = -\underline{f^{e^T}} . \underline{d^e}$$

So

$$\underline{f^e} = AL^e \rho g \int_0^1 \underline{n^e}(\xi) d\xi = AL^e \rho g \int_0^1 \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} d\xi = AL^e \rho g \begin{bmatrix} \xi - \frac{1}{2}\xi^2 \\ \frac{1}{2}\xi^2 \end{bmatrix}_0^1$$

$$\underline{f^e} = AL^e \rho g \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{AL^e \rho g}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Where $L^e = \dfrac{L}{2}$ Each element is half the total length L

3b. Bar Element Example 2 : Self weight

Global matrices

$$\underline{d} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Elemental stiffness matrices

Element (1)

$$\left[k^{(1)}\right] = \frac{EA}{\frac{1}{2}L} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array}$$

$$\underline{d^{(1)}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Applied load due to self weight

Force due to unknown reaction at node 1

$$\underline{f^{(1)}} = \frac{A\frac{L}{2}\rho g}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$$

3b. Bar Element Example 2 : Self weight

Element (2)

$$[k^{(2)}] = \frac{EA}{\frac{1}{2}L}\begin{array}{ccc} u_2 & u_3 & \\ \left[\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] & \begin{matrix} u_2 \\ u_3 \end{matrix} \end{array}$$

$$\underline{d^{(2)}} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$\underline{f^{(2)}} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_2 \\ f_3 \end{bmatrix} = \frac{AL\rho g}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a = \frac{AL\rho g}{4}$$

3b. Bar Element Example 2 : Self weight

Assembly of elemental force matrices

$$\underline{f} = \left[\begin{array}{c} a + f_1 \\ a + a \\ a \end{array}\right] \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

Element (1)

$$\underline{f^{(1)}} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} f_1 \\ 0 \end{bmatrix} \begin{array}{c} u_1 \\ u_2 \end{array}$$

Element (2)

$$\underline{f^{(2)}} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{array}{c} u_2 \\ u_3 \end{array}$$

3b. Bar Element Example 2 : Self weight

Global stiffness matrix

$$[K] = \frac{2EA}{L}\begin{array}{c c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} & \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & \boxed{1+1} & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \end{array}$$

Element (1)

$$[k^{(1)}] = \frac{2EA}{L}\begin{array}{c c} \begin{array}{cc} u_1 & u_2 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} u_1 \\ u_2 \end{array} \end{array}$$

Element (2)

$$[k^{(2)}] = \frac{2EA}{L}\begin{array}{c c} \begin{array}{cc} u_2 & u_3 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} u_2 \\ u_3 \end{array} \end{array}$$

3b. Bar Element Example 2 : Self weight

$$\boxed{[K].\underline{d} = \underline{f}} \implies \frac{2EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a + f_1 \\ 2a \\ a \end{bmatrix} \quad \boxed{Eq\ 3}$$

$$a = \frac{AL\rho g}{4}$$

As in previous example, partition out 2nd and 3rd equations where forces are known

$$\frac{2EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2a \\ a \end{bmatrix}$$

Matrix inversion (reminder)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(\mathrm{ad} - \mathrm{bc})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{La}{2EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

3b. Bar Element Example 2 : Self weight

$$\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}^{-1} = \frac{1}{(2\times 1-(-1)^2)}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} = \frac{1}{2-1}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

$$\begin{bmatrix}u_2\\ u_3\end{bmatrix} = \frac{L}{2EA}\times\overset{a}{\frac{AL\rho g}{4}}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} = \frac{L^2\rho g}{8E}\begin{bmatrix}3\\ 4\end{bmatrix}$$

First equation from Eq 3 then gives

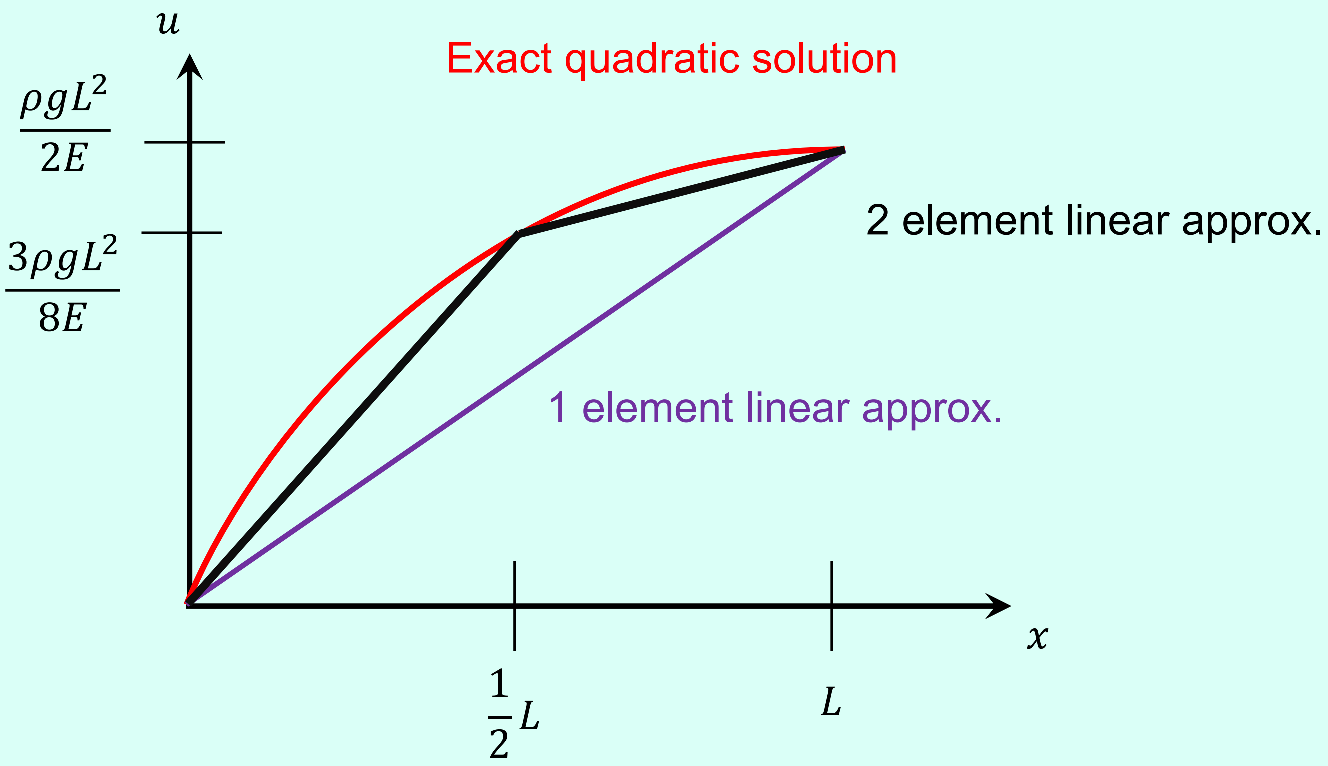
$$\frac{2EA}{L}[1.(0)-1.(u_2).0(u_3)] = a+f_1$$

$$\Rightarrow \quad a+f_1 = \frac{2EA}{L}(-u_2) = -\frac{2EA}{L}.\frac{3L^2\rho g}{8E} = -\frac{3AL\rho g}{4}$$

Reaction = weight of bar

$$\Rightarrow \quad f_1 = -\frac{3AL\rho g}{4} - \frac{AL\rho g}{4} = -AL\rho g\left(\frac{3}{4}+\frac{1}{4}\right) \quad \Rightarrow \boxed{f_1 = -AL\rho g}$$

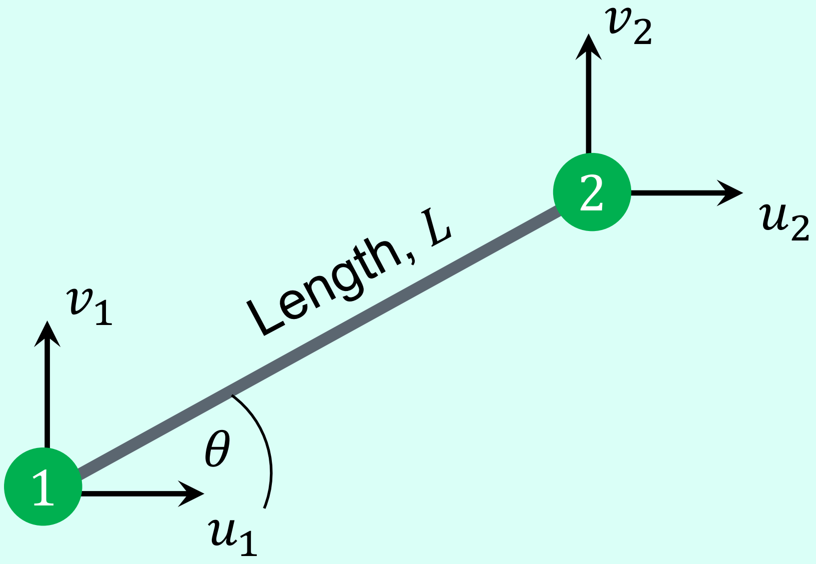
3b. Bar Element Example 2 : Self weight



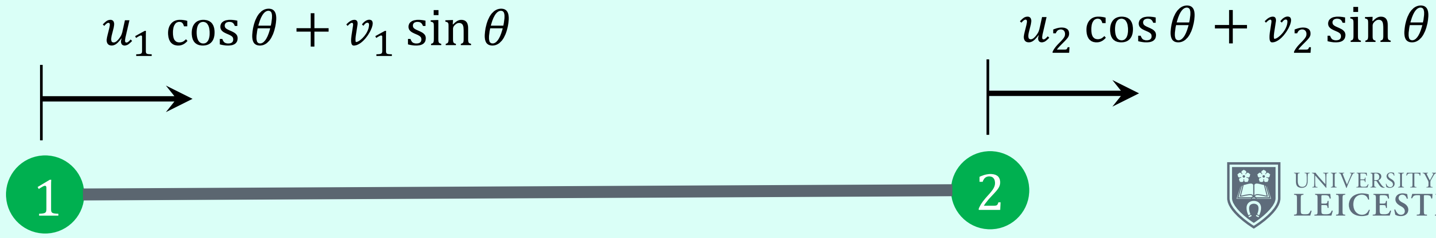
3c. Bar elements for 2D frameworks

Consider a bar element with orientation θ. In 2D the horizontal and vertical displacements are u and v.

DOF



Only displacements parallel to bar axis cause extension/compression



3c. Bar elements for 2D frameworks

Elemental strain energy

$$U^e = \frac{1}{2}\underline{d^{e^T}}.\,[k^e].\,\underline{d^e} \qquad \underline{d^e} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$[k^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \underline{d^e} = \begin{bmatrix} u_1 \cos\theta + v_1 \sin\theta \\ u_2 \cos\theta + v_2 \sin\theta \end{bmatrix}$$

$$U^e = \frac{1}{2}[u_1 \quad u_2].\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$U^e = \frac{1}{2}[u_1 \quad u_2].\frac{EA}{L}\begin{bmatrix} u_1 & -u_2 \\ -u_1 & u_2 \end{bmatrix}$$

$$U^e = \frac{1}{2}\frac{EA}{L}[u_1(u_1 - u_2) + u_2(-u_1 + u_2)]$$

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

3c. Bar elements for 2D frameworks

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

$$U^e = \frac{1}{2}\frac{EA}{L}(u_2\cos\theta + v_2\sin\theta - u_1\cos\theta - v_1\sin\theta)^2$$

$$U^e = \frac{1}{2}\underline{d^{e^T}}.\,[k^e].\,\underline{d^e}$$

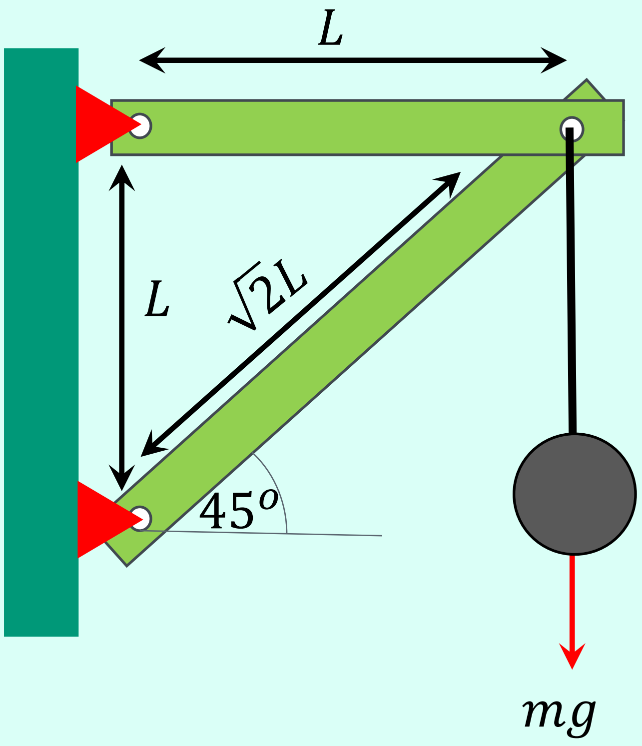
We wish to consider the four nodal DOF separately.

It is easy to show that U^e is the same with

$$[k^e] = \frac{EA}{L}\begin{array}{cc:cc|c} u_1 & v_1 & u_2 & v_2 & \\ \hline \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta & u_1 \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta & v_1 \\ \hdashline -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta & u_2 \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta & v_2 \end{array} \qquad \underline{d^e} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

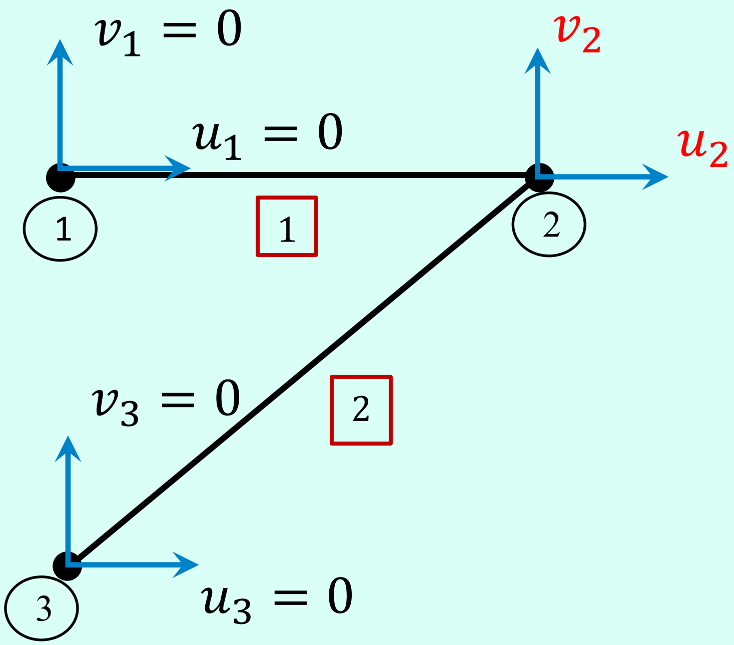
3c. Framework Example

Using two linear bar elements, find the unknown nodal displacements and forces, assuming E and A same for both bars

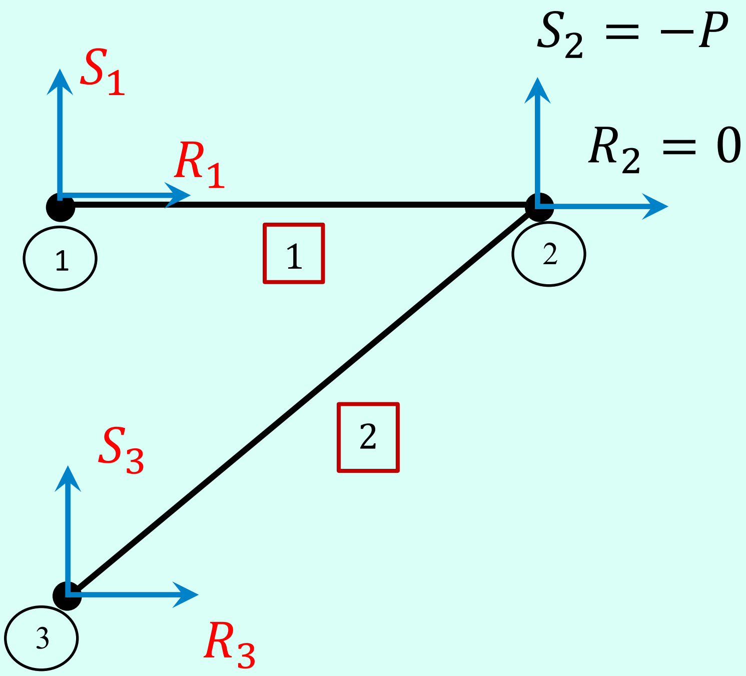


3c. Framework Example

Displacements



Forces



$P = mg$

Unknown DOF in red.

3c. Framework Example

Element (1)

$$\underline{d}^{(1)} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \end{bmatrix}$$

Element 1 $(\boldsymbol{\theta} = \mathbf{0^0})$ $\quad \cos^2 0^o = 1 \quad \cos 0^o \sin 0^o = \sin^2 0^o = 0$

$$\boxed{\left[k^{(1)}\right] = k_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad k_1 = \frac{EA}{L}}$$

3c. Framework Example

Element (2)

$$\underline{d}^{(2)} = \begin{bmatrix} u_3 \\ v_3 \\ u_2 \\ v_2 \end{bmatrix} \qquad \underline{f} = \begin{bmatrix} R_3 \\ S_3 \\ R_2 \\ S_2 \end{bmatrix}$$

Element 2 $(\boldsymbol{\theta = 45^o})$ $\quad \cos^2 45^o = \cos 45^o \sin 45^o = \sin^2 45^o = \frac{1}{2}$

$$\boxed{[k^{(2)}] = k_2 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \qquad k_2 = \frac{EA}{2\sqrt{2}L}}$$

3c. Framework Example

Assemble global matrices

$$\left[k^{(1)}\right]=k_1\begin{array}{cc} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} & \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \end{array}$$

$$\left[k^{(2)}\right]=k_2\begin{array}{cc} \begin{array}{cccc} u_3 & v_3 & u_2 & v_2 \end{array} & \\ \left[\begin{array}{cccc} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array}\right] & \begin{array}{c} u_3 \\ v_3 \\ u_2 \\ v_2 \end{array} \end{array}$$

$$k_1=\frac{EA}{L}$$

$$k_2=\frac{EA}{2\sqrt{2}L}$$

$$\underline{d}=\left[\begin{array}{c} u_1 \\ v_1 \\ \hline u_2 \\ v_2 \\ \hline u_3 \\ v_3 \end{array}\right]$$

$$[K]=\begin{array}{cc} \begin{array}{cc|cc|cc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} & \\ \left[\begin{array}{cc|cc|cc} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -k_1 & 0 & k_1+k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ \hline 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{array}\right] & \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array}$$

3c. Framework Example

Assemble global matrices

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{bmatrix}$$

$$\underline{d} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{f} = \begin{bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \\ R_3 \\ S_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \\ 0 \\ -P \\ R_3 \\ S_3 \end{bmatrix}$$

3c. Framework Example

Solution

$$\boxed{[K].\underline{d} = \underline{f}} \quad \Longrightarrow \quad \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & k_2 & k_2 & -k_2 & -k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \\ 0 & 0 & -k_2 & -k_2 & k_2 & k_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \\ 0 \\ -P \\ R_3 \\ S_3 \end{bmatrix}$$

Partition out 3rd and 4th equations where forces are known.

$$\begin{bmatrix} k_1 + k_2 & k_2 \\ k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

3c. Framework Example

Solution

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{(k_1 + k_2)k_2 - {k_2}^2} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{k_1 k_2} \begin{bmatrix} k_2 P \\ -(k_1 + k_2)P \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix}$$

3c. Framework Example

Solution

Substitute known displacements into remaining equations to find unknown forces.

Node 1

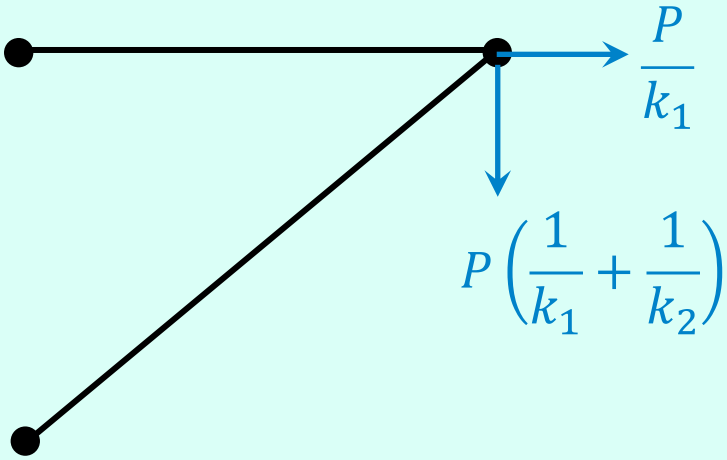
$$\begin{bmatrix} -k_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ S_1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} R_1 \\ S_1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 0 \end{bmatrix} . P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix} = P \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Node 3

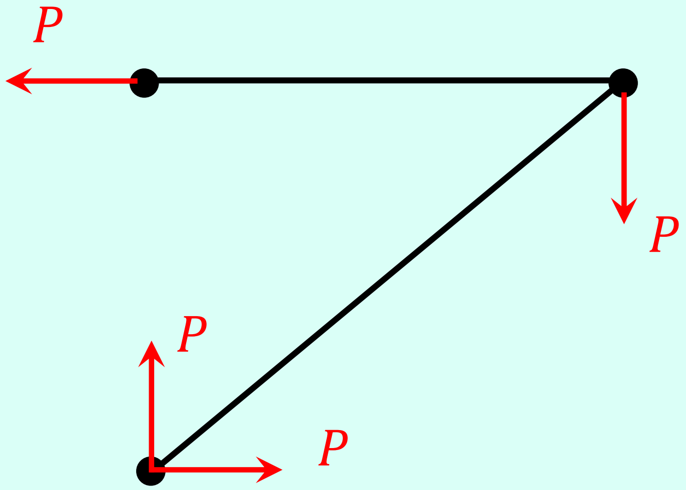
$$\begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_3 \\ S_3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} R_3 \\ S_3 \end{bmatrix} = \begin{bmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{bmatrix} . P \begin{bmatrix} \dfrac{1}{k_1} \\ -\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) \end{bmatrix} = P \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3c. Framework Example

Displacements

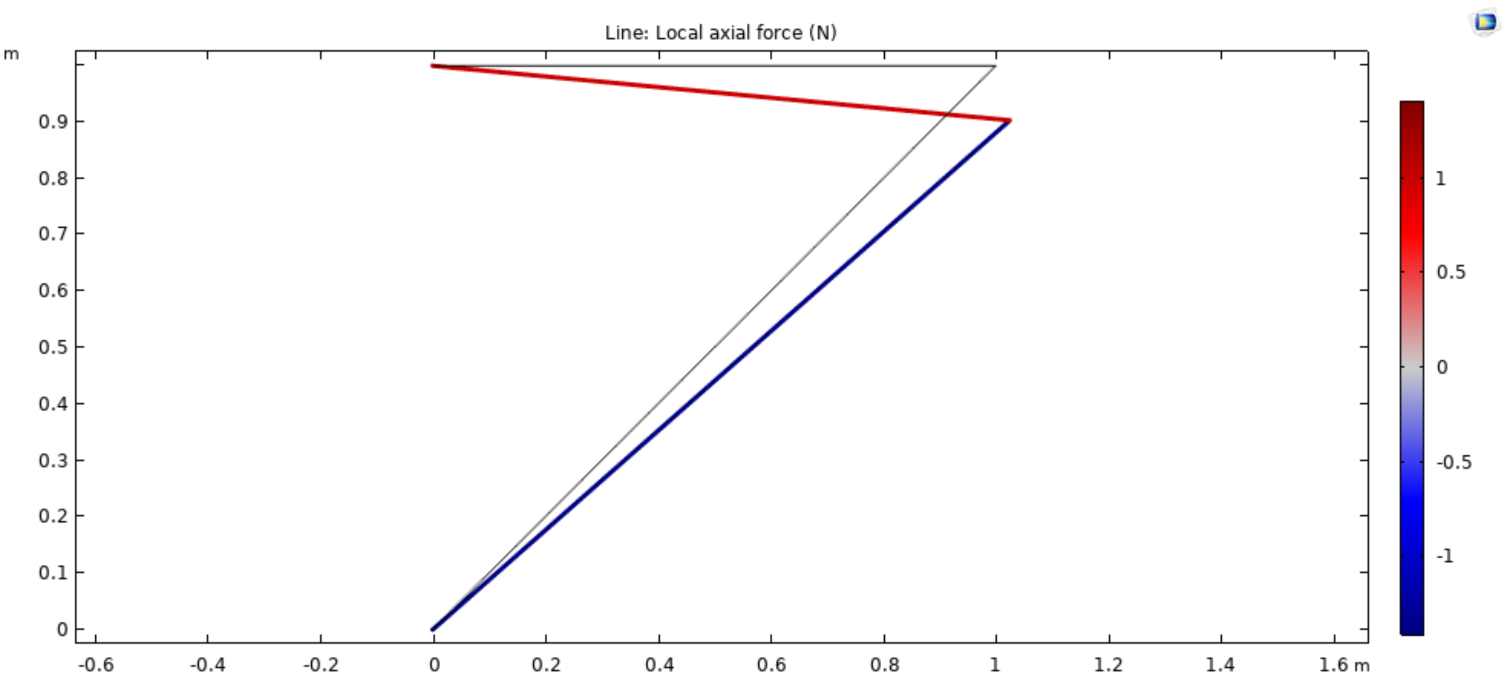


Forces



3c. COMSOL Practical #1

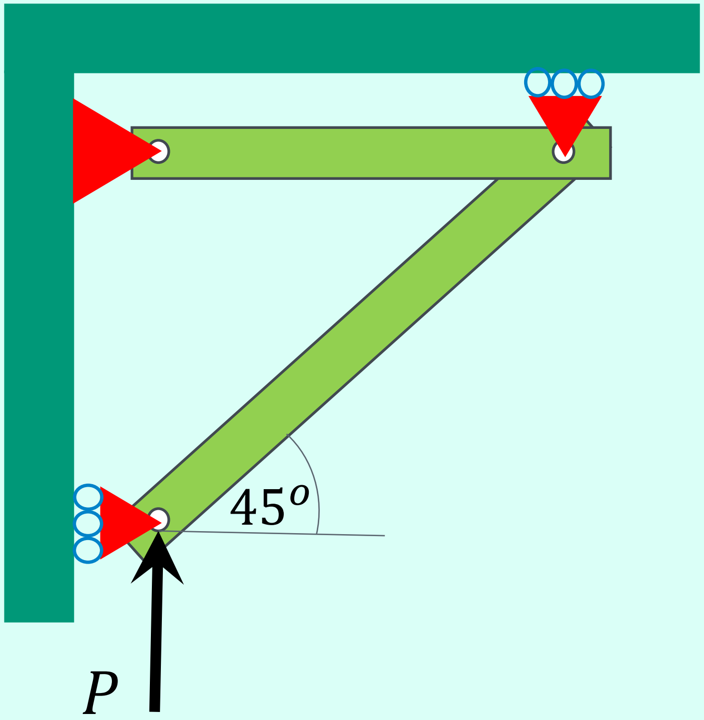
(1) Solve this example and check solution is the same

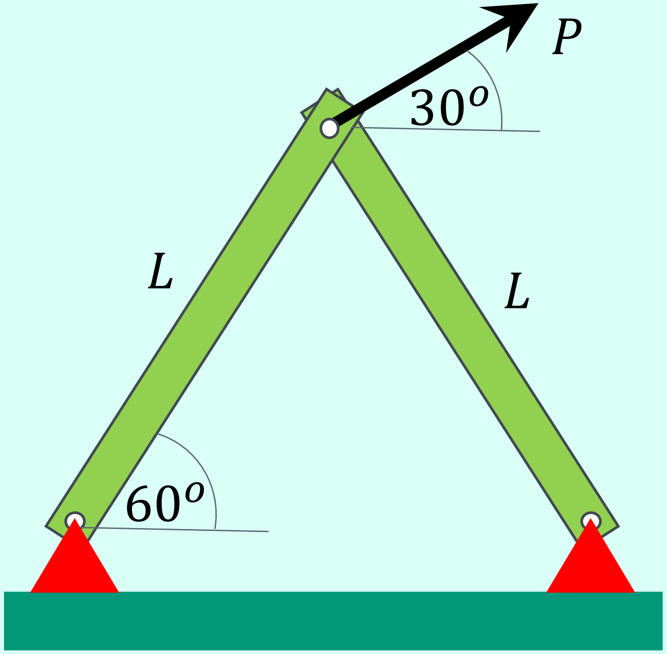


(1) Explore the effects of changing the angle from 45^o using a parametric analysis.

3c. COMSOL Practical #1 (continued)

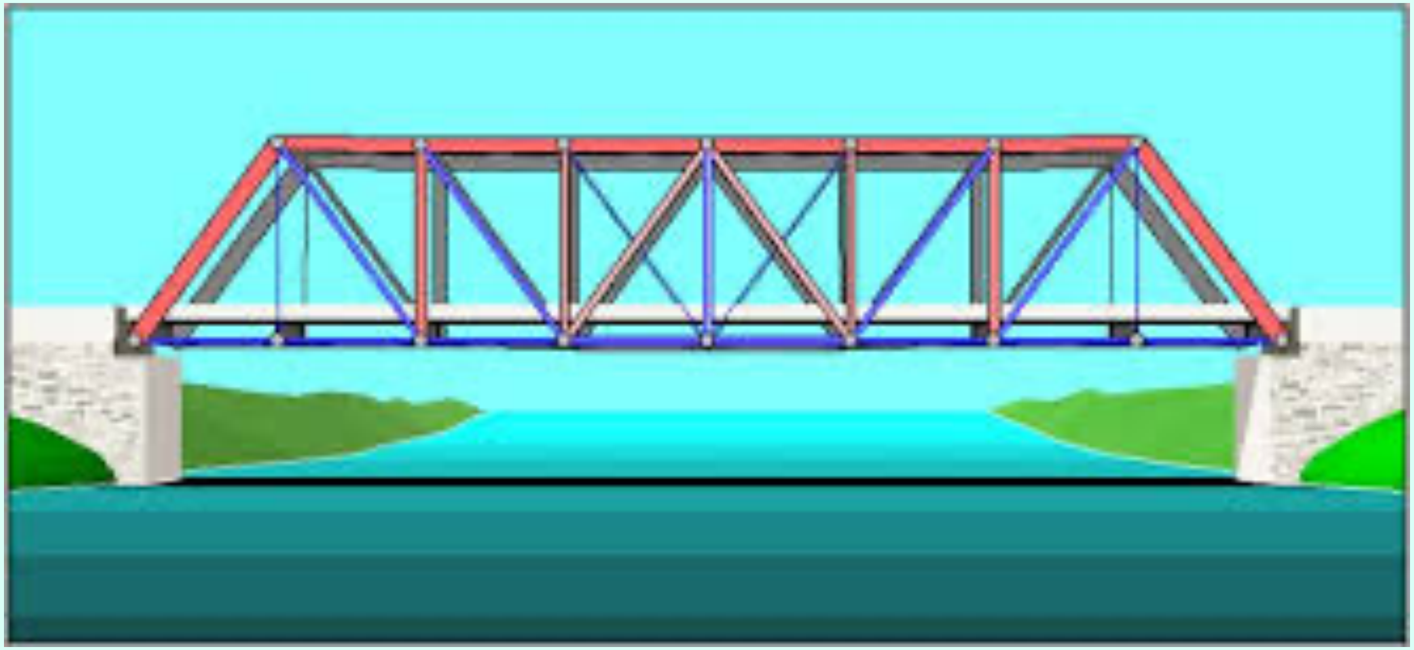
(3) Solve the examples in **Exercise Sheet #3** and check results using COMSOL.





3c. COMSOL Practical #2

Design of a truss bridge



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Next section…
(4) Beam Elements