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School of Engineering

Finite Element Analysis
what are simulations for?

Gebril El-Fallah

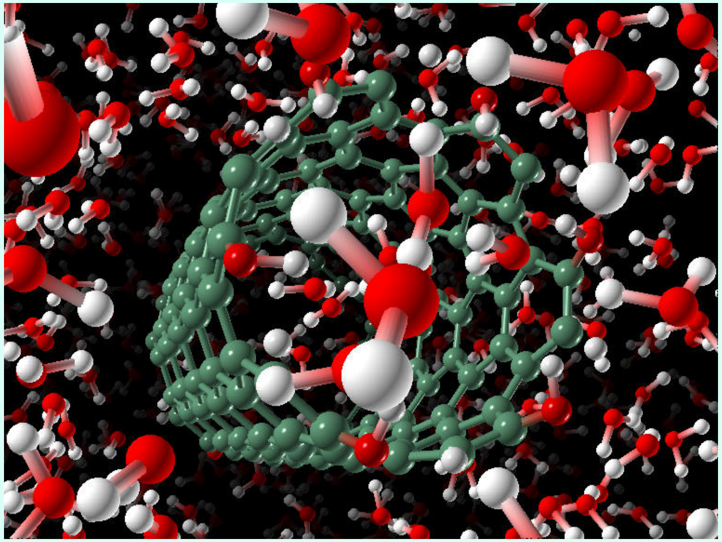
EG3111 – Finite Element Analysis and Design

gmae2@leicester.ac.uk

1a. Why should we do simulations?

- We can generate equations from simple physical laws which model the world around us.
- These can only usually be solved by computer simulation in realistic cases.
- Engineering is pushing the boundaries at both ends of the scale spectrum.

Atomic (nano) scale



Simulation of flow of water molecules through carbon nanotubes.
"www.cscs.ch"

Doping the borders of a carbon nanotube with OH groups changes its behaviour in the presence of water. The nanotube changes from hydrophobic to hydrophilic allowing a chain of water molecules to pass through it

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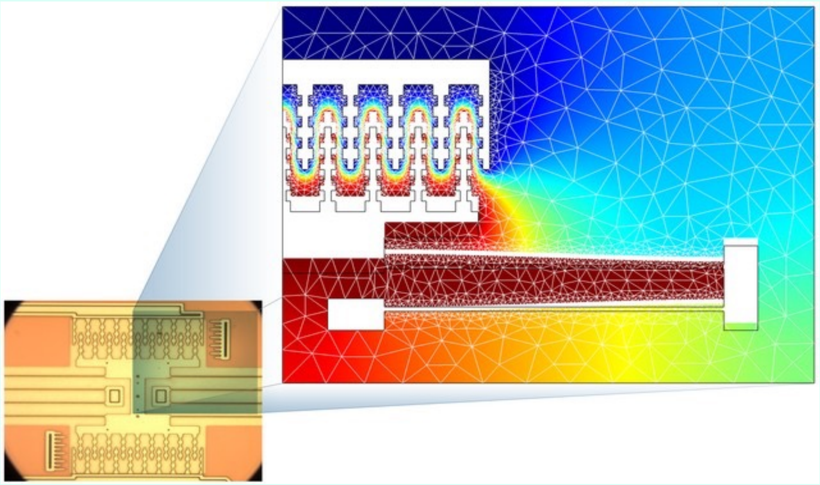
1a. Why should we do simulations?
(continued)

Micron scale



Electrostatic comb drive actuator

This is a model of an electrostatically actuated comb drive used to open and close a pair of microtweezers. Electrostatic forces attract the combs to each other. Colours show the electric field, white areas are the drive.

Kilometre scale

Simulated image of the Strait of Messina bridge, Sicily

This simulation of a new bridge built across the Strait of Messina pushes the limits of large-scale structural engineering. The bridge has a span of 3km.

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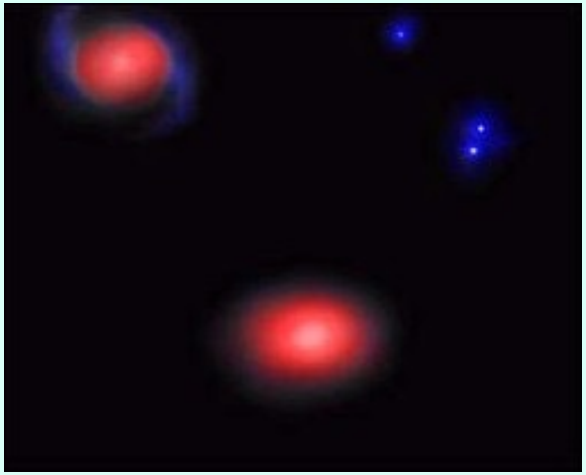
1a. Why should we do simulations?
(continued)

And beyond…….

Cosmological scale

After 0 - 0.5 billion years

Starting out with a very smooth distribution of matter directly after the Big-Bang, gravity of the more massive clumps of stars starts to attract more matter.



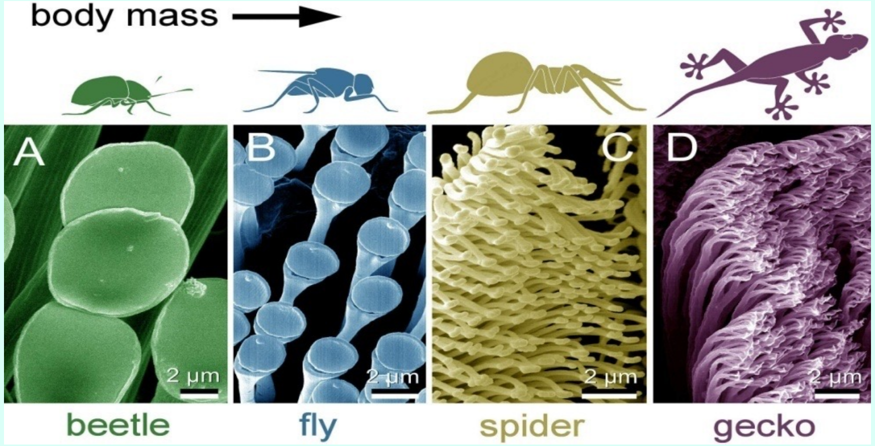
After 4 - 13 billion years

Galaxies as we see them today form, and take their final shapes.

(www.oarval.org)

1a. Why should we do simulations?
(continued)

Understanding



Understanding how biological systems do things can help engineers create new technologies. For instance, study of the adhesion of insects and reptiles to walls could lead to development new adhesives.

Optimization

A 1% increase in efficiency of a steam turbine in a power-generating plant can save millions of pounds and help environment.

1a. Why should we do simulations?
(continued)

Innovation

The Boeing Dreamliner 777-200LR (Long Range) allows direct flights from London-Sydney. This is possible because of the extensive use of carbon-fibre in its construction. It is not possible to design and build such a complicated object by trial and error.

Nearly all structural components are now designed and tested “in silico” (ie. by simulation in a computer) before a real prototype is constructed.

1a. What is Finite Element Analysis (FEA)?

There are two widely used numerical methods for finding **approximate solutions** to *Partial Differential Equations* (PDEs):

- **Finite Element Method (FEM)**
- **Finite Difference Method (FDM)**

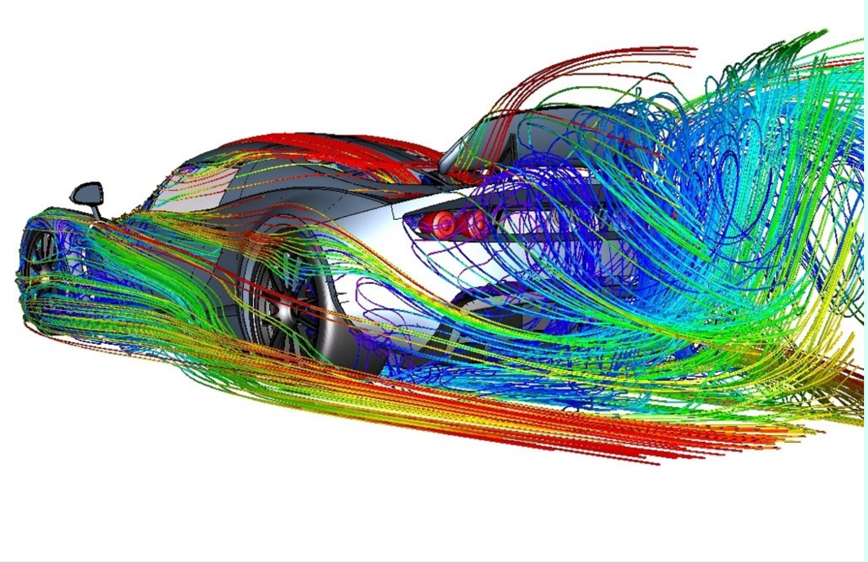
This module introduces the **FEM** and how it is used to solve practical engineering problems in **solid mechanics**.

The **FDM** is more commonly used in the field of **fluid mechanics**.

Why is that the case?

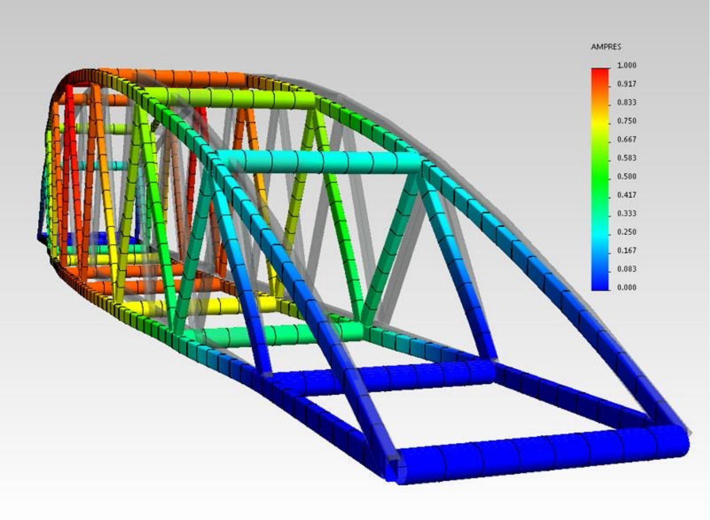
1a. FEM vs FDM

FDM
- Differential method
- Calculates values **locally** from neighbouring cells
- Good for weakly coupled problems evolving over time.
- Based on a (typically square) grid (usually
- Commonly used in Computational Fluid Dynamics (CFD), particularly for compressible air flow.



https://engineering.eckovation.com/cfd-cfd-projects/

FEM
- Integral (averaging) method
- Calculates values **globally** analysing the entire connected system as a whole.
- Finds the solution in **one** iteration.
- Based on an unstructured mesh (usually triangles)
- Commonly used in Structural Mechanics



https://www.exportersindia.com/motovated-design-and/finite-element-analysis-services-3119735.htm

Finite Element Analysis and Design

- 15 credits
  - 20 Lectures:
    - 2 Pre-recorded lectures/week
    - 1 Live lecture/week

- Assessment:
  - Exam - May 2023

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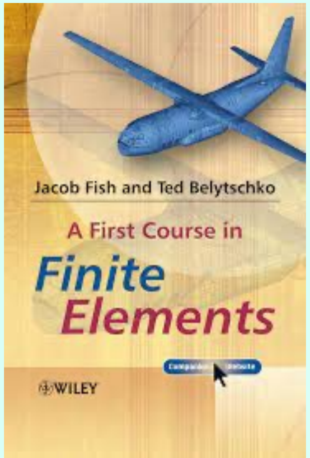
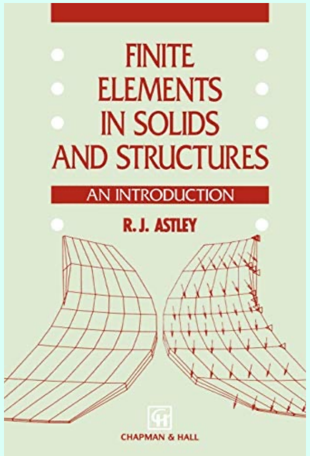
Finite Element Analysis and Design

Reference book (recommended reading):

- **A First Course in Finite Elements**
  Fish J. and Belytschko T
  Wiley Blackwell



- **Finite Elements in Solids and Structures: An Introduction.**
  Astley R.J
  Chapman and Hall



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I. Theory

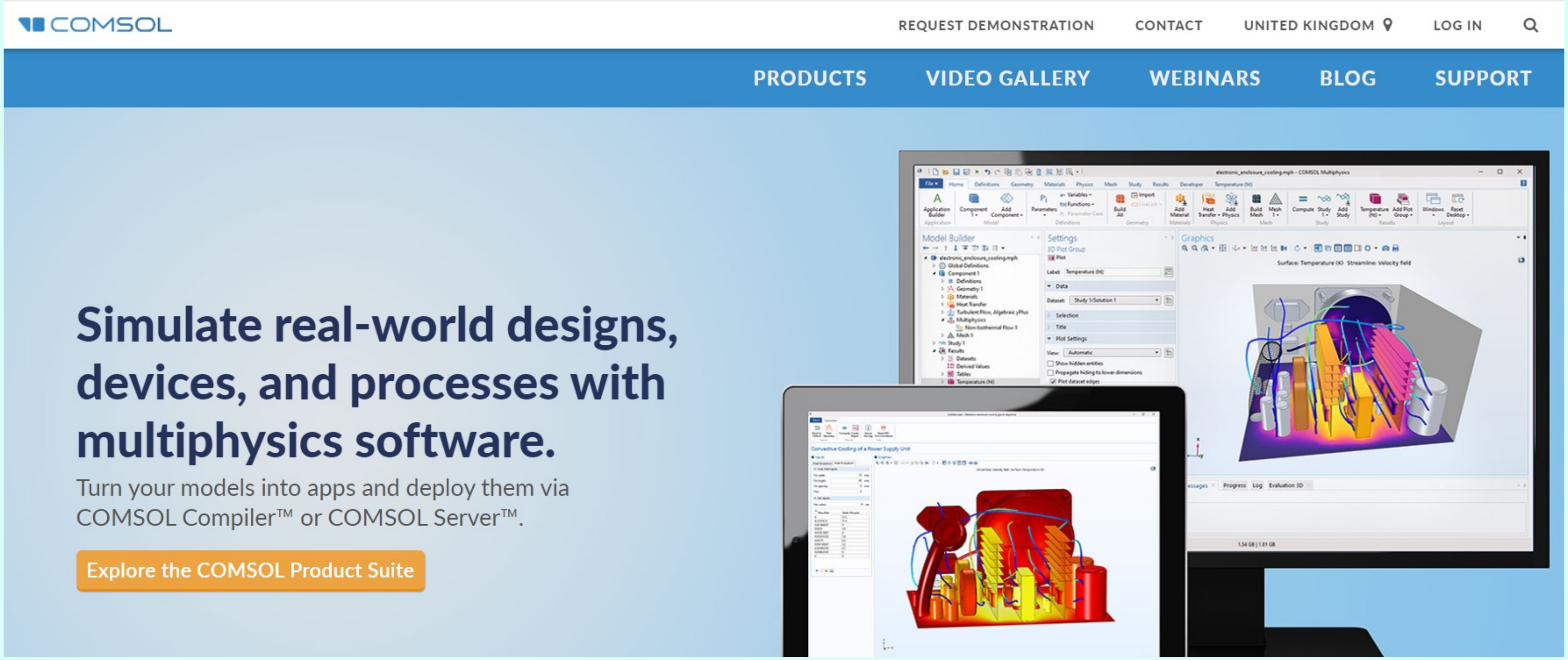
II. Application & Design

1a. Session schedule

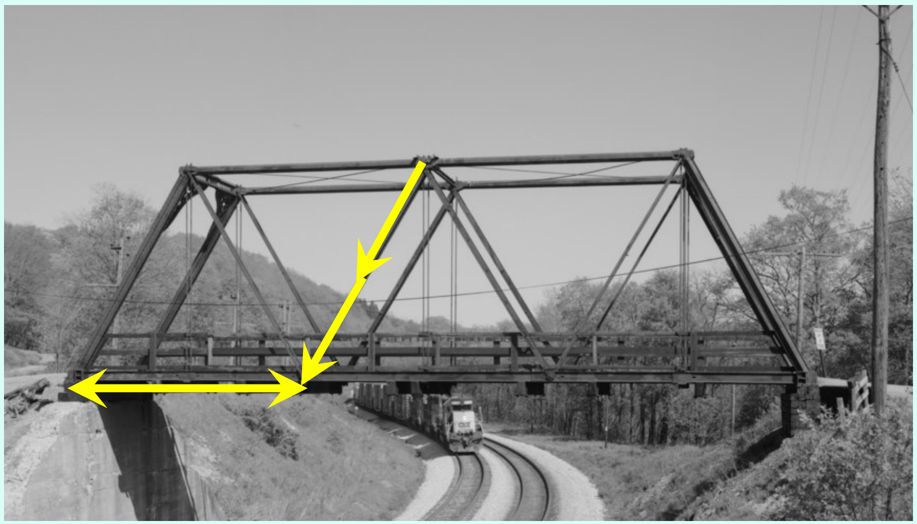
Section	Title	Approx. no. of hours
1	Introduction	1
2	Elasticity Theory	2
3	Bar Elements	2
4	Beam and Frame Elements	1
5	Solid Elements	4
6	Membrane, Plate and Shell Elements	3
7	Elastic FEA in practice	3
8	Different loading types	3
9	Non-linear analysis	2
10	Practical examples	3

1a. Eight practical sessions using COMSOL Multiphysics

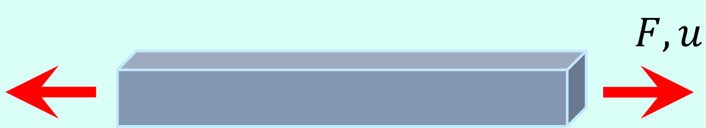
https://uk.comsol.com/



1a. 1D Elements

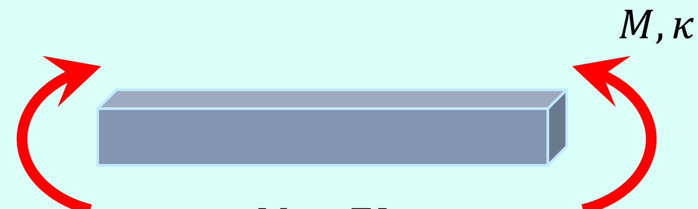


Bar Element



$$F = \frac{EA}{L} u$$

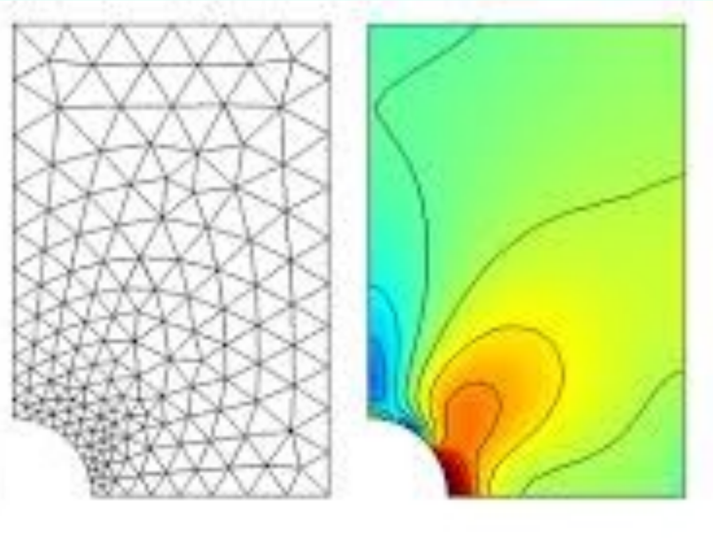
Beam Element

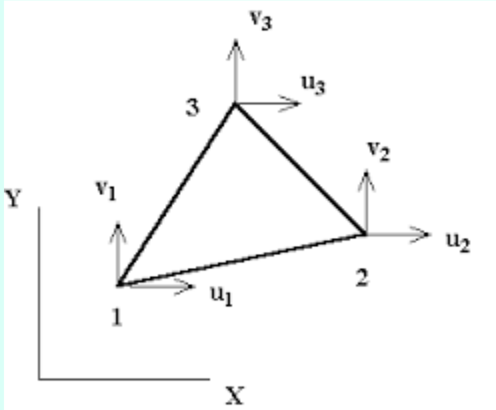


$$M = EI\kappa$$

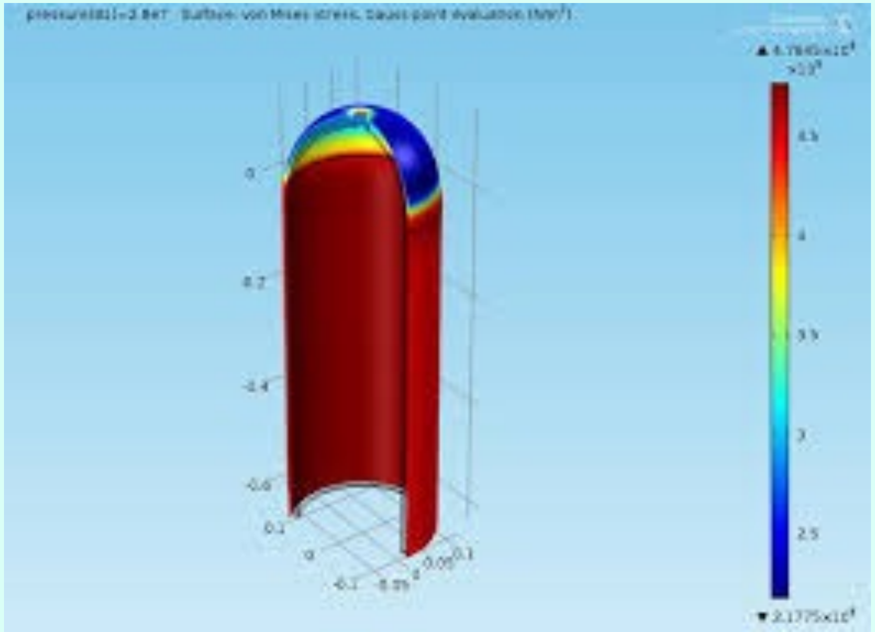
1a. 2D Elements

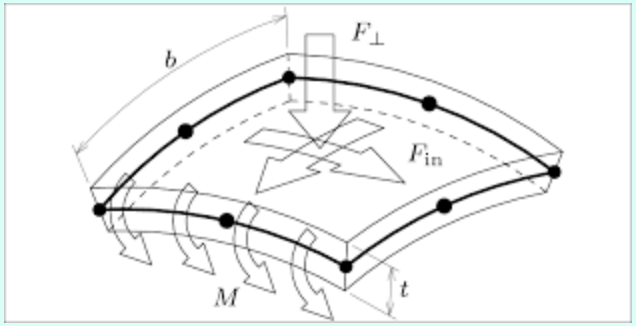
Triangular Solid Elements





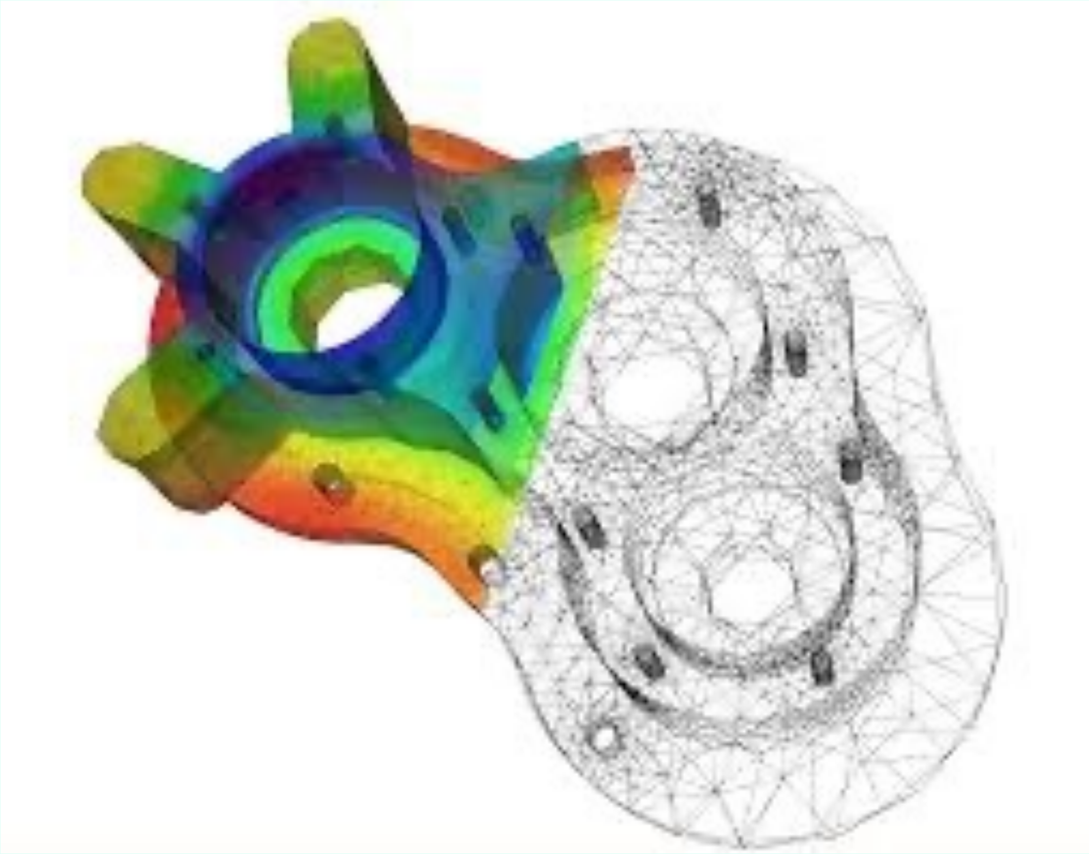
Shell Elements

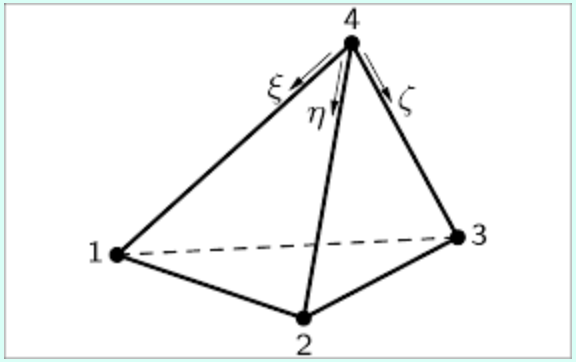




1a. 3D Elements

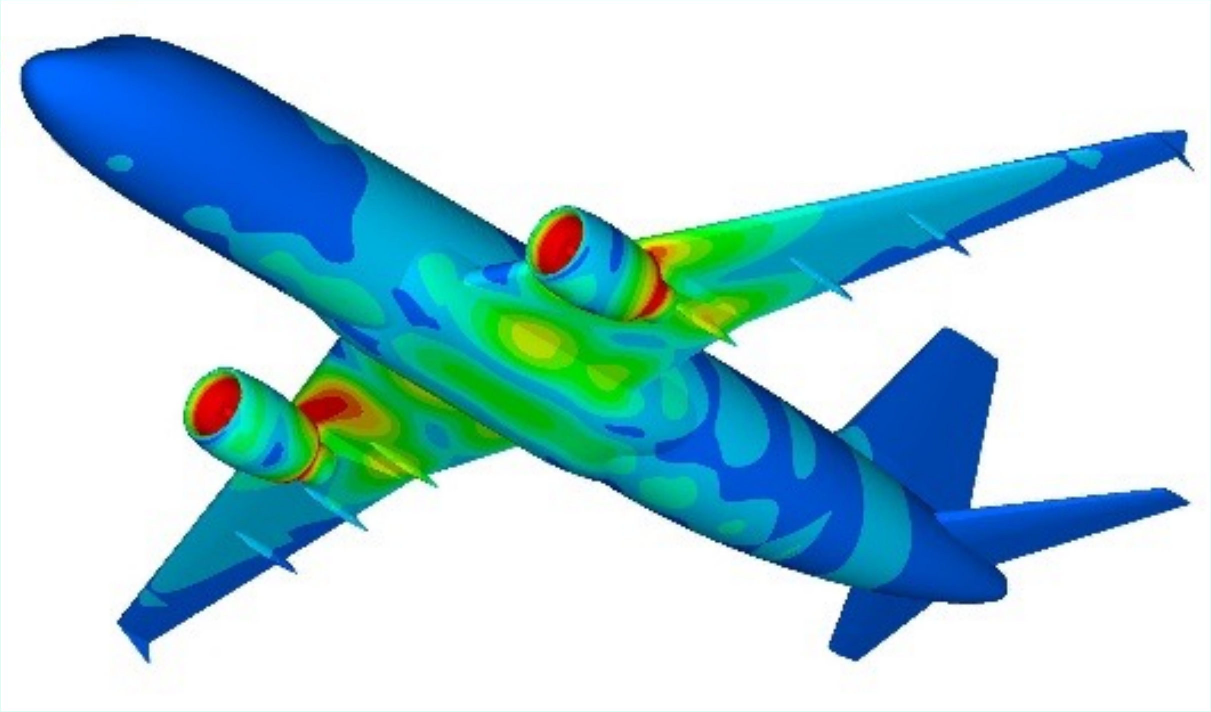
Tetrahedral Solid Elements





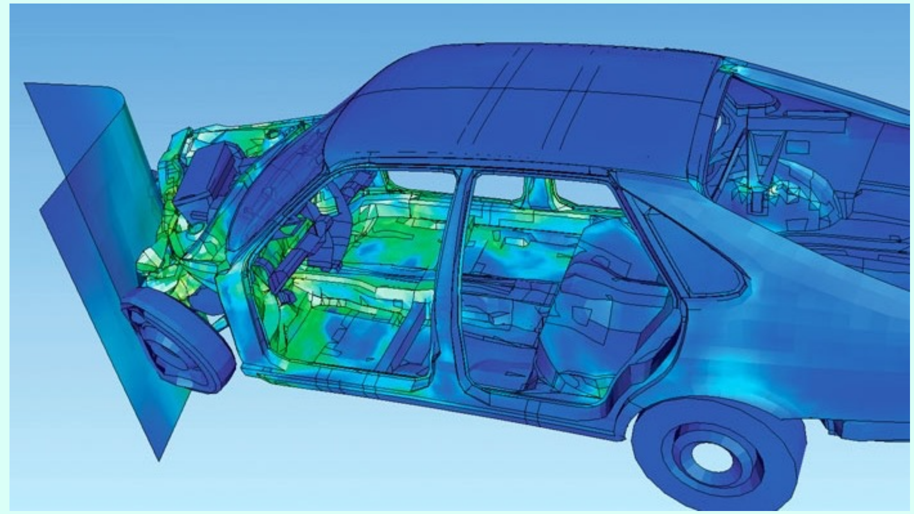
1a. Aerospace Engineering Applications

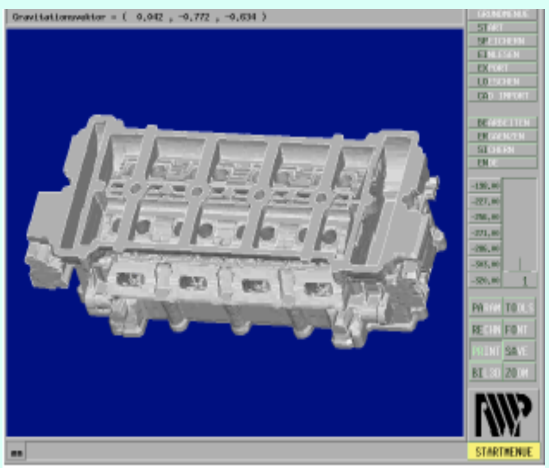
- Stress
- Fatigue
- Fracture

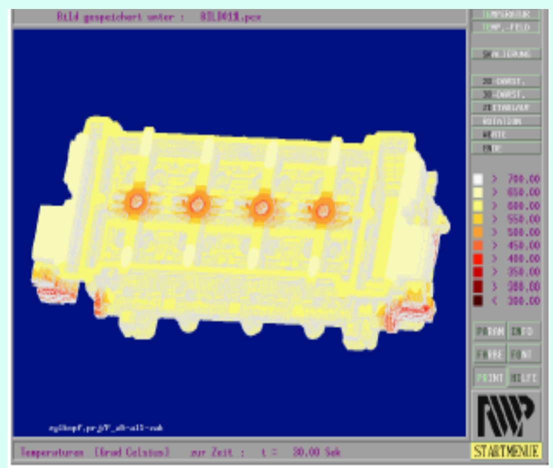


1a. Mechanical Engineering Applications

- Mechanical stresses and deformations
- Contact stresses
- Plasticity
- Thermal stresses







A FEM enmeshment has a good volume and surface approximation even with a small number of elements.

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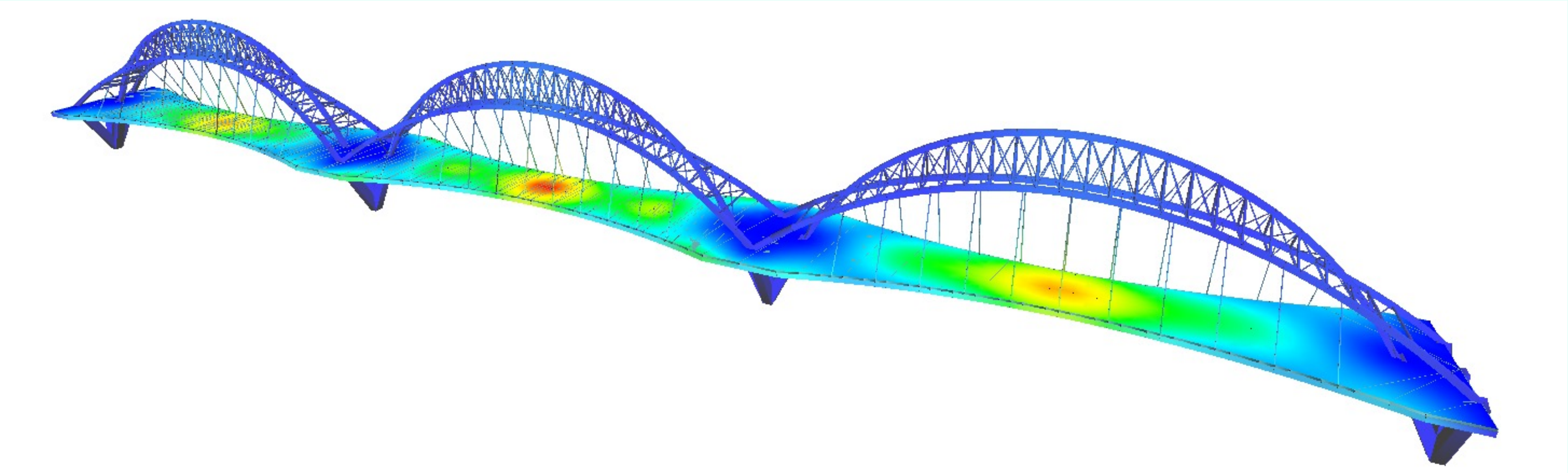
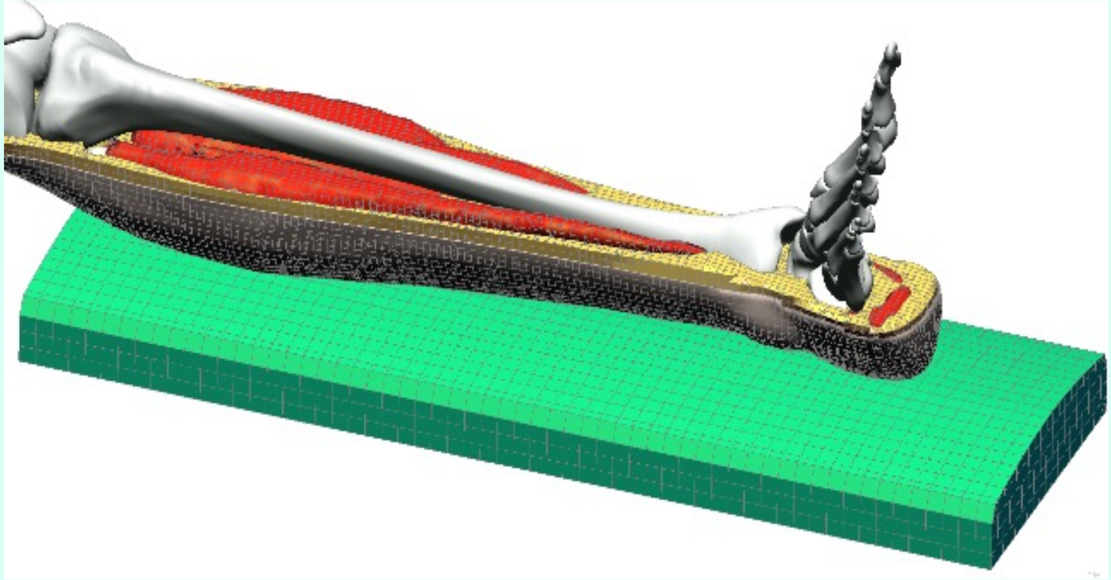
1a. Civil Engineering Applications

- Steel and concrete
- Beams, columns, shells, plates
- Soil and rock (geotechnics)



1a. Biomechanical Engineering Applications

- Soft tissues
- Prosthetics/implants
- Sports equipment
- Large deformations
- Interaction with fluids



1b. Introduction to the basis of FEM: the Weighted Residual Method

Weighted residual methods generate approximate solutions to PDEs.

This is the basis of the general FEM.

As an example, consider the ordinary differential equation

$$\frac{du}{dx} + u = 0 \qquad \text{for } 0 \leq x \geq 1$$

Subject to boundary condition

$$u(0) = 1$$

The exact solution is

$$u(x) = e^{-x}$$

1b. One Degree of Freedom (DOF) Approximation

- We will use two different weighted residual methods to generate approximations to the exact solution:
  - (i) Least squares error method
  - (ii) Galerkin method
- **Shape function** – propose a form for the approximate solution…
- The simplest approximation to $u(x)$ is a straight line

$$\tilde{u}(x) = 1 + cx$$

Approximate solution for $u(x)$

To satisfy boundary condition

Constant c is to be determined to give best approximation

- The unknown variable c is called the **degree of freedom** (DOF) of the system**.**

1b. Residual error

- As an example, consider the ordinary differential equation

$$\frac{du}{dx} + u = 0 \qquad \text{for } 0 \leq x \geq 1$$

Subject to boundary condition

$$u(0) = 1$$

- **Shape function** – propose a form for the approximate solution…
- The simplest approximation to $u(x)$ is a straight line

$$\tilde{u}(x) = 1 + cx$$

- Define the residual error $R(x)$ to be the difference between the exact value of the PDE (zero in this case) and the approximated value. In this example

$$R(x) = \left(\frac{d\tilde{u}}{dx} + \tilde{u}\right) - \cancel{\left(\frac{du}{dx} + u\right)} = \left(\frac{d\tilde{u}}{dx} + \tilde{u}\right)$$

- If the approximation is correct $(\tilde{u} = u)$ then the error $R(x) = 0$.
- For the 1 DOF approximation function

$$R(x) = \frac{d}{dx}(1 + cx) + (1 + cx) = c + 1 + cx$$

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1b. (i) Least Squares Method

- To find the best value of c minimise the sum of all the errors squared R^2
- Total error

$$I = \frac{1}{2}\int_0^1 R^2\, dx$$

- Minimum error occurs when

$$\frac{dI}{dc} = 0$$

with respect to the DOF c such that

$$\frac{dI}{dc} = \frac{1}{2}\int_0^1 \frac{d(R^2)}{dc}\, dx$$

$$\frac{dI}{dc} = \frac{1}{2}\int_0^1 \frac{2RdR}{dc} dx$$

$$\frac{dI}{dc} = \int_0^1 \frac{dR}{dc} R dx = 0 \qquad \text{Eq 1.1}$$

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1b. (i) Least Squares Method (continued)

- In this example

$$\frac{dR}{dc} = 1 + x$$

$$\frac{dI}{dc} = \int_0^1 (1 + c + cx)(1 + x)dx = 0$$

$$\frac{dI}{dc} = c\int_0^1 (1 + x)^2\,dx + (1 + x)dx = 0$$

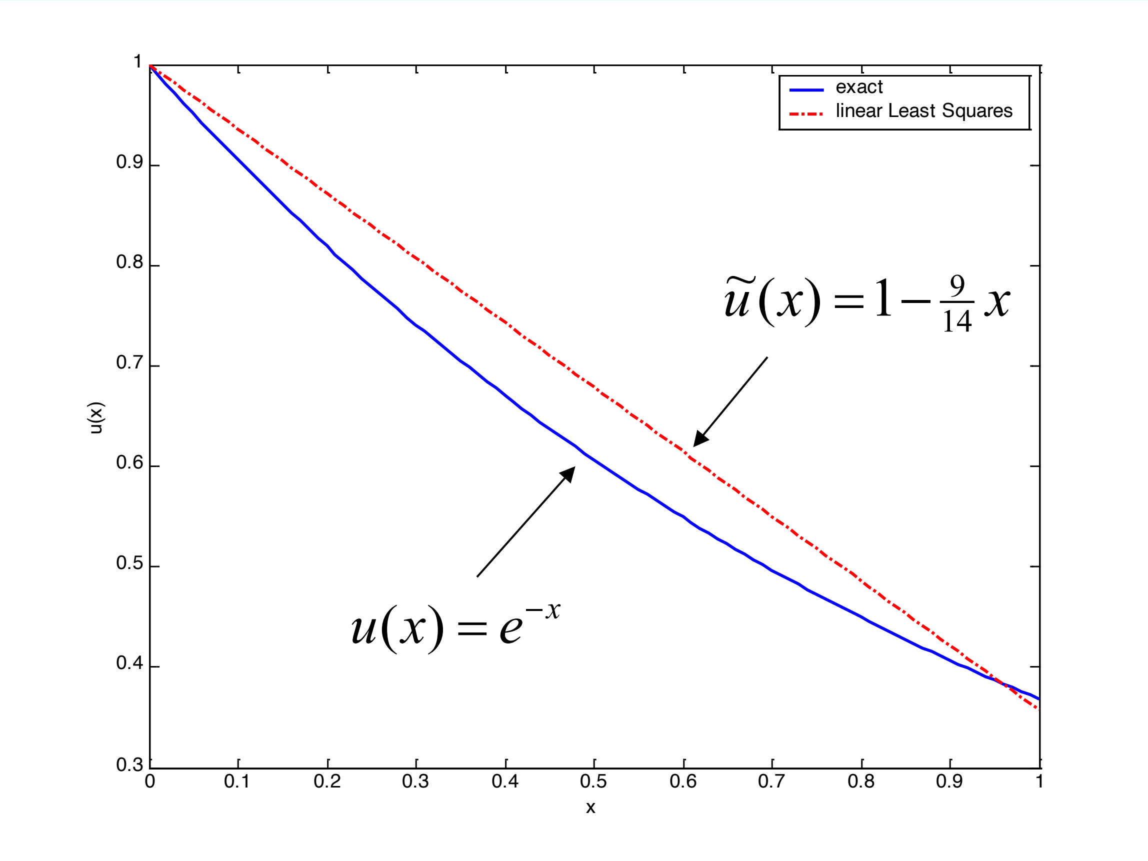
$$\frac{dI}{dc} = c\left[\frac{(1 + x)^3}{3}\right]\begin{matrix}1\\0\end{matrix} + \left[\frac{(1 + x)^2}{2}\right]\begin{matrix}1\\0\end{matrix}$$

$$\frac{dI}{dc} = c\left(\frac{8}{3} - \frac{1}{3}\right) + \left(2 - \frac{1}{2}\right)$$

$$\frac{dI}{dc} = \frac{7c}{3} + \frac{3}{2} = 0$$

$$c = -\frac{3}{3} \times \frac{3}{7} = -\frac{9}{14}$$

1b. (i) Least Squares Method
Linear (1 DOF) approximation vs exact solution



1b. (ii) Galerkin Method

- The Galerkin method is the basis of the FEM and is a generalisation of Eq 1.1

$$\int_0^1 w(x)R\,dx = 0$$

  Where $w(x)$ is a weighting function

- For the least squares method

$$w(x) = \frac{dR}{dc}$$

- For the Galerkin method

$$w(x) = \frac{d\tilde{u}}{dc}$$

1b. (ii) Galerkin Method (continued)

- In this example, here we write condition for optimal degree of freedom DOF as

$$\int_0^1 \frac{d\tilde{u}}{dc} R dx = 0$$

- We replace $(\frac{dR}{dc}\ by\ \frac{d\tilde{u}}{dc})$
- We know

$$\tilde{u}(x) = 1 + cx$$

- Gives

$$\frac{d\tilde{u}}{dc} = x$$

$$\frac{d\tilde{u}}{dc} = \int_0^1 x(1 + c + cx)dx = 0$$

$$\frac{d\tilde{u}}{dc} = c\int_0^1 x(1 + x)dx + \int_0^1 x\ dx = 0$$

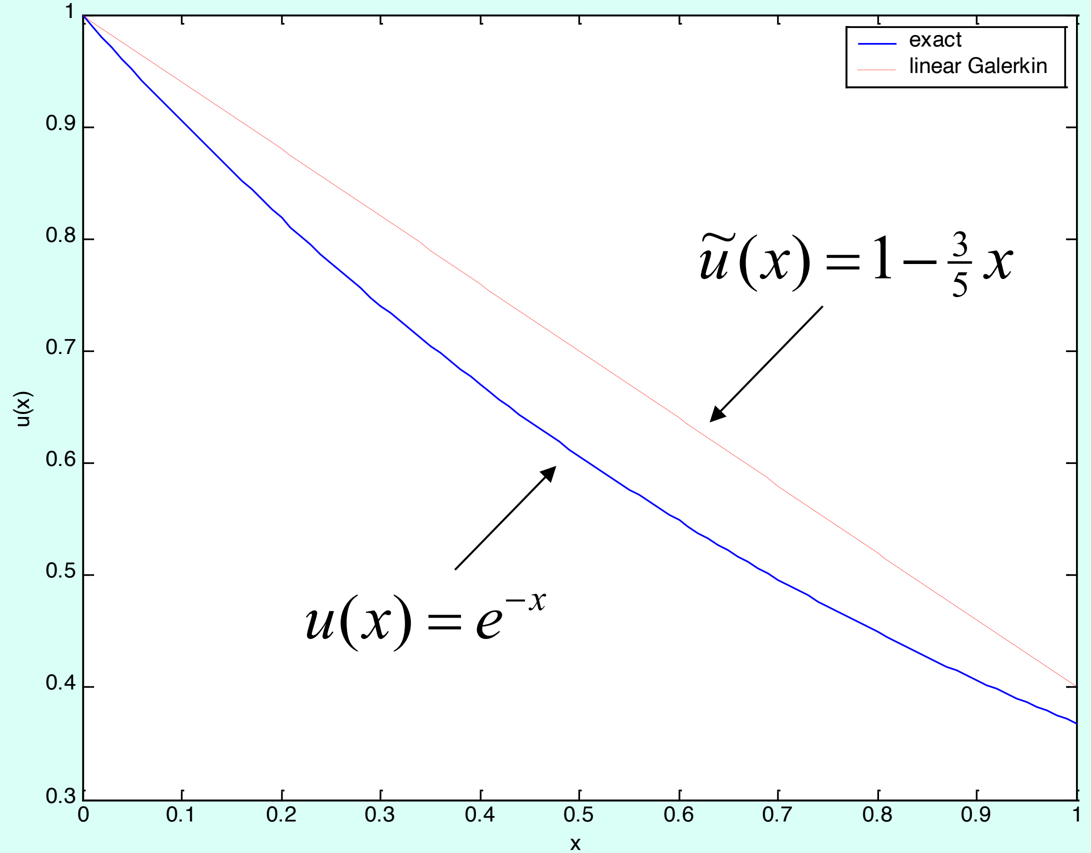
1b. (ii) Galerkin Method (continued)

$$\frac{d\tilde{u}}{dc} = c\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = 0$$

$$\frac{d\tilde{u}}{dc} = \frac{5c}{6} + \frac{1}{2} = 0$$

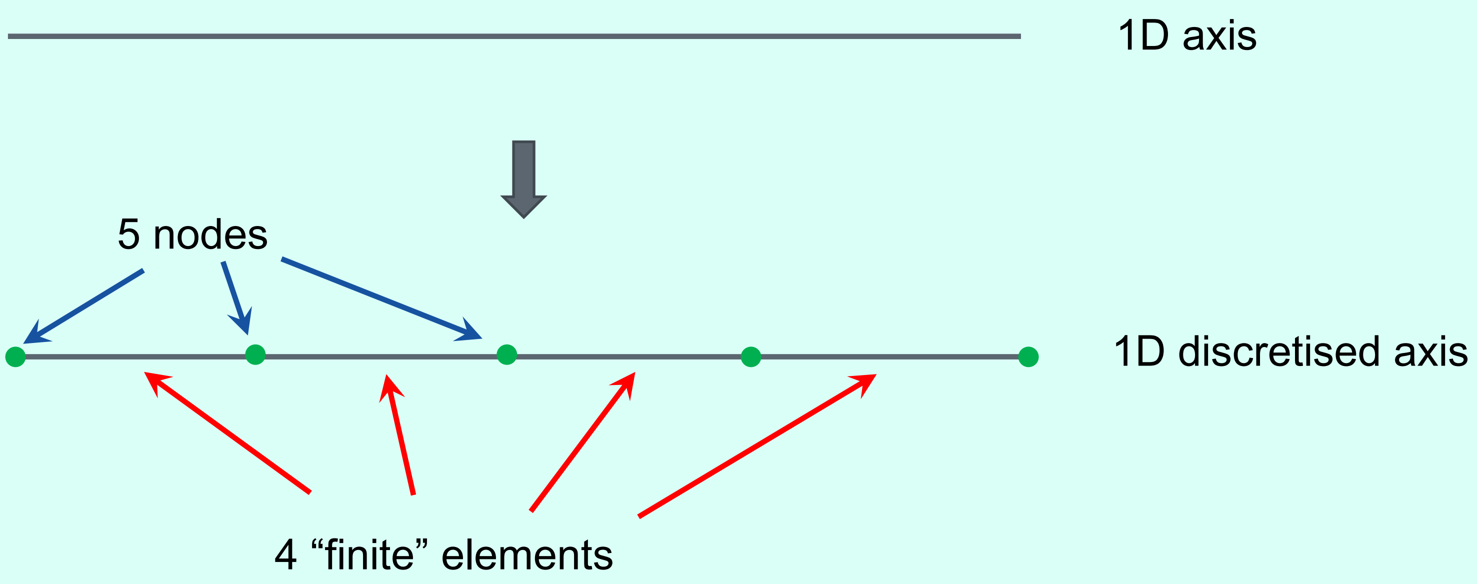
$$c = -\frac{1}{2} \times \frac{6}{5} = -\frac{3}{15}$$

1b. (ii) Galerkin
Linear (1 DOF) approximation vs exact solution



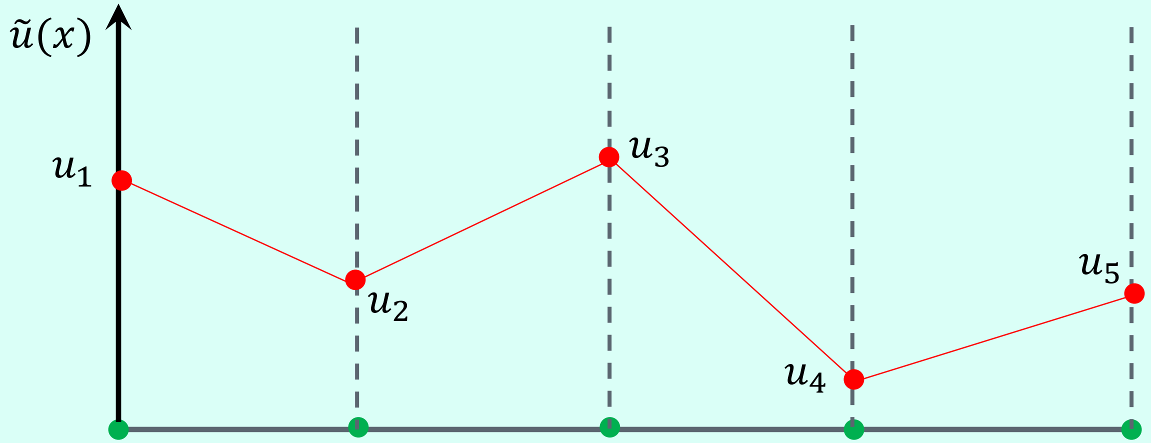
1b. Discretisation, shape functions, nodes and elements

- In general, the FEM discretises a body into elements



1b. Linear shape functions

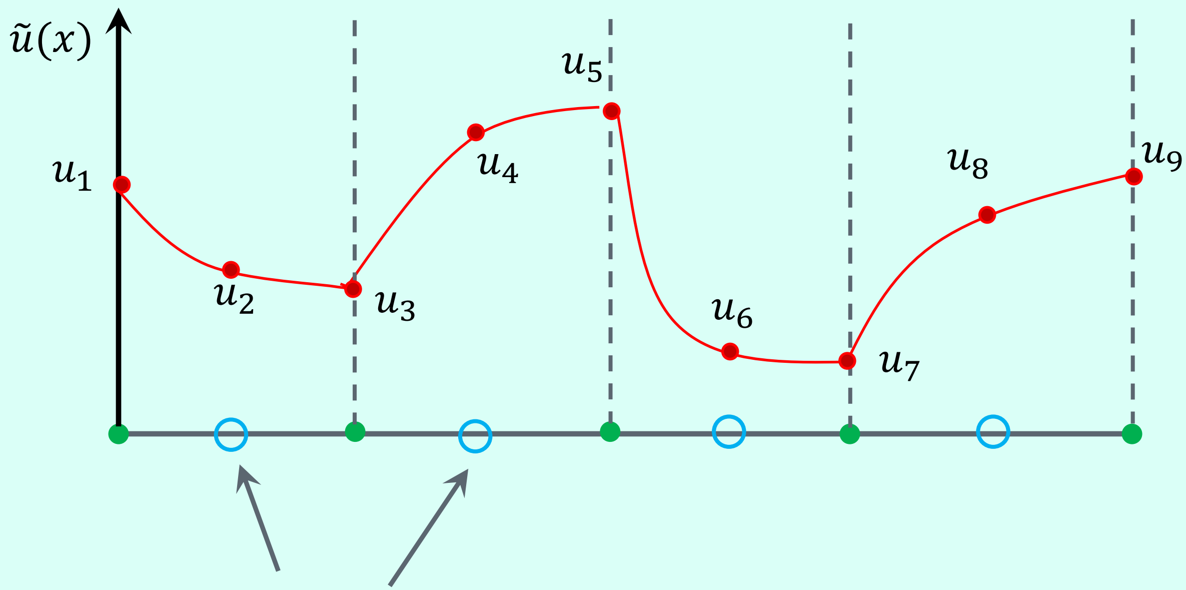
- The degrees of freedom (DOF) are the values at the nodes ($u_1, \ldots, u_5$) to ensure continuity of the approximation function $\tilde{u}(x)$ between elements



5 DOF

1b. Quadratic shape functions

Quadratics need three values per element to define them, so introduce another DOF at mid-nodes



COMSOL has:

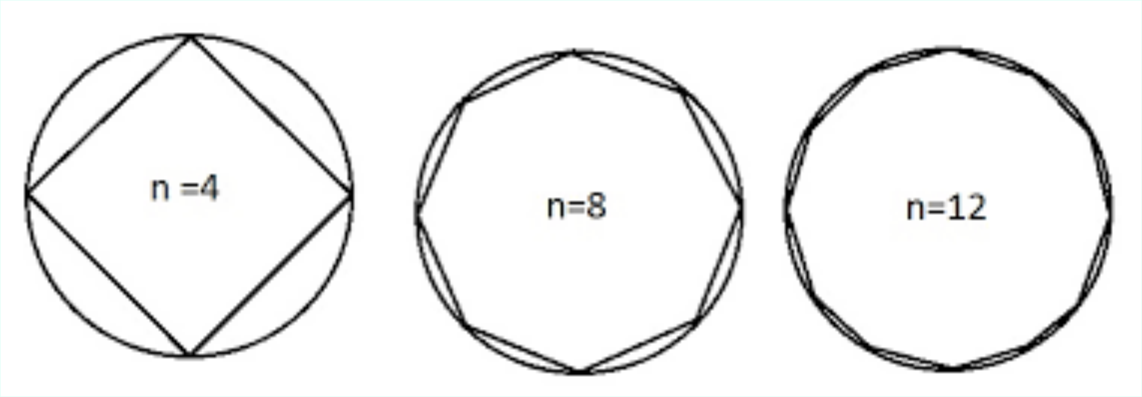
- Linear
- Quadratic
- Cubic
- Quartic
- Quintic

9 DOF

1b. Discretisation and accuracy

See solutions to Exercise sheet #1

- Increasing the number of DOF increases the accuracy of the solution
- Two ways to increase the DOF:
  - Increase the number of elements, i.e. decrease the size of the elements
  - Increase the order of the shape function, i.e. go from linear (1 DOF per element) to quadratic (2 DOF per element).



Approximating a circle with linear shape functions

Increasing DOF

Example #1: quadratic shape function

Use Galerkins method to find the approximate solution when the shape function is a second order polynomial (i.e. a quadratic curve rather than a linear curve).

$$\tilde{u}(x) = 1 + c_1 x + c_2 x^2$$

As there are now **2 DOF** (c_1 and c_2) there are also two simultaneous equations to solve

$$\int_0^1 \frac{\partial \tilde{u}}{\partial c_1} R dx = 0$$
$$\int_0^1 \frac{\partial \tilde{u}}{\partial c_2} R dx = 0$$

Plot the result against the exact solution for comparison

Exercise sheet #1

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Example #2: 2 piecewise linear shape functions

Use Galerkins method to find the approximate solution when the shape function is two straight lines such that

$$\tilde{u}(x) = \begin{cases} 1 + 2(u_1 - 1)x & for\ 0 \le x \le \frac{1}{2} \\ (2u_1 - u_2) + 2(u_2 - u_1)x & for\ \frac{1}{2} \le x \le 1 \end{cases}$$

There are now **two DOF** (u_1 and u_2) and hence two simultaneous equations to solve

$$\int_0^1 \frac{\partial \tilde{u}}{\partial u_1} R dx = 0$$

$$\int_0^1 \frac{\partial \tilde{u}}{\partial u_2} R dx = 0$$

Exercise sheet #1

Plot the result against the exact solution for comparison

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Next section…
(2) FEM and Elasticity Theory